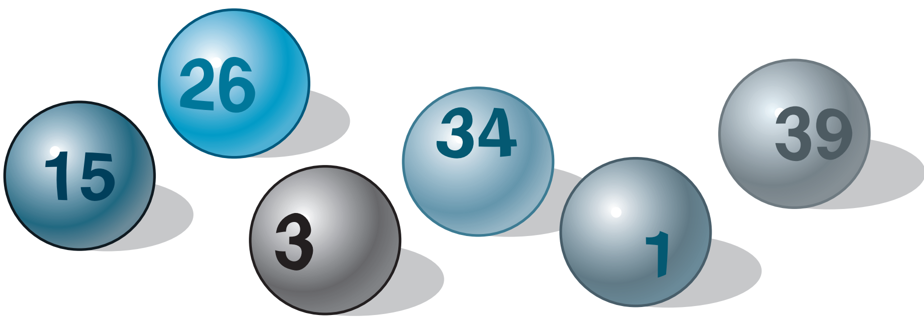
CHAPTER THREE

Interest Rate and Economic Equivalence

No Lump Sum for Lottery-Winner Grandma, 94[1] A judge denied a 94-year-old woman's attempt to force the Massachusetts Lottery Commission to pay her entire $5.6 million winnings up front on the grounds that she otherwise won't live long enough to collect it all. The ruling means that the commission can pay Louise Outing, a retired waitress, in installments over 20 years. After an initial gross payment of $283,770, Outing would be paid 19 annual gross checks of $280,000. That's about $197,000 after taxes. Lottery Executive Director Joseph Sullivan said all players are held to the same rules, which are printed on the back of Megabucks tickets. Lottery winners are allowed to "assign" their winnings to a state-approved financial company that makes the full payment—but only in return for a percentage of the total winnings. Outing, who won a Megabucks drawing in September, has seven grandchildren, nine great-grandchildren, and six great-great-grandchildren. "I'd like to get it and do what I want with it," she said. "I'm not going to live 20 years. I'll be 95 in March."



[1] "No Lump Sum for Lottery-Winner Grandma, 94," The Associated Press, December 30, 2004.

The next time you play a lottery, look at the top section of the play slip. You will see two boxes: "Cash Value" and "Annual Payments." You need to mark one of the boxes before you fill out the rest of the slip. If you don't, and you win the jackpot, you will automatically receive the jackpot as annual payments. That is what happened to Ms. Louise Outing. If you mark the "Cash Value box" and you win, you will receive the present cash value of the announced jackpot in one lump sum. This amount will be less than the announced jackpot. With the announced jackpot of $5.6 million, Ms. Outing could receive about 52.008%, or $2.912 million, in one lump sum (less withholding tax). This example is based on average market costs as of January 2005 of 20 annual payments funded by the U.S. Treasury Zero Coupon Bonds (or a 7.2% coupon rate). With this option, you can look forward to a large cash payment up front.

First, most people familiar with investments would tell Ms. Outing that receiving a lump amount of $2.912 million today is likely to prove a far better deal than receiving $280,000 a year for 20 years, even if the grandma lives long enough to collect the entire annual payments.
After losing the court appeal, Ms. Outing was able to find a buyer for her lottery in a lump-sum amount of $2.467 million. To arrive at that price, the buyer calculated the return he wanted to earn—at that time about 9.5% interest, compounded annually—and applied that rate in reverse to the $5.6 million he stood to collect over 20 years. The buyer says the deals he strikes with winners applies a basic tenet of all financial transactions, the **time value of money**: A dollar in hand today is worth more than one that will be paid to you in the future.

In engineering economics, the principles discussed in this chapter are regarded as the underpinning for nearly all project investment analysis. This is because we always need to account for the effect of interest operating on sums of cash over time. Interest formulas allow us to place different cash flows received at different times in the same time frame and to compare them. As will become apparent, almost our entire study of engineering economics is built on the principles introduced in this chapter.

CHAPTER LEARNING OBJECTIVES

After completing this chapter, you should understand the following concepts:

- The time value of money.
- The difference between simple interest and the compound interest.
- The meaning of economic equivalence and why we need it in economic analysis.
- How to compare two different money series by means of the concept of economic equivalence.
- The interest operation and the types of interest formulas used to facilitate the calculation of economic equivalence.

3.1 Interest: The Cost of Money

Most of us are familiar in a general way with the concept of interest. We know that money left in a savings account earns interest, so that the balance over time is greater than the sum of the deposits. We also know that borrowing to buy a car means repaying an amount over time, that that amount includes interest, and that it is therefore greater than the amount borrowed. What may be unfamiliar to us is the idea that, in the financial world, money itself is a commodity and, like other goods that are bought and sold, money costs money.

Market interest rate: Interest rate quoted by financial institutions.

The cost of money is established and measured by a **market interest rate**, a percentage that is periodically applied and added to an amount (or varying amounts) of money over a specified length of time. When money is borrowed, the interest paid is the charge to the borrower for the use of the lender's property; when money is lent or invested, the interest earned is the lender's gain from providing a good to another (Figure 3.1). **Interest**, then, may be defined as the cost of having money available for use. In this section, we examine how interest operates in a free-market economy and we establish a basis for understanding the more complex interest relationships that follow later on in the chapter.

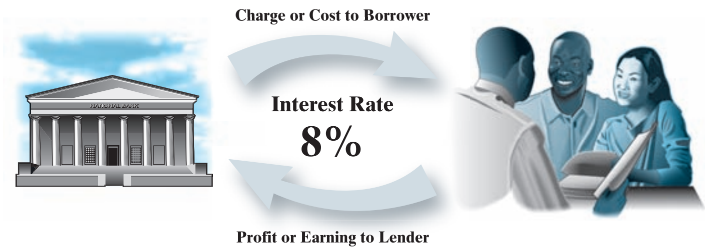


Figure 3.1 The meaning of *interest rate* to the lender (bank) and to the borrower.

	Account Value	Cost of Refrigerator
Case 1: Inflation exceeds earning power	$N = 0$ \$100 $N = 1$ \$106 (earning rate = 6%)	$N = 0$ \$100 $N = 1$ \$108 (inflation rate = 8%)
Case 2: Earning power exceeds inflation	$N = 0$ \$100 $N = 1$ \$106 (earning rate = 6%)	$N = 0$ \$100 $N = 1$ \$104 (inflation rate = 4%)

Figure 3.2 Gains achieved or losses incurred by delaying consumption.

3.1.1 The Time Value of Money

The "time value of money" seems like a sophisticated concept, yet it is a concept that you grapple with every day. Should you buy something today or save your money and buy it later? Here is a simple example of how your buying behavior can have varying results: Pretend you have \$100, and you want to buy a \$100 refrigerator for your dorm room. If you buy it now, you are broke. Suppose that you can invest money at 6% interest, but the price of the refrigerator increases only at an annual rate of 4% due to inflation. In a year you can still buy the refrigerator, and you will have \$2 left over. Well, if the price of the refrigerator increases at an annual rate of 8% instead, you will not have enough money (you will be \$2 short) to buy the refrigerator a year from now. In that case, you probably are better off buying the refrigerator now. The situation is summarized in Figure 3.2.

The time value of money: The idea that a dollar today is worth more than a dollar in the future because the dollar received today can earn interest.

Clearly, the rate at which you earn interest should be higher than the inflation rate to make any economic sense of the delayed purchase. In other words, in an inflationary economy, your purchasing power will continue to decrease as you further delay the purchase of the refrigerator. In order to make up this future loss in purchasing power, your earning interest rate should be sufficiently larger than the anticipated inflation rate. After all, time, like money, is a finite resource. There are only 24 hours in a day, so time has to be budgeted, too. What this example illustrates is that we must connect the "earning power" and the "purchasing power" to the concept of time.

When we deal with large amounts of money, long periods of time, or high interest rates, the change in the value of a sum of money over time becomes extremely significant. For example, at a current annual interest rate of 10%, \$1 million will earn \$100,000 in interest in a year; thus, to wait a year to receive \$1 million clearly involves a significant sacrifice. When deciding among alternative proposals, we must take into account the operation of interest and the time value of money in order to make valid comparisons of different amounts at various times.

The way interest operates reflects the fact that money has a time value. This is why amounts of interest depend on lengths of time; interest rates, for example, are typically given in terms of a percentage per year. We may define the principle of the time value of money as follows: The economic value of a sum depends on when it is received. Because money has both **earning** as well as **purchasing power** over time, as shown in Figure 3.3 (it can be put to work, earning more money for its owner), a dollar received today has a greater value than a dollar received at some future time.

Purchasing power: The value of a currency expressed in terms of the amount of goods or services that one unit of money can buy.

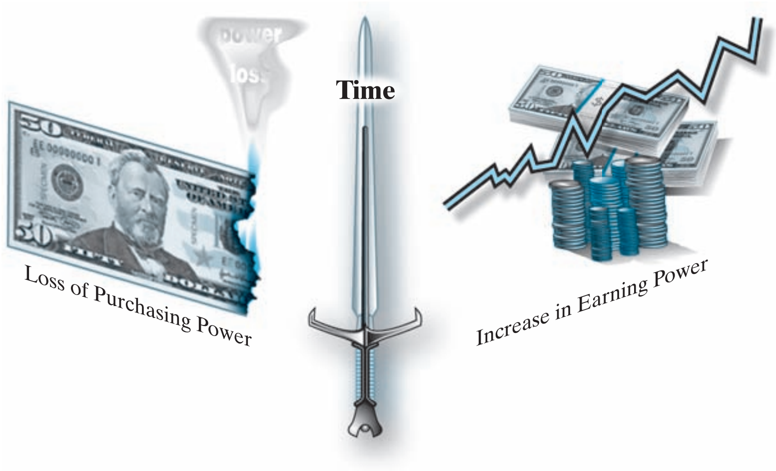


Figure 3.3 The time value of money. This is a two-edged sword whereby earning grows, but purchasing power decreases, as time goes by.

When lending or borrowing interest rates are quoted by financial institutions on the marketplace, those interest rates reflect the desired amounts to be earned, as well as any protection from loss in the future purchasing power of money because of inflation. (If we want to know the true desired earnings in isolation from inflation, we can determine the real interest rate. We consider this issue in Chapter 11. The earning power of money and its loss of value because of inflation are calculated by different analytical techniques.) In the meantime, we will assume that, unless otherwise mentioned, *the interest rate used in this book reflects the market interest rate,* which takes into account the earning power, as well as the effect of inflation perceived in the marketplace. We will also assume that all cash flow transactions are given in terms of **actual dollars**, with the effect of inflation, if any, reflected in the amount.

Actual dollars: The cash flow measured in terms of the dollars at the time of the transaction.

3.1.2 Elements of Transactions Involving Interest

Many types of transactions (e.g., borrowing or investing money or purchasing machinery on credit) involve interest, but certain elements are common to all of these types of transactions:

- An initial amount of money in transactions involving debt or investments is called the **principal**.
- The **interest rate** measures the cost or price of money and is expressed as a percentage per period of time.
- A period of time, called the **interest period**, determines how frequently interest is calculated. (Note that even though the length of time of an interest period can vary, interest rates are frequently quoted in terms of an annual percentage rate. We will discuss this potentially confusing aspect of interest in Chapter 4.)
- A specified length of time marks the duration of the transaction and thereby establishes a certain **number of interest periods**.
- A **plan for receipts or disbursements** yields a particular cash flow pattern over a specified length of time. (For example, we might have a series of equal monthly payments that repay a loan.)
- A **future amount of money** results from the cumulative effects of the interest rate over a number of interest periods.

For the purposes of calculation, these elements are represented by the following variables:

A_n = A discrete payment or receipt occurring at the end of some interest period.
i = The interest rate per interest period.
N = The total number of interest periods.
P = A sum of money at a time chosen as time zero for purposes of analysis; sometimes referred to as the **present value** or **present worth**.
F = A future sum of money at the end of the analysis period. This sum may be specified as F_N.
A = An end-of-period payment or receipt in a uniform series that continues for N periods. This is a special situation where $A_1 = A_2 = \cdots = A_N$.
V_n = An equivalent sum of money at the end of a specified period n that considers the effect of the time value of money. Note that $V_0 = P$ and $V_N = F$.

Present value: The amount that a future sum of money is worth today, given a specified rate of return.

Because frequent use of these symbols will be made in this text, it is important that you become familiar with them. Note, for example, the distinction between A, A_n, and A_N. The symbol A_n refers to a specific payment or receipt, at the end of period n, in any series of payments. A_N is the final payment in such a series, because N refers to the total number of interest periods. A refers to any series of cash flows in which all payments or receipts are equal.

Example of an Interest Transaction

As an example of how the elements we have just defined are used in a particular situation, let us suppose that an electronics manufacturing company buys a machine for $25,000 and borrows $20,000 from a bank at a 9% annual interest rate. In addition, the company pays a $200 loan origination fee when the loan commences. The bank offers two repayment plans, one with equal payments made at the end of every year for the next five years, the other with a single payment made after the loan period of five years. These two payment plans are summarized in Table 3.1.

- In Plan 1, the principal amount P is $20,000, and the interest rate i is 9%. The interest period is one year, and the duration of the transaction is five years, which means there are five interest periods $(N = 5)$. It bears repeating that whereas one year is a common interest period, interest is frequently calculated at other intervals: monthly, quarterly, or

TABLE 3.1 **Repayment Plans for Example Given in Text (for $N = 5$ years and $i = 9\%$)**

		Payments	
End of Year	**Receipts**	**Plan 1**	**Plan 2**
Year 0	$20,000.00	$ 200.00	$ 200.00
Year 1		5,141.85	0
Year 2		5,141.85	0
Year 3		5,141.85	0
Year 4		5,141.85	0
Year 5		5,141.85	30,772.48

$P = \$20,000$, $A = \$5,141.85$, $F = \$30,772.48$

Note: You actually borrow $19,800 with the origination fee of $200, but you pay back on the basis of $20,000.

semiannually, for instance. For this reason, we used the term **period** rather than **year** when we defined the preceding list of variables. The receipts and disbursements planned over the duration of this transaction yield a cash flow pattern of five equal payments A of $5,141.85 each, paid at year's end during years 1 through 5. (You'll have to accept these amounts on faith for now—the next section presents the formula used to arrive at the amount of these equal payments, given the other elements of the problem.)

- Plan 2 has most of the elements of Plan 1, except that instead of five equal repayments, we have a grace period followed by a single future repayment F of $30,772.78.

Cash Flow Diagrams

Problems involving the time value of money can be conveniently represented in graphic form with a cash flow diagram (Figure 3.4). **Cash flow diagrams** represent time by a horizontal line marked off with the number of interest periods specified. The cash flows over time are represented by arrows at relevant periods: Upward arrows denote positive flows (receipts), downward arrows negative flows (disbursements). Note, too, that the arrows actually represent **net cash flows**: Two or more receipts or disbursements made at the same time are summed and shown as a single arrow. For example, $20,000 received during the same period as a $200 payment would be recorded as an upward arrow of $19,800. Also, the lengths of the arrows can suggest the relative values of particular cash flows.

Cash flow diagrams function in a manner similar to free-body diagrams or circuit diagrams, which most engineers frequently use: Cash flow diagrams give a convenient summary of all the important elements of a problem, as well as a reference point to determine whether the statement of the problem has been converted into its appropriate parameters. The text frequently uses this graphic tool, and you are strongly encouraged to develop the habit of using well-labeled cash flow diagrams as a means to identify and summarize pertinent information in a cash flow problem. Similarly, a table such as Table 3.1 can help you organize information in another summary format.

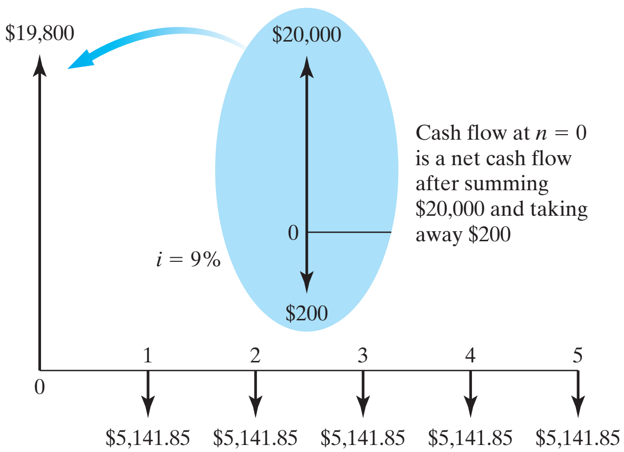


Figure 3.4 A cash flow diagram for Plan 1 of the loan repayment example summarized in Table 3.1.

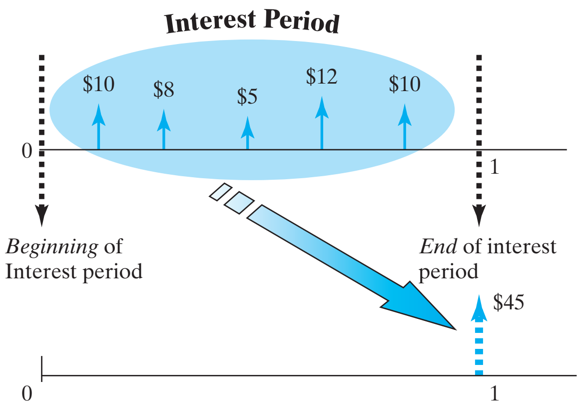


Figure 3.5 Any cash flows occurring during the interest period are summed to a single amount and placed at the end of the interest period.

End-of-Period Convention

In practice, cash flows can occur at the beginning or in the middle of an interest period—or indeed, at practically any point in time. One of the simplifying assumptions we make in engineering economic analysis is the **end-of-period convention**, which is the practice of placing all cash flow transactions at the end of an interest period. (See Figure 3.5.) This assumption relieves us of the responsibility of dealing with the effects of interest within an interest period, which would greatly complicate our calculations.

End-of-period convention: Unless otherwise mentioned, all cash flow transactions occur at the end of an interest period.

It is important to be aware of the fact that, like many of the simplifying assumptions and estimates we make in modeling engineering economic problems, the end-of-period convention inevitably leads to some discrepancies between our model and real-world results.

Suppose, for example, that $100,000 is deposited during the first month of the year in an account with an interest period of one year and an interest rate of 10% per year. In such a case, the difference of 1 month would cause an interest income loss of $10,000. This is because, under the end-of-period convention, the $100,000 deposit made during the interest period is viewed as if the deposit were made at the end of the year, as opposed to 11 months earlier. This example gives you a sense of why financial institutions choose interest periods that are less than one year, even though they usually quote their rate as an annual percentage.

Armed with an understanding of the basic elements involved in interest problems, we can now begin to look at the details of calculating interest.

3.1.3 Methods of Calculating Interest

Money can be lent and repaid in many ways, and, equally, money can earn interest in many different ways. Usually, however, at the end of each interest period, the interest earned on the principal amount is calculated according to a specified interest rate. The two computational schemes for calculating this earned interest are said to yield either **simple interest** or **compound interest**. Engineering economic analysis uses the compound-interest scheme almost exclusively.

Simple interest: The interest rate is applied only to the original principal amount in computing the amount of interest.

Simple Interest

Simple interest is interest earned on only the principal amount during each interest period. In other words, with simple interest, the interest earned during each interest period does not earn additional interest in the remaining periods, *even though you do not withdraw it*.

In general, for a deposit of P dollars at a simple interest rate of i for N periods, the total earned interest would be

$$I = (iP)N. \tag{3.1}$$

The total amount available at the end of N periods thus would be

$$F = P + I = P(1 + iN). \tag{3.2}$$

Simple interest is commonly used with add-on loans or bonds. (See Chapter 4.)

Compound: The ability of an asset to generate *earnings* that are then reinvested and generate their own earnings.

Compound Interest

Under a compound-interest scheme, the interest earned in each period is calculated on the basis of the total amount at the end of the previous period. This total amount includes the original principal plus the accumulated interest that has been left in the account. In this case, you are, in effect, increasing the deposit amount by the amount of interest earned. In general, if you deposited (invested) P dollars at interest rate i, you would have $P + iP = P(1 + i)$ dollars at the end of one period. If the entire amount (principal and interest) is reinvested at the same rate i for another period, at the end of the second period you would have

$$\begin{aligned} P(1 + i) + i[P(1 + i)] &= P(1 + i)(1 + i) \\ &= P(1 + i)^2. \end{aligned}$$

Continuing, we see that the balance after the third period is

$$P(1 + i)^2 + i[P(1 + i)^2] = P(1 + i)^3.$$

This interest-earning process repeats, and after N periods the total accumulated value (balance) F will grow to

$$F = P(1 + i)^N. \tag{3.3}$$

EXAMPLE 3.1 Compound Interest

Suppose you deposit $1,000 in a bank savings account that pays interest at a rate of 10% compounded annually. Assume that you don't withdraw the interest earned at the end of each period (one year), but let it accumulate. How much would you have at the end of year 3?

SOLUTION

Given: $P = \$1{,}000$, $N = 3$ years, and $i = 10\%$ per year.
Find: F.

Applying Eq. (3.3) to our three-year, 10% case, we obtain

$$F = \$1{,}000(1 + 0.10)^3 = \$1{,}331.$$

The total interest earned is \$331, which is \$31 more than was accumulated under the simple-interest method (Figure 3.6). We can keep track of the interest accruing process more precisely as follows:

Period	Amount at Beginning of Interest Period	Interest Earned for Period	Amount at End of Interest Period
1	\$1,000	\$1,000(0.10)	\$1,100
2	1,100	1,100(0.10)	1,210
3	1,210	1,210(0.10)	1,331

COMMENTS: At the end of the first year, you would have \$1,000, plus \$100 in interest, or a total of \$1,100. In effect, at the beginning of the second year, you would be depositing \$1,100, rather than \$1,000. Thus, at the end of the second year, the interest earned would be $0.10(\$1{,}100) = \$110$, and the balance would be $\$1{,}100 + \$110 = \$1{,}210$. This is the amount you would be depositing at the beginning of the third year, and the interest earned for that period would be $0.10(\$1{,}210) = \$121$. With a beginning principal amount of \$1,210 plus the \$121 interest, the total balance would be \$1,331 at the end of year 3.

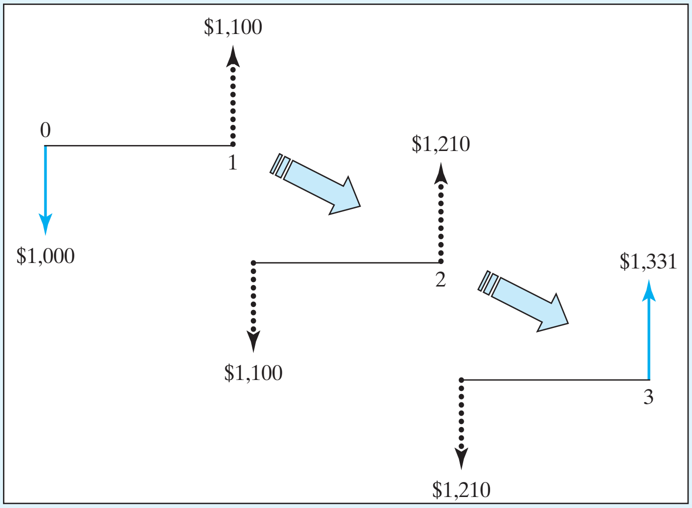


Figure 3.6 The process of computing the balance when \$1,000 at 10% is deposited for three years (Example 3.1).

3.1.4 Simple Interest versus Compound Interest

From Eq. (3.3), the total interest earned over N periods is

$$I = F - P = P[(1 + i)^N - 1]. \tag{3.4}$$

Compared with the simple-interest scheme, the additional interest earned with compound interest is

$$\Delta I = P[(1 + i)^N - 1] - (iP)N \tag{3.5}$$

$$= P[(1 + i)^N - (1 + iN)]. \tag{3.6}$$

As either i or N becomes large, the difference in interest earnings also becomes large, so the effect of compounding is further pronounced. Note that, when $N = 1$, compound interest is the same as simple interest.

Using Example 3.1, we can illustrate the difference between compound interest and the simple interest. Under the simple-interest scheme, you earn interest only on the principal amount at the end of each interest period. Under the compounding scheme, you earn interest on the principal, as well as interest on interest.

Figure 3.7 illustrates the fact that compound interest is a sum of simple interests earned on the original principal, as well as periodic simple interests earned on a series of simple interests.

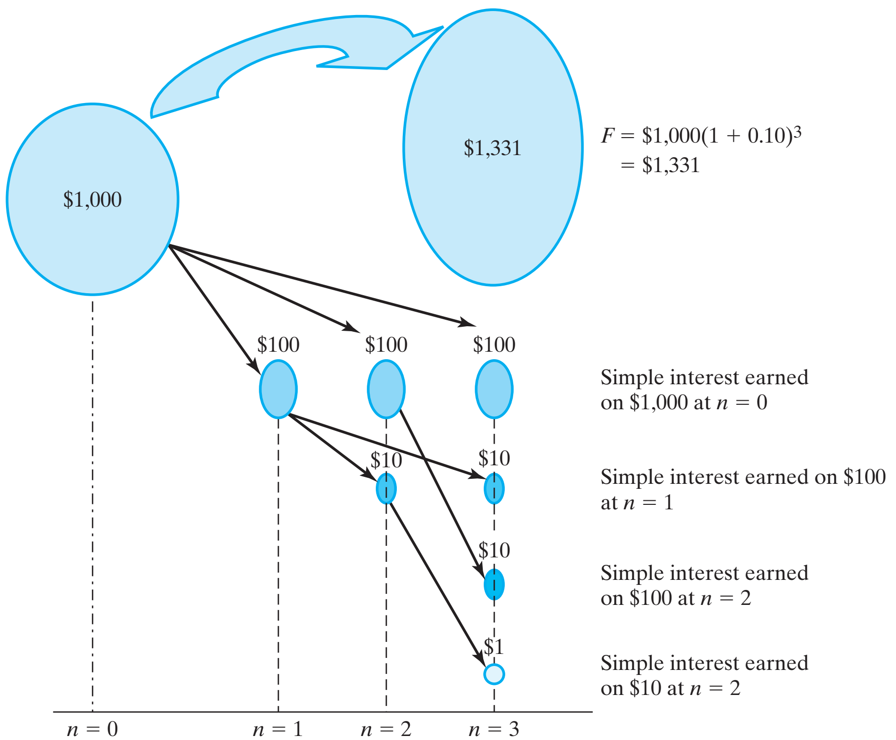


Figure 3.7 The relationship between simple interest and compound interest.

EXAMPLE 3.2 Comparing Simple with Compound Interest

In 1626, Peter Minuit of the Dutch West India Company paid \$24 to purchase Manhattan Island in New York from the Indians. In retrospect, if Minuit had invested the \$24 in a savings account that earned 8% interest, how much would it be worth in 2007?

SOLUTION

Given: $P = \$24$, $i = 8\%$ per year, and $N = 381$ years.
Find: F, based on (a) 8% simple interest and (b) 8% compound interest.

(a) With 8% simple interest,

$$F = \$24[1 + (0.08)(381)] = \$755.52.$$

(b) With 8% compound interest,

$$F = \$24(1 + 0.08)^{381} = \$130{,}215{,}319{,}909{,}015.$$

COMMENTS: The significance of compound interest is obvious in this example. Many of us can hardly comprehend the magnitude of \$130 trillion. In 2007, the total population in the United States was estimated to be around 300 million. If the money were distributed equally among the population, each individual would receive \$434,051. Certainly, there is no way of knowing exactly how much Manhattan Island is worth today, but most real-estate experts would agree that the value of the island is nowhere near \$130 trillion. (Note that the U.S. national debt as of December 31, 2007, was estimated to be \$9.19 trillion.)

3.2 Economic Equivalence

The observation that money has a time value leads us to an important question: If receiving \$100 today is not the same thing as receiving \$100 at any future point, how do we measure and compare various cash flows? How do we know, for example, whether we should prefer to have \$20,000 today and \$50,000 ten years from now, or \$8,000 each year for the next ten years? In this section, we describe the basic analytical techniques for making these comparisons. Then, in Section 3.3, we will use these techniques to develop a series of formulas that can greatly simplify our calculations.

3.2.1 Definition and Simple Calculations

The central question in deciding among alternative cash flows involves comparing their economic worth. This would be a simple matter if, in the comparison, we did not need to consider the time value of money: We could simply add the individual payments within a cash flow, treating receipts as positive cash flows and payments (disbursements) as negative cash flows. The fact that money has a time value, however, makes our calculations more complicated. We need to know more than just the size of a payment in order to determine its economic effect completely. In fact, as we will see in this section, we need to know several things:

- The magnitude of the payment.
- The direction of the payment: Is it a receipt or a disbursement?
- The timing of the payment: When is it made?
- The interest rate in operation during the period under consideration.

It follows that, to assess the economic impact of a series of payments, we must consider the impact of each payment individually.

Economic equivalence: The process of comparing two different cash amounts at different points in time.

Calculations for determining the economic effects of one or more cash flows are based on the concept of economic equivalence. **Economic equivalence** exists between cash flows that have the same economic effect and could therefore be traded for one another in the financial marketplace, which we assume to exist.

Economic equivalence refers to the fact that a cash flow—whether a single payment or a series of payments—can be converted to an *equivalent* cash flow at any point in time. For example, we could find the equivalent future value F of a present amount P at interest rate i at period n; or we could determine the equivalent present value P of N equal payments A.

The preceding strict concept of equivalence, which limits us to converting a cash flow into another equivalent cash flow, may be extended to include the comparison of alternatives. For example, we could compare the value of two proposals by finding the equivalent value of each at any common point in time. If financial proposals that appear to be quite different turn out to have the same monetary value, then we can be *economically indifferent* to choosing between them: In terms of economic effect, one would be an even exchange for the other, so no reason exists to prefer one over the other in terms of their economic value.

A way to see the concepts of equivalence and economic indifference at work in the real world is to note the variety of payment plans offered by lending institutions for consumer loans. Table 3.2 extends the example we developed earlier to include three different repayment plans for a loan of $20,000 for five years at 9% interest. You will notice, perhaps to your surprise, that the three plans require significantly different repayment patterns and

TABLE 3.2 Typical Repayment Plans for a Bank Loan of $20,000 (for $N = 5$ years and $i = 9\%$)

	Repayments		
	Plan 1	Plan 2	Plan 3
Year 1	$ 5,141.85	0	$ 1,800.00
Year 2	5,141.85	0	1,800.00
Year 3	5,141.85	0	1,800.00
Year 4	5,141.85	0	1,800.00
Year 5	5,141.85	$30,772.48	21,800.00
Total of payments	$25,709.25	$30,772.48	$29,000.00
Total interest paid	$ 5,709.25	$10,772.48	$ 9,000.00

Plan 1: Equal annual installments; Plan 2: End-of-loan-period repayment of principal and interest; Plan 3: Annual repayment of interest and end-of-loan repayment of principal

different total amounts of repayment. However, because money has a time value, these plans are equivalent, and economically, the bank is indifferent to a consumer's choice of plan. We will now discuss how such equivalence relationships are established.

Equivalence Calculations: A Simple Example

Equivalence calculations can be viewed as an application of the compound-interest relationships we developed in Section 3.1. Suppose, for example, that we invest $1,000 at 12% annual interest for five years. The formula developed for calculating compound interest, $F = P(1 + i)^N$ (Eq. 3.3), expresses the equivalence between some present amount P and a future amount F, for a given interest rate i and a number of interest periods N. Therefore, at the end of the investment period, our sums grow to

$$\$1{,}000(1 + 0.12)^5 = \$1{,}762.34.$$

Thus, we can say that at 12% interest, $1,000 received now is equivalent to $1,762.34 received in five years and that we could trade $1,000 now for the promise of receiving $1,762.34 in five years. Example 3.3 further demonstrates the application of this basic technique.

EXAMPLE 3.3 Equivalence

Suppose you are offered the alternative of receiving either $3,000 at the end of five years or P dollars today. There is no question that the $3,000 will be paid in full (no risk). Because you have no current need for the money, you would deposit the P dollars in an account that pays 8% interest. What value of P would make you indifferent to your choice between P dollars today and the promise of $3,000 at the end of five years?

STRATEGY: Our job is to determine the present amount that is economically equivalent to $3,000 in five years, given the investment potential of 8% per year. Note that the statement of the problem assumes that you would exercise the option of using the earning power of your money by depositing it. The "indifference" ascribed to you refers to economic indifference; that is, in a marketplace where 8% is the applicable interest rate, you could trade one cash flow for the other.

SOLUTION

Given: $F = \$3{,}000$, $N = 5$ years, and $i = 8\%$ per year.

Find: P.

Equation: Eq. (3.3), $F = P(1 + i)^N$.
Rearranging terms to solve for P gives

$$P = \frac{F}{(1 + i)^N}.$$

Substituting yields

$$P = \frac{\$3{,}000}{(1 + 0.08)^5} = \$2{,}042.$$

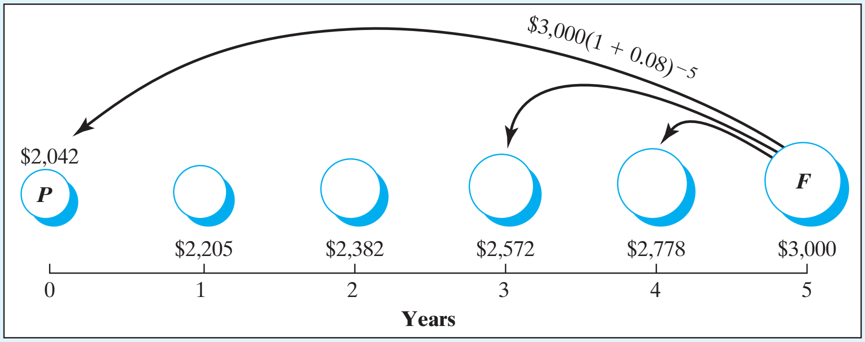


Figure 3.8 Various dollar amounts that will be economically equivalent to $3,000 in five years, given an interest rate of 8% (Example 3.3).

We summarize the problem graphically in Figure 3.8.

COMMENTS: In this example, it is clear that if P is anything less than $2,042, you would prefer the promise of $3,000 in five years to P dollars today; if P is greater than $2,042, you would prefer P. As you may have already guessed, at a lower interest rate, P must be higher to be equivalent to the future amount. For example, at $i = 4\%$, $P = \$2,466$.

3.2.2 Equivalence Calculations: General Principles

In spite of their numerical simplicity, the examples we have developed reflect several important general principles, which we will now explore.

Principle 1: Equivalence Calculations Made to Compare Alternatives Require a Common Time Basis

Common base period: To establish an economic equivalence between two cash flow amounts, a common base period must be selected.

Just as we must convert fractions to common denominators to add them together, we must also convert cash flows to a common basis to compare their value. One aspect of this basis is the choice of a single point in time at which to make our calculations. In Example 3.3, if we had been given the magnitude of each cash flow and had been asked to determine whether they were equivalent, we could have chosen any reference point and used the compound interest formula to find the value of each cash flow at that point. As you can readily see, the choice of $n = 0$ or $n = 5$ would make our problem simpler because we need to make only one set of calculations: At 8% interest, either convert $2,042 at time 0 to its equivalent value at time 5, or convert $3,000 at time 5 to its equivalent value at time 0. (To see how to choose a different reference point, take a look at Example 3.4.)

When selecting a point in time at which to compare the value of alternative cash flows, we commonly use either the present time, which yields what is called the **present worth** of the cash flows, or some point in the future, which yields their **future worth**. The choice of the point in time often depends on the circumstances surrounding a particular decision, or it may be chosen for convenience. For instance, if the present worth is known for the first two of three alternatives, all three may be compared simply by calculating the present worth of the third.

EXAMPLE 3.4 Equivalent Cash Flows Are Equivalent at Any Common Point in Time

In Example 3.3, we determined that, given an interest rate of 8% per year, receiving \$2,042 today is equivalent to receiving \$3,000 in five years. Are these cash flows also equivalent at the end of year 3?

STRATEGY: This problem is summarized in Figure 3.9. The solution consists of solving two equivalence problems: (1) What is the future value of \$2,042 after three years at 8% interest (part (a) of the solution)? (2) Given the sum of \$3,000 after five years and an interest rate of 8%, what is the equivalent sum after 3 years (part (b) of the solution)?

SOLUTION

Given:

(a) $P = \$2{,}042$; $i = 8\%$ per year; $N = 3$ years.
(b) $F = \$3{,}000$; $i = 8\%$ per year; $N = 5 - 3 = 2$ years.

Find: (1) V_3 for part (a); (2) V_3 for part (b). (3) Are these two values equivalent?

Equation:

(a) $F = P(1 + i)^N$.
(b) $P = F(1 + i)^{-N}$.

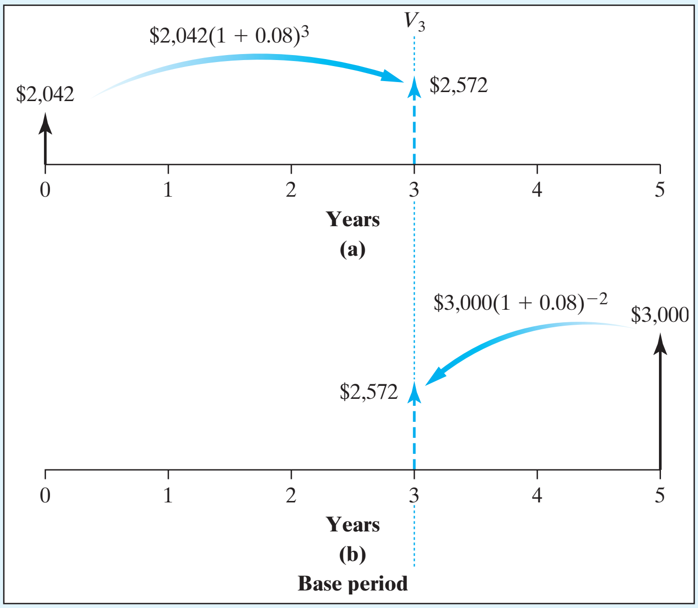


Figure 3.9 Selection of a base period for an equivalence calculation (Example 3.4).

Notation: The usual terminology of F and P is confusing in this example, since the cash flow at $n = 3$ is considered a future sum in part (a) of the solution and a past cash flow in part (b) of the solution. To simplify matters, we are free to arbitrarily designate a reference point $n = 3$ and understand that it need not to be now or the present. Therefore, we assign the equivalent cash flow at $n = 3$ to a single variable, V_3.

1. The equivalent worth of \$2,042 after three years is

$$\begin{aligned} V_3 &= 2{,}042(1 + 0.08)^3 \\ &= \$2{,}572. \end{aligned}$$

2. The equivalent worth of the sum \$3,000 two years earlier is

$$\begin{aligned} V_3 &= F(1 + i)^{-N} \\ &= \$3{,}000(1 + 0.08)^{-2} \\ &= \$2{,}572. \end{aligned}$$

(Note that $N = 2$ because that is the number of periods during which discounting is calculated in order to arrive back at year 3.)

3. While our solution doesn't strictly prove that the two cash flows are equivalent at any time, they will be equivalent at any time as long as we use an interest rate of 8%.

Principle 2: Equivalence Depends on Interest Rate

The equivalence between two cash flows is a function of the magnitude and timing of individual cash flows and the interest rate or rates that operate on those flows. This principle is easy to grasp in relation to our simple example: \$1,000 received now is equivalent to \$1,762.34 received five years from now only at a 12% interest rate. Any change in the interest rate will destroy the equivalence between these two sums, as we will demonstrate in Example 3.5.

EXAMPLE 3.5 Changing the Interest Rate Destroys Equivalence

In Example 3.3, we determined that, given an interest rate of 8% per year, receiving \$2,042 today is equivalent to receiving \$3,000 in five years. Are these cash flows equivalent at an interest rate of 10%?

SOLUTION

Given: $P = \$2{,}042$, $i = 10\%$ per year, and $N = 5$ years.
Find: F: Is it equal to \$3,000?

We first determine the base period under which an equivalence value is computed. Since we can select any period as the base period, let's select $N = 5$. Then we need to calculate the equivalent value of \$2,042 today five years from now.

$$F = \$2{,}042(1 + 0.10)^5 = \$3{,}289.$$

Since this amount is greater than \$3,000, the change in interest rate destroys the equivalence between the two cash flows.

Principle 3: Equivalence Calculations May Require the Conversion of Multiple Payment Cash Flows to a Single Cash Flow

In all the examples presented thus far, we have limited ourselves to the simplest case of converting a single payment at one time to an equivalent single payment at another time. Part of the task of comparing alternative cash flow series involves moving each individual cash flow in the series to the same single point in time and summing these values to yield a single equivalent cash flow. We perform such a calculation in Example 3.6.

EXAMPLE 3.6 Equivalence Calculations with Multiple Payments

Suppose that you borrow \$1,000 from a bank for three years at 10% annual interest. The bank offers two options: (1) repaying the interest charges for each year at the end of that year and repaying the principal at the end of year 3 or (2) repaying the loan all at once (including both interest and principal) at the end of year 3. The repayment schedules for the two options are as follows:

Options	Year 1	Year 2	Year 3
• Option 1: End-of-year repayment of interest, and principal repayment at end of loan	\$100	\$100	\$1,100
• Option 2: One end-of-loan repayment of both principal and interest	0	0	1,331

Determine whether these options are equivalent, assuming that the appropriate interest rate for the comparison is 10%.

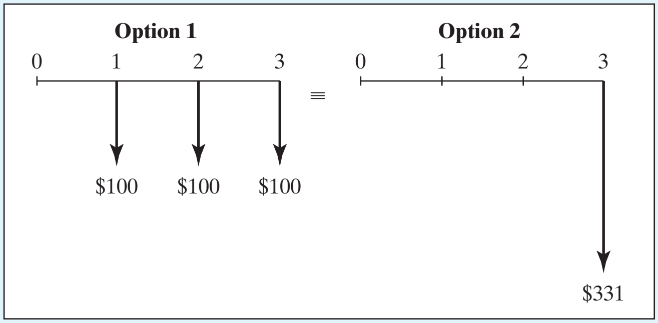
STRATEGY: Since we pay the principal after three years in either plan, the repayment of principal can be removed from our analysis. This is an important point: *We can ignore the common elements of alternatives being compared so that we can focus entirely on comparing the interest payments*. Notice that under Option 1, we will pay a total of \$300 interest, whereas under Option 2, we will pay a total of \$331. Before concluding that we prefer Option 2, remember that a comparison of the two cash flows is based on a *combination of payment amounts and the timing of those payments*. To make our comparison, we must compare the equivalent value of each option at a single point in time. Since Option 2 is already a single payment at $n = 3$ years, it is simplest to convert the cash flow pattern of Option 1 to a single value at $n = 3$. To do this, we must convert the three disbursements of Option 1 to their respective equivalent values at $n = 3$. At that point, since they share a time in common, we can simply sum them in order to compare them with the \$331 sum in Option 2.

SOLUTION

Given: Interest payment series; $i = 10\%$ per year.

Find: A single future value F of the flows in Option 1.

Equation: $F = P(1 + i)^N$, applied to each disbursement in the cash flow diagram. N in Eq. (3.3) is the number of interest periods during which interest is in effect, and n is the period number (i.e., for year 1, $n = 1$). We determine the value of F by finding the interest period for each payment. Thus, for each payment in the series, N can be calculated by subtracting n from the total number of years of the loan (3). That is, $N = 3 - n$. Once the value of each payment has been found, we sum the payments:



$$F_3 \text{ for } \$100 \text{ at } n = 1 : \$100(1 + .10)^{3-1} = \$121;$$
$$F_3 \text{ for } \$100 \text{ at } n = 2 : \$100(1 + .10)^{3-2} = \$110;$$
$$F_3 \text{ for } \$100 \text{ at } n = 3 : \$100(1 + .10)^{3-3} = \underline{\$100};$$
$$\text{Total} = \$331.$$

By converting the cash flow in Option 1 to a single future payment at year 3, we can compare Options 1 and 2. We see that the two interest payments are equivalent. Thus, the bank would be economically indifferent to a choice between the two plans. Note that the final interest payment in Option 1 does not accrue any compound interest.

Principle 4: Equivalence Is Maintained Regardless of Point of View

As long as we use the same interest rate in equivalence calculations, equivalence can be maintained regardless of point of view. In Example 3.6, the two options were equivalent at an interest rate of 10% from the banker's point of view. What about from a borrower's point of view? Suppose you borrow \$1,000 from a bank and deposit it in another bank that pays 10% interest annually. Then you make future loan repayments out of this savings account. Under Option 1, your savings account at the end of year 1 will show a balance of \$1,100 after the interest earned during the first period has been credited. Now you withdraw \$100 from this savings account (the exact amount required to pay the loan interest during the first year), and you make the first-year interest payment to the bank. This leaves only \$1,000 in your savings account. At the end of year 2, your savings account will earn another interest payment in the amount of $\$1{,}000(0.10) = \$100$, making an end-of-year balance of \$1,100. Now you withdraw another \$100 to make the required loan interest payment. After this payment, your remaining balance will be \$1,000. This balance will grow again at 10%, so you will have \$1,100 at the end of year 3. After making the last loan payment (\$1,100), you will have no money left in either account. For Option 2, you can keep track of the yearly account balances in a similar fashion. You will find that you reach a zero balance after making the lump-sum payment of \$1,331. If the borrower had used the same interest rate as the bank, the two options would be equivalent.

3.2.3 Looking Ahead

The preceding examples should have given you some insight into the basic concepts and calculations involved in the concept of economic equivalence. Obviously, the variety of financial arrangements possible for borrowing and investing money is extensive, as is the variety of time-related factors (e.g., maintenance costs over time, increased productivity over time, etc.) in alternative proposals for various engineering projects. It is important to recognize that even the most complex relationships incorporate the basic principles we have introduced in this section.

In the remainder of the chapter, we will represent all cash flow diagrams either in the context of an initial deposit with a subsequent pattern of withdrawals or in an initial borrowed amount with a subsequent pattern of repayments. If we were limited to the methods developed in this section, a comparison between the two payment options would involve a large number of calculations. Fortunately, in the analysis of many transactions, certain cash flow patterns emerge that may be categorized. For many of these patterns, we can derive formulas that can be used to simplify our work. In Section 3.3, we develop these formulas.

3.3 Development of Interest Formulas

Now that we have established some working assumptions and notations and have a preliminary understanding of the concept of equivalence, we will develop a series of interest formulas for use in more complex comparisons of cash flows.

As we begin to compare series of cash flows instead of single payments, the required analysis becomes more complicated. However, when patterns in cash flow transactions can be identified, we can take advantage of these patterns by developing concise expressions for computing either the present or future worth of the series. We will classify five major categories of cash flow transactions, develop interest formulas for them, and present several working examples of each type. Before we give the details, however, we briefly describe the five types of cash flows in the next subsection.

3.3.1 The Five Types of Cash Flows

Whenever we identify patterns in cash flow transactions, we may use those patterns to develop concise expressions for computing either the present or future worth of the series. For this purpose, we will classify cash flow transactions into five categories: (1) a single cash flow, (2) a uniform series, (3) a linear gradient series, (4) a geometric gradient series, and (5) an irregular series. To simplify the description of various interest formulas, we will use the following notation:

1. **Single Cash Flow:** The simplest case involves the equivalence of a single present amount and its future worth. Thus, the single-cash-flow formulas deal with only two amounts: a single present amount P and its future worth F (Figure 3.10a). You have already seen the derivation of one formula for this situation in Section 3.1.3, which gave us Eq. (3.3):

$$F = P(1 + i)^N.$$

2. **Equal (Uniform) Series:** Probably the most familiar category includes transactions arranged as a series of equal cash flows at regular intervals, known as an **equal payment series** (or **uniform series**) (Figure 3.10b). For example, this category describes the cash flows of the common installment loan contract, which arranges the repayment of a loan in equal periodic installments. The equal-cash-flow formulas deal with the equivalence relations P, F, and A (the constant amount of the cash flows in the series).
3. **Linear Gradient Series:** While many transactions involve series of cash flows, the amounts are not always uniform; they may, however, vary in some regular way. One common pattern of variation occurs when each cash flow in a series increases (or decreases) by a fixed amount (Figure 3.10c). A five-year loan repayment plan might specify, for example, a series of annual payments that increase by $500 each year. We call this type of cash flow pattern a **linear gradient series** because its cash flow diagram produces an ascending (or descending) straight line, as you will see in Section 3.3.5. In addition to using P, F, and A, the formulas employed in such problems involve a *constant amount* G of the change in each cash flow.
4. **Geometric Gradient Series:** Another kind of gradient series is formed when the series in a cash flow is determined not by some fixed amount like $500, but by some fixed *rate*, expressed as a percentage. For example, in a five-year financial plan for a project, the cost of a particular raw material might be budgeted to increase at a rate of 4% per year. The curving gradient in the diagram of such a series suggests its name: a **geometric gradient series** (Figure 3.10d). In the formulas dealing with such series, the rate of change is represented by a lowercase g.
5. **Irregular (Mixed) Series:** Finally, a series of cash flows may be irregular, in that it does not exhibit a regular overall pattern. Even in such a series, however, one or more of the patterns already identified may appear over segments of time in the total length of the series. The cash flows may be equal, for example, for 5 consecutive periods in a 10-period series. When such patterns appear, the formulas for dealing with them may be applied and their results included in calculating an equivalent value for the entire series.

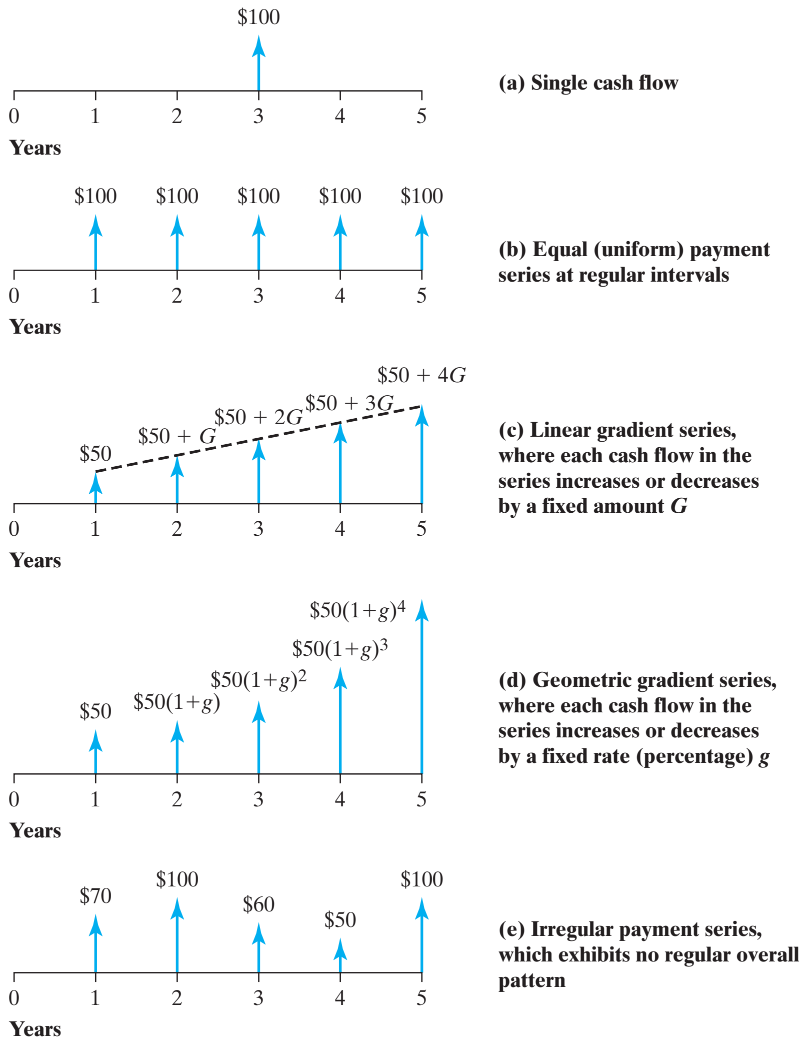


Figure 3.10 Five types of cash flows: (a) Single cash flow, (b) equal (uniform) payment series, (c) linear gradient series, (d) geometric gradient series, and (e) irregular payment series.

3.3.2 Single-Cash-Flow Formulas

We begin our coverage of interest formulas by considering the simplest of cash flows: single cash flows.

Compound Amount Factor

Given a present sum P invested for N interest periods at interest rate i, what sum will have accumulated at the end of the N periods? You probably noticed right away that this description matches the case we first encountered in describing compound interest. To solve for F (the future sum), we use Eq. (3.3):

$$F = P(1 + i)^N.$$

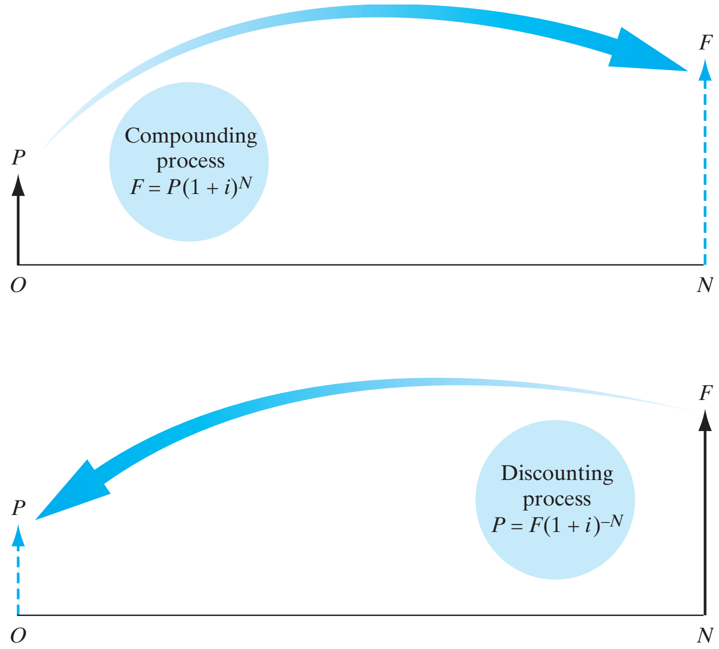


Figure 3.11 Equivalence relation between P and F.

Because of its origin in the compound-interest calculation, the factor $(1 + i)^N$ is known as the **compound-amount factor**. Like the concept of equivalence, this factor is one of the foundations of engineering economic analysis. Given the compound-amount factor, all the other important interest formulas can be derived.

Compounding process: the process of computing the future value of a current sum.

This process of finding F is often called the **compounding process**. The cash flow transaction is illustrated in Figure 3.11. (Note the time-scale convention: The first period begins at $n = 0$ and ends at $n = 1$.) If a calculator is handy, it is easy enough to calculate $(1 + i)^N$ directly.

Interest Tables

Interest formulas such as the one developed in Eq. (3.3), $F = P(1 + i)^N$, allow us to substitute known values from a particular situation into the equation and to solve for the unknown. Before the hand calculator was developed, solving these equations was very tedious. With a large value of N, for example, one might need to solve an equation such as $F = \$20{,}000(1 + 0.12)^{15}$. More complex formulas required even more involved calculations. To simplify the process, tables of compound-interest factors were developed, and these tables allow us to find the appropriate factor for a given interest rate and the number of interest periods. Even with hand calculators, it is still often convenient to use such tables, and they are included in this text in Appendix A. Take some time now to become familiar with their arrangement and, if you can, locate the compound-interest factor for the example just presented, in which we know P. Remember that, to find F, we need to know the factor by which to multiply \$20,000 when the interest rate i is 12% and the number of periods is 15:

$$F = \$20{,}000\underbrace{(1 + 0.12)^{15}}_{5.4736} = \$109{,}472.$$

Factor Notation

As we continue to develop interest formulas in the rest of this chapter, we will express the resulting compound-interest factors in a conventional notation that can be substituted in a formula to indicate precisely which table factor to use in solving an equation. In the preceding example, for instance, the formula derived as Eq. (3.3) is $F = P(1 + i)^N$. In ordinary language, this tells us that, to determine what future amount F is equivalent to a present amount P, we need to multiply P by a factor expressed as 1 plus the interest rate, raised to the power given by the number of interest periods. To specify how the interest tables are to be used, we may also express that factor in functional notation as $(F/P, i, N)$, which is read as "Find F, Given P, i, and N." This is known as the **single-payment compound-amount factor**. When we incorporate the table factor into the formula, it is expressed as

$$F = P(1 + i)^N = P(F/P, i, N).$$

Thus, in the preceding example, where we had $F = \$20{,}000(1.12)^{15}$, we can write $F = \$20{,}000(F/P, 12\%, 15)$. The table factor tells us to use the 12% interest table and find the factor in the F/P column for $N = 15$. Because using the interest tables is often the easiest way to solve an equation, this factor notation is included for each of the formulas derived in the sections that follow.

EXAMPLE 3.7 Single Amounts: Find *F*, Given *i*, *N*, and *P*

If you had \$2,000 now and invested it at 10%, how much would it be worth in eight years (Figure 3.12)?

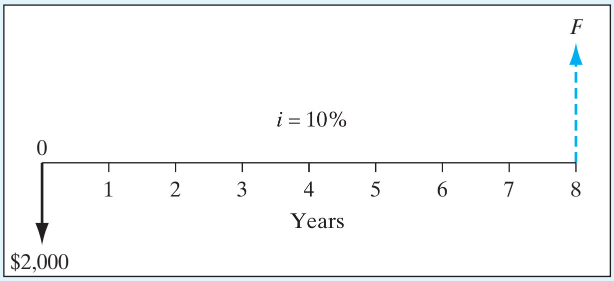


Figure 3.12 A cash flow diagram from the investor's point of view (Example 3.7).

SOLUTION

Given: $P = \$2{,}000$, $i = 10\%$ per year, and $N = 8$ years.
Find: F.

We can solve this problem in any of three ways:

1. **Using a calculator.** You can simply use a calculator to evaluate the $(1 + i)^N$ term (financial calculators are preprogrammed to solve most future-value problems):

$$F = \$2{,}000(1 + 0.10)^8$$
$$= \$4{,}287.18.$$

2. **Using compound-interest tables.** The interest tables can be used to locate the compound-amount factor for $i = 10\%$ and $N = 8$. The number you get can be substituted into the equation. Compound-interest tables are included as Appendix A of this book. From the tables, we obtain

$$F = \$2{,}000(F/P, 10\%, 8) = \$2{,}000(2.1436) = \$4{,}287.20.$$

This is essentially identical to the value obtained by the direct evaluation of the single-cash-flow compound-amount factor. This slight difference is due to rounding errors.

3. **Using Excel.** Many financial software programs for solving compound-interest problems are available for use with personal computers. Excel provides financial functions to evaluate various interest formulas, where the future-worth calculation looks like the following:

=FV(10%,8,0,−2000)

Present-Worth Factor

Discounting process: A process of calculating the present value of a future amount.

Finding the present worth of a future sum is simply the reverse of compounding and is known as the **discounting process**. In Eq. (3.3), we can see that if we were to find a present sum P, given a future sum F, we simply solve for P:

$$P = F\left[\frac{1}{(1 + i)^N}\right] = F(P/F, i, N). \tag{3.7}$$

The factor $1/(1 + i)^N$ is known as the **single-payment present-worth factor** and is designated $(P/F, i, N)$. Tables have been constructed for P/F factors and for various values of i and N. The interest rate i and the P/F factor are also referred to as the **discount rate** and **discounting factor**, respectively.

EXAMPLE 3.8 Single Amounts: Find *P*, Given *F*, *i*, and *N*

Suppose that $1,000 is to be received in five years. At an annual interest rate of 12%, what is the present worth of this amount?

SOLUTION

Given: $F = \$1{,}000$, $i = 12\%$ per year, and $N = 5$ years.
Find: P.

$$P = \$1{,}000(1 + 0.12)^{-5} = \$1{,}000(0.5674) = \$567.40.$$

Using a calculator may be the best way to make this simple calculation. To have $1,000 in your savings account at the end of five years, you must deposit $567.40 now.

We can also use the interest tables to find that

$$P = \$1{,}000\overbrace{(P/F, 12\%, 5)}^{(0.5674)} = \$567.40.$$

Again, you could use a financial calculator or a computer to find the present worth. With Excel, the present-value calculation looks like the following:

=PV(12%,5,0,−1000)

Solving for Time and Interest Rates

At this point, you should realize that the compounding and discounting processes are reciprocals of one another and that we have been dealing with one equation in two forms:

$$\text{Future-value form: } F = P(1 + i)^N;$$
$$\text{Present-value form: } P = F(1 + i)^{-N}.$$

There are four variables in these equations: P, F, N, and i. If you know the values of any three, you can find the value of the fourth. Thus far, we have always given you the interest rate i and the number of years N, plus either P or F. In many situations, though, you will need to solve for i or N, as we discuss next.

EXAMPLE 3.9 Solving for *i*

Suppose you buy a share for $10 and sell it for $20. Then your profit is $10. If that happens within a year, your rate of return is an impressive 100% ($10/$10 = 1). If it takes five years, what would be the average annual rate of return on your investment? (See Figure 3.13.)

SOLUTION

Given: $P = \$10$, $F = \$20$, and $N = 5$.

Find: i.

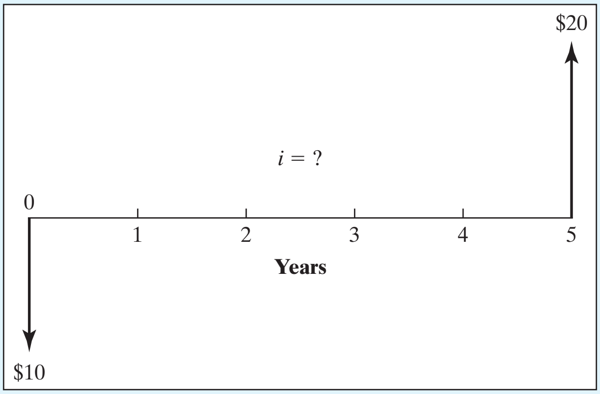


Figure 3.13 Cash flow diagram (Example 3.9).

Here, we know *P, F,* and *N,* but we do not know *i,* the interest rate you will earn on your investment. This type of rate of return is a lot easier to calculate, because you make only a one-time lump-sum investment. Problems such as this are solved as follows:

$$F = P(1 + i)^N;$$
$$\$20 = \$10(1 + i)^5; \text{ solve for } i.$$

- **Method 1.** Go through a trial-and-error process in which you insert different values of i into the equation until you find a value that "works" in the sense that the right-hand side of the equation equals \$20. The solution is $i = 14.87\%$. The trial-and-error procedure is extremely tedious and inefficient for most problems, so it is not widely practiced in the real world.
- **Method 2.** You can solve the problem by using the interest tables in Appendix A. Now look across the $N = 5$ row, under the $(F/P, i, 5)$ column, until you can locate the value of 2:

$$\$20 = \$10(1 + i)^5;$$
$$2 = (1 + i)^5 = (F/P, i, 5).$$

  This value is close to the 15% interest table with $(F/P, 15\%, 5) = 2.0114$, so the interest rate at which \$10 grows to \$20 over five years is very close to 15%. This procedure will be very tedious for fractional interest rates or when N is not a whole number, because you may have to approximate the solution by linear interpolation.
- **Method 3.** The most practical approach is to use either a financial calculator or an electronic spreadsheet such as Excel. A financial function such as RATE(*N*,0,*P*,*F*) allows us to calculate an unknown interest rate. The precise command statement would be

  =**RATE**(5,0,−10,20)=14.87%

  Note that, in Excel format, we enter the present value (P) as a negative number, indicating a cash outflow.

	A	B	C
1	P	–10	
2	F	20	
3	N	5	
4	i	14.87%	
5			

=RATE(5,0,−10,20)

EXAMPLE 3.10 Single Amounts: Find *N*, Given *P*, *F*, and *i*

You have just purchased 100 shares of General Electric stock at $60 per share. You will sell the stock when its market price has doubled. If you expect the stock price to increase 20% per year, how long do you anticipate waiting before selling the stock (Figure 3.14)?

SOLUTION

Given: $P = \$6{,}000$, $F = \$12{,}000$, and $i = 20\%$ per year.
Find: N (years).

Using the single-payment compound-amount factor, we write

$$F = P(1 + i)^N = P(F/P, i, N);$$

$$\$12{,}000 = \$6{,}000(1 + 0.20)^N = \$6{,}000(F/P, 20\%, N);$$

$$2 = (1.20)^N = (F/P, 20\%, N).$$

Again, we could use a calculator or a computer spreadsheet program to find N.

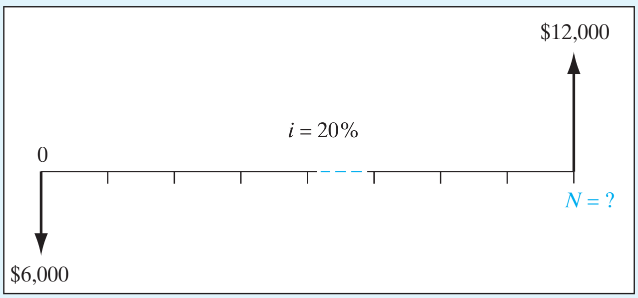


Figure 3.14

1. **Using a calculator.** Solving for N gives

$$\log 2 = N \log 1.20,$$

or

$$N = \frac{\log 2}{\log 1.20}$$
$$= 3.80 \approx 4 \text{ years.}$$

2. **Using Excel.** Within Excel, the financial function NPER(i,0,P,F) computes the number of compounding periods it will take an investment (P) to grow to a future value (F), earning a fixed interest rate (i) per compounding period. In our example, the Excel command would look like this:

$$=\text{NPER}(20\%,0,-6000,12000)$$
$$= 3.801784.$$

Rule of 72: Rule giving the approximate number of years that it will take for your investment to double.

COMMENTS: A very handy rule of thumb, called the Rule of 72, estimates approximately how long it will take for a sum of money to double. The rule states that, to find the time it takes for a present sum of money to grow by a factor of two, we divide 72 by the interest rate. In our example, the interest rate is 20%. Therefore, the Rule of 72 indicates 72/20 = 3.60, or roughly 4 years, for a sum to double. This is, in fact, relatively close to our exact solution. Figure 3.15 illustrates the number of years required to double an investment at various interest rates.

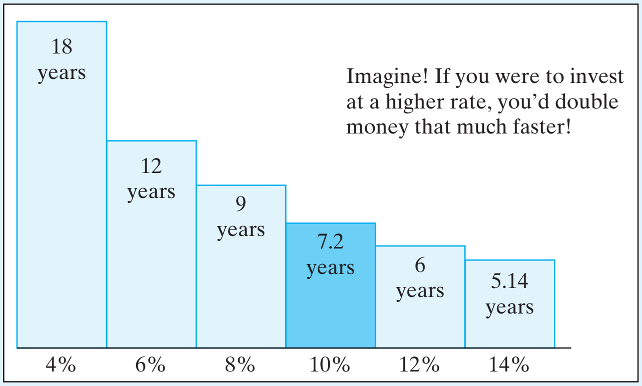


Figure 3.15 Number of years required to double an initial investment at various interest rates.

3.3.3 Uneven Payment Series

A common cash flow transaction involves a series of disbursements or receipts. Familiar examples of series payments are payment of installments on car loans and home mortgage payments. Payments on car loans and home mortgages typically involve identical sums to be paid at regular intervals. However, there is no clear pattern over the series; we call the transaction an uneven cash flow series.

We can find the present worth of any uneven stream of payments by calculating the present value of each individual payment and summing the results. Once the present worth is found, we can make other equivalence calculations (e.g., future worth can be calculated by using the interest factors developed in the previous section).

EXAMPLE 3.11 Present Values of an Uneven Series by Decomposition into Single Payments

Wilson Technology, a growing machine shop, wishes to set aside money now to invest over the next four years in automating its customer service department. The company can earn 10% on a lump sum deposited now, and it wishes to withdraw the money in the following increments:

- **Year 1:** \$25,000, to purchase a computer and database software designed for customer service use;
- **Year 2:** \$3,000, to purchase additional hardware to accommodate anticipated growth in use of the system;
- **Year 3:** No expenses; and
- **Year 4:** \$5,000, to purchase software upgrades.

How much money must be deposited now to cover the anticipated payments over the next 4 years?

STRATEGY: This problem is equivalent to asking what value of P would make you indifferent in your choice between P dollars today and the future expense stream of (\$25,000, \$3,000, \$0, \$5,000). One way to deal with an uneven series of cash flows is to calculate the equivalent present value of each single cash flow and to sum the present values to find P. In other words, the cash flow is broken into three parts as shown in Figure 3.16.

SOLUTION

Given: Uneven cash flow in Figure 3.16, with $i = 10\%$ per year.
Find: P.

$$\begin{aligned} P &= \$25{,}000(P/F, 10\%, 1) + \$3{,}000(P/F, 10\%, 2) \\ &\quad + \$5{,}000(P/F, 10\%, 4) \\ &= \$28{,}622. \end{aligned}$$

COMMENTS: To see if \$28,622 is indeed sufficient, let's calculate the balance at the end of each year. If you deposit \$28,622 now, it will grow to (1.10)(\$28,622), or \$31,484, at the end of year 1. From this balance, you pay out \$25,000. The remaining balance, \$6,484, will again grow to (1.10)(\$6,484), or \$7,132, at the end of year 2. Now you make the second payment (\$3,000) out of this balance, which will leave you with only \$4,132 at the end of year 2. Since no payment occurs in year 3, the

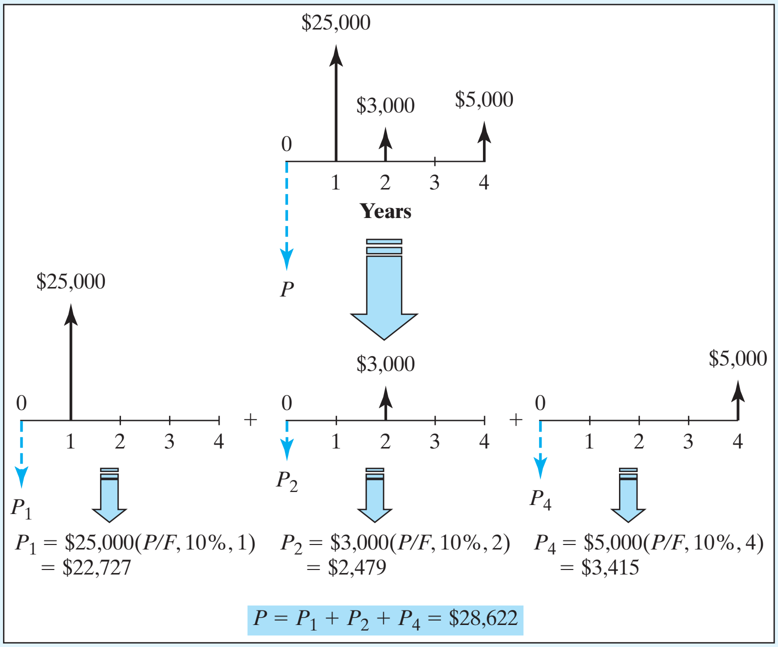


Figure 3.16 Decomposition of uneven cash flow series (Example 3.11).

balance will grow to $\$(1.10)^2(\$4,132)$, or $5,000, at the end of year 4. The final withdrawal in the amount of $5,000 will deplete the balance completely.

EXAMPLE 3.12 Calculating the Actual Worth of a Long-Term Contract of Michael Vick with Atlanta Falcons[2]

On December 23, 2004, Michael Vick became the richest player in the National Football League by agreeing to call Atlanta home for the next decade. The Falcons' quarterback signed a 10-year, $130 million contract extension Thursday that guarantees him an NFL-record $37 million in bonuses.

Base salaries for his new contract are $600,000 (2005), $1.4 million (2006), $6 million (2007), $7 million (2008), $9 million (2009), $10.5 million (2010), $13.5 million (2011), $13 million (2012), $15 million (2013), and $17 million (2014). He received an initial signing bonus of $7.5 million. Vick also received two roster bonuses in the new deal. The first is worth $22.5 million and is due in March 2005. The second is worth $7 million and is due in March 2006. Both roster bonuses will be treated as signing bonuses and prorated annually. Because 2011 is an uncapped year (the league's collective bargaining agreement (CBA) expires after the 2010 season), the initial signing bonus and 2005 roster bonus can be prorated only over the first six years of the contract. If the CBA is extended

[2] *Source:* http://www.falcfans.com/players/michael_vick.html.

prior to March 2006, then the second roster bonus of $7 million can be prorated over the final nine seasons of the contract. If the CBA is extended prior to March 2006, then his cap hits (rounded to nearest thousand) will change to $7.178 million (2006), $11.778 million (2007), $12.778 million (2008), $14.778 million (2009), $16.278 million (2010), $14.278 million (2011), $13.778 million (2012), $15.778 million (2013), and $17.778 million (2014). With the salary and signing bonus paid at the beginning of each season, the net annual payment schedule looks like the following:

Beginning of Season	Base Salary	Prorated Signing Bonus	Total Annual Payment
2005	$ 600,000	$5,000,000	$5,600,000
2006	1,400,000	5,000,000 + 778,000	7,178,000
2007	6,000,000	5,000,000 + 778,000	11,778,000
2008	7,000,000	5,000,000 + 778,000	12,778,000
2009	9,000,000	5,000,000 + 778,000	14,778,000
2010	10,500,000	778,000	16,278,000
2011	13,500,000	778,000	14,278,000
2012	13,000,000	778,000	13,778,000
2013	15,000,000	778,000	15,778,000
2014	17,000,000	778,000	17,778,000

(a) How much is Vick's contract actually worth at the time of signing? Assume that Vick's interest rate is 6% per year.

(b) For the initial signing bonus and the first year's roster bonus, suppose that the Falcons allow Vick to take either the prorated payment option as just described ($30 million over five years) or a lump-sum payment option in the amount of $23 million at the time he signs the contract. Should Vick take the lump-sum option instead of the prorated one?

SOLUTION

Given: Payment series given in Figure 3.17, with $i = 6\%$ per year.
Find: P.

(a) Actual worth of the contract at the time of signing:

$$\begin{aligned} P_{\text{contract}} &= \$5{,}600{,}000 + \$7{,}178{,}000(P/F, 6\%, 1) \\ &\quad + \$11{,}778{,}000(P/F, 6\%, 2) + \cdots \\ &\quad + \$17{,}778{,}000(P/F, 6\%, 9) \\ &= \$97{,}102{,}827. \end{aligned}$$

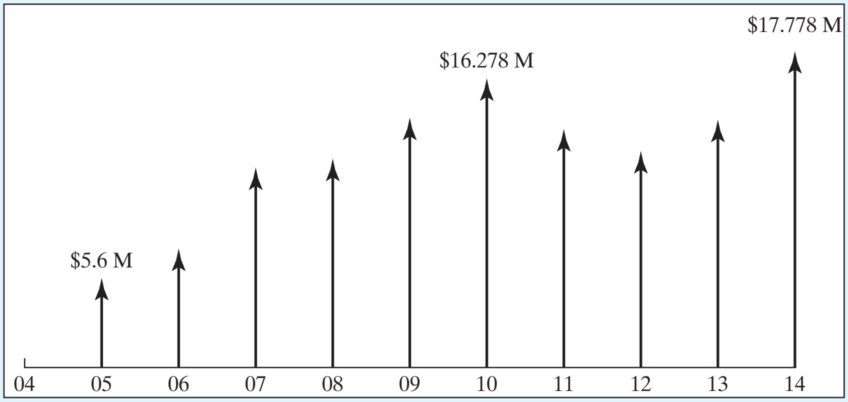


Figure 3.17 Cash flow diagram for Michael Vick's contract with Atlanta Falcons.

(b) Choice between the prorated payment option and the lump-sum payment: The equivalent present worth of the prorated payment option is

$$\begin{aligned} P_{\text{bonus}} &= \$5{,}000{,}000 + \$5{,}000{,}000(P/F, 6\%, 1) \\ &\quad + \$5{,}000{,}000(P/F, 6\%, 2) + \$5{,}000{,}000(P/F, 6\%, 3) \\ &\quad + \$5{,}000{,}000(P/F, 6\%, 4) \\ &= \$22{,}325{,}528 \end{aligned}$$

which is smaller than $23,000,000. Therefore, Vick would be better off taking the lump-sum option if, and only if, his money could be invested at 6% or higher.

COMMENTS: Note that the actual contract is worth less than the published figure of $130 million. This "brute force" approach of breaking cash flows into single amounts will always work, but it is slow and subject to error because of the many factors that must be included in the calculation. We develop more efficient methods in later sections for cash flows with certain patterns.

3.3.4 Equal Payment Series

As we learned in Example 3.12, the present worth of a stream of future cash flows can always be found by summing the present worth of each of the individual cash flows. However, if cash flow regularities are present within the stream (such as we just saw in the prorated bonus payment series in Example 3.12) then the use of shortcuts, such as finding the present worth of a uniform series, may be possible. We often encounter transactions in which a uniform series of payments exists. Rental payments, bond interest payments, and commercial installment plans are based on uniform payment series.

Compound-Amount Factor: Find *F*, Given *A*, *i*, and *N*

Suppose we are interested in the future amount F of a fund to which we contribute A dollars each period and on which we earn interest at a rate of i per period. The contributions

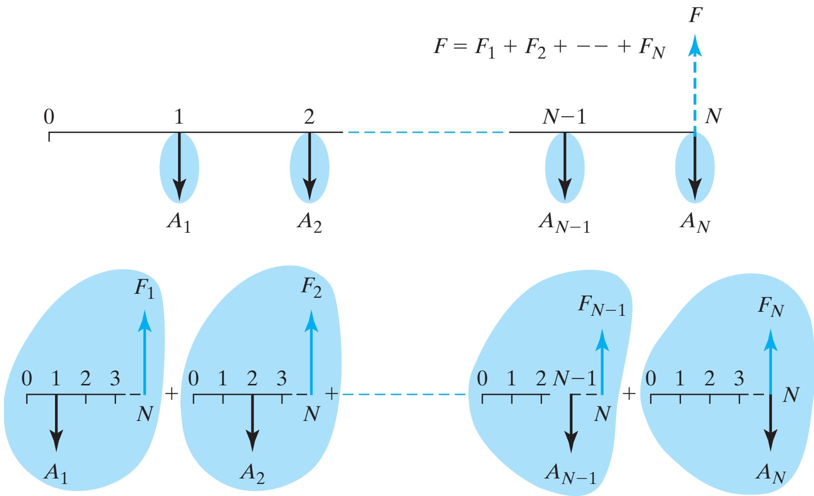


Figure 3.18 The future worth of a cash flow series obtained by summing the future-worth figures of each of the individual flows.

are made at the end of each of N equal periods. These transactions are graphically illustrated in Figure 3.18. Looking at this diagram, we see that if an amount A is invested at the end of each period, for N periods, the total amount F that can be withdrawn at the end of the N periods will be the sum of the compound amounts of the individual deposits.

As shown in Figure 3.18, the A dollars we put into the fund at the end of the first period will be worth $A(1 + i)^{N-1}$ at the end of N periods. The A dollars we put into the fund at the end of the second period will be worth $A(1 + i)^{N-2}$, and so forth. Finally, the last A dollars that we contribute at the end of the Nth period will be worth exactly A dollars at that time. This means that there exists a series of the form

$$F = A(1 + i)^{N-1} + A(1 + i)^{N-2} + \cdots + A(1 + i) + A,$$

or, expressed alternatively,

$$F = A + A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^{N-1}. \quad (3.8)$$

Multiplying Eq. (3.8) by $(1 + i)$ results in

$$(1 + i)F = A(1 + i) + A(1 + i)^2 + \cdots + A(1 + i)^N. \quad (3.9)$$

Subtracting Eq. (3.8) from Eq. (3.9) to eliminate common terms gives us

$$F(1 + i) - F = -A + A(1 + i)^N.$$

Solving for F yields

$$F = A\left[\frac{(1 + i)^N - 1}{i}\right] = A(F/A, i, N). \quad (3.10)$$

The bracketed term in Eq. (3.10) is called the **equal payment series compound-amount factor**, or the **uniform series compound-amount factor**; its factor notation is $(F/A, i, N)$. This interest factor has been calculated for various combinations of i and N in the interest tables.

EXAMPLE 3.13 Uniform Series: Find *F*, Given *i*, *A*, and *N*

Suppose you make an annual contribution of $3,000 to your savings account at the end of each year for 10 years. If the account earns 7% interest annually, how much can be withdrawn at the end of 10 years (Figure 3.19)?

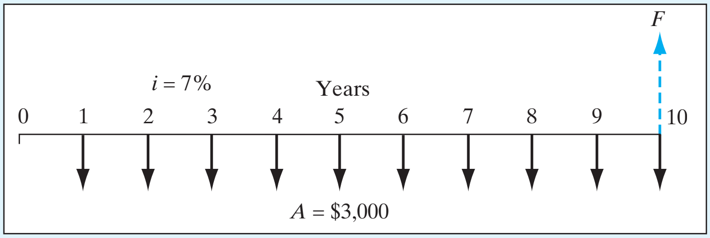


Figure 3.19 Cash flow diagram (Example 3.13).

SOLUTION

Given: $A = \$3{,}000$, $N = 10$ years, and $i = 7\%$ per year.
Find: F.

$$\begin{aligned} F &= \$3{,}000(F/A, 7\%, 10) \\ &= \$3{,}000(13.8164) \\ &= \$41{,}449.20. \end{aligned}$$

To obtain the future value of the annuity with the use of Excel, we may use the following financial command:

$$=\text{FV}(7\%,10, -3000,0,0)$$

EXAMPLE 3.14 Handling Time Shifts in a Uniform Series

In Example 3.13, the first deposit of the 10-deposit series was made at the end of period 1 and the remaining nine deposits were made at the end of each following period. Suppose that all deposits were made at the beginning of each period instead. How would you compute the balance at the end of period 10?

SOLUTION

Given: Cash flow as shown in Figure 3.20, and $i = 7\%$ per year.
Find: F_{10}.

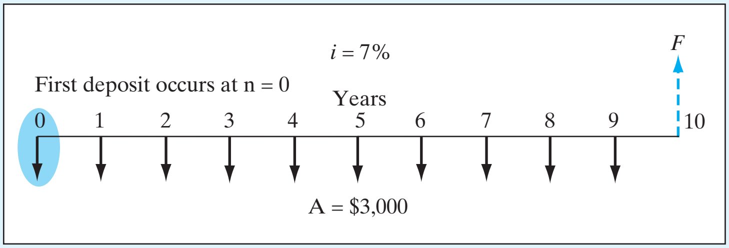


Figure 3.20 Cash Flow diagram (Example 3.14).

Compare Figure 3.20 with Figure 3.19: Each payment has been shifted to one year earlier; thus, each payment would be compounded for one extra year. Note that with the end-of-year deposit, the ending balance (*F*) was $41,449.20. With the beginning-of-year deposit, the same balance accumulates by the end of period 9. This balance can earn interest for one additional year. Therefore, we can easily calculate the resulting balance as

$$F_{10} = \$41{,}449.20(1.07) = \$44{,}350.64.$$

The annuity due can be easily evaluated with the following financial command available on Excel:

$$=\mathbf{FV}(7\%,10,-3000,0,1)$$

COMMENTS: Another way to determine the ending balance is to compare the two cash flow patterns. By adding the $3,000 deposit at period 0 to the original cash flow and subtracting the $3,000 deposit at the end of period 10, we obtain the second cash flow. Therefore, the ending balance can be found by making the following adjustment to the $41,449.20:

$$F_{10} = \$41{,}449.20 + \$3{,}000(F/P, 7\%, 10) - \$3{,}000 = \$44{,}350.64.$$

Sinking-Fund Factor: Find *A*, Given *F*, *i*, and *N*

If we solve Eq. (3.10) for *A*, we obtain

$$A = F\left[\frac{i}{(1+i)^N - 1}\right] = F(A/F, i, N). \qquad (3.11)$$

Sinking fund: A means of repaying funds advanced through a bond issue. This means *that every period, a company will pay back a portion of its bonds.*

The term within the brackets is called the **equal payment series sinking-fund factor**, or **sinking-fund factor**, and is referred to by the notation $(A/F, i, N)$. A sinking fund is an interest-bearing account into which a fixed sum is deposited each interest period; it is commonly established for the purpose of replacing fixed assets or retiring corporate bonds.

EXAMPLE 3.15 Combination of a Uniform Series and a Single Present and Future Amount

To help you reach a \$5,000 goal five years from now, your father offers to give you \$500 now. You plan to get a part-time job and make five additional deposits, one at the end of each year. (The first deposit is made at the end of the first year.) If all your money is deposited in a bank that pays 7% interest, how large must your annual deposit be?

STRATEGY: If your father reneges on his offer, the calculation of the required annual deposit is easy because your five deposits fit the standard end-of-period pattern for a uniform series. All you need to evaluate is

$$A = \$5{,}000(A/F, 7\%, 5) = \$5{,}000(0.1739) = \$869.50.$$

If you do receive the \$500 contribution from your father at $n = 0$, you may divide the deposit series into two parts: one contributed by your father at $n = 0$ and five equal annual deposit series contributed by yourself. Then you can use the F/P factor to find how much your father's contribution will be worth at the end of year 5 at a 7% interest rate. Let's call this amount F_c. The future value of your five annual deposits must then make up the difference, $\$5{,}000 - F_c$.

SOLUTION

Given: Cash flow as shown in Figure 3.21, with $i = 7\%$ per year, and $N = 5$ years.

Find: A.

$$\begin{aligned} A &= (\$5{,}000 - F_C)(A/F, 7\%, 5) \\ &= [\$5{,}000 - \$500(F/P, 7\%, 5)](A/F, 7\%, 5) \\ &= [\$5{,}000 - \$500(1.4026)](0.1739) \\ &= \$747.55. \end{aligned}$$

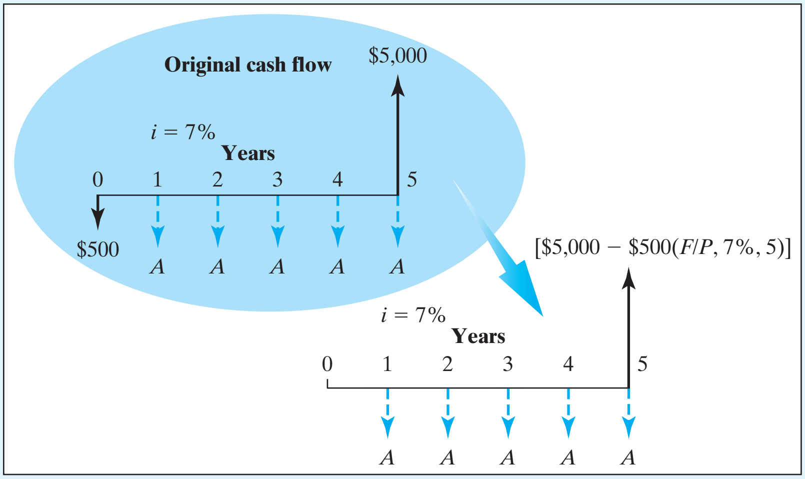


Figure 3.21 Cash flow diagram (Example 3.15).

EXAMPLE 3.16 Comparison of Three Different Investment Plans

Consider three investment plans at an annual interest rate of 9.38% (Figure 3.22):

- **Investor A.** Invest \$2,000 per year for the first 10 years of your career. At the end of 10 years, make no further investments, but reinvest the amount accumulated at the end of 10 years for the next 31 years.
- **Investor B.** Do nothing for the first 10 years. Then start investing \$2,000 per year for the next 31 years.
- **Investor C.** Invest \$2,000 per year for the entire 41 years.

Note that all investments are made at the *beginning* of each year; the first deposit will be made at the beginning of age 25 $(n = 0)$, and you want to calculate the balance at the age of 65 $(n = 41)$.

STRATEGY: Since the investments are made at the beginning of each year, we need to use the procedure outlined in Example 3.14. In other words, each deposit has one extra interest-earning period.

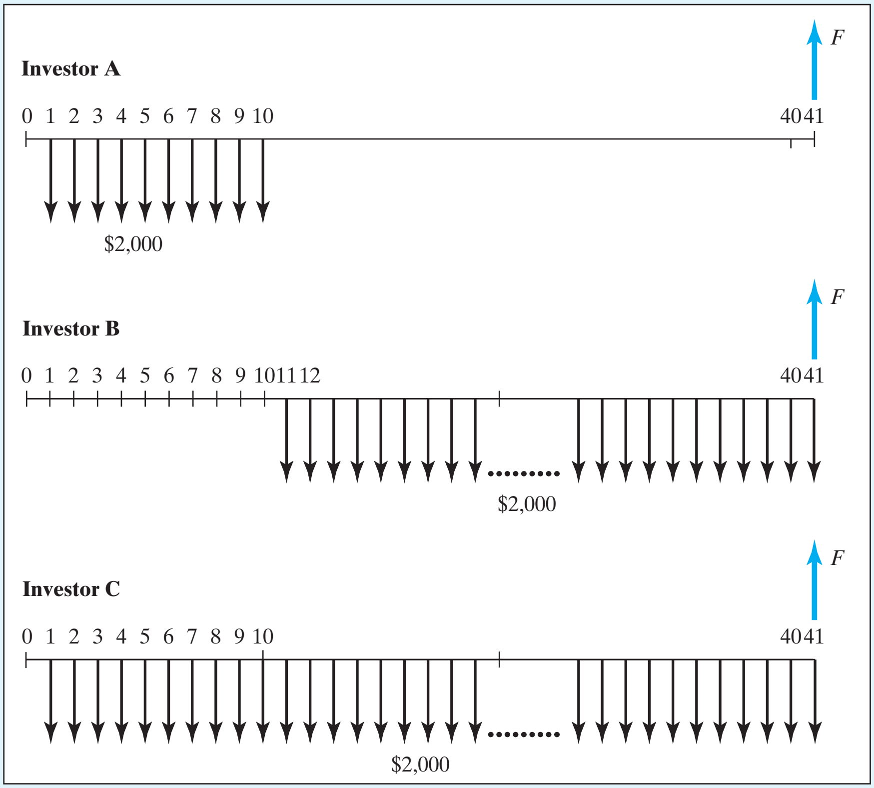


Figure 3.22 Cash flow diagrams for three investment options (Example 3.16).

SOLUTION

Given: Three different deposit scenarios with $i = 9.38\%$ and $N = 41$ years.
Find: Balance at the end of 41 years (or at the age of 65).

- Investor A:

$$F_{65} = \overbrace{\underbrace{\$2{,}000(F/A, 9.38\%, 10)(1.0938)}_{\$33{,}845}}^{\text{Balance at the end of 10 years}}(F/P, 9.38\%, 31)$$

$$= \$545{,}216.$$

- Investor B:

$$F_{65} = \underbrace{\$2{,}000(F/P, 9.38\%, 31)}_{\$322{,}159}(1.0938)$$

$$= \$352{,}377.$$

- Investor C:

$$F_{65} = \underbrace{\$2{,}000(F/P, 9.38\%, 41)}_{\$820{,}620}(1.0938)$$

$$= \$897{,}594.$$

If you know how your balance changes at the end of each year, you may want to construct a tableau such as the one shown in Table 3.3. Note that, due to rounding errors, the final balance figures are slightly off from those calculated by interest formulas.

TABLE 3.3 How Time Affects the Value of Money

		Investor A		Investor B		Investor C	
Age	Years	Contribution	Year-End Value	Contribution	Year-End Value	Contribution	Year-End Value
25	1	$2,000	$ 2,188	$0	$0	$2,000	$ 2,188
26	2	$2,000	$ 4,580	$0	$0	$2,000	$ 4,580
27	3	$2,000	$ 7,198	$0	$0	$2,000	$ 7,198
28	4	$2,000	$10,061	$0	$0	$2,000	$10,061
29	5	$2,000	$13,192	$0	$0	$2,000	$13,192
30	6	$2,000	$16,617	$0	$0	$2,000	$16,617
31	7	$2,000	$20,363	$0	$0	$2,000	$20,363
32	8	$2,000	$24,461	$0	$0	$2,000	$24,461
33	9	$2,000	$28,944	$0	$0	$2,000	$28,944
34	10	$2,000	$33,846	$0	$0	$2,000	$33,846

(*Continued*)

TABLE 3.3 Continued

		Investor A		Investor B		Investor C	
Age	Years	Contribution	Year-End Value	Contribution	Year-End Value	Contribution	Year-End Value
35	11	$0	$ 37,021	$2,000	$ 2,188	$ 2,000	$ 39,209
36	12	$0	$ 40,494	$2,000	$ 4,580	$ 2,000	$ 45,075
37	13	$0	$ 44,293	$2,000	$ 7,198	$ 2,000	$ 51,490
38	14	$0	$ 48,448	$2,000	$ 10,061	$ 2,000	$ 58,508
39	15	$0	$ 52,992	$2,000	$ 13,192	$ 2,000	$ 66,184
40	16	$0	$ 57,963	$2,000	$ 16,617	$ 2,000	$ 74,580
41	17	$0	$ 63,401	$2,000	$ 20,363	$ 2,000	$ 83,764
42	18	$0	$ 69,348	$2,000	$ 24,461	$ 2,000	$ 93,809
43	19	$0	$ 75,854	$2,000	$ 28,944	$ 2,000	$104,797
44	20	$0	$ 82,969	$2,000	$ 33,846	$ 2,000	$116,815
45	21	$0	$ 90,752	$2,000	$ 39,209	$ 2,000	$129,961
46	22	$0	$ 99,265	$2,000	$ 45,075	$ 2,000	$144,340
47	23	$0	$108,577	$2,000	$ 51,490	$ 2,000	$160,068
48	24	$0	$118,763	$2,000	$ 58,508	$ 2,000	$177,271
49	25	$0	$129,903	$2,000	$ 66,184	$ 2,000	$196,088
50	26	$0	$142,089	$2,000	$ 74,580	$ 2,000	$216,670
51	27	$0	$155,418	$2,000	$ 83,764	$ 2,000	$239,182
52	28	$0	$169,997	$2,000	$ 93,809	$ 2,000	$263,807
53	29	$0	$185,944	$2,000	$104,797	$ 2,000	$290,741
54	30	$0	$203,387	$2,000	$116,815	$ 2,000	$320,202
55	31	$0	$222,466	$2,000	$129,961	$ 2,000	$352,427
56	32	$0	$243,335	$2,000	$144,340	$ 2,000	$387,675
57	33	$0	$266,162	$2,000	$160,068	$ 2,000	$426,229
58	34	$0	$291,129	$2,000	$177,271	$ 2,000	$468,400
59	35	$0	$318,439	$2,000	$196,088	$ 2,000	$514,527
60	36	$0	$348,311	$2,000	$216,670	$ 2,000	$564,981
61	37	$0	$380,985	$2,000	$239,182	$ 2,000	$620,167
62	38	$0	$416,724	$2,000	$263,807	$ 2,000	$680,531
63	39	$0	$455,816	$2,000	$290,741	$ 2,000	$746,557
64	40	$0	$498,574	$2,000	$320,202	$ 2,000	$818,777
65	41	$0	$545,344	$2,000	$352,427	$ 2,000	$897,771
		$20,000		$62,000		$82,000	
Value at 65			**$545,344**		**$352,427**		**$897,771**
Less Total Contributions			**$20,000**		**$ 62,000**		**$82,000**
Net Earnings			**$525,344**		**$290,427**		**$815,771**

Source: Adapted from *Making Money Work for You*, UNH Cooperative Extension.

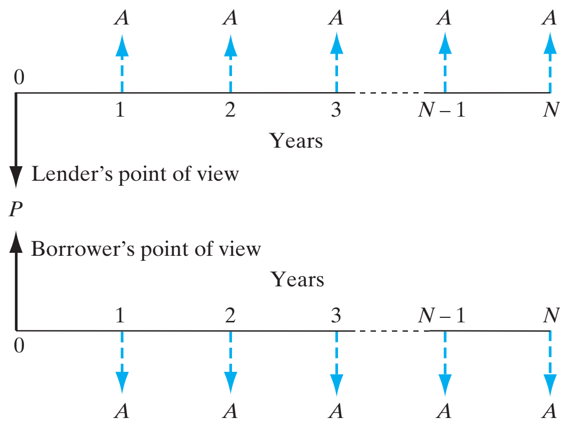


Figure 3.23

Capital Recovery Factor (Annuity Factor): Find *A*, Given *P*, *i*, and *N*

We can determine the amount of a periodic payment A if we know P, i, and N. Figure 3.23 illustrates this situation. To relate P to A, recall the relationship between P and F in Eq. (3.3), $F = P(1 + i)^N$. Replacing F in Eq. (3.11) by $P(1 + i)^N$, we get

$$A = P(1 + i)^N \left[\frac{i}{(1 + i)^N - 1} \right],$$

or

$$A = P\left[\frac{i(1 + i)^N}{(1 + i)^N - 1} \right] = P(A/P, i, N). \tag{3.12}$$

Capital recovery factor: Commonly used to determine the revenue requirements needed to address the up-front capital costs for projects.

Annuity: An annuity is essentially a level stream of *cash flows* for a fixed period of time.

Now we have an equation for determining the value of the series of end-of-period payments A when the present sum P is known. The portion within the brackets is called the **equal payment series capital recovery factor**, or simply **capital recovery factor**, which is designated $(A/P, i, N)$. In finance, this A/P factor is referred to as the **annuity factor** and indicates a series of payments of a fixed, or constant, amount for a specified number of periods.

EXAMPLE 3.17 Uniform Series: Find *A*, Given *P*, *i*, and *N*

BioGen Company, a small biotechnology firm, has borrowed \$250,000 to purchase laboratory equipment for gene splicing. The loan carries an interest rate of 8% per year and is to be repaid in equal installments over the next six years. Compute the amount of the annual installment (Figure 3.24).

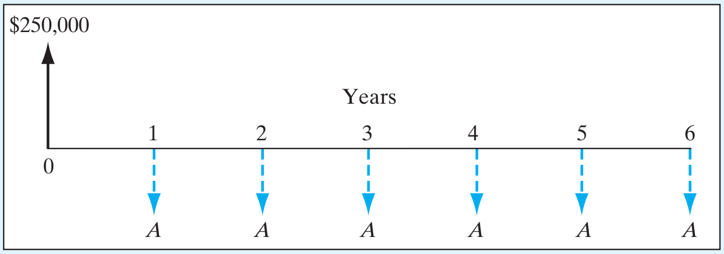


Figure 3.24 A loan cash flow diagram from BioGen's point of view.

SOLUTION

Given: $P = \$250{,}000$, $i = 8\%$ per year, and $N = 6$ years.
Find: A.

$$\begin{aligned} A &= \$250{,}000(A/P, 8\%, 6) \\ &= \$250{,}000(0.2163) \\ &= \$54{,}075. \end{aligned}$$

Here is an Excel solution using annuity function commands:

=**PMT**(i, N, P)
=**PMT**(8%,6, −250000)
=$54,075

EXAMPLE 3.18 Deferred Loan Repayment

In Example 3.17, suppose that BioGen wants to negotiate with the bank to defer the first loan repayment until the end of year 2 (but still desires to make six equal installments at 8% interest). If the bank wishes to earn the same profit, what should be the annual installment, also known as **deferred annuity** (Figure 3.25)?

Deferred annuity: A type of annuity contract that delays payments of income, installments, or a lump sum until the investor elects to receive them.

SOLUTION

Given: $P = \$250{,}000$, $i = 8\%$ per year, and $N = 6$ years, but the first payment occurs at the end of year 2.
Find: A.

By deferring the loan for year, the bank will add the interest accrued during the first year to the principal. In other words, we need to find the equivalent worth P' of $250,000 at the end of year 1:

$$\begin{aligned} P' &= \$250{,}000(F/P, 8\%, 1) \\ &= \$270{,}000. \end{aligned}$$

Grace period: The additional period of time a lender provides for a borrower to make payment on a debt without penalty.

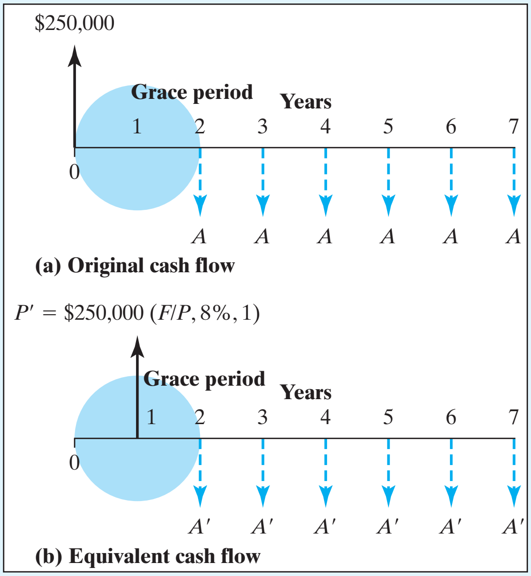


Figure 3.25 A deferred loan cash flow diagram from BioGen's point of view (Example 3.17).

In fact, BioGen is borrowing $270,000 for six years. To retire the loan with six equal installments, the deferred equal annual payment on P' will be

$$\begin{aligned} A' &= \$270{,}000(A/P, 8\%, 6) \\ &= \$58{,}401. \end{aligned}$$

By deferring the first payment for one year, BioGen needs to make additional payments of $4,326 in each year.

Present-Worth Factor: Find *P*, Given *A*, *i*, and *N*

What would you have to invest now in order to withdraw A dollars at the end of each of the next N periods? In answering this question, we face just the opposite of the equal payment capital recovery factor situation: A is known, but P has to be determined. With the capital recovery factor given in Eq. (3.12), solving for P gives us

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A(P/A, i, N). \qquad (3.13)$$

The bracketed term is referred to as the **equal payment series present-worth factor** and is designated $(P/A, i, N)$.

EXAMPLE 3.19 Uniform Series: Find *P*, Given *A*, *i*, and *N*

Let us revisit Louise Outing's lottery problem, introduced in the chapter opening. Suppose that Outing were able to find an investor who was willing to buy her lottery ticket for $2 million. Recall that after an initial gross payment of $283,770, Outing

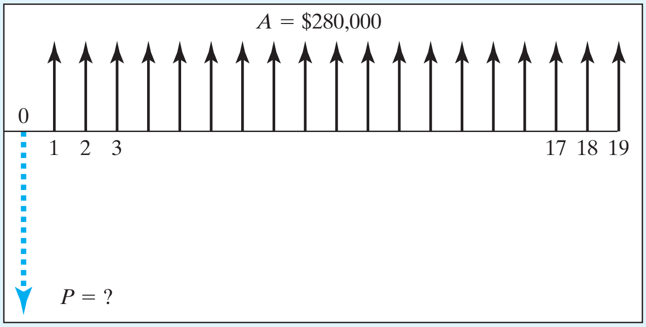


Figure 3.26 A cash flow diagram for Louise Outing's lottery winnings (Example 3.19).

would be paid 19 annual gross checks of \$280,000. (See Figure 3.26.) If she could invest her money at 8% interest, what would be the fair amount to trade her 19 future lottery receipts? (Note that she already cashed in \$283,770 after winning the lottery, so she is giving up 19 future lottery checks in the amount of \$280,000.)

SOLUTION

Given: $i = 8\%$ per year, $A = \$280{,}000$, and $N = 19$ years.
Find: P.

- Using interest factor:

$$P = \$280{,}000(P/A, 8\%, 19) = \$280{,}000(9.6036)$$
$$= \$2{,}689{,}008.$$

- Using Excel:

$$=\text{PV}(8\%,19,-280000)=\$2{,}689{,}008$$

COMMENTS: Clearly, we can tell Outing that giving up \$280,000 a year for 19 years to receive \$2 million today is a losing proposition if she can earn only an 8% return on her investment. At this point, we may be interested in knowing at just what rate of return her deal (receiving \$2 million) would in fact make sense. Since we know that $P = \$2{,}000{,}000$, $N = 19$, and $A = \$280{,}000$, we solve for i.

If you know the cash flows and the *PV* (or *FV*) of a cash flow stream, you can determine the interest rate. In this case, you are looking for the interest rate that caused the *P*/*A* factor to equal $(P/A, i, 19) = (\$2{,}000{,}000/\$280{,}000) = 7.1429$. Since we are dealing with an annuity, we could proceed as follows:

- With a financial calculator, enter $N = 19$, $PV = \$2{,}000{,}000$, $PMT = -280{,}000$, and then press the *i* key to find $i = 12.5086\%$.
- To use the interest tables, first recognize that $\$2{,}000{,}000 = \$280{,}000 \times (P/A, i, 19)$ or $(P/A, i, 19) = 7.1429$. Look up 7.1429 or a close value in

Appendix A. In the P/A column with $N = 19$ in the 12% interest table, you will find that $(P/A, 12\%, 19) = 7.3658$. If you look up the 13% interest table, you find that $(P/A, 12\%, 19) = 6.9380$, indicating that the interest rate should be closer to 12.5%.

- To use Excel's financial command, you simply evaluate the following command to solve the unknown interest rate problem for an annuity:

$$=\mathbf{RATE}(N,A,P,F,type,guess)$$
$$=\mathbf{RATE}(19{,}280000,-2000000{,}0{,}0{,}10\%)$$
$$=12.5086\%$$

It is not likely that Outing will find a financial investment which provides this high rate of return. Thus, even though the deal she has been offered is not a good one for economic reasons, she could accept it knowing that she has not much time to enjoy the future benefits.

3.3.5 Linear Gradient Series

Engineers frequently encounter situations involving periodic payments that increase or decrease by a constant amount (G) from period to period. These situations occur often enough to warrant the use of special equivalence factors that relate the arithmetic gradient to other cash flows. Figure 3.27 illustrates a **strict gradient series**, $A_n = (n - 1)G$. Note that the origin of the series is at the end of the first period with a zero value. The gradient G can be either positive or negative. If $G > 0$, the series is referred to as an *increasing* gradient series. If $G < 0$, it is a *decreasing* gradient series.

Unfortunately, the strict form of the increasing or decreasing gradient series does not correspond with the form that most engineering economic problems take. A typical problem involving a linear gradient includes an initial payment during period 1 that increases by G during some number of interest periods, a situation illustrated in Figure 3.28. This contrasts with the strict form illustrated in Figure 3.27, in which no payment is made during period 1 and the gradient is added to the previous payment beginning in period 2.

Gradient Series as Composite Series

In order to utilize the strict gradient series to solve typical problems, we must view cash flows as shown in Figure 3.28 as a **composite series**, or a set of two cash flows, each

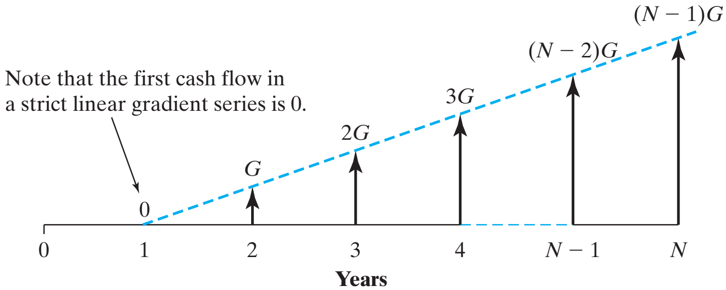


Figure 3.27 A cash flow diagram for a strict gradient series.

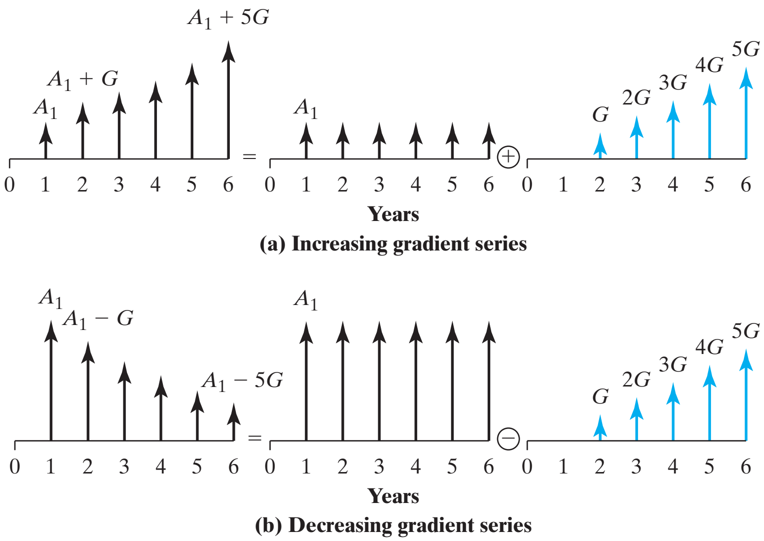


Figure 3.28 Two types of linear gradient series as composites of a uniform series of N payments of A_1 and the gradient series of increments of constant amount G.

corresponding to a form that we can recognize and easily solve: a uniform series of N payments of amount A_1 and a gradient series of increments of constant amount G. The need to view cash flows that involve linear gradient series as composites of two series is very important in solving problems, as we shall now see.

Present-Worth Factor: Linear Gradient: Find P, Given G, N, and i

How much would you have to deposit now to withdraw the gradient amounts specified in Figure 3.27? To find an expression for the present amount P, we apply the single-payment present-worth factor to each term of the series and obtain

$$P = 0 + \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \cdots + \frac{(N-1)G}{(1+i)^N},$$

or

$$P = \sum_{n=1}^{N} (n-1)G(1+i)^{-n}. \tag{3.14}$$

Letting $G = a$ and $1/(1+i) = x$ yields

$$\begin{aligned} P &= 0 + ax^2 + 2ax^3 + \cdots + (N-1)ax^N \\ &= ax[0 + x + 2x^2 + \cdots + (N-1)x^{N-1}]. \end{aligned} \tag{3.15}$$

Since an arithmetic–geometric series $\{0, x, 2x^2, \ldots, (N-1)x^{N-1}\}$ has the finite sum

$$0 + x + 2x^2 + \cdots + (N-1)x^{N-1} = x\left[\frac{1 - Nx^{N-1} + (N-1)x^N}{(1-x)^2}\right],$$

we can rewrite Eq. (3.15) as

$$P = ax^2\left[\frac{1 - Nx^{N-1} + (N - 1)x^N}{(1 - x)^2}\right]. \tag{3.16}$$

Replacing the original values for A and x, we obtain

$$P = G\left[\frac{(1 + i)^N - iN - 1}{i^2(1 + i)^N}\right] = G(P/G, i, N). \tag{3.17}$$

The resulting factor in brackets is called the **gradient series present-worth factor**, which we denote as $(P/G, i, N)$.

EXAMPLE 3.20 Linear Gradient: Find *P*, Given A_1, *G*, *i*, and *N*

A textile mill has just purchased a lift truck that has a useful life of five years. The engineer estimates that maintenance costs for the truck during the first year will be \$1,000. As the truck ages, maintenance costs are expected to increase at a rate of \$250 per year over the remaining life. Assume that the maintenance costs occur at the end of each year. The firm wants to set up a maintenance account that earns 12% annual interest. All future maintenance expenses will be paid out of this account. How much does the firm have to deposit in the account now?

SOLUTION

Given: $A_1 = \$1{,}000$, $G = \$250$, $i = 12\%$ per year, and $N = 5$ years.
Find: P.

Asking how much the firm has to deposit now is equivalent to asking what the equivalent present worth for this maintenance expenditure is if 12% interest is used. The cash flow may be broken into two components as shown in Figure 3.29.

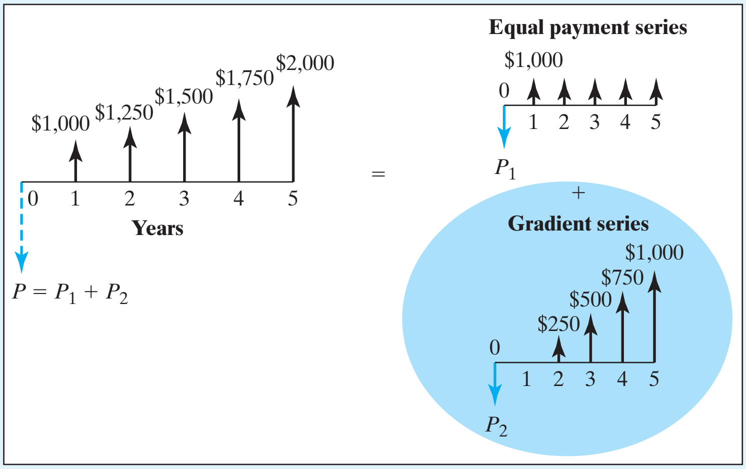


Figure 3.29 Cash flow diagram (Example 3.20).

The first component is an equal payment series (A_1), and the second is a linear gradient series (G). We have

$$P = P_1 + P_2$$

$$\begin{aligned} P &= A_1(P/A, 12\%, 5) + G(P/G, 12\%, 5) \\ &= \$1{,}000(3.6048) + \$250(6.397) \\ &= \$5{,}204. \end{aligned}$$

Note that the value of N in the gradient factor is 5, not 4. This is because, by definition of the series, the first gradient value begins at period 2.

COMMENTS: As a check, we can compute the present worth of the cash flow by using the (P/F, 12%, n) factors:

Period (n)	Cash Flow	(P/F, 12%, n)	Present Worth
1	$1,000	0.8929	$ 892.90
2	1,250	0.7972	996.50
3	1,500	0.7118	1,067.70
4	1,750	0.6355	1,112.13
5	2,000	0.5674	1,134.80
		Total	**$5,204.03**

The slight difference is caused by a rounding error.

Gradient-to-Equal-Payment Series Conversion Factor: Find *A*, Given *G*, *i*, and *N*

We can obtain an equal payment series equivalent to the gradient series, as depicted in Figure 3.30, by substituting Eq. (3.17) for P into Eq. (3.12) to obtain

$$A = G\left[\frac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right] = G(A/G, i, N), \tag{3.18}$$

where the resulting factor in brackets is referred to as the **gradient-to-equal-payment series conversion factor** and is designated (A/G, i, N).

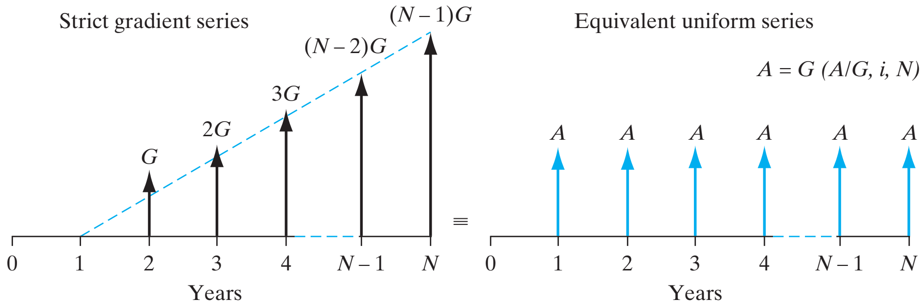


Figure 3.30

EXAMPLE 3.21 Linear Gradient: Find A, Given A_1, G, i, and N

John and Barbara have just opened two savings accounts at their credit union. The accounts earn 10% annual interest. John wants to deposit $1,000 in his account at the end of the first year and increase this amount by $300 for each of the next five years. Barbara wants to deposit an equal amount each year for the next six years. What should be the size of Barbara's annual deposit so that the two accounts will have equal balances at the end of six years (Figure 3.31)?

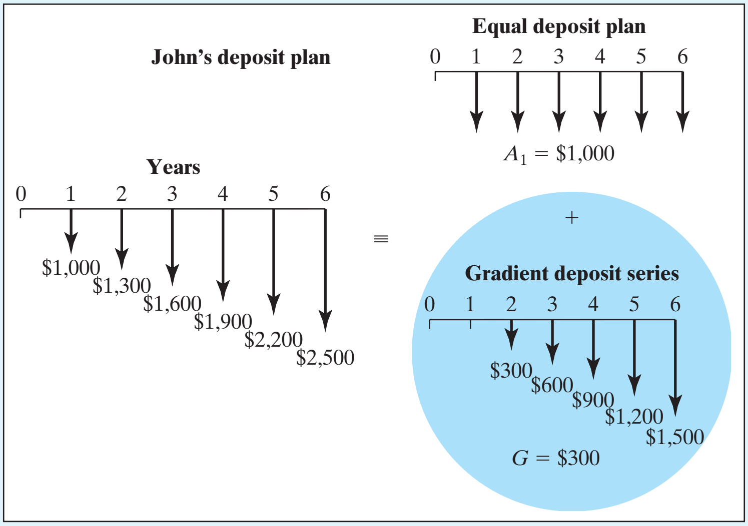


Figure 3.31 John's deposit series viewed as a combination of uniform and gradient series (Example 3.21).

SOLUTION

Given: $A_1 = \$1{,}000$, $G = \$300$, $i = 10\%$, and $N = 6$.
Find: A.

Since we use the end-of-period convention unless otherwise stated, this series begins at the end of the first year and the last contribution occurs at the end of the sixth year. We can separate the constant portion of $1,000 from the series, leaving the gradient series of 0, 0, 300, 600, ..., 1,500.

To find the equal payment series beginning at the end of year 1 and ending at year 6 that would have the same present worth as that of the gradient series, we may proceed as follows:

$$\begin{aligned} A &= \$1{,}000 + \$300(A/G, 10\%, 6) \\ &= \$1{,}000 + \$300(2.22236) \\ &= \$1{,}667.08. \end{aligned}$$

Barbara's annual contribution should be $1,667.08.

COMMENTS: Alternatively, we can compute Barbara's annual deposit by first computing the equivalent present worth of John's deposits and then finding the equivalent uniform annual amount. The present worth of this combined series is

$$\begin{aligned} P &= \$1{,}000(P/A, 10\%, 6) + \$300(P/G, 10\%, 6) \\ &= \$1{,}000(4.3553) + \$300(9.6842) \\ &= \$7{,}260.56. \end{aligned}$$

The equivalent uniform deposit is

$$A = \$7{,}260.56(A/P, 10\%, 6) = \$1{,}667.02.$$

(The slight difference in cents is caused by a rounding error.)

EXAMPLE 3.22 Declining Linear Gradient: Find *F*, Given A_1, *G*, *i*, and *N*

Suppose that you make a series of annual deposits into a bank account that pays 10% interest. The initial deposit at the end of the first year is $1,200. The deposit amounts decline by $200 in each of the next four years. How much would you have immediately after the fifth deposit?

SOLUTION

Given: Cash flow shown in Figure 3.32, $i = 10\%$ per year, and $N = 5$ years.

Find: F.

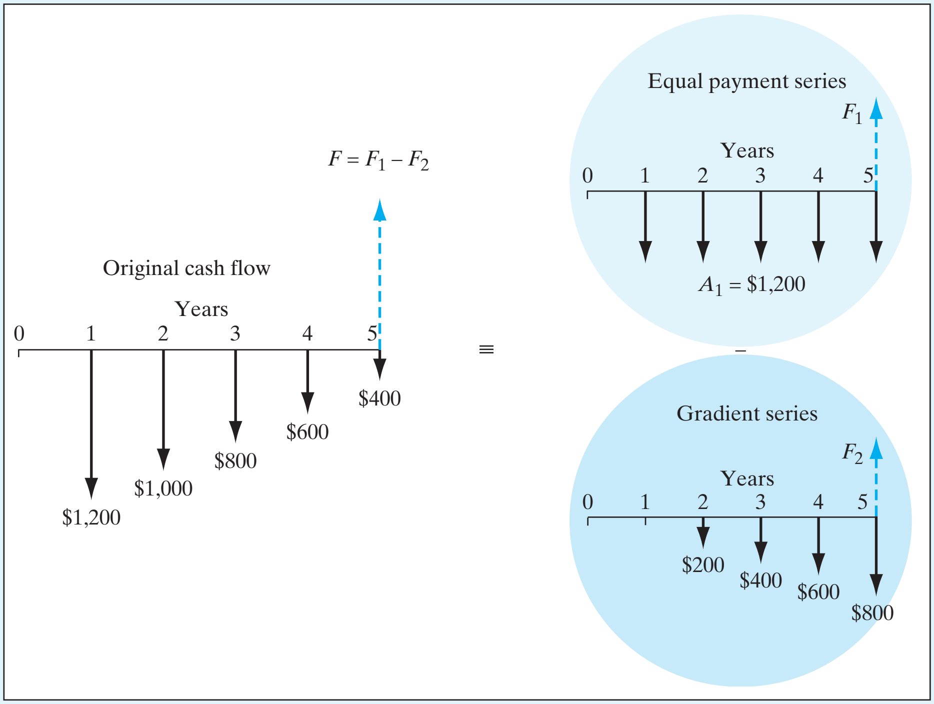


Figure 3.32

The cash flow includes a decreasing gradient series. Recall that we derived the linear gradient factors for an increasing gradient series. For a decreasing gradient series, the solution is most easily obtained by separating the flow into two components: a uniform series and an increasing gradient that is *subtracted* from the uniform series (Figure 3.32). The future value is

$$\begin{aligned} F &= F_1 - F_2 \\ &= A_1(F/A, 10\%, 5) - \$200(P/G, 10\%, 5)(F/P, 10\%, 5) \\ &= \$1{,}200(6.105) - \$200(6.862)(1.611) \\ &= \$5{,}115. \end{aligned}$$

Geometric growth: The year-over-year growth rate of an investment over a specified period of time. Compound growth is an imaginary number that describes the rate at which an investment grew as though it had grown at a steady rate.

3.3.6 Geometric Gradient Series

Many engineering economic problems—particularly those relating to construction costs—involve cash flows that increase or decrease over time, not by a constant *amount* (as with a linear gradient), but rather by a constant percentage (a **geometric gradient**). This kind of cash flow is called **compound growth**. Price changes caused by inflation are a good example of a geometric gradient series. If we use g to designate the percentage change in a payment from one period to the next, the magnitude of the nth payment, A_n, is related to the first payment A_1 by the formula

$$A_n = A_1(1 + g)^{n-1}, n = 1, 2, \ldots, N. \tag{3.19}$$

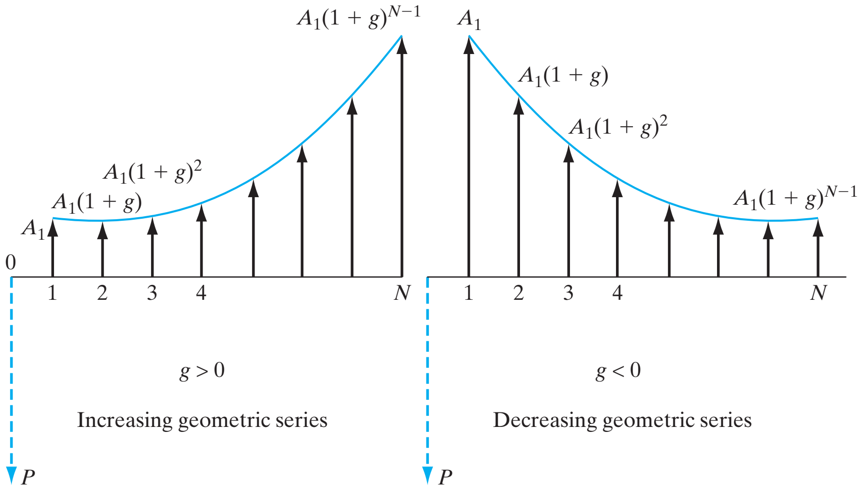


Figure 3.33 A geometrically increasing or decreasing gradient series at a constant rate g.

The variable g can take either a positive or a negative sign, depending on the type of cash flow. If $g > 0$, the series will increase, and if $g < 0$, the series will decrease. Figure 3.33 illustrates the cash flow diagram for this situation.

Present-Worth Factor: Find P, Given A_1, g, i, and N

Notice that the present worth of any cash flow A_n at interest rate i is

$$P_n = A_n(1 + i)^{-n} = A_1(1 + g)^{n-1}(1 + i)^{-n}.$$

To find an expression for the present amount P for the entire series, we apply the **single-payment present-worth factor** to each term of the series:

$$P = \sum_{n=1}^{N} A_1(1 + g)^{n-1}(1 + i)^{-n}. \tag{3.20}$$

Bringing the constant term $A_1(1 + g)^{-1}$ outside the summation yields

$$P = \frac{A_1}{(1 + g)} \sum_{n=1}^{N} \left[\frac{1 + g}{1 + i}\right]^n. \tag{3.21}$$

Let $a = \dfrac{A_1}{1 + g}$ and $x = \dfrac{1 + g}{1 + i}$. Then, rewrite Eq. (3.21) as

$$P = a(x + x^2 + x^3 + \cdots + x^N). \tag{3.22}$$

Since the summation in Eq. (3.22) represents the first N terms of a geometric series, we may obtain the closed-form expression as follows: First, multiply Eq. (3.22) by x to get

$$xP = a(x^2 + x^3 + x^4 + \cdots + x^{N+1}). \tag{3.23}$$

Then, subtract Eq. (3.23) from Eq. (3.22):

$$P - xP = a(x - x^{N+1})$$
$$P(1 - x) = a(x - x^{N+1})$$
$$P = \frac{a(x - x^{N+1})}{1 - x} \qquad (x \neq 1). \tag{3.24}$$

If we replace the original values for a and x, we obtain

$$P = \begin{cases} A_1\left[\dfrac{1 - (1 + g)^N(1 + i)^{-N}}{i - g}\right] & \text{if } i \neq g \\ NA_1/(1 + i) & \text{if } i = g \end{cases}, \tag{3.25}$$

or

$$P = A_1(P/A_1, g, i, N).$$

The factor within brackets is called the **geometric-gradient-series present-worth factor** and is designated $(P/A_1, g, i, N)$. In the special case where $i = g$, Eq. (3.21) becomes $P = [A_1/(1 + i)]N$.

EXAMPLE 3.23 Geometric Gradient: Find *P*, Given A_1, *g*, *i*, and *N*

Ansell, Inc., a medical device manufacturer, uses compressed air in solenoids and pressure switches in its machines to control various mechanical movements. Over the years, the manufacturing floor has changed layouts numerous times. With each new layout, more piping was added to the compressed-air delivery system to accommodate new locations of manufacturing machines. None of the extra, unused old pipe was capped or removed; thus, the current compressed-air delivery system is inefficient and fraught with leaks. Because of the leaks, the compressor is expected to run 70% of the time that the plant will be in operation during the upcoming year. This will require 260 kWh of electricity at a rate of \$0.05/kWh. (The plant runs 250 days a year, 24 hours per day.) If Ansell continues to operate the current air delivery system, the compressor run time will increase by 7% per year for the next five years because of ever-worsening leaks. (After five years, the current system will not be able to meet the plant's compressed-air requirement, so it will have to be replaced.) If Ansell decides to replace all of the old piping now, it will cost \$28,570.

The compressor will still run the same number of days; however, it will run 23% less (or will have $70\%(1 - 0.23) = 53.9\%$ usage during the day) because of the reduced air pressure loss. If Ansell's interest rate is 12%, is the machine worth fixing now?

SOLUTION

Given: Current power consumption, $g = 7\%$, $i = 12\%$, and $N = 5$ years.
Find: A_1 and P.

Step 1: We need to calculate the cost of power consumption of the current piping system during the first year:

$$\begin{aligned}\text{Power cost} &= \%\text{ of day operating}\\ &\quad\times \text{days operating per year}\\ &\quad\times \text{hours per day}\\ &\quad\times \text{kWh} \times \$/\text{kWh}\\ &= (70\%) \times (250\text{ days/year}) \times (24\text{ hours/day})\\ &\quad\times (260\text{ kWh}) \times (\$0.05/\text{kWh})\\ &= \$54{,}440.\end{aligned}$$

Step 2: Each year, the annual power cost will increase at the rate of 7% over the previous year's cost. The anticipated power cost over the five-year period is summarized in Figure 3.34. The equivalent present lump-sum cost at 12% for this geometric gradient series is

$$\begin{aligned}P_{\text{Old}} &= \$54{,}440(P/A_1, 7\%, 12\%, 5)\\ &= \$54{,}440\left[\frac{1-(1+0.07)^5(1+0.12)^{-5}}{0.12-0.07}\right]\\ &= \$222{,}283.\end{aligned}$$

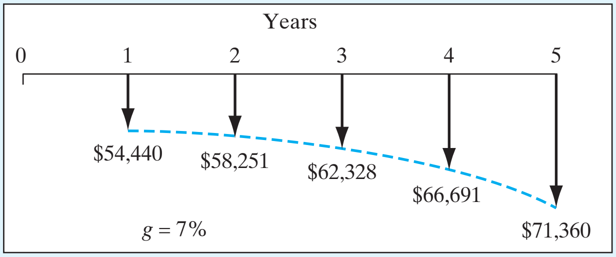


Figure 3.34 Annual power cost series if repair is not performed.

Step 3: If Ansell replaces the current compressed-air system with the new one, the annual power cost will be 23% less during the first year and will remain at that level over the next five years. The equivalent present lump-sum cost at 12% is then

$$\begin{aligned}P_{\text{New}} &= \$54{,}440(1-0.23)(P/A, 12\%, 5)\\ &= \$41{,}918.80(3.6048)\\ &= \$151{,}109.\end{aligned}$$

Step 4: The net cost for not replacing the old system now is $71,174 (= $222,283 − $151,109). Since the new system costs only $28,570, the replacement should be made now.

COMMENTS: In this example, we assumed that the cost of removing the old system was included in the cost of installing the new system. If the removed system has some salvage value, replacing it will result in even greater savings. We will consider many types of replacement issues in Chapter 14.

EXAMPLE 3.24 Geometric Gradient: Find A_1, Given F, g, i, and N

Jimmy Carpenter, a self-employed individual, is opening a retirement account at a bank. His goal is to accumulate \$1,000,000 in the account by the time he retires from work in 20 years' time. A local bank is willing to open a retirement account that pays 8% interest compounded annually throughout the 20 years. Jimmy expects that his annual income will increase 6% yearly during his working career. He wishes to start with a deposit at the end of year 1 (A_1) and increase the deposit at a rate of 6% each year thereafter. What should be the size of his first deposit (A_1)? The first deposit will occur at the end of year 1, and subsequent deposits will be made at the end of each year. The last deposit will be made at the end of year 20.

SOLUTION

Given: $F = \$1{,}000{,}000$, $g = 6\%$ per year, $i = 8\%$ per year, and $N = 20$ years.
Find: A_1 as in Figure 3.35.

We have

$$F = A_1(P/A_1, 6\%, 8\%, 20)\,(F/P, 8\%, 20)$$
$$= A_1(72.6911).$$

Solving for A_1 yields

$$A_1 = \$1{,}000{,}000/72.6911 = \$13{,}757.$$

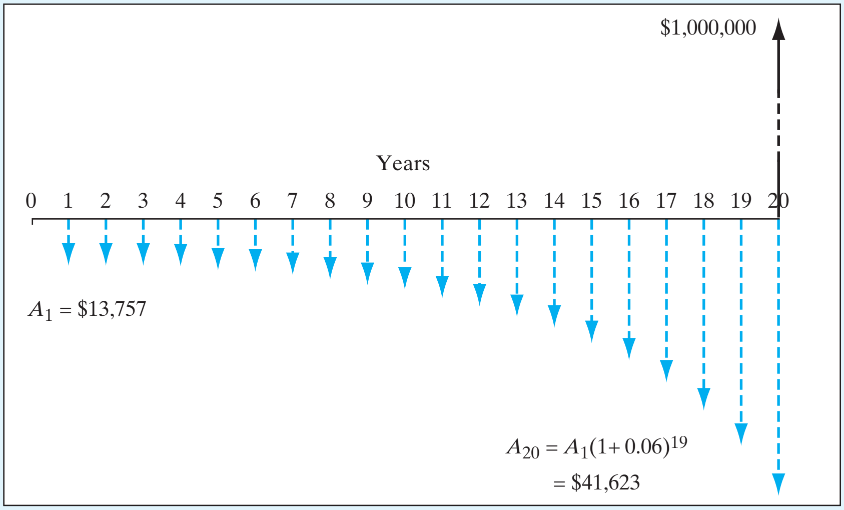


Figure 3.35 Jimmy Carpenter's retirement plan (Example 3.24).

Table 3.4 summarizes the interest formulas developed in this section and the cash flow situations in which they should be used. Recall that these formulas are applicable only to situations where the interest (compounding) period is the same as the payment period (e.g., annual compounding with annual payment). Also, we present some useful Excel financial commands in the table.

3.4 Unconventional Equivalence Calculations

In the preceding section, we occasionally presented two or more methods of attacking problems even though we had standard interest factor equations by which to solve them. It is important that you become adept at examining problems from unusual angles and that you seek out unconventional solution methods, because not all cash flow problems conform to the neat patterns for which we have discovered and developed equations. Two categories of problems that demand unconventional treatment are composite (mixed) cash flows and problems in which we must determine the interest rate implicit in a financial contract. We will begin this section by examining instances of composite cash flows.

3.4.1 Composite Cash Flows

Although many financial decisions do involve constant or systematic changes in cash flows, others contain several components of cash flows that do not exhibit an overall pattern. Consequently, it is necessary to expand our analysis to deal with these mixed types of cash flows.

To illustrate, consider the cash flow stream shown in Figure 3.36. We want to compute the equivalent present worth for this mixed payment series at an interest rate of 15%. Three different methods are presented.

Method 1. A "brute force" approach is to multiply each payment by the appropriate $(P/F, 10\%, n)$ factors and then to sum these products to obtain the present worth of the cash flows, \$543.72. Recall that this is exactly the same procedure we used to solve the category of problems called the uneven payment series, described in Section 3.3.3. Figure 3.36 illustrates this computational method.

Method 2. We may group the cash flow components according to the type of cash flow pattern that they fit, such as the single payment, equal payment series, and so forth, as shown in Figure 3.37. Then the solution procedure involves the following steps:

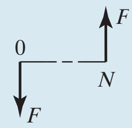
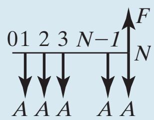
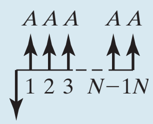
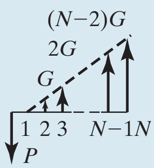
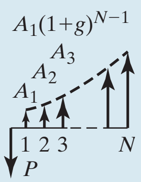
- **Group 1:** Find the present worth of \$50 due in year 1:

$$\$50(P/F, 15\%, 1) = \$43.48.$$

- **Group 2:** Find the equivalent worth of a \$100 equal payment series at year 1 (V_1), and then bring this equivalent worth at year 0 again:

$$\underbrace{\$100(P/A, 15\%, 3)}_{V_1}(P/F, 15\%, 1) = \$198.54.$$

TABLE 3.4 **Summary of Discrete Compounding Formulas with Discrete Payments**

Flow Type	Factor Notation	Formula	Excel Command	Cash Flow Diagram
SINGLE	Compound amount $(F/P, i, N)$	$F = P(1+i)^N$	$=\mathrm{FV}(i, N, P,, 0)$	
	Present worth $(P/F, i, N)$	$P = F(1+i)^{-N}$	$=\mathrm{PV}(i, N, F,, 0)$	
EQUAL PAYMENT SERIES	Compound amount $(F/A, i, N)$	$F = A\left[\dfrac{(1+i)^N - 1}{i}\right]$	$=\mathrm{PV}(i, N, A,, 0)$	
	Sinking fund $(A/F, i, N)$	$A = F\left[\dfrac{i}{(1+i)^N - 1}\right]$	$=\mathrm{PMT}(i, N, P, F, 0)$	
	Present worth $(P/A, i, N)$	$P = A\left[\dfrac{(1+i)^N - 1}{i(1+i)^N}\right]$	$=\mathrm{PV}(i, N, A,, 0)$	
	Capital recovery $(A/P, i, N)$	$A = P\left[\dfrac{i(1+i)^N}{(1+i)^N - 1}\right]$	$=\mathrm{PMT}(i, N,, P)$	
GRADIENT SERIES	Linear gradient Present worth $(P/G, i, N)$	$P = G\left[\dfrac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$		
	Conversion factor $(A/G, i, N)$	$A = G\left[\dfrac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right]$		
	Geometric gradient Present worth $(P/A_1, g, i, N)$	$P = \begin{bmatrix} A_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] \\ A_1\left(\dfrac{N}{1+i}\right)(\text{if } i = g) \end{bmatrix}$		

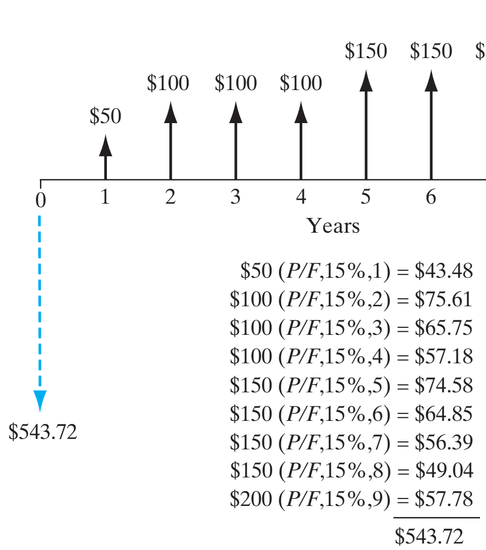


Figure 3.36 Equivalent present worth calculation using only P/F factors (Method 1 "Brute Force Approach").

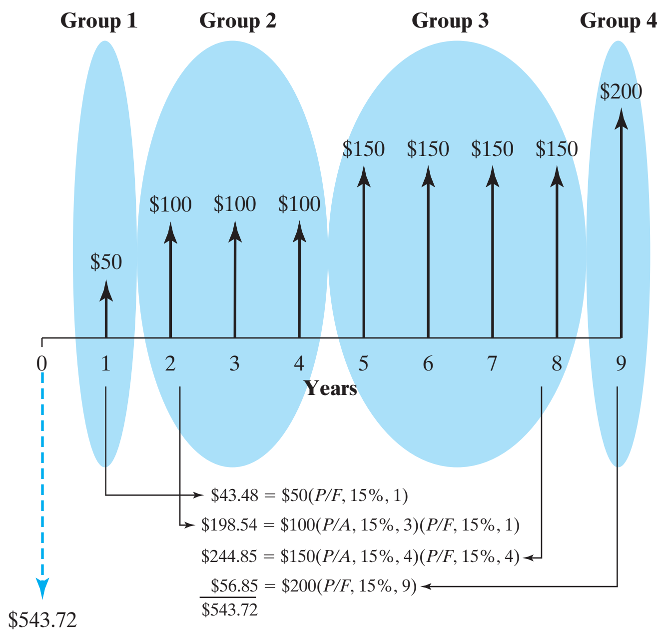


Figure 3.37 Equivalent present-worth calculation for an uneven payment series, using P/F and P/A factors (Method 2: grouping approach).

- **Group 3:** Find the equivalent worth of a \$150 equal payment series at year 4 (V_4), and then bring this equivalent worth at year 0.

$$\underbrace{\$150(P/A, 15\%, 4)}_{V_4}(P/F, 15\%, 4) = \$244.85.$$

- **Group 4:** Find the equivalent present worth of the \$200 due in year 9:

$$\$200(P/F, 15\%, 9) = \$56.85.$$

- **Group total—sum the components:**

$$P = \$43.48 + \$198.54 + \$244.85 + \$56.85 = \$543.72.$$

A pictorial view of this computational process is given in Figure 3.37.

Method 3. In computing the present worth of the equal payment series components, we may use an alternative method.

- **Group 1:** Same as in Method 2.
- **Group 2:** Recognize that a \$100 equal payment series will be received during years 2 through 4. Thus, we could determine the value of a four-year annuity, subtract the value of a one-year annuity from it, and have remaining the value of a four-year annuity whose first payment is due in year 2. This result is achieved by subtracting the (*P*/*A*, 15%, 1) for a one-year, 15% annuity from that for a four-year annuity and then multiplying the difference by \$100:

$$\begin{aligned}\$100[(P/A, 15\%, 4) - (P/A, 15\%, 1)] &= \$100(2.8550 - 0.8696)\\ &= \$198.54.\end{aligned}$$

  Thus, the equivalent present worth of the annuity component of the uneven stream is \$198.54.

- **Group 3:** We have another equal payment series that starts in year 5 and ends in year 8.

$$\begin{aligned}\$150[(P/A, 15\%, 8) - (P/A, 15\%, 4)] &= \$150(4.4873 - 2.8550)\\ &= \$244.85.\end{aligned}$$

- **Group 4:** Same as Method 2.
- **Group total—sum the components:**

$$P = \$43.48 + \$198.54 + \$244.85 + \$56.85 = \$543.72.$$

Either the "brute force" method of Figure 3.35 or the method utilizing both (*P*/*A*, *i*, *n*) and (*P*/*F*, *i*, *n*) factors can be used to solve problems of this type. However, Method 2 or Method 3 is much easier if the annuity component runs for many years. For example, the alternative solution would be clearly superior for finding the equivalent present worth of a stream consisting of \$50 in year 1, \$200 in years 2 through 19, and \$500 in year 20.

Also, note that in some instances we may want to find the equivalent value of a stream of payments at some point other than the present (year 0). In this situation, we proceed as before, but compound and discount to some other point in time—say, year 2, rather than year 0. Example 3.25 illustrates the situation.

EXAMPLE 3.25 Cash Flows with Subpatterns

The two cash flows in Figure 3.38 are equivalent at an interest rate of 12% compounded annually. Determine the unknown value C.

SOLUTION

Given: Cash flows as in Figure 3.38; $i = 12\%$ per year.

Find: C.

- **Method 1.** Compute the present worth of each cash flow at time 0:

$$P_1 = \$100(P/A, 12\%, 2) + \$300(P/A, 12\%, 3)(P/F, 12\%, 2)$$
$$= \$743.42;$$

$$P_2 = C(P/A, 12\%, 5) - C(P/F, 12\%, 3)$$
$$= 2.8930C.$$

Since the two flows are equivalent, $P_1 = P_2$, and we have

$$743.42 = 2.8930C.$$

Solving for C, we obtain $C = \$256.97$.

- **Method 2.** We may select a time point other than 0 for comparison. The best choice of a base period is determined largely by the cash flow patterns. Obviously, we want to select a base period that requires the minimum number of interest factors for the equivalence calculation. Cash flow 1 represents a combined series of two equal payment cash flows, whereas cash flow 2 can be viewed as an equal payment series with the third payment missing. For cash

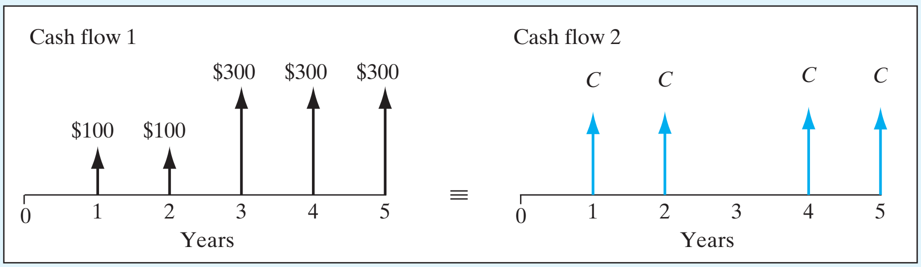


Figure 3.38 Equivalence calculation (Example 3.25).

flow 1, computing the equivalent worth at period 5 will require only two interest factors:

$$V_{5,1} = \$100(F/A, 12\%, 5) + \$200(F/A, 12\%, 3)$$
$$= \$1{,}310.16.$$

For cash flow 2, computing the equivalent worth of the equal payment series at period 5 will also require two interest factors:

$$V_{5,2} = C(F/A, 12\%, 5) - C(F/P, 12\%, 2)$$
$$= 5.0984C.$$

Therefore, the equivalence would be obtained by letting $V_{5,1} = V_{5,2}$:

$$\$1{,}310.16 = 5.0984C.$$

Solving for C yields $C = \$256.97$, which is the same result obtained from Method 1. The alternative solution of shifting the time point of comparison will require only four interest factors, whereas Method 1 requires five interest factors.

EXAMPLE 3.26 Establishing a College Fund

A couple with a newborn daughter wants to save for their child's college expenses in advance. The couple can establish a college fund that pays 7% annual interest. Assuming that the child enters college at age 18, the parents estimate that an amount of \$40,000 per year (actual dollars) will be required to support the child's college expenses for 4 years. Determine the equal annual amounts the couple must save until they send their child to college. (Assume that the first deposit will be made on the child's first birthday and the last deposit on the child's 18th birthday. The first withdrawal will be made at the beginning of the freshman year, which also is the child's 18th birthday.)

SOLUTION

Given: Deposit and withdrawal series shown in Figure 3.39; $i = 7\%$ per year.

Find: Unknown annual deposit amount (X).

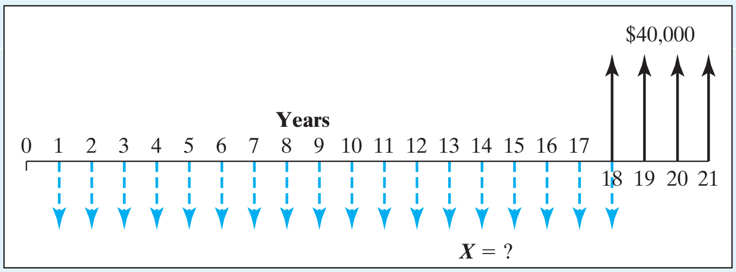


Figure 3.39 Establishing a college fund (Example 3.26). Note that the \$40,000 figure represents the actual anticipated expenditures considering the future inflation.

- **Method 1.** Establish economic equivalence at period 0:

  Step 1: Find the equivalent single lump-sum deposit now:

$$P_{\text{Deposit}} = X(P/A, 7\%, 18)$$
$$= 10.0591X.$$

  Step 2: Find the equivalent single lump-sum withdrawal now:

$$P_{\text{Withdrawal}} = \$40{,}000(P/A, 7\%, 4)(P/F, 7\%, 17)$$
$$= \$42{,}892.$$

  Step 3: Since the two amounts are equivalent, by equating $P_{\text{Deposit}} = P_{\text{Withdrawal,}}$ we obtain X:

$$10.0591X = \$42{,}892$$
$$X = \$4{,}264.$$

- **Method 2.** Establish the economic equivalence at the child's 18th birthday:

  Step 1: Find the accumulated deposit balance on the child's 18th birthday:

$$V_{18} = X(F/A, 7\%, 18)$$
$$= 33.9990X.$$

  Step 2: Find the equivalent lump-sum withdrawal on the child's 18th birthday:

$$V_{18} = \$40{,}000 + \$40{,}000(P/A, 7\%, 3)$$
$$= \$144{,}972.$$

  Step 3: Since the two amounts must be the same, we obtain

$$33.9990X = \$144{,}972$$
$$X = \$4{,}264.$$

The computational steps are summarized in Figure 3.40. In general, the second method is the more efficient way to obtain an equivalence solution to this type of decision problem.

COMMENTS: To verify whether the annual deposits of $4,264 over 18 years would be sufficient to meet the child's college expenses, we can calculate the actual year-by-year balances: With the 18 annual deposits of $4,264, the balance on the child's 18th birthday is

$$\$4{,}264(F/A, 7\%, 18) = \$144{,}972.$$

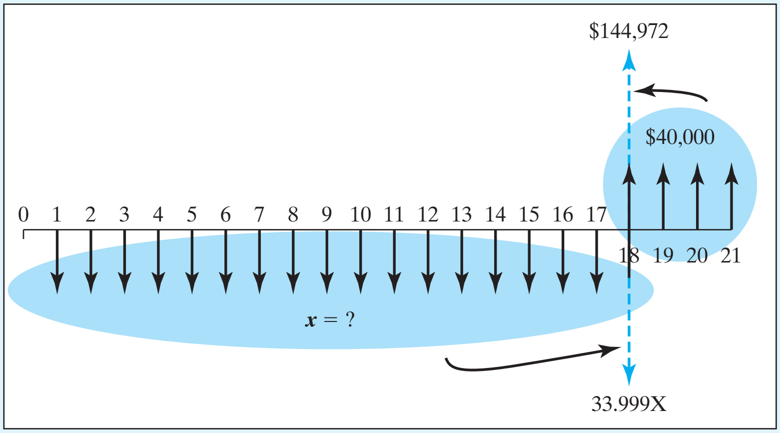


Figure 3.40 An alternative equivalence calculation (Example 3.26).

From this balance, the couple will make four annual tuition payments:

Year N	Beginning Balance	Interest Earned	Tuition Payment	Ending Balance
Freshman	$144,972	$ 0	$40,000	$104,972
Sophomore	104,972	7,348	40,000	72,320
Junior	72,320	5,062	40,000	37,382
Senior	37,382	2,618	40,000	0

3.4.2 Determining an Interest Rate to Establish Economic Equivalence

Thus far, we have assumed that, in equivalence calculations, a typical interest rate is given. Now we can use the same interest formulas that we developed earlier to determine interest rates that are explicit in equivalence problems. For most commercial loans, interest rates are already specified in the contract. However, when making some investments in financial assets, such as stocks, you may want to know the rate of growth (or rate of return) at which your asset is appreciating over the years. (This kind of calculation is the basis of rate-of-return analysis, which is covered in Chapter 7.) Although we can use interest tables to find the rate that is implicit in single payments and annuities, it is more difficult to find the rate that is implicit in an uneven series of payments. In such cases, a trial-and-error procedure or computer software may be used. To illustrate, consider Example 3.27.

EXAMPLE 3.27 Calculating an Unknown Interest Rate with Multiple Factors

You may have already won $2 million! Just peel the game piece off the Instant Winner Sweepstakes ticket, and mail it to us along with your order for subscriptions to your two favorite magazines. As a grand prize winner, you may choose between a $1 million cash prize paid immediately or $100,000 per year for 20 years—that's $2 million! Suppose that, instead of receiving one lump sum of $1 million, you decide to accept the 20 annual installments of $100,000. If you are like most jackpot winners, you will be tempted to spend your winnings to improve your lifestyle during the first several years. Only after you get this type of spending "out of your system" will you save later sums for investment purposes. Suppose that you are considering the following two options:

Option 1: You save your winnings for the first 7 years and then spend every cent of the winnings in the remaining 13 years.
Option 2: You do the reverse, spending for 7 years and then saving for 13 years.

If you can save winnings at 7% interest, how much would you have at the end of 20 years, and what interest rate on your savings will make these two options equivalent? (Cash flows into savings for the two options are shown in Figure 3.41.)

SOLUTION

Given: Cash flows in Figure 3.41.
Find: (a) F and (b) i at which the two flows are equivalent.

(a) In Option 1, the net balance at the end of year 20 can be calculated in two steps: Find the accumulated balance at the end of year 7 (V_7) first; then find the equivalent worth of V_7 at the end of year 20. For Option 2, find the equivalent worth of the 13 equal annual deposits at the end of year 20. We thus have

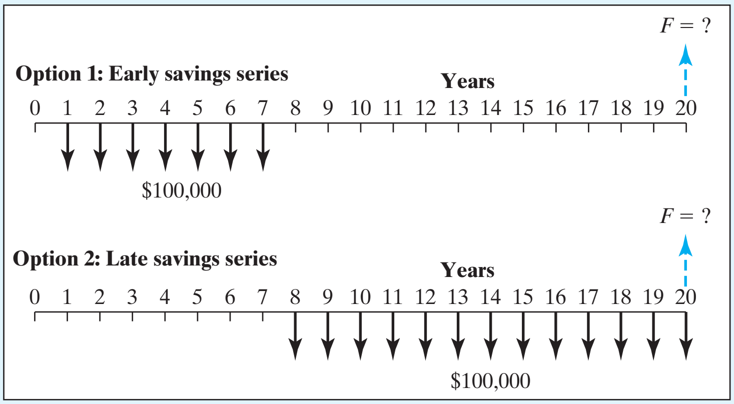


Figure 3.41 Equivalence calculation (Example 3.27).

$$
\begin{aligned}
F_{\text{Option 1}} &= \$100{,}000(F/A, 7\%, 7)(F/P, 7\%, 13) \\
&= \$2{,}085{,}485; \\
F_{\text{Option 2}} &= \$100{,}000(F/A, 7\%, 13) \\
&= \$2{,}014{,}064.
\end{aligned}
$$

Option 1 accumulates $71,421 more than Option 2.

(b) To compare the alternatives, we may compute the present worth for each option at period 0. By selecting period 7, however, we can establish the same economic equivalence with fewer interest factors. As shown in Figure 3.42, we calculate the equivalent value V_7 for each option at the end of period 7, remembering that the end of period 7 is also the beginning of period 8. (Recall from Example 3.4 that the choice of the point in time at which to compare two cash flows for equivalence is arbitrary.)

- For Option 1,

$$V_7 = \$100{,}000(F/A, i, 7).$$

- For Option 2,

$$V_7 = \$100{,}000(P/A, i, 13).$$

We equate the two values:

$$\$100{,}000(F/A, i, 7) = \$100{,}000(P/A, i, 13);$$

$$\frac{(F/A, i, 7)}{(P/A, i, 13)} = 1.$$

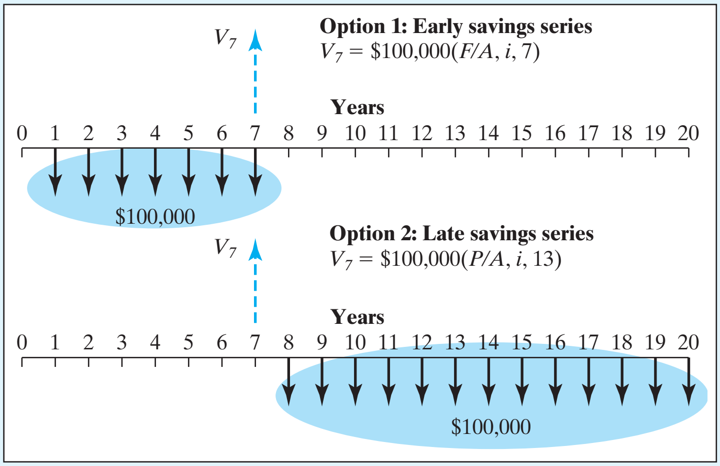


Figure 3.42 Establishing an economic equivalence at period 7 (Example 3.27).

Here, we are looking for an interest rate that gives a ratio of unity. When using the interest tables, we need to resort to a trial-and-error method. Suppose that we guess the interest rate to be 6%. Then

$$\frac{(F/A, 6\%, 7)}{(P/A, 6\%, 13)} = \frac{8.3938}{8.8527} = 0.9482.$$

This is less than unity. To increase the ratio, we need to use a value of i such that it increases the $(F/A, i, 7)$ factor value, but decreases the $(P/A, i, 13)$ value. This will happen if we use a larger interest rate. Let's try $i = 7\%$:

$$\frac{(F/A, 7\%, 7)}{(P/A, 7\%, 13)} = \frac{8.6540}{8.3577} = 1.0355.$$

Now the ratio is greater than unity.

Interest Rate	(F/A, i, 7)/(P/A, i, 13)
6%	0.9482
?	1.0000
7%	1.0355

As a result, we find that the interest rate is between 6% and 7% and may be approximated by **linear** **interpolation** as shown in Figure 3.43:

Interpolation is a method of constructing new data points from a discrete set of known data points.

$$\begin{aligned} i &= 6\% + (7\% - 6\%)\left[\frac{1 - 0.9482}{1.0355 - 0.9482}\right] \\ &= 6\% + 1\%\left[\frac{0.0518}{0.0873}\right] \\ &= 6.5934\%. \end{aligned}$$

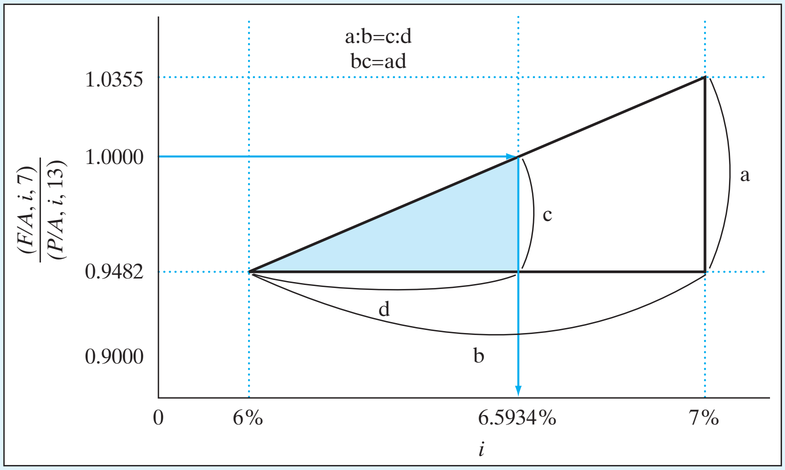


Figure 3.43 Linear interpolation to find an unknown interest rate (Example 3.27).

At 6.5934% interest, the two options are equivalent, and you may decide to indulge your desire to spend like crazy for the first 7 years. However, if you could obtain a higher interest rate, you would be wiser to save for 7 years and spend for the next 13.

COMMENTS: This example demonstrates that finding an interest rate is an iterative process that is more complicated and generally less precise than finding an equivalent worth at a known interest rate. Since computers and financial calculators can speed the process of finding unknown interest rates, such tools are highly recommended for these types of problem solving. With Excel, a more precise break-even value of 6.60219% is found by using the Goal Seek function.[3]

In Figure 3.44, the cell that contains the formula that you want to settle is called the **Set cell** (**F11 = F7-F9**). The value you want the formula to change to is called **To value (0)** and the part of the formula that you wish to change is called **By changing cell (F5, interest rate)**. The **Set cell** MUST always contain a formula or a function, whereas the **Changing cell** must contain a value only, not a formula or function.

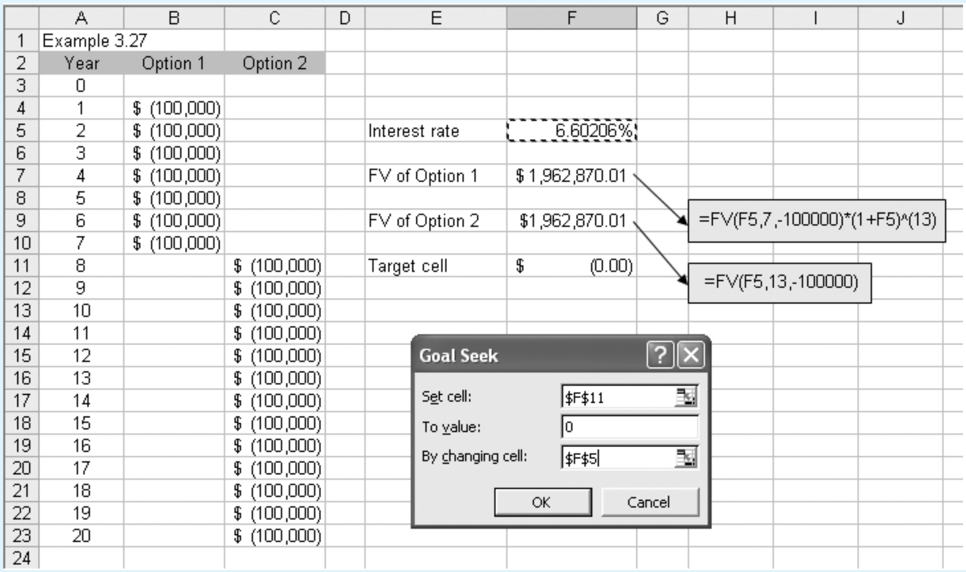


Figure 3.44 Using the Goal Seek function in Excel to find the break-even interest rate (Example 3.27). As soon as you select OK you will see that Goal Seek recalculates your formula. You then have two options, OK or Cancel. If you select OK the new term will be inserted into your worksheet. If you select Cancel, the Goal Seek box will disappear, and your worksheet will be in its original state.

[3] Goal Seek can be used when you know the result of a formula, but not the input value required by the formula to decide the result. You can change the value of a specified cell until the formula that is dependent on the changed cell returns the result you want. Goal Seek is found under the Tools menu.

SUMMARY

- Money has a time value because it can earn more money over time. A number of terms involving the time value of money were introduced in this chapter:

  Interest is the cost of money. More specifically, it is a cost to the borrower and an earning to the lender, above and beyond the initial sum borrowed or loaned.

  Interest rate is a percentage periodically applied to a sum of money to determine the amount of interest to be added to that sum.

  Simple interest is the practice of charging an interest rate only to an initial sum.

  Compound interest is the practice of charging an interest rate to an initial sum *and* to any previously accumulated interest that has not been withdrawn from the initial sum. Compound interest is by far the most commonly used system in the real world.

  Economic equivalence exists between individual cash flows and/or patterns of cash flows that have the same value. Even though the amounts and timing of the cash flows may differ, the appropriate interest rate makes them equal.

- The compound-interest formula, perhaps the single most important equation in this text, is

$$F = P(1 + i)^N,$$

  where P is a present sum, i is the interest rate, N is the number of periods for which interest is compounded, and F is the resulting future sum. All other important interest formulas are derived from this one.

- **Cash flow diagrams** are visual representations of cash inflows and outflows along a timeline. They are particularly useful for helping us detect which of the following five patterns of cash flow is represented by a particular problem:

  1. **Single payment.** A single present or future cash flow.
  2. **Uniform series.** A series of flows of equal amounts at regular intervals.
  3. **Linear gradient series.** A series of flows increasing or decreasing by a fixed amount at regular intervals.
  4. **Geometric gradient series.** A series of flows increasing or decreasing by a fixed percentage at regular intervals.
  5. **Irregular series.** A series of flows exhibiting no overall pattern. However, patterns might be detected for portions of the series.

- **Cash flow patterns** are significant because they allow us to develop **interest formulas**, which streamline the solution of equivalence problems. Table 3.4 summarizes the important interest formulas that form the foundation for all other analyses you will conduct in engineering economic analysis.

PROBLEMS

Types of Interest

3.1 You deposit $5,000 in a savings account that earns 8% simple interest per year. What is the minimum number of years you must wait to double your balance?

Suppose instead that you deposit the $5,000 in another savings account that earns 7% interest compounded yearly. How many years will it take now to double your balance?

3.2 Compare the interest earned by $1,000 for five years at 8% simple interest with that earned by the same amount for five years at 8% compounded annually.

3.3 You are considering investing $3,000 at an interest rate of 8% compounded annually for five years or investing the $3,000 at 9% per year simple interest for five years. Which option is better?

3.4 You are about to borrow $10,000 from a bank at an interest rate of 9% compounded annually. You are required to make five equal annual repayments in the amount of $2,571 per year, with the first repayment occurring at the end of year 1. Show the interest payment and principal payment in each year.

Equivalence Concept

3.5 Suppose you have the alternative of receiving either $12,000 at the end of five years or P dollars today. Currently you have no need for money, so you would deposit the P dollars in a bank that pays 5% interest. What value of P would make you indifferent in your choice between P dollars today and the promise of $12,000 at the end of five years?

3.6 Suppose that you are obtaining a personal loan from your uncle in the amount of $20,000 (now) to be repaid in two years to cover some of your college expenses. If your uncle usually earns 8% interest (annually) on his money, which is invested in various sources, what minimum lump-sum payment two years from now would make your uncle happy?

Single Payments (Use of *F/P* or *P/F* Factors)

3.7 What will be the amount accumulated by each of these present investments?
(a) $5,000 in 8 years at 5% compounded annually
(b) $2,250 in 12 years at 3% compounded annually
(c) $8,000 in 31 years at 7% compounded annually
(d) $25,000 in 7 years at 9% compounded annually

3.8 What is the present worth of these future payments?
(a) $5,500 6 years from now at 10% compounded annually
(b) $8,000 15 years from now at 6% compounded annually
(c) $30,000 5 years from now at 8% compounded annually
(d) $15,000 8 years from now at 12% compounded annually

3.9 For an interest rate of 13% compounded annually, find
(a) How much can be lent now if $10,000 will be repaid at the end of five years?
(b) How much will be required in four years to repay a $25,000 loan received now?

3.10 How many years will it take an investment to triple itself if the interest rate is 12% compounded annually?

3.11 You bought 300 shares of Microsoft (MSFT) stock at $2,600 on December 31, 2005. Your intention is to keep the stock until it doubles in value. If you expect 15% annual growth for MSFT stock, how many years do you anticipate holding onto the stock? Compare your answer with the solution obtained by the Rule of 72 (discussed in Example 3.10).

3.12 From the interest tables in the text, determine the values of the following factors by interpolation, and compare your answers with those obtained by evaluating the F/P factor or the P/F factor:

(a) The single-payment compound-amount factor for 38 periods at 9.5% interest

(b) The single-payment present-worth factor for 47 periods at 8% interest

Uneven Payment Series

3.13 If you desire to withdraw the following amounts over the next five years from a savings account that earns 8% interest compounded annually, how much do you need to deposit now?

N	Amount
2	$32,000
3	43,000
4	46,000
5	28,000

3.14 If $1,500 is invested now, $1,800 two years from now, and $2,000 four years from now at an interest rate of 6% compounded annually, what will be the total amount in 15 years?

3.15 A local newspaper headline blared, "Bo Smith Signed for $30 Million." A reading of the article revealed that on April 1, 2005, Bo Smith, the former record-breaking running back from Football University, signed a $30 million package with the Dallas Rangers. The terms of the contract were $3 million immediately, $2.4 million per year for the first five years (with the first payment after 1 year) and $3 million per year for the next five years (with the first payment at year 6). If Bo's interest rate is 8% per year, what would his contract be worth at the time he signs it?

3.16 How much invested now at 6% would be just sufficient to provide three payments, with the first payment in the amount of $7,000 occurring two years hence, then $6,000 five years hence, and finally $5,000 seven years hence?

Equal Payment Series

3.17 What is the future worth of a series of equal year-end deposits of $1,000 for 10 years in a savings account that earns 7%, annual interest if

(a) All deposits are made at the *end* of each year?

(b) All deposits are made at the *beginning* of each year?

3.18 What is the future worth of the following series of payments?

(a) $3,000 at the end of each year for 5 years at 7% compounded annually

(b) $4,000 at the end of each year for 12 years at 8.25% compounded annually

(c) $5,000 at the end of each year for 20 years at 9.4% compounded annually

(d) $6,000 at the end of each year for 12 years at 10.75% compounded annually

3.19 What equal annual series of payments must be paid into a sinking fund to accumulate the following amounts?

(a) $22,000 in 13 years at 6% compounded annually

(b) $45,000 in 8 years at 7% compounded annually

(c) \$35,000 in 25 years at 8% compounded annually

(d) \$18,000 in 8 years at 14% compounded annually

3.20 Part of the income that a machine generates is put into a sinking fund to replace the machine when it wears out. If \$1,500 is deposited annually at 7% interest, how many years must the machine be kept before a new machine costing \$30,000 can be purchased?

3.21 A no-load (commission-free) mutual fund has grown at a rate of 11% compounded annually since its beginning. If it is anticipated that it will continue to grow at that rate, how much must be invested every year so that \$15,000 will be accumulated at the end of five years?

3.22 What equal annual payment series is required to repay the following present amounts?

(a) \$10,000 in 5 years at 5% interest compounded annually

(b) \$5,500 in 4 years at 9.7% interest compounded annually

(c) \$8,500 in 3 years at 2.5% interest compounded annually

(d) \$30,000 in 20 years at 8.5% interest compounded annually

3.23 You have borrowed \$25,000 at an interest rate of 16%. Equal payments will be made over a three-year period. (The first payment will be made at the end of the first year.) What will the annual payment be, and what will the interest payment be for the second year?

3.24 What is the present worth of the following series of payments?

(a) \$800 at the end of each year for 12 years at 5.8% compounded annually

(b) \$2,500 at the end of each year for 10 years at 8.5% compounded annually

(c) \$900 at the end of each year for 5 years at 7.25% compounded annually

(d) \$5,500 at the end of each year for 8 years at 8.75% compounded annually

3.25 From the interest tables in Appendix B, determine the values of the following factors by interpolation and compare your results with those obtained from evaluating the A/P and P/A interest formulas:

(a) The capital recovery factor for 38 periods at 6.25% interest

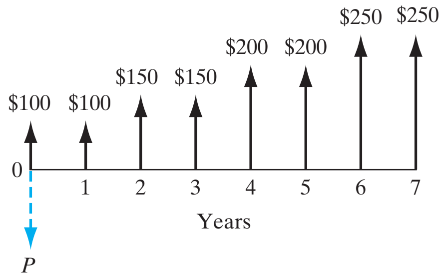
(b) The equal payment series present-worth factor for 85 periods at 9.25% interest

Linear Gradient Series

3.26 An individual deposits an annual bonus into a savings account that pays 8% interest compounded annually. The size of the bonus increases by \$2,000 each year, and the initial bonus amount was \$5,000. Determine how much will be in the account immediately after the fifth deposit.

3.27 Five annual deposits in the amounts of \$3,000, \$2,500, \$2,000, \$1,500, and \$1,000, in that order, are made into a fund that pays interest at a rate of 7% compounded annually. Determine the amount in the fund immediately after the fifth deposit.

3.28 Compute the value of P in the accompanying cash flow diagram, assuming that $i = 9\%$.



3.29 What is the equal payment series for 12 years that is equivalent to a payment series of $15,000 at the end of the first year, decreasing by $1,000 each year over 12 years? Interest is 8% compounded annually.

Geometric Gradient Series

3.30 Suppose that an oil well is expected to produce 100,000 barrels of oil during its first year in production. However, its subsequent production (yield) is expected to decrease by 10% over the previous year's production. The oil well has a proven reserve of 1,000,000 barrels.

(a) Suppose that the price of oil is expected to be $60 per barrel for the next several years. What would be the present worth of the anticipated revenue stream at an interest rate of 12% compounded annually over the next seven years?

(b) Suppose that the price of oil is expected to start at $60 per barrel during the first year, but to increase at the rate of 5% over the previous year's price. What would be the present worth of the anticipated revenue stream at an interest rate of 12% compounded annually over the next seven years?

(c) Consider part (b) again. After three years' production, you decide to sell the oil well. What would be a fair price?

3.31 A city engineer has estimated the annual toll revenues from a newly proposed highway construction over 20 years as follows:

$$A_n = (\$2{,}000{,}000)(n)(1.06)^{n-1},$$
$$n = 1, 2, \ldots, 20.$$

To validate the bond, the engineer was asked to present the estimated total present value of toll revenue at an interest rate of 6%. Assuming annual compounding, find the present value of the estimated toll revenue.

3.32 What is the amount of 10 equal annual deposits that can provide five annual withdrawals when a first withdrawal of $5,000 is made at the end of year 11 and subsequent withdrawals increase at the rate of 8% per year over the previous year's withdrawal if

(a) The interest rate is 9% compounded annually?

(b) The interest rate is 6% compounded annually?

Various Interest Factor Relationships

3.33 By using only those factors given in interest tables, find the values of the factors that follow, which are not given in your tables. Show the relationship between the factors by using factor notation, and calculate the value of the factor. Then compare the solution you obtained by using the factor formulas with a direct calculation of the factor values.

Example: $(F/P, 8\%, 38) = (F/P, 8\%, 30)(F/P, 8\%, 8) = 18.6253$

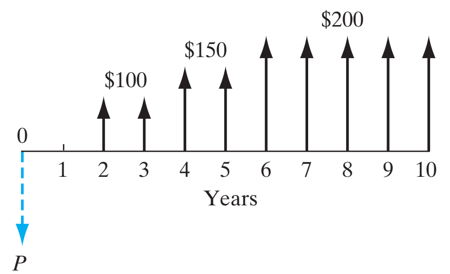
(a) $(P/F, 8\%, 67)$
(b) $(A/P, 8\%, 42)$
(c) $(P/A, 8\%, 135)$

3.34 Prove the following relationships among interest factors:

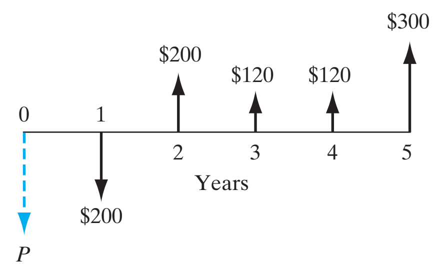
(a) $(F/P, i, N) = i(F/A, i, N) + 1$
(b) $(P/F, i, N) = 1 - (P/A, i, N)i$
(c) $(A/F, i, N) = (A/P, i, N) - i$
(d) $(A/P, i, N) = i/[1 - (P/F, i, N)]$
(e) $(P/A, i, N \rightarrow \infty) = 1/i$
(f) $(A/P, i, N \rightarrow \infty) = i$
(g) $(P/G, i, N \rightarrow \infty) = 1/i^2$

Equivalence Calculations

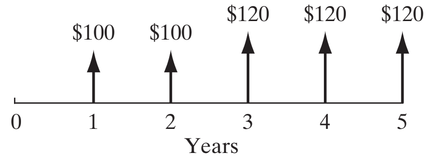
3.35 Find the present worth of the cash receipts where $i = 12\%$ compounded annually with only four interest factors.



3.36 Find the equivalent present worth of the cash receipts where $i = 8\%$. In other words, how much do you have to deposit now (with the second deposit in the amount of $200 at the end of the first year) so that you will be able to withdraw $200 at the end of second year, $120 at the end of third year, and so forth if the bank pays you a 8% annual interest on your balance?



3.37 What value of A makes two annual cash flows equivalent at 13% interest compounded annually?



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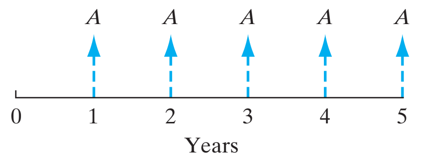
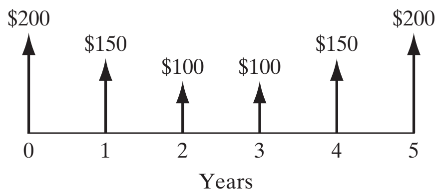


3.38 The two cash flow transactions shown in the accompanying cash flow diagram are said to be equivalent at 6% interest compounded annually. Find the unknown value of X that satisfies the equivalence.



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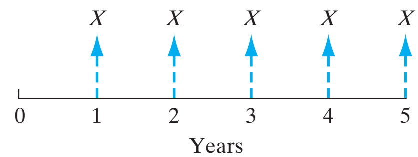
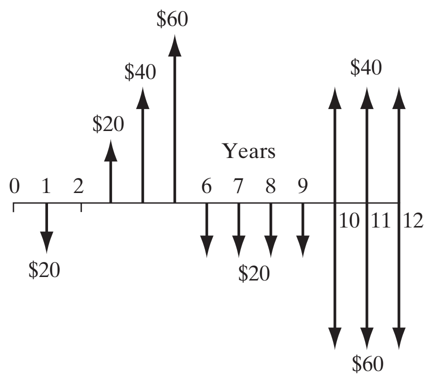
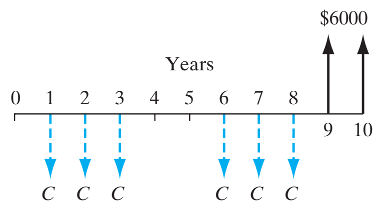


3.39 Solve for the present worth of this cash flow using at most three interest factors at 10% interest compounded annually.



3.40 From the accompanying cash flow diagram, find the value of C that will establish the economic equivalence between the deposit series and the withdrawal series at an interest rate of 8% compounded annually.



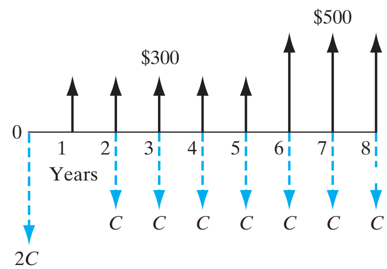
(a) \$1,335 (b) \$862
(c) \$1,283 (d) \$828

3.41 The following equation describes the conversion of a cash flow into an equivalent equal payment series with $N = 10$:

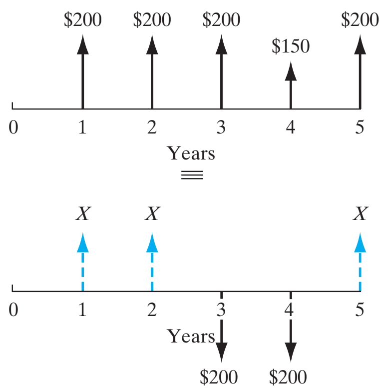
$$A = [800 + 20(A/G, 6\%, 7)] \times (P/A, 6\%, 7)(A/P, 6\%, 10) + [300(F/A, 6\%, 3) - 500](A/F, 6\%, 10).$$

Reconstruct the original cash flow diagram.

3.42 Consider the cash flow shown in the accompanying diagram. What value of C makes the inflow series equivalent to the outflow series at an interest rate of 10%?

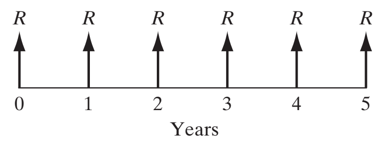


3.43 Find the value of X so that the two cash flows shown in the diagram are equivalent for an interest rate of 8%.



3.44 What single amount at the end of the fifth year is equivalent to a uniform annual series of \$3,000 per year for 10 years if the interest rate is 9% compounded annually?

3.45 From the following list, identify all the correct equations used in computing either the equivalent present worth (P) or future worth (F) for the cash flow shown at $i = 10\%$.

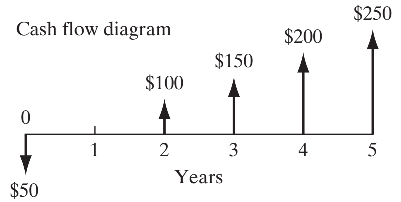


(1) $P = R(P/A, 10\%, 6)$
(2) $P = R + R(P/A, 10\%, 5)$
(3) $P = R(P/F, 10\%, 5) + R(P/A, 10\%, 5)$
(4) $F = R(F/A, 10\%, 5) + R(F/P, 10\%, 5)$
(5) $F = R + R(F/A, 10\%, 5)$
(6) $F = R(F/A, 10\%, 6)$
(7) $F = R(F/A, 10\%, 6) - R$

3.46 On the day his baby was born, a father decided to establish a savings account for the child's college education. Any money that is put into the account will earn an interest rate of 8% compounded annually. The father will make a series of annual deposits in equal amounts on each of his child's birthdays from the 1st through the 18th, so that the child can make four annual withdrawals from the account in the amount of \$30,000 on each birthday. Assuming that the first withdrawal will be made on the child's 18th birthday, which of the following equations are correctly used to calculate the required annual deposit?

(1) $A = (\$30{,}000 \times 4)/18$
(2) $A = \$30{,}000(F/A, 8\%, 4) \times (P/F, 8\%, 21)(A/P, 8\%, 18)$
(3) $A = \$30{,}000(P/A, 8\%, 18) \times (F/P, 8\%, 21)(A/F, 8\%, 4)$
(4) $A = [\$30{,}000(P/A, 8\%, 3) + \$30{,}000](A/F, 8\%, 18)$
(5) $A = \$30{,}000[(P/F, 8\%, 18) + (P/F, 8\%, 19) + (P/F, 8\%, 20) + (P/F, 8\%, 21)](A/P, 8\%, 18)$

3.47 Find the equivalent equal payment series (A) using an A/G factor such that the two cash flows are equivalent at 10% compounded annually.



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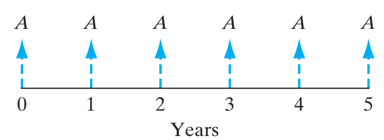


3.48 Consider the following cash flow:

Year End	Payment
0	\$500
1–5	\$1,000

In computing F at the end of year 5 at an interest rate of 12%, which of the following equations is *in*correct?

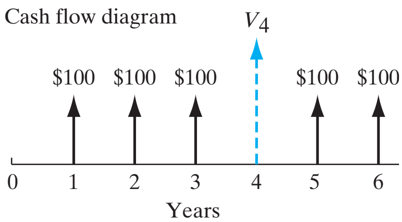
(a) $F = \$1{,}000(F/A, 12\%, 5) - \$500(F/P, 12\%, 5)$

(b) $F = \$500(F/A, 12\%, 6) + \$500(F/A, 12\%, 5)$

(c) $F = [\$500 + \$1{,}000(P/A, 12\%, 5)] \times (F/P, 12\%, 5)$

(d) $F = [\$500(A/P, 12\%, 5) + \$1{,}000] \times (F/A, 12\%, 5)$

3.49 Consider the cash flow series given. In computing the equivalent worth at $n = 4$, which of the following equations is *in*correct?



(a) $V_4 = [\$100(P/A, i, 6) - \$100(P/F, i, 4)](F/P, i, 4)$

(b) $V_4 = \$100(F/A, i, 3) + \$100(P/A, i, 2)$

(c) $V_4 = \$100(F/A, i, 4) - \$100 + \$100(P/A, i, 2)$

(d) $V_4 = [\$100(F/A, i, 6) - \$100(F/P, i, 2)](P/F, i, 2)$

3.50 Henry Cisco is planning to make two deposits: \$25,000 now and \$30,000 at the end of year 6. He wants to withdraw C each year for the first six years and $(C + \$1{,}000)$ each year for the next six years. Determine the value of C if the deposits earn 10% interest compounded annually.

(a) \$7,711

(b) \$5,794

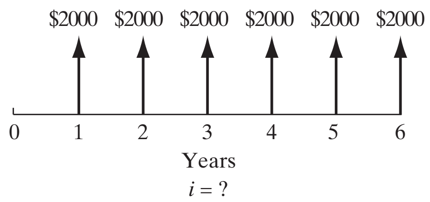
(c) \$6,934

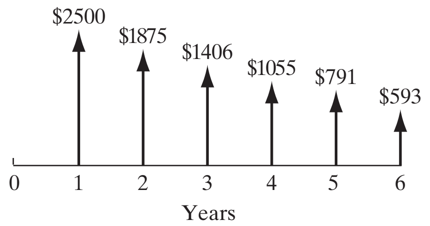
(d) \$6,522

Solving for an Unknown Interest Rate or Unknown Interest Periods

3.51 At what rate of interest compounded annually will an investment double itself in five years?

3.52 Determine the interest rate (i) that makes the pairs of cash flows shown economically equivalent.





3.53 You have $10,000 available for investment in stock. You are looking for a growth stock whose value can grow to $35,000 over five years. What kind of growth rate are you looking for?

3.54 How long will it take to save $1 million if you invest $2,000 each year at 6%?

Short Case Studies

ST3.1 Read the following letter from a magazine publisher:

Dear Parent:

Currently your *Growing Child/Growing Parent* subscription will expire with your 24-month issue. To renew on an annual basis until your child reaches 72 months would cost you a total of $63.84 ($15.96 per year). We feel it is so important for you to continue receiving this material until the 72nd month, that we offer you an opportunity to renew now for $57.12. Not only is this a savings of 10% over the regular rate, but it is an excellent inflation hedge for you against increasing rates in the future. Please act now by sending $57.12.

(a) If your money is worth 6% per year, determine whether this offer can be of any value.

(b) What rate of interest would make you indifferent between the two renewal options?

ST3.2 The State of Florida sold a total of 36.1 million lottery tickets at $1 each during the first week of January 2006. As prize money, a total of $41 million will be distributed ($1,952,381 at the *beginning* of each year) over the next 21 years. The distribution of the first-year prize money occurs now, and the remaining lottery proceeds will be put into the state's educational reserve fund, which earns interest at the rate of 6% compounded annually. After making the last prize distribution (at the beginning of year 21), how much will be left over in the reserve account?

ST3.3 A local newspaper carried the following story:

Texas Cowboys wide receiver John Young will earn either $11,406,000 over 12 years or $8,600,000 over 6 years. Young must declare which plan he prefers. The $11 million package is deferred through the year 2017, while the nondeferred arrangement ends after the 2011 season. Regardless of which plan is chosen, Young will be playing through the 2011 season. Here are the details of the two plans:

Deferred Plan		Nondeferred Plan	
2006	$2,000,000	2006	$2,000,000
2007	566,000	2007	900,000
2008	920,000	2008	1,000,000
2009	930,000	2009	1,225,000
2010	740,000	2010	1,500,000
2011	740,000	2011	1,975,000
2012	740,000		
2013	790,000		
2014	540,000		
2015	1,040,000		
2016	1,140,000		
2017	1,260,000		
Total	$11,406,000	Total	$8,600,000

(a) As it happened, Young ended up with the nondeferred plan. In retrospect, if Young's interest rate were 6%, did he make a wise decision in 2006?

(b) At what interest rate would the two plans be economically equivalent?

ST3.4 Fairmont Textile has a plant in which employees have been having trouble with carpal tunnel syndrome (CTS, an inflammation of the nerves that pass through the carpal tunnel, a tight space at the base of the palm), resulting from long-term repetitive activities, such as years of sewing. It seems as if 15 of the employees working in this facility developed signs of CTS over the last five years. DeepSouth, the company's insurance firm, has been increasing Fairmont's liability insurance steadily because of this problem. DeepSouth is willing to lower the insurance premiums to $16,000 a year (from the current $30,000 a year) for the next five years if Fairmont implements an acceptable CTS-prevention program that includes making the employees aware of CTS and how to reduce the chances of it developing. What would be the maximum amount that Fairmont should invest in the program to make it worthwhile? The firm's interest rate is 12% compounded annually.

ST3.5 Kersey Manufacturing Co., a small fabricator of plastics, needs to purchase an extrusion molding machine for $120,000. Kersey will borrow money from a bank at an interest rate of 9% over five years. Kersey expects its product sales to be slow during the first year, but to increase subsequently at an annual rate of 10%. Kersey therefore arranges with the bank to pay off the loan on a "balloon scale," which results in the lowest payment at the end of the first year and each subsequent payment being just 10% over the previous one. Determine the five annual payments.

ST3.6 Adidas will put on sale what it bills as the world's first computerized "smart shoe." But consumers will decide whether to accept the bionic running shoe's $250 price tag—four times the average shoe price at stores such as Foot Locker. Adidas uses a sensor, a microprocessor, and a motorized cable system to automatically adjust the shoe's cushioning. The sensor under the heel measures compression and decides whether the shoe is too soft or firm. That information is sent to the microprocessor and, while the shoe is in the air, the cable adjusts the heel cushion. The whole system weighs less than 40 grams. Adidas's computer-driven shoe—three years in the making—is the latest innovation in the $16.4 billion U.S. sneaker industry. The top-end running shoe from New Balance lists for $199.99. With runners typically replacing shoes by 500 miles, the $250 Adidas could push costs to 50 cents per mile. Adidas is spending an estimated $20 million on the rollout.[4]

The investment required to develop a full-scale commercial rollout cost Adidas $70 million (including the $20 million ad campaign), which will be financed at an interest rate of 10%. With a price tag of $250, Adidas will have about $100 net cash profit from each sale. The product will have a five-year market life. Assuming that the annual demand for the product remains constant over the market life, how many units does Adidas have to sell each year to pay off the initial investment and interest?

ST3.7 *Millionaire Babies: How to Save Our Social Security System.* It sounds a little wild, but that is probably the point. Former Senator Bob Kerrey, D-Nebraska, had proposed giving every newborn baby a $1,000 government savings account at birth, followed by five annual contributions of $500 each. If the money is then left untouched in an investment account, Kerrey said, by the time the baby reaches age 65, the $3,500 contribution per child would grow to $600,000, even at medium returns for a thrift savings plan. At about 9.4%, the balance would grow to be $1,005,132. (How would you calculate this number?) With about 4 million babies born each year, the proposal would cost the federal government $4 billion annually. Kerrey offered this idea in a speech devoted to tackling Social Security reform. About 90% of the total annual Social Security tax collections of more than $300 billion are used to pay current beneficiaries in the largest federal program. The remaining 10% is invested in interest-bearing government bonds that finance the day-to-day expenses of the federal government. Discuss the economics of Kerrey's Social Security savings plan.

[4] Source: "Adidas puts computer on new footing," by Michael McCarthy, USA Today—Thursday, March 3, 2006, section 5B.

ST3.8 Recently an NFL quarterback agreed to an eight-year, $50 million contract that at the time made him one of the highest paid players in professional football history. The contract included a signing bonus of $11 million. The agreement called for annual salaries of $2.5 million in 2005, $1.75 million in 2006, $4.15 million in 2007, $4.90 million in 2008, $5.25 million in 2009, $6.2 million in 2010, $6.75 million in 2011, and $7.5 million in 2012. The $11 million signing bonus was prorated over the course of the contract, so that an additional $1.375 million was paid each year over the eight-year contract period. Table ST3.8 shows the net annual payment schedule, with the salary paid at the beginning of each season.

(a) How much was the quarterback's contract actually worth at the time of signing?

(b) For the signing bonus portion, suppose that the quarterback was allowed to take either the prorated payment option as just described or a lump-sum payment option in the amount of $8 million at the time he signed the contract. Should he have taken the lump-sum option instead of the prorated one? Assume that his interest rate is 6%.

TABLE ST3.8 Net Annual Payment Schedule

Beginning of Season	Prorated Contract Salary	Actual Signing Bonus	Annual Payment
2005	$2,500,000	$1,375,000	$3,875,000
2006	1,750,000	1,375,000	3,125,000
2007	4,150,000	1,375,000	5,525,000
2008	4,900,000	1,375,000	6,275,000
2009	5,250,000	1,375,000	6,625,000
2010	6,200,000	1,375,000	7,575,000
2011	6,750,000	1,375,000	8,125,000
2012	7,500,000	1,375,000	8,875,000

ST3.9 Yuma, Arizona, resident Rosalind Setchfield won $1.3 million in a 1987 Arizona lottery drawing, to be paid in 20 annual installments of $65,277. However, in 1989, her husband, a construction worker, suffered serious injuries when a crane dropped a heavy beam on him. The couple's medical expenses and debt mounted. Six years later, in early 1995, a prize broker from Singer Asset Finance Co. telephoned Mrs. Setchfield with a promising offer. Singer would immediately give her $140,000 for one-half of each of her next 9 prize checks, an amount equal to 48% of the $293,746 she had coming over that period. A big lump sum had obvious appeal to Mrs. Setchfield at that time, and she ended up signing a contract with Singer. Did she make the right decision? The following table gives the details of the two options:

Year	Installment	Year	Installment	Reduced Payment
1988	$65,277	1995	$65,277	$32,639
1989	65,277	1996	65,277	32,639
1990	65,277	1997	65,277	32,639
1991	65,277	1998	65,277	32,639
1992	65,277	1999	65,277	32,639
1993	65,277	2000	65,277	32,639
1994	65,277	2001	65,277	32,639
		2002	65,277	32,639
		2003	65,277	32,639
		2004	65,277	
		2005	65,277	
		2006	65,277	
		2007	65,277	