2

Families of Functions

2.1 Introduction

Functions are fundamental to mathematics and its applications. Although there are many different types of functions, most of our work focuses on just a few: functions that are simple and yet sufficiently powerful to meet our needs. These types of functions can be thought of as *families of functions* because the members of each family are closely related to one another in terms of their essential properties. We have described several distinct behavior patterns already as phenomena that

1. increase at a fixed rate and so go up by the same amount each fixed time period;
2. decrease at a fixed rate and so go down by the same amount each fixed time period;
3. increase at an increasing rate and so are concave up;
4. increase at a decreasing rate and so are concave down;
5. decrease at an increasing rate and so are concave down;
6. decrease at a decreasing rate and so are concave up.

Figure 2.1 illustrates these behavior patterns.

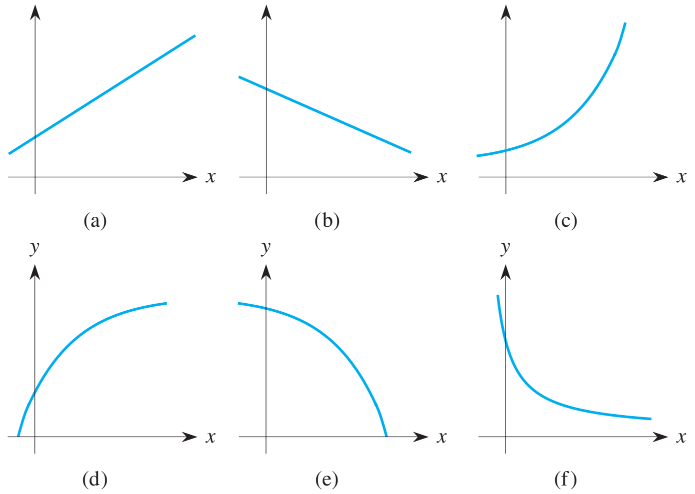


FIGURE 2.1

To model such phenomena and make predictions based on the models, you need to know families of functions that behave in each of these six ways. In this chapter, we present the families of *linear functions, exponential functions,* and *power functions,* as well as several other useful families, that possess these behavior patterns. In later chapters, we consider other families of functions, including *polynomial functions* and *trigonometric functions,* that exhibit more complex behavior patterns.

As discussed in Section 1.3, we use the letters x and y generically for the independent and dependent variables, respectively. However, in any specific context, we use letters that directly suggest the quantities under discussion.

2.2 Linear Functions

The simplest and probably most useful family of functions are the **linear functions.** These functions model any quantity that increases steadily or decreases steadily—that is, it goes up or down by a fixed amount for any fixed change in the independent variable. The graph of such a function is always a straight line.

Linear Functions That Pass Through the Origin

The simplest type of linear function is of the form $y = mx$, which can be interpreted as *y is proportional to x*. For example, suppose that you go to a deli to buy some roast beef that is selling at \$5.99 per pound. If you purchase 1 pound, the roast beef costs $C = 5.99 \times 1 = \$5.99$; if you purchase 2 pounds, it costs $C = 5.99 \times 2 = \$11.98$. If you buy N pounds, the cost is $C = 5.99N$, so the cost C of the roast beef is proportional to the number of pounds N that you buy. The multiple 5.99 is the *constant of proportionality.* We also say that the cost of the roast beef is a linear function of the number of pounds of roast beef purchased.

Similarly, the distance D that a car travels at a constant speed of 50 mph is proportional to the number of hours t driven, so the number of miles traveled is

$$D = 50t.$$

As another illustration, it is reasonable to assume that the quantity G of garbage produced in a city is proportional to the number of people P living there, so that $G = kP$, for some constant multiple k. In fact, the average amount of garbage produced annually in the United States is about 1500 pounds per person, so $k \approx 1500$. A mathematical model for the quantity G of garbage produced annually in a city whose population is P is therefore $G = 1500P$ pounds.

EXAMPLE 1

Suppose that gas costs \$1.50 per gallon. **(a)** Create a function, as a table, as a graph, and as a formula, to represent the cost C of G gallons of gas. **(b)** Create comparable functions if the price of gas rises to \$1.75/gal and to \$2.00/gal and compare them to one another.

Solution

a. If gas costs \$1.50/gal, the cost for 1 gallon is \$1.50, the cost for 2 gallons is \$3.00, the cost for 3 gallons is $3 \times 1.50 = \$4.50$, and so on. We therefore get the following table of values for this function.

***G* (gal)**	1	2	3	4	5	. . .	10	. . .	20	. . .
***C* (\$)**	1.50	3.00	4.50	6.00	7.50	. . .	15.00	. . .	30.00	. . .

When we plot these points, as shown in Figure 2.2, they all fall onto a line that passes through the origin (the cost of 0 gallons of gas is $0). To find an equation for this line, we note that, because each gallon of gas costs $1.50, the cost for buying G gallons must be

$$C = f(G) = 1.50 \times G = 1.50G.$$

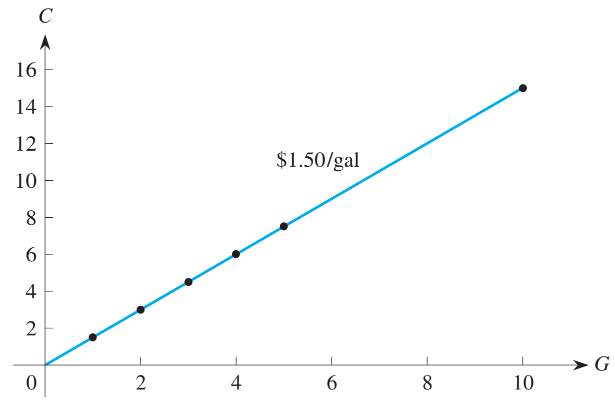


FIGURE 2.2

b. If the cost of gas rises to $1.75/gal, the corresponding function would be

$$C = g(G) = 1.75 \times G = 1.75G.$$

Similarly, if the cost of gas rises to $2.00/gal, the function would be

$$C = h(G) = 2.00 \times G = 2.00G.$$

The graphs of all three of these linear functions are shown in Figure 2.3. Note that all three lines pass through the origin, but that each is inclined at a slightly different angle. In particular, the greater the cost for a gallon of gas, the steeper the line, which makes sense because filling a tank costs more.

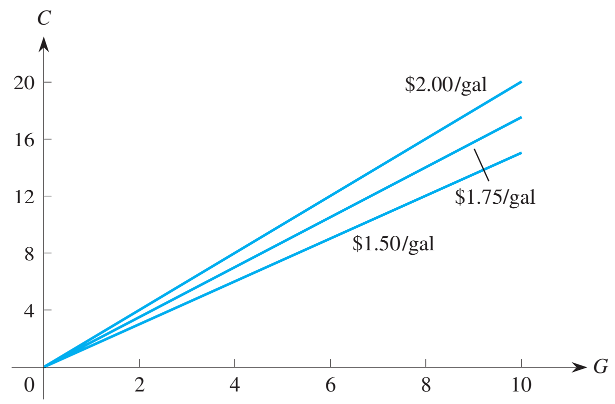


FIGURE 2.3

The Graph of a Linear Function That Passes Through the Origin

The graph of any linear function of the form $y = mx$ is a line that passes through the origin, as shown in Figure 2.4. What distinguishes one line from another is the constant m, which represents how much y changes for a given change in x. A large value for m, either positive or negative, means that y changes by a large amount for a fixed change in the variable x. A small m means that y changes relatively little for a fixed change in x. A positive value for m means that y gets larger as x gets larger. A negative value for m means that y gets smaller as x gets larger.

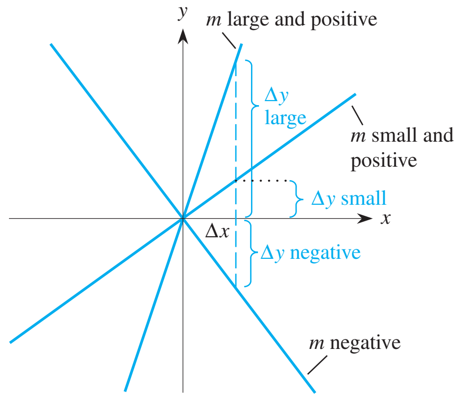


FIGURE 2.4

We use the Greek letter Δ (delta) to represent a change in any quantity. Hence Δx means the change in x, and Δy means the change in y. The quantity

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

is called the **slope** of the line. More generally, the slope of a line is

$$m = \frac{\text{change in dependent variable}}{\text{change in independent variable}}.$$

The letters used for the independent and dependent variables reflect what those quantities are and often are different from x and y. For instance, based on the data in the preceding table of values for the cost of gasoline, the independent variable G is the number of gallons of gas purchased and the dependent variable C is the cost of the gas. The slope of the line, based on the first two points, is

$$m = \frac{\Delta C}{\Delta G} = \frac{3.00 - 1.50}{2 - 1} = \frac{1.50}{1} = 1.50.$$

We get the identical value for the slope if we use any two of the points.

The Meaning of Slope

The slope of a line indicates how fast the linear function is changing. For the gasoline example $C = 1.50G$, the slope of 1.50 is the cost, \$1.50, of each additional gallon of gas. For the roast beef example $C = 5.99N$, the slope, 5.99, is the cost per pound, so each additional pound of roast beef costs an additional \$5.99. If the roast beef is on sale for \$3.99, the cost of the roast beef goes up more slowly as the weight of the purchase increases, and the slope is smaller. Similarly, if the price of roast beef goes up to \$6.99/lb, the cost goes up more rapidly, and the slope is steeper.

In general, whenever a linear function (or a line) arises in some context, the slope of that line should be given in terms of units. For instance, if we use a linear function to model the growth in the U.S. prison population over time, the units for the slope of the line might be the number of new prisoners per year.

Suppose that a car gets 25 mpg. Then

$$\text{Number of miles driven} = 25 \text{ miles per gallon} \times \text{number of gallons used},$$

or

$$25 \text{ miles per gallon} = \frac{\text{number of miles driven}}{\text{number of gallons used}}.$$

Therefore the expression "25 miles per gallon" actually describes the slope of a linear function. The units of the slope are always a ratio: the units of the dependent variable divided by the units of the independent variable.

Because the slope indicates how fast a line rises or falls, it is also known as the *average rate of change*, or simply the **rate of change.** For a linear function, the rate of change is always constant and is equal to the slope. For a nonlinear function, which may be concave up or concave down, the rate of change is not constant, as we demonstrate later in this chapter.

Lines That Don't Pass Through the Origin

Next let's consider lines that don't pass through the origin. The equation of any such line has the form

$$y = mx + b,$$

where

m is the slope of the line and

b is the vertical intercept.

The **vertical intercept** b represents the value of y when x is zero, because $y = m \cdot 0 + b = b$. The vertical intercept is sometimes called the ***y*-intercept.** A vertical intercept of zero $(b = 0)$ corresponds to the special case $y = mx$, which is the equation of a line that passes through the origin, as we discussed previously.

EXAMPLE 2

Graph the line $y = 3x - 4$ and describe it.

Solution The line has a slope of 3, so it rises 3 units for each increase of 1 unit to the right. It has a vertical intercept of -4, so the line crosses the y-axis 4 units below the origin. The graph of this line is shown in Figure 2.5.

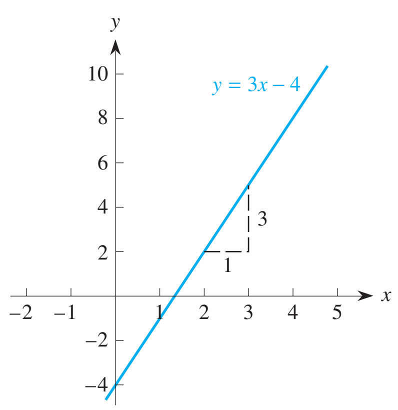


FIGURE 2.5

The graphs associated with the equations

$$y = f(x) = 2x - 1, \qquad y = g(x) = 2x + 1, \quad \text{and} \quad y = h(x) = 2x + 2$$

are shown in Figure 2.6. The three lines are parallel because they all have the same slope, $m = 2$, and so all rise 2 units for each 1 unit increase to the right. But their vertical intercepts are different: $b = -1$, $b = 1$, and $b = 2$ respectively.

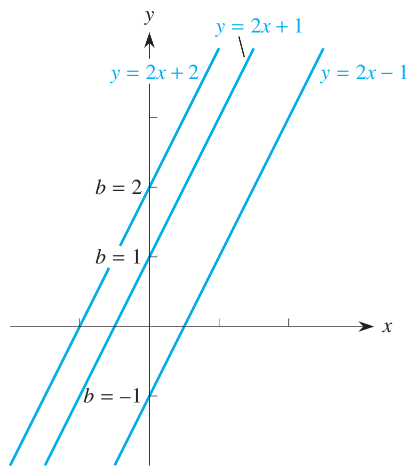


FIGURE 2.6

The graphs associated with the equations

$$y = f(x) = 2x + 1, \qquad y = g(x) = 3x + 1, \quad \text{and} \quad y = h(x) = -2x + 1$$

are shown in Figure 2.7. Note that, because all three lines cross the y-axis at the point $y = 1$, all have the same y-intercept, $b = 1$. However, the three lines have different slopes and so behave differently. The functions f and g are increasing as you move from left to right (as x increases), whereas the function h is decreasing as x increases. Moreover, the line $y = g(x) = 3x + 1$ is increasing more rapidly than the line $y = f(x) = 2x + 1$ because it has a larger slope. Again, the slope of the line determines whether a line rises or falls and how rapidly it does so.

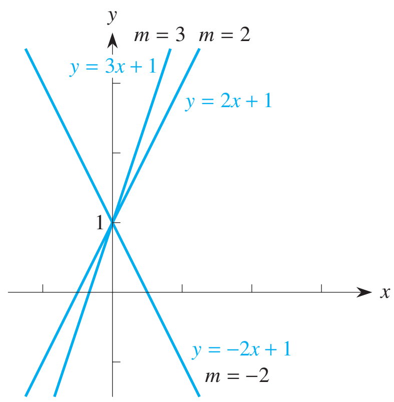


FIGURE 2.7

In summary, when the slope m is positive, the line rises as x increases from left to right and the linear function is increasing; the larger m is, the faster the line rises. When the slope m is negative, the line falls as x increases from left to right and the linear function is decreasing; the more negative the slope, the faster the line drops. (However, because a line doesn't bend, either up or down, it is neither concave up nor concave down.)

We can express the slope of a line,

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}.$$

in another way. If (x_1, y_1) and (x_2, y_2) are two points on the line, then

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1,$$

where the order of the coordinates must be the same in both Δx and Δy. Therefore the equation for the slope becomes

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Now let's explore these ideas in terms of the real world.

EXAMPLE 3

A wholesale supplier quoted the following costs, C, in dollars, for graphing calculators, depending on the number n of units ordered.

n	1	2	3	4	5	. . .
C	87	167	247	327	407	. . .

Find a linear function that models the costs and discuss the meaning of the slope and vertical intercept.

Solution Note that the cost for each additional calculator after the first is \$80. The \$87 charged for the first calculator consists of the \$80 for the calculator and shipping, plus an additional \$7 that covers the fixed cost for processing the order. This amount remains fixed no matter how many units are purchased. Therefore the cost C of buying n calculators can be written as the linear function $C = f(n) = 80n + 7$. The slope, 80, represents the increase in the total cost for each additional unit ordered—every time n increases by 1, C increases by 80. That is, the *rate* at which the cost is increasing is \$80 per calculator sold. The vertical intercept, 7, is the fixed cost for any size order. Figure 2.8 depicts the slope as the ratio

$$m = \frac{\Delta C}{\Delta n} = \frac{\text{rise}}{\text{run}} = \frac{167 - 87}{2 - 1} = 80,$$

or simply $m = \$80$ per calculator.

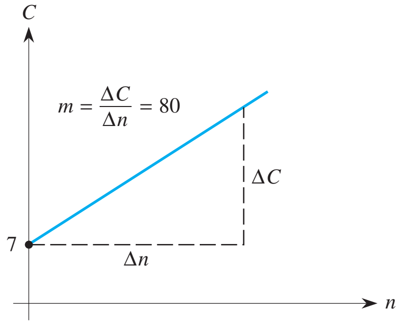


FIGURE 2.8

The value $m = 80$ was based on the two points $(1, 87)$ and $(2, 167)$. If you calculate the slope by using *any* two points on the line, you get the same value. It is this fact—that the slope, or rate of change, is the same at every point—that makes a line straight. If the rate of change varies from one point to another, then the function is not linear.

The form for the equation of the line $C = 80n + 7$ that we found in Example 3 is known as the *slope–intercept form* and usually is written as $y = mx + b$

because it highlights the *slope* $m = 80$ of the line and the *vertical intercept* $b = 7$.

The slope–intercept form is very useful for *displaying* the equation of a line. However, it is usually a poor choice for *finding* the equation of a line because the vertical intercept is often difficult to determine. Even when we can find the vertical intercept, it may have little to do with the situation we're studying. Example 4 demonstrates an easy way to apply the slope–intercept form.

EXAMPLE 4

A plumber charges \$50 for a service call to come to the job and \$70 per hour for labor. **(a)** Find a linear function for the plumber's charges for a job taking t hours (disregarding the costs for any parts). **(b)** What is the meaning of the slope in this function?

Solution

a. The plumber charges \$50 just for coming. For each hour on the job, the charge is an additional \$70, so a job lasting t hours costs an additional $70t$ dollars. Therefore the total cost is

$$C = 50 + 70t.$$

b. The slope of this line, 70, is the charge for each hour of labor and its units are dollars per hour.

Usually, a much better method for determining an equation of a line is the *point–slope form.* It is based on the idea that a line is determined by its slope m and one point (x_0, y_0) on the line. Suppose that (x, y) is any other point on the line, as shown in Figure 2.9. Because

$$m = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0},$$

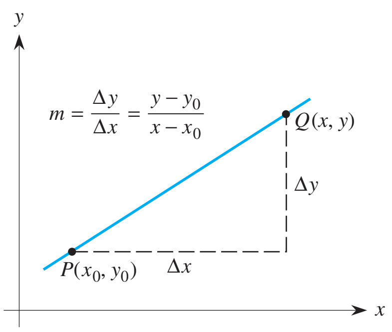


FIGURE 2.9

we can multiply both sides by $(x - x_0)$ to get

$$y - y_0 = m(x - x_0).$$

In summary we have the following formula.

Point–Slope Formula for the Equation of a Line

The equation of the line with slope m that passes through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0)$$

For instance, the line through the point $(5, 2)$ with slope 4 is

$$y - 2 = 4(x - 5).$$

You are almost always better off using the point–slope form rather than the slope–intercept form to *find* the equation of a line. In Example 5 we revisit the snow tree cricket from Example 2 in Section 1.4.

EXAMPLE 5

The following set of measurements relate the snow tree cricket's rate of chirping, in chirps per minute, to the temperature, in Fahrenheit.

Temperature, *T* (°*F*)	50	55	60	65	70	75	80
Rate, *R* (chirps/min)	40	60	80	100	120	140	160

a. Find a function that models the chirp-rate as a function of temperature.

b. Discuss the reasonableness of the model and give reasonable values for the domain and range.

Solution

a. The chirp rate is increasing steadily, so it is an increasing function of the temperature T. In particular, the chirp rate goes up 20 chirps/min for every 5°F increase in temperature. Equivalently, the chirp rate goes up 4 chirps/min for every 1°F increase in temperature. Figure 2.10 shows that the corresponding points clearly fall in a linear pattern. In other words the chirp rate R as a function of temperature T is a linear function.

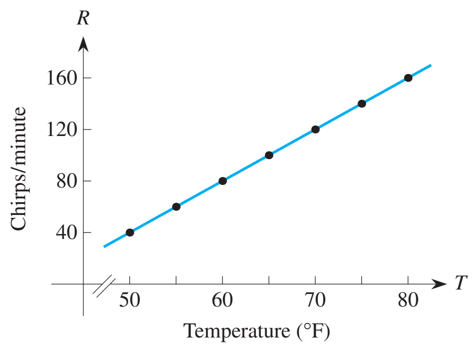


FIGURE 2.10

Because two points determine a line, we can use any two of the given points—say, $(55, 60)$ and $(75, 140)$—to find the equation for this line. Using these two points, as shown in Figure 2.11, we find that the slope of the line is

$$m = \frac{\Delta R}{\Delta T} = \frac{140 - 60}{75 - 55} = \frac{80}{20} = 4.$$

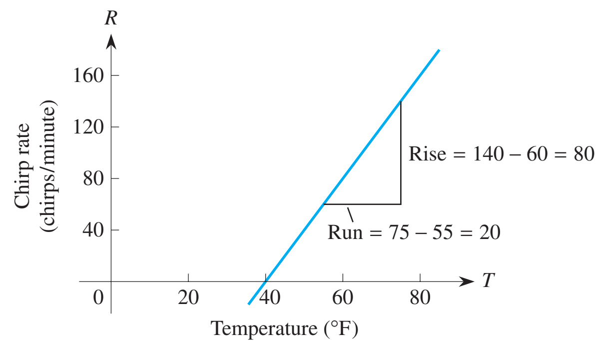


FIGURE 2.11

This value for the slope means that, for each 1°F increase in temperature, the cricket chirps 4 more times per minute. Thus, if the temperature goes up 5°F, the cricket chirps 20 more times per minute; if it goes up 10°F, the cricket chirps 40 more times per minute, and so on.

Next we apply the point–slope formula to find the equation of the line, using any point on the line. If we pick the point (55, 60) used earlier, we obtain

$$\begin{aligned} R - 60 &= 4(T - 55) \\ &= 4T - 220. \end{aligned}$$

Adding 60 to both sides of this equation, we get

$$R = f(T) = 4T - 160.$$

This equation tells us that the vertical intercept is $R = -160$ (when $T = 0$). Of course, a chirp rate of $R = -160$ is meaningless! Was the formula wrong? No. But it makes sense to describe the snow tree cricket's chirp rate only for temperatures between, or possibly near, the given set of readings—that is, from 50°F to 80°F. It does not make real-world sense to use this linear relationship far outside of this interval, such as at 0°. The formula doesn't predict sensible chirp rates for temperatures less than 40°F, when R becomes negative. It doesn't hold at temperatures high enough to cook the cricket either. Because temperatures in the Colorado Rockies aren't likely to rise above 100°F, there is a natural domain for this function:

$$\text{Domain of } f = \text{all values between } 40°\text{F and } 100°\text{F} \quad \text{or}$$
$$40 \leq \text{T} \leq 100 \quad \text{or} \quad [0, 100].$$

This function is strictly increasing, so we can find the corresponding range:

$$f(40) = 4 \cdot 40 - 160 = 0 \quad \text{and} \quad f(100) = 4 \cdot 100 - 160 = 240.$$

Thus

$$\text{Range of } f = \text{all values } R \text{ from 0 to 240} \quad \text{or} \quad 0 \leq R \leq 240 \quad \text{or} \quad [0, 240].$$

b. How reasonable are these results? At 100°F, the equation predicts that a snow tree cricket will chirp 240 times per minute, or 4 times per second, which we might decide is a bit unreasonable. Thus, even though the linear model predicts this value, we might want to rethink whether extending the linear model as far as $T = 100$°F makes sense when the upper limit of the data values is $T = 80$°F. As we've said previously, it is often misleading to extrapolate too far beyond the actual data values.

So far we have used the temperature to predict the chirp rate, and we thought of the temperature as the *independent variable* and the chirp rate as the *dependent* *variable.* However, we could reverse the role of the variables and think of temperature as a function of chirp rate. How we view a relationship determines which variable is dependent and which is independent. Thinking of temperature as a function of chirp rate would enable us to approximate temperature for given chirp rates. To do so, we again start with the formula

$$R = f(T) = 4T - 160$$

and solve it algebraically for T as a function of R. We add 160 to both sides to obtain

$$4T = R + 160$$

and then divide both sides by 4 to get

$$T = \frac{1}{4}(R + 160) = \frac{1}{4}R + 40 = g(R).$$

This linear function has slope $\frac{1}{4}$ and vertical intercept 40, except now the independent variable is R. So, if you ever encounter a snow tree cricket who is chirping merrily away, knowing this equation can help you determine the local temperature just by using your watch. Count the number of chirps in a one-minute interval and apply the formula to calculate the temperature.

We summarize the important information about linear functions as follows.

The **slope-intercept form** for the equation of a line is

$$y = mx + b,$$

where m is the **slope,** or rate of change of y with respect to x,

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}},$$

and b is the **vertical intercept,** or value of y when $x = 0$.

The **point–slope form** for the equation of a line with slope m that passes through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0).$$

Note that in the slope-intercept form for the equation of a line, there are two parameters, the slope m and the vertical intercept b. So linear functions are a *two-parameter* family of functions. We determine the equation of a particular line by finding the values of the two parameters.

EXAMPLE 6

During the early years of the Indianapolis 500 race held annually on Memorial Day, the average winning speed increased as shown in the following table. Find a formula to model these values.

Year	1919	1922	1925
Average speed (mph)	88	94.5	101

Source: *World Almanac and Book of Facts.*

Solution The average winning speed starts at 88 mph and increases at the rate of 6.5 mph each three years. Because the average winning speed S increases consistently by 6.5 mph every three years, S is a linear function of time over the period 1919 to 1925, as shown in Figure 2.12. The slope of this line is

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta S}{\Delta t} = \frac{6.5}{3} \approx 2.17,$$

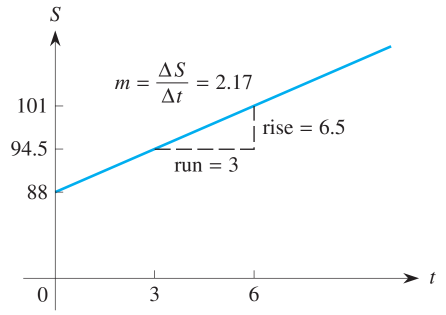


FIGURE 2.12

which shows the rate at which the winning speed increased each year. Let t be the number of years since 1919. Using the initial point (0, 88) gives the equation of this line as

$$S - 88 = 2.17(t - 0) = 2.17t,$$

or

$$S = f(t) = 2.17t + 88.$$

In this formula, t represents the number of years since 1919 and S is the speed in miles per hour. Note that we would get the same result if we used the slope–intercept form.

You may wonder whether this linear trend continued beyond 1925. Let's compare what it predicts with what actually happened. The fastest average winning speed in the Indy 500 was 186 mph in 1990, when $t = 71$ years after 1919. Using the linear equation $S = 2.17t + 88$, we predict an average speed of 242 mph in 1990. Clearly, although speeds have increased dramatically, they haven't kept up with the linear function we constructed based on just a few early data points. Further, this model again illustrates the danger of extrapolating too far from the given data.

Think About This

What does this information indicate about how long the 500-mile race takes? How much longer did it take the winning car to drive the 500 miles in 1919 than it took in 1990?

Because the data in the table are given only at specific points (every 3 years), we say that the data are *discrete*. However, because the function $S = 2.17t + 88$ makes sense for *all* possible values of t, we treat the variable t as though it were *continuous* (or defined for all points). The graph shown in Figure 2.12 is of a continuous function because it is a solid line including infinitely many points, not just the three distinct points representing the winning speeds in the race in three particular years.

EXAMPLE 7

Search and Rescue teams are often called on to find lost hikers in remote areas in the Southwest. Members of the search team walk through the search area parallel to each other at a fixed distance *d* between searchers. Experience has shown that the team's chance of finding those who are lost is related to the distance of separation *d*. The closer together the searchers are, the better are their chances of success. Based on a number of simulated missions, the percentage of lost people who were found was used to assess the probability of finding someone based on various separation distances, as shown in the following table of values. Find a formula to model these probabilities.

Distance *d* (ft)	20	40	60	80	100
Probability of success *P* (%)	90	80	70	60	50

(These values correspond to searches conducted in the relatively open terrain of the Southwest; searchers in other regions where there is dense forest or undergrowth would have to use much narrower separation distances to achieve comparable levels of success.)

Solution Because the value for the probability of success *P* decreases as distance *d* (the independent variable) increases, the function $P = f(d)$ is a decreasing function of *d*. The data indicate that each 20 foot increase in distance causes the probability of success *P* to decrease by 10%. Because this fact holds for any successive pair of points, *P* is a linear function of *d*, and the graph of the probability of success versus distance is a line, as shown in Figure 2.13. Based on the two data points (20, 90) and (100, 50), say, the slope of this line is

$$m = \frac{\Delta P}{\Delta d} = \frac{50 - 90}{100 - 20} = \frac{-40}{80} = -\frac{1}{2}.$$

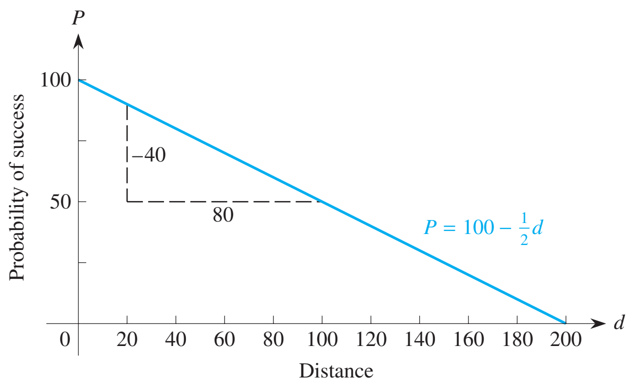


FIGURE 2.13

The negative sign reinforces the fact that *P* decreases as *d* increases. The slope is the rate at which *P* is decreasing as *d* increases.

To find the equation of the line, we use the point–slope formula. We choose any one of the given points—say, (20, 90)—and obtain

$$P - 90 = -\frac{1}{2}(d - 20)$$
$$= -\frac{1}{2}d + 10.$$

Adding 90 to both sides of this expression, we get

$$P = f(d) = -\frac{1}{2}d + 100.$$

Think About This

Pick any one of the other points in Example 7 and show that you get the same equation for P.

What is the meaning of the vertical intercept, $P = 100$? Suppose that $d = 0$ so that the searchers are walking shoulder to shoulder; we would expect everyone to be found, or $P = 100$. What is the horizontal intercept? When $P = 0$, we have $0 = -\frac{1}{2}d + 100$, or $d = 200$. According to the model, the value $d = 200$ represents the separation distance at which no one is found. This outcome is unreasonable because, even when the searchers are far apart, the search will sometimes be successful. What this situation suggests is that, somewhere outside the data given, the linear relationship ceases to hold. As in the Indy 500 example, extrapolating too far beyond the given data may not make sense.

Some Useful Facts

Several facts about lines are useful to remember.

1. *Parallel lines* have the same slope. That is, the quantities they represent are growing at the same rate.

Think About This

The lines $y = 4x + 3$, $y - 4x = 11$, and $4x - y - 15 = 0$ are all parallel. What is their common slope?

2. *Perpendicular lines* have slopes that are negative reciprocals.

The lines $y = 2x - 9$ and $y = -\frac{1}{2}x + 3$, having slopes of 2 and $-\frac{1}{2}$, respectively, are perpendicular to each other. Sketch their graphs to convince yourself of this fact. Similarly, the lines $y = 0.162x + 7.4$ and $y = -6.173x + 1.03$, which have slopes of 0.162 and $-6.173 \approx -1/0.162$, respectively, are perpendicular to each other.

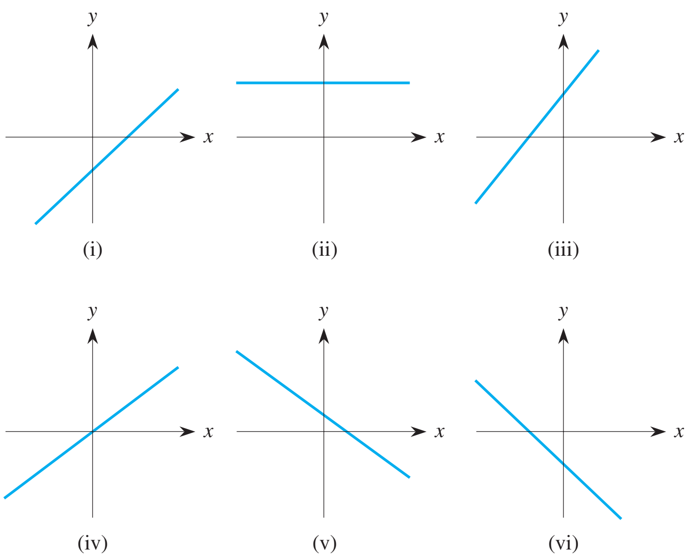
Think About This

Write the equation of a line that is perpendicular to $y = \frac{5}{4}x - 7$. (Of course, your answer will likely be different from your classmates' choices.)

3. The point where any two lines cross is known as their *point of intersection.* The x- and y-coordinates of this point must satisfy both equations simultaneously. You find the point of intersection by solving the system of simultaneous equations either algebraically or graphically. (See Appendix B and C for a discussion of ways to solve such systems of equations.)

Problems

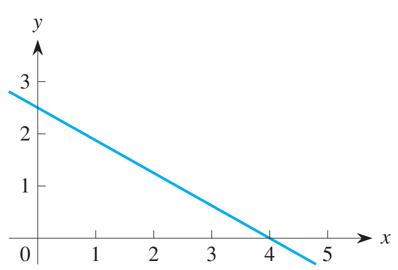
1. Match each equation with its graph. (Note that the scales of the graphs are different.)



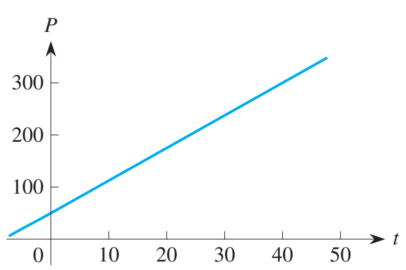
a. $y = x + 2$ **b.** $y = x - 3$
c. $y = -2x + 4$ **d.** $y = -3x - 4$
e. $y = \frac{1}{2}x$ **f.** $y = 3$

2. Estimate the slope of each line. Then use the slope to find an equation of the line.

a.



b.



3. Find the equation of the line passing through each pair of points.
   a. $(1, -2), (2, 5)$ **b.** $(1, -2), (3, -2)$
   c. $(3.52, 4.96), (-1.91, 8.36)$

4. The graph of Fahrenheit temperature F versus Celsius temperature C is a line. Water boils at 212°F and 100°C and freezes at 32°F and 0°C.
   a. Sketch the graph of the line.
   b. Find the slope of the line relating the two temperature scales.
   c. Find the equation of this line.
   d. Use the equation to find the Fahrenheit temperature that corresponds to 30°C.
   e. Use the equation to find the Celsius temperature that corresponds to 98.6°F.
   f. When is the Fahrenheit temperature the same numerical value as the Celsius temperature?

5. In 1990, 442.2 million prerecorded cassette tapes and 865.7 million CDs were sold in the United States. In 1998, 158.5 million cassettes tapes and 1,124.3 million CDs were sold. Assume (incorrectly) that the pattern of sales for both items is linear.
   a. Find the equation for the number of cassette tapes sold as a linear function of time.
   b. Find the equation for the number of CDs sold as a linear function of time.
   c. What is the practical significance of the slopes in parts (a) and (b)?
   d. If the trends in sales of both items were indeed linear, find when the number of CDs sold overtook the number of cassette tapes sold.
   e. Use the data given to find the total number of both CDs and cassette tapes sold in 1990 and 1998 and use these values to find the equation for the total number of sales of both items combined as a linear function of time.
   f. Use the fact that, in 1995, 272.6 million cassette tapes and 272.6 million CDs were sold to explain why assuming that the sales trends were linear is incorrect.

6. The charges for a taxi ride are an initial charge of \$1.80 and \$0.75 for each mile driven.
   a. Write a formula for the charge for a taxi ride as a linear function of the distance traveled.
   b. What is the meaning of the slope of this linear function?
   c. What is the cost of a 12-mile trip?
   d. Suppose that you have only \$15. How far can you go in the taxi? (Assume that you will give a \$2 tip out of the \$15 you have.)

7. A long-distance telephone company charges 40¢ to place a call from Los Angeles to London and 30¢ for each minute.

a. Write the equation of a linear function that models this situation.
b. What is the practical significance of the slope? Of the vertical intercept?
c. What is the cost of a 26-minute call?
d. Suppose that there is a 30% discount on the rates for calls made in off-peak hours. Repeat parts (a)–(c).

8. (Continuation of Problem 7) A competing long-distance company claims that it is cheaper because its rates on the Los Angeles to London call are 15¢ to place the call and 36¢ for each minute.

a. For the 26-minute call in Problem 7(c), which carrier is actually cheaper?
b. Graph both lines. What does the point where they intersect signify?
c. Find the length of call at which the second company becomes more expensive than the first.

9. A disk jockey (DJ) charges a flat fee of $120 per party plus $60 for each hour of the party. A second DJ charges $100 per party plus $75 for each hour.

a. For each DJ find a formula that gives the cost of hiring the DJ as a function of the number of hours the party lasts.
b. Sketch the graphs of both functions on the same set of axes.
c. How do you decide which DJ costs less?

10. The net income of the Apex Company was $240 million in 1980 and has been increasing by $30 million per year since. Over the same period, the net income of its chief competitor, the Best Corporation, has been growing by $20 million per year, starting with $300 million in 1980. Which company earned more in 1990? When did Apex surpass Best?

11. According to the IRS, the formula $T = 0.15I$, which gives income tax as a function of taxable income, applies only for single taxpayers with taxable incomes up to $21,450. The IRS tax table states: "If the taxable income is over $21,450, But not over $51,900, Enter on Form 1040: $3,217.50 + 28% of the amount over 21,450."

a. Rewrite this statement as an equation that can be used to calculate your taxes. What are the domain and range of the resulting function?
b. What is the practical meaning of the value you get for the slope?
c. Sketch a single graph showing both tax formulas. Is there any discrepancy?

12. When filing income tax returns, many people can claim deductions for depreciation on items such as cars and computers used for business purposes. The idea is that the value of such an asset decreases, or depreciates, over time. The simplest method used to find the depreciated value is called *straight-line depreciation,* which assumes that the item's value decreases as a linear function of time. If an $1800 computer system depreciates completely in five years, find a formula for its value as a function of time. What is it worth after three years?

13. The Athabasca glacier in southern Alberta, Canada, is part of the largest mass of ice in the Rocky Mountains. (Tourists who visit the Jasper and Banff National Parks can take a side trip out onto the actual glacier.) Over the past 120 years, the glacier has been steadily "withdrawing" at a rate of about 15 meters per year, as it slowly melts.

a. Express the approximate position of the southernmost extent of Athabasca as a function of time, measured in years from 1900. Measure its position northward from the U.S.–Canada border, which was about 300 kilometers south of the glacier in 1900.
b. If the current rate of withdrawal has been in effect indefinitely, how long ago did the toe of the glacier extend over the border?
c. Can the function in part (a) continue to apply for the next million years? Why or why not?

14. Jen is typing her term paper for Psych 101. She types the body of the paper at the rate of 35 words per minute for 30 minutes, then takes a 5-minute break, and comes back to do the references at a rate of 20 words per minute for 12 minutes.

a. Sketch the graph of Jen's typing rate as a function of time.
b. Sketch the graph of the total number of words she types as a function of time.
c. Find the equations of the different line segments you drew in part (b).

15. A bicyclist pedals at the rate of 1000 ft/min for 20 minutes, then slows to 500 ft/min for 6 minutes, then races at 1200 ft/min for 4 minutes, and cools down at 500 ft/min for 5 minutes.

a. Sketch the graph of the bicyclist's rate as a function of time.

b. Use the graph from part (a) to determine the total distance biked.
c. Sketch the graph of the distance traveled as a function of time.
d. Find the equations of the different line segments you drew in part (c).

16. The points P, Q, and R lie in order from left to right on the graph of a function f that is increasing. If the slope of line segment PQ is less than that of line segment QR, is the curve concave up or concave down? Explain your reasoning.

17. Find the equation of the line that passes through the point (6, 4) and is

a. parallel to the line $y = 5x - 3$.
b. perpendicular to this line.

18. Find the equation of the line that passes through the point (6, 4) and also passes through the point of intersection of $y = -2x + 1$ and $y = 3x + 6$.

19. The algebraic method of elimination for solving a system of linear equations involves adding a multiple of one equation to another equation to eliminate one of the variables. Consider the system of two equations in two unknowns:

$$3x - 4y = 1 \qquad \textbf{(1)}$$
$$2x + y = 8. \qquad \textbf{(2)}$$

a. Plot the two lines carefully on a sheet of graph paper and determine the point of intersection.
b. Solve the two equations algebraically.
c. Add two times Equation (2) to Equation (1) to get a new linear equation. Plot that line on the same graph you created in part (a). What do you observe about the three lines?
d. Add three times Equation (2) to Equation (1) and plot that line on the same graph. What do you observe about the four lines?
e. Add four times Equation (2) to Equation (1) and plot that line on the same graph. What do you conclude from this result?
f. Find an appropriate multiple of Equation (2) that, when added to Equation (1), will eliminate the x-term. What will the graph of the resulting line look like when x has been eliminated?

20. The point (3, 4) is on the circle $x^2 + y^2 = 25$.

a. Find the equation of the line that is tangent to the circle at this point.
b. Find the points where the line intersects the x and y axes. (*Hint*: The line tangent to a circle at a point is always perpendicular to the radius at that point.)

21. a. Of the following three linear functions, which two represent perpendicular lines?

i. $3x - 4y = 12$ **ii.** $2x + 5y = 10$
iii. $8x + 6y = 7$

b. For the two lines that are perpendicular, find the point of intersection.

Exercising Your Algebra Skills

Solve each equation for the appropriate variable.

1. $5x - 7 = 12$ **2.** $3x + 8 = -7$
3. $4x - 3 = -5$ **4.** $8x + 7 = 15$
5. $18y - 7 = 22$ **6.** $5.4x - 7.2 = 0.8$
7. $9 - 3x = 6$ **8.** $5 - 4p = -7$
9. $4.7q + 5.1 = 24.5$ **10.** $-1.3w + 12.8 = 22.7$
11. $4k + 7 = 9k - 8$ **12.** $6z - 5 = 4z + 11$
13. $3(2x - 5) = 4$ **14.** $2(4 - 3w) = 7$
15. $3.2(t - 1980) = 1700$ **16.** $1.35(t - 75) = 8$

Find **(a)** the slope and **(b)** the x and y intercepts of each line in Problems 17–20.

17. $2x - 3y = 8$ **18.** $2x + 3y = 8$
19. $4x + 7y + 5 = 0$ **20.** $3y - 2x + 4 = 0$

2.3 Linear Functions and Data

Determining Whether a Set of Data Is Linear

When you encounter tables of data relating two quantities, you will often need to determine whether a linear relationship exists between the two variables. You could plot the data points to see if the points fall into a linear pattern, but this approach is imprecise. Alternatively, you could decide whether a function $y = f(x)$ given by a table of values is linear by examining the data. If the data fall into a linear pattern, you should get the same slope no matter which pair of points you use. This reasoning gives a simple criterion for determining linearity: See whether the differences in y-values are constant for equally spaced x-values.

> If the x-values are uniformly spaced and there is a constant difference among the y-values, the data fall into a linear pattern.

You can visualize this principle by thinking of a long plank of wood and a flight of stairs. If the steps all have the same height—say, 8 inches, and the same depth, you can lay the plank on the stairs and it will touch the edge of each one, as illustrated in Figure 2.14. The plank plays the role of a line. But, if the stairs have different heights or depths, the plank won't touch every one of the edges—those edges do not fall in a linear pattern.

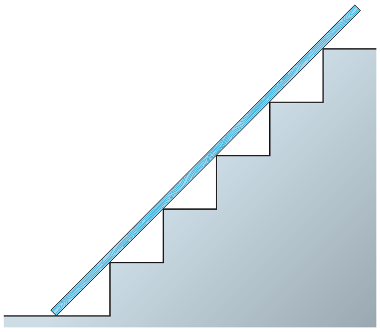


FIGURE 2.14

EXAMPLE 1

The following two sets of data represent values for a linear function and a nonlinear function. Identify which is the linear function. Find the equation of the line and the concavity of the nonlinear function.

x	$f(x)$
1.0	7.0
1.2	7.8
1.4	8.6
1.6	9.4
1.8	10.2
2.0	11.0
2.2	11.8

x	$g(x)$
1	2
2	3
3	6
4	11
5	18
6	27
7	38

Solution In both sets of values, the x-values are evenly spaced, so we can proceed to examine the successive differences in the values of the two functions, which we write as $\Delta f(x)$ and $\Delta g(x)$. For instance, for the function f the difference between the first two values is $7.8 - 7.0 = 0.8$. Continuing in this manner, we obtain the data on the next page.

x	$f(x)$	$\Delta f(x)$
1.0	7.0	
		$0.8 = 7.8 - 7.0$
1.2	7.8	
		$0.8 = 8.6 - 7.8$
1.4	8.6	
		0.8
1.6	9.4	
		0.8
1.8	10.2	
		0.8
2.0	11.0	
		0.8
2.2	11.8	

Because the difference is a constant 0.8 between the values of the function *f*, we conclude that this set of data is indeed linear. The slope of the line through these points is

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{0.8}{0.2} = 4.$$

Further, using the first point $(1, 7)$ and the point–slope form for the equation of a line, we find that the equation of the line is

$$\begin{aligned} y - 7 &= 4(x - 1) \\ &= 4x - 4. \end{aligned}$$

When we add 7 to both sides of this expression, we get

$$y = 4x + 3 = f(x).$$

Suppose that we try the same analysis on the values for the function *g*.

x	$g(x)$	$\Delta g(x)$
1	2	
		$1 = 3 - 2$
2	3	
		$3 = 6 - 3$
3	6	
		5
4	11	
		7
5	18	
		9
6	27	
		11
7	38	

The differences are not constant, so we conclude that these points don't fall into a linear pattern and hence no line passes through them. Consequently, the function *g cannot* be a linear function. In fact, because the differences are successively larger, the function is growing faster than a linear function grows. Because the function *g* is increasing at an increasing rate, it is concave up.

So far, we have given you information on some process or quantity that clearly is a linear function. In practice, however, you may face a situation in which you simply assume that one quantity grows or decays in a linear manner. Or you may even encounter a set of data that appears to be roughly linear in nature, but the particular data points do not precisely fall on a line. We illustrate both situations in Examples 2 through 4.

EXAMPLE 2

In 1990, the United States imported \$495 billion worth of goods. In 1998, the United States imported \$912 billion worth of goods. (Source: 2000 Statistical Abstract of the United States.)

a. Assuming that the growth in imports followed a linear pattern, find an equation of the linear function that models U.S. imports.

b. What is an appropriate domain for this model?

c. Use the model to predict the amount of imports in the year 2005.

d. Predict when the United States will import \$1 trillion worth of foreign goods according to this model.

Solution

a. For convenience, we take the independent variable t to be the number of years since 1990 and measure imports I in billions of dollars. We therefore have two points $(0, 495)$ and $(8, 912)$ for our linear model, as shown in Figure 2.15. The slope of the line through these points is

$$m = \frac{912 - 495}{8 - 0} \approx 52.1;$$

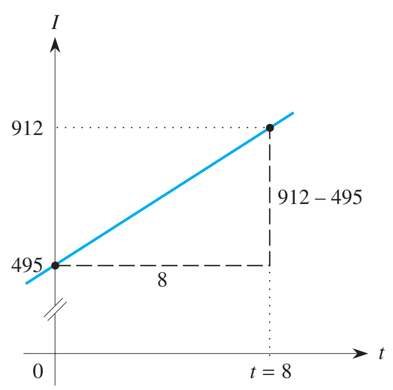


FIGURE 2.15

that is, imports have been growing at a rate of \$52.1 billion per year. Using the point–slope form for a line and the point $(0, 495)$ yields

$$I - 495 = 52.1(t - 0),$$

or equivalently the slope–intercept form

$$I = 52.1t + 495.$$

b. The data extend from 1990 when $t = 0$ to 1998 when $t = 8$. A reasonable domain might be from $t = -5$ to $t = 13$, allowing us to predict 5 years before and after the data points.

c. Assuming that this linear trend continues, we predict that the value of foreign goods that will be imported in 2003, when $t = 13$ years after 1990, is

$$I = 52.1(13) + 495 = \$1172.3 \text{ billion}.$$

d. Using this linear model, we have to solve for the value of t when

$$I = 52.1t + 495 = 1000 \text{ billion } (=1 \text{ trillion}).$$

If we subtract 495 from both sides of this equation, we obtain

$$52.1t = 505$$

so that

$$t = 505/52.1 \approx 9.7,$$

or sometime late in 1999.

In Example 2 we arbitrarily chose to count years from 1990. Let's see what happens if we choose a different baseline.

EXAMPLE 3

As in Example 2, the United States imported \$495 billion worth of goods from abroad in 1990 and \$912 billion in goods in 1998. Assuming that the growth in imports followed a linear pattern, find an equation of the linear function that models U.S. imports based on using the independent variable t to represent **(a)** the number of years since 1900 and **(b)** the number of years since year 0. For each model, state an appropriate domain and compare each model to the one constructed in Example 2. **(c)** Use each model to predict the amount of imports in 2003.

Solution In Example 2, we took the independent variable t to be the number of years since 1990, or equivalently used $t = 0$ in 1990, and constructed the linear model

$$I = 52.1t + 495.$$

a. Now suppose that the independent variable t is the number of years since 1900. We therefore have the two points (90, 495) and (98, 912). The slope of the line through these points is

$$m = \frac{912 - 495}{98 - 90} \approx 52.1,$$

which is the same value obtained before. Using the point–slope formula and the point (90, 495) gives the equation of this linear function as

$$I - 495 = 52.1(t - 90)$$

or

$$I = 52.1t - 4194.$$

Although the slope remained the same, the vertical intercept changed dramatically. The reason is that we now think of the line as "starting" in 1900, not 1990, so it has been climbing for 90 years at the rate of \$52.1 billion per year. A reasonable domain for this linear model might be from $t = 85$ to $t = 103$.

b. Now suppose that the independent variable t is the number of years since the year 0. Our two points are now (1990, 495) and (1998, 912), and the slope of the line through these points is

$$m = \frac{912 - 495}{1998 - 1990} \approx 52.1,$$

which again is the same value. Using the point–slope formula and the point (1990, 495), the equation of this linear function is

$$I - 495 = 52.1(t - 1990)$$

or

$$I = 52.1t - 103{,}184.$$

There has been a huge change in the vertical intercept because this line has been climbing at a rate of \$52.1 billion per year for almost 2000 years! An appropriate domain for this linear function might be from $t = 1985$ to $t = 2003$.

c. Using any one of the three linear models, the first with $t = 13$, the second with $t = 103$, and the third with $t = 2003$, we obtain the identical prediction for the total value of imports into the United States of about $I = \$1172$ billion in 2003.

So, which of these three models is correct? In one sense, all three are correct because they give the same predictions. In another sense, they are all wrong, because the equation by itself, without reference to what the variable t stands for, is incomplete—if we don't specify the meaning of the variable or its "starting" point, someone using the equation to make predictions may well use a different interpretation. We get very different answers from the first model,

$$I = 52.1t + 495,$$

if we use $t = 13$, $t = 103$, and $t = 2003$. Therefore we should write the three models as

$$I = 52.1t + 495, \qquad \text{where } t \text{ is the number of years since 1990,}$$

or

$$I = 52.1t - 4194, \qquad \text{where } t \text{ is the number of years since 1900,}$$

or

$$I = 52.1t - 103{,}184, \qquad \text{where } t \text{ is the number of years since the year 0.}$$

Capturing a Linear Pattern in Data

In most applications of linear functions in the real world, you will typically have far more than two points; in fact, you will often have a relatively large set of points that fall into a *roughly linear pattern.* In Example 4 we illustrate how to deal with this important situation.

EXAMPLE 4

The following table of values gives some measurements for the rate of chirping (in chirps/sec) of the striped ground cricket as a function of the temperature.

T (°F)	89	72	93	84	81	75	70	82	69	83	80	83	81	84	76
Chirps/sec	20	16	20	18	17	16	15	17	15	16	15	17	16	17	14

Source: Adapted from George W. Pierce, *The Songs of Insects.* Boston: Harvard University Press, 1948.

Even though the measurements presented for the snow tree cricket in Chapter 1 fell exactly onto a straight line, Figure 2.16(a) shows that comparable measurements for the striped ground cricket clearly do not. The difference may be due to errors in measurement; it may be that the striped ground cricket is less sensitive to temperature; or perhaps the snow tree cricket has more mathematical aptitude to get the situation right. Even though the points for the striped ground cricket do not fall precisely on a line, they do fall in a roughly linear pattern. Find an equation that captures this linear pattern.

Solution Suppose that we take a piece of black thread (or a clear plastic ruler), hold it taut, and move it back and forth over the points in Figure 2.16(a). Each possible orientation for the thread represents a different line. We can then select an orientation that seems, by eye, to give the best match or *fit* to the linear pattern in the data. Usually we want roughly half the points to be above the thread and half below it, so that the line passes "midway" between the points and follows the overall trend. Such a line superimposed over the data points is shown in Figure 2.16(b). (Obviously, different people will come up with slightly different lines.) We now estimate the equation of this line that captures the overall trend of the chirp rate function.

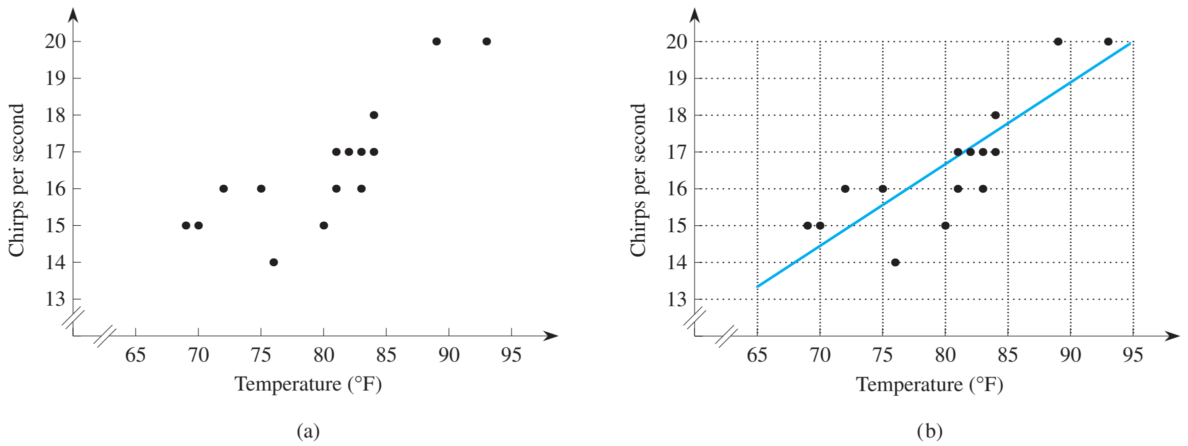


FIGURE 2.16

In Figure 2.16(b) this line seems to pass through the points (68, 14) and (90, 19). (Note that these points are *on* the line; we chose them for convenience. The points used are not necessarily actual data points. In fact, unless the line drawn happens to pass through a data point, you should not use any of the data points to estimate the slope.) Using these two points, we find that the slope of the line is approximately

$$m = \frac{19 - 14}{90 - 68} = \frac{5}{22} \approx 0.23.$$

This means that the chirp rate increases about 0.23 chirp/sec for each 1°F increase in temperature. Further, because the line apparently passes through the point (68, 14), we conclude that the equation of the line is

$$C - 14 = 0.23(T - 68),$$

or, when simplified,

$$C = 0.23T - 1.64.$$

In applying this "black thread method" to find the equation of the line, you must use two points that are *on the line* you draw. Do not use data points that are not on the line and do not force a line by drawing one that must pass through any of the data points. Note that the result you get is just an *estimate* for the equation of the line that visually best fits the linear trend in the data. In Chapter 3, we introduce methods for finding the equation of the one line that is the *best fit* to a set of data in a certain sense.

Implicit Linear Functions

We now consider a somewhat different type of situation, in which two quantities are related but in such a way that we can't necessarily identify which variable is independent and which is dependent. An ongoing debate at all levels of government concerns the allocation of money among different programs. Because typically only a fixed amount of money is available, the more that is spent on one program, the less there is to spend on other programs. Let's look at a simple case involving just two competing programs, funding road and highway repairs versus funding day-care centers.

EXAMPLE 5

Suppose that we have a total of $1,000,000 available to divide between day-care centers, which cost $200,000/center, and road repaving, which costs $50,000/mile. Find an equation of a linear function relating the number of day-care centers and the number of miles of road to be repaved. What are the domain and range of this function?

Solution Let c be the number of day-care centers and r be the number of miles of road to be repaved. Then the amount of money spent on road repaving is $50,000$r$ (it costs $50,000 to repave each mile), and the amount spent on day-care centers is $200,000$c$ (it costs $200,000 for each day-care center). Assuming that all the available money is spent, we get

$$\begin{array}{ccccc} \text{amount spent on centers} & + & \text{amount spent on repaving} & = & 1{,}000{,}000 \\ 200{,}000c & + & 50{,}000r & = & 1{,}000{,}000 \end{array}$$

or equivalently, when we divide both sides by 50,000,

$$r + 4c = 20.$$

This equation is called the *budget constraint.* To graph this equation, we first find the points at which the graph crosses the axes (the intercepts), as shown in Figure 2.17. If $c = 0$, then $r + 4(0) = 20$, so that $r = 20$. At the other extreme, if $r = 0$, we have $0 + 4c = 20$, so that $c = 5$.

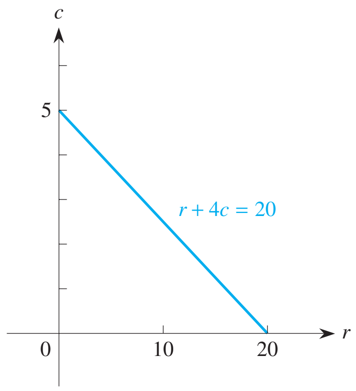


FIGURE 2.17

Because all the money not spent on road work is used for day-care centers, the number of centers funded is a function of the number of miles of roads repaved. That is, c is a function of r, and we can solve the budget constraint equation $r + 4c = 20$ for c. We subtract r from both sides of the equation and get

$$4c = 20 - r.$$

We then divide both sides by 4 and obtain

$$c = f(r) = \frac{1}{4}(20 - r) = 5 - \frac{1}{4}r.$$

Similarly, the number of miles repaved is a function of the number of centers funded, so r is a function of c. We can solve the budget constraint equation for r by subtracting $4c$ from both sides to get

$$r = g(c) = 20 - 4c.$$

To determine the applicable domain and range, we recognize that r makes sense only for values between 0 and 20, whereas c makes sense only for values between 0 and 5. Which of these is the domain and which is the range depends on which variable we think of as the independent variable and which as the dependent variable.

Note that the budget constraint equation

$$r + 4c = 20$$

is an example of a third way of writing the equation of a line, called the *normal form* of a line. In general, an equation of the form $ax + by = c$ is the **normal form** of a line. It is algebraically equivalent to either the point–slope form or the slope–intercept form. For instance, if

$$3x - 5y = 15,$$

then

$$5y = 3x - 15$$

and when we divide both sides by 5, we get

$$y = \left(\frac{1}{5}\right)(3x - 15) = \left(\frac{3}{5}\right)x - 3,$$

a line that has slope $\frac{3}{5}$ and vertical intercept -3.

To graph a line given in normal form, the easiest way is to find and plot both the vertical and the horizontal intercepts and connect them with a straight line. Thus, to find the vertical intercept of $3x - 5y = 15$, we set $x = 0$ and solve $-5y = 15$ to get $y = -3$. To find the horizontal intercept, we set $y = 0$ and solve $3x = 15$ to get $x = 5$. The resulting graph is shown in Figure 2.18.

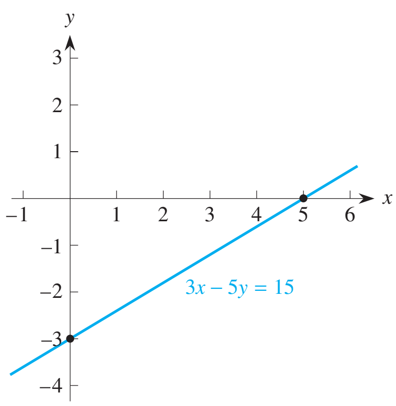


FIGURE 2.18

Incidentally, as a function, an equation such as $ax + by = c$ is called an **implicit function.** It is a function, but neither variable, x nor y, is given explicitly in terms of the other.

Problems

1. Determine which of the functions are linear. For any linear function, find the equation of the line and use it to predict the next entry to extend the table of values.

a.

x	5	6	7	8	9
y	77	71	65	59	53

b.

t	50	60	70	80	90
$L(t)$	23.2	23.9	24.6	25.2	25.9

c.

t	75	80	85	90	95
$Q(t)$	125.1	127.5	129.9	132.3	134.7

2. The data in each table of values lie along a line.
   a. For each set of data, carefully plot the points on graph paper, estimate by eye the slope and vertical intercept, and use these values to approximate the equation of the line.
   b. Then find the equation of the line algebraically. How close was your estimate?

i.

x	1	2	3	4
y	1.81	3.34	4.87	6.40

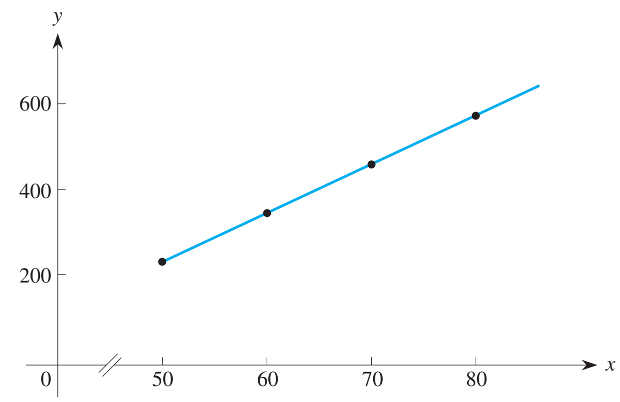
ii.

x	1	2	3	4
y	1.08	0.69	0.30	−0.09

3. Find the equation of a linear function that fits this set of values.

x	4	5	6	7
y	1.557	1.614	1.671	1.728

4. **a.** Explain why the equation of the line shown in the accompanying figure is *not* $y = 10x + 200$.



   b. Find the correct equation of the line.

5. In 1980 (when $t = 0$), \$26.5 billion were spent on water pollution prevention and cleanup in the United States. In 1990, \$33.1 billion were spent.
   a. Construct the linear function giving the amount spent on water pollution as a function of time t, where $t = 0$ in 1980.
   b. Use the linear function to estimate the amount spent in 2002.
   c. Repeat parts (a) and (b) if $t = 0$ in 1900.

6. Inspector Clueless, while investigating the murder of Mr. Jones, found the murderer's size $11\frac{1}{2}$ footprint in a flower bed. The inspector mutters something about the killer being "a man who is %#$#$&&# tall." If the equation of the best-fit line relating shoe size to height in inches is $S = 0.51H - 25.2$, decipher Clueless's muttering.

7. The table of values shows the total value, in billions of dollars, of electronics and electronic components produced in the United States during the 1990s.

t	1990	1993	1994	1995	1996	1997	1998
E	43.0	52.5	58.2	62.5	68.3	72.4	76.0

Source: *2000 Statistical Abstract of the United States.*

   a. Use graph paper to plot these data and use the black thread method to sketch the best-fit line.
   b. Estimate the slope of this line and tell what it means.

c. What is your best estimate for the equation of this line?

d. Use this line to estimate the total value of electronics and electronic components produced in 2004.

8. The table shows the height H in feet and the number of stories n in some notable buildings.

	H	n
Empire State Building (New York)	1250	102
John Hancock Tower (Boston)	788	61
Sears Tower (Chicago)	1450	110
NationsBank Plaza (Dallas)	921	72
TCBY Tower (Little Rock)	546	40
Peachtree Center (Atlanta)	374	31
Place Ville Marie (Montreal)	620	45

Source: *World Almanac and Book of Facts.*

a. Which variable, H or n, is the independent variable and which is the dependent variable?

b. Plot these points carefully on a sheet of graph paper and use the black thread method to locate and draw the line that seems to best fit the data points.

c. Estimate the equation of this line.

d. What is the meaning of the slope of this line?

e. Use your answer to part (c) to estimate the number of stories in a building 860 feet tall.

f. What is your best estimate of the height of a building that has 96 stories?

9. The table of values at the bottom of the page gives data relating a car's gas mileage to its weight.

a. Plot these points carefully on a sheet of graph paper and use the black thread method to locate and draw the line that seems to best fit the data points.

b. Estimate the equation of this line.

c. Use your answer to part (b) to estimate a car's gas mileage if it weighs 2350 pounds, 3100 pounds, 1950 pounds.

d. What is your best estimate of the weight of a car that gets 32 mpg?

10. A student who works as a waiter in a restaurant records the cost C of meals and the tip T left by couples. His data for one evening are as follows.

C ($)	28.55	31.04	32.76	33.38	36.10	38.54
T ($)	4.25	4.50	5.00	5.00	5.50	6.00

Source: Student project.

a. Plot these points on a sheet of graph paper and draw the best line you can to fit the points. Explain your choices of the independent and the dependent variable.

b. Suppose that the equation for this function is $T = 0.18C - 0.93$. In terms of this mathematical model, what is the increment in the tip for each $1 increment in the cost of the meal?

c. What does the slope of the line in part (b) represent? What significance does the vertical intercept have?

d. Suggest possible values for the domain and range of this function.

11. You have a fixed budget of $30 to spend on nuts and Gummi Bear™ candy for a party. The nuts cost $3 per pound, and the candy costs $2 per pound.

a. Write an equation expressing the relationship between the number of pounds of nuts and of Gummi Bears that you can buy if you spend your budget completely. This equation is your budget constraint.

b. Graph the budget constraint, assuming that you can buy any fractional amount of a pound. Label the intercepts.

c. What are the domain and range for this function?

d. Suppose that your roommate chips in an additional $30 for the party. Graph the new budget constraint on the same set of axes used for the budget constraint graphed in part (b).

Weight (lb)	2100	2200	2400	2500	2800	3000	3200
Mileage (mpg)	37	34	29	27	26	25	23

Source: Student project.

e. Keep the original budget at \$30 and suppose that the Gummi Bears go on sale for half the price. Sketch the new budget constraint on the same axes used in part (d).

f. Keep the original budget at \$30 and suppose that the price of nuts suddenly doubles because of a frost in the Southeast. Sketch the new budget constraint on the same axes used in part (d).

12. For the implicit equation of a line $4p - 3q = 5$, find the following.

a. An explicit function that gives p as a function of q.
b. The slope of the line in part (a).
c. An explicit function that gives q as a function of p.
d. The slope of the line in part (c).

13. a. Find the slope and vertical intercept of each line given in normal form.

$$3y - 2x = 12 \quad \text{and} \quad 4x + 5y = 20.$$

b. Draw the graphs of the two lines on the same axes.
c. Find the point of intersection of the two lines
i. graphically;
ii. numerically by trial-and-error;
iii. algebraically.

14. Repeat Problem 13 for the two lines

$$3y - 2x = 12 \quad \text{and} \quad 4x + 5y = 21.$$

15. Suppose that a function f is increasing and concave up and that $f(60) = 250$ and $f(70) = 300$. Which values are possible and which are impossible? Explain.

a. $f(65) = 270$ **b.** $f(65) = 275$
c. $f(65) = 280$ **d.** $f(100) = 400$
e. $f(100) = 450$ **f.** $f(100) = 500$
g. $f(40) = 100$ **h.** $f(40) = 150$
i. $f(40) = 200$

16. Suppose that a function f is decreasing and concave up, and that $f(10) = 80$ and $f(12) = 70$. Which values are possible and which are impossible? Explain.

a. $f(11) = 78$ **b.** $f(11) = 75$
c. $f(11) = 72$ **d.** $f(15) = 50$
e. $f(15) = 55$ **f.** $f(15) = 60$
g. $f(5) = 100$ **h.** $f(5) = 105$
i. $f(5) = 110$

17. Draw the graph of a function f that is decreasing and concave up. Mark three points on the curve: P near the left, Q near the center, and R near the right. These points determine three line segments: PQ, QR, and PR.

a. List the three line segments in the order of increasing slopes.
b. List the three segments in the order of increasing steepness.

18. Repeat Problem 17 if the function is decreasing and concave down.

Exercising Your Algebra Skills

Solve each formula for the indicated variable.

1. $A = bh$, for h **2.** $C = 2\pi r$, for r
3. $A = \pi r^2$, for r **4.** $K = \frac{1}{2}mv^2$, for m
5. $K = \frac{1}{2}mv^2$, for v **6.** $F = \frac{GmM}{d^2}$, for d
7. $T = 2\pi\sqrt{\frac{l}{g}}$, for l **8.** $F = \frac{mv^2}{r}$, for r
9. $F = \frac{mv^2}{r}$, for v

Solve each equation in normal form for y in terms of x. Identify the slope in each case.

10. $4x - 5y = 20$ **11.** $6x + 5y = 30$
12. $5x - 4y = 10$ **13.** $2x + 7y = 9$

2.4 Exponential Growth Functions

The population of Florida was 12.94 million in 1990 and has been growing as shown in the following table of values. Let's see if we can find a mathematical pattern for the way in which this population is growing. If the population grows linearly, the changes or increases in population from one year to the next, ΔP, would all be the same. Let's check these differences.

Year	Population	ΔP
1990	12.94	
		0.38 = 13.32 − 12.94
1991	13.32	
		0.38
1992	13.70	
		0.40
1993	14.10	
		0.41
1994	14.51	
		0.42
1995	14.93	
		0.43
1996	15.36	
		0.45
1997	15.81	

Not only are the successive differences not constant, they are increasing. This makes sense because as the population grows, there are more people to have babies. Consequently, Florida's population has been growing at a faster than linear rate. We therefore need a concave up function to model this population over time.

Instead of taking differences, suppose that we take ratios of successive terms. To do so, we divide the population in any year by the population in the preceding year. This quotient gives

$$\frac{\text{Population in 1991}}{\text{Population in 1990}} = \frac{13.32 \text{ million}}{12.94 \text{ million}} \approx 1.029,$$
$$\frac{\text{Population in 1992}}{\text{Population in 1991}} = \frac{13.70 \text{ million}}{13.32 \text{ million}} \approx 1.029,$$
$$\frac{\text{Population in 1993}}{\text{Population in 1992}} = \frac{14.10 \text{ million}}{13.70 \text{ million}} \approx 1.029,$$

and so on. If you check the population figures for the subsequent years through 1997, you will find that each year the population grew by the same factor of about 1.029.

Because the ratios of successive population values are constant, we have, for any year,

$$\frac{\text{Population next year}}{\text{Population this year}} = 1.029$$

or

$$\text{Population next year} = 1.029 \cdot \text{population this year}.$$

If this trend continues, we can estimate Florida's population in 1998 as

$$\text{Population in 1998} = 1.029 \cdot \text{population in 1997} = 1.029 \cdot 15.81 = 16.27.$$

The fact that Florida's population next year is 1.029 times this year's population is equivalent to saying that

$$\begin{aligned}\text{Population next year} &= 1.029 \cdot \text{population this year}\\ &= (1 + 0.029) \cdot \text{population this year}\\ &= \text{population this year} + 0.029 \cdot \text{population this year}.\end{aligned}$$

In other words, *each* year between 1990 and 1997, Florida's population grew by about $0.029 = 2.9\%$ from one year to the next. The number 2.9% is called the annual growth rate for the population.

Whenever the successive ratios are constant (here they all are 1.029), the function is an **exponential function.**

We now find an equation for this exponential function $P(t)$, where t is the number of years since 1990. The starting population, when $t = 0$, is $P(0) = 12.94$, which we write as P_0. Then

when $t = 1$, $P(1) = 1.029 \cdot P_0$, or 13.32;
when $t = 2$, $P(2) = 1.029 \cdot P(1) = 1.029 \cdot (1.029 \cdot P_0) = (1.029)^2 P_0$, or 13.70;
when $t = 3$, $P(3) = 1.029 \cdot P(2) = 1.029 \cdot (1.029)^2 P_0 = (1.029)^3 P_0$, or 14.10;

and so on. In general, after t years, the population of Florida is

$$P(t) = P_0 \cdot (1.029)^t = 12.94(1.029)^t.$$

This equation is called an *exponential growth function* with *base* 1.029. The name *exponential* is used because the independent variable (in this case, t) occurs in the exponent. The base (in this case, 1.029) is called the *growth factor.* It gives the population each year as 1.029 times the population in the preceding year. The quantity $0.029 = 2.9\%$ is the associated annual *growth rate.* Note the relationship between the growth factor and the growth rate.

$$\text{Growth factor} = 1 + \text{growth rate}$$

In this formula, you must write the growth rate as a decimal, not as a percent. For instance, if the growth rate for a process is $4\% = 0.04$ each year, the associated growth factor is $1 + 0.04 = 1.04$.

Assuming that Florida's population continues to grow with the same exponential pattern for the next 80 years, we can graph this population function as shown in Figure 2.19. The function obviously is increasing. Moreover, the graph grows faster and faster as time goes on, so the curve is concave up. This behavior is typical of an exponential growth function. Compare this function's behavior with that of an increasing linear function. Because a linear function grows at the same rate at *every* point, its graph is a line. However, exponential growth functions such as this one are curves that may seem to climb slowly at first but eventually climb extremely rapidly. This type of behavior explains why there is widespread concern about

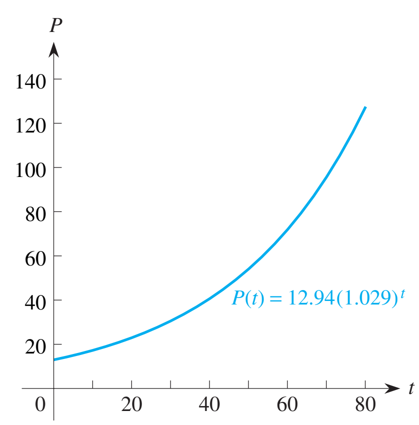


FIGURE 2.19

the exponential growth of the world's population: Eventually there won't be enough land, food, and water to sustain everyone.

The graph shown in Figure 2.19 is only an approximation to the actual graph of Florida's population. We can't have a fraction of a person, so the graph theoretically should be jagged with small steps up or down each time someone is born, dies, or moves into or out of Florida. However, on a scale of millions of people, such changes are insignificant, and our smooth curve actually is a good approximation to the population.

We summarize the formula for an exponential growth function and its parameters as follows.

Formula for an Exponential Growth Function

P is an **exponential growth function** of t with base $c > 1$, if

$$P(t) = P_0 c^t,$$

where P_0 is the initial quantity (when $t = 0$) and c is the **growth factor** by which P changes when t increases by 1 unit.

Because $c > 1$, we can write $c = 1 + a$, where a is the **growth rate** written as a decimal.

For example, if a quantity (e.g., the balance in your bank account) is growing at 5% per year, the growth rate $a = 5\% = 0.05$ and the associated growth factor is $c = 1 + a = 1.05$. The corresponding formula for the balance B in the account as a function of time t is

$$B(t) = B_0 \cdot (1.05)^t,$$

where B_0 represents the initial or starting balance.

The growth factor c in any exponential growth function $y = P_0 c^t$ plays a role similar to the slope in a linear function. The larger the growth factor c, the faster the exponential function grows. Figure 2.20 shows a series of exponential growth curves, with growth rates a of 3%, 4%, and 5% and corresponding growth factors c of 1.03, 1.04, and 1.05. All start with the same initial value at time $t = 0$, but the larger the growth factor, the faster the curve grows. A curve corresponding to a growth factor of 1.033, say, would lie between the curves for the growth factors of 1.03 and 1.04; the curve corresponding to a growth rate of 2.6% would lie below the lowest of the three curves.

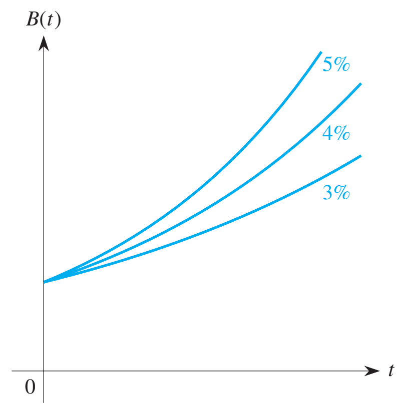


FIGURE 2.20

The coefficient P_0 in an exponential function $y = P_0c^t$ plays a role similar to the vertical intercept for a line. If we set $t = 0$, we get

$$y = P_0c^0 = P_0$$

because any number (other than 0) raised to the zero power is 1. Thus the initial or starting value P_0 of the exponential growth function represents the height at which the exponential function crosses the vertical axis (when $t = 0$). Figure 2.21 shows the graphs of three different exponential growth curves: $y = 10(1.03)^t$, $y = 30(1.03)^t$, and $y = 60(1.03)^t$. All have the same growth factor, 1.03, and so the same growth rate, $0.03 = 3\%$, but all have different initial values. Note how all three curves have similar shape, but each crosses the vertical axis (at $t = 0$) at a different height.

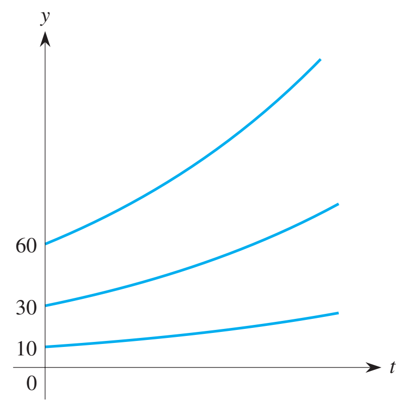


FIGURE 2.21

Comparing Linear and Exponential Growth

The key fact about linear growth is that a linear function grows at a constant rate. That is, every time the independent variable x, say, increases by 1, the linear function grows by the same amount (equal to the slope), as illustrated in Figure 2.22(a). In contrast, the key fact about exponential growth is that an exponential function grows by a fixed percentage. Suppose that the growth factor is 1.20 so that the growth rate is 20%. Then every time the independent variable x increases by 1, the exponential function grows by the same multiple, 1.2, as illustrated in Figure 2.22(b). The corresponding values are then y_0, $1.2\,y_0$, $(1.2)^2\,y_0$, $(1.2)^3\,y_0$, and so on, and these values eventually grow very rapidly as the exponent increases. As a result, eventually any exponential growth function will outstrip any linear function.

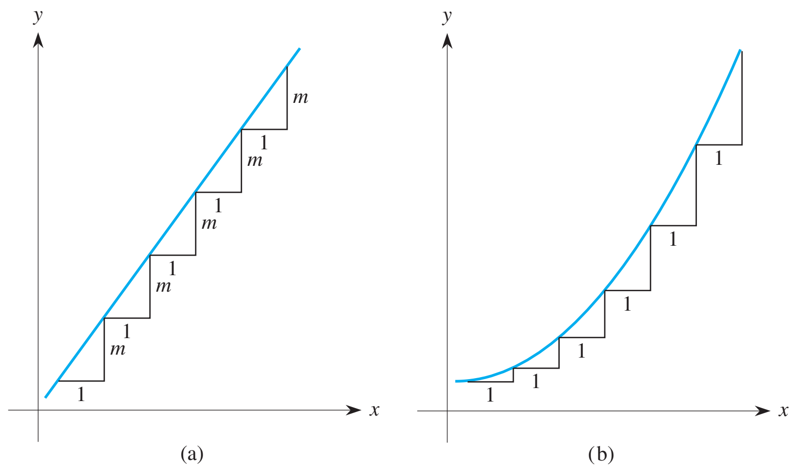


FIGURE 2.22

Applications of Exponential Growth

In Examples 1 and 2 we apply the ideas and definitions on exponential growth.

EXAMPLE 1

In 1995, the population of Mexico was 93.7 million and growing at a rate of 2.2% a year.

a. Find a formula for the population of Mexico at any time t.

b. Predict the population of Mexico in 2003.

Solution

a. Because the annual growth rate for Mexico is $2.2\% = 0.022 = a$, the corresponding growth factor is $c = 1 + 0.022 = 1.022$. Let t be the number of years since 1995 and $M(t)$ be the population of Mexico in millions. Then a formula for the Mexican population at any time t since 1995 is

$$M(t) = 93.7(1.022)^t.$$

b. Assuming that this exponential growth pattern continues until 2003, we have $t = 8$ years after 1995. We predict that the population of Mexico will be

$$M(8) = 93.7(1.022)^8 \approx 111.52 \text{ million people,}$$

as shown in Figure 2.23.

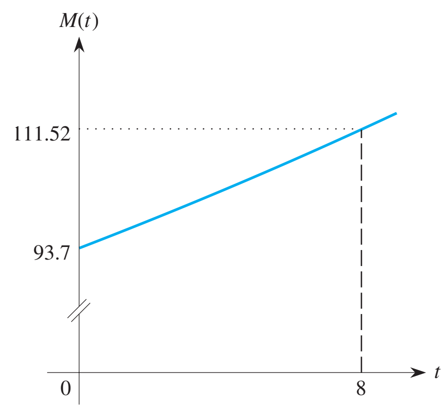


FIGURE 2.23

EXAMPLE 2

During one of New York City's recent financial crises, someone discovered a million dollar loan the city made to the U.S. Government in 1812. At first it appeared that the loan had not been repaid. For a 6% annual compound interest rate, what would this amount have become by the year 2000?

Solution The 6% growth rate corresponds to a growth factor of $c = 1.06$ so that t years after 1812, the amount would be

$$b(t) = b_0 \cdot (1.06)^t = 1{,}000{,}000(1.06)^t.$$

For 2000, $t = 2000 - 1812 = 188$, and the resulting balance would be

$$b(188) = 1{,}000{,}000(1.06)^{188}$$
$$\approx \$57{,}214{,}047{,}000.$$

As depicted in Figure 2.24, that would easily have solved the municipal finance problem for many years to come. Unfortunately for New York City, the loan was later found to have been repaid, with interest, in 1815.

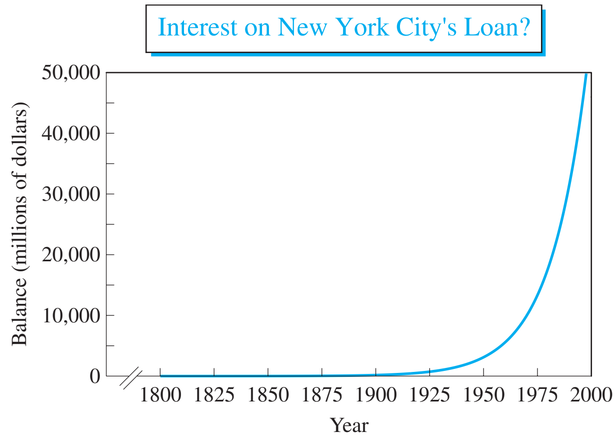


FIGURE 2.24

Think About This

How much interest did New York City receive in 1815 from this loan?

Doubling Time

One of the special characteristics of any exponential growth function is that it has a unique *doubling time*—the time needed for the exponential function to double. We illustrate this concept in Example 3.

EXAMPLE 3

Assuming that the exponential model $P(t) = 12.94(1.029)^t$ for Florida's population created at the beginning of this section continues to hold far into the future, estimate the population of Florida in **(a)** 2014, when $t = 24$; **(b)** 2038, when $t = 48$; and **(c)** 2062, when $t = 72$.

Solution We use exponential growth model

$$P(t) = 12.94(1.029)^t$$

to predict the following values.

a. $P(24) = 12.94(1.029)^{24} \approx 25.70 \approx 2 \cdot 12.94$

b. $P(48) = 12.94(1.029)^{48} \approx 51.04 \approx 4 \cdot 12.94$

c. $P(72) = 12.94(1.029)^{72} \approx 101.35 \approx 8 \cdot 12.94$

Let's look at what these predicted population values indicate. After 24 years, Florida's population has doubled. After roughly another 24 years (i.e., $t = 48$), it has doubled again. After roughly another 24 years (i.e., $t = 72$), the population has doubled yet again. Therefore we say that the doubling time of Florida's population is about 24 years: If you take the population in any given year and compare it to the population 24 years later, you will find that it has doubled.

Think About This

To extrapolate far into the future, we must assume that the population continues to grow exponentially at the same rate of 2.9% per year. The farther we project into the future, the riskier our prediction becomes because other factors can affect the growth rate. What are some?

Every population that grows exponentially has a fixed doubling time that depends only on the growth rate or the growth factor, not on the size of the population. The world's population, with an annual growth rate of about 1.5%, has a doubling time of about 38 years. (We show how to calculate doubling times later.) The current population is about 6 billion, so there will be about 12 billion people in 38 years and roughly 24 billion people in 76 years, all competing for an ever diminishing amount of resources. As another way of looking at it, if you live to be 76, the world's population will quadruple during your lifetime.

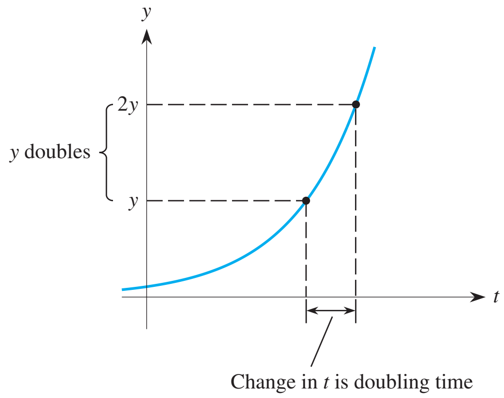


FIGURE 2.25

The doubling time T for any exponential growth process is the same at any point on the curve; that is, if you pick any point (t, y) on the exponential curve, the value for y will always increase to $2y$ (it has doubled) after T time units. You can visualize what this means by looking at Figure 2.25.

Predicting with Exponential Growth Functions

The purpose of creating an expression for an exponential growth function is to answer predictive questions about the quantity being modeled, as we illustrate in Examples 4 and 5.

EXAMPLE 4

Estimate when the population of Florida will reach 20 million.

Solution Our formula for the population of Florida is $P(t) = 12.94(1.029)^t$, and we want the value of t when the curve reaches a height of 20. We therefore must solve the equation

$$12.94(1.029)^t = 20$$

for t. We can solve this equation *numerically* by using trial-and-error by substituting different values of t until we get a value for $12.94(1.029)^t$ that is very close to 20. (Your calculator may have a table feature that allows you to generate a table of values and zoom in with smaller and smaller steps to find the value of the independent variable that produces a given value—20 in this case—for the dependent variable.)

A simpler approach is to solve the equation *graphically* by drawing the graph of the function on a function grapher and tracing along the curve to determine when the function reaches a height of 20, as shown in Figure 2.26. If necessary, this approach could also involve zooming in to increase the level of accuracy. Alternatively, we could graph the two functions, $y = 12.94(1.029)^t$ and $y = 20$, and find the point of intersection using a function grapher. Whichever way we proceed, the solution is approximately $t = 15.2$ years from 1990 or early in 2005. (We develop an algebraic approach using logarithms for solving such an equation that yields an exact answer in Section 2.8.)

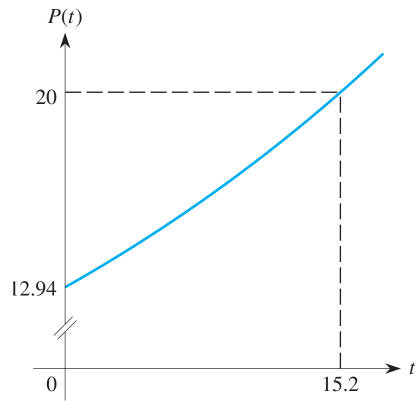


FIGURE 2.26

EXAMPLE 5

Estimate the doubling time for the population of Florida.

Solution We again use the exponential growth model $P(t) = 12.94(1.029)^t$ and we now must find how long it takes for the population to double, that is, to reach $2 \times 12.94 = 25.88$. We therefore have to solve the equation

$$12.94(1.029)^t = 2 \times 12.94 = 25.88$$

for t. We solve this equation graphically by looking for the intersection of the two curves $y = 12.94(1.029)^t$ and $y = 25.88$, as shown in Figure 2.27. This point is at $t \approx 24.2465$, so the doubling time for Florida's population is about $24\frac{1}{4}$ years.

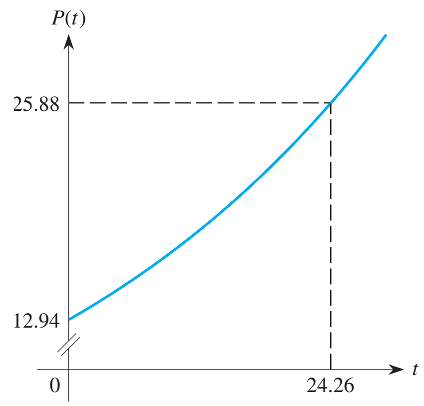


FIGURE 2.27

So far, we have thought of exponential functions as starting at time $t = 0$. In reality, the formula for any exponential function can be interpreted for negative values of the independent variable, as we demonstrate in Example 6.

EXAMPLE 6

Use the model we constructed for the population of Florida to predict what the population was in 1980.

Solution Our formula for the population of Florida is

$$P(t) = 12.94(1.029)^t,$$

where t is the number of years since 1990. The year 1980, 10 years before 1990, therefore corresponds to $t = -10$. The formula predicts that the population in 1980 was

$$P(-10) = 12.94(1.029)^{-10} \approx 12.94(0.75135) \approx 9.72 \text{ million people,}$$

as illustrated in Figure 2.28. Note that this result is considerably below the 1990 population value of 12.94 million people, as we would expect.

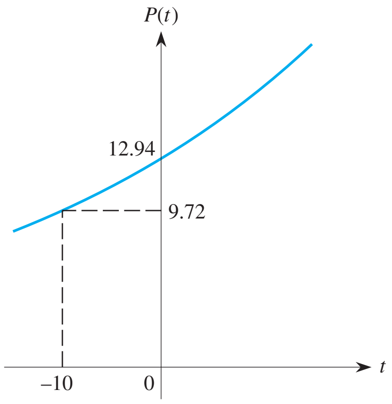


FIGURE 2.28

Just as the domain of a linear function theoretically is the set of all real numbers from $-\infty$ to $+\infty$, the domain of an exponential function is likewise theoretically from $-\infty$ to $+\infty$. Of course, in any real-world setting, there may be practical limitations to the domain. For instance, it wouldn't make sense to use the function to extrapolate the population of Florida 200 years into the past, as Florida became a state only in 1845. Moreover, as we've stated before, extrapolating far into the future or the past is risky because the trend in the data may not hold.

Also, Example 6 indicates that, when $t < 0$, the values for the exponential growth function continue to decrease from right to left. Figure 2.29 shows the typical graph of an exponential growth function $y = kc^x$, with $k > 0$. Note how it grows in the expected way toward the right and decays to 0 toward the left. The reason is that, as we move farther to the left of the vertical axis, the values of x become ever more negative. Suppose that we write $x = -z$. Recall one of the basic properties of exponents:

$$b^{-z} = \frac{1}{b^z}.$$

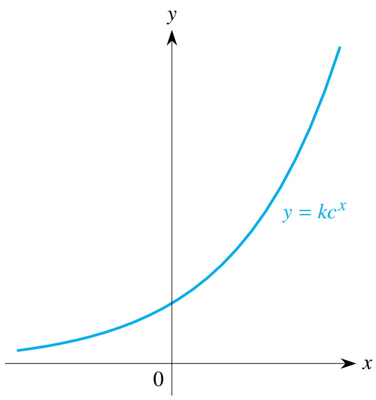


FIGURE 2.29

Consequently,

$$1.029^{-10} = \frac{1}{1.029^{10}} \approx 0.75135.$$

As the exponent becomes ever more negative,

$$1.029^{-20} \approx 0.56454,$$
$$1.029^{-30} \approx 0.42417,$$
$$1.029^{-100} \approx 0.05734,$$

the values become ever smaller and eventually approach 0. We say that the curve approaches the negative x-axis *asymptotically* because it never reaches 0 in any finite time interval. We call the horizontal axis a *horizontal asymptote* for the graph of the exponential decay function. The range of any exponential growth function $y = kc^x$, with $k > 0$, is therefore all positive values for y.

Finding an Exponential Function Through Two Points

We know that two points determine a line (because one and only one line can pass through the two points). Similarly, two points also determine an exponential function in the sense that one and only one exponential function passes through the two points, provided that the y-values for the points are either both positive or both negative. Suppose that we have any two points (x_1, y_1) and (x_2, y_2), where $x_1 < x_2$ and $0 < y_1 < y_2$, so the second point is to the right of and above the first point and both points are above the x-axis, as shown in Figure 2.30. One and only one exponential growth curve passes through the two points.

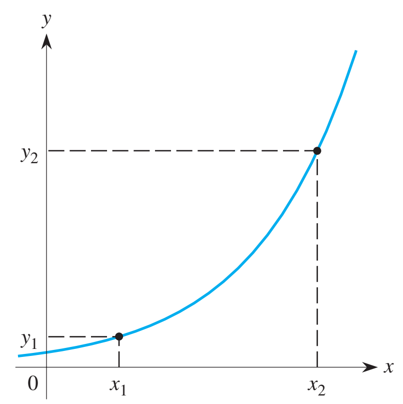


FIGURE 2.30

Think About This

By drawing several sketches, convince yourself why it is not possible to draw an exponential growth curve through two points when one is above the x-axis and the other is below the x-axis. Also, if the two points are both below the x-axis, what should you expect about the sign of the coefficient k in $y = kc^x$?

We now determine a formula for the exponential function that passes through two points. Doing so also gives us a way to find the growth rate for any exponential process. For the equation of an exponential function $y = kc^x$, values for the two parameters k and c must be determined, which is why we use two points. We demonstrate how to do so in Example 7.

EXAMPLE 7

The number of cell phones in use worldwide grew from 11 million in 1990 to 319 million in 1998.

a. Assuming that the growth pattern was exponential, find the annual growth rate for the number of cell phones in use and the equation of the exponential function that models the number of cell phones in use.

b. Predict the number of cell phones in use in 2003.

Solution

a. Let t represent the number of years since 1990 and P the number of cell phones (in millions) in use. We then have the two points $(0, 11)$ and $(8, 319)$. The exponential growth function has the form

$$P(t) = P_0 c^t,$$

where the constants P_0 and c must be determined. Substituting the coordinates of the point $(0, 11)$ into the function gives

$$P(0) = P_0 c^0 = P_0 = 11,$$

because $c^0 = 1$. Thus the exponential function becomes $P(t) = 11c^t$. Using the point $(8, 319)$ gives

$$P(8) = 11c^8 = 319.$$

Solving for c^8 gives

$$c^8 = \frac{319}{11} = 29.$$

Just as we solve $x^2 = 10$ for x by taking the square root of 10 or solve $x^3 = 10$ for x by taking the cube root of 10, we solve $c^8 = 29$ for c by taking the eighth root of 29. (We discuss the details more formally in Section 2.7.) Thus

$$c = \sqrt[8]{29} \approx 1.5234.$$

(Verify that $\sqrt[8]{29} \approx 1.5234$ by taking the eighth power of 1.5234.) For the growth factor of 1.5234, the annual growth rate in the number of cell phones in use is $0.5234 = 52.34\%$. Moreover, the exponential function that models the growth in the number of cell phones is

$$P(t) = 11(1.5234)^t,$$

where t is the number of years since 1990.

b. Because 2003 is 13 years after 1990, we set $t = 13$ as shown in Figure 2.31. We then use this exponential model to predict that the number of cell phones in use in 2003 is

$$P(13) = 11(1.5234)^{13} \approx 2618.0 \text{ million},$$

or about 2.618 billion.

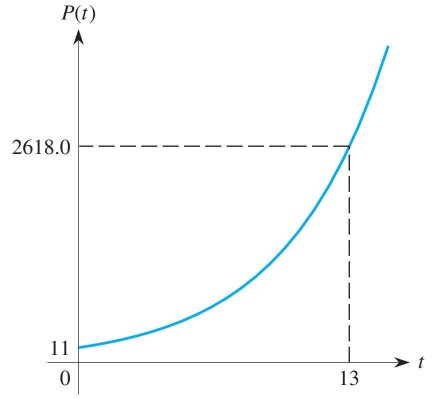


FIGURE 2.31

By letting t represent the number of years since 1990, we simplified the work in Example 7 to give the vertical intercept $(0, 11)$ as one of the points. If we can't do so, things become more complicated, as shown in Example 8.

EXAMPLE 8

Find the equation of the exponential function that passes through the points $(1, 6)$ and $(2, 9)$.

Solution The desired exponential function has the form $f(x) = kc^x$, where we must find the correct values for the parameters k and c. Using the point $(1, 6)$, we have

$$f(1) = kc^1 = kc = 6.$$

Using the point $(2, 9)$, we have

$$f(2) = kc^2 = 9.$$

From the first of these two equations, we solve for k and get $k = 6/c$. We substitute this term into the second equation to get

$$kc^2 = \left(\frac{6}{c}\right)c^2 = 6c = 9,$$

and so

$$c = \frac{9}{6} = 1.5.$$

Therefore

$$k = \frac{6}{c} = \frac{6}{1.5} = 4,$$

and the desired exponential function is $f(x) = 4(1.5)^x$, as shown in Figure 2.32.

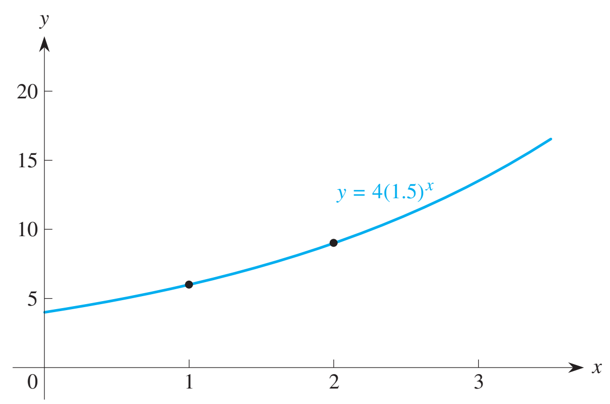


FIGURE 2.32

Determining Whether a Set of Data Is Exponential

Recall the simple criterion that determines whether a set of data follow a linear pattern: The successive differences in the dependent variable must be constant when there is a constant difference between values of the independent variable. Similarly, we can determine whether a table of data values (t, y) follows an exponential pattern by looking at the successive ratios of the y values.

If the ratios of the successive values of the dependent variable are constant for equally spaced t values, the y values follow an exponential pattern: $y = kc^t$.

The common ratio is precisely the growth factor for the exponential growth process if the t values increase by 1 unit. For instance, with Florida's population values from one year to the next, we found that the common ratio was 1.029, which is the growth factor, and that the associated growth rate is 0.029, or 2.9% per year.

EXAMPLE 9

One of the following functions is exponential and the other isn't. Determine which is the exponential function. The values are rounded to four decimal places.

x	y
0	20.0
1	21.0
2	22.10
3	23.2775
4	24.6425
5	26.2650

x	y
0	20.0
1	21.0
2	22.05
3	23.1525
4	24.3101
5	25.5256

Solution We apply the criterion for an exponential pattern and examine the ratios of successive terms for each function. For the first function the ratios are

$$\frac{21.0}{20.0} = 1.05, \quad \frac{22.10}{21.0} = 1.0524, \quad \frac{23.2775}{22.10} = 1.0533,$$

$$\frac{24.6425}{23.2775} = 1.0586, \quad \text{and} \quad \frac{26.2650}{24.6425} = 1.0658.$$

The successive ratios are not constant, so this function cannot be exponential.

For the second function the ratios are

$$\frac{21.0}{20.0} = 1.05, \quad \frac{22.05}{21.0} = 1.05, \quad \frac{23.1525}{22.05} = 1.05,$$

$$\frac{24.3101}{23.1525} = 1.04999, \quad \text{and} \quad \frac{26.5256}{24.3101} = 1.04999.$$

These ratios are essentially constant (the last two vary slightly because the entries listed in the table were rounded), so we conclude that this function is indeed exponential.

Rules for Exponents

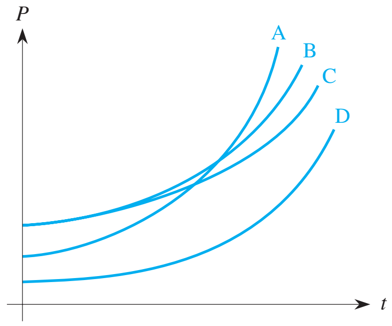
Because exponential functions involve working with exponents, all the usual algebraic rules for manipulating exponents apply. As a reminder, we list some of the fundamental definitions and algebraic rules for exponents.

Definitions and Rules for Exponents

Property	Example
1. $a^x \cdot a^y = a^{x+y}$	$10^3 \cdot 10^2 = (10 \cdot 10 \cdot 10) \cdot (10 \cdot 10) = 10^5 = 10^{3+2}$
2. $\dfrac{a^x}{a^y} = a^{x-y}, \quad a \neq 0$	$\dfrac{10^5}{10^2} = \dfrac{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}{10 \cdot 10} = 10^3 = 10^{5-2}$
3. $(a^x)^y = a^{xy}$	$(10^3)^2 = 10^3 \cdot 10^3 = 10^6 = 10^{3 \cdot 2}$
4. $a^0 = 1$	$10^0 = 1$
5. $a^{-1} = \dfrac{1}{a}, \quad a \neq 0$	$10^{-1} = \dfrac{1}{10}$
6. $a^{-n} = \dfrac{1}{a^n}, \quad a \neq 0$	$10^{-3} = \dfrac{1}{10^3} = \dfrac{1}{1000}$
7. $a^{1/n} = \sqrt[n]{a}, \quad a \geq 0$	$10^{1/2} = \sqrt{10}, \qquad 10^{1/3} = \sqrt[3]{10}$

Problems

1. The accompanying graph shows population growth curves for four different nations. Which nation
   a. has the greatest growth rate?
   b. has the smallest growth rate?
   c. has the largest initial population?
   d. has the smallest initial population?
   e. Which nations have the same growth rate?



2. Determine which of the functions are exponential. For any exponential function, find the equation of the function and use it to predict the next entry to extend the table of values.

a.

x	0	1	2	3
y	1000	1200	1440	1728

b.

t	0	1	2	3
$L(t)$	300	308	320.2	335.5

c.

t	0	10	20	30
$Q(t)$	200	208	216.32	224.97

3. Anne opens a bank account with \$1200 at 4% annual interest. Bill opens an account with \$1000 at 4.5% annual interest. Christine opens an account with \$1500 at 3.8% annual interest. Doug opens an account with \$1200 at 4.5% annual interest. Elka opens an account with \$1300 at 4.25% annual interest. Sketch a graph showing the balances in the five accounts over time on the same set of axes. Be sure to label which account belongs to which person.

4. Use the exponential growth function $f(t) = 125(1.04)^t$ to make a prediction for 2000 if **(a)** t is the number of years since 1980, **(b)** t is the number of years since 1900, **(c)** t is the number of years since the year 0.

5. In 1990, the United States imported \$495 billion worth of goods. In 1998, the United States imported \$912 billion worth of goods. Assuming that the growth in imports has been following an exponential growth pattern, find an equation of the exponential function that models U.S. imports when
   a. the independent variable t represents the number of years since 1990.
   b. the independent variable t represents the number years since 1900.
   c. the independent variable t represents the number of years since the year 0.

d. For the three functions you created in parts (a)–(c), which parameters changed and which remained the same? Explain why the changes occurred. Explain why the parameters that stayed the same didn't change.

e. Use each model from (a)–(c) to predict the amount of imports in 2005.

Source: *2000 Statistical Abstract of the United States.*

6. Match each formula with the corresponding table of values.

a. $y = a(1.1)^s$

b. $y = b(1.05)^s$

c. $y = c(1.03)^s$

i.

s	2	3	4	5	6
$f(s)$	1.06	1.09	1.13	1.16	1.19

ii.

s	1	2	3	4	5
$g(s)$	2.20	2.42	2.66	2.93	3.22

iii.

s	3	4	5	6	7
$h(s)$	3.47	3.65	3.83	4.02	4.22

7. In 1980, a total of $119 trillion was spent on food and drinks in the United States. In 1994, the total spent was $274 trillion.

a. Find the equation of the exponential function that can be used to model the total spent on food and drinks in the United States as a function of the number of years since 1980.

b. Use your model to predict the amount spent in 1990.

c. What is your prediction for the total sales of food and drink in 2004?

d. Estimate when the total sales will reach $500 trillion if this exponential trend continues.

8. The 1990 population of Arizona was 3.7 million and growing at an annual rate of 1.7%.

a. Find an expression for the population at any time *t*.

b. What will be the population in 2005?

c. Estimate the doubling time for this population.

9. The 1995 population of Venezuela was 21.8 million and growing at an annual rate of 2.6%.

a. Find an expression for the population at any time *t*.

b. What will be the population be in 2005?

c. Estimate the doubling time for this population.

10. The 1995 population of France was 58.1 million and growing at an annual rate of 0.3%.

a. Find an expression for the population at any time *t*.

b. What will be the population in 2010?

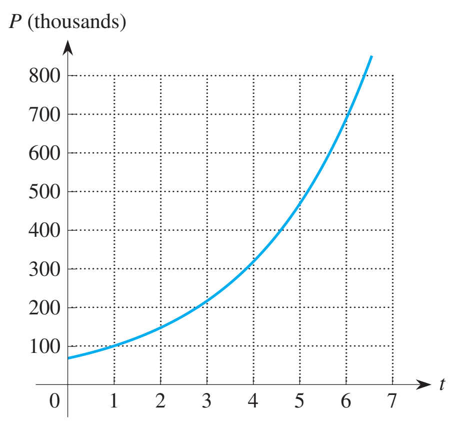
c. Estimate the doubling time for this population.

11. In 1990, 1.36 billion metric tons of carbon dioxide were emitted into the atmosphere in the United States. In 1998, 1.595 billion metric tons were emitted.

a. Construct the exponential function giving the amount of carbon dioxide emitted into the atmosphere as a function of the number of years since 1990.

b. Use the exponential function to estimate the amount emitted in 2004.

12. The population graph shown in the accompanying figure is growing exponentially.



a. Use the graph to estimate the doubling time of the population.

b. Verify graphically that the doubling time does not depend on where you start on the graph.

13. The world's population passed 6 billion in late 1999 and is increasing at a rate of about 1.5% per year.

a. Find the world's population 15 years later if this trend continues.

b. Estimate how long it will take the world's population to double.

14. Find a formula for the balance in a bank account in which $100 was deposited at 6% annual interest compounded for 10 years.

15. Find the balance after 1 year if \$100 is deposited at an annual rate of 6% compounded quarterly instead of yearly. What is the balance after 10 years? (*Hint:* What is the interest rate for each 3-month period?)

16. In 1998, the population of the United States was about 268.2 million with an annual growth rate of 0.7%. At the same time, the population of Mexico was about 100.1 million with an annual growth rate of 2.2%. If these growth rates continue, use either graphical or numerical methods to estimate when the population of Mexico will overtake that of the United States.

17. According to an article in the *New York Times* on May 27, 1990, a wealthy Pennsylvania merchant named Jacob DeHaven loaned \$450,000 to the Continental Congress in 1776 to rescue the troops at Valley Forge. The descendants of Mr. DeHaven sued the U.S. government for what they believed they were owed. The interest rate in effect in 1776 was 6% per year. How much did the family stand to collect in 1991, assuming that interest is compounded annually?

18. The lily pads in a pond grow in such a way as to double the area of the pond that they cover daily.

a. If the lily pads exactly cover the entire pond on the 25th day, how much of the pond do they cover on the 24th day?

b. Write an exponential function that models the fraction of the pond covered on any particular day.

c. If the area of the pond is 40,000 sq ft, find the area covered by the lily pads on the initial day.

d. What area of the pond is covered by the lily pads at the end of 1 week?

19. Let $f(x)$ be an exponential function of x. If $f(7) = 25.6$ and $f(8) = 28.8$, find

a. the growth factor;

b. the growth rate;

c. the value of the function when $x = 10$;

d. a formula for $f(x)$.

20. The Dow-Jones average of 30 industrial stocks is the most famous measure of performance of the New York Stock Exchange. At the beginning of 1995 the Dow was 3834, and at the beginning of 2000 it was 11,358. Assuming (incorrectly) that the Dow increased continuously over these 5 years and that the pattern is exponential, find the exponential function that models the behavior of the Dow between 1995 and 2000. What would you predict as the value for the Dow at the beginning of 2004?

21. Repeat Problem 20, using the facts that the Dow was 964 at the beginning of 1981 and was 11,358 at the beginning of 2000.

22. **a.** Suppose that you're an aggressive stockbroker who is trying to convince a little old lady to invest her life savings with you. What argument would you make based on your work on either Problem 20 or 21 to convince her.

b. Now suppose that the little old lady is your grandmother. What argument would you make based on your work on either Problem 20 or 21 to convince her to be more conservative.

23. An exponential function f is such that $f(0) = 512$ and $f(4) = 1250$. Which of the values are possible and which are impossible?

a. $f(2) = 800$ **b.** $f(2) = 881$ **c.** $f(2) = 981$

24. The net income of the Acme Company was \$240 million in 1990 and has been increasing at an annual rate of 10% per year since. Over the same period, the net income of its chief competitor, the Finest Corporation, has been growing 8% annually from an income of \$300 million in 1990. Which was the richer company in 2000? Does Acme ever surpass Finest? If so, estimate when.

25. (Extension of Problem 24) Suppose that Finest grew by a fixed amount of \$25 million per year since 1990 while Acme grew exponentially at an annual rate of 10%. By using trial and error, estimate when Acme surpassed Finest.

26. When Steven was 5 years old, his grandmother decided to set up a trust account to pay for his college education. She wanted the account to grow to \$80,000 by Steven's 18th birthday. If she was able to invest her money at 6% per year, how much did she have to put into this trust account? (*Note:* This amount is known as the *present value* of the investment. The \$80,000 is known as the *future value.*)

27. In Example 7 the number of cell phones in use increased 29-fold, from 11 million in 1990 to 319 million in 1998. This is equivalent to a 3000% increase over that 8-year period. Explain what's wrong with the reasoning that says: If the number of cell phones increased by 3000% over the 8 years, the annual growth rate is $\frac{1}{8}$ of 3000% or 375%.

28. Show that $x^{5/3} \neq \dfrac{x^5}{x^3}$

a. numerically, by finding at least one value of x for which the two expressions are different;

b. graphically, by comparing the graphs of the two functions $y = x^{5/3}$ and $y = x^5/x^3$.

Exercising Your Algebra Skills

Simplify the following.

1. $x^5 \cdot x^3$

2. $x^4 \cdot x^2$

3. $a^8 \cdot a^4$

4. $\dfrac{a^{15}}{a^6}$

5. $x^{-5} \cdot x^3$

6. $a^5 \cdot a^{-3}$

7. $\dfrac{r^8}{r^{-4}}$

8. $\dfrac{b^{15}}{b^{-6}}$

9. $\dfrac{w^{-4}}{w^{-7}}$

10. $\dfrac{w^{-7}}{w^{-4}}$

11. $x^{-1/2}x^{3/4}$

12. $y^{2/3}y^{4/3}$

13. $z^{2/3}z^{-5/3}$

14. $(x^5)^3$

15. $(x^3)^5$

16. $(a^8)^{-4}$

Perform the following operations:

17. $(a^3b^5)^4$

18. $(a^3 + b)^2$

19. $(a^3 - b)^2$

2.5 Exponential Decay Functions

Prozac is one of the most widely used drugs to treat extreme depression. Once a medication such as Prozac has been absorbed into the bloodstream, it eventually is eliminated from the body by the kidneys, which purify the blood by filtering out foreign chemicals. For now, let's assume that a person takes a single dose of Prozac and that it has been completely absorbed into the blood. It is reasonable to assume that, during any fixed time period, a fixed percentage of any medication, including Prozac, is removed from the bloodstream as the kidneys process the blood. In particular, the kidneys eliminate approximately one-fourth of the Prozac in the bloodstream during any 24-hour period, so that 75% of the drug remains. (Note that this rate is specific to Prozac and that other medications are washed out of the body at different rates.)

Suppose that the original dosage of Prozac is 80 mg (milligrams). We want to develop a formula for the amount $D(t)$ present at any time t. Clearly, it must be a decreasing function because the level of the drug in the bloodstream is decaying over time.

We start with $D(0) = 80$ mg. After the first 24 hours, one quarter of 80 mg, or 20 mg, of the Prozac is eliminated, leaving three quarters of the 80 mg, or 60 mg, of the Prozac in the bloodstream. After one 24-hour period, when $t = 1$, the amount of Prozac in the system is

$$D(1) = 0.75(80) = 60 \text{ mg}.$$

After a second 24-hour period, the kidneys remove 25% of the remaining 60 mg of Prozac, so 15 mg are eliminated, leaving 75% of the remaining 60 mg of Prozac. Thus, when $t = 2$,

$$D(2) = 0.75(60) = 0.75(0.75)(80) = (0.75)^2(80) \text{ mg}.$$

After the third 24-hour period, 25% of the remaining Prozac is eliminated, leaving

$$D(3) = 0.75D(2) = 0.75(0.75)^2(80) = (0.75)^3(80).$$

Similarly,

$$D(4) = 0.75D(3) = (0.75)^4(80)$$

and

$$D(5) = 0.75D(4) = (0.75)^5(80).$$

In general, after t days the amount of Prozac in the bloodstream is given by

$$D(t) = 80(0.75)^t.$$

This function has the same form, $y = kc^t$, as the exponential growth functions presented in Section 2.4 except that the base c is 0.75, which is less than 1. It is an example of an *exponential decay function,* and its graph is shown in Figure 2.33. Note that the behavior is that of a decreasing, concave up function. Each step down is smaller than the previous one. This result makes sense because, as the amount of Prozac remaining in the bloodstream gets smaller, there is less of the drug left to eliminate, and the amount of decrease in drug strength diminishes every successive day.

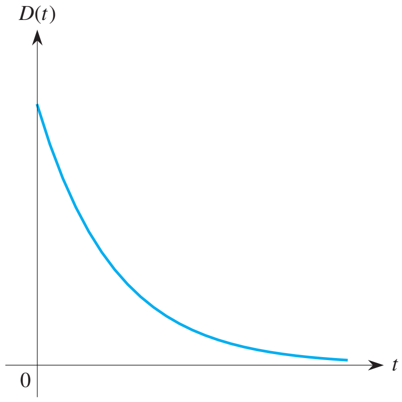


FIGURE 2.33

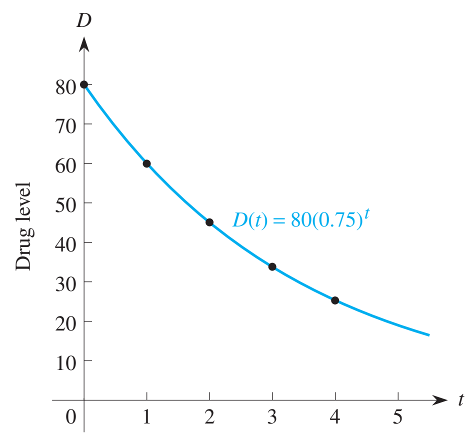


FIGURE 2.34

You can see this numerically by calculating the values of the function previously given:

$$D(0) = 80, \quad D(1) = 60, \quad D(2) = 45, \quad D(3) = 33.75, \quad D(4) = 25.3125, \ldots,$$

which is a decreasing, concave up pattern. If you continue these calculations, you will find that the values eventually approach 0 *asymptotically;* that is, the drug level never reaches 0 in any finite time interval, as illustrated in Figure 2.34. Thus, the horizontal axis is a horizontal asymptote for the graph of the exponential decay function.

In general, the graph of any exponential decay function, $y = kc^t$, with $0 < c < 1$, is a decreasing, concave up curve that approaches 0 as t gets larger and larger. In comparison, the graph of any exponential growth function, $y = kc^t$, with $c > 1$, is an increasing, concave up curve. Because the base c for an exponential decay function is between 0 and 1, we call it the *decay factor.*

Often, we are told that a process is decaying at a given rate—say, 12% per year. The 12% $= 0.12$ is known as the *decay rate* and the associated decay factor c is

$$\text{Decay factor} = 1 - \text{decay rate},$$

where the decay rate must be written as a decimal. Thus

$$c = 1 - 0.12 = 0.88,$$

because 88% (or 0.88) of the original amount is left. By comparison, for exponential growth, recall that

$$\text{Growth factor} = 1 + \text{growth rate}.$$

Note that whether we have an exponential decay function such as $y = 80(0.75)^t$ for the level of Prozac or an exponential growth function such as $y = 12.94(1.029)^t$ for the population of Florida, it is still an exponential function and the same techniques that we introduced in Section 2.4 apply. The only difference is that, for an exponential growth function, $c > 1$, whereas for an exponential decay function, $0 < c < 1$.

EXAMPLE 1

Find the amount of Prozac in the bloodstream after 1 week.

Solution We use the formula for the exponential decay function,

$$D(t) = 80(0.75)^t,$$

we previously constructed. After 1 week, $t = 7$ days, so the level of Prozac will be

$$D(7) = 80(0.75)^7 \approx 10.679,$$

or about $10\frac{2}{3}$ mg.

EXAMPLE 2

Estimate how long it takes until the level of Prozac in the bloodstream drops to 2 mg.

Solution Using the formula for the level of Prozac, we have to find t so that

$$D(t) = 80(0.75)^t = 2.$$

Using either numerical or graphical methods, we find that $t \approx 12.8$ days, as shown in Figure 2.35.

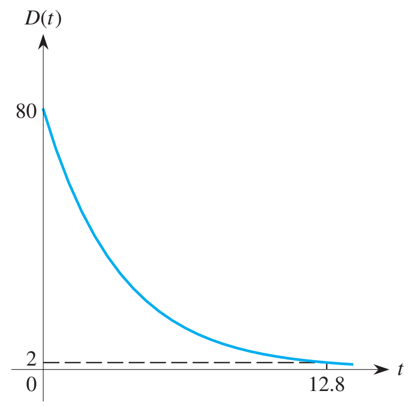


FIGURE 2.35

We summarize the formula for an exponential decay function and its parameters as follows.

Formula for an Exponential Decay Function

P is an **exponential decay function** of t with base c, $0 < c < 1$, if

$$P(t) = P_0c^t,$$

where P_0 is the initial quantity (when $t = 0$) and c is the **decay factor** by which P changes when t increases by 1 unit. Because $0 < c < 1$, we write $c = 1 - a$, where a is the **decay rate,** written as a decimal.

The larger the decay rate a, and hence the smaller the decay factor c, the faster the exponential decay function approaches 0, as illustrated in Figure 2.36.

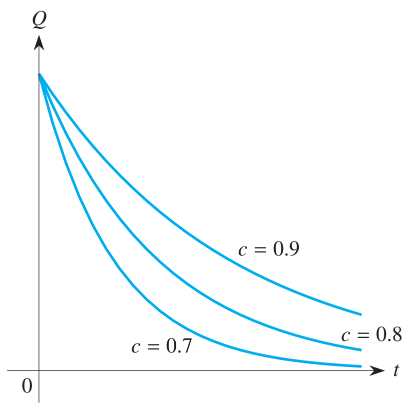


FIGURE 2.36

For example, if a quantity is decreasing at the rate of 12% per hour (e.g., the effectiveness of a medication in the body), the decay rate is $a = 0.12$ and the decay factor is $c = 1 - a = 0.88$. This reflects the fact that, if 12% of the quantity is removed each hour, then 88% of the quantity remains at the end of the hour. The corresponding formula for the exponential decay function that models the quantity Q is

$$Q(t) = Q_0 \cdot (0.88)^t,$$

where Q_0 is the initial amount of the quantity at time $t = 0$.

Half-life

Just as the doubling time for an exponential growth process is the time needed for the quantity to double, the *half-life* for an exponential decay process is the time T needed for the quantity to be reduced by half. You can visualize what this means by looking at Figure 2.37.

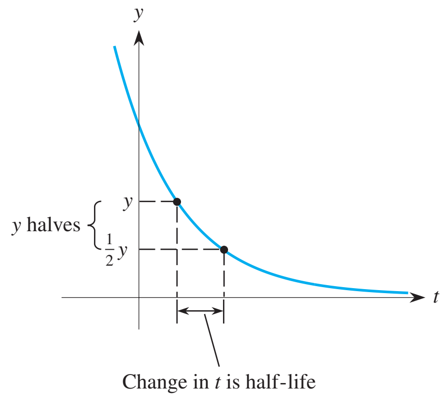


FIGURE 2.37

Note that the half-life T for any specific process is the same at any quantity level; no matter which point (t, y) you select, the quantity will decrease to $\frac{1}{2}y$ after T time units.

EXAMPLE 3

Estimate the half-life of Prozac in the bloodstream following an 80 mg dose.

Solution The exponential decay function that models the amount of Prozac in the bloodstream is

$$D(t) = 80(0.75)^t.$$

We want to find the time t needed for this level to drop to $(\frac{1}{2})80 = 40$ mg, so we must solve the equation

$$80(0.75)^t = 40.$$

Using either numerical or graphical methods, as shown in Figure 2.38, we get $t \approx 2.4$ days. Therefore, no matter what level of Prozac is in the blood at any specific time, the level will be down by half about 2.4 days, or 58 hours, later.

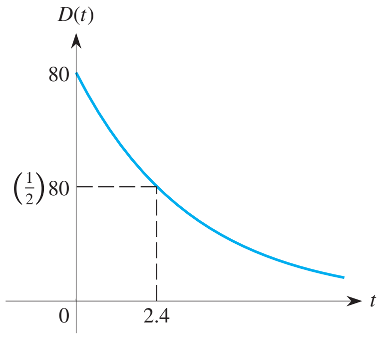


FIGURE 2.38

Radioactive Decay

One of the characteristics of any radioactive substance, such as radium or uranium, is that it transforms, or decays, to some other element, often lead, as time progresses. This decay is accompanied by the release of energy, called radioactivity, which can be detected and measured. More specifically, the rate at which an element decays is distinctive for that element. That is, during any fixed length of time, the same percentage of the mass of a radioactive element will decay. For instance, over the course of any 100-year period, approximately 4.3% of any radium present will decay to lead, leaving 95.7% of the radium at the end of 100 years, as illustrated in Figure 2.39. Thus, if someone had put aside $R_0 = 100$ grams of radium in the

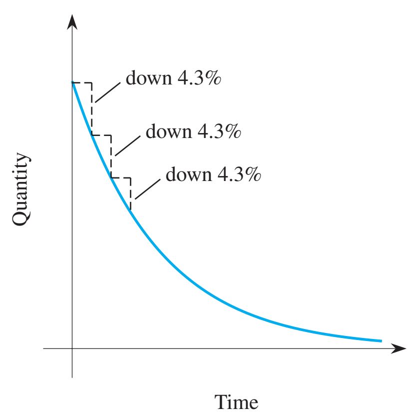


FIGURE 2.39

year 1900, we would expect to find only $R(1) = 95.7$ grams by the year 2000. By the end of a second century, the amount of radium left would be

$$R(2) = 0.957\,R(1) = (0.957)^2\,R_0$$

and by the end of a third century it would be

$$R(3) = 0.957\,R(2) = (0.957)^3\,R_0.$$

In general, the amount of radium present after t centuries is modeled by the exponential decay function

$$R(t) = (0.957)^t\,R_0$$

for any t.

Alternatively, because $4.3\% = 0.043$ is the decay rate for this exponential decay process, the decay factor is $1 - 0.043 = 0.957$. Thus, if the initial amount of radium is R_0, we can use the general formula for an exponential decay function to get $R(t) = (0.957)^t\,R_0$.

Figure 2.40 shows a graph of the amount of radium as a decaying exponential function of time. The amount of radium begins decreasing relatively rapidly, then decreases more slowly, and eventually approaches the time axis as a horizontal asymptote.

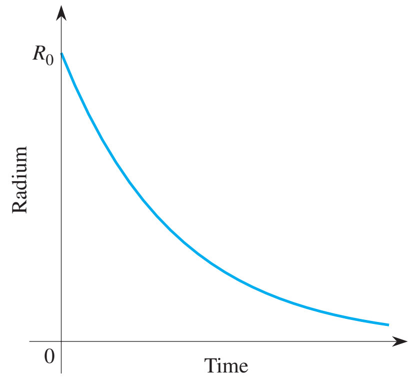


FIGURE 2.40

EXAMPLE 4

Estimate the half-life of radium.

Solution We want to determine the value of t for which

$$R(t) = (0.957)^t\,R_0 = \frac{1}{2}R_0.$$

We first divide both sides of this equation by R_0 to obtain

$$(0.957)^t = \frac{1}{2}.$$

If we now use either numerical or graphical methods, as shown in Figure 2.41, we find that $t \approx 15.77$ centuries. That is, the half-life for radium is approximately 1577 years. (The actual value for its half-life is closer to 1590 years; our calculations were based on the fact that *approximately* 4.3% of the radium decays to lead each century, and this rounding produced an error.)

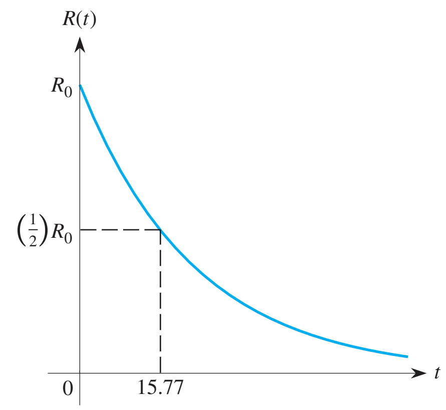


FIGURE 2.41

Think About This What is the actual percentage of radium that decays into lead each year, based on its half-life of 1590 years?

In all of the examples so far, time is the independent variable. Example 5 illustrates a situation in which the independent variable in an exponential function may represent some other quantity.

EXAMPLE 5

The strength of any signal in a fiber-optic cable, such as the type used for telephone and other communication lines, diminishes 15% every 10 miles.

a. Find an expression for the strength of a signal remaining after a given number of 10-mile lengths.

b. How much of the signal is left after 100 miles?

c. How far does a signal go until its strength is down to 1% of the original level?

Solution

a. If the signal diminishes by 15% every 10 miles of cable, after each 10-mile stretch, only 85% of the original signal strength remains. Let S_0 be the initial strength of some signal and let $S(n)$ be the strength of the signal remaining after n 10-mile lengths. Therefore, after the first 10-mile length of cable ($n = 1$), 85% of S_0 is left, so

$$S(1) = 0.85S_0.$$

Similarly, after the second 10-mile length ($n = 2$), 85% of $S(1)$,the signal strength remaining after the first 10-mile length, is left. That is,

$$S(2) = 0.85S(1).$$

Continuing this pattern, we get

$$\begin{aligned} S(0) &= S_0, \\ S(1) &= (0.85)S_0, \\ S(2) &= (0.85)S(1) = (0.85)(0.85)S_0 = (0.85)^2S_0, \\ S(3) &= (0.85)S(2) = (0.85)(0.85)^2S_0 = (0.85)^3S_0, \end{aligned}$$

and so on. After n 10-mile lengths of a cable,

$$S(n) = S_0 \cdot (0.85)^n,$$

which is an exponential decay function with decay factor $c = 0.85$.

b. After 100 miles, or $n = 10$ ten-mile lengths, the fraction of the original signal strength remaining is

$$S(10) = S_0 \cdot (0.85)^{10} = 0.1969\, S_0,$$

so just under 20% of the original signal strength is left.

c. To find out how far the cable can go until only 1% of the signal strength is left, we must find the value of n for which the strength remaining is $0.01\, S_0$, or

$$S(n) = S_0 \cdot (0.85)^n = 0.01 S_0.$$

If we divide both sides of this equation by the initial signal strength S_0, we get

$$(0.85)^n = 0.01.$$

Solving this equation either numerically or graphically, as shown in Figure 2.42, we find that $n \approx 28$. Therefore the signal deteriorates by 99% after about 28 ten-mile lengths, or about 280 miles.

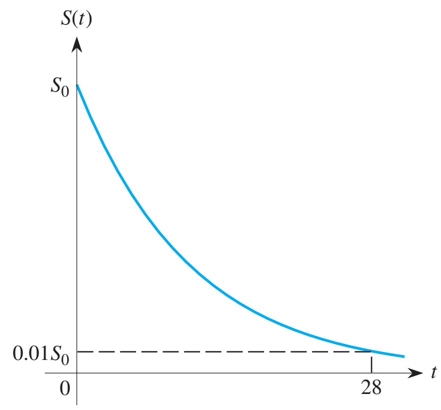


FIGURE 2.42

Think About This

In practice, this model suggests that fiber-optic signals need to be boosted if they are to go any great distance. For instance, if a booster station can clearly detect a signal at 1% of its original strength, such stations would have to be located every 280 miles. Suppose that the equipment used can clearly detect a signal at 0.1% of its original level. How far apart would the booster stations have to be?

Determining Whether a Set of Data Is Exponential Growth or Exponential Decay

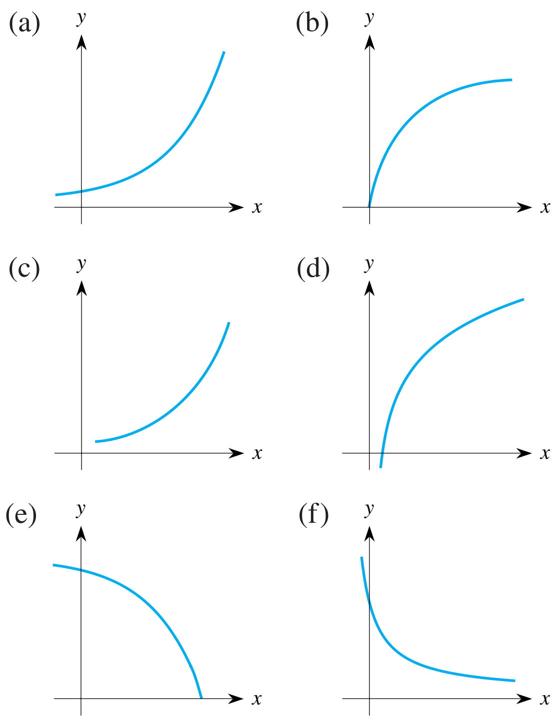
In Section 2.4, we presented a simple criterion for recognizing that a set of data follows an exponential growth pattern: The successive ratios of the values of the dependent variable y are constant for equally spaced t values. The same criterion applies if the values of y are decreasing in an exponential decay pattern. In this case, the common ratio is precisely the decay factor for the process if the values of t increase by 1 unit. In general, the ratio criterion works whether the data values are increasing or decreasing. A common ratio greater than 1 gives the growth factor for an exponential growth process; a common ratio less than 1 gives the decay factor for an exponential decay process.

Finally, we consider some parallels between the family of linear functions and the family of exponential functions. The general formula for a linear function is $y = mx + b$, and the general formula for an exponential function is $y = kc^x$, so both are two-parameter families. For linear functions the more important parameter usually is the slope, and its sign determines whether the function increases or decreases. For exponential functions the more important parameter is the growth or decay factor c, and whether its value is greater than 1 or less than 1 determines whether the exponential function increases or decreases.

The following problems include both exponential growth and exponential decay situations because you need to learn to distinguish between them.

Problems

1. Determine which of the six functions could be exponential functions of the form $f(x) = kc^x$ and which cannot be exponential. Explain your reasoning.



2. Determine which of the functions are exponential. For any exponential function, find the equation of the function and use it to predict the next entry to extend the table of values.

a.

x	0	1	2	3
y	2000	1800	1620	1458

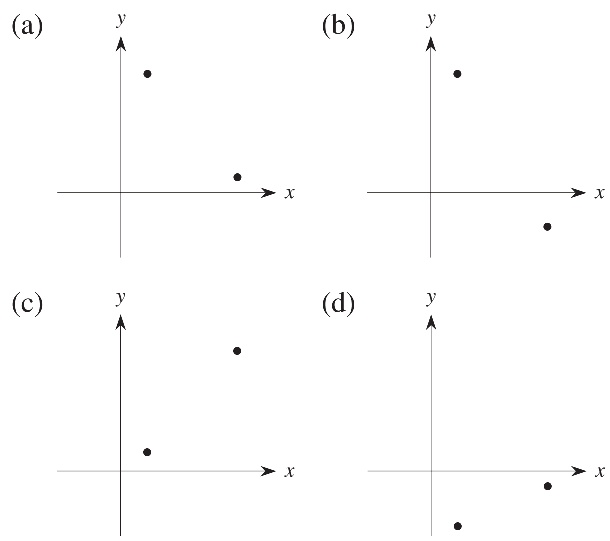
b.

t	0	1	2	3
$L(t)$	300	240	190	150

c.

t	0	10	20	30
$Q(t)$	400	288	207.36	149.30

3. Which of the following pairs of points can determine an exponential function of the form $y = Ac^x$ and which cannot. For those that can, sketch the graph of the exponential function and indicate the sign of A and whether the growth or decay factor c is greater than or less than 1.



4. Decide which situations represent exponential growth, exponential decay, linear increase or decrease, or none of these patterns.

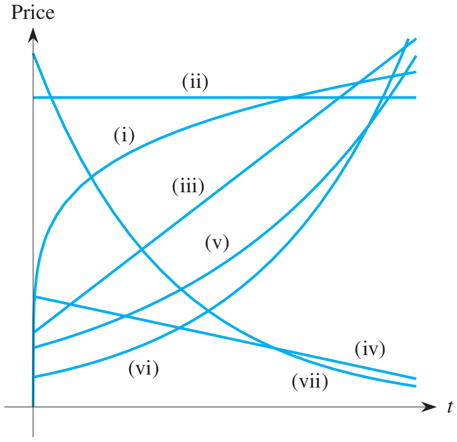
a. The value for a rare bottle of wine goes up \$50 each year.

b. The value for a piece of sculpture increases 15% each year.

c. A 3-year labor contract calls for yearly increases of \$800.

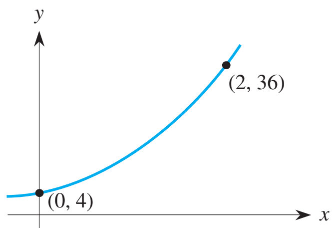
d. A 3-year labor contract calls for an increase of 5% the first year, 4% the second year, and 3% the third year.
e. The value of a car drops by 40% each year.
f. The average cost of a home computer for the first-time buyer has been dropping by $300 each year.
g. The number of new cases of a disease reported over the last decade has been dropping by 12% each year.

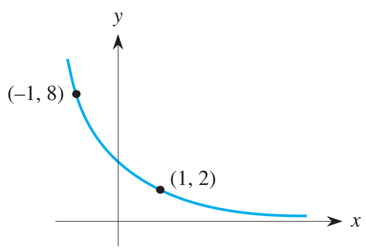
5. The accompanying figure shows the graph of the price of each of seven collectible toys as a function of time. Match each scenario with one of the graphs and write a brief scenario for each of the remaining graphs.

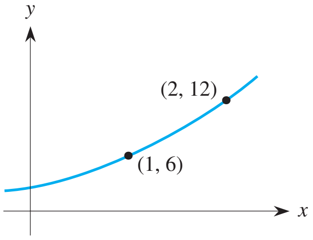


a. The price of the toy increased by 10% each year.
b. The price of the toy increased by 6% each year.
c. The price of the toy dropped by $5 each year.
d. The price of the toy remained steady.

6. Find possible equations for the exponential functions graphed in **(a)–(c)**.

(a)


(b)


(c)


7. In 1980, about 27,700 cases of tuberculosis were reported in the United States. In 1997, there were 19,900 such cases. Source: U.S. Centers for Disease Control and Prevention.

a. Write an exponential decay function that models the number of reported cases of TB as a function of time.
b. Predict the number of cases in 2004.
c. Estimate how long it will take for the number of reported cases to drop to 10,000.

8. In 1940, there were 6,102,000 farms in the United States. By 1997, the number of farms had dropped to 1,912,000.

Source: *2000 Statistical Abstract of the United States.*

a. Assuming that the pattern of decay is exponential, find the equation of a model that can be used to predict the number of farms.
b. Use your model to predict the number of farms in 1980. How close is your prediction to the correct value of 2,440,000 farms?
c. Predict the number of farms in 2005.
d. If the trend continues, estimate when there will be 1 million farms.
e. Write a paragraph describing the long-term implications if this trend continues.

9. When a person smokes a cigarette, about 0.4 mg of nicotine is absorbed into the blood. About 35% of the nicotine is washed out of the blood every hour.
a. Find the equation of a function that models the level of nicotine in the blood after a single cigarette.
b. Use your model to estimate how long it takes for the amount of nicotine in the blood to drop to 0.005 mg.

10. The amount of the drug ampicillin (a form of penicillin) in the bloodstream decreases by about 42% every hour.

a. If the dosage of ampicillin is 250 mg, write a function that can be used to model the level of ampicillin in the blood as a function of time, if one dose is taken.

b. How much ampicillin is left in the blood after 5 hours?

c. Estimate how long it will take for the level of ampicillin to drop to 1 mg.

11. A hospital patient is administered 3 mg of morphine to control his pain. About 31% of the morphine in the blood is washed out every hour.

a. Construct a function that models the level of morphine in the blood after one dose.

b. How much morphine remains in the blood after 4 hours?

c. Estimate how long it will take for the amount of morphine left to drop to 0.2 mg.

12. The level of pollution in the Great Lakes is a major concern to environmentalists.

a. In Lake Erie, about 38% of the pollutants are washed out each year if no pollutants are added. Write a function that models the level of pollutants in the lake as a function of time.

b. How long will it take for 90% of the pollutants to be washed out of Lake Erie if no further pollutants are added?

c. In Lake Superior, about 0.053% of the pollutants are washed out each year if no further pollutants are added. Write a function to model the level of pollutants in Lake Superior as a function of time.

d. How long will it take for 90% of the pollutants to be washed out of Lake Superior if no further pollutants are added?

13. One of the major concerns about above-ground nuclear testing is that it produces strontium-90, a radioactive element whose half-life is 29 years and which has worked its way into the food chain. That is, strontium-90 from fallout is deposited on grass, eaten by cows, carried into their milk, and eventually finds its way onto the kitchen table. Suppose that, as a result of a single nuclear explosion, the amount of strontium-90 in a particular valley exceeds health limits by a factor of 10. Estimate how long it will take for the strontium-90 to decay to the safety level.

14. Carbon-14, a radioactive form of carbon, is used in the carbon-dating process to measure the age of objects. About 0.012% of the carbon-14 decays into carbon-12 every century.

a. Write a function for the amount of carbon-14 remaining in an object that originally contained C_0 grams of carbon-14.

b. What percentage of the carbon-14 remains after a thousand years.

15. The filter in a swimming pool removes 30% of all impurities in the water every hour it operates.

a. Find an expression for the level of impurities left in the pool after n hours, if no further impurities are added.

b. How much is left after 5 hours?

16. Use the information in Example 5 to estimate the half-life of a signal in a fiber-optic cable. What does it mean?

17. One of the major problems associated with any organ transplant is the long-term risk of rejection, despite patients' taking anti-rejection drugs for the rest of their lives. The percentage of individuals who have not rejected a transplanted organ can be modeled by an exponential decay function as a function of time in years. According to one study, the half-life of kidney transplants done in 1988 was 9.1 years; according to another study, the half-life of kidney transplants done in 1996 was projected to be 13.3 years. Is this later result good or bad news? Explain your reasoning.

18. According to a medical study, the half-life of kidney transplants was 13.3 years.

a. Write a formula for an exponential function that can be used to model the percentage of kidney transplant recipients who haven't rejected the kidney as a function of time.

b. What percentage of kidney transplant recipients do you predict will still have their new kidneys functioning after 10 years?

c. How long will it take until the percentage of kidney transplant recipients having their new kidneys will be down to 20%?

19. Treatments for different kinds of cancer are usually reported in terms of the percentage of patients who survive for 5 years after receiving the treatment, be it surgery, chemotherapy, or radiation therapy. The percentage who survive can be modeled by a exponential decay function. The 5-year survival rate for early stage malignant melanoma, a particularly severe type of skin cancer, is 80%.

a. What percentage of patients having this treatment will survive 10 years?

b. Use the information given to write an exponential decay function that models the percentage of patients treated for melanoma who survive any given length of time t in years.

c. What is the half-life for survival among patients having this treatment?

20. The 5-year survival rate for stage I lung cancer (the mildest and earliest form) treated by surgery is 60% to 70%.

a. Use the middle value of 65% to write an exponential decay function that models the percentage of patients treated for stage I lung cancer who survive any given length of time t in years.

b. What is the half-life for survival among patients having this treatment?

c. Repeat parts (a) and (b), using the lowest survival rate of 60%.

d. Repeat parts (a) and (b), using the highest survival rate of 70%.

21. In 1990, 442.2 million prerecorded cassette tapes and 865.7 million CDs were sold in the United States. In 1998, 158.5 million cassettes tapes and 1,124.3 million CDs were sold. Assume for now that the patterns of sales for both items are exponential functions.

a. Find the equation for the number of cassette tapes sold as an exponential function of time.

b. Find the equation for the number of CDs sold as an exponential function of time.

c. What is the practical significance of the growth or decay factors and growth or decay rates in parts (a) and (b)?

d. If the trends in sales of both items were indeed exponential functions, estimate when the number of CDs sold overtook the number of cassette tapes sold.

22. An exponential function f is such that $f(1) = 96$ and $f(5) = 6$. Which of the values are possible and which are impossible.

a. $f(3) = 24$ **b.** $f(3) = 51$ **c.** $f(3) = 65$

23. Suppose that a scientist has some initial amount R_0 of a radioactive substance whose half-life is measured on a scale of days.

a. Sketch the graph of the amount of this substance present as a function of time.

Use the concavity of your graph from part (a) to answer the following questions.

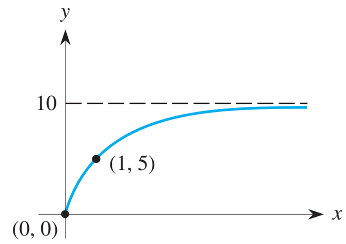
b. Suppose that you measure the amount of the substance after 10 days and find that 800 grams are left and after 11 days that 750 grams are left. Use this information to estimate the number of grams remaining after 20 days. Is the actual value higher or lower than your estimate? How do you know?

c. Suppose that you are told that the amount of the substance present after 30 days is 400 grams. Use this information and the amount left after 10 days to estimate the amount present after 20 days. Is the actual value higher or lower than your estimate? How do you know?

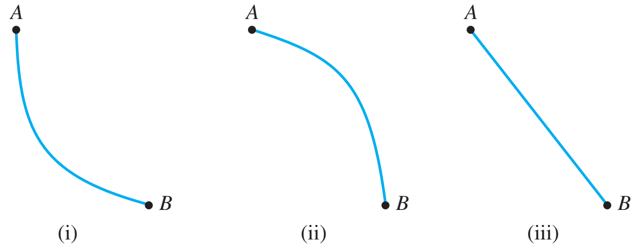
d. How might you use the results from (b) and (c) to come up with a better estimate of the amount of radioactive material present after 20 days?

24. A certain radioactive isotope has a half-life of 20 days. Suppose that 800 mg are present initially and consider a 60-day time period. Let r_1 represent the average daily rate of decrease of the isotope over the full 60-day period, let r_2 be the average daily rate of decrease over the first 30-day period, and let r_3 be the average daily rate of decrease over the last 30 days. List these three rates in increasing order without calculating their values.

25. The function shown in the accompanying figure is a modified exponential function of the form $y = A + B \cdot c^x$, with $c < 1$. Find appropriate values for the three constants A, B, and c.



26. You have been asked to design a slide at a water amusement park that extends vertically from point A to point B. A person sliding down it will speed up due to the force of gravity. For the three possible shapes of the slide shown, along which will a person make the trip from A to B most rapidly? Give reasons for your answer. (The specific curve along which an object will slide without friction from A to B in the shortest possible time is known as the *brachistochrone* and was first solved by Jacques Bernoulli in about 1700.)



Exercising Your Algebra Skills

Simplify the expressions by using properties of exponents.

1. $2^m \cdot 2^n$
2. $\frac{1}{2^u} \cdot \frac{1}{2^v}$
3. $5^3 \cdot 5^x$
4. $4^{-2} \cdot 4^{3x}$
5. $3^5 \cdot 3^{-2a}$
6. $2^{-4} \cdot 2^{-3w}$
7. $\frac{10^{-3x}}{10^{2x}}$
8. $\frac{4^{3x}}{4^{-3x}}$
9. $(2^x)^5$
10. $(0.7^x)^{10}$

Write each expression as the product of two terms, one an exponential term and the other a constant.

11. 3^{x+2}
12. 5^{x-2}
13. 10^{3x+1}
14. $\left(\frac{1}{2}\right)^{4x+3}$

2.6 Logarithmic Functions

In Section 2.4, we constructed an exponential function to approximate the population (in millions) of Florida as

$$P = f(t) = 12.94(1.029)^t,$$

where t is the number of years since 1990. Using this model, we can predict Florida's population at any given time, assuming that the growth rate remains 2.9% each year.

In Example 4 of Section 2.4, we estimated (using both numerical and graphical methods) that Florida's population will reach 20 million in early 2005, when $t \approx 15.2$. This problem involved finding the value of t for which

$$f(t) = 12.94(1.029)^t = 20.$$

Because this exponential function is always increasing, we know that there must be only one value of t when $P = 20$. We can always find an approximate value for t numerically or graphically, as we did in Sections 2.4 and 2.5. We now develop an algebraic approach for solving such equations exactly rather than approximately. We want a process that extracts the variable t from the exponent in $P = kc^t$. This process involves a new function called the **logarithm.** As with exponential functions, logarithms have a base b. Although logarithms can have any base b (denoted $\log_b$), we work primarily with logarithms to base 10.

Definition of Logarithms to Base 10

$$\log_{10} x = y \quad \text{means} \quad 10^y = x.$$

The logarithm to the base 10 of x is that power of 10 needed to produce x.

For instance,

$$\log_{10} 100 = \log_{10} 10^2 = 2$$

because 2 is that power of 10 needed to produce 100, or $10^2 = 100$. Also,

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

because 3 is that power of 10 needed to produce 1000, or $10^3 = 1000$. Similarly,

$$\log_{10}(0.1) = \log_{10} 10^{-1} = -1$$

because $0.1 = 1/10 = 1/10^1 = 10^{-1}$ and -1 is the power to which 10 must be raised to produce 0.1, or $10^{-1} = 0.1$.

The logarithm to the base 10 of x, $\log_{10} x$, is usually written simply as $\log x$. Because the logarithm is a function, it would be preferable to write $\log(x)$ rather than just $\log x$. But $\log(x)$ is not standard usage, so we avoid it. However, we do use parentheses for expressions such as $\log(5x)$.

The definition of the logarithm also gives two useful formulas.

Fundamental Logarithmic-Exponential Identities

$$\log(10^x) = x, \qquad \text{for all real } x$$

$$10^{\log x} = x, \qquad \text{for all } x > 0$$

Because these formulas hold for all appropriate values of x, they are called **identities.** Think about the two results to be sure that you understand them thoroughly. For the first identity, $\log(10^x)$ is that power of 10 needed to produce 10^x. Clearly, that power must be x itself. For instance, $\log(10^{1.234}) = 1.234$. For the second identity, the exponent in $10^{\log x}$ is $\log x$. In other words, $\log x$ is the power of 10 that gives the number x. For instance, $10^{\log 50.7} = 50.7$. The second property allows us to undo an equation involving logarithms. We discuss why the second identity is restricted to positive values of x later in this section.

Using the Logarithm

The principal reason for introducing logarithms here is to solve equations of the form

$$c^x = A$$

for the variable x in the exponent when the quantities c and A are known. For instance, we might want to solve $3^x = 8$ for the variable x in the exponent. To do so, we apply the following fundamental property of logarithms.

$$\log(c^x) = x \cdot \log c, \qquad c > 0$$

A proof of this formula can be found in any algebra textbook.

EXAMPLE 1

Solve for x in the equation $3^x = 8$.

Solution To extract x from the exponent, we take the logarithm of both sides of the equation:

$$\log(3^x) = \log 8.$$

We use the above fundamental property of logarithms to get

$$\log(3^x) = x \cdot \log 3 = \log 8,$$

where log 3 and log 8 are both just numbers. Finally, we divide both sides by log 3 to obtain

$$x = \frac{\log 8}{\log 3} \approx 1.893,$$

using a calculator. The graph of the function is depicted in Figure 2.43. To verify this result, we substitute $x = 1.893$ into the original equation and get

$$3^{1.893} \approx 8.0018.$$

We would have gotten 8 exactly if we hadn't rounded $\log 8/\log 3 \approx 1.893$.

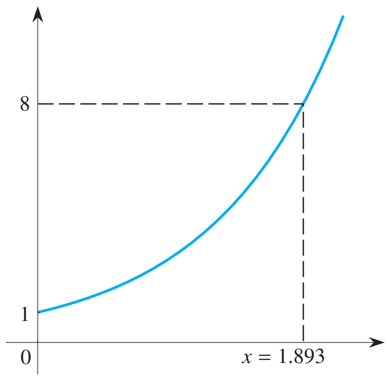


FIGURE 2.43

We now show how to obtain an exact solution to the question on the growth of Florida's population posed at the beginning of this section.

EXAMPLE 2

Determine when the population of Florida will reach 20 million.

Solution We begin with the equation

$$P = f(t) = 12.94(1.029)^t = 20.$$

Dividing both sides of the equation by 12.94, we get

$$(1.029)^t = \frac{20}{12.94}.$$

We now take logarithms of both sides and use the fundamental property of logarithms to get

$$\log(1.029^t) = t\log(1.029) = \log\left(\frac{20}{12.94}\right).$$

Dividing both sides by $\log(1.029)$ gives

$$t = \frac{\log(20/12.94)}{\log(1.029)},$$

which is the exact solution. When we perform the actual calculations, we get $t \approx 15.23$. Thus Florida's population will reach 20 million about $15\frac{1}{4}$ years after 1990, or early in 2005, as depicted in Figure 2.44.

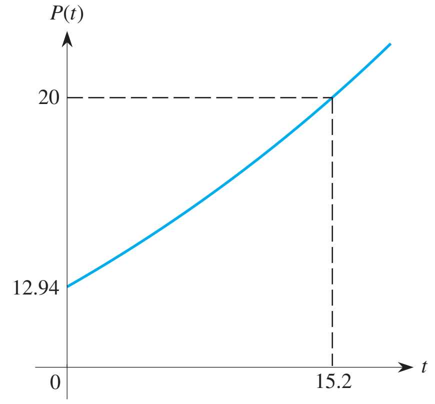


FIGURE 2.44

Think About This In the solution to Example 2, we intentionally left the quantity in the form $20/12.94$ to avoid possible rounding errors when dividing it out. What happens to the final answer if you perform the division operation early in the solution and round differently? See what happens, for instance, if you use $20/12.94 \approx 1.5$ or $20/12.94 \approx 1.55$ or $20/12.94 \approx 1.546$.

In Example 3 of Section 2.5, we estimated numerically and graphically that the half-life of Prozac is approximately 2.4 days based on the exponential decay function

$$D(t) = 80(0.75)^t$$

that models the amount of Prozac in the bloodstream following an 80 mg dose. We now show how to obtain the exact answer using logarithms.

EXAMPLE 3

Determine the half-life of Prozac in the bloodstream.

Solution To find the half-life exactly, we must find the time t needed until the original 80 mg drug level falls to 40 mg. Therefore we must solve the equation

$$80(0.75)^t = 40.$$

If we divide both sides by 80, we get

$$(0.75)^t = \frac{40}{80} = 0.5.$$

We now take logarithms of both sides and use the fundamental property of logarithms to find

$$\log(0.75)^t = t\log(0.75) = \log(0.5).$$

When we divide both sides by $\log(0.75)$, we get

$$t = \frac{\log(0.5)}{\log(0.75)} \approx 2.4094.$$

Thus, no matter what level of Prozac is in the blood at any given time, the level will drop by half about 2.4 days later, as illustrated in Figure 2.45.

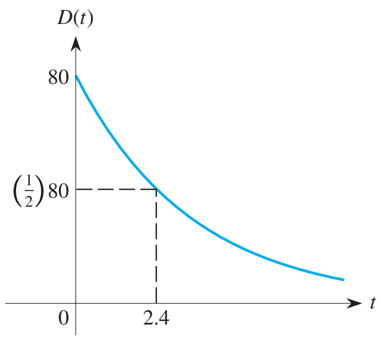


FIGURE 2.45

Properties of Logarithms

To use logarithms, you need to know their basic properties.

Properties of Logarithms

1. $\log(A^x) = x \cdot \log A$
2. $\log(A \cdot B) = \log A + \log B$
3. $\log(A/B) = \log A - \log B$

Proofs of all three of these properties can be found in any algebra textbook.

The first property is the tool we used to extract a variable from the exponent. The second property lets us simplify the *logarithm of a product* by writing it as the *sum of the individual logarithms;* for instance,

$$\log(5 \cdot 12) = \log 5 + \log 12.$$

Check this result on your calculator. Also,

$$\log(100x) = \log 100 + \log x = 2 + \log x.$$

The third property lets us simplify the *logarithm of a quotient* by writing it as the *difference of the individual logarithms;* for instance,

$$\log\left(\frac{9}{4}\right) = \log 9 - \log 4.$$

Check this result on your calculator. Also,

$$\log\left(\frac{1000}{x}\right) = \log 1000 - \log x = 3 - \log x.$$

Think About This Is $\log(9/4)$ the same as $(\log 9)/(\log 4)$? Why or why not? Try it on your calculator to see.

Think About This Is $\log(1000)/\log(x) = 3/\log x$ the same as $\log 1000 - \log x$? Why or why not? Graph both $y = \log(1000)/\log(x)$ and $y = 3/\log x$ to see whether it is true.

Note that all three properties of logarithms apply to logarithms with any base b, not just the base 10. Furthermore, these properties give us some alternative tools for solving some of the problems that we have already encountered. In Example 4, we show how to use the second and third properties of logarithms to determine once more when the population of Florida will reach 20 million.

EXAMPLE 4

Find when Florida's population will reach 20 million by using properties of logarithms.

Solution We again have to solve the equation

$$12.94(1.029)^t = 20.$$

In Example 2, we began by dividing both sides by 12.94. Instead, suppose that we start by taking logarithms of both sides of the equation:

$$\log[12.94(1.029)^t] = \log 20;$$

$$\log 12.94 + \log(1.029)^t = \log 20; \qquad \log(AB) = \log A + \log B$$

$$\log 12.94 + t \cdot \log 1.029 = \log 20. \qquad \log(A^x) = x \log A$$

To solve for t, we subtract log 12.94 from both sides of the equation:

$$t \cdot \log(1.029) = \log 20 - \log 12.94 = \log\left(\frac{20}{12.94}\right). \qquad \log A - \log B = \log\left(\frac{A}{B}\right)$$

Dividing by $\log(1.029)$, we get

$$t = \frac{\log(20/12.94)}{\log(1.029)} \approx 15.23,$$

which is the same result as in Example 2.

Behavior of the Logarithmic Function

For any value of $x > 0$, there is a single corresponding value of $\log x$, so $\log x$ is a function of x. We call this function the **logarithmic function** or simply the **log function.**

Let's now consider the behavior of the log function. Recall that the logarithm of a number x represents that power of 10 needed to produce x. Because no power of 10 ever produces 0 (10 raised to what power is 0?), log 0 is undefined. Similarly, because every power of 10 is positive, $\log x$ is not defined for negative values of x. Thus the domain of the logarithmic function is $x > 0$.

However, it is possible to have a negative power of 10, so $\log x$ can be negative. For instance, $10^{-0.25} \approx 0.56234$ means that $\log 0.56234 \approx -0.25$. Thus the range of the logarithmic function includes both positive and negative values. Figure 2.46 demonstrates that the logarithm of a number between 0 and 1 is negative and that

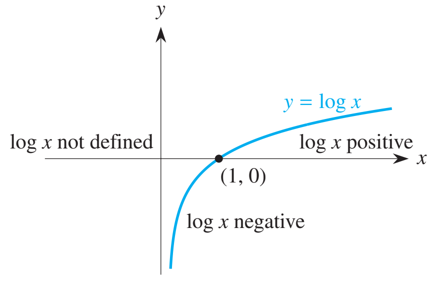


FIGURE 2.46

the logarithm of any number larger than 1 is positive. Finally, because $\log 1 = 0$ (since $10^0 = 1$), the range of the log function consists of all real numbers.

We use these ideas to examine the behavior of the log function $f(x) = \log x$. First, let's look at some values for this function. We know that

$$\begin{aligned} \log 1 &= 0, \\ \log 10 &= 1, \\ \log 100 &= 2, \\ \log 1000 &= 3, \\ &\vdots \\ \log 1{,}000{,}000 &= 6, \end{aligned}$$

and so on. Clearly, $\log x$ is an increasing function, at least for $x \geq 1$. Because the successive values grow more and more slowly, it is concave down. In fact, the most significant feature of the growth pattern for the logarithmic function is that it grows extremely slowly. Note what happens with the above values for the log function—to gain just one unit vertically requires going 10 times as far horizontally. Thus you need an extremely large value of x to make $\log x$ large. For instance, what value of x is needed to make $\log x = 100$? By the definition, x must be 10^{100} because $\log(10^{100}) = 100$. Consequently, it takes an incredibly long time for the log curve to reach a height of 100. The log function goes to infinity as x increases, although it does so exceedingly slowly.

The log function is not defined for $x = 0$ or for negative values. But what happens for small positive values of x? Consider the values

$$\begin{aligned} \log(1) &= 0, \\ \log(0.1) &= \log(10^{-1}) = -1, \\ \log(0.01) &= \log(10^{-2}) = -2, \\ \log(0.001) &= \log(10^{-3}) = -3, \\ &\vdots \\ \log(0.000001) &= \log(10^{-6}) = -6, \end{aligned}$$

and so on. As x gets closer and closer to 0, $\log x$ becomes more and more negative. Thus the line $x = 0$ (the y-axis) is a *vertical asymptote* of the graph of $y = \log x$ because the curve gets closer and closer to this line as x gets closer and closer to 0, but the curve never reaches it. This vertical asymptote reinforces the fact that the logarithmic function is not defined for $x = 0$, and so the graph of $y = \log x$ has no y-intercept. It does, however, have an x-intercept at $x = 1$ because $\log(1) = 0$, as illustrated in Figure 2.46.

Comparing Exponential and Logarithmic Functions

From the definition of the logarithm, it is clear that there is a close interrelationship between $y = \log x$ and the exponential function $y = 10^x$. To investigate this relationship, we start with the graphs of the two functions shown in Figure 2.47. Both are clearly increasing functions. The exponential function is concave up, whereas the log function is concave down. However, the main difference in their growth patterns is that the exponential function $y = 10^x$ grows extremely rapidly but the logarithm function $y = \log x$ grows extremely slowly.

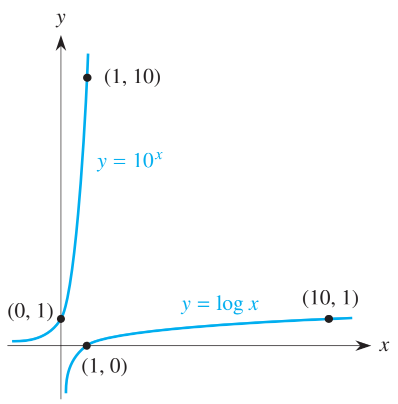


FIGURE 2.47

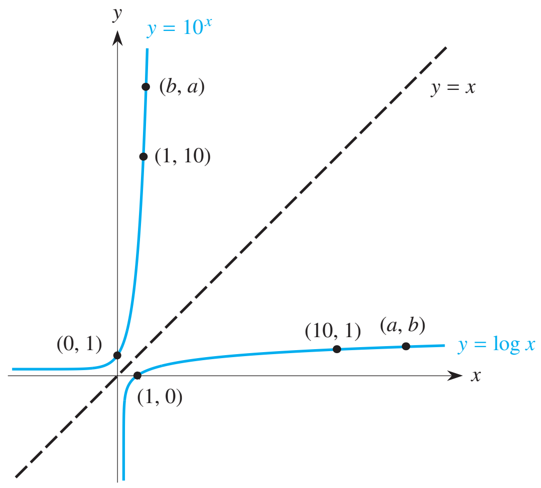


FIGURE 2.48

Figure 2.48 shows something striking about the graphs of the two functions $y = 10^x$ and $y = \log x$: They are reflections of each other about the diagonal line $y = x$ and thus are *symmetric* about this line (see Appendix D). Let's see why.

We know that

$$\log 10 = 1,$$

so the point $(10, 1)$ is on the graph of $y = \log x$. By the definition of the logarithm,

$$\log 10 = 1 \quad \text{means} \quad 10^1 = 10.$$

But $10^1 = 10$ tells us that the point $(1, 10)$ satisfies the equation $10^x = y$, so the point $(1, 10)$ is on the exponential graph $y = 10^x$. The points $(10, 1)$ and $(1, 10)$ are reflections of each other about the line $y = x$. In general, if the point (a, b) is on the graph of $y = \log x$, then

$$\log a = b.$$

This expression is equivalent to saying that

$$10^b = a,$$

which means that the point (b, a) is on the graph of the exponential function $y = 10^x$. Hence the log graph and the exponential graph are reflections of each other about the line $y = x$. We investigate this phenomenon in more depth in Section 2.9.

Applications of Logarithmic Functions

Logarithms have many applications. For instance, chemists use a quantity known as the pH to measure how acidic a water solution is. The pH is based on the concentration of hydrogen ions (measured in moles per liter) in the solution. The hydrogen-ion concentration of pure water is 10^{-7} moles per liter. Thus the pH of pure water is

$$\text{pH} = -\log(\text{concentration}) = -\log(10^{-7}) = -(-7)\log 10 = 7 \cdot 1 = 7,$$

which is used as the reference point for a neutral solution. Water solutions whose pH values are less than 7 are said to be acidic, whereas water solutions with pH values greater than 7 are said to be basic or alkaline. The lower the pH, the more acidic the solution; the higher the pH, the more basic the solution. Orange juice, which is somewhat acidic, has a hydrogen-ion concentration of 2×10^{-4} moles per liter and so its pH is

$$\begin{aligned} \text{pH} = -\log(2 \times 10^{-4}) &= -[\log 2 + \log(10^{-4})] \\ &= -\log 2 - \log(10^{-4}) \\ &= -\log 2 - (-4)\log 10 \\ &\approx -0.301 + 4 \cdot 1 \approx 3.7. \end{aligned}$$

Hydrochloric acid, with a hydrogen-ion concentration of 10^{-1} moles per liter, has a pH of 1, which indicates that it is extremely acidic. In comparison, household ammonia, with a pH of 11.5, is extremely basic.

Think About This Human blood has a hydrogen-ion concentration of 4×10^{-8}. What is its pH? Is blood slightly or extremely basic?

Note that each one-point decrease in pH represents a tenfold increase in the hydrogen-ion concentration.

The crust of the Earth is composed of about 20 rigid plates that "float" on liquid magma (the molten material beneath the Earth's crust). The study of this phenomenon is called *plate tectonics.* A geologic fault, such as the famous San Andreas Fault in California, is the space between two plates. As plates move, they bump into one another and sometimes one plate passes slightly under another, causing the upper plate to shift and heave. The result is an earthquake on the Earth's surface. There are about a million earthquakes, mostly very minor, each year. American seismologist Charles Richter developed a way of measuring the intensity of an earthquake. The *Richter scale* is based on the idea that there is a minimum noticeable, or threshold, level of earthquake intensity, denoted by I_0. The energy involved in a threshold level earthquake is approximately equal to the energy released by 10,000 atomic bombs. Any stronger quake has an intensity denoted by I. The Richter scale relates the magnitude R of an earthquake to its intensity, or

$$R = \log\left(\frac{I}{I_0}\right).$$

That is, the magnitude given by the Richter scale measurement is the logarithm of the ratio of the actual intensity to the threshold level.

The largest recorded earthquake, which occurred in Japan in 1933, had magnitude $R = 8.9$ on the Richter scale. Let's see just how powerful this quake was. We have

$$R = \log\left(\frac{I}{I_0}\right) = 8.9,$$

so when we take powers of 10 of both sides of the equation,

$$10^{\log(I/I_0)} = \frac{I}{I_0} = 10^{8.9} \approx 794{,}328{,}235.$$

Therefore this quake had an intensity almost 800 million times greater than the threshold level!

How are different measurements on the Richter scale related? For instance, if the measurement for one earthquake is double that of another, how much greater is it? What does a one point change in magnitude represent?

EXAMPLE 5

How does a magnitude 5 earthquake on the Richter scale compare to a magnitude 6 quake?

Solution If $R = 5$, we have

$$\log\left(\frac{I}{I_0}\right) = 5.$$

We undo the logarithm by taking powers of 10 of both sides of the equation and use the fundamental identity

$$10^{\log x} = x$$

to get

$$10^{\log(I/I_0)} = \frac{I}{I_0} = 10^5 = 100{,}000.$$

For $I/I_0 = 100{,}000$, we get $I = 100{,}000I_0$, so the intensity of a magnitude 5 quake is 100,000 times the threshold level. This quake's energy is equivalent to roughly $100{,}000 \times 10{,}000 = 10^9$, or 1 billion, atomic bombs exploding simultaneously.

Similarly, consider an earthquake measuring $R = 6$ on the Richter scale. We now get

$$I/I_0 = 10^6 = 1{,}000{,}000.$$

The intensity of this quake is 1 million times the threshold level. Thus an increase of 1 Richter unit corresponds to a tenfold increase in the intensity of the earthquake.

Suppose that one earthquake has a reading twice that of another on the Richter scale. How much stronger is it? Is it four times as strong? Is the relative intensity the same? Does it depend on the value for R? Let's compare $R = 4$ to $R = 2$ to see what happens. With $R = 4$, we have

$$\log\left(\frac{I}{I_0}\right) = 4,$$

so when we take powers of 10 of both sides of the equation,

$$10^{\log(I/I_0)} = \frac{I}{I_0} = 10^4$$

and therefore

$$I = 10^4 \cdot I_0.$$

Hence a magnitude 4 quake is 10,000 times the intensity of the threshold level. For $R = 2$, we have

$$\log\left(\frac{I}{I_0}\right) = 2$$

so that

$$10^{\log(I/I_0)} = \frac{I}{I_0} = 10^2$$

and therefore

$$I = 10^2 \cdot I_0.$$

Hence a magnitude 2 quake is 100 times the intensity of the threshold level. Consequently, a magnitude 4 quake is actually $10^4/10^2 = 100$ times stronger than a magnitude 2 quake.

Changing Bases

Throughout this book, we work with logarithms to the base 10 to undo exponential functions of the form $y = k \cdot 10^x$. However, as we stated earlier, it is possible to have bases other than 10—say, $c = 2$ or $c = 1.029$—as the base for an exponential function $y = kc^x$. Each possible base gives rise to a corresponding logarithmic function. For instance, we could work with logarithms to the base 2, written $\log_2 x$.

Definition of Logarithms to Base *c*

$$\log_c x = y \quad \text{means} \quad c^y = x, \qquad x > 0.$$

The logarithm to the base c of x is that power of c needed to produce x.

In practice, there is one particular base other than 10 that is widely used: the number $e = 2.71828\ldots$. The logarithm corresponding to base e is called the **natural logarithm.** It is especially important in calculus and many of the sciences. Even though we could write the natural logarithm as $\log_e x$, it is customarily written $\ln x$.

Although $\log_{10} 10 = 1$, we have $\ln 10 \approx 2.3026$ because $e^{2.3026} \approx 2.71828^{2.3026} \approx 10.0001$. Similarly, whereas $\log_{10} 100 = 2$, we have $\ln 100 \approx 4.6052$ because $e^{4.6052} \approx 2.71828^{4.6052} \approx 100.003$.

We previously said that all the properties of logarithms apply no matter what base is used. Thus the following are properties of the natural logarithm.

Properties and Identities for the Natural Logarithm

1. $\ln(A^p) = p \ln A$
2. $\ln(A \cdot B) = \ln A + \ln B$
3. $\ln(A/B) = \ln A - \ln B$
4. $\ln e^x = \log_e e^x = x$
5. $e^{\ln x} = x, \qquad \text{if } x > 0$

If different bases are used, we must be able to convert either an exponential function or a logarithm in one base into an exponential function or a logarithm in a different base. That is, for any number x, how do we convert from c^x to 10^x or convert $\log x$ to $\ln x$ and vice versa? Let's first look at converting bases of exponential functions.

EXAMPLE 6

We found that the population of Florida can be modeled by the exponential function $P(t) = 12.94(1.029)^t$. Convert this function to an equivalent expression that involves **(a)** base 10 and **(b)** base e.

Solution

a. Suppose that we try to find the appropriate power q so that

$$(1.029)^t = 10^q.$$

Using properties of logarithms, we take logs of both sides and get

$$t \log(1.029) = q \log 10 = q \cdot 1 = q.$$

The formula for the population of Florida becomes

$$P(t) = 12.94(1.029)^t = 12.94(10^q) = 12.94(10^{t \log(1.029)}) = 12.94(10^{0.0124t}),$$

since $\log(1.029) = 0.0124$. Alternatively, we might think of this result as coming from

$$10^{t \log(1.029)} = (10^{\log(1.029)})^t = (1.029)^t.$$

b. To convert the formula for the Florida population to an equivalent formula using base e, we use the property $e^{\ln x} = x$:

$$(1.029)^t = (e^{\ln 1.029})^t = (e^{0.0286})^t = e^{0.0286t}.$$

Therefore

$$P(t) = 12.94(1.029)^t = 12.94(e^{0.0286t}).$$

We have three formulas for the population of Florida:

$$P(t) = 12.94(1.029)^t;$$
$$P(t) = 12.94(10^{0.0124t});$$
$$P(t) = 12.94(e^{0.0286t}).$$

These three formulas are mathematically equivalent—only the bases are different. Graph these three functions using your function grapher to convince yourself that they are identical, except perhaps for slight differences due to rounding.

In general, to convert an exponential function $y = kc^x$ from base c to base 10, where $y = k \cdot 10^{mx}$, we write $c = 10^m$ so that $m = \log c$. To convert an exponential function $y = kc^x$ from base c to base e, where $y = ke^{mx}$, we write $c = e^m$, so that $m = \ln c$.

Now let's consider how to convert a logarithm in one base to a logarithm in another base. We begin by looking at some typical values of $\log x$ and $\ln x$, rounded to four decimal places, as shown in the following table. To see if there is any clear relationship between the two sets of logarithmic values, we plot the values of $\ln x$ versus $\log x$, as shown in Figure 2.49.

The graph shows that there is a linear pattern. The line passes through the origin, so its vertical intercept is 0 and we can write

$$\ln x = m \log x$$

for some constant of proportionality m. You can also see this from the ratio of $\ln x$ and $\log x$ for any value of x—in every case, the ratio is approximately 2.3026. Check this

x	$\log x$	$\ln x$
1	0	0
2	0.3010	0.6932
3	0.4771	1.0986
4	0.6021	1.3863
5	0.6990	1.6094
6	0.7782	1.7918
7	0.8451	1.9459
8	0.9031	2.0794
9	0.9542	2.1972
10	1	2.3026

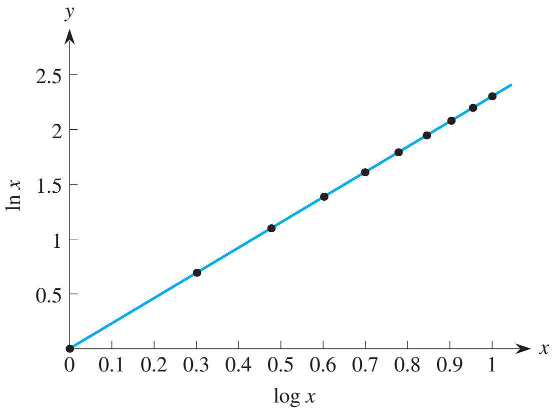


FIGURE 2.49

on your calculator by using several different values of x. Note that Figure 2.49 shows the values of $\ln x$ plotted against those of $\log x$. If we plotted either $\ln x$ against x or $\log x$ against x, we would get the usual graph of a logarithmic function—one that is increasing and concave down. Only the graph of $\ln x$ versus $\log x$ results in a line.

The value of the constant of proportionality $m = 2.3026$ is also the slope of the line through the points shown in Figure 2.49. Thus we can write

$$\ln x = 2.3026 \log x$$

for any x. Because 2.3026 appears in the last row of the preceding table as the value of ln 10, we can rewrite this relationship as

$$\ln x = (\ln 10)\log x,$$

or equivalently,

$$\log x = \frac{\ln x}{\ln 10}.$$

Rewriting this equation to highlight the base of the logarithm, we get

$$\log_{10} x = \frac{\log_e x}{\log_e 10}.$$

In fact, if we perform the identical analysis with any other base (say, c instead of e), we obtain the comparable result for changing between base 10 and base c, for any c; or

$$\log_{10} x = \frac{\log_c x}{\log_c 10}.$$

Problems

1. The graphs of the following functions may surprise you. Use your function grapher to graph each function and then explain what you see and why, using the properties of logarithms.
   a. $y = \log 10^{2x}$ **b.** $y = \log(2x) - \log(x)$
   c. $y = \log 10^{x^2}$ **d.** $y = 10^{\log(x^2)}$
   e. $y = \log 3^x$ **f.** $y = \log(10/6^x)$
2. Use your function grapher to draw simultaneously the graphs of $y = \log(2^x)$, $y = \log(3^x)$, and $y = \log(5^x)$. For each function, use the properties

of logarithms to explain why you get the graphs you do.

3. The population of Argentina was 34.6 million in 1995 and was growing exponentially at an annual rate of 1.3%.
   a. Find an expression for Argentina's population at any time t, where t is the number of years since 1995.
   b. What population would you predict for 2005 if the present trend continues?
   c. Use logarithms to find the doubling time exactly.
4. The population of Kenya is growing exponentially. Its population was 23.3 million people in 1988 and 28.3 million in 1995.
   a. Find an expression for the population at any time t, where t is the number of years since 1988.
   b. What would be the population in 2005?
   c. Use logarithms to find the doubling time.
5. Because of ardent fishermen during the summer months, the population of fish in a lake is reduced by 10% each week. Find the half-life of this dwindling fish population, using logarithms.
6. The Best Company earned \$50 million in 2000, and its income is growing at a rate of 2% per year. The Acme Corporation earned \$30 million that year, and its income is growing at a rate of 6.5% per year. When will Acme overtake Best in annual income?
7. a. Find the doubling time for annual growth rates of 3%, 4%, 5%, 6%, and 7%.
   b. Consider the doubling time d as a function of growth rate r. Plot your results from part (a) and decide what type of function seems to fit the behavior pattern you observe.
8. Bankers use a technique called the Rule of 70 to estimate the doubling time for money invested at different interest rates, dividing 70 by 100 times the interest rate. Thus for an interest rate of $10\% = 0.10$, bankers estimate the doubling time to be
$$\frac{70}{100 \cdot 0.10} = \frac{70}{10} = 7 \text{ years.}$$
   Use your results from Problem 7 to test the accuracy of this method.
9. Determine when the cost of first-class postage for a letter will reach \$1, given that first class postage rose to 29¢ in 1990 and to 37¢ in 2002.

Problems 10–13 are based on the carbon dating process to measure the age of objects. Carbon-14, a radioactive form of carbon, decays into carbon-12 with a half-life of 5730 years.

10. The famous Cro-Magnon cave paintings are found in the Lascaux Cave in France. If the level of carbon-14 radioactivity in charcoal in the cave is approximately 14% of that of living wood, estimate the date when the paintings were made.
11. The level of carbon-14 in a charred roof beam found in a 1950 excavation of an ancient Babylonian city is about 61% of the level in living wood. Estimate when the fire occurred.
12. The well-preserved body of a Stone Age man was found in melting snow in the northern Italian Alps in 1991. Examination of a tissue sample from the body indicated that 47% of the carbon-14 present in the body at the time of death had decayed. When did the man die?
13. Several groups of scientists were allowed to test the Shroud of Turin, the supposed burial cloth of Jesus, in 1991. They found that the cloth contained 91% of the amount of carbon 14 contained in newly made cloth of the same material. Based on this information, how old is the Shroud of Turin?
14. A radioactive substance decays exponentially so that after 10 years, 40% of the initial amount R_0 remains.
   a. Find an expression for the quantity remaining after t years.
   b. How much will be present after 25 years?
   c. What is the half-life of the substance?
   d. How long will it be before only 2% of the original amount is left?
15. In an effort to reduce the breeding rate of a strain of pesticide-resistant mosquitoes in the southeastern United States, a group of scientists released large numbers of sterilized male mosquitoes to mate with the fertile females who would consequently produce no offspring. Suppose that effort reduced the mosquito population by 2% per month.
   a. What percentage of the original population P_0 would remain after 1 year?
   b. How long would it take to lower the population by half?
   c. How long would it take for the population to fall to 10% of the original level?
16. In computer science, the efficiency of algorithms (methods for accomplishing a certain task) are often analyzed by how long it takes to perform the operation with n objects. Typically, as n increases, the time involved for the operation increases signif-

icantly. Two algorithms used to put a set of names in alphabetical order are compared. For one algorithm, the time needed to order n names, as a function of n, is $B(n) = \frac{1}{2}n^2$. The time for the second algorithm, as a function of n, is $S(n) = n \log n$. Which method is faster? Explain.

17. How much stronger is a magnitude 6 earthquake than a magnitude 3 earthquake?

18. How much stronger is

a. a magnitude 7 quake than a magnitude 5 quake?
b. a magnitude 7 quake than a magnitude 4 quake?

19. Let I_0 be the minimum (or threshold) level of sound that can be heard by human beings. If the intensity of a particular sound is I, the magnitude of the sound, measures in decibels d, is given by

$$d = 10 \log\left(\frac{I}{I_0}\right).$$

a. Normal conversation measures about 60 decibels. How much more intense is this level than the threshold level?
b. A loud noise of about 150 decibels will cause deafness. How much more intense is this level than the threshold level?
c. An aircraft taking off has a loudness level of about 120 decibels. How much more intense is this level than the threshold level?
d. How loud (that is, how many decibels) is a sound whose intensity is 1 million times the threshold level?
e. The noise level from a rock band is about 100 billion times higher than the threshold level. What is the decibel measure of this noise level?

20. Convert the formula $D(t) = 80(0.75)^t$ for the level of Prozac in the bloodstream following an initial dose of 80 mg to equivalent formulas with base 10 and base e.

21. The population of the world can be modeled by the function $P(t) = 6(1.015)^t$, where t is the number of years since 1999. Convert this formula to equivalent formulas with base 10 and base e.

Exercising Your Algebra Skills

Simplify each expression by using the properties of logarithms.

1. $\log x + \log x^2 + \log x^3$ **2.** $\log x + \log \sqrt{x}$

3. $\log x^2 + \log y^3 - \log x - \log y^2$

4. $\log(x/y) - \log(y/x)$ **5.** $\log 10^{x^2}$

6. $10^{\log(x^2)}$

Solve each expression for x.

7. $7^x = 11$ **8.** $1.05^x = 2$

9. $0.4^x = 0.6$ **10.** $3(1.04)^x = 5$

11. $12(0.86)^x = 3$ **12.** $9(0.17)^x = 0.25$

13. $4(1.05)^x = 5(1.04)^x$ **14.** $3(0.7)^x = 6(0.5)^x$

15. $\log x = 2$ **16.** $\log x = 0.5$

Without using a calculator, evaluate each term.

17. $\log 1{,}000{,}000$ **18.** $\log 0.001$

19. $\log(1/10{,}000)$ **20.** $\log \sqrt{10}$

Determine which of the following are true for all values of x and which are not by using algebra.

21. $\log(x^2 - 1) = \log(x - 1) + \log(x + 1)$

22. $\log(x^2 + 1) = \log(x^2) + \log(1)$

23. $\log(1/x) = -\log x + \log(1)$

2.7 Power Functions

The area A of a circle with radius r is given by

$$A = f(r) = \pi r^2.$$

The surface area S of a sphere with radius r is given by

$$S = g(r) = 4\pi r^2.$$

(Picture a tennis ball whose surface is made up of four roughly circular regions, as shown in Figure 2.50). The volume V of the sphere is given by

$$V = h(r) = \frac{4}{3}\pi r^3.$$

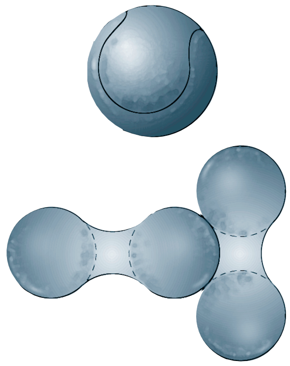


FIGURE 2.50 Tennis ball cover split apart

Similarly, the inverse square law of gravitation describes how the force of gravity of one object on any other object in the universe varies with distance. In particular, the gravitational force F on a unit mass at a distance d from the center of the Earth is given by

$$F = \frac{k}{d^2} \quad \text{or} \quad F = kd^{-2},$$

where k is a positive constant.

All four of these functions are examples of *power functions,* so called because the independent variable is raised to a constant power. In each case, the dependent variable is a constant multiple of some power of the independent variable. In general, a **power function** is any function of the form

$$y = f(x) = kx^p,$$

where k and p are any constants, positive or negative. (Compare this expression for a power function with an exponential function of the form $y = kc^x$, where the independent variable x is the exponent and the base c is a constant, as shown in Figure 2.51.) Note that the family of power functions $y = kx^p$ is a two-parameter family with parameters p and k.

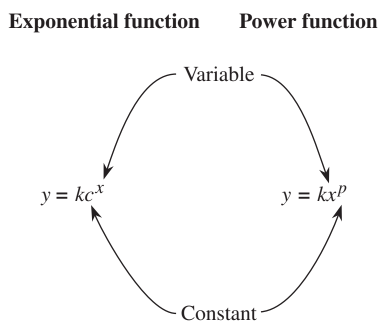


FIGURE 2.51

The definition of a power function includes the special case where $p = 1$, which gives the linear function $y = kx^1 = kx$ that passes through the origin. The definition also includes functions such as

$$y = x^2, \qquad y = x^3, \quad \text{and} \quad y = x^4,$$

as well as the case where p is a positive or negative fraction or a decimal, such as

$$y = x^{1/2}, \qquad y = x^{1/3}, \qquad y = x^{3/2}, \quad \text{and} \quad y = x^{-2.83}.$$

In algebra, fractional exponents are usually introduced purely as a means for simplifying operations with terms involving radicals. Recall that

$$x^{1/2} = \sqrt{x}, \qquad x^{1/3} = \sqrt[3]{x}, \qquad x^{5/8} = \sqrt[8]{x^5},$$

and, in general,

$$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m.$$

Power functions of the form $y = f(x) = kx^{m/n}$ arise naturally in many applications. For instance, biologists have found a relationship between the weight W of large flying birds and their wingspan S. This relationship can be modeled by the power function

$$W = f(s) = 0.15S^{9/4}.$$

This function gives the weight that can be supported by a given wingspan. For example, the wingspan S of a condor is about 10 feet. According to this model, its weight is approximately

$$W = 0.15(10)^{9/4} \approx 27 \text{ lb}.$$

To perform this calculation we must either first raise $S = 10$ to the ninth power and then take the fourth root of the result, so that

$$W = 0.15(10)^{9/4} = 0.15[(10)^9]^{1/4} = 0.15[1{,}000{,}000{,}000]^{1/4} \approx 27$$

or first take the fourth root of S and then raise the result to the ninth power, so that

$$W = 0.15(10)^{9/4} = 0.15[(10)^{1/4}]^9 = 0.15[1.778279]^9 \approx 27.$$

Symbolically, we write

$$S^{9/4} = \sqrt[4]{S^9} = (\sqrt[4]{S})^9.$$

When you use a calculator to evaluate such an expression, be careful to use parentheses around the fractional exponent, as in

```
0.15*10^(9/4);
```

without the parentheses, the rules for the order of operations will give you a very different answer.

The graph of the power function $W = 0.15S^{9/4}$ is shown in Figure 2.52. Note that the pattern is that of an increasing, concave up function; thus, as the wingspan S of a bird increases, its weight W increases even more rapidly. Consequently, heavier birds require relatively smaller wingspans in order to fly, which is likely contrary to intuition.

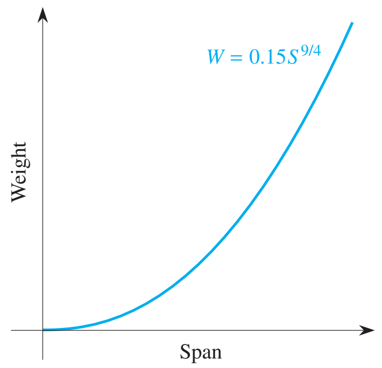


FIGURE 2.52

Think About This

The largest known bird is the Steller's eagle, with a wingspan of 8 feet. Estimate the weight of an adult Steller's eagle.

Aeronautical engineers use the same principle—based on a similar relationship between wingspan and the weight of a plane—when designing new aircraft.

Behavior of Power Functions for $x > 0$

Recall that for exponential functions the value of the base c in $y = kc^x$ leads to different behavior patterns—exponential growth when $c > 1$ and exponential decay when $0 < c < 1$. Similarly, the behavior of power functions depends on the size of the constant power p in $y = kx^p$. To simplify things initially, we let $k = 1$ so that we can consider the more basic power function $y = x^p$.

The three different behavior patterns for power functions are illustrated in Figure 2.53: the graphs of $y = x^2$, $y = x^{1/2}$, and $y = x^{-2}$ (along with the graph of $y = x$ for reference) for $x > 0$. Note how the graph of $y = x^2$ (with $p = 2$) is increasing and concave up; that of $y = x^{1/2}$ (with $p = \frac{1}{2}$) is increasing and concave down, and that of $y = x^{-2}$ (with $p = -2$) is decreasing and concave up. The specific values $p = 1$ (when we get a line through the origin) and $p = 0$ (when we get a horizontal line at height $y = 1$) are critical in separating one kind of behavior from another.

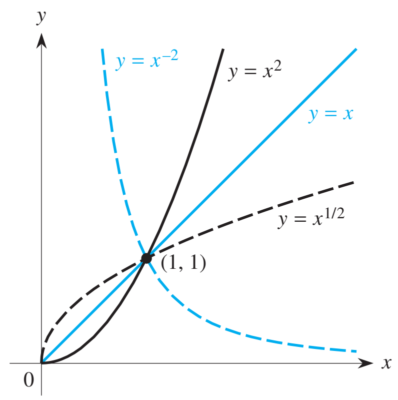


FIGURE 2.53

Let's investigate these cases in more detail by looking at a variety of different power functions of each type. Figure 2.54(a) shows the graphs of three related

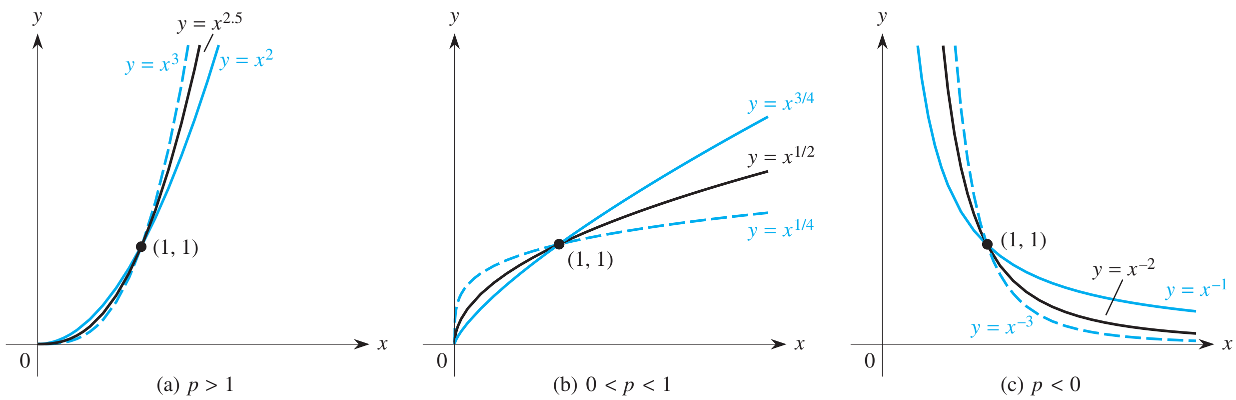


FIGURE 2.54

power functions, $y = x^2$, $y = x^{2.5}$, and $y = x^3$, all of which behave in the same manner as $y = x^2$. Similarly, Figure 2.54(b) shows the graphs of three other power functions, $y = x^{1/4}$, $y = x^{1/2}$, and $y = x^{3/4}$, all of which behave in a pattern similar to that for $y = x^{1/2}$. Finally, Figure 2.54(c) shows the graphs of $y = x^{-1}$, $y = x^{-2}$, and $y = x^{-3}$, all of which behave in a third manner, similar to $y = x^{-2}$.

These graphs suggest the following facts about power functions for $x \geq 0$:

1. If $p > 1$, the power function $y = x^p$ is increasing and concave up.

   If $0 < p < 1$, the power function $y = x^p$ is increasing and concave down.

   If $p < 0$, the power function $y = x^p$ is decreasing and concave up.
2. Every power function $y = x^p$ passes through the point (1, 1).
3. If $p > 0$, the power function $y = x^p$ passes through the origin.

   If $p < 0$, the power function $y = x^p$ rises toward the positive y-axis as x gets closer to 0 (the y-axis is a vertical asymptote) and decays toward the positive x-axis as x approaches ∞.

Statement 2 is true because 1 raised to any power is 1 (that is, $1^p = 1$ for any p). The first part of Statement 3 is obvious because 0 raised to any positive power will be 0 (that is, $0^p = 0$ for $p > 0$). As for the second part of Statement 3, if the power p is negative, we can write it as $p = -q$ so that

$$x^p = x^{-q} = \frac{1}{x^q},$$

using one of the basic rules for exponents from algebra. Obviously, we can't have $x = 0$ because the quotient $1/x^q$ is not defined at 0. However, the closer x is to 0, the closer x^q is to 0 also, and therefore the larger $1/x^q$ is. That is, the graph of any power function of the form $y = x^p = x^{-q}$ must always rise and approach the positive y-axis as x approaches 0. The y-axis is a vertical asymptote for these curves because they approach it more and more closely but never reach it. Also, if $p < 0$, as x increases, $x^p = x^{-q} = 1/x^q$ becomes smaller and eventually approaches 0. So the x-axis is a horizontal asymptote for any power function with $p < 0$.

Behavior of Power Functions for $x \geq 1$

It is also evident from Figure 2.54 that the behavior patterns for these power functions are different when x is between $x = 0$ and $x = 1$ compared to when $x > 1$. Again, the point (1, 1) serves as a demarcation between the different behaviors. To see the differences more clearly, in Figure 2.55 we zoom in on all the graphs shown in Figure 2.54 to examine what happens when $x > 1$ for different values of p. (Incidentally, in all these cases, you should think of x as the variable and p as a parameter that takes on a fixed value to produce a particular curve whose values depend on x.)

Figure 2.55 suggests the following additional fact about power functions.

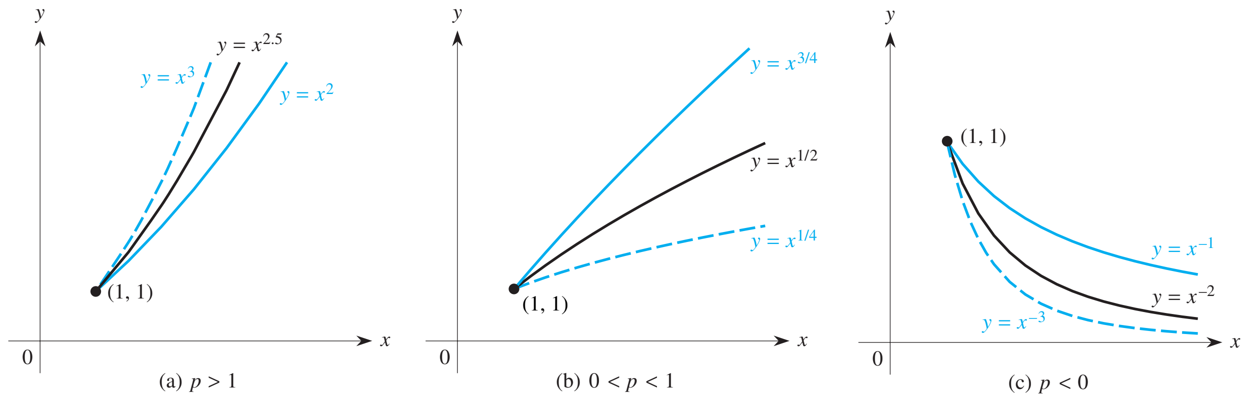


FIGURE 2.55

> **4.** If $p > 0$, the larger p is, the larger x^p is when $x > 1$.
> If $p < 0$, the more negative p is, the more rapidly x^p dies out as x increases.

Thus, for instance, when $x > 1$,

$$x^5 > x^4 > x^3 > x^2,$$

and also

$$x^{-5} < x^{-4} < x^{-3} < x^{-2}.$$

These relationships are shown numerically in Table 2.1 for $p > 0$. For each value of $x > 1$, not only do the higher powers of x get larger, but they get *larger much faster*.

TABLE 2.1

	$x = 2$	$x = 5$	$x = 10$	$x = 20$
$y = x^2$	4	25	100	400
$y = x^3$	8	125	1000	8000
$y = x^4$	16	625	10,000	160,000
$y = x^5$	32	3125	100,000	3,200,000

We can prove this fact algebraically. For instance, if $x > 1$ and we multiply both sides of this inequality by the positive term x^3, we get

$$x^3 \cdot x > x^3 \cdot 1 \quad \text{so that} \quad x^4 > x^3.$$

This comparison gets more pronounced as x gets larger and larger. Compare the values in the table for $x = 10$ and $x = 20$. As x gets ever larger (i.e., as x approaches infinity, denoted by $x \to \infty$), any positive power completely overwhelms, or dominates, any smaller power.

Now let's look at the case when $p < 0$. For each value of $x > 1$, Table 2.2 shows that, not only do the negative powers of x get smaller as x increases, but the more negative the power, the faster the function $y = x^p$ dies out.

TABLE 2.2

	$x = 2$	$x = 5$	$x = 10$	$x = 20$
$y = x^{-2}$	0.25	0.04	0.01	0.0025
$y = x^{-3}$	0.125	0.008	0.001	0.000125
$y = x^{-4}$	0.0625	0.0016	0.0001	0.00000625
$y = x^{-5}$	0.03125	0.00032	0.00001	0.0000003125

Behavior of Power Functions for $0 < x < 1$

The preceding conclusions are based on what happens to power functions when $x > 1$. Now let's see what happens to these same power functions when $0 < x < 1$, as shown in Figure 2.56. Note that the behavior patterns are reversed from those when $x > 1$. In particular, the graphs suggest the following fact about power functions.

> **5.** If $p > 1$ or $p < 0$, the larger p is, the smaller x^p is when $0 < x < 1$. If $0 < p < 1$, the larger p is, the larger x^p is when $0 < x < 1$.

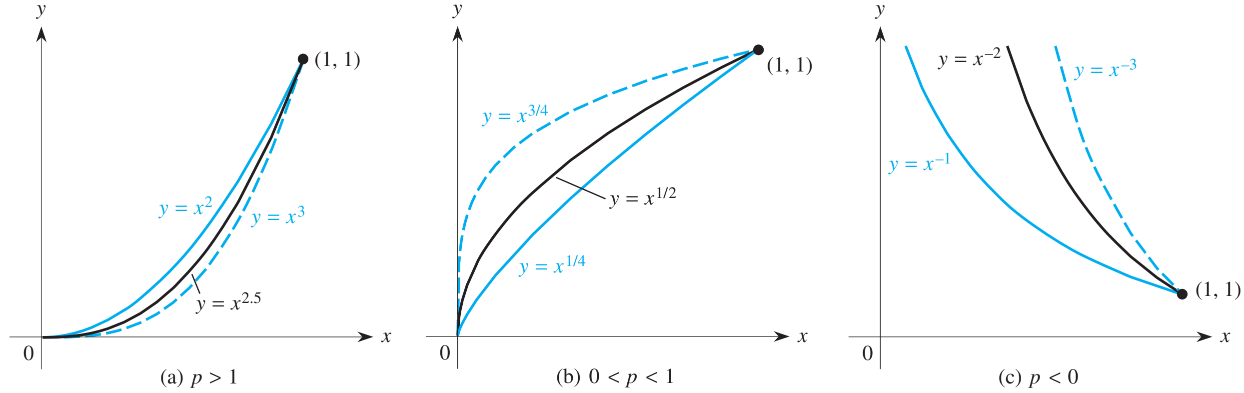


FIGURE 2.56

Thus, for instance, when x is between 0 and 1,

$$x^5 < x^4 < x^3 < x^2,$$

and

$$x^{-2} < x^{-3} < x^{-4} < x^{-5}.$$

Thus, as x approaches 0, the more positive the power p, the faster a power function dies out near the origin. Similarly, the more negative the power p, the faster a power function grows toward infinity as x approaches 0.

Think About This What do the graphs of $y = x^{0.99} = x^{(99/100)}$ and $y = x^{1.01}$ look like? How do they behave compared to the line $y = x$? ❐

Think About This What do the graphs of $y = x^{0.01}$, $y = x^0$, and $y = x^{-0.01}$ look like? How do they compare to one another? ❐

Applications of Power Functions

In Sections 2.2 and 2.3, we dealt with many situations that led to equations involving linear functions. In Sections 2.4–2.6, we similarly explored some situations leading to equations involving exponential functions of the form

$$y = kc^t.$$

In many of these situations, we had a value for y and had to solve for the unknown t in the exponent by using logarithms. Similarly, situations often arise that lead to equations involving power functions of the form

$$y = kx^p,$$

where a value for y is given and we have to solve for the unknown x, which in this case is raised to a constant power.

As we discussed in several simple examples in previous sections, we solve the power function equation $x^2 = 25$ by taking the square root of both sides of the equation to get $x = \pm 5$. (To envision the two solutions, imagine the graph of the function $y = x^2$: It reaches a height of 25 in two places, one for $x < 0$ and another for $x > 0$.) Similarly, we solve the equation $x^3 = 27$ by taking the cube root of both sides of the equation to get

$$x = \sqrt[3]{27} = 3.$$

Similarly, we solve the equation $x^7 = 50$ by taking the 7th root of both sides of the equation (equivalently by raising both sides to the $\frac{1}{7}$ power):

$$x = \sqrt[7]{50} = 50^{1/7} \approx 1.7487.$$

EXAMPLE 1

A bald eagle weighs about 16 pounds. Use the relationship $W = 0.15S^{9/4}$ to estimate its wingspan.

Solution We have to solve for the eagle's wingspan S corresponding to $W = 16$ pounds in the equation

$$0.15S^{9/4} = 16.$$

We first divide both sides by 0.15 to get

$$S^{9/4} = 106.667.$$

To solve for S, we have to undo the $\frac{9}{4}$ power, which we do by raising both sides of this equation to the $\frac{4}{9}$ power:

$$(S^{9/4})^{4/9} = S^1 = S = 106.667^{4/9} \approx 7.968,$$

or about 8 feet.

In Example 2 we show how these ideas about power functions arise in the context of one of the most useful applications of radioactive decay. Scientists routinely use a process known as carbon-dating to establish the age of fossils. It is based on the fact that carbon-14 decays to carbon-12 with a half-life of 5730 years.

EXAMPLE 2

Crater Lake in Oregon was formed as the result of a volcanic eruption. A charcoal sample from a tree that burned during the eruption contains about 46% of the carbon-14 found in live trees.

a. What is the decay factor for carbon-14?

b. What was the approximate date for the formation of Crater Lake?

Solution Because the half-life of carbon-14 is 5730 years and slightly more than 50% of the radioactive carbon has disintegrated, we expect that the time involved is somewhat more than 5730 years; we might estimate, say, about 6000 years. Now let's find out more precisely.

a. The exponential decay function that models the radioactive decay process is

$$R(t) = R_0 c^t$$

for some initial quantity R_0 of the radioactive carbon and some decay factor $c < 1$. Because the half-life of carbon-14 is 5730 years, we substitute $t = 5730$ into the expression for the function to get

$$R(5730) = R_0 c^{5730} = \frac{1}{2} R_0$$

so that

$$c^{5730} = \frac{1}{2}.$$

We solve this equation for c by extracting the 5730th root of $\frac{1}{2}$, to get the decay factor:

$$c = \left(\frac{1}{2}\right)^{1/5730} \approx 0.99988.$$

b. We know that

$$R(t) = R_0 \cdot (0.99988)^t.$$

We now have to find how long it takes for the carbon-14 to decay to the point where only 46% of R_0 is present. Thus we want to find t when

$$R(t) = R_0 \cdot (0.99988)^t = 0.46 R_0.$$

Dividing through by R_0 gives

$$(0.99988)^t = 0.46.$$

To solve this exponential equation, we take logs of both sides and get

$$\log(0.99988)^t = t \log(0.99988) = \log(0.46).$$

Therefore

$$t = \log(0.46)/\log(0.99988) \approx 6470.685,$$

or about 6471 years ago, as shown in Figure 2.57. We thus conclude that Crater Lake was formed in roughly 4471 B.C.

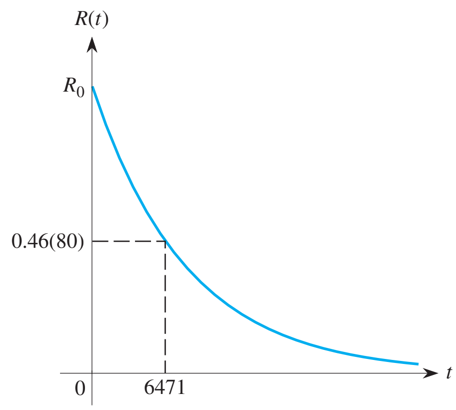


FIGURE 2.57

Note that we solved two very different mathematical problems in Example 2. In part (a), we had the power function equation $c^{5730} = \frac{1}{2}$ and solved for c by taking the 5730th root. In part (b), we had the exponential function equation $(0.99988)^t = 0.46$ and solved for t by using logarithms. In general, we take roots to undo powers and so solve equations of the form $x^n = A$ involving power functions. Similarly, we use logarithms to extract variables from an exponent and so solve equations of the form $b^x = A$ involving exponential functions.

Unfortunately, neither operation will do anything useful to solve an equation such as

$$3^x = x^4,$$

which involves both an exponential function and a power function. This equation can be solved numerically or graphically to find an approximate solution that is accurate to any desired degree of accuracy, by using a graphing calculator or a computer graphics program. (We ask you to solve it as a problem at the end of the section.) But, the equation *cannot* be solved algebraically to find an exact solution.

Fitting Power Functions to Two Points

We have seen that two points determine a line or an exponential function because only one line or one exponential function passes through those points. Similarly, two points also determine a power function, which we illustrate in Examples 3–5.

EXAMPLE 3

Find the power function that passes through the points $(1, 5)$ and $(4, 60)$.

Solution A power function is of the form $y = kx^p$, for some constants k and p. We substitute the coordinates of the first point $(1, 5)$ into this expression to obtain

$$k \cdot (1)^p = k = 5.$$

Therefore the expression for the power function reduces to $y = 5x^p$. We use the second point, $(4, 60)$, to determine the power p. When we substitute the coordinates $x = 4$ and $y = 60$, we get

$$5(4)^p = 60.$$

We divide both sides of this equation by 5 to obtain

$$4^p = 12.$$

To extract the unknown p from the exponent, we take logs of both sides to get

$$\log(4^p) = p \log 4 = \log 12,$$

so that

$$p = \frac{\log 12}{\log 4} \approx 1.792.$$

Consequently, the power function that passes through the two points $(1, 5)$ and $(4, 60)$ is

$$y = 5x^{1.792}.$$

As depicted in Figure 2.58, this function is increasing and concave up.

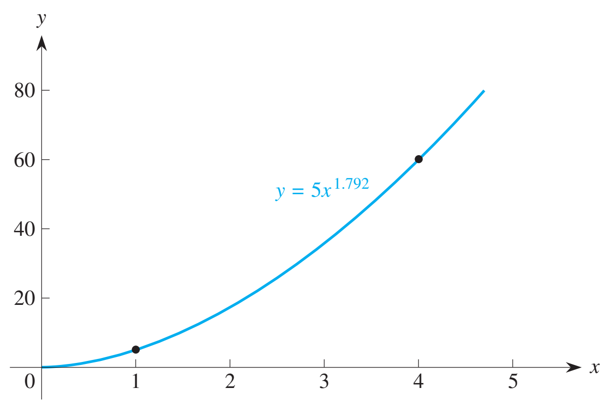


FIGURE 2.58

Example 4 illustrates a somewhat more complicated situation.

EXAMPLE 4

Find the power function that passes through the points $(2, 5)$ and $(4, 60)$.

Solution When we substitute the coordinates of the first point $(2, 5)$ into the equation of a power function $y = kx^p$, we get

$$k \cdot 2^p = 5,$$

which involves both unknowns k and p. Similarly, when we substitute the coordinates of the second point $(4, 60)$, we get

$$k \cdot 4^p = 60.$$

To eliminate one of the unknowns, we divide the second equation by the first:

$$\frac{k \cdot 4^p}{k \cdot 2^p} = \frac{4^p}{2^p} = \frac{60}{5} = 12.$$

Because

$$\frac{4^p}{2^p} = \left(\frac{4}{2}\right)^p = 2^p,$$

the preceding equation becomes

$$2^p = 12.$$

We solve for p by using logarithms:

$$\log 2^p = p \log 2 = \log 12$$

so that

$$p = \frac{\log 12}{\log 2} \approx 3.584963.$$

The desired power function is therefore $y = k \cdot x^{3.584963}$. To determine the value of k, we use the first point $(2, 5)$ and obtain

$$5 = k \cdot 2^{3.584963}.$$

Therefore

$$k = \frac{5}{2^{3.584963}} \approx 0.41667,$$

and the power function, which is shown in Figure 2.59, is

$$y = 0.41667x^{3.584963}.$$

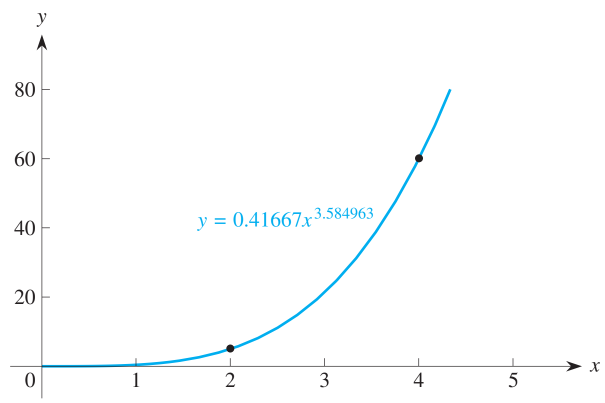


FIGURE 2.59

EXAMPLE 5

Biologists have long observed that the larger the area of a region, the more species inhabit it. The relationship is best modeled by a power function. The island of Puerto Rico contains 40 species of reptiles and amphibians on its 3459 square miles. The nearby island of Hispaniola (comprising Haiti and the Dominican Republic) contains 84 species on 29,418 square miles.

a. Determine a power function that relates the number of species of reptiles and amphibians on a Caribbean island to its area.

b. Use the relationship from part (a) to predict the number of species of reptiles and amphibians on Cuba, which measures 44,218 square miles.

Solution

a. We want a power function of the form $S = kA^p$, where S is the number of species, A is the area in square miles, and k and p are constants that must be determined. Using the information on Puerto Rico, where $S = 40$ and $A = 3459$, we have

$$k \cdot (3459)^p = 40, \qquad \textbf{(1)}$$

which involves both p and k. The data on Hispaniola, $A = 29{,}418$ and $S = 84$, give

$$k \cdot (29{,}418)^p = 84, \qquad \textbf{(2)}$$

so we now have two *nonlinear* equations in the two unknowns k and p. We can eliminate the unknown k by dividing Equation (2) by Equation (1). We then get

$$\frac{k \cdot (29418)^p}{k \cdot (3459)^p} = \frac{84}{40} = 2.1.$$

We cancel the common factor k to get

$$\frac{29418^p}{3459^p} = 2.1.$$

Using one of the properties of exponents, we find that this equation reduces to

$$\left(\frac{29418}{3459}\right)^p = (8.505)^p = 2.1.$$

To solve for p, we take logs of both sides of this exponential equation to get

$$\log(8.505)^p = p \log(8.505) = \log(2.1),$$

from which we find that

$$p = \frac{\log(2.1)}{\log(8.505)} \approx 0.3466.$$

Substituting this value into Equation (1) gives

$$k \cdot (3459)^{0.3466} = 40,$$

so that

$$k = \frac{40}{(3459)^{0.3466}} \approx 2.3739.$$

Thus the power function that models the number of species of reptiles and amphibians on a Caribbean island having area A is

$$S = 2.3739A^{0.3466}.$$

Note from the graph of this function shown in Figure 2.60 that it is an increasing, concave down function, which is what we would expect from a power function with $p = 0.3466$.

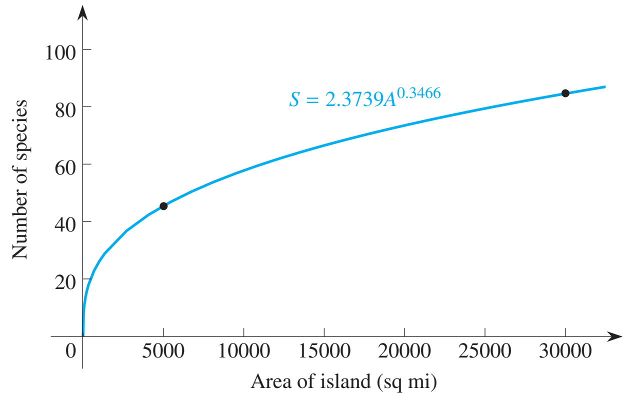


FIGURE 2.60

b. For the area of Cuba, 44,218 square miles, we use this formula to estimate that there are

$$S = 2.3739(44218)^{0.3466} \approx 96.745,$$

or about 97 reptile and amphibian species on Cuba.

Power Functions with Integer Powers

So far, we have restricted our attention to power functions with $x \geq 0$ because most power functions of the form $y = x^{m/n}$ with rational exponents are not defined when $x < 0$. However, if the power is an integer, either positive or negative, the power function $y = x^n$ (n an integer) is well defined for all values of x, both positive and negative. So, when the power is an integer, we extend the domain to include all real x.

Let's look at the case where the power n is a positive integer so that these power functions include $y = x$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, . . . , for all real x. Note from Figure 2.61 that the graphs of these power functions fall into two groups: functions with odd powers and functions with even powers.

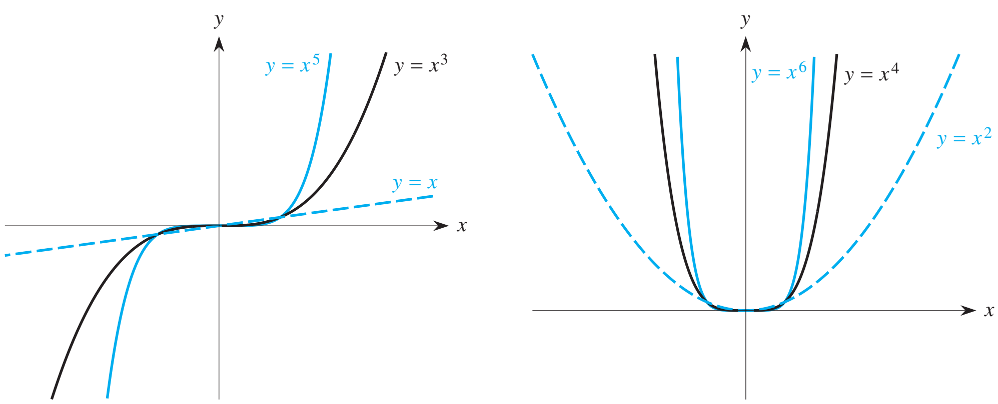


FIGURE 2.61 (a) Odd powers (b) Even powers

Even Positive Integer Powers

We look at the even power functions: $y = x^2$, $y = x^4$, $y = x^6$, . . . , as shown in Figure 2.61(b). Their common characteristics are the following.

- All even power functions decrease at first (until $x = 0$) and then increase as x increases from left to right, so all are U-shaped with a turning point at the origin.
- The higher the power n, the flatter the curve is as it passes through the origin.
- All the even powers are concave up everywhere, so they do not have a point of inflection.
- Not only does each curve pass through the origin and the point (1, 1), but each one also passes through the point $(-1, 1)$ [because $(-1)^n = 1$ if n is an even integer].
- All even power functions are symmetric about the y-axis (the left and right halves of the curves are mirror images.) (See Appendix D.)

Examine some of the even power functions on your own, using your function grapher, to convince yourself that these properties are valid.

Odd Positive Integer Powers

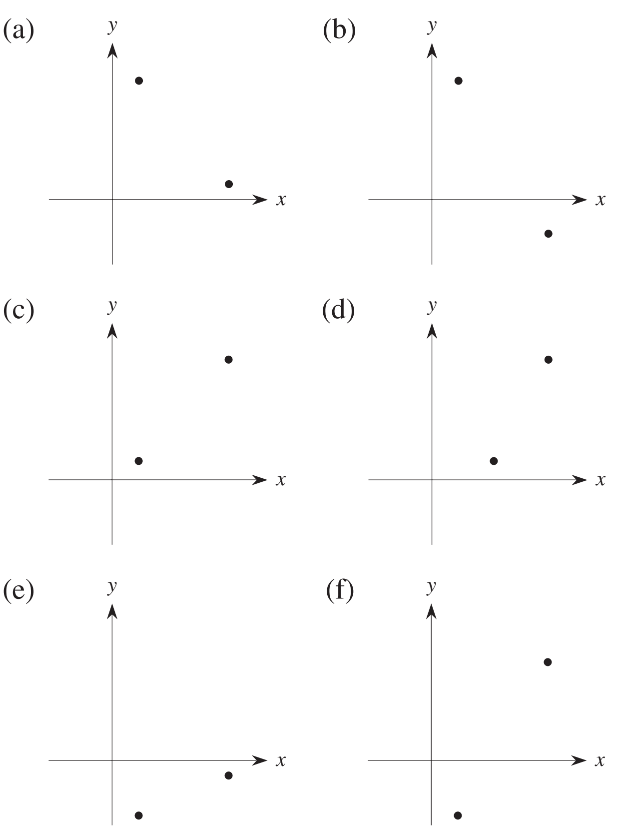
Next, let's examine the power functions with odd powers: $y = x$, $y = x^3$, $y = x^5$, . . . , as shown in Figure 2.61(a). They have the following characteristics in common.

- They all are increasing everywhere as x increases from left to right.
- If $n > 1$, all the odd power functions are concave down when $x < 0$ and concave up when $x > 0$. This change in concavity at $x = 0$ means that every odd power function except the line $y = x^1 = x$ has a point of inflection at the origin, where it is growing most slowly.
- The higher the power n, the flatter the curve is as it passes through the origin.
- Not only does each curve pass through the origin and the point $(1, 1)$, but each one also passes through the point $(-1, -1)$ [because $(-1)^n = -1$ if n is an odd integer].
- All odd power functions are symmetric about the origin; that is, the portion of each curve in the third quadrant is the mirror image of the corresponding portion in the first quadrant.

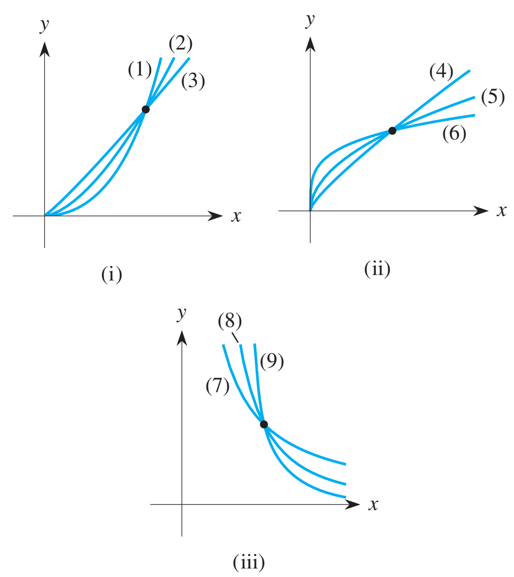
Examine some of the odd power functions on your own, using your function grapher, to convince yourself that these properties are valid.

Problems

1. Which of the pairs of points shown can determine a power function of the form $y = kx^p$ and which cannot. For those that do, sketch the graph of the power function, indicate the sign of the coefficient k, and tell whether the power p is less than 0, between 0 and 1, or greater than 1.



2. Match each formula for a power function with one of the graphs. Explain the reasons for your decisions.



a. $y = x^{-0.8}$ **b.** $y = x^{-0.2}$ **c.** $y = x^{-1.4}$
d. $y = x^{1.8}$ **e.** $y = x^{1.6}$ **f.** $y = x^{1.4}$
g. $y = x^{0.8}$ **h.** $y = x^{0.6}$ **i.** $y = x^{0.4}$

3. Identify which of the functions in parts (a)–(n) are exponential functions, which are power functions, and which are neither. Give the reasons for your decisions.

a. $f(x) = 40x^{1.05}$ **b.** $f(x) = 40(1.05)^x$
c. $f(x) = \dfrac{1}{(1.4)^x}$ **d.** $f(x) = -\dfrac{3}{x^{2.4}}$
e. $f(t) = 5t^{-3.7}$ **f.** $f(q) = 1.09q - 4.37$

g. $f(t) = 12(0.35)^{-t}$ **h.** $f(t) = 5\sqrt{t}$
i. $f(s) = \sqrt{s^2 + 3}$ **j.** $f(r) = \frac{4}{3}\pi r^3$
k. $f(z) = z \cdot z^{3/5}$ **l.** $f(x) = x^x$
m. $f(w) = w^2 \cdot 3^w$ **n.** $f(u) = 7(1.62)^{-u}$

4. Match each formula with its corresponding table of values.

a. $y = 4x^{1.2}$ **b.** $y = 5x^{0.8}$ **c.** $y = 4(1.2)^x$.

i.

x	2	3	4	5	6
$f(x)$	8.71	12.04	15.16	18.12	20.96

ii.

x	1	2	3	4	5
$g(x)$	4.80	5.76	6.91	8.29	9.95

iii.

x	2	4	6	8	10
$h(x)$	9.19	21.11	34.34	48.50	63.40

5. Data from three different functions are shown in the tables of values. One function is exponential, one has the form $y = ax^2$, and one has the form $y = bx^3$. Which function is which?

i.

x	3	3.5	4	4.5	5
$f(x)$	28.8	39.2	51.2	64.8	80.0

ii.

x	3	3.5	4	4.5	5
$g(x)$	4.39	5.01	5.71	6.51	7.42

iii.

x	3	3.5	4	4.5	5
$h(x)$	10.80	17.15	25.60	36.45	50.00

6. For each relationship, **(i)** identify which of the quantities should be considered the independent variable and which the dependent variable; **(ii)** write an equation expressing the dependent variable in terms of the independent variable to create a power function that represents the relationship; and **(iii)** sketch a rough graph of the function based on the value of the power p.

a. If a car is traveling at a constant rate, the distance d that it travels is proportional to the time t that it travels.

b. The distance d that an object falls under the influence of gravity is proportional to the square of the time t that it is falling.

c. According to the ideal gas law, when a gas is kept at a constant temperature, the pressure P is inversely proportional to the volume V.

d. The force F of gravity between two objects is inversely proportional to the square of the distance d between them.

e. The square of the diameter d of the long bone in the leg of many animals is proportional to the cube of the length L of the bone.

f. The cube of the surface area S of many vertebrate mammals is proportional to the square of their body mass m.

g. The fourth power of the rate R at which air flows into and out of the lungs of many vertebrate mammals is proportional to the cube of their body mass m.

h. The fourth power of the speed s at which many mammals can trot is proportional to their body mass m.

i. The cube of the speed s at which most birds fly is proportional to the square of their body mass m.

j. The square of the swimming speed s of most species of fish is proportional to the length L of their bodies.

k. The fifth power of the radius r of the shock wave after the explosion of a nuclear bomb is proportional to the square of the time t since the bomb exploded.

7. Use the formula relating the weight of a large flying bird to its wingspan to explain why a 15 pound turkey with a wingspan of about 2.5 feet can't soar like an eagle.

8. A full grown African vulture has a 9-foot wingspan. Based on the model relating weight to wingspan, how much does a vulture weigh?

9. The largest known flying creature, with a wingspan of 40 feet, was the pterosaur that lived 65 million years ago. Assuming that the formula for birds also applies to flying dinosaurs, estimate the weight of an adult pterosaur. What can you conclude from your answer?

10. Find a formula expressing the volume V of a sphere as a function of its surface area S.

11. Find the power function that passes through the following pairs of points.

a. (1, 3) and (4, 6) **b.** (1, 3) and (4, 8)
c. (1, 3) and (4, 10) **d.** (5, 20) and (6, 30)
e. (1, 10) and (4, 5) **f.** (2, 20) and (5, 8)

12. Police sometimes use the formula $s = \sqrt{30kd}$ to estimate the speed s in miles per hour that a car was

going if it left a set of skid marks d feet long. The coefficient k depends on the road conditions (dry or wet) and the type of pavement. For instance, $k = 0.8$ for dry concrete, $k = 0.4$ for wet concrete, $k = 1.0$ for dry tar, and $k = 0.5$ for wet tar.

a. A car left a set of skid marks 120 feet long on dry concrete. How fast was it going?
b. Suppose that the concrete pavement in part (a) was wet. How fast was the car going?
c. If the car in part (a) left skid marks 240 feet long, how fast was it going?
d. Suppose that a car is going 50 mph on a dry tar surface when the driver slams on the brakes. How far will it skid?
e. Suppose that the tar pavement in part (d) was wet. How far will the car skid?

13. Scientists are actively investigating the potential of using windmills to generate electricity. They have found that, for moderate wind speeds, the power P in watts generated by a windmill is related to the wind speed v in miles per hour according to the equation

$$P = 0.015v^3.$$

a. How much power is generated by a steady wind at 10 mph?
b. How much power is generated by a steady wind at 20 mph?
c. Based on your results in parts (a) and (b), by what factor does doubling the wind speed increase the power generated?
d. Compare the power generated by a steady wind at 5 mph to that of a steady wind at 10 mph. Does doubling of the wind speed increase the power generated by the same factor found in part (c)?
e. Suppose that a certain community has power needs for an additional 250 kilowatts of electricity and can anticipate winds on the average of 12 mph. How many windmills would be needed to meet the added electric demand?
f. What wind speed would be needed to light a 100-watt light bulb?

14. a. Use your function grapher to plot on the same screen the graphs of the power functions $y = x^2$, x^5, and x^8 for the interval $-0.2 \le x \le 0.2$. Determine an appropriate range for y so that all powers will be distinguishable in the viewing rectangle.
b. Plot the same graphs for $-2 \le x \le 2$ and determine an appropriate range for y.
c. Plot the same graphs for $-20 \le x \le 20$ and determine an appropriate range for y.

15. a. Use your function grapher to plot on the same screen the graphs of the power functions $y = x^{1/2}$, $x^{1/3}$, and $x^{1/4}$ for the interval $0 \le x \le 0.2$. Determine an appropriate range for y so that all powers will be distinguishable in the viewing rectangle.
b. Plot the same graphs for $0 \le x \le 2$ and determine an appropriate range for y.
c. Plot the same graphs for $0 \le x \le 20$ and determine an appropriate range for y.
d. What happens if you use the interval $-2 \le x \le 2$?

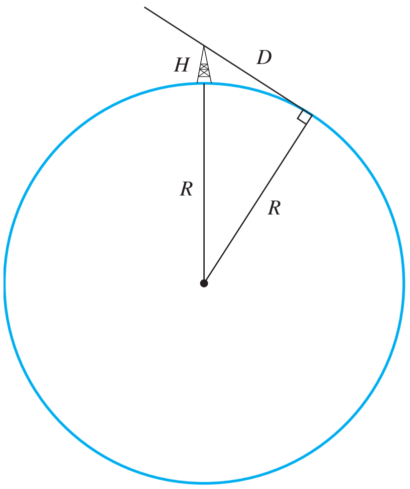
16. What happens to

a. x^3 as $x \to \infty$? as $x \to -\infty$?
b. $-x^3$ as $x \to \infty$? as $x \to -\infty$?
c. $x^{1/3}$ as $x \to \infty$? as $x \to -\infty$?
d. $-x^{1/3}$ as $x \to \infty$? as $x \to -\infty$?
e. x^{-3} as $x \to \infty$? as $x \to -\infty$?
f. x^{-3} as $x \to 0$?

17. In 1990, 442.2 million prerecorded audio cassette tapes were sold, and 865.7 million CDs were sold in the United States. In 1998, 158.5 million cassette tapes were sold, and 1,124.3 million CDs were sold. Assume for now that the patterns of sales for both items are power functions.

a. Find the equation for the number of cassette tapes sold as a power function of time.
b. Find the equation for the number of CDs sold as a power function of time.
c. If the trends in sales of both items were indeed power functions, find when the number of CDs sold overtook the number of cassette tapes sold.

18. In the accompanying figure let R be the radius of the Earth (about 3960 miles). Find an expression for the distance D to the horizon from a point at a height of H miles above the Earth's surface. (*Hint:*



Recall that, for a circle, any tangent line is perpendicular to a radius.)

19. The observation deck of the Empire State Building in New York is 1250 feet high. If you're standing there, complete the phrase: "On a clear day, you can see. . . ." (*Hint:* Use the formula you created in Problem 18.)

20. Ultra high frequency (UHF) TV transmissions travel along a line of sight from a transmitter as far as the horizon. In the Chicago area, the UHF stations broadcast from a transmitter atop the 1454-foot $(= 0.275 \text{ mile} \approx \frac{1}{4} \text{ mile})$ high Sears tower. What is the greatest distance that someone could receive a UHF signal from the tower?

21. Suppose that a mast 250 feet (about $\frac{1}{20}$ mile) high is being planned for the Sears tower to extend the broadcast range of UHF stations. How much farther would the signal extend? How much larger a receiving area would be covered?

22. NASA's space shuttles orbit the Earth at altitudes of about 200 miles. Find the maximum line-of-sight transmission distance from the shuttle to the surface of the Earth. Approximately how large a receiving area on the Earth is in range of this shuttle?

23. Communications satellites orbit the Earth in geosynchronous orbits (carefully chosen heights and velocities so that they appear to be permanently above a fixed point on the surface of the Earth as the Earth rotates). Suppose that such a satellite is in orbit at a height of 23,000 miles above a point on the equator. The radius of the Earth is about $R = 3960$ miles, so the distance around the equator is approximately $2\pi R = 24{,}880$ miles. Consequently, a point on the equator is rotating at a velocity of about 1037 mph. Find the orbital velocity of such a communication satellite in a geosynchronous orbit.

24. Explain why it isn't possible to have a communications satellite whose signals cover a full half the Earth's surface.

25. Using $R = 3960$ miles for the radius of the Earth, the formula you found in Problem 18 for the line-of-sight distance to the horizon from a height of H miles is $D(H) = \sqrt{H^2 + 2RH} = \sqrt{H^2 + 7920H}$. When H is small, the term H^2 seemingly has little effect on the value of D, so you might be tempted to approximate the distance D using the simpler formula $D \approx \sqrt{7920H} \approx 89\sqrt{H}$. Determine whether using this approximation is reasonable by completing the following table comparing the estimated value for this distance with the actual value.

H	$D \approx 89\sqrt{H}$	$D = \sqrt{H^2 + 7920H}$
0.1 mile		
1 mile		
10 miles		
100 miles		

26. **a.** Find, correct to three decimal places, *all* values of x for which $x^4 = 3^x$ by graphing the two functions $y = x^4$ and $y = 3^x$. (*Hint:* Use different windows to convince yourself that you have located all points of intersection of the two curves.)
    b. Repeat part (a) by creating the function $y = x^4 - 3^x$ and looking for all the points where $y = 0$.

27. Consider the function $f(x) = x^2$ and let P be the point on the curve where $x = 0$, R be the point where $x = 2$, and Q be the midpoint where $x = 1$. Find the slopes of the three line segments PQ, QR, and PR. How does the slope of PR compare to the slopes of the other two segments?

28. Repeat Problem 27, using the function $g(x) = x^3$. Does the relationship among the three slopes you found in Problem 27 also hold for g?

29. Consider the function $f(x) = x^2$ and let P be the point where $x = a$, Q be the point where $x = a + h$, and R be the point where $x = a + 2h$, for any quantity $h > 0$. Find the slopes of the three line segments PQ, QR, and PR. Show that the slope of PR is the average of the other two slopes.

30. Consider the sequence of values 10^5, 10^4, 10^3, 10^2, and 10^1 and use it to provide a reason for defining $10^0 = 1$. What about 10^{-1}?

31. By trial and error, determine the largest power of 10 that your calculator can handle. What is the smallest positive number?

Exercising Your Algebra Skills

Use the properties of exponents to evaluate each term (do not use a calculator).

1. $9^{1/2}$
2. $9^{-1/2}$
3. $8^{4/3}$
4. $8^{-4/3}$

Simplify.

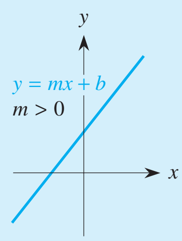
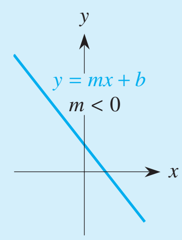
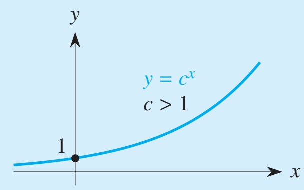
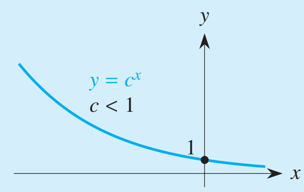
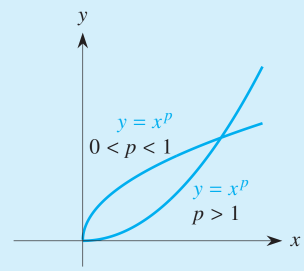
5. $x^4 \cdot x^3$
6. $x^6 \cdot x^{-8}$

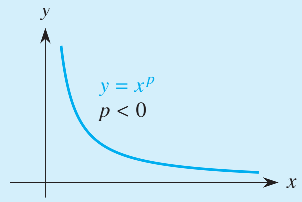
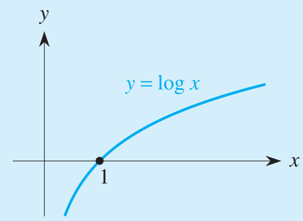
7. $\dfrac{r^8}{r^4}$ **8.** $\dfrac{z^{12}}{z^{-9}}$ **11.** $\dfrac{x^{3/4}}{x^{5/4}}$ **12.** $\dfrac{a^{-2/3}}{a^{5/3}}$

9. $x^{3/4} \cdot x^{5/4}$ **10.** $a^{-2/3} \cdot a^{5/3}$ **13.** $\dfrac{a^{5/3}}{a^{-2/3}}$

2.8 Comparing Rates of Growth and Decay

Most of the families of functions that we've discussed in this chapter—linear, exponential, and logarithmic—are either strictly increasing or strictly decreasing. If we restrict our attention to nonnegative values of x, power functions also are either strictly increasing or strictly decreasing. Let's summarize what we know so far.

Function	Equation	Behavior	Graph
Linear	$y = mx + b$	Strictly increasing when $m > 0$. The more positive the slope, the faster the rate of increase. Strictly decreasing when $m < 0$. The more negative the slope, the greater the rate of decrease.	
Exponential	$y = c^x$ (for $c > 0$)	Strictly increasing when the growth factor $c > 1$. The larger c is, the faster the function grows. Strictly decreasing when the decay factor $0 < c < 1$. The smaller c is, the faster the function decays toward zero. Exponential graphs are always concave up.	
Power	$y = x^p$	Strictly increasing when $p > 0$. The larger p is, the faster the function grows beyond $x = 1$. If $p > 1$, the graph is concave up—it grows more and more rapidly. If $0 < p < 1$, the graph is concave down—it grows more and more slowly.	

		Strictly decreasing for $x > 0$ when $p < 0$. The more negative p is, the faster the function decays toward zero. If $p < 0$, the graph is concave up, for $x > 0$.	
Logarithmic	$y = \log x$ $x > 0$	Strictly increasing. Logarithmic graphs are always concave down.	

This summary of information compares the growth or decay rate of one function in a family to that of other functions in the same family. In this section, we look at two other ways to compare rates of growth. At a local level, we look at how fast a single function is growing or decaying at different points. At a global level, we look at how quickly functions in one family grow or decay compared to how quickly functions in a different family grow or decay. In particular, we want to answer two questions: (1) which family of functions grows fastest? and (2) which family of functions decays to zero fastest?

Exponential Versus Linear Functions: Which Grow Faster?

We saw in Section 2.4 that any exponential growth function $y = kc^x$ with $c > 1$ will eventually grow faster than any linear function with positive slope. The reason is that the multiplicative effect of the growth factor c in an exponential function is greater than the additive effect of the slope in a linear function. Similarly, any exponential decay function will eventually decrease more slowly than any linear function with a negative slope.

Exponential Versus Power Functions: Which Grows Faster?

Power functions of the form $y = x^p$, $p > 1$ and exponential growth functions of the form $y = c^x$, $c > 1$ both grow rapidly as x increases. But, do they grow at roughly the same rate, or does one grow much faster than the other?

Consider $y = 2^x$ and $y = x^4$ for $x \geq 0$. We know that every power function of the form $y = x^p$ with $p > 0$ passes through the origin and that every exponential curve of the form $y = c^x$ crosses the vertical axis at $y = 1$. So let's begin by comparing these two functions for small values of x. The local, or close-up, view, in Figure 2.62 shows that, between $x = 0$ and $x = 1$, the graph of $y = 2^x$ is above the graph of $y = x^4$ but that the power function seems to be growing more rapidly. If we extend the interval somewhat, we find that by $x = 2$ the power function has surpassed the exponential function. (Where does that happen?) On a somewhat larger scale, Figure 2.63 reveals that the power function continues to pull away from the exponential function.

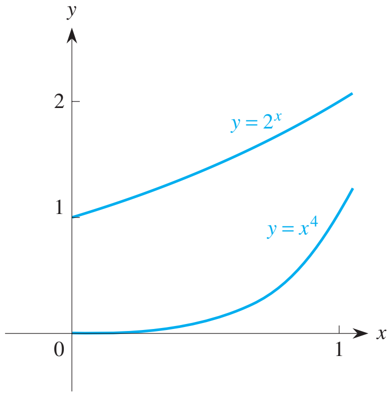


FIGURE 2.62

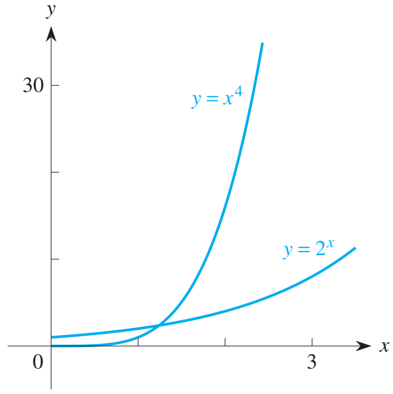


FIGURE 2.63

However, in Figure 2.64, which shows the interval from $x = 0$ to $x = 20$, the exponential curve has again overtaken the power curve. (Where does that happen?) The still larger view from $x = 0$ to $x = 25$ in Figure 2.65 shows that, for large x-values, $y = x^4$ is insignificant compared to $y = 2^x$. In fact, $y = 2^x$ is growing so much faster than $y = x^4$ that its graph appears almost vertical in comparison to the relatively slow growth of $y = x^4$. Verify this comparison numerically by trying several different values of x—say, $x = 1$, $x = 2$, $x = 10$, and $x = 50$ (but don't go too far because you may exceed your calculator's capacity).

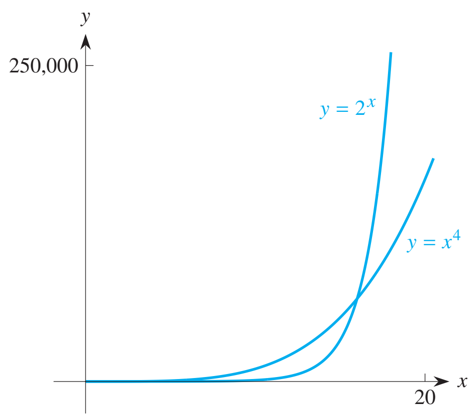


FIGURE 2.64

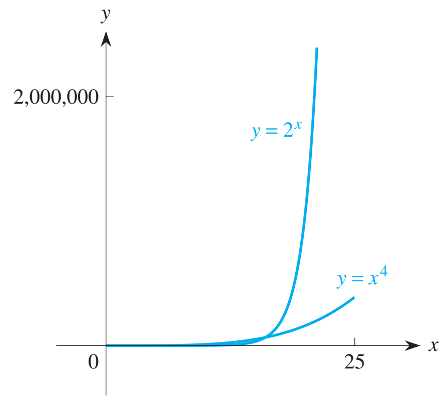


FIGURE 2.65

Think About This

Use your function grapher to find, correct to two decimal places, all the points where $y = 2^x$ and $y = x^4$ intersect.

This behavior pattern is typical of any power function $y = x^p$, with $p > 1$ compared to any exponential function $y = c^x$, for $c > 1$. Although the exponential function starts more slowly than the power function for small values of x, the exponential function eventually dominates $y = x^p$ for any value of $p > 1$, and so it always wins the race toward infinity.

Think About This

Plot $y = 3^x$ and $y = x^5$ for $0 \le x \le 20$ with $0 \le y \le 300{,}000$ to see where the exponential function overtakes the power function.

You have already seen that a positive constant multiple doesn't change the overall shape or behavior of a function. For instance, the power function $y = 5000x^4$ has the same shape as the power function $y = x^4$, but it grows more rapidly because the first is 5000 times larger for any x value. We've already shown that the exponential function $y = 2^x$ eventually overtakes the power function $y = x^4$. It also eventually overtakes the power function $y = 5000x^4$; it just takes longer. The only question is: Where does that happen? In the long run, the exponential function invariably wins the race to infinity.

EXAMPLE 1

Estimate the point x where $f(x) = 1.05^x$ finally overtakes $g(x) = x^{10}$.

Solution We know that the power function $g(x) = x^{10}$ grows very rapidly and that the exponential function $f(x) = 1.05^x$ has a fairly small growth factor of 1.05. Let's look at their respective function values numerically for different values of x.

x	$f(x) = (1.05)^x$	$g(x) = x^{10}$
10	1.62889	10^{10}
100	131.501	$100^{10} = (10^2)^{10} = 10^{20}$
1000	1.5463×10^{21}	$1000^{10} = (10^3)^{10} = 10^{30}$
10,000	7.816×10^{211}	$10{,}000^{10} = (10^4)^{10} = 10^{40}$

From this comparison, it is evident that the exponential function has overtaken the power function sometime after $x = 1000$ but long before $x = 10{,}000$, where $f(x) = (1.05)^x$ has far exceeded the value of $g(x) = x^{10}$. Suppose we narrow our search by trying a few additional values of x.

x	$f(x) = (1.05)^x$	$g(x) = x^{10}$
2000	2.3911×10^{42}	1.024×10^{33}
1500	6.0806×10^{31}	5.7665×10^{31}
1499	5.7911×10^{31}	5.7282×10^{31}
1498	5.5153×10^{31}	5.6901×10^{31}

We conclude that the exponential function $f(x) = (1.05)^x$ finally overtakes the power function $g(x) = x^{10}$ just before $x = 1499$, as illustrated in Figure 2.66. If we zoom in, either numerically on the table or geometrically on the graph, we find that the point of intersection of the two functions occurs near $x = 1498.718$ and $y = 5.71169 \times 10^{31}$.

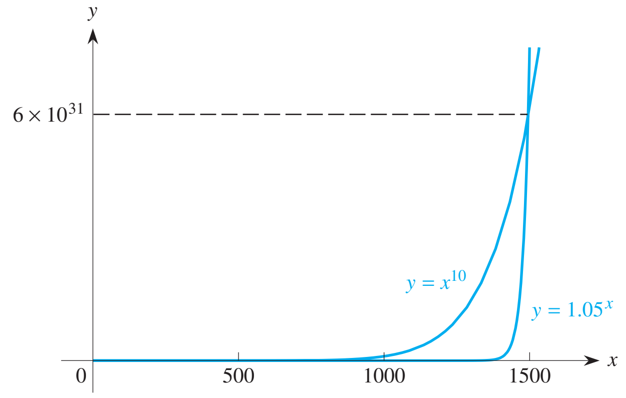


FIGURE 2.66

Now let's compare a decaying exponential function and a power function with a negative exponent. Consider $y = (\frac{1}{2})^x = (0.5)^x$ and $y = x^{-2} = 1/x^2$. Both graphs eventually approach the x-axis as a horizontal asymptote, but which one approaches the x-axis faster? We could compare them graphically or numerically by trying several large values of x, such as $x = 100$ or $x = 1000$. Alternatively, we can reason that, because 2^x is eventually larger than x^2, we know that $(0.5)^x = (\frac{1}{2})^x = 1/2^x$ is eventually smaller than $x^{-2} = 1/x^2$. So the graph of $y = (\frac{1}{2})^x$ eventually falls below the graph of $y = x^{-2}$, as shown in Figure 2.67.

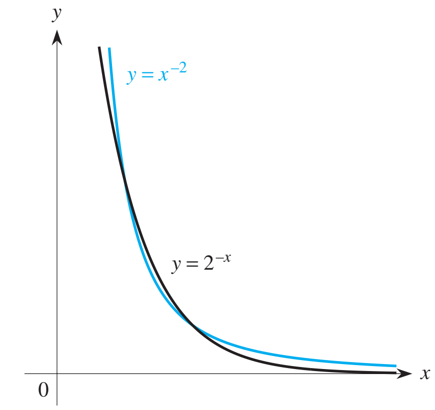


FIGURE 2.67

This behavior pattern is typical. All decaying exponential functions invariably approach 0 faster than any power function with a negative exponent as $x \to \infty$. A power function could begin dropping at a faster rate when compared to an exponential function; for instance, compare $y = x^{-100}$ with $y = 0.9^x$. However, as $x \to \infty$, the exponential function eventually decays faster than the power function to win the race toward 0. The only question is: When does the decaying exponential function overtake the decaying power function on the way to zero? As we have demonstrated, this point of intersection can be approximated with any desired degree of accuracy by using either numerical or graphical methods.

Logarithmic Functions Versus Power Functions: Which Grows Faster?

We know that a logarithmic function and a power function with power $0 < p < 1$ both increase and are concave down. A table of values for $f(x) = \log x$ reveals that the logarithm grows very slowly as x increases beyond $x = 1$. Figure 2.68 shows the

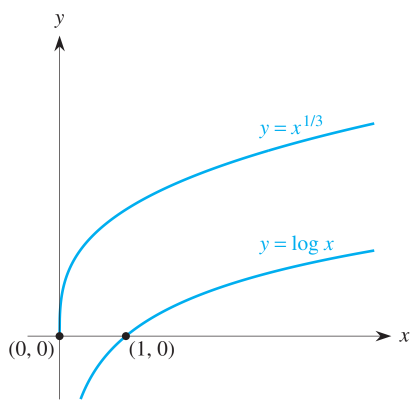


FIGURE 2.68

graphs of $y = \log x$ and $y = x^{1/3}$. In fact, $\log x$ grows more slowly than any positive power of x. In the long run, any power function with $0 < p < 1$ beats the logarithmic function in the race toward infinity.

Think About This

Compare the behavior of different power functions (for example, $y = x^{1/2}$ or $y = x^{1/10}$) to $y = \log x$ on your function grapher to convince yourself of how slowly the log function grows.

Furthermore, both logarithmic functions and power functions with power $0 < p < 1$ grow more slowly than linear functions with slope $m > 0$. The reason is that the linear function $y = x$ is a power function with $p = 1$, which grows faster than any power function with a smaller power p.

In summary, we have the following facts:

Comparisons When x is Large

Concave Up Growth Functions

Power functions with power $p > 1$ grow faster than linear functions with slope $m > 0$.

Exponential functions with growth factor $c > 1$ grow faster than power functions with power $p > 1$.

Concave Down Growth Functions

Power functions with power $0 < p < 1$ grow more slowly than linear functions with slope $m > 0$.

Logarithmic functions grow more slowly than power functions with power $0 < p < 1$.

Decay Functions

Exponential functions with decay factor $0 < c < 1$ decay more rapidly than power functions with power $p < 0$.

Average Rate of Growth

We have often described an increasing, concave up function as "increasing faster and faster" or "increasing at an increasing rate," although we never precisely defined what this means. We now formalize this concept by building on what we know about lines. One of the main characteristics of a linear function is that it grows (or decays) at a constant rate. That is, for each fixed increase (say, Δx) in the independent variable x, the line rises (or falls) the same amount Δy no matter what point on the line we use, as shown in Figure 2.69. The constant ratio $\Delta y/\Delta x$ is the slope of the line.

Now let's try this with an exponential growth function $y = f(x) = kc^x$. At different points on the curve, move the same horizontal distance Δx to the right and determine the corresponding vertical change Δy, as shown in Figure 2.70. At point P, there is a relatively small change in y; at point Q, the corresponding change in y is somewhat larger; and by the time we get to point S, the corresponding change in y is considerably larger.

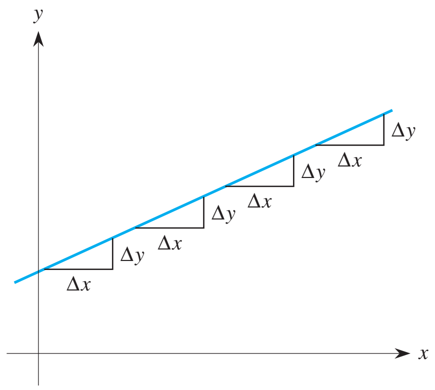


FIGURE 2.69

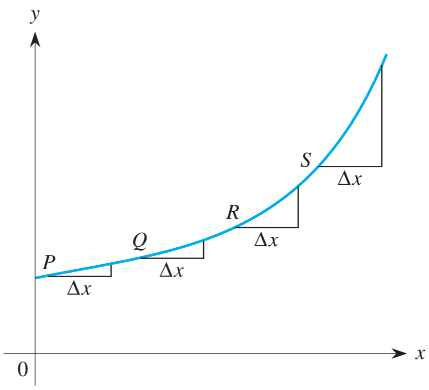


FIGURE 2.70

For instance, suppose that $y = f(x) = 1.2^x$ and that we take steps of size $\Delta x = 0.5$. We start at $x = 0$, where $y = 1.2^0 = 1$. When we move $\Delta x = 0.5$ to the right, we get to $x = 0.5$ at a height of $1.2^{0.5} = 1.095$. The change in height is $\Delta y = 1.095 - 1 = 0.095$, as shown in Figure 2.71.

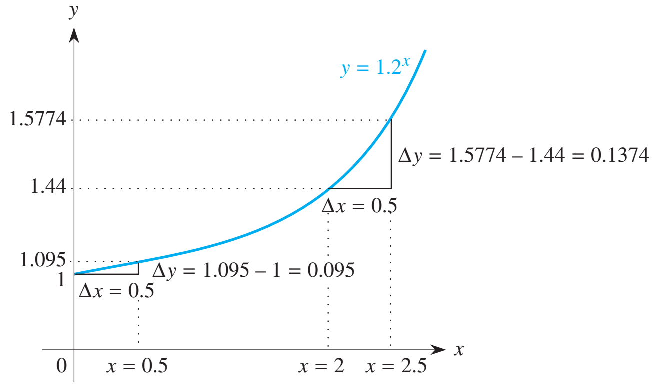


FIGURE 2.71

We now repeat this process, starting at $x = 2$ where $y = f(2) = 1.2^2 = 1.44$. We again move $\Delta x = 0.5$ to the right, getting to the point $x = 2.5$ and $y = f(2.5) = 1.2^{2.5} = 1.5774$. The corresponding change in height is now $\Delta y = 1.5774 - 1.44 = 0.1374$, as shown in Figure 2.71. Thus the same size step to the right has resulted in a considerably larger increase in the change in height of the exponential growth function.

Think About This

Repeat the preceding argument by starting from the point $x = 3$ and moving to the right by $\Delta x = 0.5$. How does the change in y compare to the values of Δy that we found? □

To measure how rapidly this, or any other function $y = f(x)$ is increasing (or decreasing), we consider the ratio $\Delta y/\Delta x$, called the *average rate of change* of the function over an interval, which we discussed briefly in Section 1.1. We have

$$\text{Average rate of change of } f \text{ from } x_1 \text{ to } x_2 = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

In the special case where the function is linear, this ratio is simply the slope of the line.

EXAMPLE 1

Find the average rate of change of the linear function $f(x) = 3x + 5$ between $x = 1$ and $x = 9$.

Solution The average rate of change is

$$\frac{f(9)-f(1)}{\Delta x} = \frac{(3\cdot 9+5)-(3\cdot 1+5)}{9-1} = \frac{32-8}{8} = \frac{24}{8} = 3,$$

which is the slope of the line.

EXAMPLE 2

Find the average rate of change of the exponential function $f(x) = 1.2^x$ **(a)** between $x = 0$ and $x = 0.5$ and **(b)** between $x = 2$ and $x = 2.5$.

Solution

a. The average rate of change of $f(x) = 1.2^x$ between $x = 0$ and $x = 0.5$ is

$$\frac{f(0.5)-f(0)}{\Delta x} = \frac{1.095-1}{0.5} = 0.19.$$

b. The average rate of change of $f(x) = 1.2^x$ between $x = 2$ and $x = 2.5$ is

$$\frac{f(2.5)-f(2)}{\Delta x} = \frac{1.5774-1.44}{0.5} = \frac{0.13744}{0.5} = 0.27488.$$

Note how the average rate of change between $x = 2$ and 2.5 is considerably larger than that between $x = 0$ and 0.5. The function is growing ever faster as we move to the right.

Instead of thinking of the average rate of change of a function $y = f(x)$ from x_1 to x_2, we can also think of it as the average rate of change from any point $x = x_0$ to $x = x_0 + \Delta x$. We then have

$$\text{Average rate of change} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x},$$

as illustrated in Figure 2.72. Geometrically, the average rate of change is the slope of the line segment through the two points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$. The more positive this slope is, the faster the curve is increasing. Thus, for an increasing, concave up function, the average rate of change increases as we move from left to right.

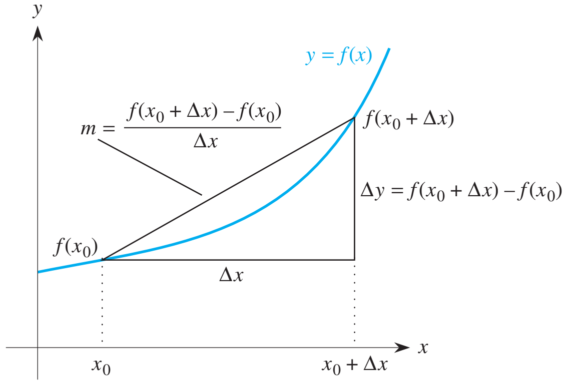


FIGURE 2.72

If $f(x)$ is a decreasing function, the average rate of change will be negative, just as the slope is negative for a decreasing linear function.

Think About This What is the average rate of change of $f(x) = x^{-1} = 1/x$ from $x = 1$ to $x = 1.02$?

EXAMPLE 3

Find the average rate of change of $f(x) = \log x$ between $x = 2$ and $x = 2.4$.

Solution In going from $x = 2$ to $x = 2.4$, we have a step of $\Delta x = 0.4$, as depicted in Figure 2.73. Therefore the average rate of change between $x = 2$ and $x = 2.4$ is

$$\frac{f(2.4) - f(2)}{\Delta x} = \frac{\log 2.4 - \log 2}{0.4} \approx 0.19795.$$

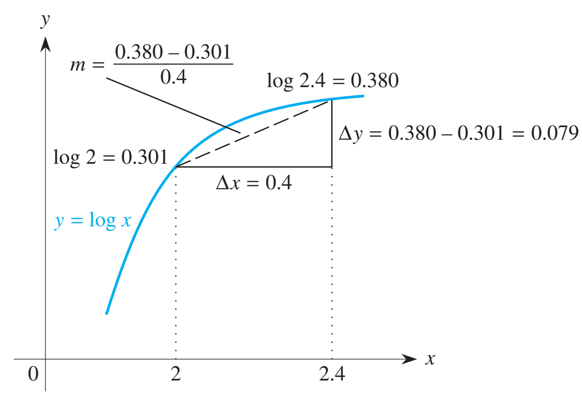
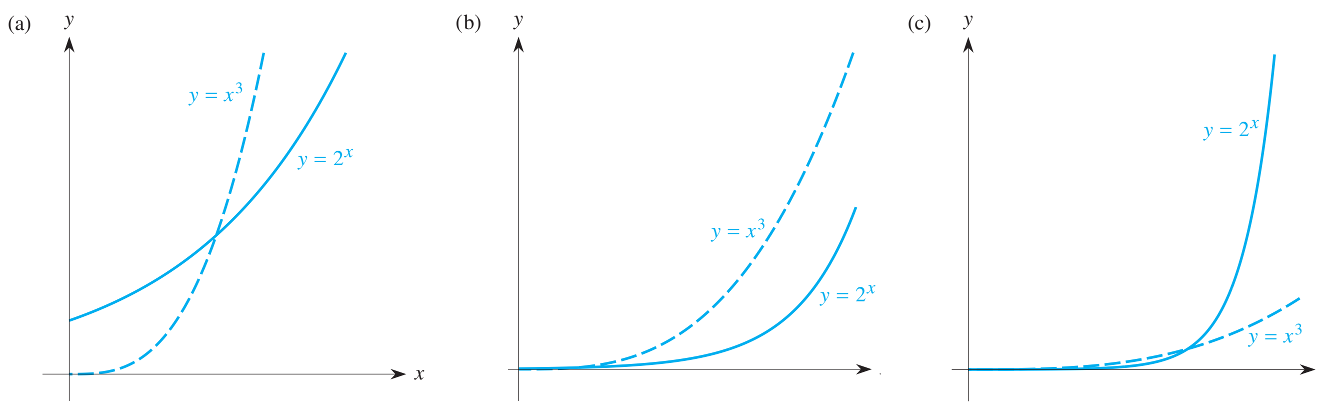


FIGURE 2.73

Problems

1. Use your function grapher to graph $y = x^3$ and $y = 2^x$. Determine appropriate x- and y-scales to obtain the diagrams shown below.
2. **a.** For what values of x is $2^x > x^2$?
   b. For what values of x is $3^x > x^3$?
   c. For what values of x is $4^x > x^4$?
3. Use your function grapher to estimate when $y = (0.6)^x$ overtakes $y = x^{-6}$ as they both decay to zero.
4. Estimate where $f(x) = x^{0.15}$ finally overtakes $g(x) = \log x$ on their "turtle versus snail" race toward infinity.
5. For each linear function, find the average rate of change on the indicated intervals.
   a. $f(x) = 4x - 9$ between $x = 2$ and $x = 5$
   b. $f(x) = 4x - 9$ between $x = -2$ and $x = 3$
   c. $f(x) = -3x + 4$ between $x = 2$ and $x = 5$
   d. $f(x) = -3x + 4$ between $x = -2$ and $x = 3$
   e. $4x - 3y = 12$ between $x = 2$ and $x = 5$
6. Prove that the average rate of change for any linear function $f(x) = mx + b$ on any interval from x_1 to x_2 is equal to the slope m.
7. Find the average rate of change of $f(x) = x^2$ **(a)** between $x = 0$ and $x = 1$, **(b)** between $x = 0$ and $x = 2$, and **(c)** between $x = 1$ and $x = 2$. Put them in ascending numerical order.

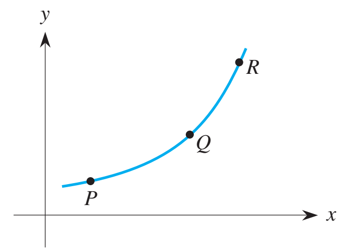


8. Consider the function $f(x) = x^2$.
   a. Find the average rate of change of f between $x = 0$ and 0.01.
   b. Find the average rate of change of f between $x = 1$ and 1.01.
   c. Find the average rate of change of f between $x = 2$ and 2.01.
   d. Based on your results from parts (a)–(c), can you predict the average rate of change of f between $x = 3$ and 3.01? between $x = 4$ and 4.01?
   e. What happens to your answers in parts (a)–(c) if $\Delta x = 0.001$ instead of 0.01?

9. a. In parts (b), (c), and (d), you are asked to find the average rate of change of $f(x) = \sqrt{x}$ between $x = 0$ and 1, between 1 and 2, and between 0 and 2. Before calculating these values, predict the numerical order, from smallest to largest, of these three quantities.
   b. Find the average rate of change of f between $x = 0$ and 1.
   c. Find the average rate of change of f between $x = 1$ and 2.
   d. Find the average rate of change of f between $x = 0$ and 2.

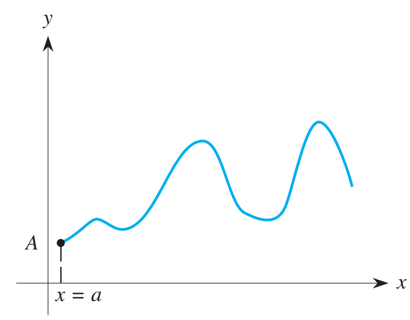
10. The functions $y = x^2$ and $y = 2^x$ intersect at $x = 2$ and at $x = 4$; the functions $y = x^3$ and $y = 3^x$ intersect at $x = 3$ and at $x \approx 2.478$. In general, for $x \geq 0$ the graphs of $f(x) = x^p$ and $g(x) = p^x$ intersect at two points except for one specific value of p (with $p > 1$) for which the curves intersect at only one point. Use your function grapher and trial and error to locate the one special value of p (accurate to two decimal places) for which the curves $y = x^p$ and $y = p^x$ intersect at only one point.

11. a. Use the three points P, Q, and R shown on the accompanying graph of $y = f(x)$ to determine the three line segments PQ, QR, and PR. List these line segments in the order of *increasing* slope (smallest to largest).



   b. Repeat part (a) if the function is increasing and concave down instead.

12. Consider the function f shown and the point A at $x = a$ on the curve. Determine the point:



   a. B at $x = b$ giving the interval from a to b over which the change in f is least.
   b. C at $x = c$ giving the interval from a to c over which the change in f is greatest.
   c. D at $x = d$ giving the interval from a to d over which the average rate of change in f is least.
   d. E at $x = e$ giving the interval from a to d over which the average rate of change in f is greatest.

2.9 Inverse Functions

Countries using the metric system report temperatures in degrees Celsius. Thus an American visiting Canada who wants to know the temperature in degrees Fahrenheit must be able to convert the Celsius readings to Fahrenheit readings by using the formula

$$F = \frac{9}{5}C + 32.$$

In this relationship the Fahrenheit measurement is a function of the Celsius measurement. Canadian visitors to the United States face the reverse problem: They must convert Fahrenheit readings to Celsius readings. They can do so by solving the preceding formula algebraically for C as a function of F, getting the related function

$$C = \frac{5}{9}(F - 32).$$

These two functions have the effect of undoing each other. For that reason, they are called **inverse functions.**

In general, we write the inverse of a function f as f^{-1} and read it as "f inverse." The inverse of a function f is a function that reverses or undoes f. The two functions relating F and C, which give temperature conversions between the two systems of measurements, represent a pair of inverse functions.

Suppose that we have a function $y = f(x)$ that represents some quantity or process of interest to us. Typically, we can ask two types of predictive questions. The first question is: Determine the value of y corresponding to a particular value of x. All we need do is substitute the value of x in the expression for the function. The second question is: Determine when the quantity achieves a particular level—that is, find the value of the independent variable x that produces a given value for the dependent variable y. Here we must undo the given function, which is what the inverse function is all about. To be able to answer this question requires the existence of an inverse function and the ability either to find its equation algebraically or to estimate its values graphically or numerically.

When a function is given in a table, finding the inverse function is trivial, as we demonstrate in Example 1.

EXAMPLE 1

Table 2.3 gives the average distance D from the sun (in millions of miles) for each of the planets as a function of its average speed S (in miles per hour). Find the inverse function.

Solution For this function, we think of the average speed S of each planet as the independent variable and the average distance D from the sun as the dependent variable, so

TABLE 2.3

Planet	Speed	Distance $D = f(S)$
Mercury	4,110	36.0
Venus	7,671	67.2
Earth	10,605	92.9
Mars	16,153	141.5
Jupiter	55,171	483.3
Saturn	101,164	886.2
Uranus	203,459	1782.3
Neptune	318,790	2792.6
Pluto	418,744	3668.2

$D = f(S)$. The corresponding inverse function f^{-1} simply reverses the role of the two variables. Thus the average distance D from the sun of each planet is now the independent variable and the average speed S of each planet is the dependent variable. We write $S = f^{-1}(D)$. Table 2.4 simply interchanges the columns for S and D from Table 2.3, as shown.

TABLE 2.4

Planet	Distance	Speed $S = f^{-1}(D)$
Mercury	36.0	4,110
Venus	67.2	7,671
Earth	92.9	10,605
Mars	141.5	16,153
Jupiter	483.3	55,171
Saturn	886.2	101,164
Uranus	1782.3	203,459
Neptune	2792.6	318,790
Pluto	3668.2	418,744

In general, if (a, b) is a point on the graph of a function $y = f(x)$, then (b, a) must be a point on the inverse function $x = f^{-1}(y)$. Thus, finding the inverse for any function given in a table is trivial, assuming that the inverse function exists.

For relatively simple functions given by formulas, determining the inverse function f^{-1} for a given function f is straightforward. We just solve the original expression algebraically for the independent variable in terms of the dependent variable, as illustrated in Example 2.

EXAMPLE 2

Find the inverse function to the Celsius to Fahrenheit conversion function $F = \frac{9}{5}C + 32$.

Solution To find the inverse function, we first subtract 32 from both sides of the equation and get

$$\frac{9}{5}C = F - 32.$$

Multiplying both sides of the equation by $\frac{5}{9}$ gives

$$C = \frac{5}{9}(F - 32).$$

So $C = \frac{5}{9}(F - 32)$ is the inverse function.

Many functions, however, do not have an inverse, as we show later. Even when a function f does have an inverse, it is not always possible to find a formula for the inverse f^{-1} algebraically. Fortunately, as Examples 3 and 4 indicate, most of the common functions that we have discussed do have inverses and we can find expressions for them algebraically.

EXAMPLE 3

a. Find the inverse function for the exponential function $P(t) = 12.94(1.029)^t$ that models the population of Florida, where t is the number of years since 1990 and P is the population in millions.

b. What does this inverse function tell us?

c. What are reasonable values for the domain and range of this inverse function?

Solution

a. Since $P = 12.94(1.029)^t$, we have to solve for t as a function of P. First, we divide both sides by 12.94:

$$\frac{P}{12.94} = (1.029)^t.$$

To solve for t, we take logs of both sides of this equation:

$$\log\left(\frac{P}{12.94}\right) = \log(1.029)^t = t\log(1.029).$$

Therefore

$$t = \frac{\log(P/12.94)}{\log(1.029)} = f^{-1}(P).$$

We can stop here or we can use properties of logarithms to simplify this expression. Using the property that the logarithm of a quotient is the difference of the logs, we get

$$t = f^{-1}(P) = \frac{\log(P) - \log(12.94)}{\log(1.029)}$$

or, using the approximate values of log 12.94 and log 1.029, we have

$$t \approx \frac{\log(P) - 1.1119}{0.0124} \approx 80.545\log(P) - 89.669.$$

This logarithmic function is the inverse to the function modeling Florida's population.

b. The inverse function gives the number of years since 1990 (the value of t) that it takes for the population of Florida to reach any given level P.

c. For the inverse function, the independent variable is the value of the population P and the dependent variable is the number of years t since 1990. Therefore the domain of the inverse function consists of all reasonable values for P—say, from 5 million to a maximum of 50 million people. The range consists of all corresponding values of t. To find these values, we use the equation for the inverse function that we obtained in part a. If we substitute $P = 5$ into the preceding equation, we get

$$t = f^{-1}(5) \approx 80.545\log(5) - 89.669 \approx -33.4.$$

According to this model, the population of Florida was 5 million about 33 years before 1990, or in 1957. Similarly, substituting $P = 50$ yields

$$t = f^{-1}(50) \approx 80.545\log(50) - 89.669 \approx 47.2,$$

so the population of Florida will reach 50 million about 47 years after 1990, or in early 2037. Consequently, a reasonable range for the inverse function is t from -33.4 to 47.2 years, which corresponds to about 1957 to about 2037.

In general, the inverse of any exponential function will be a logarithmic function (and vice versa) because the logarithm undoes the exponential function.

EXAMPLE 4

We have shown that the power function $W = f(S) = 0.15S^{9/4}$ can be used to model the weight W of large birds as a function of their wingspan S.

a. Find the inverse function for f.
b. What does the inverse function tell us?
c. What wingspan would allow a 15 pound turkey to fly?
d. What is a realistic domain and range for the inverse function?

Solution

a. We have

$$W = f(S) = 0.15S^{9/4}.$$

To solve for S, we first divide both sides of the equation by 0.15:

$$S^{9/4} = W/0.15 \approx 6.667\ W.$$

To find S, we must undo the $\frac{9}{4}$ power, so we raise both sides of this equation to the $\frac{4}{9}$ power:

$$(S^{9/4})^{4/9} = S = (6.667W)^{4/9},$$

using properties of exponents. Therefore the inverse function is

$$S = (6.667)^{4/9}\, W^{4/9} \approx 2.324\ W^{4/9}.$$

b. The inverse function gives the wingspan S in feet needed to support in flight a bird that weighs W pounds.

c. If a turkey weighs 15 pounds, this formula predicts that, in order for the turkey to fly, it would need a wingspan of

$$S = 2.324(15)^{4/9} \approx 7.7436 \text{ feet}.$$

Since this is about three times the actual wingspan of a turkey, it isn't able to fly.

d. For the original function f, the independent variable is the wingspan S and the dependent variable is the weight W of a bird. For the inverse function f^{-1}, the independent variable is the weight W and the dependent variable is the wingspan S. If we consider reasonably large birds that weigh between 2 pounds and 20 pounds, say, the domain for the inverse function f^{-1} will be between $W = 2$ and $W = 20$. To find the corresponding range, we use the equation for S we obtained in part (a):

$$f^{-1}(2) = 2.324(2)^{4/9} \approx 3.16;$$
$$f^{-1}(20) = 2.324(20)^{4/9} \approx 8.80.$$

Thus the range of the inverse function is from about 3 feet to almost 9 feet.

Note that the inverse function to the power function $W = f(S) = 0.15S^{9/4}$ turned out to be another power function, $S = f^{-1}(W) = 2.324W^{4/9}$. In general, the inverse of any power function, if it exists, is a power function.

Further, note how the powers of the two functions $W = f(S) = 0.15S^{9/4}$ and $S = f^{-1}(W) = 2.324W^{4/9}$ compare algebraically: Each power is the reciprocal of the other. This result is analogous to what happens with $y = x^3$: We solve for x by extracting the cube root to get $x = y^{1/3} = \sqrt[3]{y}$. Thus the inverse function for $y = f(x) = x^3$ is $x = y^{1/3} = f^{-1}(y)$. Similarly, if we need to find the value of x for which

$$x^{175} = 200, \quad \text{then} \quad x = \sqrt[175]{200} = 200^{1/175} \approx 1.03074.$$

If we need to determine the value of the base b for which

$$b^{2004} = \frac{1}{2}, \quad \text{then} \quad b = \sqrt[2004]{\frac{1}{2}} = \left(\frac{1}{2}\right)^{1/2004} \approx 0.99965.$$

To verify these results, just calculate $(1.03074)^{175}$ and $(0.99965)^{2004}$. Although these numbers may seem bizarre to you, we perform such operations routinely in later chapters because they allow us to answer some interesting questions.

Examples 3 and 4 illustrate two important, though different, situations. To extract an unknown variable that appears as the base in a power function $y = x^p$, take the corresponding pth root of both sides of the equation. To extract an unknown variable that appears in the exponent of an exponential function $y = c^x$, take the logarithm of both sides of the equation. We summarize this information as follows. Be sure that you understand the difference between these two situations.

> To solve for the variable x from a power function $y = x^p$, we extract the pth root. That is,
>
> if $y = f(x) = x^p$, then $x = f^{-1}(y) = y^{1/p}$.
>
> If p is a fraction whose denominator is even, we must have $x \geq 0$ as the domain for f.
>
> The inverse of a power function is another power function.
>
> To solve for the variable x from an exponential function $y = c^x$, we take logarithms. That is, for $c = 10$,
>
> if $y = f(x) = 10^x$, then $x = f^{-1}(y) = \log y$.
>
> The inverse of an exponential function is a logarithmic function and vice versa.

Extracting the appropriate root from the preceding functions was quite simple. Unfortunately, complications can arise, depending on the behavior of the function, as we illustrate in Example 5.

EXAMPLE 5

The function $y = f(t) = -16t^2 + 48t + 6$ models the height y of a ball thrown straight up as a function of time t. Find how long it takes the ball to reach a height of 35 feet.

Solution For the desired height of $y = 35$ feet, we want to find the corresponding value of t, so we have to solve the equation

$$-16t^2 + 48t + 6 = 35.$$

Figure 2.74 shows the graph of the function $f(t)$. Note that the horizontal line corresponding to the height $y = 35$ crosses the curve twice, once when $t \approx 0.84$ seconds (on the way up) and again when $t \approx 2.16$ seconds (on the way down).

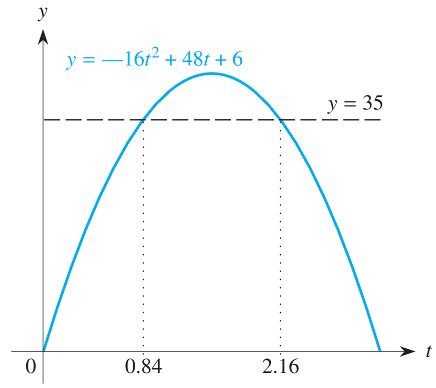


FIGURE 2.74

Example 5 illustrates the fact that not every function has an inverse. In this case, two different values of t correspond to a height of 35 feet. Consequently, the function $y = f(t) = -16t^2 + 48t + 6$ does not have an inverse because t is *not* a function of y—at least one value of y ($y = 35$) leads to two different values of t. That is, we can't undo the effects of the original function f uniquely. We discuss this situation in more detail later.

We have said that a function f and its inverse f^{-1} undo each other. To show what this means, suppose that we start with a number x in the domain of f. The function f carries x into the corresponding value of $y = f(x)$ in the range of f, as illustrated in Figure 2.75. Similarly, if we start with any value of y in the range of f, then f^{-1} maps y into the value of x associated with it so that $f^{-1}(y) = x$. That is,

$$x = f^{-1}(y) \quad \text{and} \quad y = f(x).$$

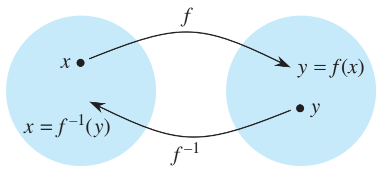


FIGURE 2.75

Again, consider Figure 2.75. For the original function f, the circle on the left represents the domain of f (the allowable values of x) and the circle on the right represents the range of f (the corresponding values of y). For the inverse function f^{-1}, the circle on the right represents the domain of f^{-1} (the allowable values of y) and the circle on the left represents the range of f^{-1} (the corresponding values of x).

In particular, suppose that x_0 is any specific value of x and that $y_0 = f(x_0)$, so that f transforms x_0 into y_0. If we follow this by applying f^{-1} to y_0, we get $f^{-1}(y_0) = x_0$, returning to the original x_0 value. That is, f^{-1} undoes the effect of f on any value x_0. (We consider the idea of applying one function after another in detail in Section 4.6.)

Similarly, if y_0 is any specific value of y and $x_0 = f^{-1}(y_0)$, then $f(x_0) = y_0$. That is, f undoes the effect of f^{-1} on any value y_0.

We can represent these ideas pictorially as follows.

> If f and f^{-1} are inverse functions,
>
> for any x: $\quad x \xrightarrow{f} y = f(x) \xrightarrow{f^{-1}} x = f^{-1}(y);$
>
> for any y: $\quad y \xrightarrow{f^{-1}} x = f^{-1}(x) \xrightarrow{f} y = f(x).$

For instance, the exponential function and the logarithmic function are inverses of each other, which simply restates the relationships

$$\log(10^x) = x \quad \text{and} \quad 10^{\log y} = y.$$

Think About This

Consider the model for the population of Florida $P = f(t) = 12.94(1.029)^t$ and its inverse

$$t = f^{-1}(P) = \frac{\log(P) - \log(12.94)}{\log(1.029)}$$

(from Example 3.) Select any year—say, 1996 when $t = 6$—and verify that if $P = f(t)$, then $t = f^{-1}(P)$.

Determining the Existence of an Inverse Function

Based on the results of Example 5 on the height of a ball thrown vertically upward, we know that not every function f has an inverse f^{-1}. In particular, no power function with an even power can have an inverse. To see why, consider the function $f(x) = x^2$, whose graph is the parabola shown in Figure 2.76. The inverse function, if it exists, should give the value of x that produces any given height y. Suppose that we start at a height of $y = 25$. Clearly, there are two different points on the parabola at a height of 25, one corresponding to $x = -5$ and the other corresponding to $x = +5$. We cannot reverse the process uniquely to get only one value of x for a given y, so y is *not* a function of x and $f(x) = x^2$ does not have an inverse.

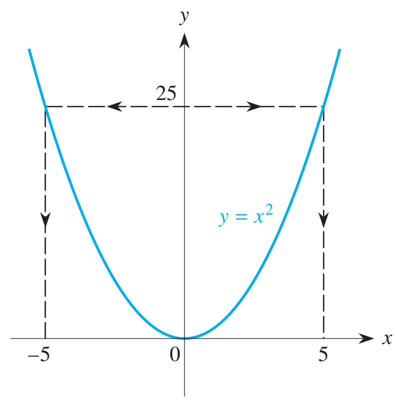


FIGURE 2.76

However, we can restrict the domain of this function to produce a partial inverse. Suppose that we limit our attention to nonnegative values of x and consider the function $g(x) = x^2$, for $x \geq 0$. In this case, if we take any positive value for y

(say, $y = 25$,) we can undo the function g by taking the square root and accepting only the nonnegative value, $x = 5$. Thus the function $g(x) = x^2$ with domain restricted to $x \geq 0$ has an inverse, $g^{-1}(y) = \sqrt{y}$. Alternatively, if we restrict our domain to values of $x \leq 0$, we could also uniquely undo the results of squaring and get the inverse $h^{-1}(y) = -\sqrt{y}$.

Is there a simple criterion to determine whether a function f has an inverse? Definitely! Again, we know that the function $f(x) = x^2$ has an inverse if we restrict its domain to either $x \geq 0$ or $x \leq 0$. It does not have an inverse if we allow the domain to include both positive and negative values for x. When we restrict the domain to $x \geq 0$, we consider only the right-hand side of the parabola where the function is strictly increasing. When we restrict the domain to $x \leq 0$, we consider only the left-hand side of the parabola where the function is strictly decreasing. In both instances, the restricted function has an inverse. On the one hand, when we allow both positive and negative values for x, the function first decreases and then increases and thus does not have an inverse. On the other hand, the function $h(x) = x^3$ has an inverse $x = h^{-1}(y) = \sqrt[3]{y}$ without any restrictions on x. We also know that this function is strictly increasing for all values of x, as shown in Figure 2.77.

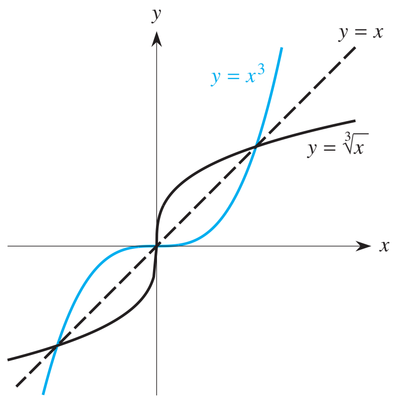


FIGURE 2.77

These observations suggest a simple criterion for functions to have an inverse: The function must be either strictly increasing or strictly decreasing. We call such a function *monotonic.* Compare the two functions shown in Figure 2.78. The one on the left is strictly increasing and, for any desired height y, we can undo the effect of the function and come back to a unique x that produced that particular y. In contrast, the one on the right increases and decreases over different intervals. There are some heights, such as y_3 and y_4, that occur several times, so different x-values correspond to the same y-value. Thus it is not possible to find the unique x that produces a particular y-value.

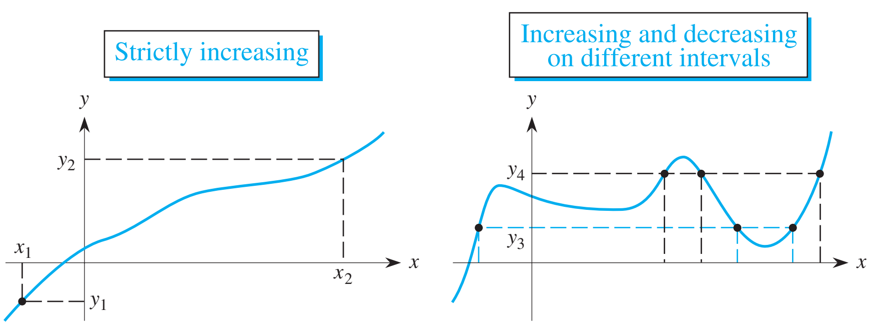


FIGURE 2.78

In summary, all linear, exponential, and logarithmic functions have inverses because they are either strictly increasing or strictly decreasing. Any power function that is restricted to $x > 0$ also has an inverse.

There is an alternative criterion analogous to the Vertical Line Test that we discussed in Section 1.4 for determining whether a curve represents a function. Recall that a curve represents a function if every vertical line crosses the curve at most once. In other words, for any value of x, one and only one value of y corresponds to that x. In an analogous way, we can use a Horizontal Line Test to determine whether a function has an inverse: If every horizontal line crosses the curve at most once, a function f has an inverse f^{-1}. In other words, for any height y, one and only one value of x corresponds to that height.

Behavior of the Inverse Function

At times, expressing both f and f^{-1} as functions of the same variable is desirable so that their graphs can be drawn on the same set of axes and their behaviors compared easily. For instance, we previously showed that

$$y = f(x) = x^3 \quad \text{and} \quad x = f^{-1}(y) = \sqrt[3]{y}$$

are inverse functions. Instead, let's write these two related functions so that both are functions of the same independent variable, x:

$$y = f(x) = x^3 \quad \text{and} \quad y = f^{-1}(x) = \sqrt[3]{x}.$$

Figure 2.77 displayed the graphs of both f and f^{-1}. Notice that the graphs of the function $f(x) = x^3$ and its inverse $f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$ are mirror images of each other about the line $y = x$. (Note that, if we didn't interchange x and y for the inverse function, the two formulas $y = x^3$ and $x = \sqrt[3]{y} = y^{1/3}$ would represent identical curves, so that we would see only one curve.)

Similarly, the exponential function $y = 10^x$ and the logarithmic function $y = \log x$ are inverse functions. If $y = f(x) = 10^x$, we can solve for x by taking the logarithm of both sides to get

$$\log y = \log 10^x = x = f^{-1}(y).$$

We now interchange the variables so that x is also the independent variable for the inverse function and write $y = f^{-1}(x) = \log x$. Again, note that the graphs of these two functions are mirror images of each other about the line $y = x$, as shown in Figure 2.79.

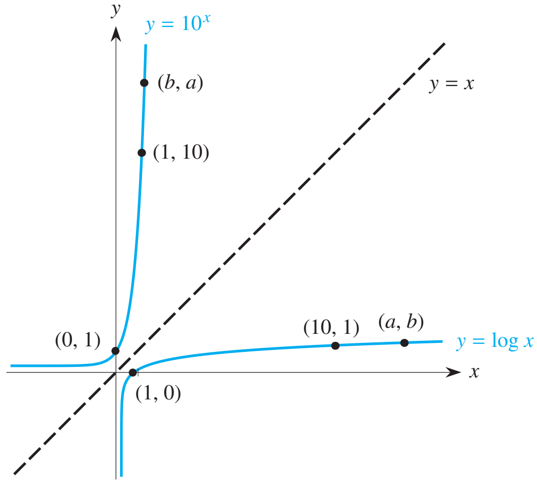


FIGURE 2.79

In general, the graphs of a function f and its inverse f^{-1} are always mirror images of each other about the line $y = x$. For the inverse to exist, the function f must be either strictly increasing or strictly decreasing. Consequently, the graph of the inverse function f^{-1} also is monotonic, and both functions increase or both decrease.

However, there are no clear patterns for the concavity of f and f^{-1}. Both f and f^{-1} can be concave up, both can be concave down, each can have opposite concavity, or both can have no concavity (if their graphs are lines).

Note that $f^{-1}(x)$ is not the same as $1/f(x)$. For instance,

$$\text{if } f(x) = x^3, \quad \text{then} \quad f^{-1}(x) = x^{1/3},$$

whereas $1/f(x) = 1/x^3 = x^{-3}$, which is not the same as $x^{1/3}$. (Check their graphs to convince yourself.) Similarly,

$$\text{if } g(x) = 10^x, \quad \text{then} \quad g^{-1}(x) = \log x,$$

but $1/g(x) = 1/10^x = 10^{-x}$, which is not the same as $\log x$. (Check their graphs to convince yourself.) Figure 2.80 shows the graphs of $g(x) = 10^x$ and $1/g(x) = 10^{-x}$. The graphs are not mirror images of each other about the line $y = x$. Using the symmetry condition, describe where the graph of the inverse function of g would be in Figure 2.80.

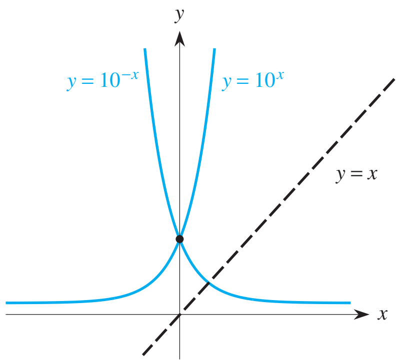


FIGURE 2.80

Finding the Inverse Function

Suppose that we know that a function f has an inverse because it is either strictly increasing or strictly decreasing. Can we always find the inverse function f^{-1}? If the formula for the function is quite simple, we might be able to undo the equation algebraically to obtain a formula for f^{-1}. For instance, we demonstrated earlier that we can undo the conversion formulas between the Fahrenheit and Celsius temperature scales, getting

$$f(x) = \frac{9}{5}x + 32 \quad \text{and} \quad f^{-1}(x) = \frac{5}{9}(x - 32);$$

the algebra is simple because the relationship is linear. Similarly, we can undo the equation of an exponential function to get the logarithmic function and vice versa so that

$$g(x) = 10^x \quad \text{and} \quad g^{-1}(x) = \log x, \qquad \text{for } x > 0.$$

Also, we can undo the relationship between the squaring and square root functions algebraically, obtaining

$$h(x) = x^2 \quad \text{and} \quad h^{-1}(x) = \sqrt{x}, \qquad \text{for } x > 0.$$

In Example 6, we illustrate these ideas with a somewhat more complicated function.

EXAMPLE 6

For the function $y = g(x) = 1 + 1/x$ with domain $x > 0$, find g^{-1}. Analyze the behavior of g and g^{-1}.

Solution As shown in Figure 2.81, the graph of the function g is strictly decreasing, so its inverse exists. And because the domain of g is $x > 0$, we know that $1/x > 0$. Thus $1 + 1/x$ must be larger than 1, and therefore the range of g must be $y > 1$.

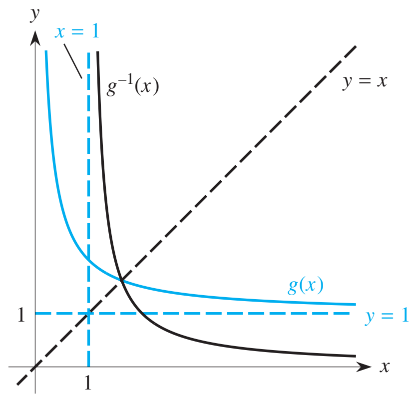


FIGURE 2.81

Next, let's find a formula for the inverse function. If $y = 1 + 1/x$,

$$\frac{1}{x} = y - 1.$$

Taking the reciprocal of both sides, we get

$$x = \frac{1}{y-1} = g^{-1}(y), \qquad y > 1.$$

Interchanging the roles of x and y to use the same independent variable yields

$$y = \frac{1}{x-1} = g^{-1}(x), \qquad x > 1.$$

The graphs of the function g and its inverse g^{-1} are also shown in Figure 2.81. As expected, they are mirror images of the other about the line $y = x$.

Further, the original function g is a decreasing function; it decays from a vertical asymptote at $x = 0$ toward a horizontal asymptote of $y = 1$. The graph of g^{-1} also decreases from a vertical asymptote at $x = 1$ toward a horizontal asymptote of $y = 0$. Note that the vertical and horizontal asymptotes are interchanged and that both curves are concave up.

Unfortunately, solving for an inverse function algebraically, as we did in Example 6, usually is not possible. Hence, we usually have to resort to numerical or graphical methods to estimate values for the inverse function; that is, given a particular value for y, we can determine the corresponding value of x by examining either the graph of the original function or successive numerical estimates.

EXAMPLE 7

For the function $f(x) = 2^x + 3^x$, **(a)** explain why f^{-1} exists and **(b)** estimate the value of $f^{-1}(10)$.

Solution

a. As shown in the graph of f in Figure 2.82, the function appears to be strictly increasing, which means that the inverse function exists.

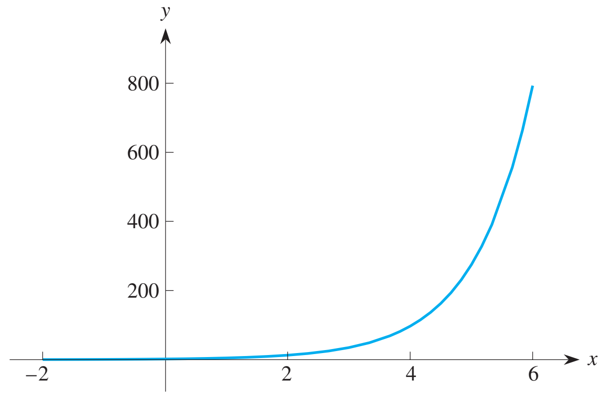


FIGURE 2.82

b. Unfortunately, for the expression $y = 2^x + 3^x$, it is not possible to solve for x in terms of y, so we are unable to find a formula for f^{-1}, even though we do know that f^{-1} exists. Graphically, the curve clearly passes the level $y = 10$ at some point x, so we must estimate, either numerically or graphically, the value of x for which $f(x) = 10$. We know that $f(1) = 2 + 3 = 5$ and that $f(2) = 2^2 + 3^2 = 4 + 9 = 13$, so the desired value of x must be between 1 and 2. We can zoom in either by checking further numerical values or by examining the graph of the function between 1 and 2, as shown in Figure 2.83. Either way, we find that $x \approx 1.73$, so $f(1.73) = 2^{1.73} + 3^{1.73} \approx 10.007$.

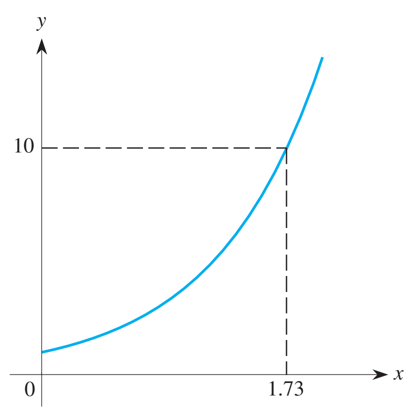


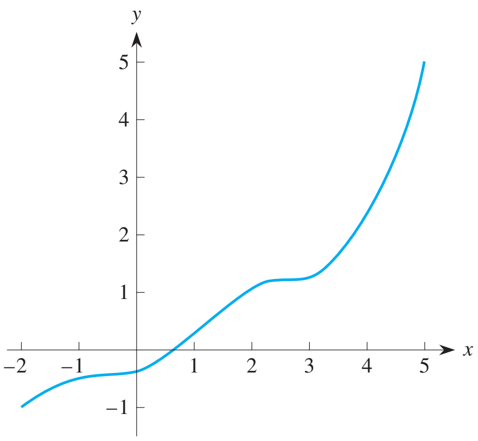
FIGURE 2.83

Problems

1. Which of the following functions have inverses? Explain why or why not. For any function having an inverse, describe what the inverse function tells you.
   a. The height of water after t minutes in a child's pool that you are filling at a steady rate, using a garden hose.
   b. Your distance from New York on an airplane flight from New York to San Francisco as a function of the time t since takeoff.
   c. The height of the student who is numbered n on your instructor's class roster.
   d. The amount that the nth customer in line at Burger Heaven pays for lunch.

e. The length of the fingernail on your right index finger if t is the number of hours since you last clipped your nails.
f. The amount spent by a family to heat their home if T is the temperature at which they set the thermostat.
g. The depth of the snow on a person's front lawn in Buffalo as a function of the time t elapsed from October 1 to the following March 1.
h. The total amount of snow that falls on the person's lawn in part (g) as a function of the time t elapsed from October 1 to the following March 1.

2. For the function f shown in the accompanying figure, estimate the value for x that corresponds to



a. $y = 0$; **b.** $y = 2$;
c. $y = 5$; **d.** $y = -1$.

Then plot the resulting points and use them to sketch the graph of the inverse function f^{-1}.

3. Consider the function f with values given in the following table.

x	0	1	2	3	4	5
$f(x)$	2.94	2.48	2.05	1.84	1.44	1.12

a. What is the domain of f? What is the range?
b. Create a table of values for f^{-1}. What are its domain and range?

4. Use your function grapher to decide which functions have inverses. For those functions that do, estimate the value for $f^{-1}(10)$.

a. $f(x) = x^3 - 9x^2 + 5x - 5$
b. $f(x) = x^3 - 2x^2 + 5x - 5$
c. $f(x) = 2x + x^2$ **d.** $f(x) = 2x - x^2$
e. $f(x) = 2x + x^3$ **f.** $f(x) = 2x - x^3$

5. The table of values gives the time T needed for a Trans Am to accelerate from zero to the indicated final speed v.

Final speed, v (mph)	30	40	50	60	70
Time, T (sec)	3.00	4.29	5.52	7.38	9.81

a. Explain why this set of data represents a function and why it has an inverse.
b. Explain what the inverse function tells you. What is $f^{-1}(5.52)$? Estimate the value of $f^{-1}(7)$.

6. We know that 1 inch is equivalent to about 2.54 centimeters.

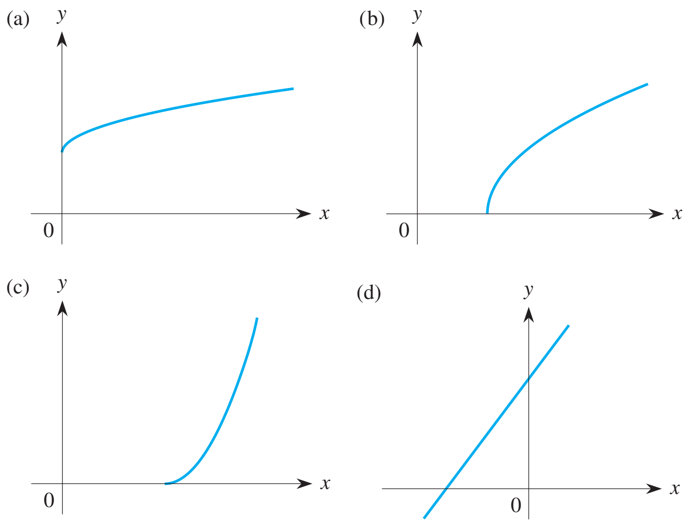
a. Write a formula for the function f that gives an object's length C in centimeters as a function of its length I in inches.
b. Find a formula for the inverse function f^{-1} and explain what f^{-1} tells you, in practical terms.

7. Find the inverse function of $p(t) = (1.04)^t$.

8. Find the inverse function of $f(t) = 50(10)^{0.1t}$.

9. Suppose that the temperature of an object is being measured to the nearest degree on both the Fahrenheit and Celsius scales. In general, which reading would you expect to be more accurate? Why?

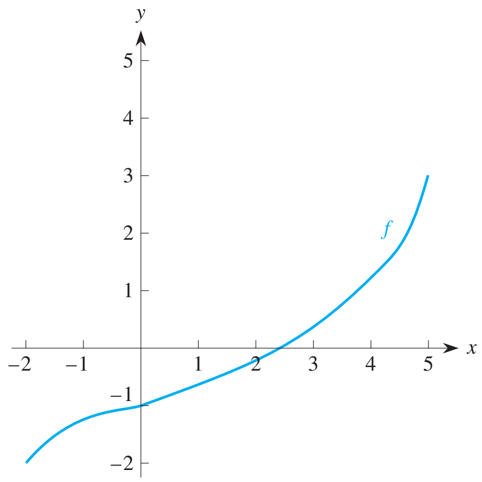
10. For each function f shown, sketch the graph of the inverse function f^{-1} on the same set of axes.



11. Suppose that a function f is increasing and concave up. By thinking of its inverse as the reflection about the line $y = x$, explain why f^{-1} is also increasing. Is it concave up or concave down? What happens if f is increasing and concave down?

12. Repeat Problem 11 if the function f is decreasing and concave up; the function f is decreasing and concave down.

13. On the same set of axes, sketch the graph of f^{-1} that corresponds to the function f shown in the accompanying figure.



14. Use the quadratic formula

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve the quadratic equation $-16t^2 + 48t + 6 = 35$ from Example 5 modeling the height of a ball. What is the significance of the two roots of this equation?

15. The level of Prozac in the blood can be modeled by the function $P(t) = 80(0.75)^t$.
   a. Find a formula for the inverse function.
   b. Use the inverse function to determine how long it will take until the level of Prozac drops to 25 mg.

16. The temperature of a chicken cooking in an oven can be modeled by the function $T(t) = 350 - 310(0.99)^t$.
   a. Find a formula for the inverse function.
   b. Use the inverse function to determine how long it will take until the temperature of the chicken reaches 175°.

17. In Problem 20 of Section 1.3 we introduced a function f that represents a simple replacement code in which each letter of the alphabet is replaced by a different letter according to $f(A) = M$, $f(B) = D$, $f(C) = K$, $f(D) = V$, $f(E) = X$, $f(F) = B$, $f(G) = P$, $f(H) = T$, $f(I) = J$, $f(J) = S$, $f(K) = Z$, $f(L) = Q$, $f(M) = H$, $f(N) = O$, $f(O) = A$, $f(P) = L$, $f(Q) = W$, $f(R) = C$, $f(S) = F$, $f(T) = Y$, $f(U) = R$, $f(V) = G$, $f(W) = I$, $f(X) = U$, $f(Y) = N$, and $f(Z) = E$.
   a. Explain why this function has an inverse f^{-1}.
   b. Use the inverse to decode the message

   JF YTJF HMYT?

Exercising Your Algebra Skills

Solve for the unknown in each equation.

1. $c^{25} = 14$
2. $0.07^t = 3$
3. $0.84^k = 0.20$
4. $m^{1995} = 4$
5. $17b^8 = 32$
6. $25c^9 = 8$
7. $4(1.02)^x = 7$
8. $2(0.75)^t = 1$

For Problems 9–12, solve for the independent variable in terms of the dependent variable for each function.

9. $y = f(x) = 12x^{7/2}$
10. $y = f(t) = 12(1.06)^t$
11. $Q = g(w) = 27w^{-3/4}$
12. $L = g(t) = 125(0.92)^t$

For Problems 13–16, solve for the indicated variable in each formula to find the inverse function.

13. $F = ma$, for a
14. $E = mc^2$, for m
15. $P = kVT$, for V
16. $K = \frac{1}{2}mv^2$, for v

Chapter Summary

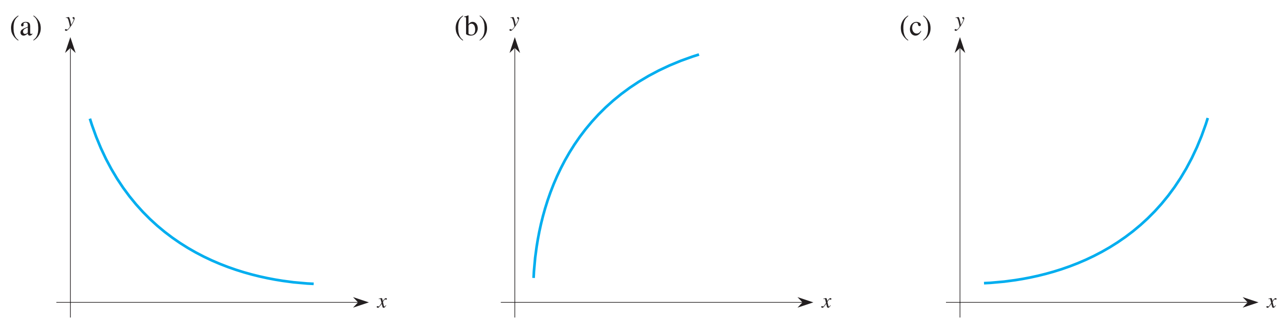
In this chapter, we covered the following ideas and approaches relating to families of functions.

- Important behavior characteristics of four important families of functions—linear functions, exponential functions, logarithmic functions, and power functions.

- How to find the slope and the equation of a line.
- What the slope of a line means.
- A criterion for knowing when a set of data follows a linear pattern.
- How to estimate the equation of a line that captures the linear pattern in a set of data.
- How to set up and solve problems involving linear processes.
- The behavior of exponential growth and exponential decay functions.
- What the growth and decay rates and the growth and decay factors mean.
- A criterion for knowing when a set of data follows an exponential pattern.
- How to find the doubling time for an exponential growth process and the half-life for an exponential decay process.
- How exponential behavior compares to linear behavior.
- How to find the exponential function that passes through two points.
- How to set up and solve problems involving exponential processes.
- The behavior of logarithmic functions.
- How to set up and solve problems involving logarithmic functions.
- How to use logarithms with bases other than 10.
- The behavior of power functions when the power is greater than 1, between 0 and 1, and negative.
- How to find the power function that passes through two points.
- How to set up and solve problems involving power functions.
- How power function behavior compares to exponential behavior.
- How logarithmic function behavior compares to power function behavior.
- How to determine whether a function has an inverse.
- What the inverse function tells you.
- How to set up and solve problems involving inverse functions.

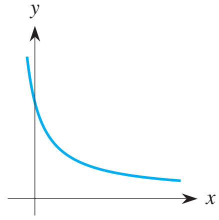
Review Problems

1. For each of the curves shown, suggest any types of functions that might have the indicated behavior pattern. If you suggest an exponential function, indicate whether the base c is greater than 1 or less than 1. If you suggest a power function, indicate whether the power p is positive or negative and whether p is greater than 1 or less than 1.

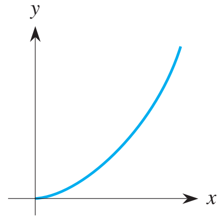


2. Identify each function as linear, exponential, logarithmic, or power. In each case, explain your reasoning.

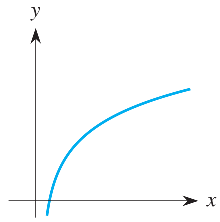
a.



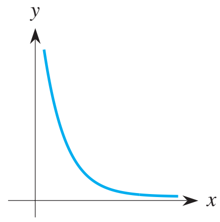
b.



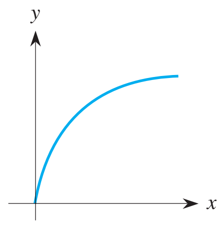
c.



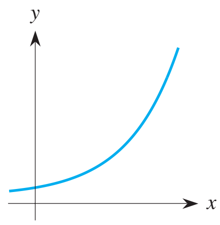
d.



e.



f.



g. $y = 1.05^x$

h. $y = x^{1.05}$

i. $y = (0.7)^x$

j. $y = x^{0.7}$

k. $y = 1/\sqrt{x}$

l. $5x - 3y = 15$

m.

x	y
0	3
1	5.1
2	7.2
3	9.3

n.

x	y
0	5
1	7
2	9.8
3	13.72

3. Match each formula for a function with one of the graphs (*A*)–(*L*). Because more than one function from the same family appears, match each member of that family to the most appropriate graph.

a. $y = 3x + 3$ **b.** $y = 2x - 3$

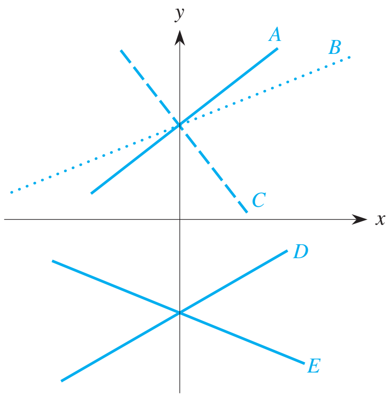
c. $y = 3 - 2x$ **d.** $y = 2x + 3$

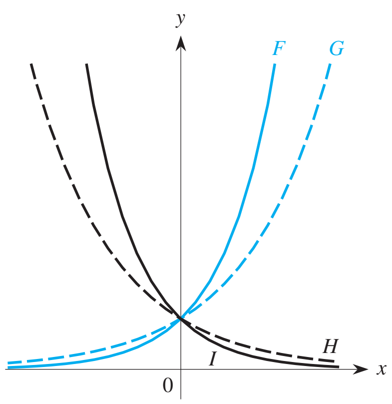
e. $y = -x - 3$ **f.** $y = 5(0.92)^x$

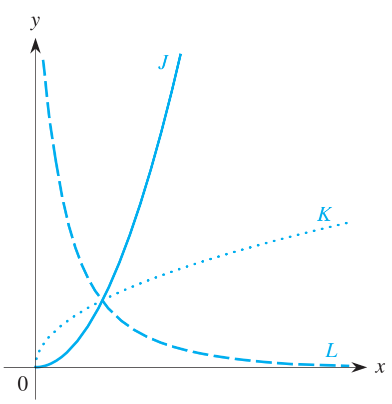
g. $y = 5(0.97)^x$ **h.** $y = 5(1.03)^x$

i. $y = 5(1.08)^x$ **j.** $y = x^{2.5}$

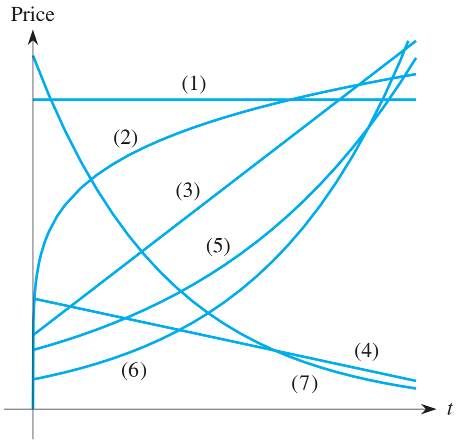
k. $y = x^{-2.5}$ **l.** $y = x^{0.25}$





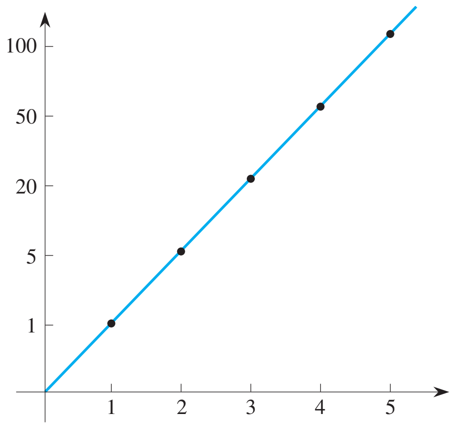


4. The accompanying figure shows the graphs of the values of shares of 7 stocks as functions of time. Match each scenario with one of the graphs and write a brief scenario for each of the remaining graphs.



a. The value of the stock increased 12% per year.
b. The value of the stock increased 8% per year.
c. The value of the stock dropped by $4 each year.
d. The value of the stock increased by $6 each year.
e. The value of the stock remained steady.

5. The points $(1, 1)$, $(2, 5)$, $(3, 20)$, $(4, 50)$, and $(5, 100)$ are plotted in the accompanying figure and a line is drawn through them.



a. What is wrong with the graph as it is drawn?
b. Draw a correct graph of these points and sketch a smooth curve that passes through them.
c. Describe the behavior pattern in the function based on these points. Identify a possible type of function that might be an appropriate model for these values. Explain your reasoning.

6. A function $F(t)$ is exponential with known values $F(0) = 5$ and $F(3) = 8.5$. Determine the function and give the growth factor.

7. **a.** The profits of Alamo Paper Company are growing by $100,000 each year. In 1990, its profits were $1.5 million. Determine the profit function $P(t)$, where t represents the number of years since 1990. Draw a graph of $y = P(t)$ and determine the year in which profits first exceed $2 million.
b. Ord Paper Company had profits of $950,000 in 1990, and its profits are growing at the rate of 10% each year. Determine when its profits first exceed $2 million.
c. Use your function grapher to determine when the profits of Ord first exceed the profits of Alamo.

8. When the bald eagle was formally put on the endangered species list in 1967, there were about 800 of the eagles in the United States. As a result of eagle protection and restoration efforts, the bald eagle was removed from the list in 1994 when its population was just over 8000.
a. Determine the doubling time of the eagle population assuming the growth pattern is exponential.
b. Estimate the number of bald eagles in the United States in 2005, assuming that the growth trend continues.
c. When can you expect the eagle population to reach 20,000?

9. In preparing a holiday cranberry mold, a cook added boiling water at 212°F to the fruit and gelatin mixture, which was then poured into the mold and put into a 40° refrigerator. After 30 minutes, the temperature of the mixture was 148°. The temperature $F(t)$ at time t (in minutes) is given by $F(t) = 172(1 + a)^t + 40$, where a is a constant. What is the temperature of the mixture 3 hours after the mold was put in the refrigerator?

10. From 1980 to 1998, the number of workers, in millions, covered by Social Security can be approximated by a linear function of time t. Use the data in the table below and the black thread method to find the equation of a line that fits the data.

Use this linear function to estimate the number of workers covered by Social Security in 2005.

Year	1980	1985	1990	1992	1993	1994	1995	1996	1997	1998
Workers (millions)	140.4	150.9	164.0	167.5	169.1	170.7	172.9	174.8	177.0	179.1

Source: *2000 Statistical Abstract of the United States*

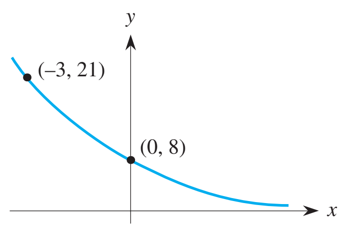
11. The following table shows the number of deaths in the United States resulting from accidental poisoning by drugs and medications in various years. Use the black thread method to find the equation of a linear function that fits the data. What is the slope of the line? What does this slope represent? Use the equation to predict the number of deaths from accidental poisoning in 2003.

Year	1980	1990	1994	1995	1996
Deaths	2492	4506	7828	8000	8431

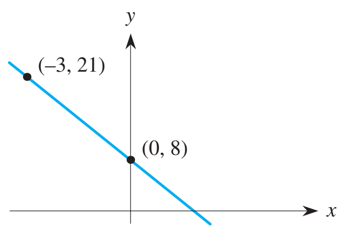
Source: *2000 Statistical Abstract of the United States*

12. Find possible equations for the function represented by each graph.

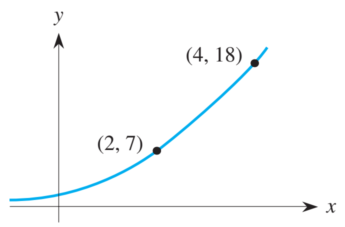
a.



b.



c.



13. In 1990 (when $t = 0$), the IRS collected \$1055 billion in taxes. In 1995, the IRS collected \$1573 billion.

a. Construct the linear function giving the amount of taxes collected by the IRS as a function of time t.

b. Use the linear function to estimate the amount of taxes collected in 2003.

c. Construct the exponential function giving the amount of taxes collected as a function of time t.

d. Use the exponential function to estimate the amount of taxes collected in 2003.

e. Use both models to predict the amount of taxes collected in 2010. Which model seems more accurate? Explain.

14. The median family income I in the United States was about \$17,700 in 1980 and rose to about \$37,000 in 1997. Let t be the number of years since 1980.

t	**Linear Model**	**Exponential Model**
0	\$17,700	\$17,700
5	?	?
10	?	?
15	?	?
17	\$37,000	\$37,000
25	?	?
30	?	?

a. If you assume that the increase in family income has been linear, find an equation for the line representing income I in terms of t. Use this equation to complete the second column of the table.

b. If you assume that the increase in family income has been exponential, find an equation of the form $I = I_0c^t$ to represent family income levels since 1980 and complete the third column of the table.

c. On the same set of axes, sketch the graphs of the functions you obtained in parts (a) and (b).

d. Use the equations from parts (a) and (b) to predict the median family income in 2003 for both types of growth.

e. Suppose that both predictions from part (d) seem unreasonable. Can you suggest any other types of functions that might be a better fit?

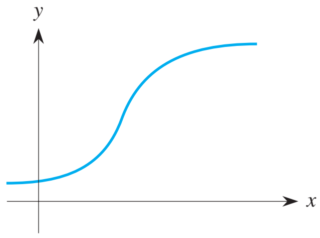
15. (Continuation of Problem 14) Suppose that you learn that the median family income in 1990 was about \$28,900.

a. Which of the two models in Problem 14 now seems more accurate?

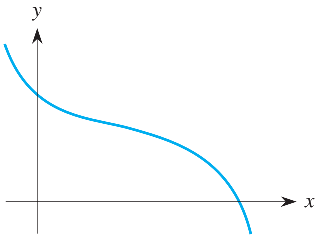
b. If you plot the three data points corresponding to 1980, 1990, and 1997, how would you describe the shape of the graph of median family income as a function of time? What is the significance of this shape?

16. For each function, draw the inverse function, if one exists, on the same axes. If the function has no inverse, explain why.

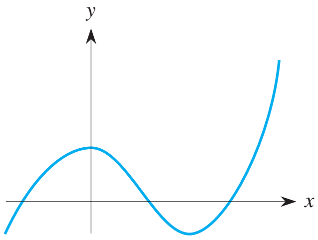
a.



b.



c.



17. For each function, give its domain and find the inverse function.

a. $f(t) = 0.5\log(2t - 4)$ **b.** $g(x) = x^3 + 6$

18. The function $F(x)$ is either linear or exponential. From the values in the table, decide which is the correct type and find a formula for F.

x	1	2	3	4	5
$F(x)$	6	9	13.5	20.25	30.375

19. Match each of the functions f, g, and h in the table to the behavior described as

a. increasing, straight line.
b. increasing, concave down.
c. increasing, concave up.

x	1	2	3	4	5
$f(x)$	2.70	3.64	4.92	6.34	8.96
$g(x)$	1.4	4.6	7.8	11.0	14.2
$h(x)$	5.10	5.19	5.27	5.34	5.40

20. **a.** Since 1960, the price of an ice cream cone in one southern city has been growing approximately exponentially according to the function $f(t) = Ac^t$. If the price of a one-scoop cone was 20¢ in 1960 and \$1.80 in 2000, (i) determine the function f and (ii) predict what the price of such a cone will be in 2005.

b. The average price of a ticket to a first-run movie was \$2.00 in 1960. This price has been growing exponentially and in 2000 was \$9.00. Which of the prices, for ice cream or for movies, is growing faster?

c. When can you expect the ice cream and movie prices to be the same if they each continue to grow in the same way?

d. How much would a ticket to the movies cost at the time you found in part (c)?

e. Would you use your model to predict the answer you got in parts (c) and (d)? Explain.

21. The aim of a college administration is to reduce the number of students who need remedial work in English by 10% each year. At the time the policy was put into place, 1600 students were enrolled in remedial English classes. If this program is successful, how many students will be enrolled in remedial English in 3 years? How long will it take for the number of students enrolled in such classes to be reduced to one section of 15 students?

22. The level of a drug in the bloodstream decreases at a rate of 30% of the drug per hour. Assume that the initial dose is 150 mg. How long does it take to bring the drug level down to under 20 mg? How long does it take to bring the drug level down to 5% of the original level?