6

Introduction to Trigonometry

6.1 The Tangent of an Angle

Historically, trigonometry was developed to solve problems involving right triangles. Thus, for the right triangle shown in Figure 6.1, if we know any two of the three sides a, b, and c, we can easily find the third side with the Pythagorean theorem

$$c^2 = a^2 + b^2,$$

where c is the length of the hypotenuse. But, there is no simple way to find the two unknown angles. To find them, we need to use trigonometry.

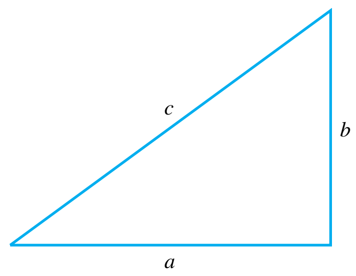


FIGURE 6.1

Similarly, if only one of the three sides and one angle are known, we can easily find the other angle (the two nonright angles must sum to 90° because they are *complementary angles*). However, without trigonometry, there is no simple way to find the lengths of the other two sides.

The basic idea behind trigonometry is a fundamental geometric fact about right triangles. The two right triangles shown in Figure 6.2 share the angle θ (lowercase Greek letter theta). Therefore the remaining angle in both triangles must be the same. We denote it ϕ (the lowercase Greek letter phi). Because all three angles in both triangles are the same, the triangles are *similar* (see Appendix A4). As a consequence, once an angle θ (other than the right angle) has been specified in a right triangle, that triangle is similar to every other right triangle having the same angle θ.

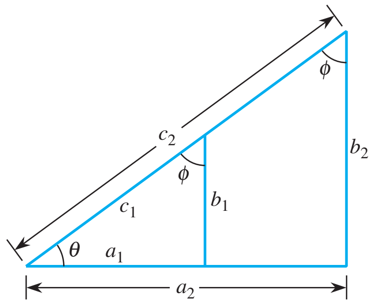


FIGURE 6.2

In the smaller triangle shown in Figure 6.2, from the point of view of the angle θ, there is an *adjacent side,* denoted by a_1; an *opposite side,* denoted by b_1; and the *hypotenuse,* denoted by c_1. In the larger triangle, also from the point of view of the angle θ, the *adjacent side* is a_2, the *opposite side* is b_2, and the *hypotenuse* is c_2. The triangles are similar, so that their corresponding sides are proportional, and

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Equivalently, once an angle θ has been specified, the ratio of corresponding sides of these right triangles will be the same. In particular, among several other comparable ratios,

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}, \quad \frac{a_1}{c_1} = \frac{a_2}{c_2}, \quad \text{and} \quad \frac{b_1}{c_1} = \frac{b_2}{c_2}.$$

That is, each of these ratios depends solely on the angle θ, not on the dimensions of the triangle. It is these ratios, and their dependence on the angle θ, that form the basis of trigonometry. In this section, we begin by examining one of these three ratios.

The Tangent of an Angle

Suppose that your math instructor has assigned you the task of calculating the height of a tall flagpole in the middle of campus. The direct approach would be to climb to the top, release a string until the bottom reaches the ground, and then measure the length of string. Obviously, this method presents some practical difficulties, and you would likely try to come up with some less physical approach.

Assume that, when you go out to the flagpole, you notice that the pole is casting a 66-foot-long shadow. How can you use this piece of extra information to determine the height of the pole? Suppose that you enlist the aid of a friend Ron, who is exactly six feet tall. Have him stand in the shadow cast by the pole so that the tip of his shadow falls exactly on the same spot *A* as the tip of the shadow of the flagpole, as illustrated in Figure 6.3. Also, suppose that the length of his shadow is $8\frac{1}{4}$, or 8.25, feet. The two triangles *ABC* and *ADE* are similar because the angles are the same, and so the corresponding sides are proportional. Therefore

$$\frac{\text{Ron's height}}{\text{length of his shadow}} = \frac{\text{height of pole}}{\text{length of pole's shadow}}$$

$$\frac{6}{8.25} = \frac{\text{height of pole}}{66}.$$

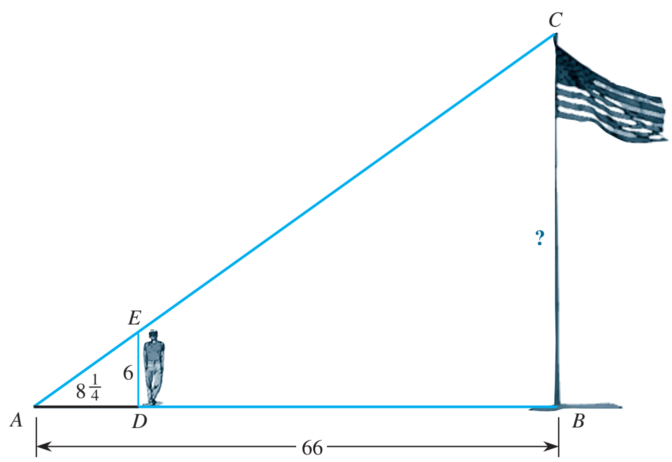


FIGURE 6.3

Multiplying both sides by 66 yields

$$\text{Height of the pole} = \frac{6(66)}{8.25} = 48 \text{ feet.}$$

Is this result correct? You can check it with the help of another friend, Sue, who is five feet tall. Have her stand so that the tip of her shadow matches the end of the pole's shadow. Suppose that the length of her shadow is $6\frac{7}{8}$, or 6.875, feet, which leads to right triangle *AFG* that is similar to the previous two, as illustrated in Figure 6.4. Because the corresponding sides are proportional, we get

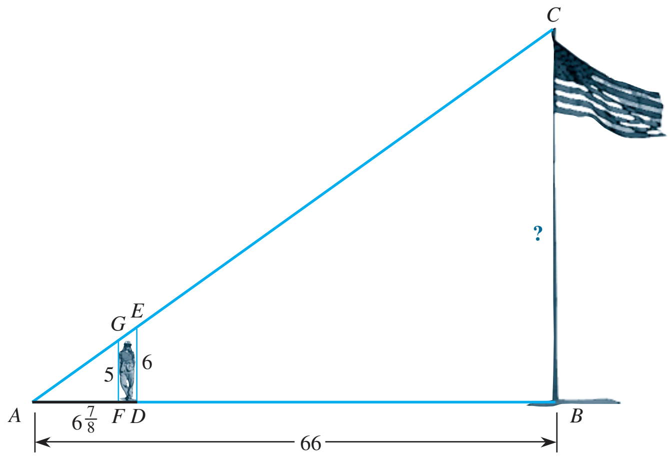


FIGURE 6.4

$$\frac{\text{Sue's height}}{\text{length of her shadow}} = \frac{\text{height of pole}}{\text{length of pole's shadow}}$$

$$\frac{5}{6.875} = \frac{\text{height of pole}}{66}.$$

Again, we find that

$$\text{Height of the pole} = \frac{5(66)}{6.875} = 48 \text{ feet.}$$

Let's look at this situation from a slightly more sophisticated point of view. In each of the three right triangles shown in Figure 6.5, the various lengths are different but the angles in corresponding positions are all the same, so all three triangles are similar. The angle θ (which is the same as angle CAB, angle EAD, and angle GAF) is called the *angle of inclination*. Using a protractor, we measure this angle and find that θ is about 36°. In fact, in *any* right triangle where the angle of inclination is 36°, the ratio of the vertical height (the opposite side) to the horizontal distance or width (the adjacent side) will always be the same; in this case,

$$\frac{\text{Height}}{\text{Width}} = \frac{6}{8.25} = \frac{5}{6\frac{7}{8}} \approx 0.727.$$

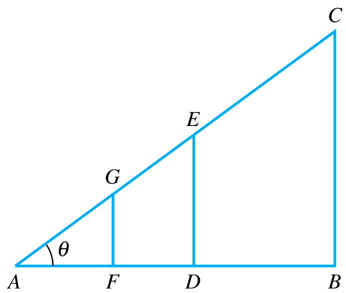


FIGURE 6.5

Of course, if the angle θ has a different value—say, $\theta = 40°$—the configuration of height and width is different and their ratio therefore is different. The ratio of height to width, or opposite side to adjacent side, in a right triangle depends only on the size of the angle θ, so this ratio is a function of the angle. We call this function the **tangent of the angle,** the **tangent ratio,** or the **tangent function** and write it as

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{height}}{\text{width}}.$$

Use your calculator, in `Degree` mode, to verify that $\tan 36° = 0.7265$. (Note that the values of the tangent function, as well as the other trigonometric functions that we discuss in Section 6.2, typically are irrational numbers, but we usually give the values to three or four decimal places.)

Because we are concerned exclusively with right triangles here, the angle θ must be between 0° and 90°, and so for now the domain of the tangent function consists of all angles $0° < \theta < 90°$. (Later we show how we can extend it to a larger domain.) Also, we can have a right triangle in any possible orientation, as shown in Figure 6.6, so the words *height* and *width* may not be appropriate. Instead, we typically think of the tangent ratio for an angle θ as follows.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

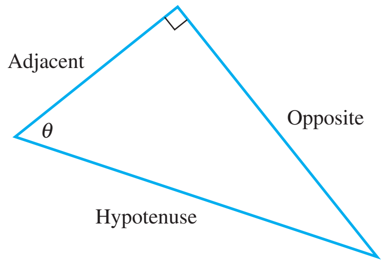


FIGURE 6.6

From the point of view of the other angle ϕ in the triangle, the opposite and adjacent sides are reversed, as depicted in Figure 6.7. Note also that the angles θ and ϕ are complementary angles.

The Tangent of Some "Special" Angles

Recall from geometry that in any 45°–45°–90° right triangle the two sides flanking the hypotenuse are equal and, by the Pythagorean theorem,

$$c^2 = a^2 + a^2 = 2a^2,$$

so $c = \sqrt{2}a$. That is, the hypotenuse must be $\sqrt{2}$ times the length of either side, as illustrated in Figure 6.8. In this triangle with angle $\theta = 45°$ and sides a, a, and $\sqrt{2}a$,

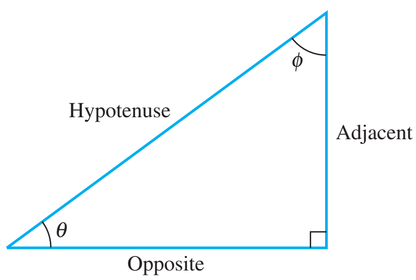


FIGURE 6.7

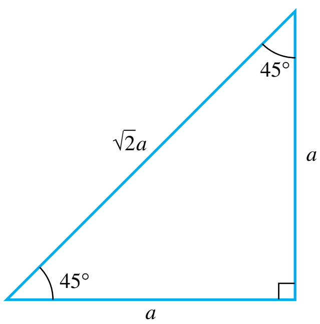


FIGURE 6.8

$$\tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{a} = 1.$$

You can easily verify that tan 45° = 1 on your calculator. (Be sure that your calculator is set in `Degree` mode.)

Similarly, recall from geometry that in any 30°–60°–90° right triangle, the side opposite the 30° angle is one-half the hypotenuse, or, equivalently, the hypotenuse is twice the side opposite the 30° angle. In such a triangle, suppose that the side opposite the 30° angle has length a so that the hypotenuse has length $2a$, as shown in Figure 6.9. We find the length of the third side from the Pythagorean theorem. Because $a^2 + b^2 = c^2$, we have $b^2 = c^2 - a^2$, so

$$b^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2 \quad \text{so that} \quad b = \sqrt{3}a.$$

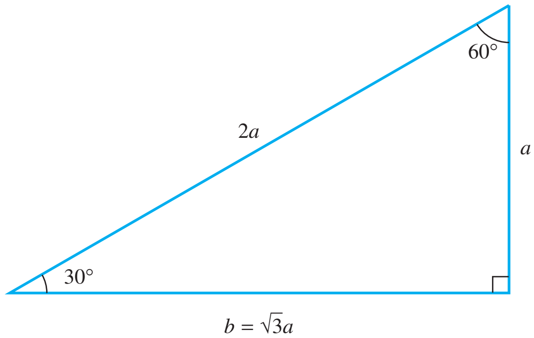


FIGURE 6.9

Consequently, for an angle of 30°, the ratio of the opposite side to the adjacent side is

$$\tan 30° = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}} \approx 0.577.$$

Alternatively, using a calculator, we find tan 30° ≈ 0.577.

Similarly, to find the tangent of 60°, we see from Figure 6.9 that the side opposite the 60° angle is $\sqrt{3}a$ and the side adjacent to it is a, so that

$$\tan 60° = \frac{\sqrt{3}a}{a} = \sqrt{3} \approx 1.732,$$

which you can also check on your calculator.

For any angle θ between 0° and 90°, you can use a calculator to obtain the corresponding value for $\tan\theta$. For instance, to three decimal place accuracy,

$$\tan 10° \approx 0.176,$$
$$\tan 20° \approx 0.364,$$
$$\tan 50° \approx 1.192,$$
$$\tan 80° \approx 5.671.$$

Note that as θ increases toward 90°, the value of $\tan\theta$ also increases; that is, the tangent is an increasing function of θ, at least between 0° and 90°. Does that make sense? Imagine walking toward the 556-foot-high Washington Monument while keeping your eye fixed on the top of the monument, as illustrated in Figure 6.10.

The opposite side (the vertical height) remains the same, 556 feet, while the adjacent side (the horizontal distance) gets smaller and smaller. The closer you get to the monument, the larger the angle of inclination and the larger the ratio of the fixed vertical height to the diminishing horizontal distance. By the time your eye is practically touching the side of the monument, and the angle is virtually 90°, the value for the tangent function has gotten very large indeed. The tangent function is *not defined* for $\theta = 90°$ because the length of the adjacent side would be zero.

What about the tangent of 0°? Suppose that you're standing across the street from a glass elevator that is descending along the outside of a tall building, as illustrated in Figure 6.11. Now the adjacent side (the horizontal distance) is fixed, the opposite side (the vertical height) is decreasing, and the angle θ is decreasing toward 0°. Therefore the value of the tangent function is likewise diminishing because it is the ratio of the decreasing vertical height and the fixed horizontal distance. Clearly, tan 0° is 0. We therefore conclude that the domain of the tangent function can be extended at least to $0° \leq \theta < 90°$.

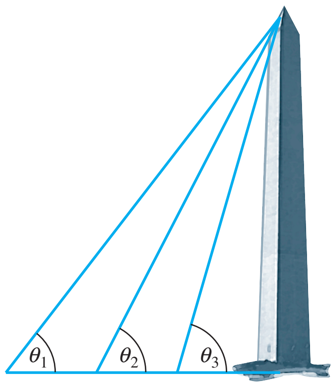


FIGURE 6.10

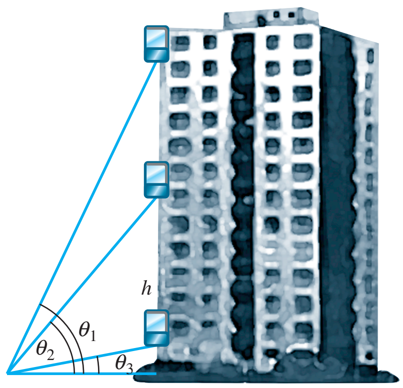


FIGURE 6.11

Behavior of the Tangent Function

Let's consider the values for the tangent function and investigate their growth pattern. Using a calculator, we obtain the following values.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\tan\theta$	0	0.176	0.364	0.577	0.839	1.192	1.732	2.747	5.671	UNDEF

Note that, as the angle θ increases from 0° to 10° to 20°, and so on, the tangent function is growing ever more quickly, so the function is concave up. The graph of the tangent function $y = \tan\theta$ for angles between 0° and 90° is shown in Figure 6.12. It passes through the origin and grows in a concave up pattern, approaching a vertical asymptote as θ approaches 90°.

How does the growth pattern compare to that of an exponential function? If you examine the successive ratios of the values of tan θ, you will find that they are

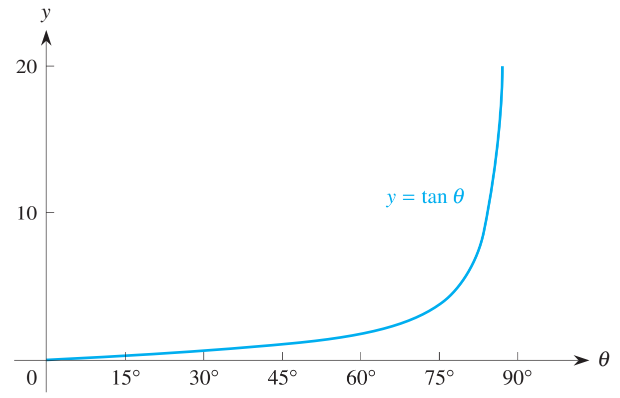


FIGURE 6.12

not constant, but rather are increasing considerably. In fact, the tangent function grows extremely rapidly near $\theta = 90°$ because $\theta = 90°$ is a vertical asymptote for the function. You might want to look at its graph on your function grapher for angles between 0° and somewhat less than 90°. We examine the properties of the tangent function in considerably more detail in Section 7.4.

Think About This Construct a table of values for the tangent function $y = \tan\theta$ for $\theta = 80°, 81°, 82°, \ldots, 89°$ and plot the points. Repeat for $\theta = 89°, 89.1°, 89.2°, \ldots, 89.9°$.

Using the Tangent Ratio

Suppose that we have a right triangle in which we can measure one of the angles other than the 90° angle with a protractor. We can find the tangent of that angle with a calculator. Then, if we know the length of either the adjacent side or the opposite side, we can easily find the length of the other side without involving Ron, Sue, or anyone else to solve the type of problem we used to begin this discussion.

EXAMPLE 1

A flagpole casts a shadow of length 66 feet. If the angle of inclination from the tip of the shadow to the top of the flagpole is 36°, find the height of the flagpole.

Solution Figure 6.13 shows that

$$\tan 36° = \frac{\text{opposite}}{\text{adjacent}} = \frac{H}{66},$$

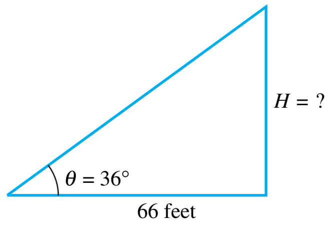


FIGURE 6.13

so

$$H = 66 \tan 36° = 47.952,$$

or about 48 feet high.

Note the approach used in Example 1. The first, and key, step was to draw a sketch of the situation, in which we identified all known parts of the right triangle, and marked the unknown parts. We then set up the tangent ratio and used it to find the unknown quantity.

EXAMPLE 2

While hiking through the mountains, you come to the edge of a deep gorge and wonder how far it is to the other side. A vertical tree is rooted on your side at the edge of the gorge. From a point 15 feet up in the tree, you find that the angle of depression (measured down from the horizontal at eye level) to the opposite edge of the gorge is 22°. How far is it across the gorge?

Solution The height (15 feet) to the point in the tree and the unknown distance D across the gorge form two sides of a right triangle, as depicted in Figure 6.14. Note that the 22° angle of depression is not an angle of the triangle. However, it does determine the measures of the triangle's angles θ and ϕ, based on some simple geometry. First, the angle $\theta = 68°$ because it is the complement of 22°. Second, the angle $\phi = 22°$ because ϕ is the complement of $\theta = 68°$. We therefore have

$$\tan\theta = \tan 68° \approx 2.475.$$

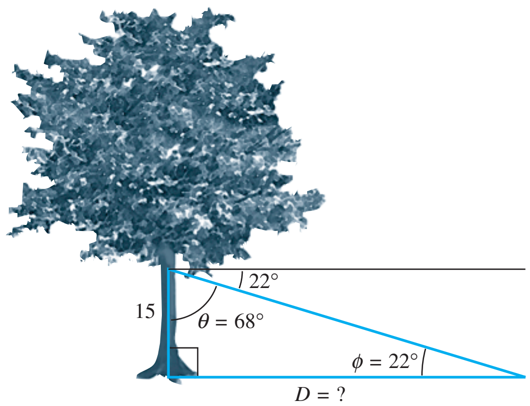


FIGURE 6.14

From the triangle shown in Figure 6.14,

$$\tan\theta = \frac{D}{15} \approx 2.475$$

so that

$$D = 15(2.475) \approx 37.125.$$

Therefore the gorge is about 37 feet across.

Note that if we worked with the angle ϕ instead, we would obtain the same result:

$$\tan\phi = \tan 22° = \frac{15}{D}$$

so that

$$D = \frac{15}{\tan 22°} \approx 37.13.$$

Finding an Angle in a Triangle

We often face the problem of determining an angle θ in a right triangle when we know two of the sides. For example, let the two sides of the right triangle shown in Figure 6.15 be $a = 20$ and $b = 13$, so that

$$\tan\theta = \frac{b}{a} = \frac{13}{20} = 0.65.$$

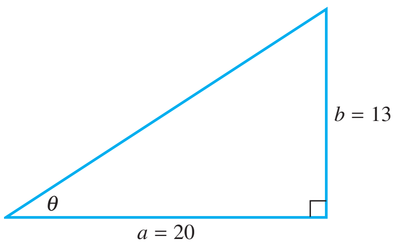


FIGURE 6.15

Suppose that we want to find θ. We know from the table of values we constructed previously for the tangent function that $\tan 30° = 0.577$ and $\tan 40° = 0.839$. Because the values for the tangent are strictly increasing, we expect θ to be between 30° and 40°. We can improve on these rough estimates by trial and error. For instance, using a calculator, we might find that $\tan 35° = 0.7002$ (too high), $\tan 32° = 0.625$ (too low), $\tan 34° = 0.6745$ (slightly too high), and so on.

A far more effective method is to use the inverse of the tangent function, which gives the angle whose tangent has a particular value. (We discuss this inverse function in detail in Section 7.4.) For now, on your calculator simply press either `2nd` or `INV` followed by `TAN` and then the known tangent value. For this example, `INV TAN 0.65` returns 33.024. That is, 33.024° is the angle whose tangent value is 0.65. You can check on your calculator that $\tan 33.024° \approx 0.65$.

The inverse tangent of a number x is usually written as either $\arctan x$ or $\mathrm{Tan}^{-1} x$. We will use the first notation, $\arctan x$.

EXAMPLE 3

A ski slope drops 1500 feet vertically in the process of covering 4300 feet horizontally.

a. What is the angle of inclination of the ski slope?

b. What is the actual distance that a skier will ski down the slope?

Solution

a. We start with a sketch of the ski slope, as shown in Figure 6.16. From geometry, the angle θ equals the angle inside the triangle at the end of the ski run (they are alternate angles between parallel lines). Therefore

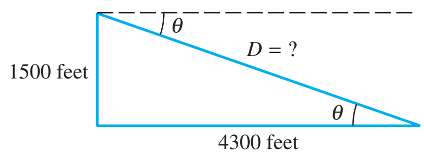


FIGURE 6.16

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{1500}{4300} \approx 0.3488,$$

so that

$$\theta = \arctan(0.3488) = 19.2288,$$

or about 19.2°.

b. The actual distance skied D is simply the length of the hypotenuse. Therefore, from the Pythagorean theorem,

$$D^2 = 1500^2 + 4300^2,$$

so that

$$\sqrt{1500^2 + 4300^2} = \sqrt{20{,}740{,}000} \approx 4554.12,$$

or about 4554 feet.

In general, problems in right angle trigonometry typically involve knowing a small amount of information about a right triangle and using that information intelligently to determine values for the other parts (either the sides or the angles) of the triangle. In fact, there are only a limited number of possibilities. We list these cases (based on the right triangle shown in Figure 6.17) in the following table. We leave the last column for you to complete. Decide on an appropriate strategy for finding each of the missing pieces, based on the information given or previously determined.

Given	Objective	Strategy
a and b	Find c.	
	Find θ.	
a and c	Find b.	
	Find θ.	
b and c	Find a.	
	Find θ.	
a and θ	Find b.	
	Find c.	
b and θ	Find a.	

(continued)

Given	Objective	Strategy
	Find c.	
c and θ	Find a.	Cannot be done simply by using $\tan\theta$.
	Find b.	Cannot be done simply by using $\tan\theta$.

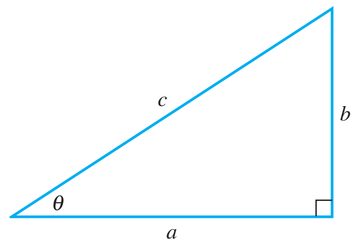


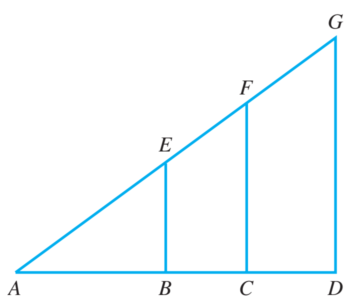
FIGURE 6.17

We examine the last case when c and θ are known—which cannot be solved by using the tangent of an angle—in Section 6.2.

Whenever you face any problem involving a right triangle, your first step should always be to draw a simple picture of the situation to identify the different parts of the triangle and see how they are related. Your drawing will help you determine which strategy, if any, to use to solve for the remaining parts of the triangle.

Problems

1. **a.** Use a ruler to measure, as accurately as possible, the lengths AB, AC, AD, AE, AF, AG, BE, CF, and DG.

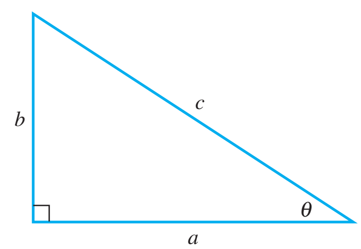


   b. Using your values from part (a), calculate the ratios $\frac{AB}{AE}$, $\frac{AC}{AF}$, $\frac{AD}{AG}$, $\frac{BE}{AB}$, $\frac{BE}{AE}$, $\frac{CF}{AF}$, $\frac{DG}{AG}$, $\frac{CF}{AC}$, and $\frac{DG}{AD}$.

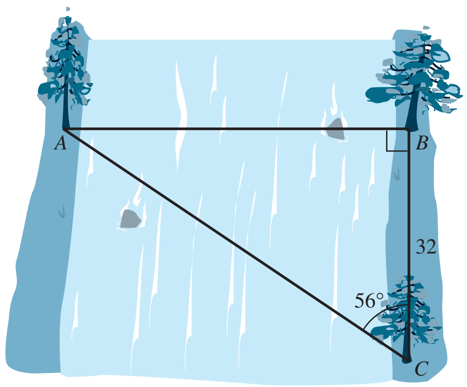
   c. Group the ratios that you calculated in part (b) that appear to be equal. Identify any patterns that help you explain why certain ratios have the same values.

For Problems 2–6, refer to the accompanying figure. Use the information given to find all other parts of the triangle.

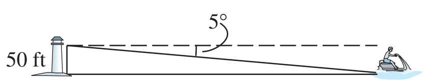
2. $\theta = 52°$ and $b = 12$
3. $\theta = 16°$ and $a = 12$
4. $c = 15$ and $a = 6$
5. $c = 30$ and $b = 18$
6. $a = 72$ and $b = 47$



7. The shadow of a flagpole is 50 feet long. A line of sight from the tip of the shadow to the tip of the pole makes an angle of 28° with the ground. How high is the pole?
8. You want to find the distance across a straight, fast-flowing river. You find two vertical trees that are directly across the river from one another at points A and B so that the angle at B is a right angle, as shown in the accompanying diagram. You then measure a distance of 32 feet to another tree at point C on the edge of the river on your side. From the tree at C, you find that the angle ACB is 56°. Find the distance across the river.

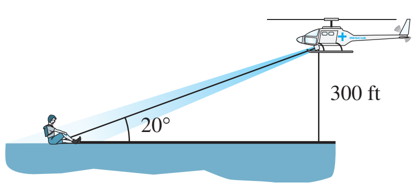


9. The line of sight from the top of a lighthouse to a Jet-Ski out on the water makes an angle of depression of 5° with the horizontal. The lighthouse is 50 feet high.



a. How far is the Jet-Ski from the base of the lighthouse?
b. What is the straight-line distance from the top of the lighthouse to the Jet-Ski?

10. A helicopter is hovering over a particular spot with its searchlight trained on an injured hiker on the ground. Because of tricky wind currents, the pilot can't get the copter any closer to the hiker. The angle that the searchlight makes with the ground is 20°. If the copter pilot estimates that she is at a height of 300 feet above the ground, how far away, horizontally, is the injured hiker?



11. A wheelchair ramp is to be built from ground level to a platform 7 feet above the ground. The angle of inclination with the ground is required to be no greater than 15°.
a. What is the shortest length for a ramp that meets this requirement?
b. How far is the start of the ramp from the base of the platform?

12. Jill is standing at the top of a vertical cliff and Jack is standing 25 feet away from the foot of the cliff and estimates that the angle of elevation θ from his position to Jill's is 40°. Approximately how high is the cliff?

13. Suppose that Jack's measurement in Problem 12 of the distance to the cliff is off by 1 foot. How much difference does this error make in the calculated height of the cliff? (*Hint*: Recalculate your answer to Problem 12 two ways, once for a distance of 24 feet and then for a distance of 26 feet.)

14. Suppose that Jack's estimate of the angle of elevation in Problem 12 is off by 1°. How much difference does this error make in the calculated height of the cliff?

15. Suppose that Jill, at the top of the cliff, wants to find the horizontal distance from the foot of the cliff to where Jack is standing without climbing down and measuring it directly. She drops a rock at the end of a long measuring tape down the cliff and finds that the height of the cliff is about 75 feet. Next, she measures the angle of depression from her position to Jack's to be approximately 70°. How far is Jack from the foot of the cliff?

16. The installation instructions for a TV satellite receiver at a particular location call for it to be aimed at an angle of 68° from the horizontal. Unfortunately, your protractor is broken. Devise a strategy that will help you aim the dish in the proper direction.

17. a. Find the missing entries in the table.

$\tan\theta$	0	0.5	1	1.5	2	2.5	3
θ							

b. Plot the points $(\tan\theta, \theta)$ and connect them with a smooth curve. (This is part of the graph of the function $y = \arctan\theta$.)

6.2 The Sine and Cosine of an Angle

Suppose that you're flying a kite at the end of 400 feet of string and are curious about how high the kite is. How can you find its height? Figure 6.18 shows that the length of string is simply the hypotenuse of the right triangle. You can measure the angle of inclination θ that the kite string makes with the horizontal—say, $\theta = 37°$. Recall that this case is the last one presented in the strategy table in Section 6.1, where we pointed out that the tangent function is of no help. We must devise a different strategy to determine the height y of the kite.

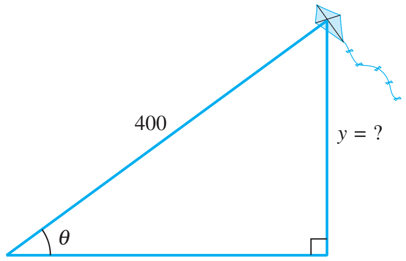


FIGURE 6.18

Using a yardstick, you could measure 5 feet along the kite string and then measure the height from the horizontal to that point on the string; say that you get 3 feet. As shown in Figure 6.19, you have a pair of similar right triangles *ABC* and *ADE*, so you know that their corresponding sides are proportional. Consequently,

$$\frac{\text{height of kite}}{\text{length of hypotenuse}} = \frac{\text{height to point on string}}{\text{length of string to that point}}$$

$$\frac{y}{400} = \frac{3}{5}.$$

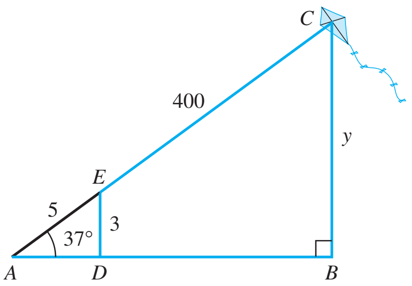


FIGURE 6.19

Therefore the height of the kite is

$$y = 400\left(\frac{3}{5}\right) = 240,$$

or 240 feet above the "horizontal." (If, in fact, you hold the kite string chest-high, say, 4 feet above the ground, the kite is 240 feet above your hand. Hence the kite is actually 244 feet above the ground.)

The Sine of an Angle

The key to solving this problem is to construct the ratio of the height and the hypotenuse of the right triangle. In our discussion of the tangent of an angle, we indicated that, for any angle θ, we can construct infinitely many right triangles that are all similar. Thus the ratio of the height and the hypotenuse will be the same for all these similar triangles, as illustrated in Figure 6.20. Because the ratio changes as the angle changes, this ratio is a function of the angle θ. We define this ratio to be the **sine of the angle,** or the **sine function,** and write it as follows.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

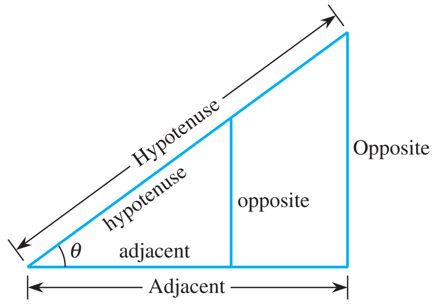


FIGURE 6.20

As with the tangent function, *opposite* refers to the side opposite the angle θ regardless of the orientation of the right triangle. You must think of the opposite side in terms of the angle θ, not as the side of a triangle that is in some particular location, such as the vertical position.

The Behavior of the Sine Function

For now we're concerned only with right triangles, so the domain of the sine function consists of angles between 0° and 90°. The following comments apply only to this situation. In Section 6.3, we consider cases for which this restriction is lifted, leading to more interesting and useful behavior patterns for the sine function.

As with the tangent ratio, you can get the values for the sine of any angle in a right triangle using your calculator in `Degree` mode. For instance,

$$\sin 10^\circ = 0.174,$$
$$\sin 20^\circ = 0.342,$$
$$\sin 30^\circ = 0.5,$$
$$\sin 40^\circ = 0.643,$$
$$\sin 75^\circ = 0.966.$$

Note that the values for the sine function $y = \sin\theta$ are increasing as the angle θ increases from 0° to 90°. Further, the sine function grows more rapidly for small angles and less rapidly as angles get closer to 90°, so that these values follow a concave down pattern. Figure 6.21 shows a graph of the sine function for θ between 0° and 90°. (We discuss the sine's behavior for angles outside this interval in Section 6.3.)

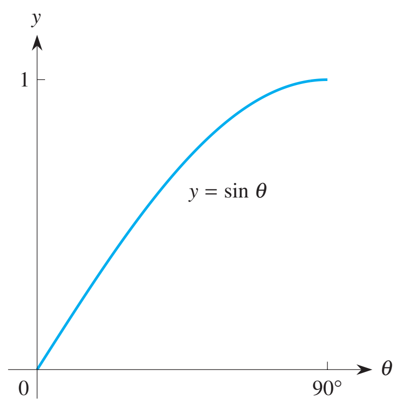


FIGURE 6.21

The Sine of Some "Special" Angles

As we did with the tangent function in Section 6.1, let's consider some of the special angles, notably $\theta = 0°, 30°, 45°, 60°,$ and $90°$, from the point of view of the sine function. To begin, think about what happens in a right triangle as the angle shrinks to 0° for a fixed hypotenuse c. (Imagine the kite nosediving toward the ground at the end of the taut string as shown in Figure 6.22.) The length of the opposite side also decreases to 0, so

$$\sin 0° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{0}{c} = 0.$$

What about sin 90°? Again, for a fixed hypotenuse c, think about what happens in a right triangle as the angle increases to 90°. (Although improbable, imagine the kite moving directly overhead so that the height of the kite becomes equal to the length of the string, as illustrated in Figure 6.23.) The length of the opposite side in the triangle grows until it approaches the length of the hypotenuse, so

$$\sin 90° = \frac{c}{c} = 1.$$

Verify these two facts on your calculator.

Now let's look at the other special angles. As shown in Figure 6.24, when $\theta = 45°$, we have a right triangle with two angles of 45° and the two corresponding sides of equal length, say a. Recall that, by the Pythagorean theorem, the length of

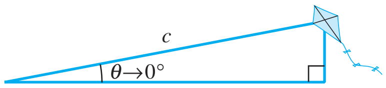


FIGURE 6.22

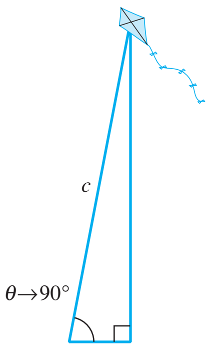


FIGURE 6.23

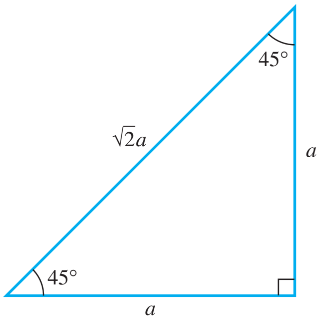


FIGURE 6.24

the hypotenuse is $\sqrt{2}a$. Hence

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} \approx 0.707.$$

Similarly, as shown in Figure 6.25, when $\theta = 30°$, the remaining angle is 60°. Recall that, in any 30°–60°–90° right triangle, the length of the side opposite the 30° angle is half the length of the hypotenuse. If the hypotenuse has length $2a$, the opposite side has length a. Consequently,

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{2a} = 0.5.$$

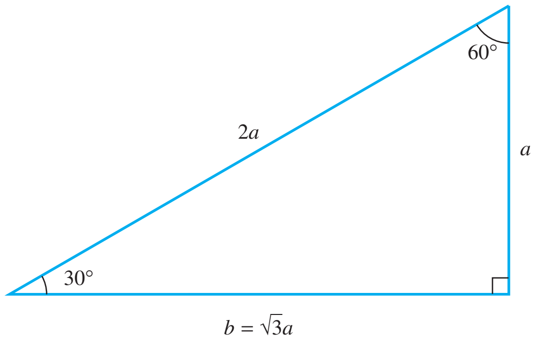


FIGURE 6.25

By the Pythagorean theorem, if the remaining side has length b, then

$$b^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2,$$

so $b = \sqrt{3}a$. We therefore have

$$\sin 60° = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2} \approx 0.866,$$

as we found previously by using a calculator.

We summarize these findings as follows

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	0.5	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{\sqrt{3}}{2} \approx 0.866$	1

Note that the values for the sine of any angle in a right triangle must always lie between 0 and 1. The reason is that the sine is the ratio of the opposite side and the hypotenuse, and in any right triangle the hypotenuse is always the longest side.

Applications of the Sine Function

We apply the sine function in Examples 1–3 to illustrate its use in solving different types of everyday problems.

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EXAMPLE 1

A highway through the mountains has a stretch that drops at a grade of 5°. If you drive a distance of 12 miles along this road, how far do you descend vertically?

Solution To help visualize the situation, we "straighten out" all curves in the road and sketch the situation, as shown in Figure 6.26, which is not to scale. Note that a 5° grade also can be thought of as a 5° angle of descent or a 5° angle of declination. We know that the length of the hypotenuse is 12 miles. We let y be the vertical drop, and get

$$\sin 5° = \frac{y}{12} \quad \text{so that} \quad y = 12(\sin 5°) \approx 1.05.$$

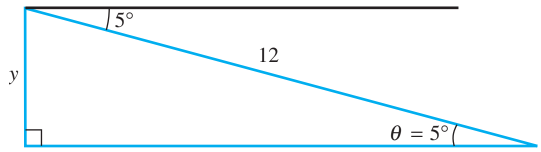


FIGURE 6.26

Consequently, along this stretch of highway, the road drops about 1.05 miles, or about 5544 feet.

EXAMPLE 2

A tall tree has been uprooted during a storm. It is tilted over and supported near its top by a vertical wall, as shown in Figure 6.27. The actual horizontal distance from the tree's roots to the wall is 42 feet and the angle of elevation of the tree is estimated to be 35°.

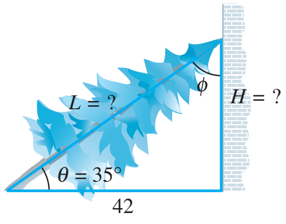


FIGURE 6.27

a. Estimate the length of the tree.

b. Estimate how high on the wall the top of the tree is lodged.

Solution

a. Note that

$$\sin \theta = \sin 35° = \frac{H}{L},$$

which involves two unknowns; thus we cannot solve the equation. Instead, we must work with the remaining angle ϕ in the triangle, which is $\phi = 90° - 35° = 55°$. Therefore we have

$$\sin 55° = \frac{42}{L},$$

so that

$$L = \frac{42}{\sin 55°} \approx 51.27 \text{ feet.}$$

(Note that we could also have used the tangent of 35° to determine the height H of the triangle and then used the Pythagorean theorem to find the length of the hypotenuse.)

b. We now use the Pythagorean theorem to find the height of the triangle:

$$H^2 = 51.27^2 - 42^2 = 864.613,$$

so that

$$H = 29.4 \text{ feet.}$$

Think About This Suppose that the estimate of the angle in Example 2 is off by 5°, either high or low. How much difference would this error make in the answers to parts (a) and (b) of Example 2.

Often, we face the problem of determining an angle when we know the value of the sine of that angle. For instance, if the hypotenuse of a right triangle is 20 and the side opposite the angle θ is 15, as shown in Figure 6.28, then $\sin \theta = 0.75$. What is the angle θ? We could find it by trial and error (we know that $\sin 45° = 0.707$ and $\sin 60° = 0.866$, so we might try 50°, and so on). A far more effective approach is to use the *inverse sine function,* which gives the angle whose sine has a particular value. We write this inverse function as $\arcsin x$, for any given value x (although $\sin^{-1} x$ is also used). We discuss the inverse sine function in detail in Section 7.3. For now, with your calculator, simply press either `2nd` or `INV`, followed by `SIN`, and then the known value of the sine function—say, 0.75: the calculator returns 48.590. To verify that it is the correct angle, we check that

$$\sin 48.59 = 0.749996 \approx 0.75.$$

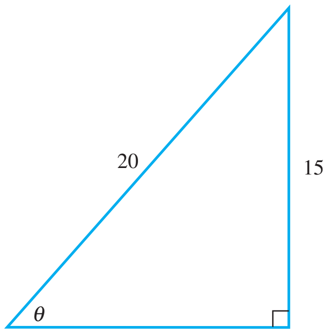


FIGURE 6.28

EXAMPLE 3

A 12-foot-long ladder is leaning against a wall. If the foot of the ladder is 4 feet from the wall, what is the angle of inclination of the ladder?

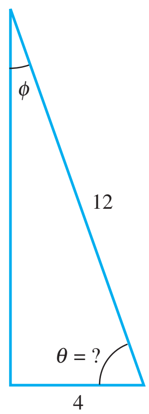


FIGURE 6.29

Solution We start with a sketch of the situation, as shown in Figure 6.29. To use the sine function, we have to consider the point of view of the angle ϕ, so

$$\sin \phi = \frac{4}{12} = \frac{1}{3}$$

and therefore

$$\phi = \arcsin \frac{1}{3} = 19.47°.$$

Hence the angle of inclination of the ladder is

$$\theta = 90° - 19.47° = 70.53°.$$

In Section 6.1, we asked you to complete a table outlining strategies for solving for all the parts of a right triangle given various combinations of sides and angles. In all but one of those cases, you could determine all the other parts of the triangle by using the tangent. Now we ask you to complete the table again by deciding on appropriate strategies to determine the parts of a right triangle by using the sine instead of the tangent. Refer to Figure 6.30.

Given	Objective	Strategy
a and b	Find c.	
	Find θ.	
a and c	Find b.	
	Find θ.	
b and c	Find a.	
	Find θ.	
a and θ	Find b.	
	Find c.	
b and θ	Find a.	
	Find c.	
c and θ	Find a.	
	Find b.	

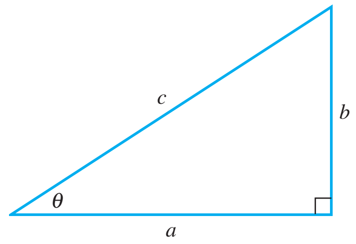


FIGURE 6.30

In practice, which function you apply doesn't matter so long as you use a correct strategy. Thus for most of the cases, a variety of different approaches will give the correct answers. Incidentally, together the tangent function and the sine function allow you to solve for all the parts of any right triangle in all six cases.

The Cosine of an Angle

So far, we've considered two of the six possible ratios among the sides of a right triangle:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \quad \text{and} \quad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

One other ratio is very useful: the ratio of the adjacent side and the hypotenuse, which also depends only on the angle θ. We now define a third trigonometric function, the **cosine of an angle,** or the **cosine function,** as follows.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

The Behavior of the Cosine Function

As with the sine and the tangent, you can get the values for the cosine of any angle in a right triangle using your calculator. For instance,

$$\begin{aligned}\cos 10° &\approx 0.985,\\ \cos 20° &\approx 0.940,\\ \cos 30° &\approx 0.866,\\ \cos 40° &\approx 0.766,\\ \cos 50° &\approx 0.643.\end{aligned}$$

These values are decreasing more slowly for small angles and more rapidly as angles get closer to 90°. Therefore the values for the cosine function decrease in a concave down pattern for angles between 0° and 90°. Figure 6.31 shows a graph of the cosine function $y = \cos\theta$ for θ between 0° and 90°. Note that it starts at a height of 1 and decreases toward 0 in a concave down pattern as θ approaches 90°.

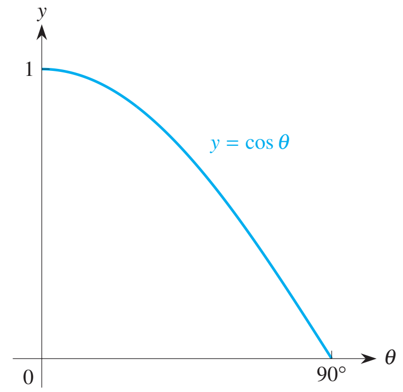


FIGURE 6.31

The Cosine of Some “Special” Angles

Again, let’s consider the special angles $\theta = 0°, 30°, 45°, 60°$, and $90°$. To begin, what is cos 0°? Think about a right triangle in which the hypotenuse remains constant and the angle θ shrinks to 0°. (Imagine again the kite as it nosedives toward the ground on a windy day so that the string remains taut, as shown in Figure 6.32.) The hypotenuse gets closer and closer to the adjacent side, so

$$\cos 0° = \frac{\text{adjacent}}{\text{hypotenuse}} = 1.$$

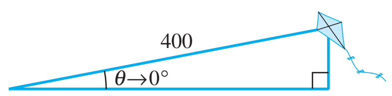


FIGURE 6.32

Similarly, think about a right triangle in which the hypotenuse remains fixed and the angle approaches 90°. (Imagine the kite moving directly overhead, as shown in Figure 6.33.) The adjacent side gets closer to 0, so

$$\cos 90° = 0.$$

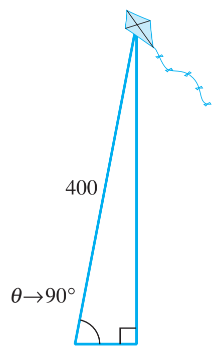


FIGURE 6.33

Next, let's look at the other special angles. As shown in Figure 6.34, when $\theta = 45°$,

$$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} \approx 0.707.$$

Think About This

This result is the same value we found for the sine of 45°. Explain why they are the same.

Also, when $\theta = 30°$, Figure 6.35 shows that

$$\cos 30° = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2} \approx 0.866,$$

which is the same as sin 60°. Similarly,

$$\cos 60° = \frac{a}{2a} = \frac{1}{2},$$

which is the same as sin 30°.

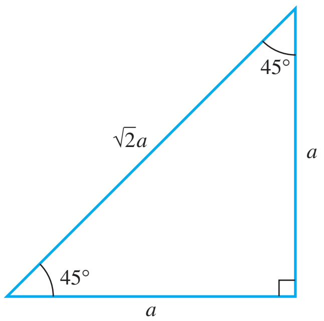


FIGURE 6.34

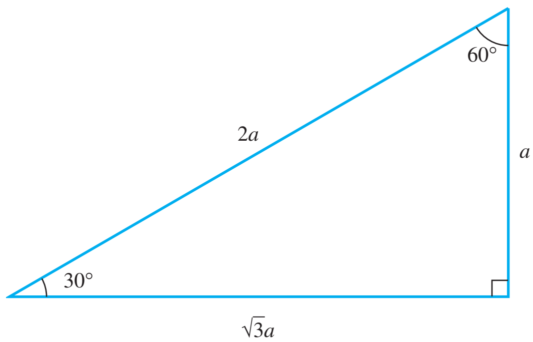


FIGURE 6.35

We summarize these key values for the cosine function as follows.

θ	0°	30°	45°	60°	90°
$\cos\theta$	1	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{\sqrt{2}} \approx 0.707$	0.5	1

Think About This

Explain why $\cos 30° = \sin 60°$ and $\cos 60° = \sin 30°$.

Applications of the Cosine Function

We use the cosine function in Examples 4 and 5 to illustrate its value in solving a couple of rather simple problems.

EXAMPLE 4

To get onto a straight water slide at an amusement park requires climbing a flight of steps 60 feet high. The slide itself is inclined downward at a 42° angle. How long is the actual slide?

Solution Figure 6.36 indicates that the angle in the right triangle is 48° and that the adjacent side is 60 feet long. Therefore, to find the length L of the slide, we use

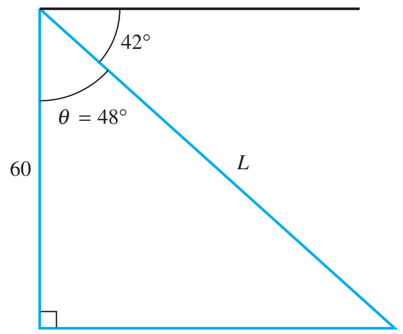


FIGURE 6.36

$$\cos 48° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{60}{L}.$$

Thus

$$L = \frac{60}{\cos 48°} \approx 89.67.$$

So the slide is almost 90 feet long.

Think About This Can you solve Example 4 by using the sine function instead? the tangent function?

EXAMPLE 5

A 30 foot ramp extends 24 feet horizontally.

a. What is the angle of elevation of the ramp?

b. How high does the ramp extend?

Solution

a. We start with a sketch of the situation, as shown in Figure 6.37. The hypotenuse has length 30 feet and the base (which is the adjacent side from the point of view of the unknown angle θ) is 24 feet. Therefore

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{24}{30}.$$

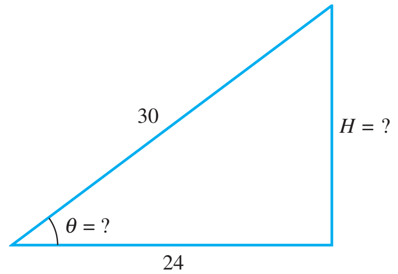


FIGURE 6.37

To find the angle θ, we undo the cosine using the inverse cosine (see Section 7.3), so that

$$\theta = \arccos\left(\frac{24}{30}\right) = 36.87°,$$

or about 37°.

b. We can solve for the height H in a variety of ways. Probably the simplest is to use the Pythagorean theorem, which gives

$$H^2 = 30^2 - 24^2 = 324,$$

so that

$$H = 18 \text{ feet.}$$

Applications from Physics

The trigonometric functions arise frequently in applications of the physical sciences. Many physical quantities, such as force and velocity, involve both a direction and a size. Such quantities are known as *vectors,* and we look at them more formally in Section 10.1. For now, we consider some physical applications informally to illustrate the use of trigonometry.

Imagine pushing against a window that is stuck in order to open it. You exert a certain force, but the effect of that force depends on the angle θ at which you exert it. If the angle is primarily vertical, most of the effect of your effort is applied to push the window upward, as shown in Figure 6.38(a). If the angle is more horizontal, as shown in Figure 6.38(b), only a small portion of your effort is applied to moving the window upward while most of your effort is wasted in the horizontal direction, effectively pushing the window outward. The total force exerted can be broken into two parts, one *horizontal* and the other *vertical,* based on the angle at which the force is applied. We illustrate this principle in Examples 6–8.

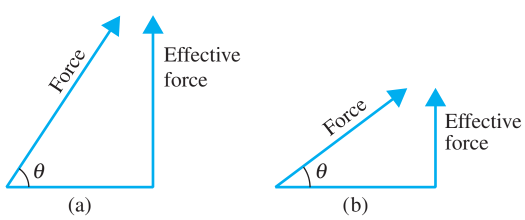


FIGURE 6.38

EXAMPLE 6

A 30 pound force is exerted at an angle of 20° with the vertical to push a stuck window upward. Find the effective value of the force actually exerted to move the window vertically.

Solution We begin with a sketch of the situation, as shown in Figure 6.39, where the force being exerted is represented by the hypotenuse of the right triangle. We let the lengths of the sides equal the sizes of the forces. Thus the hypotenuse has length 30. The portion F of the force effectively applied to move the window vertically upward is the vertical side of this triangle. From the point of view of the 20° angle, the effective force is exerted along the adjacent side of the triangle, which suggests using the cosine function. In particular,

$$\cos 20° = \frac{F}{30} = 0.9397$$

so that

$$F = 30 \cos 20° = 28.19,$$

or slightly more than 28 pounds of the 30 pounds of the force is applied to moving the window.

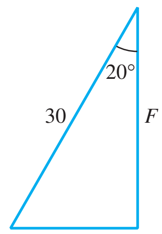


FIGURE 6.39

EXAMPLE 7

A sailboat is out on a still lake where the wind is blowing at a speed of 16 mph from the northeast, as shown in Figure 6.40. How fast is the sailboat moving toward the west? toward the south?

Solution Because the wind is blowing from the northeast, the angle it makes with the horizontal is 45°. The wind is actually pushing the sailboat toward the southwest at 16 mph, as represented by the hypotenuse of the right triangle shown in Figure 6.40. We want to find the speed w in the westward direction, as indicated by the horizontal side of the triangle, and the speed s in the southward direction, as indicated by the vertical side of the triangle.

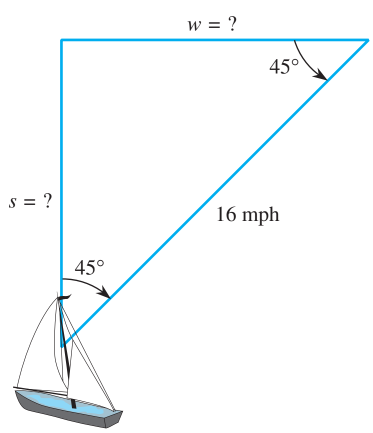


FIGURE 6.40

Let's first determine the sailboat's speed toward the west. Because the side opposite that angle is the unknown and we have the hypotenuse, we use the sine function. Thus

$$\sin 45° = \frac{w}{16}$$

so that

$$w = 16 \sin 45° \approx 11.314.$$

That is, the sailboat is moving at slightly more than 11 mph toward the west.

To find the speed toward the south, we simply observe that, because the angle in this right triangle is 45°, the two sides are equal and so $s \approx 11.314$ also. Thus the sailboat is also moving at slightly more than 11 mph toward the south.

Suppose that the wind is not quite blowing from the northeast but from some angle other than 45°. Example 8 demonstrates how the solution changes accordingly.

EXAMPLE 8

A sailboat is out on a still lake where the wind is blowing at a speed of 16 mph from the northeasterly direction of 32° east of north, as shown in Figure 6.41. How fast is the sailboat moving toward the west? toward the south?

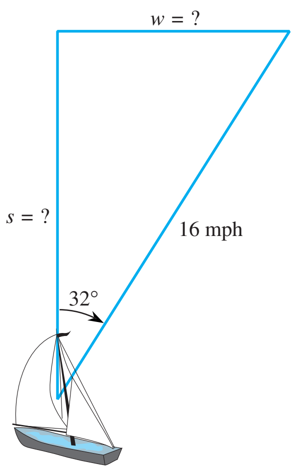


FIGURE 6.41

Solution As in Example 7, the wind is actually pushing the sailboat in a southwesterly direction at 16 mph, as represented by the hypotenuse in the right triangle in Figure 6.41. But now the angle is 32° instead of 45°. We again indicate the westward speed w along the horizontal side of the triangle and the southward speed s along the vertical side of the triangle.

We first determine the speed toward the west. Using the sine function, we have

$$\sin 32° = \frac{w}{16}$$

so that

$$w = 16 \sin 32° = 8.479.$$

That is, the sailboat is moving at about $8\frac{1}{2}$ mph toward the west.

To find the speed toward the south, we need to determine the remaining side of the right triangle. We do so by using the Pythagorean theorem:

$$s^2 = 16^2 - w^2 = 16^2 - 8.479^2 = 184.107$$

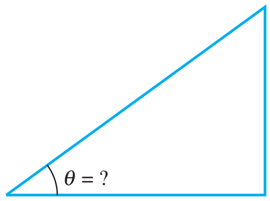
so that

$$s \approx 13.569.$$

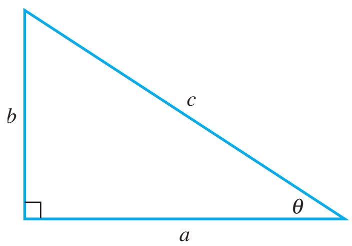
Thus the sailboat is moving at about $13\frac{1}{2}$ mph toward the south.

Problems

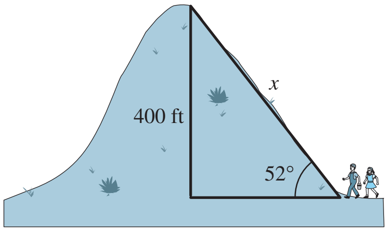
1. Use a ruler to measure the three sides of the triangle shown. Based on the measurements, what are your best estimates for $\sin\theta$, $\cos\theta$, and $\tan\theta$? What is your estimate for the angle θ?



Problems 2–7 refer to the accompanying figure. Use the information given to find all other parts of the triangle.

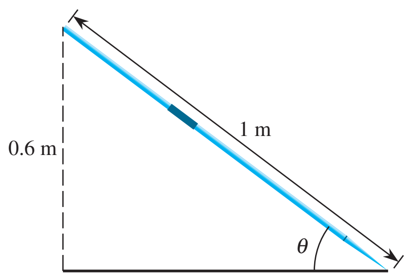


2. $\theta = 52°$ and $b = 15$
3. $\theta = 16°$ and $c = 12$
4. $c = 22$ and $a = 16$
5. $c = 30$ and $b = 8$
6. $a = 12$ and $b = 9$
7. $a = 42$ and $\theta = 72°$
8. A road up a hill is inclined at 11° to the horizontal. A driver starts driving up this hill and, by checking the odometer, discovers that the steep portion of the road extends for three-quarters of a mile. How much has the car gained in altitude?
9. With its radar, an aircraft spots another aircraft 10,000 feet away at an angle of depression of 15°. Find the horizontal distance from one aircraft to the other.
10. As a pendulum of length 21 inches swings back and forth, the maximum angle it makes from the vertical is $\theta = 18°$. What is the greatest height that the end of the pendulum reaches compared to its lowest height when it passes the vertical?
11. From takeoff, an airplane reaches a height of 2 miles (10,560 feet) in the process of covering 20 miles horizontally.
    - **a.** Find the average angle of ascent of the airplane as it climbs.
    - **b.** Is the actual path upward of the airplane a straight line, or is the path curved in a concave up pattern or in a concave down pattern? Explain your reasoning.
    - **c.** If the airplane were to climb along a straight-line path, find the distance it would travel as it goes from the ground to the 2-mile height. Is the distance that the airplane actually travels greater than or less than the distance you calculated? Why?
12. When the space shuttle comes in for a landing at Cape Canaveral, its descent to the ground for the final 10,000 feet of height is at an angle of 19° with the horizontal.
    - **a.** What actual distance does the shuttle traverse along this final glide path?
    - **b.** How far from touchdown, horizontally, should the shuttle be when it passes the 10,000-foot altitude?
13. Jack and Jill are about to climb a 400-foot-high hill. If the angle of ascent is 52° from the horizontal, what is the actual distance they will cover to reach the summit on a straight track?



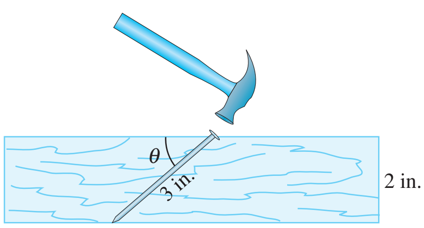
14. When Jack and Jill came tumbling down from the top of the hill in Problem 13, their angle of descent was 61° from the horizontal. What is the actual distance they covered while tumbling down?

15. A javelin is 1 meter long. When it lands after being thrown, its base is 0.6 meters (= 60 cm) vertically above the ground. What angle does the javelin make?



16. Problem 15 is unrealistic because the point of the javelin is going to be embedded in the ground. Suppose that 92 cm of the javelin is visible above the ground, and that its base is still 60 cm vertically above ground level. What angle does the javelin make with the ground?

17. You must hammer a 3-inch-long nail into a piece of wood 2 inches thick. Find the steepest angle at which you can hammer the nail all the way into the wood without it coming out the opposite side.



18. When an airplane takes off, it climbs at an angle of 16° at a speed of 180 feet per second. How high is the plane after 1 second? after 2 seconds?

19. The cranberry sauce to go with your holiday turkey comes out of a can and has a diameter of 3 inches. When you slice the roll of cranberry sauce at an angle, most of the slices will be ellipses with a minor axis of 3 inches. Suppose that you slice the roll at an angle of 27° to the vertical. Find the length of the major axis of each elliptical slice. (See Appendix A7; ellipses and their properties are covered in detail in Section 9.3.)

20. An escalator rises at a 26° angle with the horizontal. If it rises 28 feet vertically, what is its length?

21. A safety regulation limits the maximum angle of inclination for the ladder on a fire truck to 72°. If a hook-and-ladder fire truck has a ladder that can extend to a length of 90 feet, what is the maximum height that it can reach?

22. A balloonist is trying to cross the Atlantic Ocean. If the wind is blowing at 40 mph from the northwest, what is the actual airspeed at which the balloon is traveling eastward toward Europe?

23. The wind in Problem 22 now shifts slightly and increases in speed so that it is now blowing at 50 mph from 40° north of west. What is the actual airspeed at which the balloon is moving eastward?

24. **a.** Find the missing entries in the table.

$\sin\theta$	0	0.2	0.4	0.6	0.8	1
θ						

b. Plot the points ($\sin\theta$, θ) and connect them with a smooth curve.

c. This curve is part of the graph of what function?

25. **a.** Find the missing entries in the table using only your answers to Problem 24.

$\cos\theta$	1	0.8	0.6	0.4	0.2	0
θ						

b. Plot the points ($\cos\theta$, θ) and connect them with a smooth curve.

c. This curve is part of the graph of what function?

6.3 The Sine, Cosine, and Tangent in General

So far, we've considered the trigonometric functions only for angles in a right triangle—that is, angles between 0° and 90°. However, we often encounter situations in which we need to consider angles larger than 90°. A natural question is: How do we adapt the ideas previously discussed to such cases? Before we address that question, let's choose such an angle—say, $\theta = 125°$—and find out what happens when we use a calculator. We get

$$\sin 125° = 0.819, \qquad \cos 125° = -0.574 \quad \text{and} \quad \tan 125° = -1.428.$$

Let's see how these values are defined and why they have the indicated signs.

Angles Between 90° and 180°

Consider any angle θ between 90° and 180°; imagine it with one side, its **initial side,** along the positive x-axis and the other side, its **terminal side,** in the second quadrant, as depicted in Figure 6.42. By convention, we measure such an angle starting along the positive x-axis and rotating counterclockwise. The terminal side forms the hypotenuse of a right triangle in the second quadrant when we drop a vertical line from the terminal side to the x-axis. Suppose that the hypotenuse of this right triangle is h, the length of the vertical side is y, and the length of the horizontal side is x. By convention, because y extends up from the horizontal axis, we think of it as positive. Because x extends to the left of the vertical axis, we think of it as negative. By convention, the hypotenuse is always considered positive. The angle ϕ in this right triangle is the *supplement* of the angle θ because $\theta + \phi = 180°$; thus $\theta = 180° - \phi$, or $\phi = 180° - \theta$. The angle ϕ is sometimes called the *reference angle.*

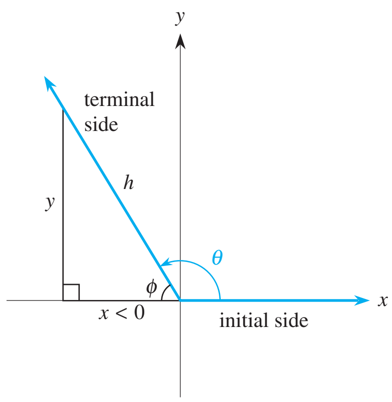


FIGURE 6.42

We define the trigonometric functions when the angle θ is between 90° and 180° in terms of the comparable values for the angle ϕ. Therefore

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{h},$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{h}, \qquad x < 0$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}, \qquad x < 0.$$

As previously mentioned, for any angle θ between 90° and 180° with its terminal side in the second quadrant, x is negative and y and h are positive. Thus the cosine and tangent of that angle are negative, whereas the sine is positive, as we saw with $\sin 125° = 0.819$, $\cos 125° = -0.574$, and $\tan 125° = -1.428$.

If $90° < \theta < 180°$,

$$\sin\theta > 0,$$
$$\cos\theta < 0,$$
$$\tan\theta < 0.$$

Angles Between 180° and 270°

What about an angle θ between 180° and 270° whose terminal side is in the third quadrant, as depicted in Figure 6.43. We construct a right triangle in the third quadrant by drawing a vertical line from the terminal side to the x-axis that determines a reference angle ϕ. Now both the x- and y-values are negative, and $\theta = 180° + \phi$. As before, we define the trigonometric functions for θ in terms of this reference angle ϕ by using the appropriate lengths in the right triangle so that

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{h}, \qquad y < 0$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{h}, \qquad x < 0$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}, \qquad x < 0 \quad \text{and} \quad y < 0.$$

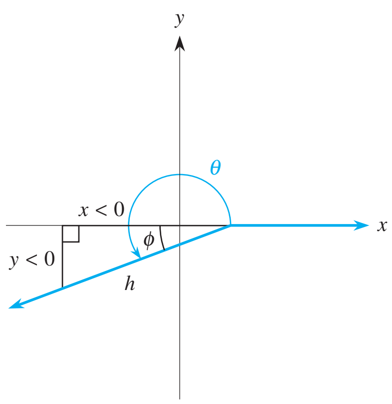


FIGURE 6.43

For any angle between 180° and 270°, the values of x and y are negative, so the sine and cosine are negative. However, for these angles, the tangent is positive because it is the quotient of two negative quantities.

If $180° < \theta < 270°$,

$$\sin\theta < 0,$$
$$\cos\theta < 0,$$
$$\tan\theta > 0.$$

Think About This

Suppose that $\theta = 211°$ so that $\phi = 31°$. Use your calculator to find the values for sin 211°, cos 211°, and tan 211°. How do they compare with sin 31°, cos 31°, and tan 31°?

Angles Between 270° and 360°

Next, consider an angle θ between 270° and 360° whose terminal side is in the fourth quadrant, as shown in Figure 6.44. Once more, we construct a right triangle by drawing a vertical line from the terminal side to the x-axis. We define each of the trigonometric functions in terms of the reference angle ϕ in that triangle. Now the y-value is negative, the x-value is positive, and $\theta + \phi = 360°$, so $\theta = 360° - \phi$. Also,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{h}, \qquad y < 0$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{h},$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}, \qquad y < 0.$$

Therefore, because y is negative, for any angle between 270° and 360°, the cosine is positive and the sine and the tangent are both negative.

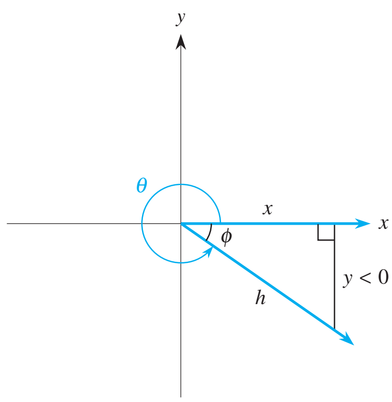


FIGURE 6.44

If $270° < \theta < 360°$,

$$\sin\theta < 0,$$
$$\cos\theta > 0,$$
$$\tan\theta < 0.$$

Angles Greater than 360°

What happens if θ is greater than 360°—say, 410°? As shown in Figure 6.45, we can construct such an angle by looping around a full 360° and then an additional 50°; essentially, this angle is equivalent to an angle of $\phi = 50°$ in the first quadrant. Using a calculator, we find that

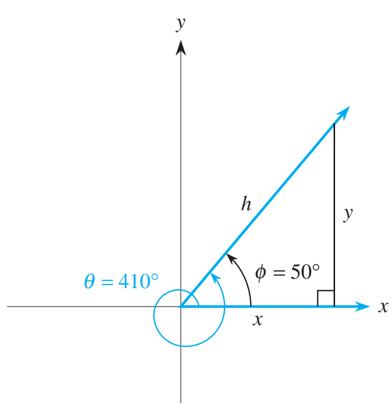


FIGURE 6.45

$$\sin 410° = 0.766 = \sin 50°,$$
$$\cos 410° = 0.643 = \cos 50°,$$
$$\tan 410° = 1.192 = \tan 50°.$$

Similarly, if $\theta = 775°$, we make two full rotations (accounting for $2 \times 360° = 720°$), which leaves an angle $\phi = 55°$, and so

$$\sin 775° = 0.819 = \sin 55°,$$
$$\cos 775° = 0.574 = \cos 55°,$$
$$\tan 775° = 1.428 = \tan 55°.$$

Note that the values for the three trigonometric functions repeat every 360°. Therefore they are **periodic functions** because their behavior repeats. The smallest interval over which the pattern repeats is called the **period.** The periods of the sine and cosine functions are both 360°. In general, for any angle θ, we have the following.

$$\sin(\theta + 360°) = \sin \theta$$
$$\cos(\theta + 360°) = \cos \theta$$

However, the period of the tangent function is 180° because its values repeat every 180°. Thus, for any angle θ, we have the following.

$$\tan(\theta + 180°) = \tan \theta$$

Check these identities on your calculator either numerically with a variety of different values for θ or graphically by comparing the graphs of $y = \tan x$ and $y = \tan(x + 180)$.

Angles Less Than 0°

Finally consider a negative angle—say, $\theta = -30°$—drawn clockwise, as shown in Figure 6.46. This angle is equivalent to a positive angle of 330° because both angles have the same terminal side. Note that y is negative and that x is positive. We therefore have

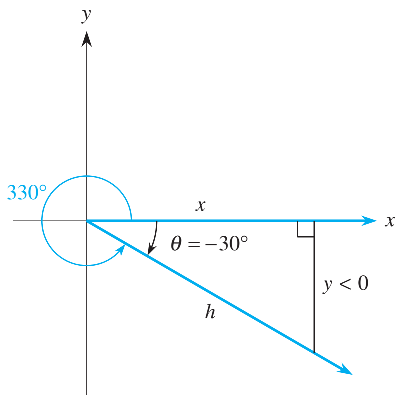


FIGURE 6.46

$$\sin\theta = \frac{y}{h}, \qquad y < 0$$
$$\cos\theta = \frac{x}{h},$$
$$\tan\theta = \frac{y}{x}, \qquad y < 0,$$

as we have already discussed for angles in the fourth quadrant.

Figure 6.47 summarizes the information about the signs of the three trigonometric functions, based on the quadrant containing the terminal side.

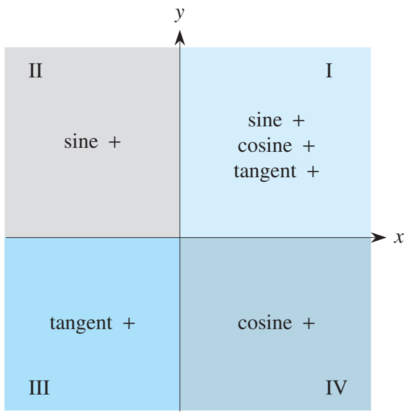


FIGURE 6.47

The Graph of the Sine Function

Let's summarize what we already know about the sine function to determine its overall behavior pattern. For θ between 0° and 90°, $y = \sin\theta$ increases from 0 to 1. For θ between 90° and 180°, $y = \sin\theta$ decreases from 1 to 0. For θ between 180° and 270°, $y = \sin\theta$ continues to decrease from 0 to −1. For θ between 270° and 360°, $y = \sin\theta$ increases from −1 to 0. Thus the sine function has a maximum value of 1 and a minimum value of −1. This oscillatory pattern continues indefinitely in both directions (for $\theta > 360°$ and for $\theta < 0°$). Use your function grapher with θ between −500° and 500°, say, to observe this pattern.

Think About This Give a similar summary for the behavior of the cosine function.

You can visualize the behavior of the trigonometric functions by looking at their graphs. Figure 6.48(a) shows the graph of the function $y = \sin\theta$ for θ between 0° and 360° and how the graph relates to the signs of $\sin\theta$ in the four quadrants shown in Figure 6.47. In Figure 6.48(b) we expand the graph of $y = \sin\theta$ to show its behavior between −360° and 720°. This portion of the curve consists of three full *cycles,* or repetitions, of the *basic sine curve* that occurs between 0° and 360°, which is one full period of the function. Also, the curve oscillates between a minimum height of $y = -1$ and a maximum height of $y = 1$. In particular, the sine curve reaches its maximum when

$$\theta = \ldots, -270°, 90°, 450°, \ldots,$$

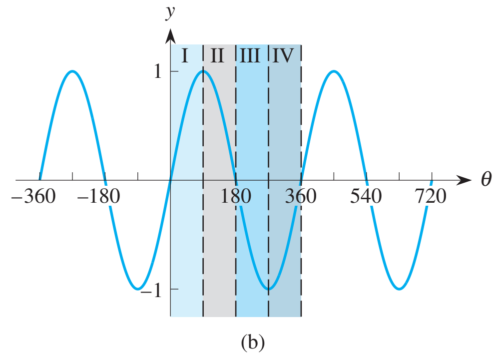


FIGURE 6.48

and it reaches its minimum when

$$\theta = \ldots, -90°, 270°, 630°, \ldots$$

In addition, the sine curve is concave down between $\theta = 0°$ and $\theta = 180°$ and is concave up from $\theta = 180°$ to $\theta = 360°$ and again in every other cycle.

You should carefully distinguish between the information shown in Figure 6.47 regarding the *signs* of the sine function in different quadrants and what happens in the *graph* of the sine function shown in Figure 6.48(a). The quadrants referred to in Figure 6.47 are based on a coordinate system with y versus x, and values for the angle θ are measured by rotating the terminal side of the angle. These are not the same quadrants shown in Figure 6.48(a) because that graph shows y as a function of θ, with θ measured horizontally as it takes on values in the different quadrants. In particular, angles in the first quadrant in Figure 6.47 correspond to the portion of the θ-axis in Figure 6.48(a) between $\theta = 0°$ and $\theta = 90°$; the second quadrant in Figure 6.47 corresponds to the portion of the θ-axis between $\theta = 90°$ and $\theta = 180°$ in Figure 6.48(a); and so on. We have marked these differences in Figures 6.48(a) and (b) with Roman numerals and corresponding shadings for the different quadrants to help make the point. Be sure that you understand these subtle differences before going on.

The Graph of the Cosine Function

Figure 6.49 shows the graph of the cosine function $y = \cos\theta$ from $\theta = -360°$ to $\theta = 720°$. Use the graph to answer the following questions: Where is the cosine function increasing? Where is it decreasing? What are its maximum and minimum values? Where do they occur? Where is the cosine function concave up? Where is it concave down? Also, be sure that you understand how the information shown in Figure 6.47 on the sign of the cosine function in the different quadrants relates to the behavior depicted in the graph of the cosine function.

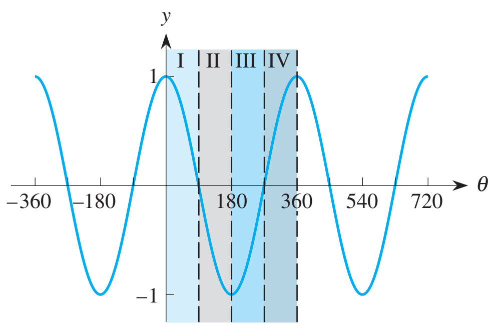


FIGURE 6.49

Problems

In Problems 1–12, find the value of each quantity by using only the information in the table. Do not use the trigonometric function keys on your calculator. (*Hint:* Start by drawing a picture of each angle.)

θ	0°	30°	45°	60°
sin θ	0	0.5	0.707	0.866
cos θ	1	0.866	0.707	0.5
tan θ	0	0.577	1	1.732

1. sin 225°
2. cos 210°
3. tan 135°
4. sin 150°
5. sin 330°
6. cos 390°
7. cos 315°
8. tan 240°
9. cos 840°
10. sin (−450°)
11. cos (−240°)
12. tan (−225°)

Decide whether each quantity is positive, negative, or zero without calculating its value. Give a reason for your answer.

13. sin 300°
14. cos 450°
15. cos 240°
16. sin 210°
17. tan 300°
18. tan 450°
19. cos 270°
20. sin 225°
21. sin 215°
22. cos 320°
23. cos 520°
24. sin 885°
25. tan 925°
26. sin 1000°
27. cos 1000°
28. tan 1000°
29. sin (−480°)
30. cos (−500°)
31. tan (−500°)
32. sin (−1000°)
33. Consider the function $f(x) = \sin x \cos x$.
    a. Determine the sign of $f(x)$ for x between 0° and 90°. between 90° and 180°. between 450° and 540°.
    b. For what values of x between 0° and 540° is $f(x) = 0$?
    c. Use the results of part (a) to sketch a rough graph of $f(x)$.
    d. Does the function appear to be periodic? If so, what is its period?
34. **a.** Find the missing entries in the table below.
    b. Plot the points $(\sin\theta, \theta)$ and connect them with a smooth curve.
    c. This curve is the graph of what function?

sin θ	−1	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	1
θ									

35. **a.** Find the missing entries in the table below using only your answers to Problem 34.
    b. Plot the points $(\cos\theta, \theta)$ and connect them with a smooth curve.
    c. This curve is the graph of what function?

cos θ	1	0.75	0.5	0.25	0	−0.25	−0.5	−0.75	−1
θ									

6.4 Relationships among Trigonometric Functions

Consider the following values for the sine and cosine of the special angles between 0° and 90°.

θ	0°	30°	45°	60°	90°
sin θ	0	0.5	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{\sqrt{3}}{2} \approx 0.866$	0
cos θ	1	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{\sqrt{2}} \approx 0.707$	0.5	1

Comparing the values of the sine and cosine, you will notice that the values associated with 0° and 90° are reversed, as are the values for 30° and 60°, so that $\sin 30° = \cos 60°$ and $\sin 60° = \cos 30°$. This is no coincidence. In general, for any angle θ,

$$\cos\theta = \sin(90° - \theta) \quad \text{and} \quad \sin\theta = \cos(90° - \theta).$$

Why are these two relationships true? Consider the right triangle shown in Figure 6.50. Side a, opposite angle θ, is the side adjacent to the angle $90° - \theta$. Similarly, side b, adjacent to angle θ, is the side opposite angle $90° - \theta$. That is, their roles are reversed, depending on which angle, θ or $90° - \theta$, you consider.

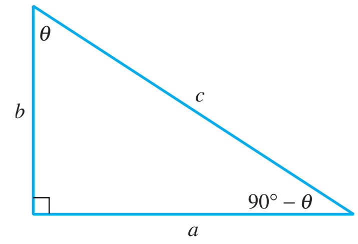


FIGURE 6.50

One reason why the trigonometric functions are so useful is that there are many interrelationships among them, as the two preceding formulas demonstrate. They are called **identities** because they hold for every possible value of the variable θ. But other identities involving relationships between the trigonometric functions are far more important. Let's consider again the right triangle shown in Figure 6.50 and the definitions of the sine and cosine:

$$\sin\theta = \frac{a}{c} \quad \text{and} \quad \cos\theta = \frac{b}{c}.$$

If we multiply both sides of these equations by c, we obtain

$$a = c\sin\theta \quad \text{and} \quad b = c\cos\theta.$$

We substitute these expressions into the formula for the tangent function to get

$$\tan\theta = \frac{a}{b} = \frac{c\sin\theta}{c\cos\theta},$$

provided that $\cos\theta \neq 0$. Therefore, for any such angle θ, we have the following identity.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \text{if } \cos\theta \neq 0$$

For instance, if $\theta = 74°$, $\sin 74° = 0.9613$, $\cos 74° = 0.2756$, and

$$\tan 74° = 3.4874 = \frac{\sin 74°}{\cos 74°} \approx 3.488.$$

If we use more than four digits for sin 74° and cos 74°, the result would be more accurate.

Next let's apply the Pythagorean theorem to the right triangle shown in Figure 6.50:

$$a^2 + b^2 = c^2.$$

When we substitute $a = c \sin\theta$ and $b = c\cos\theta$, we have

$$(c\sin\theta)^2 + (c\cos\theta)^2 = c^2$$

or

$$c^2(\sin\theta)^2 + c^2(\cos\theta)^2 = c^2.$$

Dividing both sides by c^2 (which is not 0) gives

$$(\sin\theta)^2 + (\cos\theta)^2 = 1,$$

which holds for any angle θ. For convenience, it is customary to write

$$(\sin\theta)^2 = \sin^2\theta \quad \text{and} \quad (\cos\theta)^2 = \cos^2\theta$$

and we have the following identity.

The Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

Check this result on your calculator by using different values for θ. Be careful to enter the expressions as `(SIN X)^2` and `(COS X)^2`. For instance, if $\theta = 74°$ again, we have

$$\sin^2(74°) + \cos^2(74°) = (0.9613)^2 + (0.2756)^2 = 1.000053.$$

The discrepancy is due to rounding errors. If we use more digits in sin 74° and cos 74°, the result would be even closer to 1.

Let's now start with the Pythagorean identity and divide both sides by $\cos^2\theta$. We then get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta},$$

or, equivalently, because $\sin\theta/\cos\theta = \tan\theta$,

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}.$$

This relationship is more commonly written in the following form.

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

We summarize the definitions and special relationships among the three trigonometric functions as follows.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

We investigate many other relationships between these three functions in Section 8.1.

EXAMPLE 1

Suppose that the SIN and TAN keys on your calculator are broken. You can use the COS key to find that $\cos 20° = 0.940$. Determine the values for $\sin 20°$ and $\tan 20°$.

Solution We illustrate three different approaches to solving this problem.

Method 1

Using the Pythagorean relationship,

$$\sin^2\theta + \cos^2\theta = 1,$$

we find that

$$\sin^2\theta = 1 - \cos^2\theta$$

for any angle θ. Therefore, when $\theta = 20°$,

$$\sin^2(20°) = 1 - \cos^2(20°) = 1 - (0.940)^2 = 0.1170.$$

We take the square root of both sides to find

$$\sin 20° = \sqrt{0.1170} = 0.342.$$

Further, we have

$$\tan 20° = \frac{\sin 20°}{\cos 20°} = \frac{0.342}{0.940} = 0.364.$$

Method 2

Figure 6.51 shows that $\sin 20° = b/c$. However, $\cos 70° = b/c$ also, and we can use the "broken" calculator to find $\cos 70° = 0.342$. Therefore $\sin 20° = 0.342$ also. Knowing $\sin 20°$ and $\cos 20°$, we now can find $\tan 20° = 0.364$, as we did in Method 1.

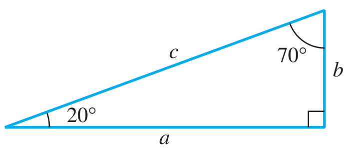


FIGURE 6.51

Method 3

We know that

$$\cos 20° = 0.940 = \frac{\text{adjacent}}{\text{hypotenuse}},$$

so from the triangle shown in Figure 6.52, the ratio a/c must be 0.940. Thus we can assume, for instance, that $a = 94$ and $c = 100$. (There are infinitely many other possibilities; another is $a = 0.940$ and $c = 1$.) Consequently, using the Pythagorean theorem, we can find the third side b:

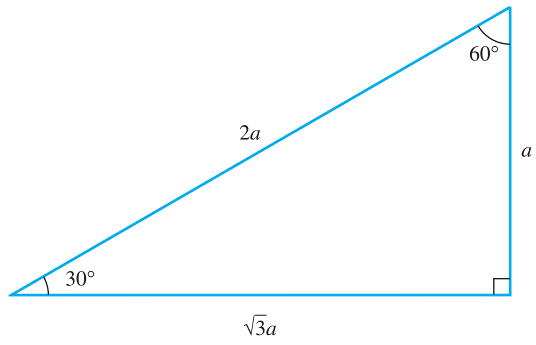


FIGURE 6.52

$$b^2 = c^2 - a^2 = 100^2 - 94^2 = 1164;$$
$$b = \sqrt{1164} = 34.117.$$

As a result, we have

$$\sin 20° = \frac{b}{c} = \frac{34.117}{100} \approx 0.341;$$
$$\tan 20° = \frac{b}{a} = \frac{34.117}{94} \approx 0.363.$$

Both of these values differ slightly from the results in Methods 1 and 2 because of rounding.

EXAMPLE 2

Suppose that the `SIN` and `COS` keys on your calculator are broken. Using the `TAN` key, you find that $\tan 25° = 0.466$. Find the sine and cosine of this angle **(a)** by using trigonometric identities and **(b)** by constructing an appropriate triangle.

Solution

a. We have the relationship

$$1 + \tan^2(25°) = \frac{1}{\cos^2(25°)}$$

so that

$$1 + (0.466)^2 = 1.2172 = \frac{1}{\cos^2(25°)}.$$

Consequently,

$$\cos^2(25°) = \frac{1}{1.2172} = 0.8216.$$

Taking the positive square root, we have

$$\cos 25° = 0.906.$$

Furthermore, because $\cos^2(25°) = 0.8216$, we use the Pythagorean identity to find that

$$\sin^2(25°) = 1 - \cos^2(25°) = 0.178.$$

Taking the positive square root, we have

$$\sin 25° = 0.422.$$

b. Because

$$\tan 25° = \frac{\text{opposite}}{\text{adjacent}} = 0.466$$

we can construct a right triangle in which the length of the side opposite the 25° angle is 466, say, and the adjacent side is 1000, as shown in Figure 6.53. To find the hypotenuse H of this triangle, we have

$$H^2 = 466^2 + 1000^2 = 1{,}217{,}156$$

so that, when we take the positive square root,

$$H \approx 1103.25.$$

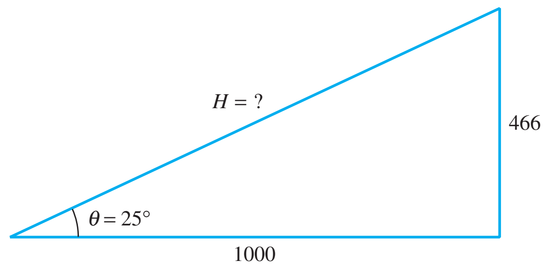


FIGURE 6.53

For this triangle, we now have the desired values

$$\sin 25° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{466}{1103.25} = 0.422$$

and

$$\cos 25° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1000}{1103.25} = 0.906.$$

EXAMPLE 3

Simplify the expression $\sin^3 x + \sin x \cos^2 x$ by using one of the trigonometric identities.

Solution We first factor out the common factor of $\sin x$ to get

$$\sin^3 x + \sin x \cos^2 x = \sin x(\sin^2 x + \cos^2 x).$$

Using the Pythagorean identity yields

$$\sin^3 x + \sin x \cos^2 x = \sin x \cdot (1) = \sin x.$$

Think About This Verify graphically that the given expression $\sin^3 x + \sin x \cos^2 x$ and the final expression $\sin x$ in Example 3 are identically equal for all values of x by graphing both.

Problems

1. Suppose that the COS and TAN keys on your calculator are broken. You can use your SIN key to find that, for some angle θ in the first quadrant, $\sin\theta = 0.3$. Determine the values for $\cos\theta$ and $\tan\theta$. What is the angle θ?
2. Suppose that the SIN and TAN keys on your calculator are broken. You can use your COS key to find that, for some angle θ in the first quadrant, $\cos\theta = 0.4$. Determine the values for $\sin\theta$ and $\tan\theta$ algebraically. What is the angle θ?
3. Suppose that, for a certain angle θ in the first quadrant, $\sin\theta = 0.6$. Using paper and pencil only, find the cosine and tangent of θ.
4. Suppose that, for a certain angle θ in the first quadrant, $\cos\theta = 0.6$. Using paper and pencil only, find the sine and tangent of θ.
5. Suppose that, for a certain angle θ in the first quadrant, $\tan\theta = \frac{3}{4}$. Using paper and pencil only, find the sine and cosine of θ.
6. Suppose that, for a certain angle θ in the first quadrant, $\tan\theta = 1.2$. Find the cosine and sine of θ algebraically.
7. Suppose that, for a certain angle in the second quadrant, $\sin\theta = 0.52$. Find the cosine and tangent of θ algebraically.
8. Suppose that, for a certain angle in the third quadrant, $\tan\theta = 0.75$. Find the cosine and sine of θ algebraically.
9. Suppose that, for a certain angle in the fourth quadrant, $\sin\theta = -0.7$. Find the cosine and tangent of θ algebraically.
10. Simplify the expression $\sin^2 x \cos x + \cos^3 x$ by using one of the trigonometric identities.
11. Consider the two equations:

    i. $\dfrac{\tan x}{\cos x} = \sin x$

    ii. $\dfrac{\sin x}{\tan x} = \cos x$

    a. Determine graphically which of these equations represents an identity that is true for every value of x, except for those points where the denominator is 0, and which is not an identity.

    b. Prove algebraically, using trigonometric identities, that the identity is indeed true.

    c. For the equation that is not an identity, find two different values of x that satisfy the equation.

Exercising Your Algebra Skills

Use appropriate trigonometric identities to simplify each expression.

1. $\cos x \tan x$
2. $(1 - \sin x)(1 + \sin x)$
3. $(1 - \cos x)(1 + \cos x)$
4. $(\sin\theta + \cos\theta)^2$
5. $(\sin\theta - \cos\theta)^2$
6. $\cos^3 x + \sin^2 x \cos x$
7. $\cos x + \tan^2 x \cos x$
8. $\tan^2\theta - \dfrac{1}{\cos^2\theta}$
9. $\left(1 - \dfrac{1}{\cos x}\right)\left(1 + \dfrac{1}{\cos x}\right)$

6.5 The Law of Sines and the Law of Cosines

When we first introduced trigonometry, our original development was presented in terms of angles in right triangles. We subsequently extended the definitions of the sine, cosine, and tangent to angles larger than 90°. We now consider some additional properties of the sine and cosine in any triangle, not just in a right triangle.

The Law of Sines

Consider the triangle *ABC* shown in Figure 6.54 where the sides opposite the angles *A*, *B*, and *C* are denoted by *a*, *b*, and *c*, respectively. All three angles are acute; that is, each is less than 90°. Later we consider the case where one angle is greater than 90°.

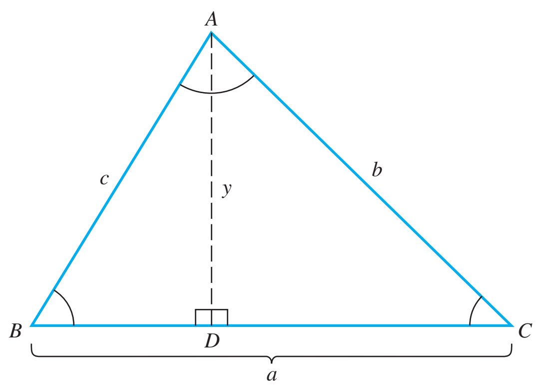


FIGURE 6.54

Suppose that we drop a perpendicular from the vertex at angle *A* to point *D* on base *a*. That perpendicular line *AD*, whose length we call *y*, produces two right triangles. In triangle *ABD* we have

$$\sin B = \frac{y}{c} \quad \text{so that} \quad y = c \sin B.$$

Similarly, in triangle *ACD* we have

$$\sin C = \frac{y}{b} \quad \text{so that} \quad y = b \sin C.$$

These two expressions for *y* must be equal, so

$$y = c \sin B = b \sin C,$$

and therefore

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

However, we could just as easily have drawn a perpendicular from the vertex at angle *B* to the opposite side *b*. In that case, using the same reasoning, we get

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Together, these results yield

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

for any triangle with three acute angles.

What about a triangle with an angle greater than 90°? Consider the one shown in Figure 6.55. We can still drop a perpendicular of length *y* from the vertex at angle *A* to point *D* on an extension of side *a*, as shown. This line forms two right triangles. Clearly, in the large right triangle *ACD*,

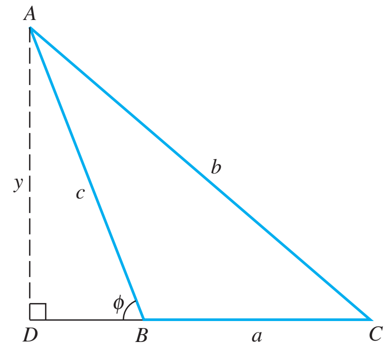


FIGURE 6.55

$$\sin C = \frac{y}{b} \quad \text{so that} \quad y = b \sin C.$$

To determine the sine of angle *B*, we use the angle ϕ in the smaller right triangle *ABD*. We thus find that

$$\sin B = \frac{y}{c} \quad \text{so that} \quad y = c \sin B.$$

Consequently,

$$y = c \sin B = b \sin C,$$

so that again we have

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

We can similarly drop a perpendicular either from the vertex at angle *B* onto side *b* or from the vertex at angle *C* onto an extension of side *c* and obtain a similar relationship involving

$$\frac{\sin A}{a}.$$

What we have just proved is called the **law of sines.**

The Law of Sines

In any triangle,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The law of sines can be used to find all the remaining sides and angles in any triangle if two sides and one angle are known or one side and two angles are known, provided that the known combination of sides and angles includes one angle and the side opposite it.

Think About This

Draw a triangle in which two sides and one angle are known and the law of sines will not apply. Then draw a triangle in which one side and two angles are known and the law of sines will not apply.

We illustrate use of the law of sines in Example 1.

EXAMPLE 1

The Federal Communications Commission (FCC) is attempting to locate a pirate radio station by a method called *triangulation.* The FCC set up two monitoring stations 30 miles apart on an east–west line and took simultaneous readings on the direction of the radio signal. The westernmost monitor measured the signal as coming from a direction 42° north of east; the other monitor measured the signal as coming from a point 56° north of west. Where is the pirate station located?

Solution The information recorded determines the triangle shown in Figure 6.56. The two monitoring stations are located 30 miles apart at the points A and B. The signal directions determine the angles of 42° and 56°. The pirate is located at point C. Hence the angle at C must be $180° - 42° - 56° = 82°$.

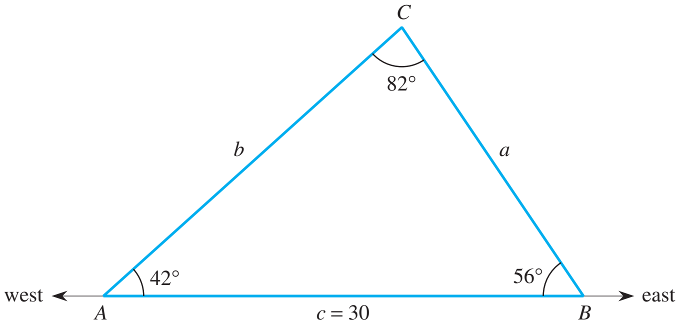


FIGURE 6.56

We now apply the law of sines to find the lengths of sides a and b. Using angles $A = 42°$ and $C = 82°$, we find

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin 42°}{a} = \frac{\sin 82°}{30}$$

so that

$$a = \frac{30 \sin 42°}{\sin 82°} \approx 20.27.$$

Similarly, to find b we apply the law of sines, using the angles B and C, to get

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin 56°}{b} = \frac{\sin 82°}{30}$$

so that

$$b = \frac{30 \sin 56°}{\sin 82°} \approx 25.12.$$

Therefore the pirate station is located 25.12 miles from station A in a direction of 42° toward the northeast and 20.27 miles from station B in a direction of 56° toward the northwest. The point C is determined precisely by these two facts.

In Example 1 we used the law of sines when two angles and one side of a triangle are known. The law of sines can also be used when two sides (say, a and b) and the angle opposite one of these sides (either A or B) are known. However, depending on the sizes of the two known sides, it is possible to obtain either a unique answer or two distinct configurations for the triangle. This *ambiguous case* occurs when we try to find the angle from its sine. Recall that there will be two angles—one less than 90° and the other greater than 90°—that both have the same sine value. We ask you to explore possible ambiguous cases in the Problems at the end of this section.

Another complication may arise when we're using the law of sines if we know two sides and the angle opposite one of them. If in the midst of such a set of calculations, we obtain a sine value greater than 1, it indicates that the values we're working with could not have come from a real triangle. Again, you will encounter such a case in the Problems at the end of this section.

The Law of Cosines

There is one useful relationship involving the cosine function that relates the three sides of any triangle and any one of its three angles. Consider the triangle shown in Figure 6.57 with sides a, b, and c, where the angle opposite side c is C. The sides and angle are related by the **law of cosines.**

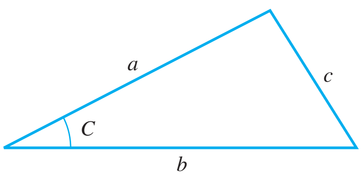


FIGURE 6.57

Law of Cosines

In any triangle,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note that the triangle need not be a right triangle; the law of cosines applies to any triangle. The law of cosines allows us to determine (1) the length of the side opposite a known angle if the other two sides are known, or (2) any angle if the three sides of the triangle are known.

We prove this formula for the case where the triangle has three acute angles; a similar argument applies if one of the angles is greater than 90°. Also, to make things easier we assume that one of the vertices is at the origin and that one of the sides of the triangle lies on the x-axis, as shown in Figure 6.58. Note that the coordinates of the point P are $(a, 0)$ and that the coordinates of the point Q are at $x = b\cos C$ and $y = b\sin C$. As a result, the length of side c is just the distance from P to Q and we can find it by using the usual formula for the distance between two points (see Appendix A5):

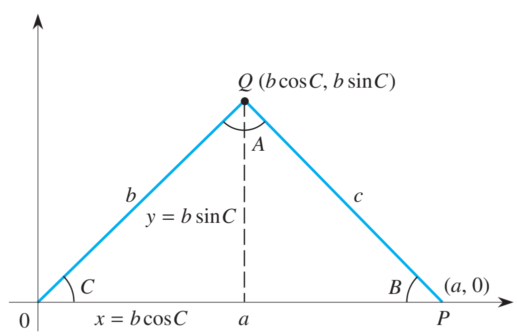


FIGURE 6.58

$$c = \text{distance from } P \text{ to } Q = \sqrt{(b\cos C - a)^2 + (b\sin C)^2}.$$

We square both sides to eliminate the square root and obtain

$$\begin{aligned} c^2 &= (b\cos C - a)^2 + (b\sin C)^2 \\ &= (b^2\cos^2 C - 2ab\cos C + a^2) + b^2\sin^2 C \\ &= (b^2\cos^2 C + b^2\sin^2 C) + a^2 - 2ab\cos C \\ &= b^2(\cos^2 C + \sin^2 C) + a^2 - 2ab\cos C \\ &= b^2 + a^2 - 2ab\cos C && \text{Pythagorean identity} \\ &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

If we have a right triangle in which angle $C = 90°$, then $\cos C = 0$ and the law of cosines reduces to the Pythagorean theorem $c^2 = a^2 + b^2$.

We use the law of cosines in Examples 2–4.

EXAMPLE 2

Let ABC be a triangle with sides $a = 5$ and $b = 7$, and let the angle C between the two sides a and b be 60°.

a. Find the third side c.

b. Find the other two angles A and B.

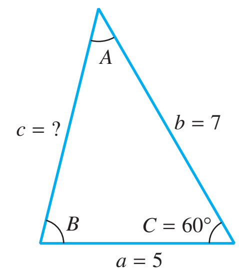


FIGURE 6.59

Solution

a. We begin with Figure 6.59. Using the law of cosines, we have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 5^2 + 7^2 - 2(5)(7)\cos 60° \\ &= 25 + 49 - 70(\tfrac{1}{2}) = 39. \end{aligned}$$

Thus the third side is $c = \sqrt{39} \approx 6.245$.

b. To find the angle A, we use the law of sines:

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{5} = \frac{\sin 60°}{6.245}.$$

Therefore

$$\sin A = \frac{5\sin 60°}{6.245} \approx 0.6934,$$

so that

$$A = \arcsin(0.6934) = 43.9°.$$

Consequently,

$$B = 180° - 60° - 43.9° = 76.1°$$

EXAMPLE 3

In a standard baseball infield, the four bases are at the corners of a square whose sides are 90 feet in length. The pitcher's mound is 60 feet, 6 inches, or 60.5 feet, from home plate on a line through second base, as illustrated in Figure 6.60. The distance from the pitcher's mound to second base is about 67 feet.

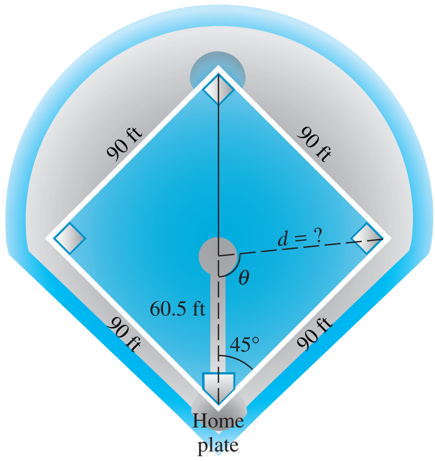


FIGURE 6.60

a. How far is the pitcher's mound from first base?

b. What is the angle at the pitcher's mound between home plate and first base?

Solution

a. Note that the angle at home plate between the mound and first base is 45°, so we know the angle opposite the unknown length d. Therefore, using the law of cosines, we have

$$d^2 = 60.5^2 + 90^2 - 2(60.5)(90)\cos 45°$$
$$= 3660.25 + 8100 - 2(60.5)(90)\cos 45° \approx 4059.86.$$

When we take the square root of both sides, we find that

$$d \approx 63.72 \text{ feet.}$$

b. Note that the angle θ at the pitcher's mound between home plate and first base must be more than 90°. We use the law of sines:

$$\frac{\sin\theta}{90} = \frac{\sin 45°}{d}$$

so that

$$\sin\theta = \frac{90\sin 45°}{63.72} \approx 0.9987.$$

Therefore, using the inverse sine function, we get $\theta = \arcsin(0.9987) = 87.08°$. Because this is less than 90°, it must be the angle at the pitcher's mound between first base and second base. The desired angle is the supplement, $180° - 87.08° = 92.92°$.

In Section 6.2, we considered some physical examples in which a force or a velocity could be broken into two parts, one horizontal and the other vertical. At the time, we were limited to particularly simple examples where, for instance, a boat was out on a still lake with the wind blowing, but there was no mention of a current. We now look at a more complicated situation.

EXAMPLE 4

The pilot of a small plane is flying due east at its top speed of 200 mph. The wind is blowing out of the northwest at a speed of 40 mph. The wind pushes the plane in a direction south of east and increases its airspeed to more than the 200 mph, as illustrated in Figure 6.61.

a. What is the actual airspeed of the plane due to its own engines and the wind?

b. What is the actual direction that the plane flies?

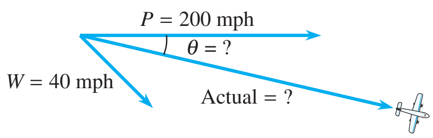


FIGURE 6.61

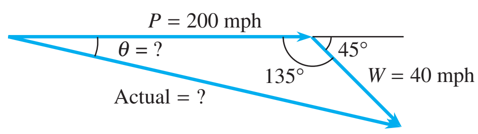


FIGURE 6.62

Solution

a. We first consider the speeds. The plane itself contributes a horizontal airspeed of $P = 200$ mph. The wind comes from the northwest at $W = 40$ mph, so the associated angle inside the triangle in Figure 6.62 is $180° - 45° = 135°$. The actual airspeed s of the plane is the length of the remaining side in the triangle, and we find it by using the law of cosines:

$$s^2 = P^2 + W^2 - 2PW\cos 135°$$
$$= 200^2 + 40^2 - 2(200)(40)(-0.707) = 52{,}912.$$

Taking the positive square root yields

$$s \approx 230.026,$$

or about 230 mph.

b. To find the angle θ in the triangle, we use the law of sines:

$$\frac{\sin\theta}{40} = \frac{\sin 135°}{230},$$

so that

$$\sin\theta = \frac{40\sin 135°}{230} \approx 0.1230.$$

Therefore

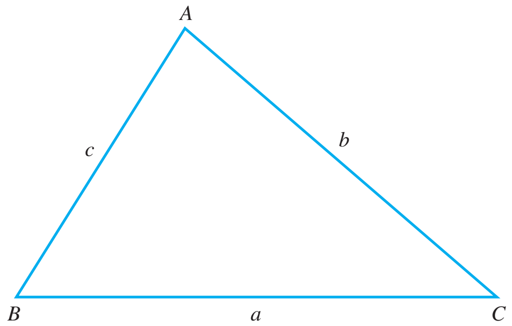
$$\theta = \arcsin(0.1230) = 7.07°,$$

or the plane is actually flying about 7° south of east.

Problems

For Problems 1–6, refer to the notation for the sides and angles in the accompanying figure. Use the information given to find all other parts of the triangle.

1. $A = 26°, B = 63°, b = 12$
2. $A = 47°, C = 72°, c = 60$
3. $A = 35°, B = 65°, c = 24$



4. $A = 40°$, $a = 10$, $b = 6$ (*Hint*: Is it possible to have two different values for B?)

5. $A = 40°$, $a = 10$, $b = 12$ (*Hint*: Is it possible to have two different values for B?)

6. $A = 40°$, $a = 10$, $b = 18$

7. Two ships at sea are 50 miles apart on a north–south line when they both receive an SOS signal from a third ship in trouble. One ship receives the SOS from a direction of 41° north of east. The other ship receives the signal from a direction of 54° south of east. Where is the third ship?

8. You want to find the distance across a fast-flowing river. You pick two large trees, at points A and B, that are 35 feet apart along the edge of the river on your side. You then spot another tree on the opposite side of the river at point C. The angle CAB at point A is 43°; the angle CBA at point B is 52°. Find the distance across the river.

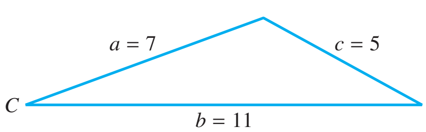
9. Problems 4–6 involved three cases in which the law of sines works very differently because of the relative sizes of sides a and b. Based on those results, explain the following statements.

a. Given a value for angle A, there will always be one triangle whenever $b < a$.

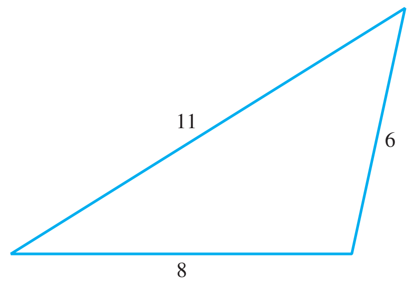
b. There will be two different possible triangles whenever b is somewhat larger than a.

c. There will be no triangle whenever b is much larger than a.

10. Find the angle C if $a = 7$, $b = 11$, and $c = 5$, as shown in the accompanying figure.

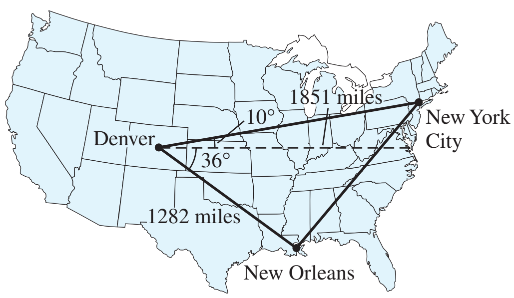


11. Find the angle opposite the side of a triangle whose length is 6 if the lengths of the other two sides are 11 and 8, as shown in the accompanying figure.



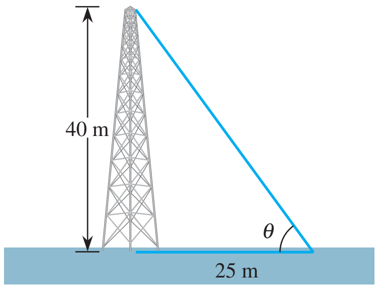
12. A TV camera is positioned 100 feet behind home plate. The center fielder is standing 300 feet from home plate and directly on the line from home plate through second base. The batter hits a long fly ball toward right-center field that the center fielder catches against the wall, 380 feet from home plate. If the TV camera had to pan through an angle of 8° in following the center fielder from the point where he was standing to the point where he made the spectacular catch, how far did he have to run?

13. Using a map of the United States (which ignores the fact that the earth is round), New York City is 1851 miles from Denver and about 10° north of east from Denver. New Orleans is 1282 miles from Denver and about 36° south of east from Denver. Estimate the distance from New York City to New Orleans.

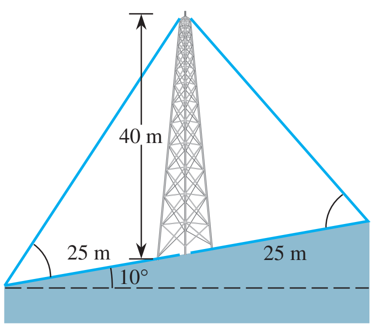


14. Chicago is 695 miles from Atlanta, and Seattle is 2756 miles from Atlanta. The same map of the United States as in Problem 13 shows that Chicago lies at an angle of about 65° north of west from Atlanta and that Seattle lies at an angle of about 29° north of west from Atlanta. Estimate the distance from Chicago to Seattle.

15. A 40 meter vertical tower is to be built and supported by several guy wires anchored to the ground 25 meters from the base of the tower on flat land. Find the length of the guy wires and the angle they make with the ground.



16. The 40 meter tower in Problem 15 is to be built on the side of a hill that slopes upward at an angle of 10° from the horizontal. One guy wire will be positioned directly uphill from the tower and another will be positioned directly downhill from the tower. Each guy wire will be anchored 25 meters from the base of the tower. Find the lengths of the two guy wires and the angles they make with the sloping ground.



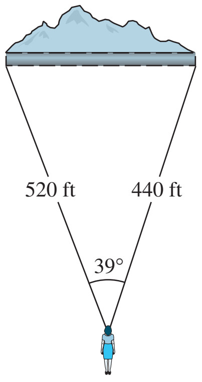
17. A communications satellite passes directly over Phoenix and Los Angeles, which are 340 miles apart. At some point in its orbit when the satellite is between Phoenix and Los Angeles, its angle of elevation from Phoenix is 52° and its angle of elevation from Los Angles is 72°.

a. How far is the satellite from Phoenix at that moment?
b. How high is the satellite above the Earth?

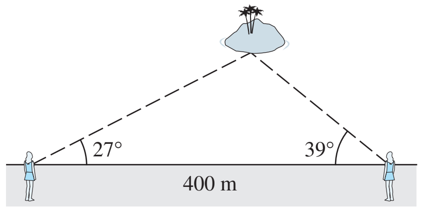
18. Al and Bob are driving toward a moored hot air balloon from opposite sides of the balloon and are in contact via cell phones. When they are 4 miles apart, they both take sightings on the balloon. (Of course, everyone who is about to take a balloon ride carries a protractor.) From Al's position, the angle of elevation to the balloon is 28°; from Bob's position, the angle of elevation is 35°.

a. How high is the balloon?
b. How far is each of them from where the balloon is moored?

19. A straight tunnel passes through a mountain. An observer has a clear view of the two ends of the tunnel. The distance from her position to the tunnel entrance toward her left is 520 meters, and the distance to the entrance toward her right is 440 meters. If the angle subtended by the two tunnel entrances is 39°, how long is the tunnel?



20. Meryl and Bernice are walking along a straight beach when they observe a small island in the distance and wonder if they can swim out to it. To estimate the distance, they separate and walk 400 meters apart. From Meryl's perspective, the angle to the island is 27° and from Bernice's perspective the angle is 39°. How far is the island from the shore?



Problems 21 and 22 refer to Figure 6.57 in the text.

21. Find c if $a = 5$, $b = 3$, and $C = 20°$.

22. Find c if $a = 7$, $b = 4$, and $C = 25°$.

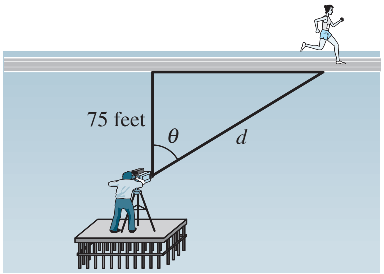
Chapter Summary

In this chapter we introduced some of the fundamental ideas and applications of trigonometry as they apply to right triangles. In particular, we discussed:

- The definition of the tangent ratio in terms of the sides of a right triangle.
- How to use the tangent ratio to solve problems involving right triangles.
- The graph of the tangent function between 0° and 90°.
- The definition of the sine and cosine ratios in terms of the sides of a right triangle.
- How to use the sine and cosine to solve problems involving right triangles.
- The graphs of the sine and cosine functions between 0° and 90°.
- How to extend the sine, cosine, and tangent functions to angles beyond 0° to 90°.
- The fundamental identities that relate the sine, the cosine, and the tangent functions.
- How to use the law of sines and the law of cosines to solve various types of problems.

Review Problems

1. A TV cameraman is standing on a platform 75 feet from a straight portion of a race track and is focusing on the runner in the lead as she runs from left to right.

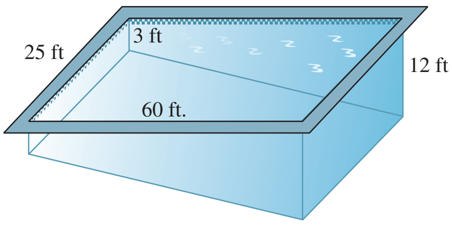


   a. Write a formula for the distance d from the camera to the runner as a function of the angle θ, as shown in the accompanying diagram.
   b. Suppose that the maximum distance for which the camera lens can get a good image is 240 feet. Through what interval of angles can the cameraman pan while focusing on the runner?
   c. What might be appropriate values for the domain of this function?

2. The camera in Problem 1 is again focused on the lead runner in the race.
   a. Write a formula for the distance that the runner covers from the instant that she passes closest to the cameraman as a function of the angle θ.
   b. If $\theta = 25°$, how far has the runner gone since she passed the point closest to the cameraman?
   c. When the runner has gone 150 feet past the point closest to the cameraman, through what interval of angles has the cameraman panned while focusing on her?

3. The next assignment for the TV cameraman in Problems 1 and 2 is to videotape the liftoff of the space shuttle. The cameraman is positioned at ground level 500 meters from the launch pad.
   a. Write a formula for the height y of the shuttle as a function of the angle of inclination α.
   b. Find the height of the shuttle when $\alpha = 20°$.
   c. Find the height of the shuttle when $\alpha = 40°$.
   d. Find the angle of inclination when the shuttle is at a height of 2000 meters.

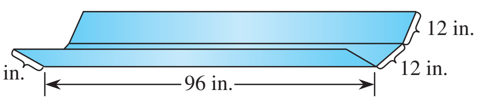
4. A swimming pool is 60 feet long and 25 feet wide. It is 3 feet deep at the shallow end and 12 feet deep at the deep end.



a. Find the angle of depression of the bottom of the pool.
b. Find the equation of the line along the bottom of the pool extending from the shallow end to the deep end along one of the long sides of the pool.
c. Find the equation of the line along the bottom of the pool that extends from one corner to the opposite corner of the pool.

5. A salami is 4 inches in diameter. However, the man in the deli department slices it at an angle of 28° so that each slice comes out oval for a fancier presentation. What is the longest length of each slice of the salami?

6. A piece of metal 96 inches long by 36 inches wide is to be made into a watering trough by bending up 12 inches of the metal along each long side, as shown in the accompanying figure.



a. If the two metal sides are bent up at angles of 35°, how deep is the trough?
b. If the two metal sides are bent up at angles of 55°, how deep is the trough?

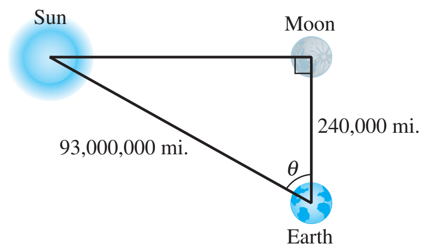
7. The shape of the trough in Problem 6 has a trapezoidal cross section because the top and bottom are parallel. The volume of water that the trough can hold is then its length, 96 inches, times its cross-sectional area, $\frac{1}{2}(b_1 + b_2)h$, where h is the height of the trough and b_1 and b_2 are the horizontal lengths of the top and bottom of each cross section.

a. What volume of water can the trough hold if the two edges are bent up at angles of 35°?
b. What volume of water can the trough hold if the two edges are bent up at angles of 55°?
c. Write a formula for the total volume of water that the trough can hold as a function of the angle θ at which the two sides are bent up.
d. Use your function grapher to estimate the angle that produces a trough that will hold the maximum amount of water.

8. A tall building stands across the street from a hotel—a distance of 220 feet. From one of the hotel windows, a guest in the hotel observes that the angle of inclination to the roof of the building is 36° and that the angle of depression to the base of the building is 23°.

a. How tall is the building?
b. How high is the window in the hotel?

9. The Earth is 93 million miles from the sun, and the moon is 240,000 miles from Earth. When the moon is exactly half full, the Earth, the moon, and the sun form a right triangle with the right angle at the moon. Calculate, correct to two decimal places, the angle at the Earth in this triangle.



10. To calculate the height of a mountain above a level plain, two measurements are necessary. Suppose that, from a certain point on the plain, the angle of elevation to the top of a mountain is $\alpha = 34°$. The observer then moves 1000 meters closer to the mountain, takes a second reading, and gets an angle of elevation of $\beta = 37°$. How tall is the mountain?

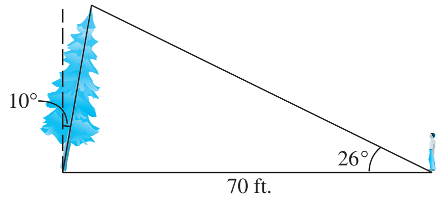
11. As you sit in class waiting for the end of the period, you notice that the length of the minute hand on the wall clock is 10 inches.

a. How far vertically does the point of the minute hand rise from 45 minutes after the hour until 50 minutes after the hour?
b. Explain why the point of the minute hand cannot rise by the same amount from 50 minutes after the hour until 55 minutes after the hour.
c. During what other 5-minute time intervals over the course of an hour does the minute hand either rise or fall vertically by that same amount, as in part (a)?

12. The tangent of some angle in the first quadrant is 1.20.

a. Find the sine and cosine of that angle, using only appropriate trigonometric identities.
b. Find the sine and cosine of that angle by constructing an appropriate triangle.

13. Because of a storm, a tree is inclined at an angle of 10° from the vertical. From a point 70 feet from the base of the tree, the angle of elevation to the top of the tree is 26°, with the tree leaning toward the observer, as shown in the figure on the next page.



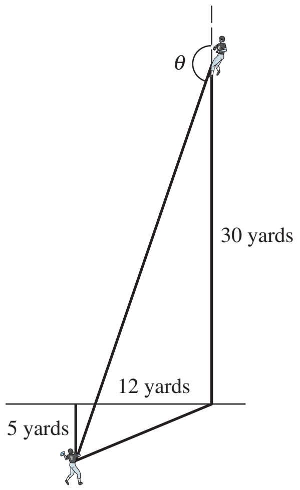
a. Find the height of the tree.

b. Find how high the top of the tree is above ground level.

c. How do your answers to parts (a) and (b) change if the tree is leaning away from instead of toward the observer?

14. Two forest rangers are in observation towers 16 miles apart on an east–west line, and each spots a fire. From the point of view of the ranger in tower *A*, the direction of the fire is 41° north of east. From the point of view of the ranger in tower *B*, the fire is 35° north of west. Find the distance to the fire from each tower.

15. In a football passing play, the wide receiver lines up 12 yards to the right of the quarterback on the line of scrimmage. The quarterback drops straight back 5 yards before throwing the football. The wide receiver runs straight down the field 30 yards before turning to catch the pass.



a. At the moment the quarterback throws the ball, what is the distance from the quarterback to the receiver's original position on the line of scrimmage just before the play started?

b. At the moment the quarterback throws the ball, what is the angle between the line of scrimmage and the line from the quarterback to the receiver's original position on the line of scrimmage?

c. Find the distance that the ball travels from the quarterback to the receiver.

d. To catch the ball straight on, the receiver has to turn toward the quarterback. Through what angle θ must the receiver turn in order to face directly toward the quarterback?

16. A motorboat leaves its dock and motors 6 miles due north. It then changes course, heading northeasterly at an angle of 58° east of north for 14 miles. At that point, the pilot decides to turn back and head directly to the dock.

a. How far is it from the turnaround point back to the dock?

b. Through what angle does the motorboat have to turn in order to be pointed directly back to the dock?

c. If the motorboat moves at a roughly constant speed of 18 mph, how long will the return trip take? How long does the entire outing take?

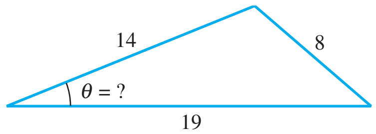
d. The motorboat gets 6 mpg. If the refueling station at the dock charges $2.85 per gallon, how much did the entire outing cost?

17. A steep, snow-covered mountain rises 2700 feet above the surrounding plain and rises at an angle of 68° to the horizontal. A ski lift is to be built from a point 750 feet from the base of the mountain to the summit.

a. What will be the angle of inclination of the cable for the ski lift?

b. What will be the length of the cable?

18. The sides of a triangle have lengths 8 cm, 14 cm, and 19 cm.



a. What is the angle θ?

b. What is the height of the triangle?

c. What is the area of the triangle?

d. Write a formula for the area of any triangle given the lengths *a*, *b*, and *c* of the three sides and one of the angles θ.