7

Modeling Periodic Behavior

7.1 Introduction to the Sine and Cosine Functions

One of the most common behavior patterns in nature is a *periodic oscillatory effect*—a pattern that repeats over and over. For instance, think about how the ocean level varies at a beach between low tide and high tide approximately every 12 hours. If low tide occurs at midnight, high tide will occur at about 6 A.M., low tide will occur again at about noon, and so on indefinitely. This periodic oscillatory behavior is shown in Figure 7.1. Recall that the word *periodic* refers to the fact that this phenomenon repeats indefinitely and that the *period* is the time needed to complete one full cycle. If it takes 12 hours to complete a full cycle, the period is 12 hours.

Similarly, consider the number of hours of daylight each day in a particular location. The minimum number of hours of daylight occurs on the winter solstice, December 21, the "shortest" day of the year. The number of hours of daylight increases slowly until the maximum daylight occurs on the summer solstice, June 21, the "longest" day, and then decreases to the same minimum the following December 21. This oscillatory behavior repeats year after year. For instance, suppose that, at some location, there are 10 hours of daylight on the shortest day of the year and 14 hours of daylight on the longest day. The number of hours of daylight over the course of several years can be represented by the graph shown in Figure 7.2, which has the same shape as that in Figure 7.1.

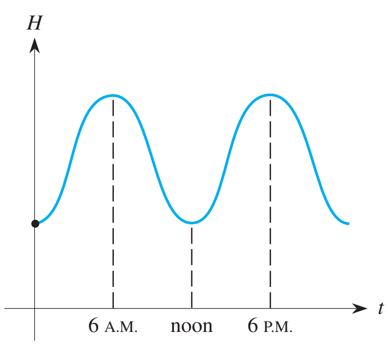


FIGURE 7.1

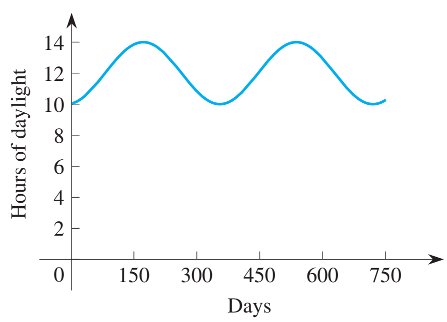


FIGURE 7.2

In this chapter, we consider how to model such periodic phenomena by introducing some new functions that have this type of periodic, oscillatory behavior. We can then use the ideas on stretching and shifting functions from Section 4.7 to create two families of functions that are used to model any such periodic phenomenon.

To begin, imagine a clock on a wall that is running backward, so that the hands move counterclockwise, as shown in Figure 7.3. (Unfortunately, the people who originally developed the mathematics relating to periodic, oscillatory behavior chose counterclockwise as their convention and we're stuck with it). In particular, picture the motion of the arrowhead on the minute hand. Let the horizontal axis be the line through the 3 and 9 positions on the clock. Suppose that the minute hand is 5 inches long and that we start the process at the instant the minute hand is pointing straight up. Every 60 minutes, the point of the minute hand moves from a maximum height of 5 inches above the center (when it is pointing straight up) to a minimum height of 5 inches below the center (when it is pointing straight down) and then back up toward to a maximum height of 5 inches again—and it repeats this cycle indefinitely. Thus the height of the arrowhead on the minute hand, as a function of time t, repeatedly traces a path that is periodic and oscillatory. The arrowhead traces the path shown in Figure 7.4 over the first 3 hours, or 180 minutes—it oscillates between -5 and $+5$ every 60 minutes. This type of function is just what we need to model periodic phenomena.

FIGURE 7.3

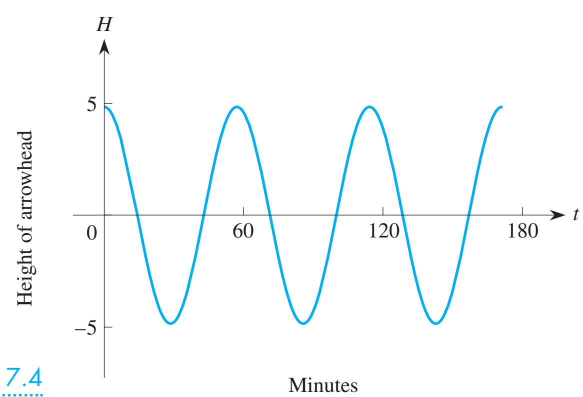


FIGURE 7.4

To develop this process more formally, we use the **unit circle**—a circle with radius 1 centered at the origin—as shown on the left in Figure 7.5. A point P with coordinates (x, y) lies on this circle if

$$x^2 + y^2 = 1.$$

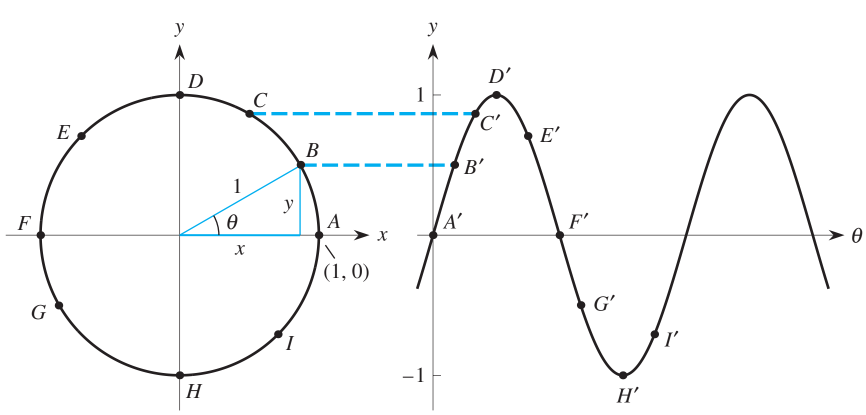


FIGURE 7.5

This point P corresponds to an angle θ measured counterclockwise from the positive x-axis. For instance, when $\theta = 0°$, P is on the positive x-axis; when $\theta = 90°$, P is on the positive y-axis. We consider separately the horizontal distance x to the left or right of the y-axis and the vertical height y above or below the x-axis. Each of these quantities is actually a function of θ, the angle at the center of the circle.

Let's start with the height y. In the triangle at the left in Figure 7.5,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y,$$

so the sine of any angle θ precisely equals the y-coordinate of the corresponding point on the unit circle. For instance, consider the point A at $(1, 0)$ at the extreme right of the circle as the starting point; it has height $y = 0$ and an angle of inclination $\theta = 0$, so $y = \sin 0 = 0$. We set up a second coordinate system where y is a function of θ, as shown in the graph on the right in Figure 7.5. When we think of y as a function of θ, the pair $(\theta, y) = (0, 0)$ corresponds to the point A' at the origin in the graph on the right.

Be sure that you distinguish carefully between the coordinate systems in the two diagrams—the circle on the left, where we think of the height y as a function of the horizontal distance x, and the associated graph being created on the right showing the height y as a function of the angle θ. In the circle, x is measured horizontally, whereas in the graph θ is measured horizontally. The position of any point—say, B—on the circle can be specified either (1) in terms of its x- and y-coordinates or (2) in terms of its angle of inclination θ and its vertical distance y above or below the horizontal axis of the circle. It is these (θ, y) pairs that are graphed on the right.

The Sine Function

We know that there are 360° in a circle, so we can trace the complete circle as θ runs from 0° to 360°. Also, we can continue tracing the circle over and over as θ increases beyond 360°; in fact, every 360° we complete another full revolution around the circle.

Let's consider all the points P on the circle. We labeled several specific points as $A, B, C, D, E, \ldots, I$ in Figure 7.5. Each point corresponds to a particular angle θ, and we plot the heights y corresponding to these angles θ in the graph at the right. For instance, the angle θ corresponding to point B is 30° and its height is $\frac{1}{2}$, or 0.5. As point P traces the circle, starting at point A and passing through the points $B, C, D, E, \ldots, I, \ldots$, the height y on the circle rises from 0 (at A) to a maximum of 1 (at D when $\theta = 90°$), then decreases past 0 (at F when $\theta = 180°$) to a minimum height of -1 (at H when $\theta = 270°$), and then back up to 0 as P finally returns to the starting point A, having gone through a full 360°. This motion of P continues as P traces the circle again and again, and the identical pattern of heights recurs repeatedly, every 360°. The oscillatory pattern shown in the right-hand graph is periodic; it repeats forever with a period of 360° and is part of the graph of the **sine function,** $f(\theta) = \sin\theta$, for any angle θ. It is this periodic, oscillatory effect that we need to model periodic phenomena.

Think About This

You can observe the development of the sine function dynamically by using your graphing calculator. Set the mode for radians (we discuss this topic shortly), for parametric graphing (we discuss this topic in Chapter 9), and for simultaneous plotting. The independent variable used is now typically t instead of x. See the instructions for your particular calculator if necessary. Go to the `Y=` menu and enter

$$X1 = \cos t$$
$$Y1 = \sin t$$
$$X2 = t$$
$$Y2 = \sin t$$

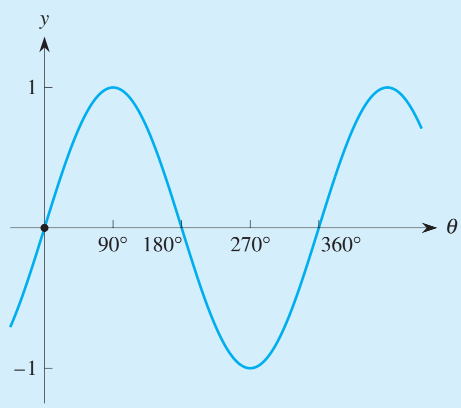
For the viewing window, set t between 0 and 7, $\Delta t = 0.1$, set x between -1.5 and 7, and set y between -2 and 2. When the graphs are drawn, the circle (somewhat flattened because of the screen dimensions) and the sine curve are produced simultaneously. Watch how the heights of points on the circle precisely match the heights on the sine curve for any angle t. You may also want to trace this behavior with your fingers on the graphs in Figure 7.5.

We summarize these ideas about the sine function as follows.

The **sine function**

$$y = \sin \theta$$

represents the height that a point on the unit circle is above or below the horizontal axis as a function of the angle θ. The graph of the sine function is shown at the right.



Again, note that the graph of the sine function oscillates between a maximum height of 1 and a minimum height of -1. Also, the basic shape repeats every 360°, so the behavior pattern you see from 0° to 360° occurs again from $\theta = 360°$ to $\theta = 720°$, again from 720° to 1080°, and so on. Similarly, the same pattern occurs between $\theta = -360°$ and 0°, between $\theta = -720°$ and $-360°$, and so on. Thus the sine function is a periodic function and its period is 360°, as shown in Figure 7.6.

The graph of the sine function passes through the origin and oscillates between -1 and 1 every 360°.

In addition, the sine curve reaches its maximum height of 1 at $\theta = 90°$ and again at $\theta = 450°$ $(= 90° + 360°)$, $810°$ $(= 90° + 2 \times 360°)$, . . . , as well as at $\theta = -270°$ $(= 90° - 360°)$, $-630°$ $(= 90° - 2 \times 360°)$, . . . Similarly, the sine curve reaches its minimum height of -1 at $\theta = 270°, 630°, \ldots$, and at $\theta = -90°$, $-450°, -810°, \ldots$ Note also that the sine curve crosses the horizontal axis at the origin where $\theta = 0°$, again at $\theta = 180°$ (corresponding to the extreme left-hand point on the unit circle), yet again at $\theta = 360°$, and so on indefinitely. In fact, the sine function has zeros at every integer multiple of 180°.

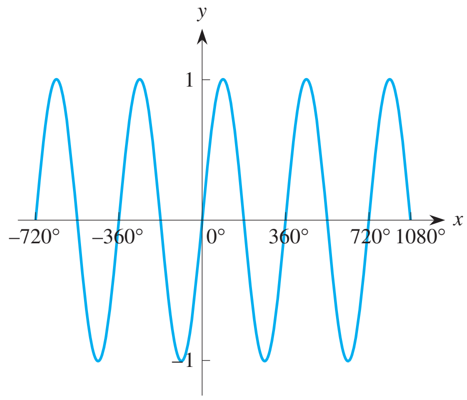


FIGURE 7.6

The Cosine Function

The sine function reflects the *vertical* height y above or below the horizontal axis in the unit circle. Next, we consider the *horizontal* distance x from the vertical axis of the unit circle to points P at (x, y) on the circle, as shown on the left in Figure 7.7. We now treat x as a function of θ. Thus, in the graph of (θ, x) on the right in Figure 7.7, the angle θ is measured along the horizontal axis and the distance x is measured vertically.

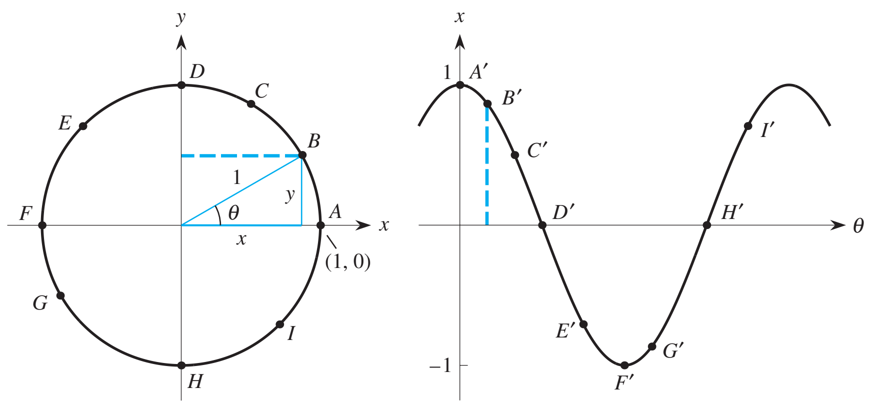


FIGURE 7.7

The initial point A at $(1, 0)$ on the circle lies at a distance of $+1$ to the right of the vertical axis, so the corresponding point A' on the graph at the right has a height of $x = 1$. The succeeding points B and C are closer to the vertical axis of the circle and so, because the x-values are smaller than 1, the corresponding heights on the graph are smaller. The point D is on the vertical axis, so its distance from the vertical axis is 0. The points E and F are to the left of the vertical axis of the circle, so the corresponding heights on the graph are negative. In fact, F is located where $\theta = 180°$ at the extreme left point of the circle at a distance of -1 from the vertical axis, so the corresponding point F' on the graph is a minimum.

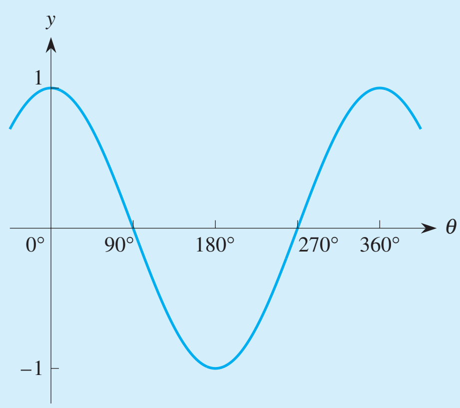
As the angle θ continues to increase, the points on the circle approach the vertical axis from the left, and the corresponding points on the graph now rise toward 0. Eventually, the tracing point P on the circle passes the vertical axis at H and approaches the initial point A where $\theta = 360°$. The horizontal distance that P is from the axis changes from negative, to 0, to positive and approaches the distance 1 to the right of the vertical axis, which is where we started. Simultaneously, the graph on the right crosses the θ-axis and rises to its initial starting height of 1. Allowing θ to continue beyond $\theta = 360°$ we see that the previous pattern repeats exactly and indefinitely.

The graph shown on the right in Figure 7.7 is also a periodic, oscillatory curve. This curve, which corresponds to the horizontal distances from the vertical axis of the unit circle to points on the circle is the graph of the **cosine function,** $g(\theta) = \cos\theta$. The cosine function, like the sine function, is periodic and repeats every 360°, so its period is also 360°. The maximum value of the cosine function is 1, which occurs at $\theta = 0°, 360°, 720°, \ldots$, as well as at $\theta = -360°, -720°, \ldots$ The minimum value of the cosine function is -1, which occurs at $\theta = \pm180°, \pm540°, \ldots$ The cosine function has zeros when $\theta = \pm90°, \pm270°, \pm450°, \ldots$

> The cosine curves passes through the point $(0, 1)$ and oscillates between -1 and 1 every 360°.

We summarize these ideas about the cosine function as follows.

> The **cosine function,**
>
> $$y = \cos\theta$$
>
> represents the horizontal distance that a point on the unit circle is to the right or left of the vertical axis as a function of the angle θ. The graph of the cosine function is shown at the right.



You may find the preceding construction of the cosine curve somewhat easier to visualize by using the following trick. Rotate the circle shown in Figure 7.7 through an angle of 90° counterclockwise, as shown in Figure 7.8. Each horizontal

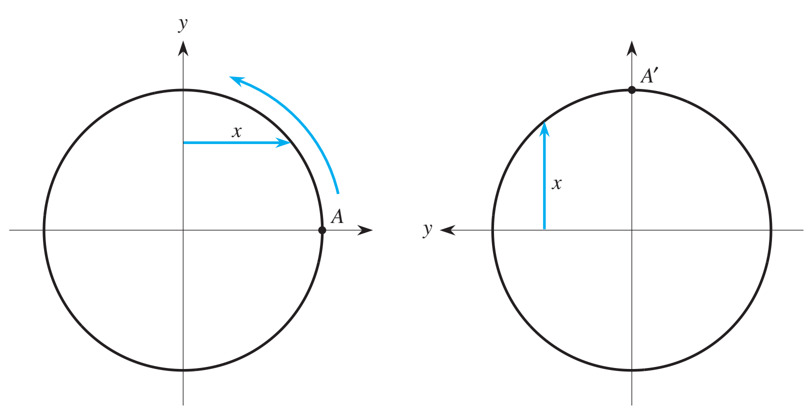


FIGURE 7.8

distance x is then transformed into an equivalent "height" above or below the new horizontal axis. These "heights" produce the heights of the points on the cosine curve shown in the graph on the right in Figure 7.7.

Because of the way that the sine and cosine functions can be defined in terms of the unit circle, they are sometimes called *circular functions.*

Figure 7.9 shows both the sine and cosine graphs from $\theta = 0°$ to $\theta = 540°$. Clearly, these two functions are closely related. Both have the same shape and each can be thought of as arising from the other by an appropriate horizontal shift. If we shift the sine curve to the left by 90°, we get the cosine curve, so

$$\cos\theta = \sin(\theta + 90°).$$

Alternatively, if we shift the cosine curve to the right by 90°, we get the sine curve, so

$$\sin\theta = \cos(\theta - 90°).$$

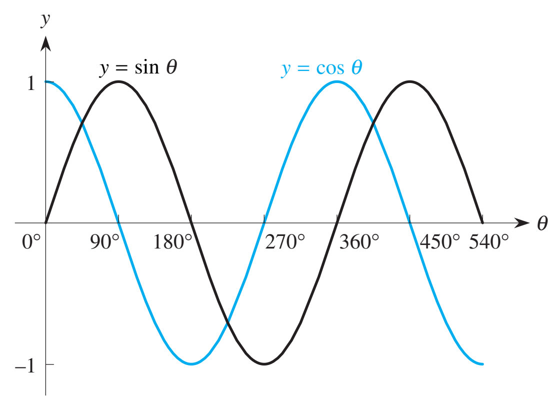


FIGURE 7.9

Moreover, the unit-circle definition suggests what is perhaps the most important relationship between the two functions. Figure 7.10 shows that the vertical height y to a point (x, y) on the unit circle equals $\sin\theta$. Similarly, the horizontal distance x from the vertical axis to the same point equals $\cos\theta$. Because $x = \cos\theta$ and $y = \sin\theta$ must satisfy the equation of the unit circle,

$$x^2 + y^2 = 1,$$

it follows that

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$

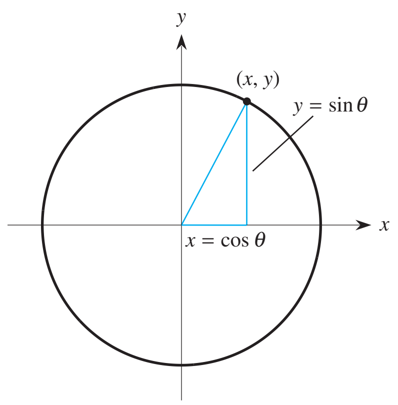


FIGURE 7.10

Recall that we write

$$(\cos\theta)^2 \quad \text{as} \quad \cos^2\theta \qquad \text{and} \qquad (\sin\theta)^2 \quad \text{as} \quad \sin^2\theta.$$

Note that this is another way to prove the *Pythagorean identity* that we presented in Section 6.4.

The **Pythagorean identity** is

$$\sin^2\theta + \cos^2\theta = 1$$

for *any* angle θ.

Think About This

Use your function grapher to graph the function $f(x) = \sin^2 x + \cos^2 x$ for any interval of x values. What does it look like? [You will likely have to enter the function as `(sin x)^2+(cos x)^2`.] ❐

Radian Measure

Because we want to use the sine and cosine functions to model phenomena that are periodic over time, such as the heights of tides or the number of hours of daylight, we need a function of time t rather than a function of an angle θ. Therefore we need a way to avoid angles measured in degrees in our definitions of these functions. To do so, we introduce an alternative unit, called the **radian,** for measuring an angle. In the circle of radius 1 shown in Figure 7.11, we begin on the horizontal axis at the point A at $(1, 0)$. We move counterclockwise around this circle and measure off a distance equal to the radius, or 1. This distance produces an angle α whose size is defined as *one radian.* In degrees, this angle is approximately 57°, as we show shortly.

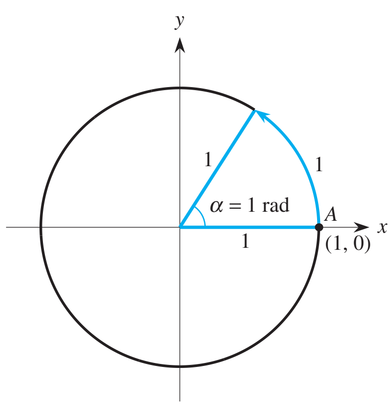


FIGURE 7.11

We next develop a way to convert between radians and degrees. The length of the arc that defines one radian equals the radius of the unit circle. Because $r = 1$, the total circumference of the circle is $2\pi r = 2\pi$. Moreover, the angle α represents a fraction of the full 360° in the circle. As a result, we can set up the proportion

$$\frac{\text{Fraction of the total angle}}{\text{Total angle}} = \frac{\text{Fraction of the total circumference}}{\text{Total circumference}}$$

or

$$\frac{1 \text{ radian}}{360°} = \frac{1}{2\pi}.$$

Cross-multiplying, we get

$$2\pi \text{ radians} = 360°$$

or

$$\boxed{\pi \text{ radians} = 180°}$$

Alternatively, dividing both sides by π gives

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ} \approx 57.29578°,$$

or about 57.3°. Furthermore, because π radians $= 180°$, we can divide both sides by 180 to get

$$1° = \frac{\pi}{180} \text{ radians}.$$

If we perform the same construction in any circle with radius r—that is, if we measure an arc whose length equals the radius r—the corresponding angle would be the same 1 radian, or about 57.3°. Thus an angle measured in radians is the same no matter what the size of the circle. More important, radians are not tied directly to angles the way degrees are. Using radians, we can consider any variable and apply the sine and cosine functions to it. Thus we can use a variable representing time, height, or any other desired quantity as the independent variable with either the sine or the cosine function.

EXAMPLE

Use the fact that $180° = \pi$ radians to obtain the radian measure for the common angles 90°, 60°, 45°, and 30°.

Solution If we divide 180° by 2, we get

$$90° = \frac{180°}{2} = \frac{\pi}{2} \text{ radians} = \frac{\pi}{2}.$$

Similarly,

$$60° = \frac{180°}{3} = \frac{\pi}{3} \text{ radians} = \frac{\pi}{3};$$

$$45° = \frac{180°}{4} = \frac{\pi}{4} \text{ radians} = \frac{\pi}{4};$$

$$30° = \frac{180°}{6} = \frac{\pi}{6} \text{ radians} = \frac{\pi}{6}.$$

To summarize, we have the following relationships.

$$180° = \pi \text{ radians}$$

In particular,

$$30° = \frac{\pi}{6}, \quad 45° = \frac{\pi}{4}, \quad 60° = \frac{\pi}{3}, \quad \text{and} \quad 90° = \frac{\pi}{2}.$$

These results occur often in applications of the trigonometric functions, and you need to know them.

For these standard, or special, angles, we have the following values for the sine and cosine functions (which we derived in Chapter 6).

$$\sin 30° = \sin\left(\frac{\pi}{6}\right) = 0.5 = \frac{1}{2} \qquad \cos 30° = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$$

$$\sin 45° = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \approx 0.707 \qquad \cos 45° = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \approx 0.707$$

$$\sin 60° = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866 \qquad \cos 60° = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = 0.5$$

$$\sin 90° = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \cos 90° = \cos\left(\frac{\pi}{2}\right) = 0$$

Be sure that you know how to use your calculator to obtain the value for the sine or cosine of any argument, both in degrees and radians. We strongly recommend that you permanently set your calculator mode to `Radians`; we work with radians almost exclusively from this point on.

Note that radians are not always given in terms of π. For instance, we might have $x = 0.5$ radians or $x = 2$ radians or $x = -4.27$ radians.

The Behavior of the Sine and Cosine Functions

Finally, let's consider the important aspects of the behavior of the sine and cosine functions. In general, we want to answer several questions for any function.

1. Where is it increasing?
2. Where is it decreasing?
3. Where is it concave up?
4. Where is it concave down?
5. Where are its points of inflection?
6. Where are its zeros?
7. Where does it achieve its maximum value, and what is that maximum value?
8. Where does it achieve its minimum value, and what is that minimum value?
9. Is it periodic? If so, what is its period?

We can answer all these questions about the sine and cosine functions by examining their graphs and applying ideas developed earlier in this book. However, for other functions that may not be as well known, answering some of these questions requires the use of calculus.

Let's consider the behavior of the sine function $y = \sin x$. It is evident from its graph that the sine curve increases for x between 0 and $\pi/2$, then decreases from $\pi/2$ to $3\pi/2$, and then increases from $3\pi/2$ to 2π—and repeats this cycle thereafter. This behavior is also clear from the unit circle definition of the sine.

Further, the sine curve is concave down for x between 0 and π, concave up for x between π and 2π, and then repeats this cycle thereafter. Consequently, the sine curve has points of inflection at $x = 0, \pm\pi, \pm 2\pi, \ldots$, where its concavity changes.

In addition, the sine function has zeros when $x = 0, \pm\pi, \pm 2\pi, \ldots$ A special characteristic of the function $f(x) = \sin x$ is that its zeros and its points of inflection are identical, which is not the case for most other common functions.

Finally, the sine function achieves its maximum value of 1 at $x = \pi/2$, at $x = 5\pi/2$, at $x = 9\pi/2, \ldots$ and at $x = -3\pi/2$, at $x = -7\pi/2, \ldots$ The sine function achieves its minimum value of -1 at $x = 3\pi/2$, at $x = 7\pi/2$, at $x = 11\pi/2, \ldots$ and at $x = -\pi/2$, at $x = -5\pi/2, \ldots$

We ask you to describe the behavior of the cosine function in the Problems at the end of this section.

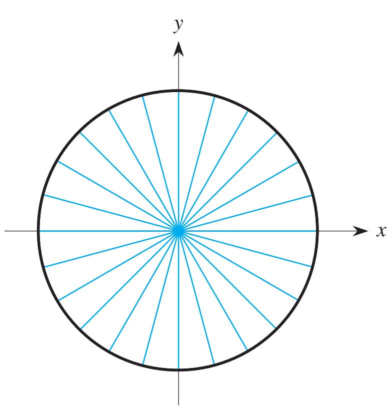
In summary, the key points about the sine and cosine functions are:

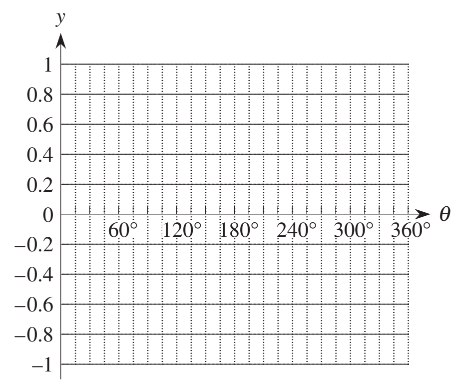
The sine function passes through the origin and oscillates between -1 and $+1$ every 2π.

The cosine function passes through the point $(0, 1)$ and oscillates between -1 and $+1$ every 2π.

Problems

1. Janis trims her fingernails every Saturday morning. Sketch the graph of the length of her nails as a function of time. Can this process be modeled by a periodic function? If it is periodic, what is the period?
2. Harry gets a haircut on the first of every month. Sketch the graph of the length of his hair as a function of time. Can this process be modeled by a periodic function? If it is periodic, what is the period?
3. In the accompanying figure, the circle on the left has been subdivided every 15° from $\theta = 0°$ to $\theta = 360°$. Use the heights from the horizontal axis to the associated points on the circle to construct the graph of the sine function for $\theta = 0°$ to $\theta = 360°$ on the axes at the right.





4. Convert each angle from degrees to radians.

   a. 15° **b.** 75° **c.** 120°
   d. 150° **e.** 225° **f.** 315°
   g. 270° **h.** 240° **i.** −135°
   j. −210°

5. Convert each angle from radians to degrees.

   a. $\frac{3\pi}{4}$ **b.** $\frac{4\pi}{5}$ **c.** $\frac{2\pi}{3}$
   d. 1.5 **e.** 2.5 **f.** 3
   g. $\frac{\pi}{8}$ **h.** $\frac{5\pi}{3}$ **i.** $-\frac{3\pi}{2}$
   j. $-\frac{5\pi}{3}$

6. For $f(\theta) = 5 \sin \theta$, evaluate each function.

   a. $f(30°)$ **b.** $f(45°)$ **c.** $f(60°)$

d. $f(120°)$ **e.** $f(-15°)$ **f.** $f(873°)$

g. $f\left(\frac{\pi}{4}\right)$ **h.** $f\left(\frac{\pi}{3}\right)$ **i.** $f\left(\frac{\pi}{12}\right)$

j. $f\left(-\frac{\pi}{6}\right)$ **k.** $f(5.27)$ **l.** $f(-25.614)$

7. For $f(\theta) = \sin 2\theta$, evaluate each function.

a. $f(30°)$ **b.** $f(45°)$ **c.** $f(120°)$
d. $f(225°)$ **e.** $f(\pi/3)$ **f.** $f(\pi/12)$
g. $f(3\pi/8)$ **h.** $f(2\pi/7)$

8. At the end of this section we posed nine questions about the behavior of any function. Answer these questions for the cosine function.

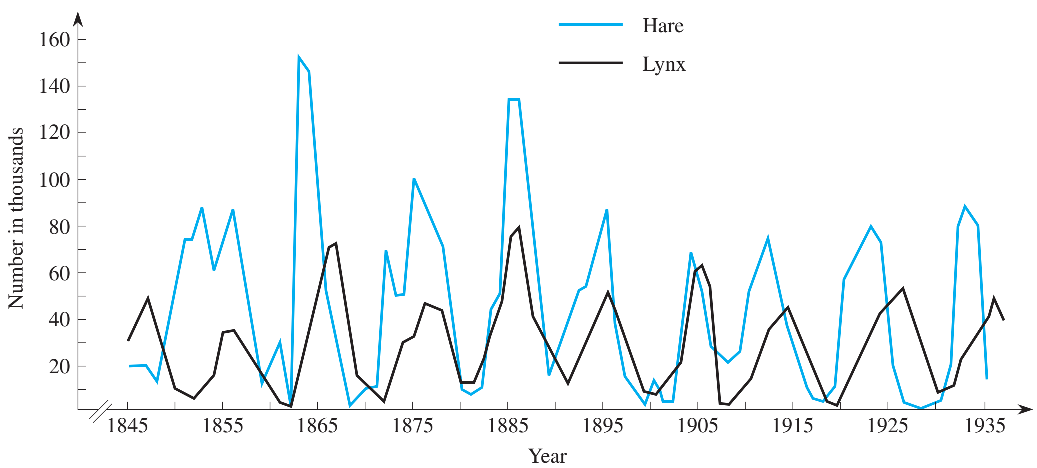
9. With your calculator set in radians, graph the two functions $y = \sin x$ and $y = \cos(x - \pi/2)$. What do you observe? Explain what you observed.

10. Plot the functions $y = \cos x$ and $y = \sin(x + \pi/2)$. Explain why you see only one graph. (If you see two graphs, check that your calculator `MODE` is set for `Radians`.)

11. The population growth patterns of two species are interrelated when one species preys on the other. This situation occurs in northern Canada where lynxes are the predators and hares are the prey. The figure below is based on records kept by the Hudson's Bay Trading Company on the number of each species caught by fur trappers from 1845 through 1935. The graphs indicate that both populations change in roughly periodic cycles.

a. Estimate the period of the cycle for the lynxes.
b. Estimate the period of the cycle for the hares.
c. Estimate the years in which the lynx population reached its maximum and minimum values.
d. Estimate the years in which the hare population reached its maximum and minimum values.
e. Can you find any relationship between the lengths of the periods in parts (a) and (b) and the times in parts (c) and (d)? If so, what is it?
f. Estimate the years when the hare population passed its points of inflection. How do they compare to any of the times you found in parts (c) and (d)?

12. Consider the functions $y = \cos x$ and $y = \sin(\#\$\%\#\$)$. What could #$%#$ represent so that the two graphs are identical? Is there only one correct answer to this question? Explain.



7.2 Modeling Periodic Behavior with the Sine and Cosine

Because of their behavior patterns, the sine and cosine functions are used as the mathematical models to represent most periodic phenomena. For example, the number of hours of daylight H any day of the year in San Diego can be modeled by the function

$$H(t) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right],$$

where t is the number of days from the first of the year ($t = 1$ on January 1). We begin by using the formula for some predictions and then see where the formula comes from.

EXAMPLE 1

Based on the model, how many hours of daylight are there in San Diego on (**a**) February 15? (**b**) March 21? (**c**) June 21?

Solution

a. February 15 is the 46th day of the year (31 days in January plus 15 more in February), so $t = 46$. Using a calculator set to radian mode, we find that

$$H(46) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(46 - 80)\right]$$
$$= 12 + 2.4 \sin\left[\frac{2\pi}{365}(-34)\right] = 10.67 \text{ hours.}$$

b. March 21 is the $t = 31 + 28 + 21 = 80$th day of the year, so

$$H(80) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(80 - 80)\right] = 12 \text{ hours.}$$

Incidentally, March 21 is the spring equinox, which means that there are 12 hours of daylight and 12 hours of darkness, so the model gives the right prediction.

c. June 21 is the $t = 31 + 28 + 31 + 30 + 31 + 21 = 172$nd day of the year, so the number of hours of daylight on June 21(the first day of summer and so the "longest" day of the year) is

$$H(172) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(172 - 80)\right] = 14.40 \text{ hours.}$$

Think About This Without using a calculator, find the number of hours of daylight in San Diego on December 21, the 355th (and "shortest") day.

Later in this section we show how a similar formula can be developed for any city. For now, let's see what the different numbers 12, 2.4, 365, and 80 in the formula

$$H(t) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

for the number of hours of daylight in San Diego actually represent. Obviously, 365 represents the number of days in a year; it tells us how long it takes for a full cycle to be completed. Over a full year, the average number of hours of daylight is 12 hours per day—the days when there are more than 12 hours of daylight exactly counterbalance the days when there are fewer than 12 hours of daylight. Alternatively, averaging the number of hours of daylight for all 365 days gives 12 hours. So the 12 represents the average, or middle, value for the sine function.

Next, as we found in Example 1, the longest day in San Diego has 14.4 hours of daylight and the shortest day has 9.6 hours of daylight. Note that 14.4 is 2.4 hours more than the average level of 12 and that 9.6 is 2.4 hours less than 12. Of course, the maximum and minimum number of hours of daylight at any particular location, as well as the number on any specific date, depend on the location itself and are therefore modeled by a function slightly different from H; think about how long a "day" is during the winter or the summer in the far north, the so-called "land of the midnight sun".

The graph of H, the number of hours of daylight in San Diego over a 3-year interval, is shown in Figure 7.12. It has the same *shape* as the graph of the basic sine or cosine function. However, it does not oscillate about the horizontal axis; rather it oscillates about the horizontal line $H = 12$, which represents the average number of hours of daylight over a full year, so it is shifted up by 12 hours. Also, its maximum and minimum "heights" above the horizontal line $H = 12$ are no longer $+1$ and -1 as with the basic sine and cosine functions. Instead, the graph varies from a minimum of 9.6 hours to a maximum of 14.4 hours, which is 2.4 hours either side of the average 12, so the sine function has been stretched by a factor of 2.4.

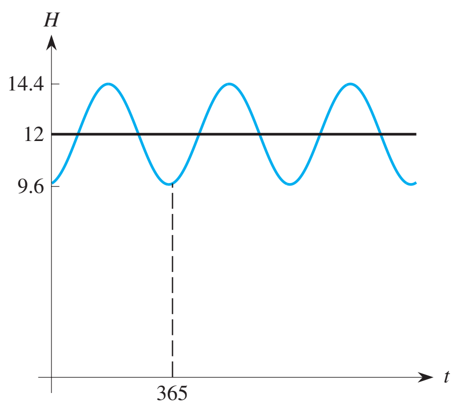


FIGURE 7.12

Note some additional differences: The graph is shifted to the right compared to the sine curve (the curve does not "start" at the vertical axis where $t = 0$ and $H = 12$). Also, the period is 365 days, rather than the usual 2π radians, or 360°. (Incidentally, the ancient Babylonians believed that the length of a year was 360 days. That's why we divide a circle into 360 degrees.)

This particular function $H(t)$ differs from the standard, or base, function $y = \sin x$ that we discussed in Section 7.1 in four ways:

1. a vertical shift,
2. an oscillation other than from -1 to $+1$ (a stretch),
3. the length of a cycle, and
4. the "starting" point of the cycle (a horizontal shift).

Understanding how to incorporate these variations is crucial for applying the sine and cosine functions to describe periodic phenomena. We therefore focus on each in detail.

The equation for the number of hours of daylight in San Diego is

$$H = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right].$$

Consider the more general *sinusoidal function*

$$S(x) = D + A \sin[B(x - C)],$$

where A, B, C, and D are all constants and x is the independent variable. In the San Diego situation, $D = 12$, $A = 2.4$, $B = 2\pi/365$, and $C = 80$. Let's investigate how each of these four *parameters* affects the graph of the basic sine curve. To do so, we consider each parameter separately.

The Vertical Shift or Midline

To show the significance of the D term, we consider the simpler function with $A = 1$, $B = 1$, and $C = 0$:

$$S(x) = D + \sin x.$$

We know that $y = \sin x$ oscillates repeatedly between -1 and $+1$. What is the effect of adding a constant D? From our discussion in Sections 4.6 and 4.7, we know that D raises or lowers the basic sine curve by the amount D. The graph of $S(x) = 2 + \sin x$ has the same shape as the basic sine function but is shifted up 2 units; it oscillates about the horizontal line $y = 2$, between 1 and 3 units above the x-axis as shown in Figure 7.13. Similarly, the graph of $S(x) = -5 + \sin x$ oscillates about the horizontal line $y = -5$, between -6 and -4.

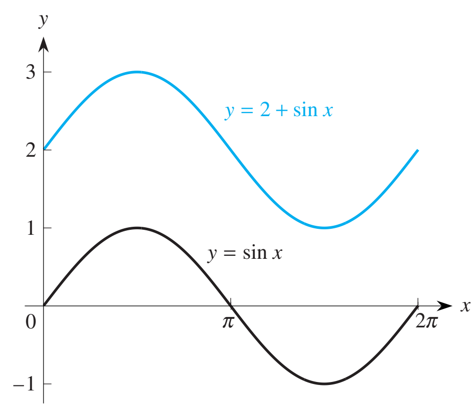


FIGURE 7.13

Thus the effect of the constant D in

$$S(x) = D + \sin x$$

is to produce a sinusoidal curve that oscillates about the horizontal line $y = D$, between $D - 1$ and $D + 1$. If D is positive, the sine curve is shifted upward D units; if D is negative, the curve is shifted downward D units. The number D is the **vertical shift** or the **midline.** In the formula for the number of hours of daylight in San Diego, the vertical shift, or midline, is 12.

The Amplitude

We next investigate the effect of the multiplicative constant A in the general equation of a sinusoidal function $y = D + A\sin[B(x - C)]$. We set $D = 0$, $B = 1$, and $C = 0$ to consider the simpler function

$$S(x) = A\sin x.$$

For example, if $A = 2$, we get $S(x) = 2\sin x$, whose graph is shown in Figure 7.14, where it is compared to the basic curve for the sine function, $y = \sin x$ (for which

$A = 1$). For comparison, we also show the graph of $T(x) = \frac{1}{2}\sin x$. Although the basic sine function oscillates between -1 and $+1$, the transformed function $S(x) = 2\sin x$ oscillates between -2 and $+2$ and the transformed function $T(x) = \frac{1}{2}\sin x$ oscillates between $-\frac{1}{2}$ and $+\frac{1}{2}$. In general, the effect of multiplying the sine function by a constant A is to increase its vertical height above and below the midline by the factor $|A|$.

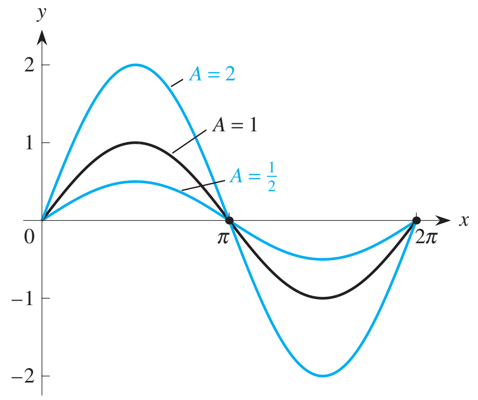


FIGURE 7.14

To show why the absolute value is necessary, we consider the graph of

$$S(x) = -4\sin x,$$

which has the same shape as the basic sine curve, but oscillates between -4 and $+4$. The main difference is that the graph of this function is flipped over the x-axis compared to the graph of $y = \sin x$. We naturally think of it as being four times as high as the base curve, not -4 times as high. (Draw simultaneously the graphs of $y = \sin x$ and $y = -4\sin x$ using your function grapher.) This is the same effect of the negative multiple that we encountered in Section 4.6. However, this curve has the same period ($2\pi = 360°$) and the same zeros ($x = 0, \pm\pi, \pm 2\pi, \ldots$) as the basic sine curve.

The quantity $|A|$ is called the **amplitude** of the sine function. In the expression for $H(t)$ modeling the number of hours of daylight in San Diego, the amplitude is 2.4.

In Example 2 we show what happens when we combine the two transformations to construct a new function.

EXAMPLE 2

Analyze the graph of

$$S(x) = 2 + 3\sin x.$$

Solution In this formula, 2 is the vertical shift, or midline, and 3 is the amplitude. The effect of multiplying the sine function by 3 is to stretch it vertically by a factor of 3, so that $3\sin x$ oscillates between -3 and 3. Adding the constant 2 to the function $3\sin x$ simply raises the entire curve 2 units vertically. Consequently, the combined effect is to produce a sinusoidal function that oscillates from 3 units below the horizontal line $y = 2$ to 3 units above the line; that is, from -1 to $+5$, as shown in Figure 7.15.

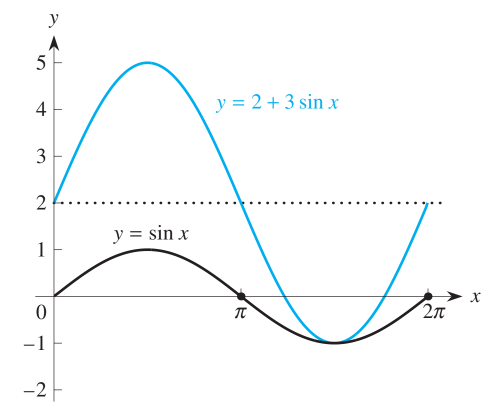


FIGURE 7.15

Incidentally, the function $y = 2 + 3 \sin x$ is not the same as $y = 5 \sin x$; the coefficients cannot be combined because 2 and $3 \sin x$ are not like terms. Graph both functions to see that they produce very different results. Also, $y = 2 + 3 \sin x$ is not the same as $y = 3 + 2 \sin x$—each parameter has its own role to play.

Use your function grapher to examine the graphs of several functions of the form $y = D + A \sin x$ for different values of A and D. Predict and then observe how the different constant values are reflected in the corresponding sinusoidal curve.

The Frequency and the Period

We next consider the effect of the parameter B, which multiplies the term $(x - C)$ in

$$S(x) = D + A \sin[B(x - C)].$$

To concentrate on B only, we take $C = 0$, $D = 0$, and $A = 1$. We also assume that $B > 0$. Consider how the function

$$S(x) = \sin(2x)$$

compares to the basic curve $y = \sin x$, as shown in Figure 7.16. The resulting sinusoidal curve $y = \sin(2x)$ completes two full cycles between $x = 0$ and $x = 2\pi$, compared to one full cycle for the basic sine curve.

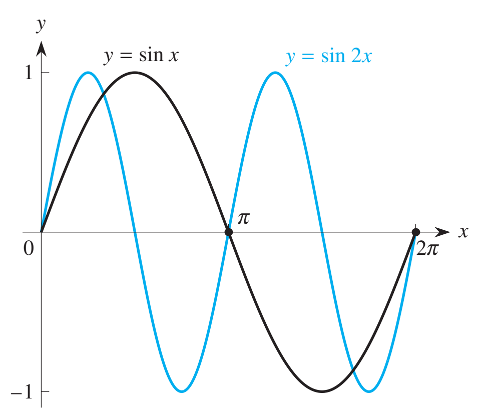


FIGURE 7.16

Similarly, the graph of

$$S(x) = \sin(3x)$$

shown in Figure 7.17 completes three full cycles across the interval from 0 to 2π. Based on these two results, we expect that the graph of

$$S(x) = \sin(nx),$$

for any positive integer n, will complete n full cycles between $x = 0$ and $x = 2\pi$. Try this with your function grapher for values of n such as 5 or 8.

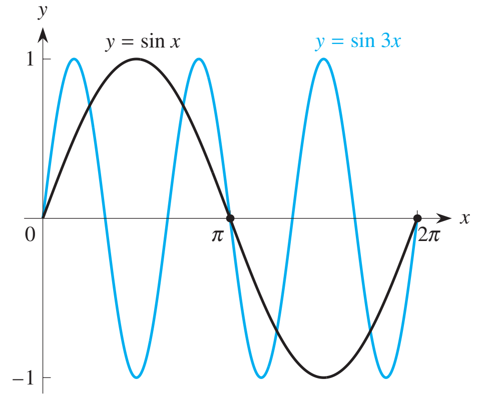


FIGURE 7.17

What happens if the positive multiple B is not an integer? The graph of $y = \sin(\frac{1}{2}x)$ is shown in Figure 7.18. The function $y = \sin(\frac{1}{2}x)$ completes half a complete cycle between 0 and 2π; it actually requires an interval of values for x from 0 to 4π to complete a full cycle.

Similarly, Figure 7.19 shows that the function $y = \sin(2.5x)$ completes 2.5 full cycles between 0 and 2π. (Trace the graph with your finger and count the cycles.) It therefore completes one full cycle in $1/2.5 = 0.4 = 2/5$ of this interval.

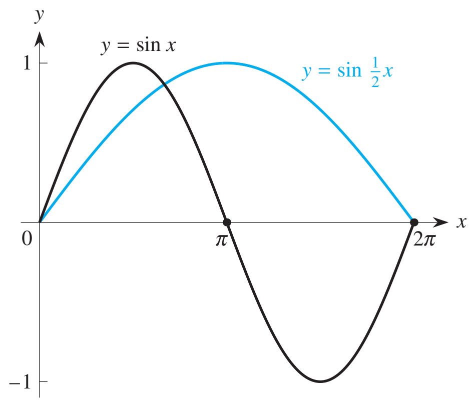


FIGURE 7.18

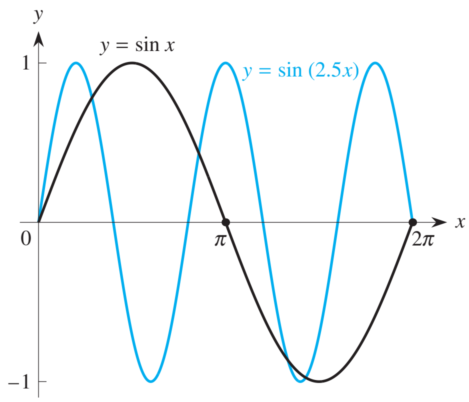


FIGURE 7.19

Mathematicians call the parameter B in $y = \sin(Bx)$ the **frequency** of the sinusoidal function. It tells us the number of complete cycles that occur between $x = 0$ and $x = 2\pi = 360°$. For instance, the function $y = \sin(6x)$ completes six full cycles across this interval, whereas the function $y = \sin(\frac{3}{8}x)$ completes $3/8$ of one cycle.

Note that standard usage is to write $y = \sin 2x$ or $y = \sin 5x$, say, rather than $y = \sin(2x)$ or $y = \sin(5x)$, even though it is less precise a notation. But parentheses are essential on calculators and computers.

As with any periodic function, the *period* of a sinusoidal function $y = \sin Bx$ is the length of the interval needed to complete one full cycle. For $y = \sin 2x$, a full cycle is completed in any interval of x-values of length π (see Figure 7.16), so the period is π radians $= 180°$. For $y = \sin 3x$, the period is

$$\left(\frac{1}{3}\right)2\pi = \frac{2\pi}{3} \quad \text{or} \quad \left(\frac{1}{3}\right)360° = 120°,$$

(see Figure 7.17). For $y = \sin\left(\frac{1}{2}x\right)$, the period is

$$\frac{1}{\frac{1}{2}}(2\pi) = 4\pi \text{ radians},$$

or 720° (see Figure 7.18).

In general, the period of $y = \sin Bx$ is

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{\text{frequency}}.$$

Thus, for instance, for $y = \sin(2.5x)$, the period is

$$\text{Period} = \frac{2\pi}{2.5} = \frac{2\pi}{5/2} = \frac{2}{5}(2\pi) = \frac{4}{5}\pi$$

because it is the length of the interval needed for this sinusoidal function to complete one full cycle (see Figure 7.19). This result agrees with our earlier statement that the function $y = \sin(2.5x)$ completes one full cycle in 2/5 of the interval from 0 to 2π.

We have shown that the period of any periodic function is the length of the interval needed to complete one full cycle. Alternatively, if we start with the period B,

$$\text{Frequency} = \frac{2\pi}{\text{period}}.$$

In the formula for the number of hours of daylight in San Diego

$$H = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right],$$

the frequency of the sinusoidal curve is

$$\text{Frequency} = \frac{2\pi}{365} \approx 0.0172.$$

The period of the sinusoidal curve is

$$\text{Period} = \frac{2\pi}{\text{frequency}} = \frac{2\pi}{(2\pi/365)} = 365 \text{ days}.$$

Thus, as we would expect, the period is 1 year.

In summary we have the following.

$$\text{period} = \frac{2\pi}{\text{frequency}} \qquad \text{frequency} = \frac{2\pi}{\text{period}}$$

Note that in engineering, the frequency is defined somewhat differently. Instead of meaning the number of cycles in 2π radians, engineers consider the number of cycles in a given length of time—say, cycles per second. They then write

$$\text{Frequency} = \frac{1}{\text{period}}$$

and write the sinusoidal function in the form $y = \sin(2\pi Bt)$, where B, not $2\pi B$, is the frequency. Unfortunately, this slight difference in terminology is so deeply embedded in the two fields that it is not possible for either field to change to match the other.

The Phase Shift

Finally, we consider the role of the parameter C in

$$S(x) = D + A\sin[B(x - C)].$$

We simplify the discussion by taking $A = 1$, $B = 1$, and $D = 0$ so that we consider only

$$S(x) = \sin(x - C).$$

From Section 4.7, we know that the term $(x - C)$ should have the effect of a horizontal shift to the right when C is positive and to the left when C is negative.

Figure 7.20 compares the graph of $S(x) = \sin(x + \pi/4)$ to the basic curve $y = \sin x$. The two curves appear similar, but $S(x)$ is shifted to the left (backward) by $\pi/4$, or $\frac{1}{8}$ of 2π (which is one-eighth of a full cycle). Similarly, Figure 7.21 shows the graph of $T(x) = \sin(x - \pi/3)$. It has been shifted to the right (forward) by $\pi/3$, or $\frac{1}{6}$ of 2π (which is one-sixth of a full cycle).

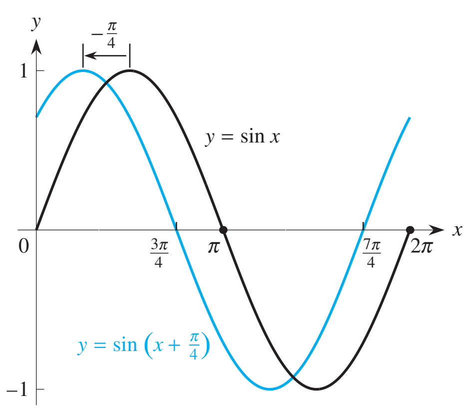


FIGURE 7.20

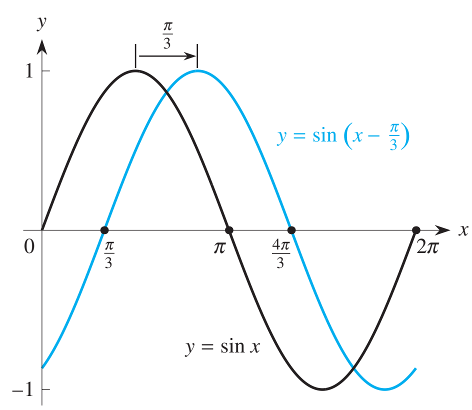


FIGURE 7.21

In general, the parameter C shifts a sinusoidal curve to the left or the right by the amount C. If C is positive in the term $(x - C)$, as in $y = \sin(x - \pi/3)$, the curve is shifted to the right by C; if C is negative in $(x - C)$, as in $y = \sin(x + \pi/4)$, the curve is shifted to the left by C. This parameter is called the **phase shift,** instead of the horizontal shift, in the context of sinusoidal functions. In the expression for the daylight function for San Diego

$$H = 12 + 2.4\sin\left[\frac{2\pi}{365}(t - 80)\right],$$

the phase shift is 80 days.

EXAMPLE 3

What is the significance of the phase shift in the formula for H?

Solution The phase shift shifts the curve to the right by 80 days. Recall that the 80th day of the year is March 21, which is the spring equinox (the day when there are equal numbers of hours of daylight and darkness). On this day, the graph for the sinusoidal function crosses the midline, or average level, of $D = 12$ hours.

In general, the phase shift for a sine function corresponds to the first point to the right of the origin where the curve crosses the midline while increasing. Equivalently, it occurs midway, horizontally, from a minimum point to a maximum.

We summarize all the results for the San Diego daylight function in Figure 7.22.

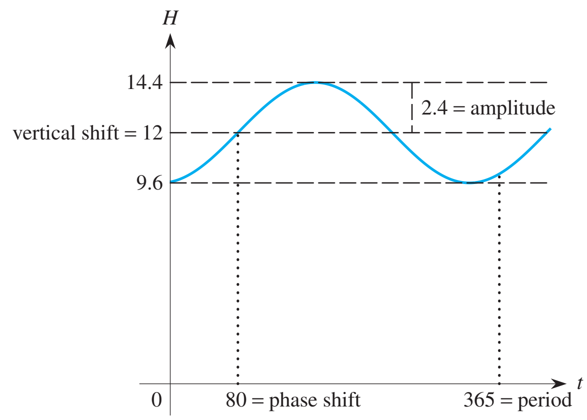


FIGURE 7.22

EXAMPLE 4

The water at a boat dock is 7 feet deep at low tide and 11 feet deep at high tide. On a certain day, low tide occurs at 4 A.M. and high tide at 10 A.M. Find an equation for the height of the tide y as a function of time t.

Solution We use the given information to sketch the graph of a sinusoidal curve in Figure 7.23.

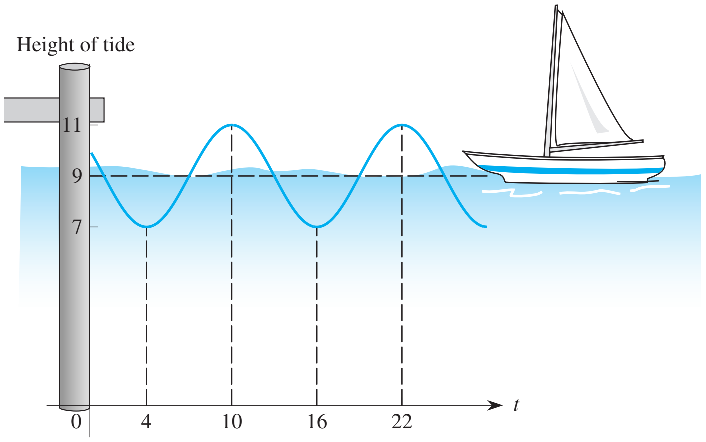


FIGURE 7.23

Because the tide ranges from a minimum height of 7 feet to a maximum height of 11 feet above sea bottom, the curve oscillates about the middle value of 9 feet, which is the vertical shift, or midline. Also, the amplitude of this sinusoidal curve is 2. Further, the time interval between the minimum and maximum heights of the water level is 6 hours; consequently, a complete tide cycle takes 12 hours, so the period is 12 hours. As a result,

$$\text{Frequency} = \frac{2\pi}{\text{period}} = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Finally, because the tide level increases from 4 A.M. to 10 A.M., the curve passes across the middle height of 9 feet halfway between 4 A.M. and 10 A.M. (or at 7 A.M.), which gives the phase shift. (The graph shows that, even though the tide function also crosses the 9-foot level at 1 A.M., the function is decreasing there and so this does not give the phase shift.) Therefore the height y of the water at any time t is modeled by

$$y = 9 + 2\sin\left[\frac{\pi}{6}(t - 7)\right].$$

EXAMPLE 5

The air conditioning in a home is set to go on when the temperature reaches 74°F and to go off when the temperature drops to 68°. This cycle repeats every 20 minutes. If the temperature in the house at noon is 71° and rising, write a sinusoidal function to model the temperature as a function of the number of minutes t since noon.

Solution A sinusoidal function is of the form $T = D + A\sin[B(t - C)]$, where A, B, C, and D must be determined. We know that the temperature oscillates between 68° and 74°, so it is centered about 71°, which is the vertical shift, or midline, D. Further, because the size of the oscillation above and below this midline is 3, we know that the amplitude $A = 3$. We also know that the length of the cycle is 20 minutes, so the period is 20 and therefore the frequency $B = 2\pi/20 = \pi/10$. Finally, because the house temperature reaches 71°—the level of the vertical shift—at noon when $t = 0$, the phase shift is 0, so $C = 0$. Therefore our model for the temperature of the house as a function of time is

$$T = 71 + 3\sin\left(\frac{\pi}{10}t\right).$$

Figure 7.24 shows the graph of this sinusoidal function for the first 60 minutes.

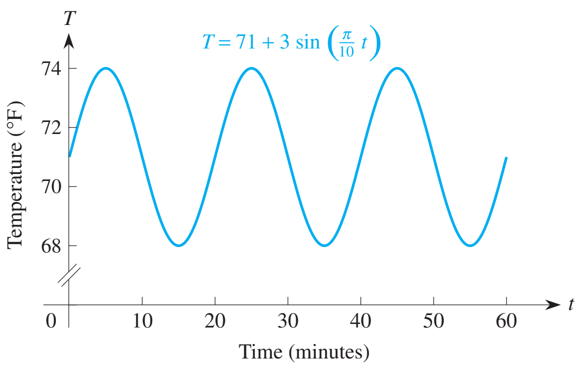


FIGURE 7.24

EXAMPLE 6

For part of the year, the temperature T in the Colorado Rockies can be modeled by the function

$$T(t) = 60 + 10\sin\left(\frac{\pi}{12}t\right),$$

where t is measured in hours and $t = 0$ is at 9 A.M. In Example 5 of Section 2.2, we used the linear function

$$C(T) = 4T - 160$$

to model the chirp rate C (in chirps per minute) of the snow tree cricket as a function of the air temperature T (in °F).

a. Express the chirp rate as a function of time.

b. How fast is the cricket chirping at 5 P.M.?

c. What are the domain and range of this function?

Solution

a. The chirp rate in $C = 4T - 160$ is measured in chirps per minute and the time t in the formula for the temperature as a function of time is measured in hours. To make things consistent, we convert the chirp rate to chirps per hour by multiplying by 60 (minutes per hour) to get

$$C = 60(4T - 160) = 240T - 9600.$$

We now have C as a function of T, where T is a function of t, so C is a composite function,

$$\begin{aligned} C = f(t) &= 240T - 9600 \\ &= 240\left[60 + 10\sin\left(\frac{\pi}{12}t\right)\right] - 9600 \\ &= 14{,}400 + 2400\sin\left(\frac{\pi}{12}t\right) - 9600 \\ &= 4800 + 2400\sin\left(\frac{\pi}{12}t\right), \end{aligned}$$

where t is measured in hours since 9 A.M. and C is chirps per hour, as illustrated in Figure 7.25. (Note that we could have gone the other way and converted everything to minutes, but then the numbers get quite large.)

b. At 5 P.M., when $t = 8$ hours after 9 A.M., the chirp rate is

$$C = f(8) = 4800 + 2400\sin\left(\frac{\pi}{12}\cdot 8\right) \approx 6878 \text{ chirps per hour.}$$

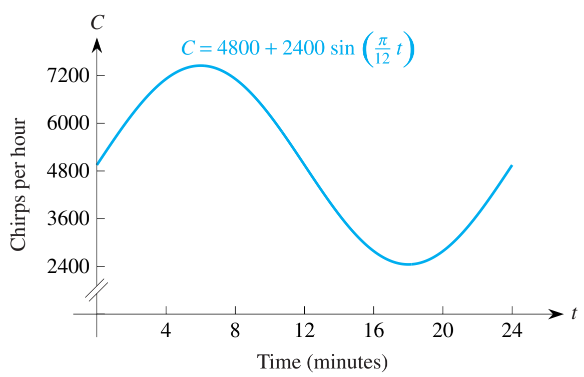


FIGURE 7.25

c. The independent variable is the time t in hours from 9 A.M. on a particular day. The practical domain of the function f depends on how long the functions that are the component models make sense. The model for the temperature is good for only part of the year; let's assume that it applies only for a 30-day period. If the initial time is at the beginning of that time period, the domain would last for the following 30 days; if the initial time is at the middle of that time period, the domain would extend from 15 days before to 15 days after. As for the range, look at the function f. It oscillates above and below 4800 chirps per hour, from a minimum of $4800 - 2400 = 2400$ to a maximum of $4800 + 2400 = 7200$ chirps per hour. So the range is 2400 to 7200.

Identical ideas about vertical shift, amplitude, period, frequency, and phase shift apply to cosine functions of the form

$$y = D + A\cos[B(x - C)],$$

whose behavior also is described as *sinusoidal.* The midline D serves to raise or lower the "center" of the cosine curve; the amplitude A stretches or shrinks the cosine curve vertically about the midline; the frequency B represents the number of cycles over an interval of 2π; and the parameter C is the phase shift, which shifts the cosine curve to the left or the right, depending on the sign of C. The only difference between working with sines and cosines lies in finding the phase shift. For a cosine function, the phase shift corresponds to the first point to the right of the origin where the curve reaches its maximum.

EXAMPLE 7

Describe the graph of the sinusoidal function $y = 5 + 3\cos\left[2\left(x - \frac{\pi}{4}\right)\right]$.

Solution The basic cosine curve is multiplied by 3, so its amplitude is 3. Because the midline is 5, the function extends from a minimum of $5 - 3 = 2$ to a maximum of $5 + 3 = 8$. Because the frequency is 2, there are two complete cycles between 0 and 2π, so the period is π. Finally, the phase shift is $\pi/4$, so the curve is shifted to the right by $\pi/4$. The graph of this function from $x = 0$ to $x = 2\pi$, shown in Figure 7.26, illustrates all these effects.

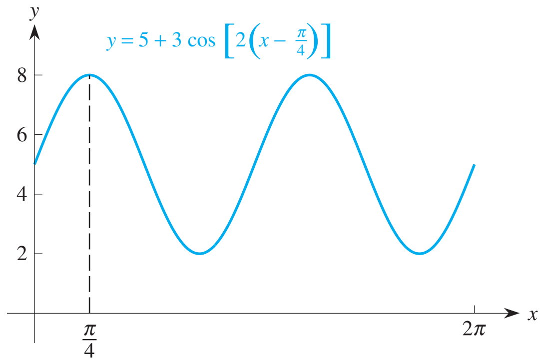


FIGURE 7.26

Combining Sinusoidal and Exponential Functions

Consider a spring hanging vertically from the ceiling with a weight attached at the bottom, as shown in Figure 7.27. The weight is pulled down and then released, so the weight bobs up and down with smaller and smaller oscillations until it settles to a stop in the original rest position, called its equilibrium. Figure 7.28 displays a graph of this vertical displacement y as a function of time t.

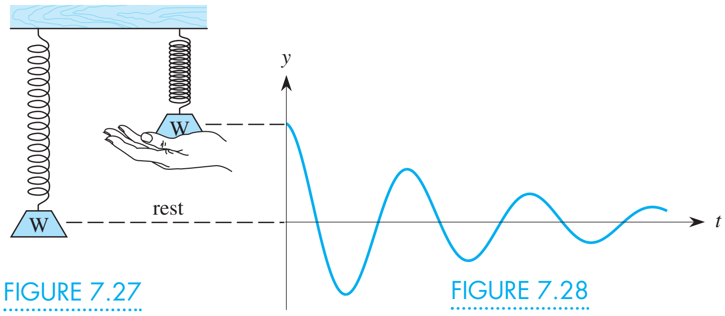


FIGURE 7.27

FIGURE 7.28

EXAMPLE 8

Figure 7.29 shows the results of recording the vertical oscillations of an object attached to a particular spring as a function of time from $t = 0$ to $t = 2\pi$. Construct a function that models this behavior pattern.

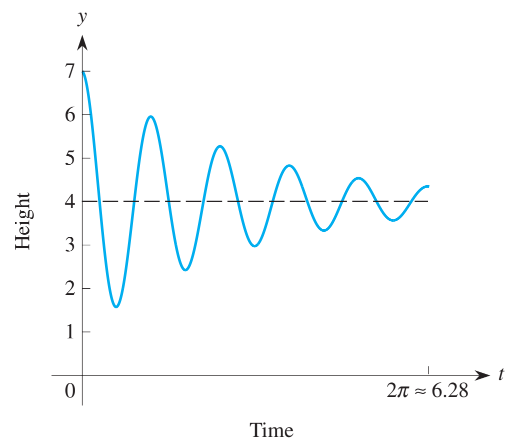


FIGURE 7.29

Solution What type of function could have this kind of behavior pattern? The oscillatory effect certainly suggests a sinusoidal function, either a sine or a cosine, but the amplitude is not constant. In fact, the oscillations eventually die out. The overall effect of the decreasing amplitude might suggest either a decaying power function or an exponential decay function. Because there is a finite starting value, not a vertical asymptote, at time $t = 0$, a power function is not appropriate. An exponential decay term makes more sense. Moreover, based on our discussion in Section 4.6, we might be tempted to consider the product of such an exponential decay function and a sinusoidal function. Two possible formulas for functions that combine these two behavior patterns are

$$y = Ab^t \sin ct \quad \text{or} \quad y = Ab^t \cos ct,$$

with $b < 1$. You can think of the decaying exponential function as a variable amplitude that decays to 0 over time. In addition, there is a vertical shift, so the possible functions are $y = D + Ab^t \sin ct$ or $y = D + Ab^t \cos ct$. Our task is to determine values for the four parameters A, b, c, and D.

First, we note that the initial height of the object is approximately $y = 7$. Starting from there, the object drops at first. Its height decreases from a maximum, which suggests a cosine function rather than a sine function.

Second, we note that the final, or equilibrium, height for the object is about 4, so the object seems to be oscillating about a height of $y = 4$. The maximum height is 7, or 3 units above this equilibrium level, so the form for the function seems to be

$$y = f(t) = 4 + 3b^t \cos ct,$$

where $f(0) = 4 + 3b^0 \cos 0 = 4 + 3(1)(1) = 7$.

Further, we estimate from the graph that, between $t = 0$ and $t = 2\pi \approx 6.28$, there are about five complete diminishing cycles. Thus the frequency for the cosine function is approximately 5, giving the equation

$$y = f(t) = 4 + 3b^t \cos 5t.$$

Finally, consider the exponential decay curve $g(t) = 3b^t$ that is superimposed over the successive peaks of the decaying sinusoidal function in Figure 7.30. It starts with an initial height of 7 and decays to a final level of 4. Using a ruler, we can estimate that it has dropped halfway (to a height of 5.5) at about $t = 2$, Therefore we use $t = 2$ as an approximation for the half-life of the pure exponential decay function $g(t) = 3b^t$. Thus we must solve the equation

$$g(2) = 3b^2 = \tfrac{1}{2}(3),$$

which gives

$$b^2 = 0.5 \quad \text{so that} \quad b = \sqrt{0.5} \approx 0.707.$$

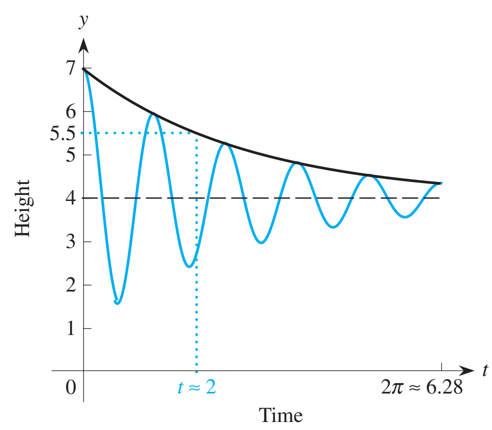


FIGURE 7.30

We can now estimate that the desired function is given by

$$y = f(t) = 4 + 3(0.707)^t \cos 5t.$$

Verify that this function matches the required pattern for the oscillation shown in Figure 7.29 by using your function grapher to graph f between 0 and 2π.

Incidentally, we have implicitly assumed throughout this section that all periodic processes follow a sinusoidal pattern (either a sine or a cosine curve) precisely. In practice, this assumption may be expecting a lot. Think about the length of

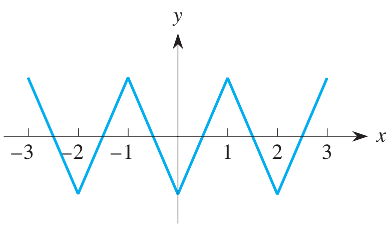
Janis's fingernails in Problem 1 from Section 7.1. The nail length is a periodic function, but it is not sinusoidal. Even if you observe that the overall pattern for some periodic process is smooth and appears to be that of a sine curve, you have no guarantee that the behavior is exactly sinusoidal. Nevertheless, sinusoidal functions are your best models for such types of periodic phenomena and consequently are the models that you should use when faced with such behavior.

Finally, just as you can construct linear, exponential, power, and other functions to fit a set of data, you often will be faced with the problem of having a set of data that exhibits a periodic pattern and wanting to find the periodic function that best fits the data. We ask you to explore several such cases in the following Problems.

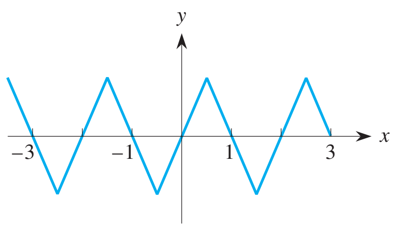
Problems

1. Decide which of the following functions are periodic. For those that are periodic, what is the period? (Assume that each graph continues in the same pattern indefinitely to the left and right.)

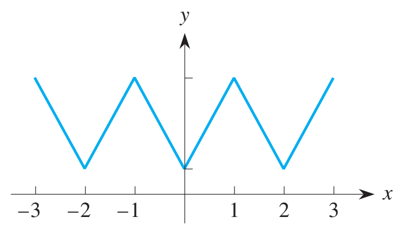
2. Find the number of hours of daylight in San Diego on March 1, on May 12, on July 4.



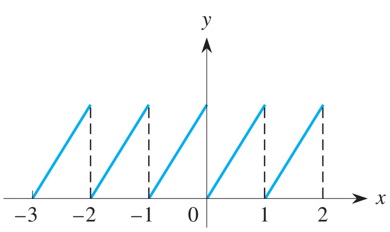
(a)



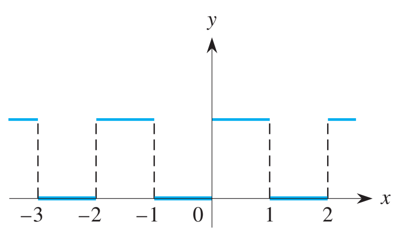
(b)



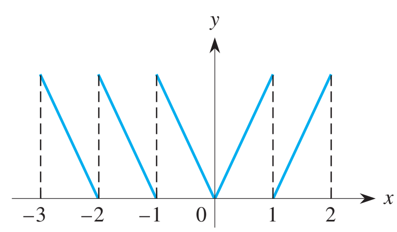
(c)



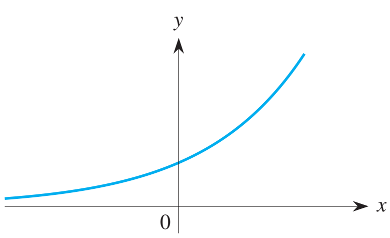
(d)



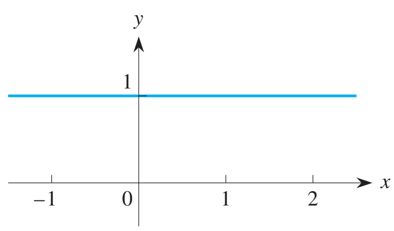
(e)



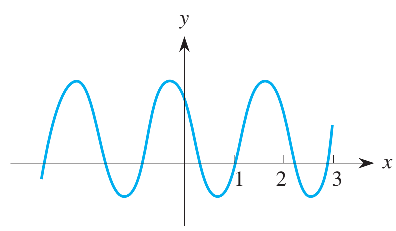
(f)



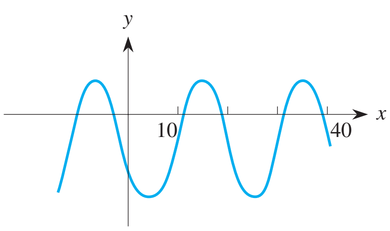
(g)



(h)



(i)



(j)

3. The number of hours of daylight in Montreal is given by

$$H(t) = 12 + 3.6 \sin\left[\frac{2\pi}{365}(t - 80)\right],$$

where t is the number of days from the 1st of the year.

a. What is the amplitude of this function?
b. What is the period of this function?
c. What is the number of hours of daylight on the shortest day of the year?
d. What is the number of hours of daylight on the longest day of the year?

4. The shortest day of the year in Fairbanks, Alaska, has 3.70 hours of daylight. Find a formula for the number of hours of daylight there on any day of the year.

5. Write a formula giving the number of hours of darkness in San Diego as a function of the day of the year.

6. Consider Example 1 regarding the height of the tide at a dock. Suppose that low tide still occurs at 4 A.M. but that high tide actually occurs at 10:30 A.M. Find an equation for the height of the tide as a function of time t.

7. The Bay of Fundy in eastern Canada is known for the highest tides in the world. The tides there rise and fall by as much as 50 feet. If the tidal cycle takes 11 hours, find a sinusoidal function that models the tides in the bay. For convenience, assume that low tide corresponds to a height of 0.

8. The thermostat in Sylvia's home in Baltimore is set at 66°F. Whenever the temperature drops to 66° (roughly every 30 minutes), the furnace comes on and stays on until the temperature reaches 70°.

a. Write a sinusoidal function that models this situation.
b. Gary's thermostat in upper New York State is set the same way. How would the model you created in part (a) change to reflect Gary's climate?
c. Jodi, who lives in central Florida, likewise has her thermostat set to come on at 66°F. How would you change the models you created for parts (a) and (b) to reflect her climate?
d. Is a sinusoidal function necessarily a good model? Explain. (*Hint:* Think about the *rates* at which the temperature increases and decreases.)

9. Ocean waves move in a roughly sinusoidal pattern. As a rule of thumb, the length of a wave (crest to crest, say) on the open seas is about 20 times the height of the wave (trough to crest). (This rule doesn't apply near coastlines where waves are much choppier and their intervals shorter.)

a. Write a formula for ocean waves that are 4 feet high in moderately calm seas.
b. Write a formula for ocean waves that are 15 feet high in rough seas.

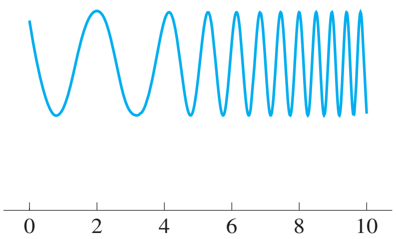
10. Meryl is a normal individual with a pulse rate of 72 beats per minute and a blood pressure of 120 over 80. Thus her heart is beating 72 times each minute and her blood pressure is oscillating between a low (diastolic) reading of 80 and a high (systolic) reading of 120. Assume that the oscillation in Meryl's blood pressure can be modeled by a sinusoidal function.

a. What is the period of this sinusoid?
b. What is the frequency of this sinusoid?
c. What is the equation of this sinusoid?

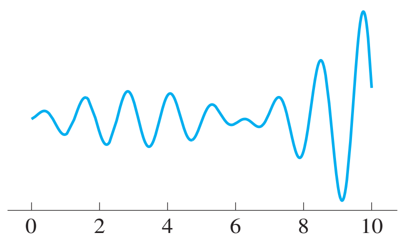
11. Your Thanksgiving turkey is taken from a refrigerator at 40°F and placed in an oven set at 350°. Suppose that the temperature of the bird is 130° after 60 minutes. You know that an oven cycles on and off as some of the heat escapes. Suppose that the cycle occurs every 10 minutes and that the actual temperatures inside the oven oscillate between 340° and 360°.

a. Use this information to construct a sinusoidal function to model the temperature of the oven as a function of time t.
b. Use Newton's law of heating from Section 5.4 to estimate how large a variation is possible in the temperature of the turkey after 60 minutes, and after 100 minutes. (*Hint:* Solve the problem with the minimum and maximum oven temperatures.)

12. A standard radio has two bands—the AM (*amplitude modulation* band) and the FM (*frequency modulation* band). In one case, the amplitude of a sinusoidal wave is modulated (varied) to produce the desired output sounds; in the other, the frequency of a sinusoidal wave is modulated. Which of the following represents an AM sound and which represents an FM sound?



(a)



(b)

13. Two successive turning points of a sinusoidal function are at (8, 72) and (20, 30).

a. Write a possible formula for this function, using a sine function.

b. Write a possible formula for this function, using a cosine function.

14. Two successive inflection points of a sinusoidal function are at (6, 20) and (18, 20); the maximum attained by the function is 43.

a. Write a possible formula for this function, using a sine function.

b. Write a possible formula for this function, using a cosine function.

15. Suppose that the historical average daytime high temperature in Fairbanks ranges from a low of −20°F to a high of 64°F and that the coldest day of the year, historically, is the 40th day. Write a formula for a sinusoidal function that can be used to model the average daytime high temperature in Fairbanks as a function of the day of the year.

16. Sketch by hand the graph of each function. Draw the basic curve $y = \sin x$ or $y = \cos x$ on the same set of axes for comparison. (Do not use your function grapher.)

a. $y = 3 \sin 4x$ **b.** $y = 3 \sin\left(\frac{1}{2}x\right)$

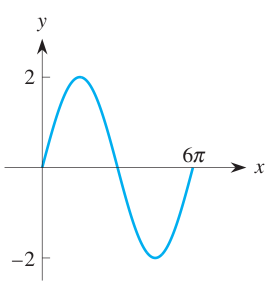
c. $y = 2 \sin 3x$ **d.** $y = 4 \cos 2x$

e. $y = -3 \cos 2x$ **f.** $y = 4 + 2 \sin x$

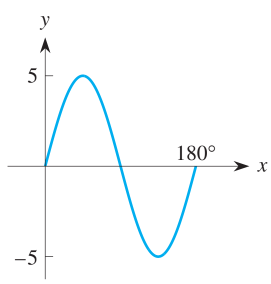
g. $y = \sin\left(x - \frac{\pi}{4}\right)$ **h.** $y = 3 \sin\left(2x - \frac{\pi}{6}\right)$

i. $y = 4 + 2 \cos\left(x + \frac{\pi}{3}\right)$

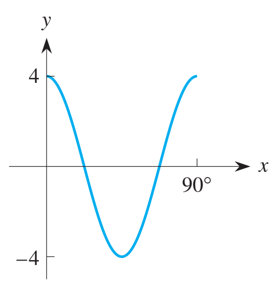
17. Write a possible formula for each sinusoidal function (*a*)–(*l*) from its graph.



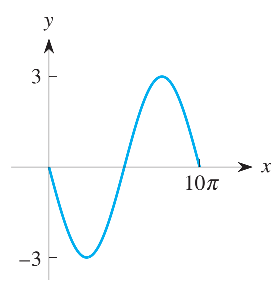
(a)



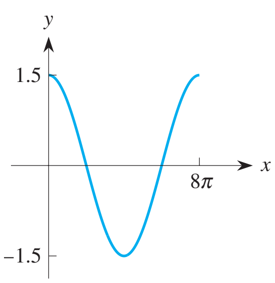
(b)



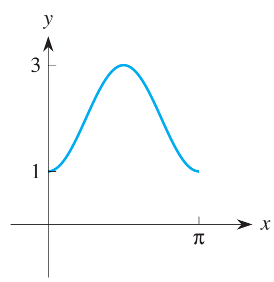
(c)



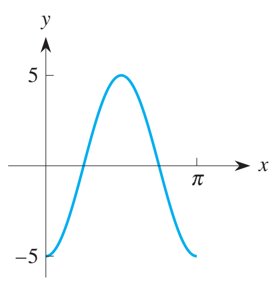
(d)



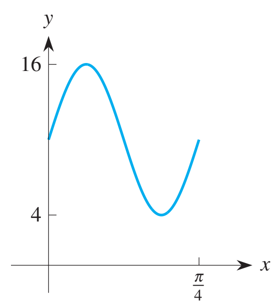
(e)



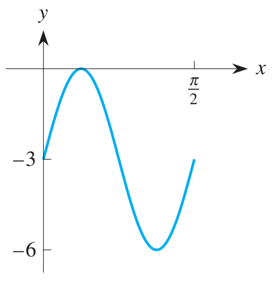
(f)



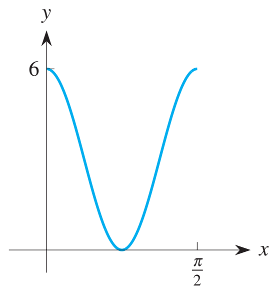
(g)



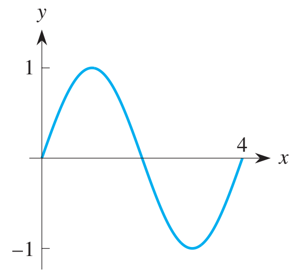
(h)



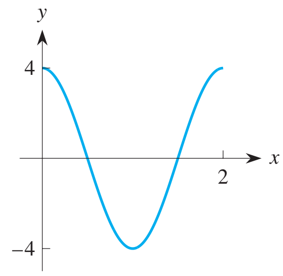
(i)



(j)



(k)



(l)

18. Identify a possible formula for each of the six sinusoidal functions $f_1, f_2, \ldots, f_6$ whose values are given in the table. Note that there are many possible correct answers.

x	f_1	f_2	f_3	f_4	f_5	f_6
−6	0.279	2.279	0.559	0.537	−0.141	0.721
−5	0.959	2.959	1.918	0.544	−0.598	0.041
−4	0.757	2.757	1.514	−0.989	−0.909	0.243
−3	−0.141	1.859	−0.282	0.279	−0.997	1.141
−2	−0.909	1.091	−1.819	0.757	−0.841	1.909
−1	−0.841	1.159	−1.683	−0.909	−0.479	1.841
0	0.000	2.000	0.000	0.000	0.000	1.000
1	0.841	2.841	1.683	0.909	0.479	0.159
2	0.909	2.909	1.819	−0.757	0.841	0.091
3	0.141	2.141	0.282	−0.279	0.997	0.859
4	−0.757	1.243	−1.514	0.989	0.909	1.757
5	−0.959	1.041	−1.918	−0.544	0.598	1.959
6	−0.279	1.721	−0.559	−0.537	0.141	1.279

19. The table gives the outdoor temperatures in °F in Chicago during one 24-hour period.

Midnight	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	
53	48	47	49	53	59	
Noon	2 P.M.	4 P.M.	6 P.M.	8 P.M.	10 P.M.	Midnight
66	71	68	65	58	54	53

If you were to fit a sinusoidal function to this set of data, what is the vertical shift? the amplitude? the period? the frequency? What is the equation of the resulting sinusoidal function?

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Avg. daily high temp. (°F)	65.2	64.4	65.9	67.8	68.6	71.3	75.6	77.6	76.8	74.6	69.9	66.1

20. The table above shows the average daytime high temperature each month in San Diego.

a. Construct a sinusoidal function that best fits these data.

b. How does the phase shift for this function compare to the phase shift used in the text for the number of hours of daylight in San Diego? In particular, explain in practical terms why the sinusoidal function for air temperature lags behind the function for hours of daylight.

21. The table gives the average daytime high temperature in Dallas on different days of the year (roughly every 2 weeks), based on historical weather records.

a. Assuming that the temperature behavior in Dallas is periodic from year to year, determine a sinusoidal function that models the average daytime high temperature in Dallas.

b. The values shown in the table are temperatures roughly every 2 weeks, but two entries are missing. Use your model from part (a) to predict the average daytime high temperature in Dallas on the missing dates.

Day	1	15	32	46	60	74	91	105	121	135	152
Avg. daily high temp. (°F)	55	53	56	59	63	67	72	77	81	84	89
Day	196	213	227	244	258	274	288	305	319	335	349
Avg. daily high temp. (°F)	98	99	98	94	90	85	80	72	66	61	58

22. The table shows the average number of tornados reported in the United States per month based on historical records.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Tornados	16	24	60	111	191	179	96	66	41	26	31	22

Determine a sinusoidal function that models the monthly number of tornados as a function of time.

23. For a normal adult at rest, the rate R, in liters per second, at which air flows in and out of the lungs can be modeled by the function $R(t) = 0.85 \sin[(2\pi/5)t]$, where t is measured in seconds. The person is inhaling when $R > 0$ and exhaling when $R < 0$. How many times does the person breathe per minute?

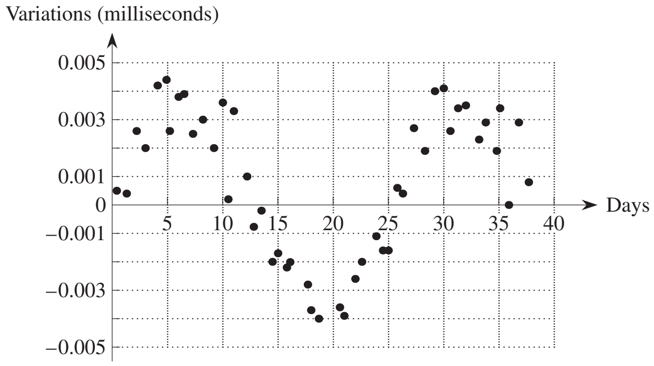
24. Astronomers recently reported the discovery of the first known planets outside the solar system. They found three worlds orbiting around a pulsar, a rotating star that emits radiation with constant frequency. For this pulsar, the astronomers detected slight variations in the intensity of the radiation, as shown in the accompanying figure. This variation would be the effect of a planet in orbit about the pulsar.

a. From the figure, estimate the length of the year for the planet.

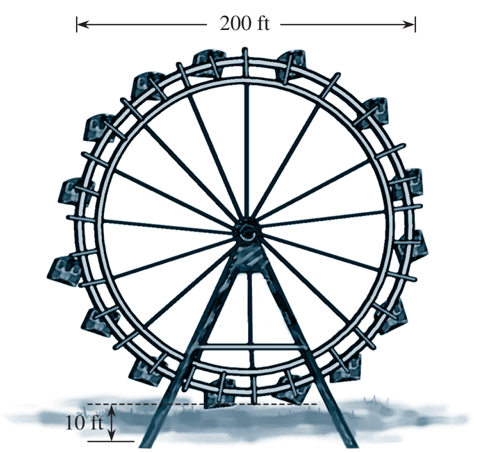
b. Use Kepler's law from Example 4 in Section 3.6 (assuming that the same coefficient applies) to calculate the distance from this planet to its star.

c. Assuming that the orbit of this new planet is circular, how fast is it moving in its orbit about the pulsar?

d. For comparison, Earth takes 365 days to complete one revolution about the sun at a distance of about 93 million miles. How fast is Earth moving in its orbit about the sun?



25. You are taking a ride on a Ferris wheel that is 200 feet in diameter and has a bottom point 10 feet above the ground. Suppose that the Ferris wheel rotates twice every minute and, from your friend's viewpoint on the ground, is rotating clockwise.



a. Sketch your height y above the ground as a function of time t.

b. Find a formula for your height y as a function of t. Does it agree with your rough sketch in part (a)?

c. Find a formula for the horizontal distance x from the center of the wheel as a function of t.

d. Find all intervals of t values for which you are moving forward. Indicate these intervals on the graph of the function in part (c). What do you observe?

e. Suppose that the Ferris wheel rotates in the opposite direction, so it is now moving counterclockwise. How do your answers to parts (c) and (d) change?

f. Find a formula relating your height y above the ground and the horizontal distance x from the vertical axis through the center of the wheel.

26. A Ferris wheel is 12 meters in diameter and completes one full revolution every 20 seconds. If the bottom of the Ferris wheel is 2 meters above the ground, write a formula for the height above ground of a person on the Ferris wheel as a function of time.

27. Certain stars are called *variable stars* because their brightness increases and decreases in a periodic manner. The brightest variable star that can be observed from Earth is Delta Cephei, whose brightness varies between a minimum brightness (or magnitude) level of 3.65 and a maximum brightness of 4.35, with a cycle of 5.4 days. Write an equation that represents the brightness of Delta Cephei as a function of time t based on $t = 0$ at an instant when the star has minimal brightness.

28. Many people believe that a person's life is determined by three independent cycles, called *biorhythms.* One cycle, with a period of 23 days, represents the physical or health dimension of a person, $H(t) = \sin(2\pi t/23)$, where time t is measured in days starting at birth. A second cycle, with a period of 28 days, represents the emotional or sensitivity aspects of a person, $E(t) = \sin(2\pi t/28)$. A third cycle, with a period of 33 days, represents the mental or intellectual aspects of an individual, $M(t) = \sin(2\pi t/33)$.

a. Suppose that Tony was born on January 1. Consider the 60-day period immediately following his 20th birthday. What set of values for t are appropriate?

b. Which days would you recommend as being suitable for Tony to compete in a track-and-field meet?

c. Which days would you recommend as being good days for Tony to ask his girlfriend to marry him?

d. Which days could you suggest as days on which Tony could hope to have a major exam at school?

e. Are there any days when you would recommend that Tony simply not get out of bed?

f. Are there any days when all the signs are highly positive?

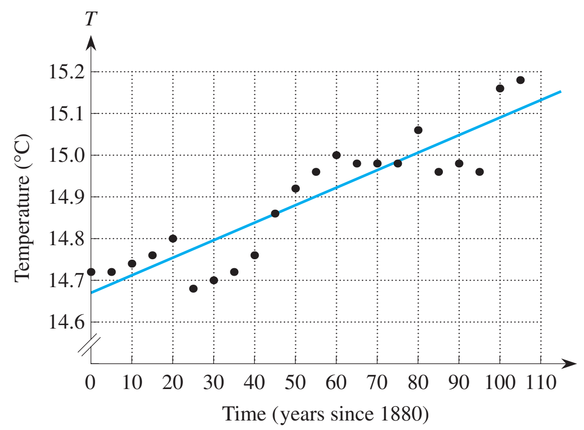
29. As part of a study on the possibility of global warming at a National Science Foundation math modeling workshop at Pellissippi State College, the accompanying scatterplot was produced. It suggests that the average global temperature values appear to oscillate about the regression line,

$T = 0.0042t + 14.67$, where t represents years since 1880.

a. Use the scatterplot to estimate the parameters for a sinusoidal function that oscillates above and below the indicated line. What is the equation of the resulting function?

b. Use your function grapher to draw the graph of that function. Does it have the correct shape?

c. What is your prediction for the average global temperature in 2005, based on the combination of the given linear function and the sinusoidal function you created?



7.3 Solving Equations with Sine and Cosine: The Inverse Functions

We have shown that the number of hours H of daylight in San Diego as a function of the day of the year t is given by

$$H(t) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right].$$

Suppose that we now ask: When will there be 13 hours of daylight? That is, on which day t of the year will $H = 13$? To find this date, as illustrated in Figure 7.31, we must solve for the independent variable t in the equation

$$H(t) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right] = 13.$$

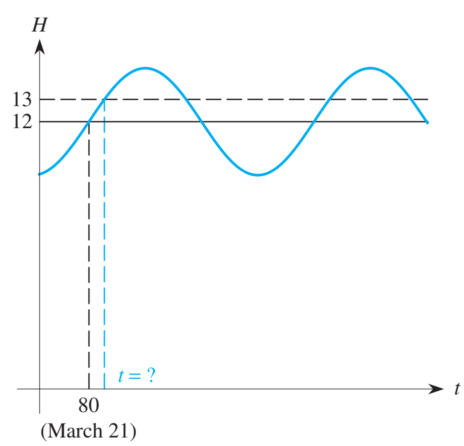


FIGURE 7.31

We can solve such an equation in a variety of ways, as we illustrate in Examples 1 and 2.

EXAMPLE 1

Determine graphically all days on which there will be 13 hours of daylight in San Diego.

Solution The function $H(t)$ oscillates between 9.6 and 14.4, so there are 13 hours of daylight on two different days each year, as indicated in Figure 7.31. One of these days occurs in the spring when the days are lengthening; the other occurs during the fall when the days are shortening. When we trace the graph of this function, we find that the two solutions are $t \approx 105$ and $t \approx 238$. The 105th day of the year is April 15 (31 days in January + 28 in February + 31 in March + 15 in April = 105). The 238th day of the year is August 26. Moreover, these same values will occur *every* year because the function is periodic. Check this result on your calculator by evaluating $H(105)$ and $H(238)$.

EXAMPLE 2

Determine algebraically when there will be 13 hours of daylight in San Diego.

Solution We start with the equation

$$H(t) = 12 + 2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right] = 13.$$

To solve algebraically for t, we first subtract 12 from both sides:

$$2.4 \sin\left[\frac{2\pi}{365}(t - 80)\right] = 1.$$

We next divide both sides by 2.4:

$$\sin\left[\frac{2\pi}{365}(t - 80)\right] = \frac{1}{2.4} = 0.417.$$

Our task now is to extract the variable t from the argument of the sine function.

Compare this problem to the situation we repeatedly faced of extracting the variable from an exponential function, such as 10^x. We solved that problem by using logarithms to undo the exponential function. The reason that this method works is because the exponential and logarithmic functions are inverse functions of one another.

Similarly, we can undo the sine function by using the *inverse sine function*. You can do so on your calculator by pressing either `INV` or `2nd` followed by `SIN` to get the arcsine function. When you do this in radian mode, you will find that

$$\arcsin 0.417 \approx 0.430;$$

that is, the value whose sine is 0.417 is 0.430 radians. Therefore

$$\frac{2\pi}{365}(t - 80) = \arcsin 0.417 \approx 0.430.$$

To solve for t, we now multiply both sides by 365 and get

$$2\pi(t - 80) = (0.430)(365) = 156.95.$$

Dividing both sides by 2π yields

$$t - 80 = \frac{156.95}{2\pi} = 24.98 \approx 25.$$

Hence

$$t = 25 + 80 = 105.$$

That is, there will be 13 hours of daylight in San Diego on approximately the 105th day of the year (April 15), which is the same answer we obtained graphically in Example 1.

Actually, the result in Example 2 is not complete because it is only one of the two possible days each year on which the sinusoidal function H passes across the 13-hour level. However, this is the only solution that you can get directly from a calculator or computer when you use the inverse sine function. You can determine the other day when 13 hours of daylight occurs by using the following line of reasoning, based on some key facts about the sine function. We found that the solution $t = 105$ corresponds to April 15, which is 25 days *after* the spring equinox on March 21. Using the symmetry of the sine curve, we should expect that there will also be 13 hours of daylight 25 days *before* the fall equinox on September 21. But 25 days before September 21 is August 27, which is roughly the other solution we found in Example 1. Finally, because of the periodicity of the sine function, there will be 13 hours of daylight in San Diego on April 15 and August 27 every year.

The Inverse Sine Function

Let's now examine more carefully what the inverse sine function is all about and the reason for the limitation in Example 2. Recall that a continuous function f has an inverse f^{-1} when it is either strictly increasing or strictly decreasing or, equivalently, if it satisfies the horizontal line test. Obviously, the sine function does not fulfill either of these conditions because of its shape. The only way to obtain an inverse for the sine function is to restrict its domain, as we did for the parabola in Section 2.9 when we considered only the right side of the parabola $y = x^2$. We thus use only a small portion of the sine curve $y = \sin\theta$—where the function is strictly increasing. By convention, the restricted domain for the sine function is from $\theta = -\pi/2$ to $\theta = \pi/2$, as depicted in Figure 7.32.

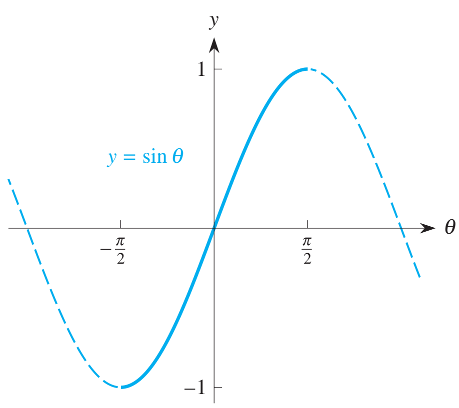


FIGURE 7.32

Let's work temporarily in degrees. Suppose that

$$\sin\theta = 0.825$$

and θ is some unknown value. From the graph of the sine function, we expect that θ will be closer to 90° than to 0°. The inverse sine function, arcsin y or $\text{Sin}^{-1}y$, allows us to solve for the correct value of θ. That is, if

$$\sin\theta = 0.825,$$

then

$$\theta = \arcsin(0.825) = 55.59°.$$

Equivalently, if we use radians instead of degrees, then

$$\theta = \arcsin(0.825) = 0.9702 \text{ radians.}$$

Thus the inverse sine function undoes the sine function to extract any value θ between $-90°$ and $90°$ or, equivalently, between $-\pi/2$ and $\pi/2$ radians. Check this result by taking the sine of 55.59° or 0.9702 radians. It may be helpful to think of arcsin y as *the angle whose sine is y.*

In general, for any function f, if $y = f(x)$ and $x = g(y) = f^{-1}(y)$ are inverse functions, we know that

$$f^{-1}(f(x)) = x, \qquad \text{for all } x \text{ in the domain of } f;$$
$$f(f^{-1}(y)) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1}.$$

In the case of the sine function and its inverse, we have

$$\arcsin(\sin\theta) = \theta, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2};$$
$$\sin(\arcsin y) = y, \qquad -1 \le y \le 1.$$

As we found previously, when we wanted to find the day on which there would be 13 hours of daylight in San Diego, the inverse sine function returned only $t = 105$. This is the only answer we get because we must restrict the domain in order to have an inverse function.

Let's try a few values for y to see the effects of the inverse sine function.

If $y = 0.2$, $\arcsin 0.2 = 11.5°$.
If $y = 0.6$, $\arcsin 0.6 = 36.9°$.
If $y = 0.95$, $\arcsin 0.95 = 71.8°$.
If $y = -0.4$, $\arcsin(-0.4) = -23.6°$.
If $y = -0.88$, $\arcsin(-0.88) = -61.6°$.

What if you try to find arcsin(2.5)? Most calculators will return an error message, usually indicating a problem with the domain. The reason is that you are trying to find a number whose sine is 2.5. But the only permissible values for the sine function are between -1 and 1. Thus, if you try to use any value outside this interval, the inverse sine function is not defined and you will get an error message on most calculators. (Some models return a complex number instead of an error message, but that is a topic for considerably more advanced mathematics courses.)

You have seen that when you use any value for y between -1 and 1, the inverse sine function returns an answer between $-90°$ and $90°$ in degree mode or between $-\pi/2$ and $\pi/2$ in radian mode. These values are called the *principal values* of the inverse sine. The inverse sine function does not give any values larger than 90° (or $\pi/2$) or smaller than $-90°$ (or $-\pi/2$). It is up to you to realize that, for any real number k between -1 and 1, there are infinitely many values for θ whose sine is k. You can find these values by visualizing the graphs of the sine function and the horizontal line $y = k$ and determining all points where they intersect, that is, where $\sin\theta = k$. You can also use what you know about the symmetry of each arch of the sine curve. Finally, you can always estimate these values graphically, an approach that is usually straightforward. In Examples 3–5 we demonstrate the algebraic approach and the associated reasoning, because it is more difficult than the graphical approach.

EXAMPLE 3

Find all values of θ, in degrees, for which $\sin\theta = 0.6$.

Solution If $y = \sin\theta = 0.6$,

$$\arcsin 0.6 = 36.9° \approx 37°,$$

which occurs while the sine curve is increasing. We have to find the second solution which occurs while the sine curve is decreasing. We know that the sine curve reaches its maximum height of 1 at 90° and is symmetric about 90°. Therefore, because 37° is 53° *before* 90°, a second value at which the sine function reaches the same height of 0.6 occurs 53° *after* 90°, or at $\theta = 143°$, as shown in Figure 7.33. (Verify that $\sin 143° \approx 0.6$ with your calculator.)

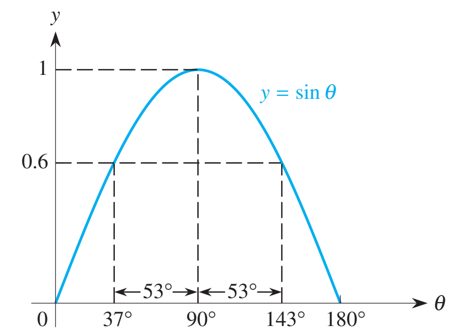


FIGURE 7.33

Further, because of the periodicity of the sine curve, we know that the same behavior pattern will repeat every 360°. Therefore, other values of θ whose sine is 0.6 are at $37° + 360° = 397°$, at $143° + 360° = 503°$, at $37° + 2(360°) = 757°$, at $143° + 2(360°) = 863°$, and so on. Verify some of these values with your calculator as well.

In Example 5 of Section 7.2, we created the sinusoidal function

$$T = 71 + 3\sin\left(\frac{\pi}{10}t\right)$$

to model the temperature in a house where the air conditioning control is set to turn on the air conditioner when the temperature rises to 74°F and to turn it off when the temperature drops to 68°F, a cycle that repeats every 20 minutes. Also, we were told that, at noon, the temperature was 71°F and rising. We now consider some inverse predictions based on this model.

EXAMPLE 4

Use the sinusoidal model to determine all times between noon and 1 P.M. when the temperature in the house is 70°F.

Solution We need to solve the equation

$$T = 71 + 3\sin\left(\frac{\pi}{10}t\right) = 70.$$

We show an algebraic solution. We first subtract 71 from both sides and get

$$3\sin\left(\frac{\pi}{10}t\right) = -1 \quad \text{so that} \quad \sin\left(\frac{\pi}{10}t\right) = -\frac{1}{3}.$$

Consequently, in radians

$$\frac{\pi}{10}t = \arcsin\left(-\frac{1}{3}\right) \approx -0.3398,$$

so that

$$t \approx \frac{10}{\pi}(-0.3398) \approx -1.08,$$

or about 1 minute *before* noon. The air conditioning cycle takes 20 minutes, so we conclude that the temperature will be 70° again about one minute *before* 12:20, or at about 12:19, again at about 12:39, and once more at about 12:59. All these values occur while the sine curve is increasing, as shown in Figure 7.34.

To find the times that the temperature is 70° while the sine curve is decreasing, we reason as follows. A complete cycle takes 20 minutes, so a half cycle takes 10 minutes. The first time the temperature reaches 70° while the curve is increasing is at about $t = 19$ minutes, or 1 minute *before* the end of the cycle. Therefore, from the symmetry of the sine curve over the first 20 minutes, as illustrated in Figure 7.35, the first time the sine curve passes the 70° level while the curve is decreasing must occur about 1 minute after the middle ($t = 10$ minutes) of the cycle. That is, the other solution is at about $t = 11$ minutes, or at about 12:11. Because of the periodic nature of the sine function, the 70° temperature will also occur at about 12:31 and at about 12:51.

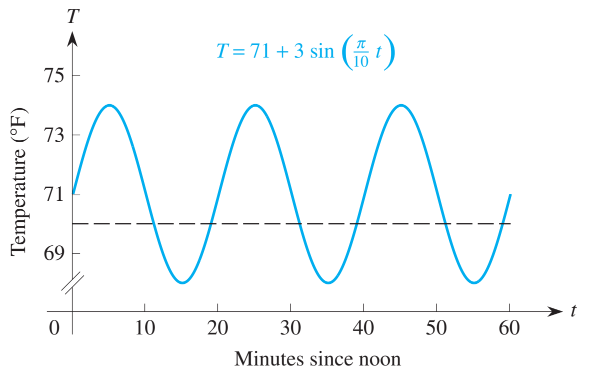


FIGURE 7.34

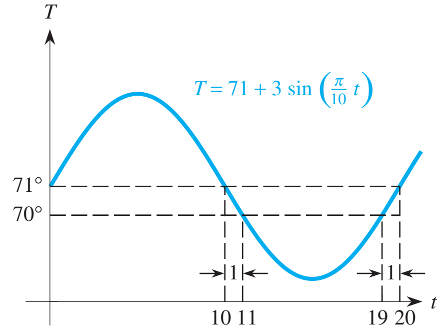


FIGURE 7.35

EXAMPLE 5

In Example 6 of Section 7.2, we created the sinusoidal function

$$C = 4800 + 2400 \sin\left(\frac{\pi}{12}t\right)$$

to model a cricket's chirp rate C in chirps per hour, where t is measured in hours since 9 A.M. At what times, if any, does the cricket chirp at a rate of 6000 times per hour?

Solution We first solve this problem graphically. The graph of C over a 24-hour period starting at 9 A.M. (when $t = 0$) is shown in Figure 7.36. Note the horizontal line at a height of 6000. The times when the cricket chirps at a rate of 6000 times per hour are the points at which the curve intersects the line. If we zoom in on the graph about these two points, we find that t is approximately $t = 2$ hours after 9 A.M. (or about 11 A.M.) and $t = 10$ hours after 9 A.M. (or about 7 P.M.).

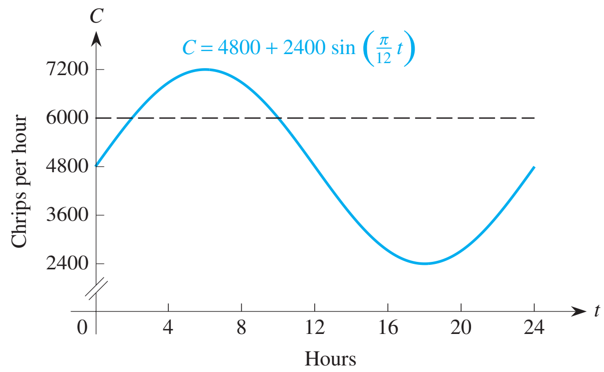


FIGURE 7.36

Alternatively, let's solve for t algebraically from

$$C = 4800 + 2400\sin\left(\frac{\pi}{12}t\right) = 6000.$$

We subtract 4800 from both sides to get

$$2400\sin\left(\frac{\pi}{12}t\right) = 1200$$

so that

$$\sin\left(\frac{\pi}{12}t\right) = \frac{1200}{2400} = 0.5.$$

Using the inverse sine function in radian mode, we get

$$\frac{\pi}{12}t = \arcsin(0.5) \approx 0.5236$$

so that

$$t \approx \frac{12}{\pi}(0.5236) \approx 2.0000,$$

or 2 hours after 9 A.M., which is 11 A.M., as we found graphically.

To find the other time of day when the cricket is chirping 6000 times per hour, we note that the period of the sinusoidal function is 24 hours, starting at 9 A.M., so a half cycle takes 12 hours. The symmetry of the sinusoidal function, as depicted in Figure 7.37, shows that the other time that the curve passes a height of 6000 must be 2 hours before the 12-hour mark at 9 P.M. (or 10 hours after 9 A.M.), which is at 7 P.M.

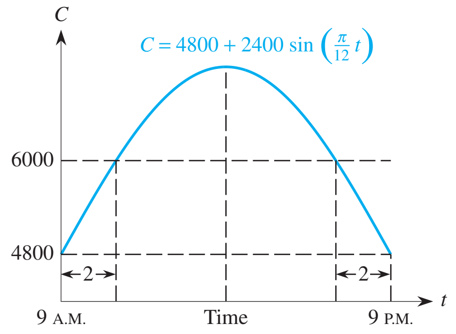


FIGURE 7.37

We summarize the important properties of the inverse sine function as follows.

Properties of the Inverse Sine Function $y = \arcsin x$

1. $\arcsin x$ is defined only for values of x (the domain) between -1 and 1.
2. The principal values for y (the range) lie between $-\pi/2$ and $\pi/2$ radians.
3. $\arcsin(\sin y) = y,\quad$ for $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.
4. $\sin(\arcsin x) = x,\quad$ for $-1 \le x \le 1$.

In general, suppose that f^{-1} is the inverse of a function f. If we write the two functions in terms of the same independent variable x so that $y = f(x)$ and $y = f^{-1}(x)$, their graphs are mirror images of each other about the line $y = x$, as we showed in Section 2.9. (Visualize the graphs of the exponential and logarithmic functions.) Using this fact, we can easily construct the graph of the inverse sine function from our knowledge of the graph of the sine function. We begin with the sine curve shown in Figure 7.38(a), in which the portion corresponding to the domain from $-\pi/2$ to $\pi/2$ is highlighted. When we reflect the highlighted portion of the sine curve about the line $y = x$, we get the graph of the inverse sine function; both graphs are shown in Figure 7.38(b). Finally, the graph of the inverse sine function alone is shown in Figure 7.38(c). Note that $y = \arcsin x$ exists only for values of x between -1 and 1 and that the corresponding heights range from $-\pi/2$ to $\pi/2$.

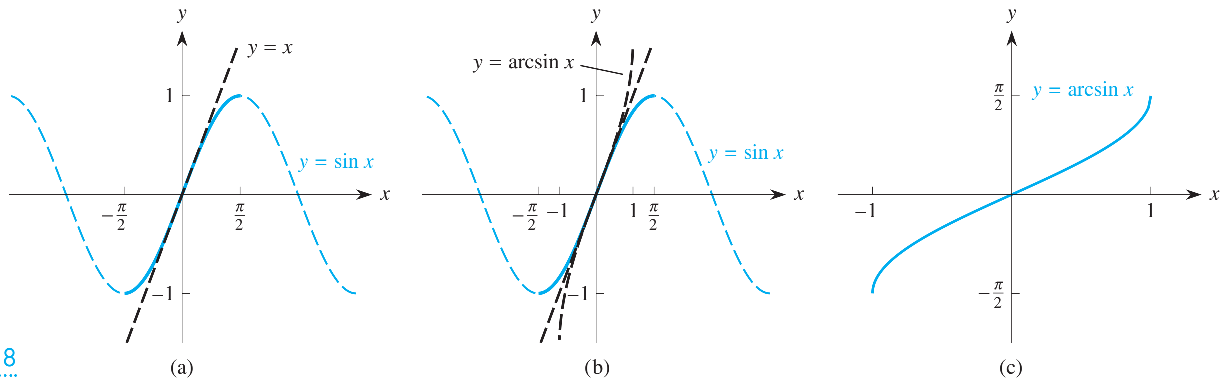


FIGURE 7.38

The Inverse Cosine Function

Suppose that

$$\cos\theta = 0.3$$

and we want to find a number θ whose cosine has this value. To solve this equation, we introduce the *inverse cosine function*, $y = \arccos x$, using ideas that are similar to those that led to the inverse sine function. We use the inverse cosine (press `2nd COS` or `INV COS` on your calculator, in `Degree` mode) to get

$$\theta = \arccos 0.3 \approx 72.5°.$$

Verify that $\cos 72.5° \approx 0.3$.

We know that the only possible values for $y = \cos\theta$ lie between -1 and $+1$. For the inverse cosine function $\theta = \arccos y$, we must restrict the domain to a suitable portion of the cosine curve where the cosine function is either strictly increasing or strictly decreasing. By convention, we consider only values of θ between 0° and 180° (or equivalently between 0 and π radians) where the cosine function is strictly decreasing, as shown in Figure 7.39. These are the principal values for the inverse cosine and they are the only values that your calculator will return when you use the inverse cosine function. As with the inverse sine function, you will have to use what you know about the behavior of the cosine graph, including the symmetry of the arches on the graph, if you want to determine all other numbers having the specified cosine value.

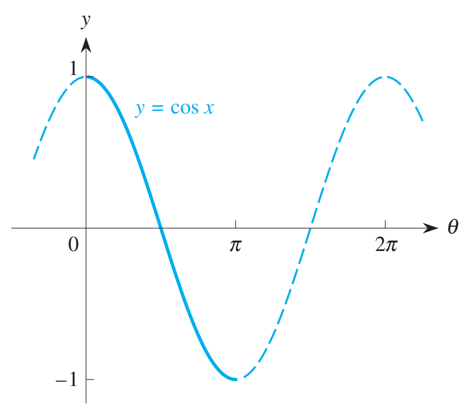


FIGURE 7.39

EXAMPLE 6

Find all values of θ in both degrees and radians for which $\cos\theta = 0.92$.

Solution Although we can solve this problem graphically, we illustrate the details of the algebraic solution. Because $\cos\theta = 0.92$,

$$\theta = \arccos(0.92) \approx 23°$$

in degree mode. Because of the periodicity of the cosine function, we also know that the value of 0.92 will repeat every 360°, so the solutions include

$$\theta = 23°,\quad 23° + 360°,\quad 23° + 2(360°),\quad 23° + 3(360°), \ldots,$$

or, in general,

$$\theta = 23° + n\cdot(360°), \qquad \text{for any integer } n \geq 0.$$

Moreover the graph of the cosine function shown in Figure 7.40 indicates that there must be another value of θ just before 360° whose cosine is also 0.92. In particular, because our first solution is $\theta = 23°$, the other value must be 23° before 360°, or $360° - 23° = 337°$. Thus the solutions also include

$$\theta = 337°,\quad 337° + 360°,\quad 337° + 2(360°),\quad 337° + 3(360°), \ldots,$$

or, in general,

$$\theta = 337° + n\cdot(360°), \qquad \text{for any integer } n \geq 0.$$

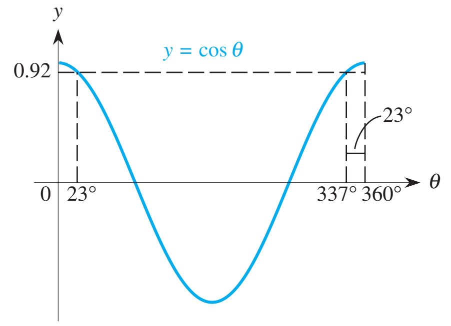


FIGURE 7.40

Using radian measure gives the equivalent solutions

$$\theta = 0.401 + 2n\pi \text{ radians} \quad \text{and} \quad \theta = 5.882 + 2n\pi \text{ radians}, \qquad \text{for any integer } n \geq 0.$$

Moreover, the cosine function is symmetric about the vertical axis $\theta = 0$ (it is an even function; see Appendix D). Therefore we know that the same patterns repeat with negative values for θ, so all the other solutions are

$$\theta = -23° - n \cdot (360°) \quad \text{and} \quad \theta = -337° - n \cdot (360°), \qquad \text{for any integer } n \geq 0.$$

Equivalently, with radian measure, $\theta = -0.401 - 2n\pi$ radians and $\theta = -5.882 - 2n\pi$, for any integer $n \geq 0$.

EXAMPLE 7

A clock is mounted on the wall with its center 7 feet (or 84 inches) above the floor. Suppose that the minute hand is 5 inches long.

a. Write a formula for the vertical height y of the arrowhead on the minute hand above or below the horizontal line through the center of the clock as a function of the time t in minutes from the instant that the minute hand is pointing vertically upward to the 12.

b. Determine all times during the first hour when the arrowhead on the minute hand is 2 inches above that horizontal line.

Solution

a. The midline, or vertical shift, is 84 inches above floor level. The minute hand is 5 inches long, so the height of the arrowhead on the hand oscillates between $84 - 5 = 79$ and $84 + 5 = 89$ inches above the floor, giving an amplitude of 5. This cycle repeats every 60 minutes, so the period is 60 and the frequency is $2\pi/60 = \pi/30$. Because $t = 0$ corresponds to the instant when the hand is pointing vertically upward, the initial height for the arrowhead is $y = 84 + 5 = 89$ inches. This suggests that we use a cosine function with a phase shift of 0 as our model. The resulting function is

$$y = 84 + 5\cos\left(\frac{\pi}{30}t\right).$$

b. To find all times when the arrowhead is 2 inches above the midline (equivalently, when the arrowhead is $84 + 2 = 86$ inches above the floor), we need to solve the equation

$$84 + 5\cos\left(\frac{\pi}{30}t\right) = 86.$$

Proceeding algebraically, we have

$$5\cos\left(\frac{\pi}{30}t\right) = 2 \quad \text{so that} \quad \cos\left(\frac{\pi}{30}t\right) = \frac{2}{5} = 0.4.$$

Therefore, in radians,

$$\frac{\pi}{30}t = \arccos 0.4 \approx 1.1593$$

so that

$$t \approx \frac{30}{\pi}(1.1593) \approx 11.07,$$

or about 11 minutes after the hour, as illustrated in Figure 7.41.

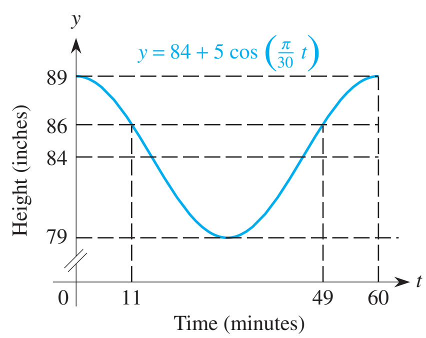


FIGURE 7.41

Because the first solution occurs at about 11 minutes after the hour, we know from the symmetry of the cosine curve that the same height must occur at about 11 minutes before the next hour, or at $t = 49$.

We summarize the important properties of the inverse cosine function as follows.

Properties of the Inverse Cosine Function $y = \arccos x$

1. $\arccos x$ is defined only for values of x (the domain) between -1 and 1.
2. The principal values for y (the range) lie between 0 and π radians.
3. $\arccos(\cos y) = y$, for $0 \le y \le \pi$.
4. $\cos(\arccos x) = x$, for $-1 \le x \le 1$.

Finally, as with the graph of the inverse sine function, the graph of the inverse cosine function $y = \arccos x$ is the mirror image of the cosine graph about the line $y = x$, as shown in Figure 7.42. Note that the inverse cosine is defined only for x between -1 and 1 and that the inverse cosine values lie between 0 and π.

As a final note, you may find the names arcsine and arccosine to be rather strange. To see where they come from, think about how we defined radians. In a unit circle, we measured a length of 1 and defined the corresponding angle to be 1 radian. The same is true for any angle—its measure in radians equals the length of arc along the unit circle. So, to solve $\sin\theta = a$, say, we find the angle θ that equals the length of an arc on the circle corresponding to the value of a. Thus we have arcsin a.

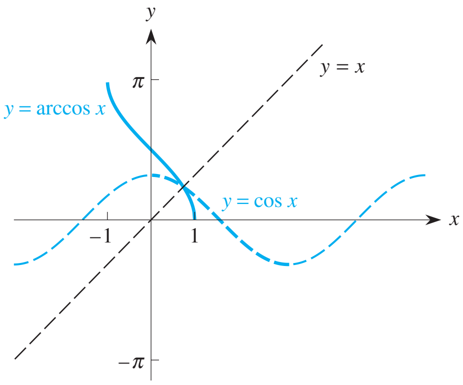


FIGURE 7.42

Problems

1. On what days of the year will San Diego have 11 hours of daylight? 10 hours of daylight? 9 hours of daylight?
2. The height of water at a dock is given by the formula $h(t) = 10 + 4\sin(\pi t/6)$, where t is measured in hours since midnight.
   - **a.** When does high tide occur?
   - **b.** When does low tide occur?
   - **c.** When does the water level reach 8 feet? 10 feet? 11 feet? 12 feet?
3. An air conditioner is being used to cool a room. The temperature T oscillates according to the formula $T(t) = 69 + 3\sin(\pi t/10)$, where t is measured in minutes after 9 A.M.
   - **a.** At about what temperature is the thermostat set? (i.e., when does the air conditioner kick in?)
   - **b.** At about what temperature does the air conditioner kick out?
   - **c.** When does the room temperature reach 70°F? 67°F?
4. The thermostat in an apartment is set to turn the heat on when the temperature falls to 64°F and to turn it off when the temperature rises to 70°F. This cycle takes 15 minutes.
   - **a.** Write a formula for the temperature T as a function of time t, where t is the number of minutes after noon. Assume that the temperature at noon is 70°.
   - **b.** Determine all times between noon and 1 P.M. when the temperature is 66°.
   - **c.** Suppose that the temperature at noon is 67°. Repeat parts (a) and (b).
   - **d.** Suppose that the temperature at noon is 68°. Repeat parts (a) and (b).
5. One of the dangers at places that have very high tides, such as Canada's Bay of Fundy, is the rate at which the tide can come in and potentially trap unwary visitors. Use the formula you devised for a sinusoidal function that models the heights of the tides at the Bay of Fundy in Problem 7 of Section 7.2 to determine how long it takes for the water level to rise 5 feet
   - **a.** from a point of low tide.
   - **b.** from a point at the average tide level.
6. A Ferris wheel is 12 meters in diameter and completes one full revolution every 20 seconds. The bottom of the Ferris wheel is 2 meters above the ground. In Problem 26 of Section 7.2, you were asked to write a formula for the height above ground of a person on the Ferris wheel as a function of time. Use that model to determine the times at which a person is 10 meters above the ground.
7. The historical average daytime high temperature in Fairbanks ranges from a low of −20°F to a high of 64°F, and the coldest day of the year, historically, is the 40th day. In Problem 15 of Section 7.2, you were asked to write a formula for a sinusoidal function that can be used to model the average daytime high temperature in Fairbanks as a function of the day of the year. Use this model to determine the days on which the high temperature in Fairbanks will be 0°.
8. The table below gives the outdoor temperatures in Chicago during one 24-hour period:

Time	Midnight	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	Noon	2 P.M.	4 P.M.	6 P.M.	8 P.M.	10 P.M.	Midnight
Temp.(°F)	53	48	47	49	53	59	66	71	68	65	58	54	53

In Problem 19 of Section 7.2, you were asked to create the equation of the sinusoidal function that fits these data. Use that model to determine the times at which the temperature in Chicago will be (**a**) 50° and (**b**) 60°.

9. The table below shows the average daytime high temperature each month in San Diego. In Problem 20 of Section 7.2, you were asked to construct a sinusoidal function that fits these data. Use that model to determine the months on which the average daytime high temperature in San Diego will be (**a**) 65°, (**b**) 70°, and (**c**) 80°.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Avg. daily high temp. (°F)	65.2	64.4	65.9	67.8	68.6	71.3	75.6	77.6	76.8	74.6	69.9	66.1

10. A 25-foot ladder is leaning against the side of a building and begins to slip. Write a formula for the angle θ that the ladder makes with the ground as a function of the distance x from the foot of the ladder to the building. Use your function grapher to draw the graph of this function. What are appropriate values for the domain of the function? Is the graph concave up or concave down? When is it maximum?

Exercising Your Algebra Skills

Several of the following equations do not have solutions. By inspection, decide which ones do not (give reasons) and then find the solutions to the remaining equations.

1. $\sin\theta = 4$
2. $\sin\theta = 0.4$
3. $3\sin\theta = 4$
4. $4\sin\theta = 3$
5. $4\sin\theta = -3$
6. $5\sin 2x = 3$
7. $5\cos 2x = -3$
8. $3\cos x = 2$
9. $5\sin 2x = 3\cos x$

7.4 The Tangent Function

We have considered many situations that can be modeled with either the sine or cosine function. In this section we return to the third trigonometric function, the *tangent,* and consider its properties and some applications. Recall from Section 6.1 that the tangent is defined by the ratio

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

of sides in a right triangle in terms of an angle θ, as shown in Figure 7.43.

The Graph of $y = \tan x$

As with any function, our first concern is to determine the graph of the tangent function to help us understand its behavior. The graph of the tangent function is shown in Figure 7.44, from which we observe the following characteristics.

1. The tangent function is periodic with period π; the tangent graph completes one full cycle between $x = -3\pi/2$ and $-\pi/2$, another full cycle between $-\pi/2$ and $\pi/2$, a third cycle between $\pi/2$ and $3\pi/2$, and so on. Each segment is called a *branch* of the graph.
2. The tangent function has zeros at $x = 0, \pm\pi, \pm 2\pi, \ldots$

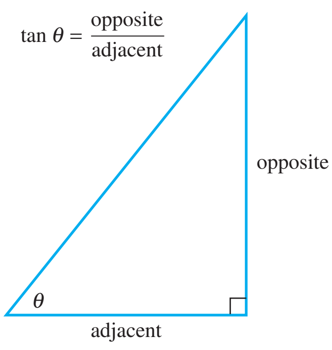


FIGURE 7.43

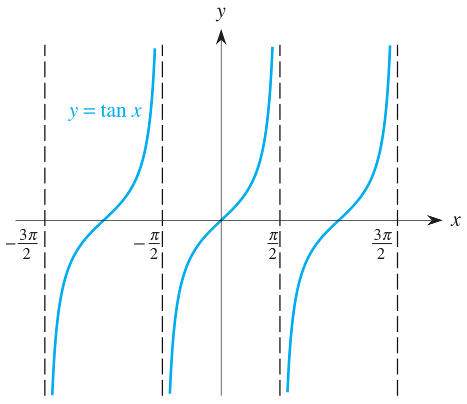


FIGURE 7.44

3. The tangent graph has vertical asymptotes at $x = \pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2, \ldots$, so the tangent function is not defined at these points (which separate the branches).
4. The tangent is an increasing function for all intervals between successive vertical asymptotes.
5. The tangent curve is first concave down and then concave up on each branch. The point of inflection on each branch occurs at the point where the tangent curve crosses the x-axis: at $x = 0, \pm\pi, \pm 2\pi, \ldots$

To understand the behavior of the tangent function, we consider the fundamental relationship.

$$\tan x = \frac{\sin x}{\cos x}, \qquad \text{for any value of } x \text{ for which } \cos x \neq 0.$$

Because the tangent is the quotient of two functions, we can analyze this relationship in the same way that we analyzed the behavior of rational functions in Section 4.6. First, the tangent function must have a zero wherever the numerator, $\sin x$, is 0. This corresponds to $x = 0, \pm\pi, \pm 2\pi, \ldots$, which clearly agrees with what the graph of the tangent function shows. Second, the tangent function is undefined and therefore has a vertical asymptote wherever the denominator, $\cos x$, is 0. This occurs at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$, which again agrees with what the graph of the tangent function shows.

Now let's see how these ideas help in understanding the graph of the tangent function. Consider what happens between $x = 0$ and $\pi/2$. The sine function (the numerator for $y = \tan x$) is positive and increasing toward 1, whereas the cosine function (the denominator for $y = \tan x$) is positive and decreasing toward 0. Because both are positive, $\tan x$ must be positive between 0 and $\pi/2$. Also, the ratio involves a numerator that is getting larger and a denominator that is getting smaller and approaching 0, so there is a vertical asymptote at $x = \pi/2$. The tangent is a positive function that increases toward ∞ as x approaches $\pi/2$ from the left.

Similarly, between $x = -\pi/2$ and $x = 0$, the sine function is negative and increasing toward 0, whereas the cosine function is positive and increasing toward 1.

Thus their ratio is negative and increases toward 0 as x increases toward 0. Moreover, as x approaches $-\pi/2$ from the right, the tangent ratio becomes ever more negative and eventually approaches $-\infty$.

Next, why is the period of the tangent function π when the periods for the sine and the cosine are both 2π? Visualize the sine curve from 0 to π and then from π to 2π, as shown in Figure 7.45. If you flip the second half of the curve over the x-axis, you get a curve identical to the first half. So the values for sin x between π and 2π are the same as those between 0 and π, but with the signs reversed. The same is true for the cosine between π and 2π—its values repeat those for the cosine between 0 and π, but with the signs reversed, as shown in Figure 7.46.

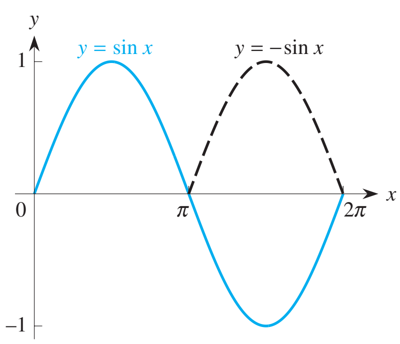


FIGURE 7.45

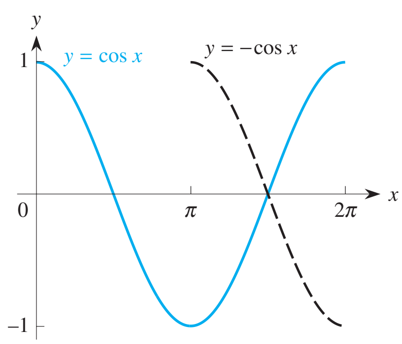


FIGURE 7.46

So, when we take the ratio $\sin x/\cos x$ for x between π and 2π $(x \neq 3\pi/2)$, the numerator and the denominator have the same numerical values as the ratio of $\sin x/\cos x$ for x between 0 and π $(x \neq \pi/2)$, but the signs of both the numerator and the denominator are reversed. In the quotient, these reversed signs cancel, so that the values for tan x from π to 2π match the corresponding values for tan x from 0 to π. But, if the same values are repeated, the function is periodic and therefore its period is π.

Finally, because the tangent function is periodic with period π, it repeats the behavior that we've outlined here, leading to the graph previously shown in Figure 7.44. For reference purposes, you should know that

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0 \quad \text{and} \quad \tan\frac{\pi}{4} = \tan 45° = 1.$$

EXAMPLE 1

A video cameraman is taping a 100 meter dash down a straight track. He is positioned halfway along the track 40 meters from the inside lane where the race's favorite is running. He plans to focus his camera on the favorite throughout the race.

a. Write a formula for a function that models the distance d from the runner to the point A on the track as a function of the angle θ, as illustrated in Figure 7.47.

b. What is the runner's distance from the line extending from the cameraman to the middle of the track when the angle $\theta = 30°$?

c. What is the runner's distance from that line when $\theta = 0.6$ radian?

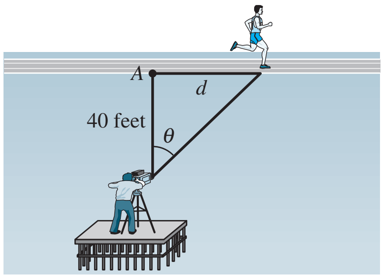


FIGURE 7.47

Solution

a. From Figure 7.47,

$$\tan\theta = \frac{d}{40} \quad \text{so that} \quad d = 40\tan\theta = f(\theta).$$

b. When $\theta = 30°$,

$$d = f(30°) = 40\tan 30° \approx 23.09 \text{ meters.}$$

c. When $\theta = 0.6$ radian,

$$d = f(0.6) = 40\tan 0.6 \approx 27.37 \text{ meters.}$$

We can write more general tangent functions of the form

$$y = D + A\tan[B(x - C)],$$

where D is the vertical shift or midline, A is the amplitude, B is the frequency, and C is the phase shift. These ideas are the same as those that we encountered with general sinusoidal functions in Section 7.2. We explore some of these ideas for the tangent function in the Problems at the end of this section.

The Inverse Tangent

Suppose that we have an equation such as $\tan\theta = 1.5$. To find a value of θ that satisfies this equation, we use the *inverse tangent function,* arctan x, that gives the number whose tangent value is x. Using a calculator, we find

$$\theta = \arctan 1.5 \approx 0.9828 \text{ radian} \approx 56.31°.$$

As with the inverse sine and inverse cosine functions, we have to restrict the domain of the tangent function in order to define the inverse tangent function. By convention, the principal values for the tangent function are from $-\pi/2$ to $\pi/2$ where the tangent function is strictly increasing. Accordingly, a calculator returns a value only between $-\pi/2$ and $\pi/2$ (or between $-90°$ and $90°$) for the inverse tangent.

We summarize the important properties of the inverse tangent function as follows.

Properties of the Inverse Tangent Function $y = \arctan x$

1. $\arctan x$ is defined for values of x (the domain) between $-\infty$ and ∞.
2. The principal values for y (the range) lie between $-\pi/2$ and $\pi/2$ radians.
3. $\arctan(\tan y) = y$, for $-\pi/2 < y < \pi/2$.
4. $\tan(\arctan x) = x$, for $-\infty < x < \infty$.

Finally, as with the graphs of the other inverse functions, the graph of the inverse tangent function $y = \arctan x$ is the mirror image of the tangent graph about the line $y = x$, as shown in Figure 7.48, where x goes from -15 radians to 15 radians. Note how the curve levels off to the right at a height of $\pi/2 \approx 1.57$ and to the left at a height of about $-\pi/2 \approx -1.57$. These are a pair of horizontal asymptotes.

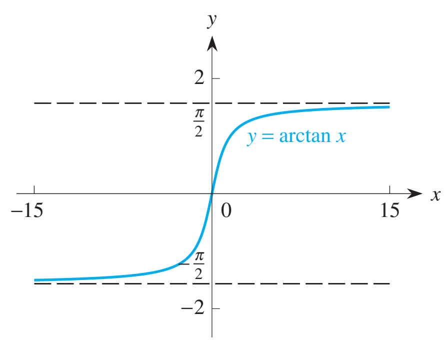


FIGURE 7.48

If you need other values of θ outside the interval from $-\pi/2$ to $\pi/2$, you will have to determine them by using what you know about the symmetry of the graph of the tangent function.

EXAMPLE 2

You enter a movie theater that has a screen 20 feet high positioned 5 feet above your eye level. If you sit too far back in the theater, the screen appears too small because your viewing angle is too small. If you sit too close to the screen, the picture will seem distorted because your viewing angle is again too small.

a. Find a formula giving the viewing angle θ as a function of your distance d from the screen, as illustrated in Figure 7.49.

b. What is your viewing angle θ if you sit 40 feet back from the screen?

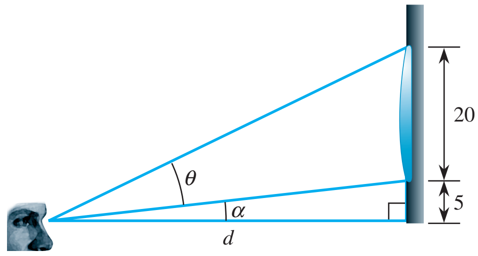


FIGURE 7.49

c. Does the viewing angle increase or decrease if you move farther back? In particular, estimate the distance you should sit from the screen to get the largest possible viewing angle.

Solution

a. Your viewing angle θ is the angle subtended by the screen. There is no direct way to get an expression for θ, because the associated triangle shown in Figure 7.49 is not a right triangle. Thus you have to get an expression for θ in a somewhat indirect manner. To do so, introduce the angle α shown in Figure 7.49 representing the angle from your eye level vertically upward to the bottom of the screen. This gives you two right triangles. In the smaller triangle,

$$\tan \alpha = \frac{5}{d} \quad \text{so that} \quad \alpha = \arctan \frac{5}{d}.$$

In the larger triangle,

$$\tan(\theta + \alpha) = \frac{25}{d} \quad \text{so that} \quad \theta + \alpha = \arctan \frac{25}{d}.$$

Consequently, the desired expression for θ is

$$\theta = (\theta + \alpha) - \alpha = \arctan \frac{25}{d} - \arctan \frac{5}{d} = f(d).$$

(Note that these two terms *cannot* be combined algebraically.)

b. For $d = 40$ feet,

$$\theta = f(40) = \arctan \frac{25}{40} - \arctan \frac{5}{40} \approx 0.4342 \text{ radians},$$

or about 24.9°.

c. To determine what happens to this viewing angle θ as you move farther back, just replace the 40-foot distance with somewhat larger values—say, 41 or 45 feet. Alternatively, graph the function f that gives the angle θ as a function of the distance d. If you graph this function on the interval from 0 to 50, say, as shown in Figure 7.50, you can determine the behavior of this function for θ more thoroughly. The function increases rapidly, starting at $d = 0$, and rises to a maximum viewing angle when d is approximately 11 feet. You can verify this result on your calculator. Then the function slowly decreases as d increases thereafter. Therefore if you move farther back from the screen, the viewing angle will decrease.

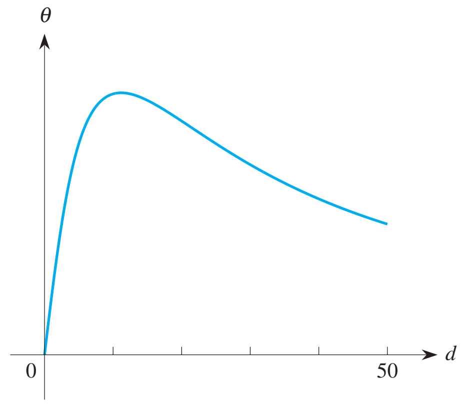


FIGURE 7.50

Finally, we note that it is possible to consider three other trigonometric functions—the *cotangent,* the *secant,* and the *cosecant*—which are just reciprocals of the tangent, the cosine, and the sine functions, respectively. These functions are defined as

$$\cot\theta = \frac{1}{\tan\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \text{and} \qquad \csc\theta = \frac{1}{\sin\theta}.$$

Be sure that you understand that these are reciprocals of the basic trigonometric functions. They have nothing to do with the associated inverse functions.

The cosecant, secant, and cotangent functions have been useful in the past because they simplified the hand calculations required in working with certain trigonometric problems. However, with technology, working with the actual reciprocals is just as easy as using $\csc\theta$, $\sec\theta$, and $\cot\theta$. Consequently, the cotangent, secant, and cosecant functions gradually are being laid to rest, and we don't consider them further.

Problems

1. Each of the figures (a)–(f) shows the graph of $y = \tan x$, where x is in degrees.

a. Write an equation for a tangent function with frequency 2 and sketch its graph superimposed over the graph of $y = \tan x$ in Figure (a).

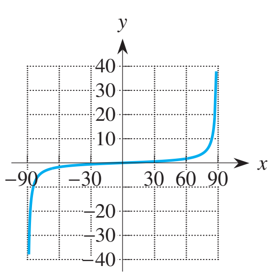
b. Write an equation for a tangent function with frequency $\frac{1}{2}$ and sketch its graph superimposed over the graph of $y = \tan x$ in Figure (b).

c. Write an equation for a tangent function with amplitude 3 and sketch its graph superimposed over the graph of $y = \tan x$ in Figure (c).

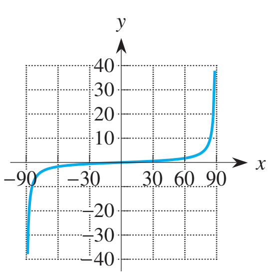
d. Write an equation for a tangent function with amplitude -2 and sketch its graph superimposed over the graph of $y = \tan x$ in Figure (d).

e. Write an equation for a tangent function with phase shift of 30° and sketch its graph superimposed over the graph of $y = \tan x$ in Figure (e).

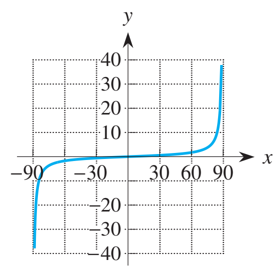
f. Write an equation for a tangent function with vertical shift of -10 and sketch its graph superimposed over the graph of $y = \tan x$ in Figure (f).



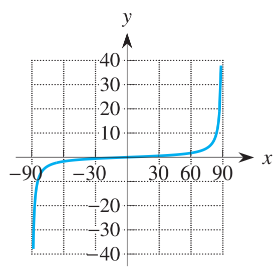
(a)



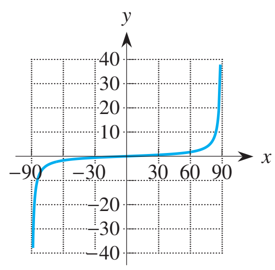
(b)



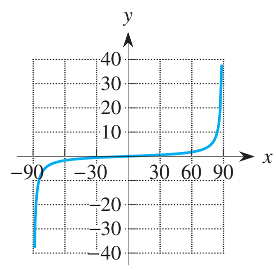
(c)



(d)

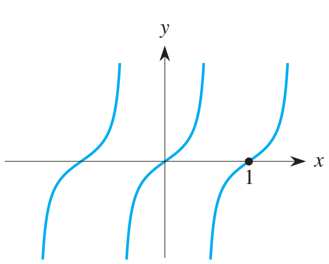


(e)

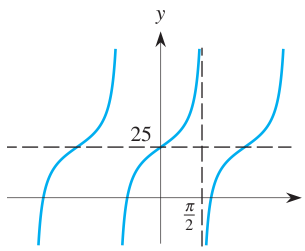


(f)

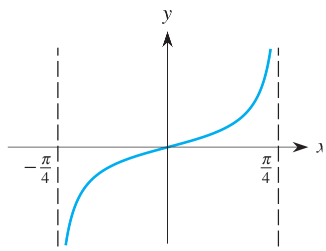
2. Write a possible formula involving tangent functions for each function (a)–(c) shown.



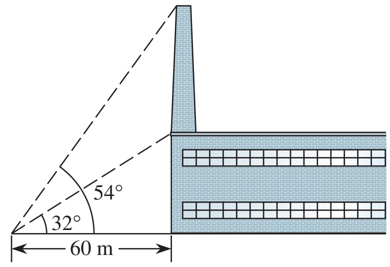
(a)



(b)



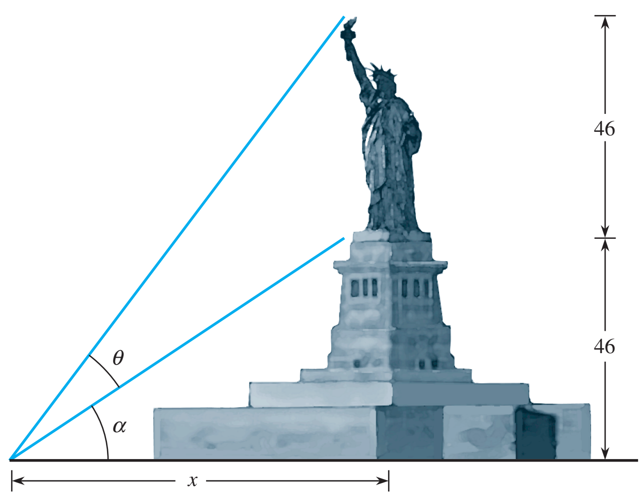
(c)



3. Write a possible formula involving a tangent function for the function whose values are given in the table.

θ	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$
$f(\theta)$	UNDEF	−2.414	−1	0	1	2.414	UNDEF

4. The Statue of Liberty is 46 meters tall and stands on a base that is also 46 meters tall. Find an expression for the angle subtended by the statue from ground level as a function of distance from the base of the statue. Use this function to estimate graphically the distance when the angle is maximum. Approximately what is this maximum angle?



5. A tall smokestack extends from the roof of a large industrial plant. At a point 60 meters from the base of the building, the angle of elevation to the roof (the bottom of the smokestack) is 32°, and the angle of elevation to the top of the smokestack is 54°. Find both the height of the building and the height of the smokestack.

6. The lines $y = x$, $y = 2x$, $y = 3x$, and $y = 4x$ all pass through the origin. Find the angle each line makes with the x-axis.

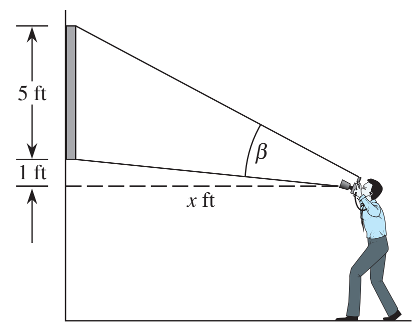
7. **a.** For the general equation of a line through the origin $y = mx$, interpret the meaning of the slope m in terms of trig functions.
   b. What is the significance of the slope m in $y = mx + b$ from this point of view?

8. Use the graph of $y = \sin x$ to sketch the graph of its reciprocal function $y = 1/\sin x$. (This is the cosecant function.)

9. Use the graph of $y = \cos x$ to help you sketch the graph of its reciprocal function $y = 1/\cos x$. (This is the secant function.)

10. Use the graph of $y = \tan x$ to sketch the graph of $y = 1/\tan x$. (This is the cotangent function.)

11. A 5-foot high painting is hanging on the wall of an art museum when a photographer takes a picture of it. The lens of his camera is 1 foot below the bottom of the painting when he snaps the picture.



   a. Find a formula for the angle β subtended by the painting at the camera's lens at a distance of x feet from the wall.
   b. Using your function grapher, estimate the distance x from the wall at which the photographer

should position his camera to subtend the greatest possible angle with the painting.

12. A TV cameraman is videotaping the liftoff of the space shuttle. The cameraman is positioned at ground level 500 meters from the launch pad and is tracking the shuttle as it rises.
 a. Write a formula for the angle of inclination α to the shuttle as a function of the height y of the shuttle above the ground.
 b. Find the angle of inclination α when the shuttle is 1000 meters high.
 c. Find the angle of inclination α when the shuttle is 2000 meters high.

13. According to Einstein's theory of relativity, the mass M of an object increases as its speed v increases according to the formula

$$M = f(v) = \frac{M_0}{\sqrt{1 - \frac{v^2}{c^2}}} = M_0 \cdot \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

where M_0 is the mass of the object at rest ($v = 0$) and c is the speed of light (about 186,282 miles per second). Suppose that an object has a rest mass of $M_0 = 1$ unit.
 a. Construct a table of values for the mass of the object for each of the following speeds expressed as a fraction of the speed of light: $v = 0$, $0.5c$, $0.9c$, $0.95c$, $0.99c$, and $0.999c$.
 b. Sketch a graph showing the behavior of the mass M of an object as its speed approaches the speed of light.
 c. The speed of light is the physical equivalent of a vertical asymptote. Write the formula for a function involving the tangent that can be used to model the mass of an object as a function of its speed expressed as a fraction of the speed of light.

Exercising Your Algebra Skills

Solve each trigonometric equation.

1. $4 \tan\theta = 5$
2. $5 \tan\theta = 4$
3. $\cos\theta = -\sin\theta$
4. $2\cos\theta = \sin\theta$
5. $5\sin\theta = 4\cos\theta$
6. $4\sin\theta = 5\cos\theta$
7. $\sin\theta + \cos\theta = 0$
8. $4\sin\theta - 3\cos\theta = 0$

Chapter Summary

In this chapter, we introduced the use of the sine and cosine functions for modeling periodic phenomena and the tangent function. In particular, we discussed the following:

- The behavior of the sine and cosine functions.
- How to convert between radian measure and degree measure.
- What the *vertical shift* or *midline* means for the sine and cosine functions.
- What *amplitude* means for the sine and cosine functions.
- What *frequency* means for the sine and cosine functions.
- What *period* means for the sine and cosine functions.
- What *phase shift* means for the sine and cosine functions.
- How to use the sine and cosine functions to model periodic behavior.
- How to fit sine and cosine functions to data.
- The behavior of the inverse sine and inverse cosine functions.

- How to solve trigonometric equations, using the inverse sine and inverse cosine functions.
- The behavior of the tangent function.
- The behavior of the inverse tangent function.
- How to solve trigonometric equations, using the inverse tangent function.

Review Problems

1. The student with whom you are working finishes a problem and announces her answer is $\cos(5.70)$. You get an answer in the form $\sin(7.2708)$. Under what circumstances are these answers the same?

2. Suppose that θ is 60°.

a. Find two positive angles and two negative angles that have the same sine as θ.

b. Write the angles from part (a) in radian form.

3. Let $\theta = 45°$.

a. Find two positive angles and two negative angles with the same cosine as θ.

b. Write the radian form of the angles from part (a).

For each sinusoidal function in Problems 4–11, identify the vertical shift, the amplitude, the frequency, the period, and the phase shift.

4. $y = 325 + 10\sin\left(\frac{2\pi}{9}t\right)$

5. $y = 63 + 3\sin\left(\frac{2\pi}{25}t\right)$

6. $y = 71 + 2\cos\left(\frac{2\pi}{15}t\right)$

7. $y = 80 + 13\cos\left[\frac{2\pi}{24}(t - 15)\right]$

8. $y = 38 + 8\sin\left[\frac{2\pi}{24}(t - 5)\right]$

9. $y = 100 + 25\sin\left(\frac{2\pi}{72}t\right)$

10. $y = 100 + 25\sin\left(\frac{2\pi}{97}t\right)$

11. $y = 145 + 40\sin\left(\frac{2\pi}{83}t\right)$

12–19. Each of the functions in Problems 4–11 can be a model for a common periodic phenomenon. For each function,

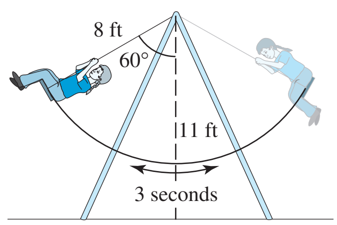
a. describe a phenomenon that each function could model.

b. What do the variables represent?

c. What are the units?

d. What are possible values for the domain and range?

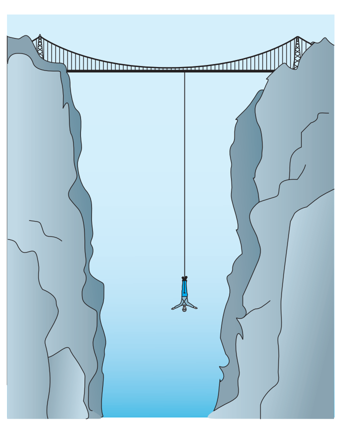
20. Bernice is swinging on a playground swing whose supporting crossbar is 11 feet above the ground and the length of the chain to her seat is 8 feet. At the end of each swing, she makes an angle of 60° with the vertical and it takes her 3 seconds to complete each full cycle.



a. Write a sinusoidal function that can be used to model the height of the seat above the ground as a function of time *t*.

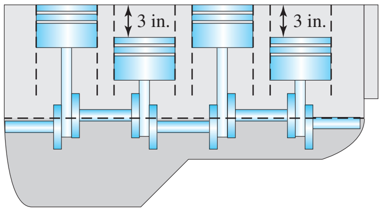
b. Write a sinusoidal function that can be used to model the horizontal displacement from directly under the crossbar as a function of time *t*.

21. A bungee jumper dives off a bridge that spans across a deep gorge. The bungee cord initially stretches to a maximum length of 200 feet before the jumper begins her first rebound. Over the course of the next



60 seconds, she bounces up and down with ever-diminishing oscillations, each lasting about 6 seconds, until she comes to rest about 160 feet below the bridge. Write the equation of a decaying oscillatory function that models the height of the bungee jumper as a function of time as measured from the instant the cord is extended to its maximum stretch.

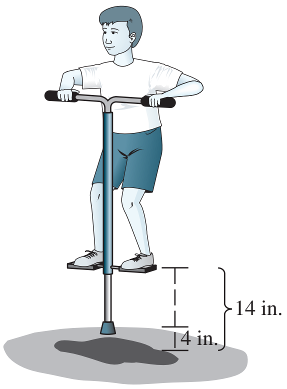
22. If a car's engine is operating at 2000 rpm, its pistons are moving up and down 2000 times per minute. Thus, in a four-cylinder engine, each piston moves up and down 500 times per minute. Suppose that the total vertical distance that a piston moves is 3 inches.



a. Write a sinusoidal function that models the height of the piston as a function of time in minutes, based on the midline for the height.

b. Write a sinusoidal function that models the height of the piston as a function of time in minutes, based on the lowest height of the piston.

23. A pogo stick consists of a spring in a vertical tube with two fixed pedals on which a person stands and jumps up and down. Suppose that a child on a pogo stick hops up and down every 3 seconds and that the height of the pedals varies from 4 inches above the ground to 14 inches above the ground. Write a sinusoidal function to model the height of the pedals above ground level as a function of time. (*Hint:* Assume that the pedals are at the midline level at the start.)



24. The child in Problem 23 is also moving forward 20 inches with each bounce of the pogo stick.

a. Write a sinusoidal function to model the path of the child's feet—that is, the height y above the ground as a function of the horizontal distance x covered.

b. By comparing the graph of the function you created in part (a) to your image of what is actually happening, explain why the sinusoidal model may not make sense.

c. Look at the graph of the absolute value of the function you created in part (a). Is it a better or worse model for the behavior you envision?

To solve Problems 25–27, use the fact that, if an arc of length s on a circle of radius r subtends an angle of θ radians, then $s = r\theta$.

25. The distance between two points P and Q on the Earth is measured as the distance along the arc of the circle through P and Q and centered at the center of the Earth O. The radius of the Earth is about 4000 miles. Find the distance from P to Q if the angle POQ has the following measurements.

a. $\frac{\pi}{4}$ **b.** $\frac{\pi}{3}$

c. $\frac{5\pi}{6}$ **d.** $15°$

26. A wheel of radius 2 feet rotates at a constant rate of 180 revolutions per minute.

a. How many radians per minute are swept by the wheel?

b. How far does a point on the rim of the wheel travel in 1 minute?

27. Find the diameter of the tires on your car. Assume that the car is traveling at 60 mph and determine the number of revolutions the tire makes every minute.

28. On the same set of axes, graph the functions

$$S(x) = 2\sin x, \qquad R(x) = 2\sin 3x$$
$$\text{and} \quad T(x) = 2\sin 0.5x.$$

Clearly mark the zeros of each function.

29. For each function give the frequency, period, amplitude, and phase shift.

a. $y = 5 + 2\cos\left(\frac{3}{4}x\right)$

b. $y = 5 - 2\cos\left(\frac{3}{4}x + \pi\right)$

c. $y = 5 - 2\cos\left(\pi x + \frac{3}{4}\right)$

d. $y = 5 - 2\cos\left[\frac{3}{4}(\pi x - 1)\right]$

30. Determine the values of x for which each function in Problem 29 equals 6.

31. **a.** Graph the function $y = \arcsin[\sin(x)]$ for x between -10 and 10 radians. Explain why you get the pattern you do.

b. Repeat part (a) with the function $y = \sin[\arcsin(x)]$ for x between -1 and 1.

32. Solve for θ.

a. $-4\sin\theta = 6\cos\theta$

b. $2\cos\theta = \sin\theta$

c. $3\tan\theta - 21 = 0$

33. Solve for x.

a. $\arctan x = 1.35$

b. $\arcsin x = 0.5$