SERIES AND PARALLEL AC CIRCUITS

Objectives

- *Become familiar with the characteristics of series and parallel ac networks and be able to find current, voltage, and power levels for each element.*

- *Be able to find the total impedance of any series or parallel ac network and sketch the impedance and admittance diagram of each.*

- *Develop confidence in applying Kirchhoff's current and voltage law to any series or parallel configuration.*

- *Be able to apply the voltage divider rule or current divider rule to any ac network.*

- *Become adept at finding the frequency response of a series or parallel combination of elements.*

15.1 INTRODUCTION

In this chapter, phasor algebra is used to develop a quick, direct method for solving both series and parallel ac circuits. The close relationship that exists between this method for solving for unknown quantities and the approach used for dc circuits will become apparent after a few simple examples are considered. Once this association is established, many of the rules (current divider rule, voltage divider rule, and so on) for dc circuits can be readily applied to ac circuits.
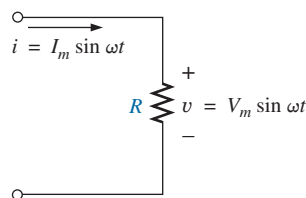
SERIES ac CIRCUITS

15.2 IMPEDANCE AND THE PHASOR DIAGRAM
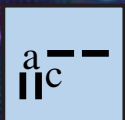
Resistive Elements

In Chapter 14, we found, for the purely resistive circuit in Fig. 15.1, that v and i were in phase, and the magnitude
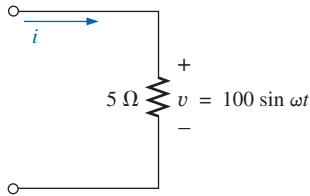
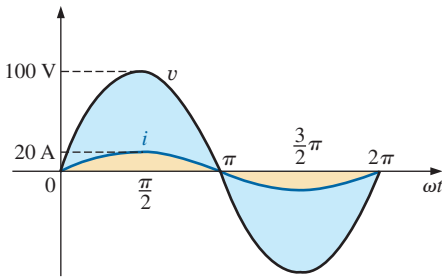$$I_m = \frac{V_m}{R} \quad \text{or} \quad V_m = I_m R$$

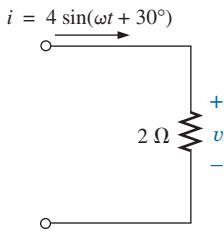**FIG. 15.1**
Resistive ac circuit.

In phasor form,

$$v = V_m \sin \omega t \Rightarrow \mathbf{V} = V \angle 0°$$

where $V = 0.707V_m$.

Applying Ohm's law and using phasor algebra, we have

$$\mathbf{I} = \frac{V \angle 0°}{R \angle \theta_R} = \frac{V}{R} \underline{/0° - \theta_R}$$

Since i and v are in phase, the angle associated with i also must be 0°. To satisfy this condition, θ_R must equal 0°. Substituting $\theta_R = 0°$, we find

$$\mathbf{I} = \frac{V \angle 0°}{R \angle 0°} = \frac{V}{R} \underline{/0° - 0°} = \frac{V}{R} \angle 0°$$

so that in the time domain,

$$i = \sqrt{2}\left(\frac{V}{R}\right) \sin \omega t$$

We use the fact that $\theta_R = 0°$ in the following polar format to ensure the proper phase relationship between the voltage and current of a resistor:

$$\boxed{\mathbf{Z}_R = R \angle 0°} \tag{15.1}$$

The boldface roman quantity $\mathbf{Z}_R$, having both magnitude and an associated angle, is referred to as the *impedance* of a resistive element. It is measured in ohms and is a measure of how much the element will "impede" the flow of charge through the network. The above format will prove to be a useful "tool" when the networks become more complex and phase relationships become less obvious. It is important to realize, however, that $\mathbf{Z}_R$ is *not a phasor*, even though the format $R \angle 0°$ is very similar to the phasor notation for sinusoidal currents and voltages. The term *phasor* is reserved for quantities that vary with time, and R and its associated angle of 0° are fixed, nonvarying quantities.
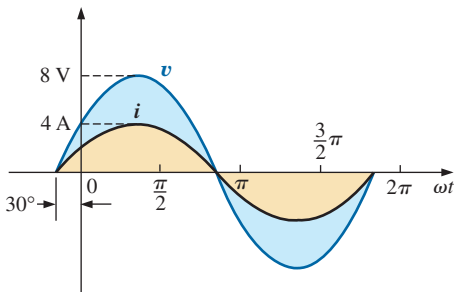

FIG. 15.2
Example 15.1.

FIG. 15.3
Waveforms for Example 15.1.

FIG. 15.4
Example 15.2.

FIG. 15.5
Waveforms for Example 15.2.

EXAMPLE 15.1 Using complex algebra, find the current i for the circuit in Fig. 15.2. Sketch the waveforms of v and i.

Solution: Note Fig. 15.3:

$$v = 100 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 70.71 \text{ V} \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{V \angle \theta}{R \angle 0°} = \frac{70.71 \text{ V} \angle 0°}{5 \text{ } \Omega \angle 0°} = 14.14 \text{ A} \angle 0°$$

and
$$i = \sqrt{2}(14.14) \sin \omega t = \mathbf{20 \sin \omega t}$$

EXAMPLE 15.2 Using complex algebra, find the voltage v for the circuit in Fig. 15.4. Sketch the waveforms of v and i.

Solution: Note Fig. 15.5:

$$i = 4 \sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 2.828 \text{ A} \angle 30°$$
$$\mathbf{V} = \mathbf{I}\mathbf{Z}_R = (I \angle \theta)(R \angle 0°) = (2.828 \text{ A} \angle 30°)(2 \text{ } \Omega \angle 0°)$$
$$= 5.656 \text{ V} \angle 30°$$

and $\quad v = \sqrt{2}(5.656) \sin(\omega t + 30°) = \mathbf{8.0 \sin(\omega t + 30°)}$

It is often helpful in the analysis of networks to have a **phasor dia-gram,** which shows at a glance the *magnitudes* and *phase relations* among the various quantities within the network. For example, the pha-sor diagrams of the circuits considered in the two preceding examples would be as shown in Fig. 15.6. In both cases, it is immediately obvious that v and i are in phase since they both have the same phase angle.

Inductive Reactance

We learned in Chapter 13 that for the pure inductor in Fig. 15.7, the volt-age leads the current by 90° and that the reactance of the coil X_L is deter-mined by ωL.

$$v = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V \angle 0°$$
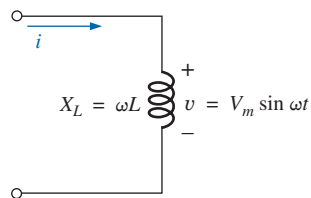
FIG. 15.7
Inductive ac circuit.

By Ohm's law,

$$\mathbf{I} = \frac{V \angle 0°}{X_L \angle \theta_L} = \frac{V}{X_L} \angle 0° - \theta_L$$

Since v leads i by 90°, i must have an angle of $-90°$ associated with it. To satisfy this condition, θ_L must equal $+90°$. Substituting $\theta_L = 90°$, we obtain

$$\mathbf{I} = \frac{V \angle 0°}{X_L \angle 90°} = \frac{V}{X_L} \angle 0° - 90° = \frac{V}{X_L} \angle -90°$$

so that in the time domain,

$$i = \sqrt{2} \left(\frac{V}{X_L} \right) \sin(\omega t - 90°)$$

We use the fact that $\theta_L = 90°$ in the following polar format for induc-tive reactance to ensure the proper phase relationship between the volt-age and current of an inductor:

$$\boxed{\mathbf{Z}_L = X_L \angle 90°} \tag{15.2}$$

The boldface roman quantity $\mathbf{Z}_L$, having both magnitude and an asso-ciated angle, is referred to as the *impedance* of an inductive element. It is measured in ohms and is a measure of how much the inductive element "controls or impedes" the level of current through the network (always keep in mind that inductive elements are storage devices and do not dis-sipate like resistors). The above format, like that defined for the resistive element, will prove to be a useful tool in the analysis of ac networks. Again, be aware that $\mathbf{Z}_L$ is not a phasor quantity, for the same reasons in-dicated for a resistive element.
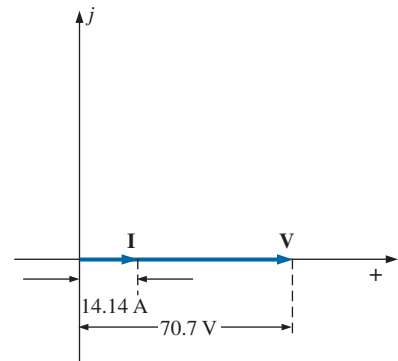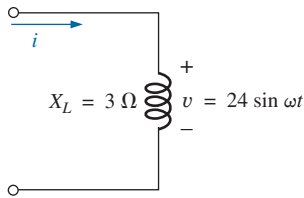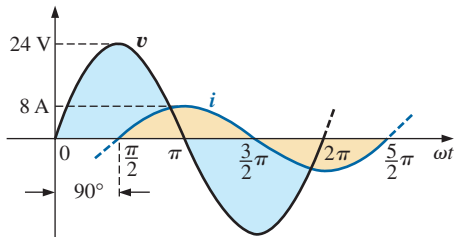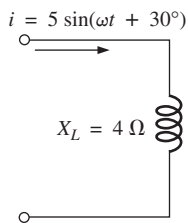

(a)
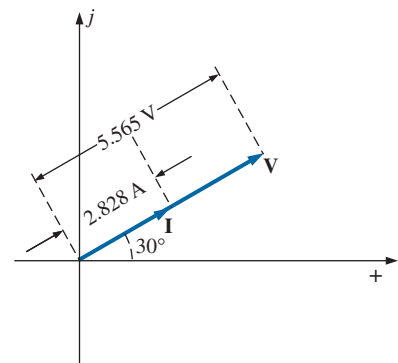
(b)

FIG. 15.6
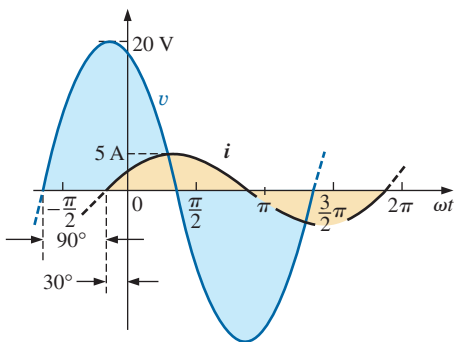*Phasor diagrams for Examples 15.1 and 15.2.*

FIG. 15.8
Example 15.3.

FIG. 15.9
Waveforms for Example 15.3.

FIG. 15.10
Example 15.4.

FIG. 15.11
Waveforms for Example 15.4.

EXAMPLE 15.3 Using complex algebra, find the current i for the circuit in Fig. 15.8. Sketch the v and i curves.

Solution: Note Fig. 15.9:

$$v = 24 \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = 16.968 \text{ V } \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{V \angle \theta}{X_L \angle 90°} = \frac{16.968 \text{ V} \angle 0°}{3 \text{ } \Omega \angle 90°} = 5.656 \text{ A } \angle -90°$$

and $\quad i = \sqrt{2}(5.656) \sin(\omega t - 90°) = \mathbf{8.0 \sin(\omega t - 90°)}$

EXAMPLE 15.4 Using complex algebra, find the voltage v for the circuit in Fig. 15.10. Sketch the v and i curves.
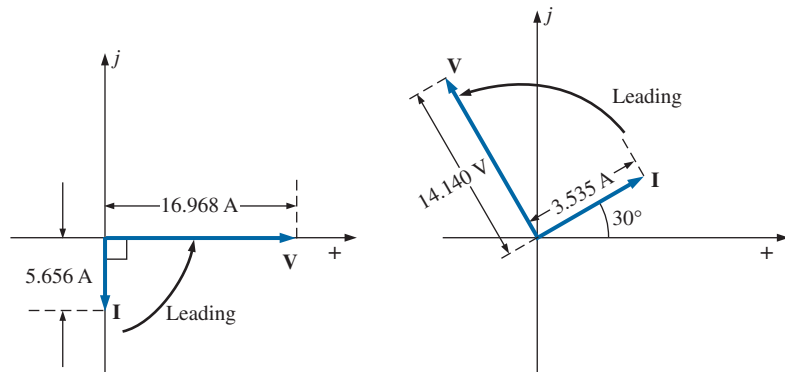
Solution: Note Fig. 15.11:

$$i = 5 \sin(\omega t + 30°) \Rightarrow \text{phasor form } \mathbf{I} = 3.535 \text{ A } \angle 30°$$

$$\mathbf{V} = \mathbf{IZ}_L = (I \angle \theta)(X_L \angle 90°) = (3.535 \text{ A } \angle 30°)(4 \text{ } \Omega \angle +90°)$$

$$= 14.140 \text{ V } \angle 120°$$

and $\quad v = \sqrt{2}(14.140) \sin(\omega t + 120°) = \mathbf{20 \sin(\omega t + 120°)}$

The phasor diagrams for the two circuits of the two preceding examples are shown in Fig. 15.12. Both indicate quite clearly that the voltage leads the current by 90°.

FIG. 15.12
Phasor diagrams for Examples 15.3 and 15.4.

Capacitive Reactance

We learned in Chapter 13 that for the pure capacitor in Fig. 15.13, the current leads the voltage by 90° and that the reactance of the capacitor X_C is determined by $1/\omega C$.
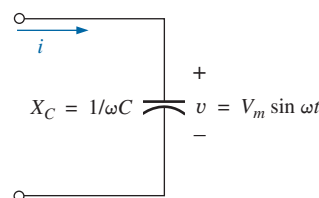
FIG. 15.13
Capacitive ac circuit.

$$v = V_m \sin \omega t \Rightarrow \text{phasor form } \mathbf{V} = V \angle 0°$$

Applying Ohm's law and using phasor algebra, we find

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle \theta_C} = \frac{V}{X_C} \angle 0° - \theta_C$$

Since i leads v by 90°, i must have an angle of $+90°$ associated with it. To satisfy this condition, θ_C must equal $-90°$. Substituting $\theta_C = -90°$ yields

$$\mathbf{I} = \frac{V \angle 0°}{X_C \angle -90°} = \frac{V}{X_C} \angle 0° - (-90°) = \frac{V}{X_C} \angle 90°$$

so, in the time domain,

$$i = \sqrt{2}\left(\frac{V}{X_C}\right) \sin(\omega t + 90°)$$

We use the fact that $\theta_C = -90°$ in the following polar format for capacitive reactance to ensure the proper phase relationship between the voltage and current of a capacitor:

$$\boxed{\mathbf{Z}_C = X_C \angle -90°} \qquad (15.3)$$

The boldface roman quantity $\mathbf{Z}_C$, having both magnitude and an associated angle, is referred to as the *impedance* of a capacitive element. It is measured in ohms and is a measure of how much the capacitive element "controls or impedes" the level of current through the network (always keep in mind that capacitive elements are storage devices and do not dissipate like resistors). The above format, like that defined for the resistive element, will prove a very useful tool in the analysis of ac networks. Again, be aware that $\mathbf{Z}_C$ is not a phasor quantity, for the same reasons indicated for a resistive element.

EXAMPLE 15.5 Using complex algebra, find the current i for the circuit in Fig. 15.14. Sketch the v and i curves.

Solution: Note Fig. 15.15:

$$v = 15 \sin \omega t \Rightarrow \text{phasor notation } \mathbf{V} = 10.605 \text{ V} \angle 0°$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_C} = \frac{V \angle \theta}{X_C \angle -90°} = \frac{10.605 \text{ V} \angle 0°}{2 \ \Omega \angle -90°} = 5.303 \text{ A} \angle 90°$$

and $\quad i = \sqrt{2}(5.303) \sin(\omega t + 90°) = \textbf{7.5} \sin(\omega t + \textbf{90°})$

FIG. 15.14

Example 15.5.

FIG. 15.15

Waveforms for Example 15.5.

$i = 6 \sin(\omega t - 60°)$

$X_C = 0.5 \, \Omega$
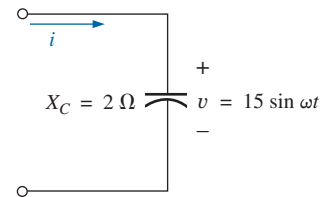
$+$
v
$-$

FIG. 15.16
Example 15.6.

EXAMPLE 15.6 Using complex algebra, find the voltage v for the circuit in Fig. 15.16. Sketch the v and i curves.
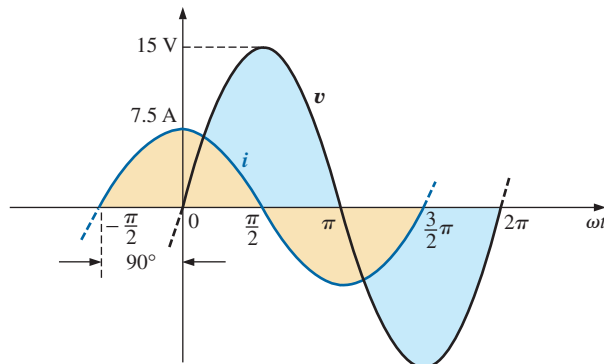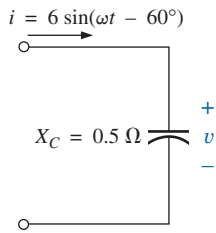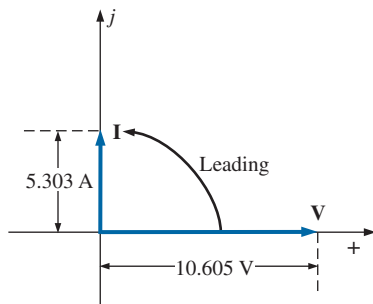
Solution: Note Fig. 15.17:

$$i = 6 \sin(\omega t - 60°) \Rightarrow \text{phasor notation } \mathbf{I} = 4.242 \text{ A } \angle -60°$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = (I \angle \theta)(X_C \angle -90°) = (4.242 \text{ A } \angle -60°)(0.5 \, \Omega \angle -90°)$$
$$= 2.121 \text{ V } \angle -150°$$

and

$$v = \sqrt{2}(2.121) \sin(\omega t - 150°) = \mathbf{3.0 \sin(\omega t - 150°)}$$
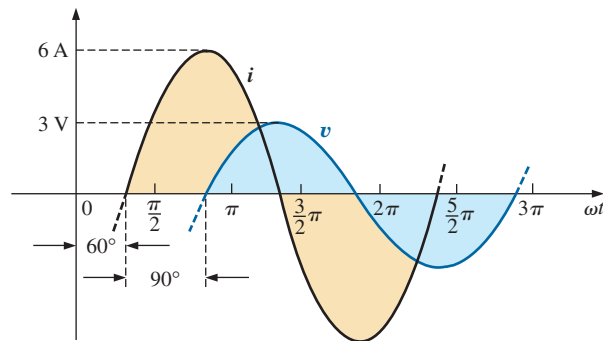
FIG. 15.17
Waveforms for Example 15.6.

(a)

(b)

FIG. 15.18
Phasor diagrams for Examples 15.5 and 15.6.

The phasor diagrams for the two circuits of the two preceding examples are shown in Fig. 15.18. Both indicate quite clearly that the current i leads the voltage v by 90°.

Impedance Diagram

Now that an angle is associated with resistance, inductive reactance, and capacitive reactance, each can be placed on a complex plane diagram, as shown in Fig. 15.19. For any network, the resistance will *always* appear on the positive real axis, the inductive reactance on the positive imaginary axis, and the capacitive reactance on the negative imaginary axis.

FIG. 15.19
Impedance diagram.

The result is an **impedance diagram** that can reflect the individual and total impedance levels of an ac network.

We will find in the rest of this text that networks combining different types of elements will have total impedances that extend from $-90°$ to $+90°$. If the total impedance has an angle of $0°$, it is said to be resistive in nature. If it is closer to $90°$, it is inductive in nature. If it is closer to $-90°$, it is capacitive in nature.

Of course, for single-element networks, the angle associated with the impedance will be the same as that of the resistive or reactive element, as revealed by Eqs. (15.1) through (15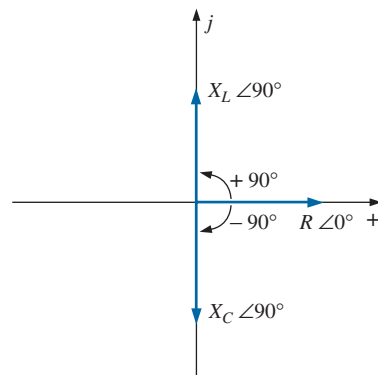.3). It is important to remember that impedance, like resistance or reactance, is not a phasor quantity representing a time-varying function with a particular phase shift. It is simply an operating tool that is extremely useful in determining the magnitude and angle of quantities in a sinusoidal ac network.

Once the total impedance of a network is determined, its magnitude will define the resulting current level (through Ohm's law), whereas its angle will reveal whether the network is primarily inductive or capacitive or simply resistive.

For any configuration (series, parallel, series-parallel, and so on), the angle associated with the total impedance is the angle by which the applied voltage leads the source current. For inductive networks, θ_T will be positive, whereas for capacitive networks, θ_T will be negative.

15.3 SERIES CONFIGURATION

The overall properties of series ac circuits (Fig. 15.20) are the same as those for dc circuits. For instance, the total impedance of a system is the sum of the individual impedances:

$$\boxed{\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_N} \qquad \textbf{(15.4)}$$
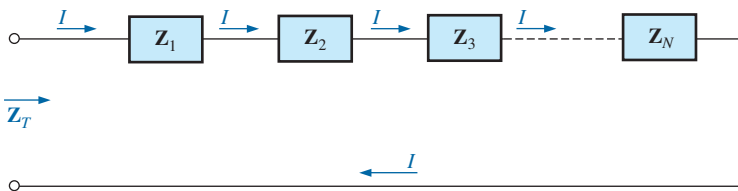
FIG. 15.20

Series impedances.
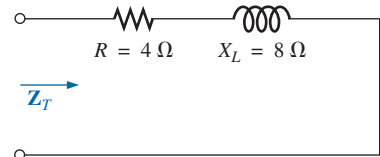
FIG. 15.21

Example 15.7.

EXAMPLE 15.7 Draw the impedance diagram for the circuit in Fig. 15.21, and find the total impedance.

Solution: As indicated by Fig. 15.22, the input impedance can be found graphically from the impedance diagram by properly scaling the real and imaginary axes and finding the length of the resultant vector Z_T and angle θ_T. Or, by using vector algebra, we obtain

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2$$
$$= R \angle 0° + X_L \angle 90°$$
$$= R + jX_L = 4\ \Omega + j8\ \Omega$$
$$\mathbf{Z}_T = \textbf{8.94}\ \boldsymbol{\Omega}\ \boldsymbol{\angle} \textbf{63.43°}$$
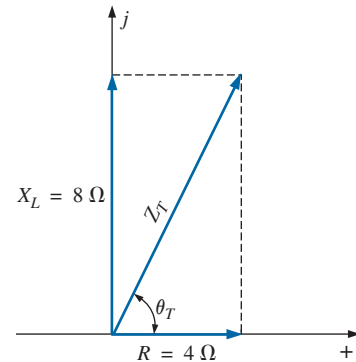
FIG. 15.22

Impedance diagram for Example 15.7.
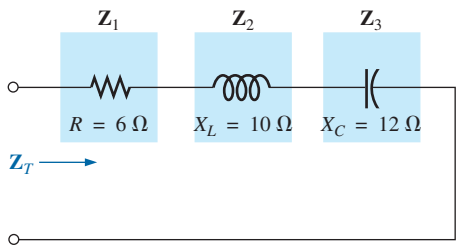
FIG. 15.23

Example 15.8

FIG. 15.24

Impedance diagram for Example 15.8.

FIG. 15.25

Series ac circuit.

EXAMPLE 15.8 Determine the input impedance to the series network in Fig. 15.23. Draw the impedance diagram.

Solution:

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$
$$= R \angle 0° + X_L \angle 90° + X_C \angle -90°$$
$$= R + jX_L - jX_C$$
$$= R + j(X_L - X_C) = 6\ \Omega + j(10\ \Omega - 12\ \Omega) = 6\ \Omega - j2\ \Omega$$
$$\mathbf{Z}_T = \mathbf{6.32\ \Omega\ \angle -18.43°}$$

The impedance diagram appears in Fig. 15.24. Note that in this example, series inductive and capacitive reactances are in direct opposition. For the circuit in Fig. 15.23, if the inductive reactance were equal to the capacitive reactance, the input impedance would be purely resistive. We will have more to say about this particular condition in a later chapter.

For the representative **series ac configuration** in Fig. 15.25 having two impedances, *the current is the same through each element* (as it was for the series dc circuits) and is determined by Ohm's law:
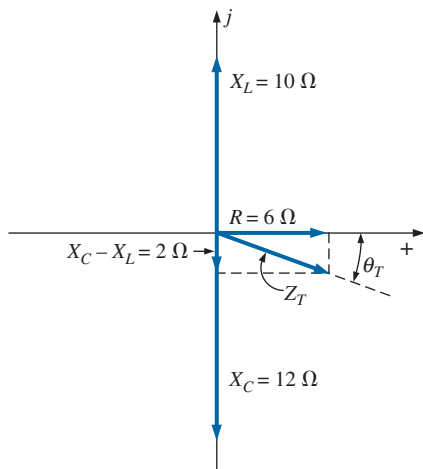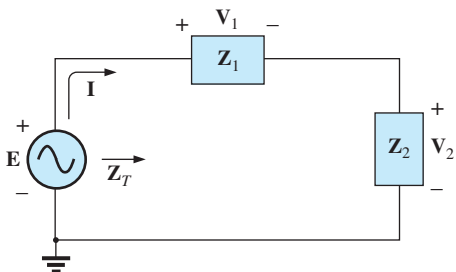
$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2$$

and

$$\boxed{\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T}} \tag{15.5}$$

The voltage across each element can then be found by another application of Ohm's law:

$$\boxed{\mathbf{V}_1 = \mathbf{IZ}_1} \tag{15.6a}$$

$$\boxed{\mathbf{V}_2 = \mathbf{IZ}_2} \tag{15.6b}$$

Kirchhoff's voltage law can then be applied in the same manner as it is employed for dc circuits. However, keep in mind that we are now dealing with the algebraic manipulation of quantities that have both magnitude and direction.

$$\mathbf{E} - \mathbf{V}_1 - \mathbf{V}_2 = 0$$

or

$$\boxed{\mathbf{E} = \mathbf{V}_1 + \mathbf{V}_2} \tag{15.7}$$

The power to the circuit can be determined by

$$\boxed{P = EI \cos \theta_T} \tag{15.8}$$

where θ_T is the phase angle between **E** and **I**.

Now that a general approach has been introduced, the simplest of series configurations will be investigated in detail to further emphasize the similarities in the analysis of dc circuits. In many of the circuits to be considered, $3 + j4 = 5 \angle 53.13°$ and $4 + j3 = 5 \angle 36.87°$ are used quite frequently to ensure that the approach is as clear as possible and not lost in mathematical complexity. Of course, the problems at the end of the chapter will provide plenty of experience with random values.

R-L

Refer to Fig. 15.26.

Phasor Notation

$$e = 141.4 \sin \omega t \Rightarrow \mathbf{E} = 100 \text{ V } \angle 0°$$

Note Fig. 15.27.

Z$_T$

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 = 3 \text{ } \Omega \angle 0° + 4 \text{ } \Omega \angle 90° = 3 \text{ } \Omega + j 4 \text{ } \Omega$$

and $\mathbf{Z}_T = \mathbf{5} \text{ } \Omega \angle \mathbf{53.13°}$

Impedance diagram: See Fig. 15.28.

I

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{100 \text{ V } \angle 0°}{5 \text{ } \Omega \angle 53.13°} = \mathbf{20} \text{ A } \angle -\mathbf{53.13°}$$

V$_R$ and V$_L$
  Ohm's law:

$$\mathbf{V}_R = \mathbf{IZ}_R = (20 \text{ A } \angle -53.13°)(3 \text{ } \Omega \angle 0°)$$
$$= \mathbf{60} \text{ V } \angle -\mathbf{53.13°}$$
$$\mathbf{V}_L = \mathbf{IZ}_L = (20 \text{ A } \angle -53.13°)(4 \text{ } \Omega \angle 90°)$$
$$= \mathbf{80} \text{ V } \angle \mathbf{36.87°}$$

  Kirchhoff's voltage law:

$$\Sigma_\circlearrowright \mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_L = 0$$

or $\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L$

In rectangular form,

$$\mathbf{V}_R = 60 \text{ V } \angle -53.13° = 36 \text{ V } - j 48 \text{ V}$$
$$\mathbf{V}_L = 80 \text{ V } \angle +36.87° = 64 \text{ V } + j 48 \text{ V}$$

and

$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L = (36 \text{ V } - j 48 \text{ V}) + (64 \text{ V } + j 48 \text{ V}) = 100 \text{ V } + j 0$$
$$= 100 \text{ V } \angle 0°$$

as applied.
  Phasor diagram: Note that for the phasor diagram in Fig. 15.29, **I** is in phase with the voltage across the resistor and lags the voltage across the inductor by 90°.
  Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta_T$$
$$= (100 \text{ V})(20 \text{ A}) \cos 53.13° = (2000 \text{ W})(0.6)$$
$$= \mathbf{1200} \text{ W}$$

where E and I are effective values and θ_T is the phase angle between E and I, or

$$P_T = I^2 R$$
$$= (20 \text{ A})^2 (3 \text{ } \Omega) = (400)(3)$$
$$= \mathbf{1200} \text{ W}$$
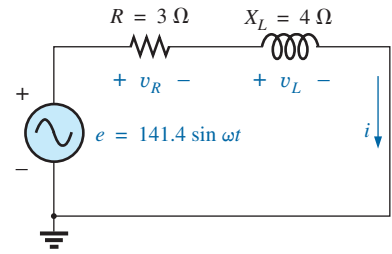
FIG. 15.26

Series R-L circuit.
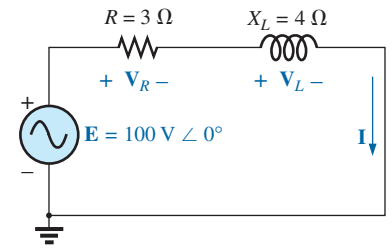
FIG. 15.27

Applying phasor notation to the network in Fig. 15.26.
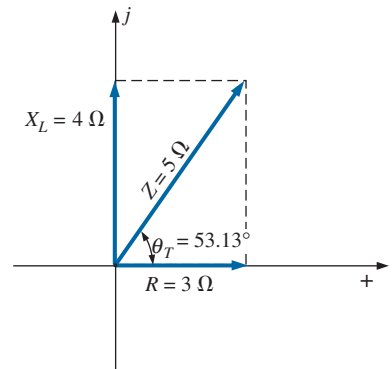
FIG. 15.28

Impedance diagram for the series R-L circuit in Fig. 15.26.
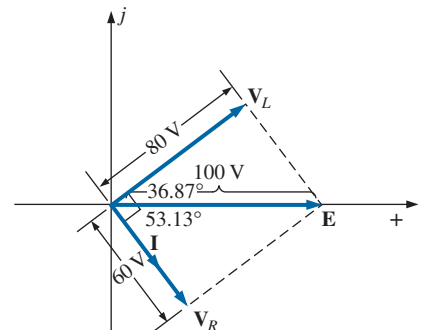
FIG. 15.29

Phasor diagram for the series R-L circuit in Fig. 15.26.

where I is the effective value, or, finally,

$$P_T = P_R + P_L = V_R I \cos \theta_R + V_L I \cos \theta_L$$
$$= (60 \text{ V})(20 \text{ A}) \cos 0° + (80 \text{ V})(20 \text{ A}) \cos 90°$$
$$= 1200 \text{ W} + 0$$
$$= \mathbf{1200 \text{ W}}$$

where θ_R is the phase angle between $\mathbf{V}_R$ and $\mathbf{I}$, and θ_L is the phase angle between $\mathbf{V}_L$ and $\mathbf{I}$.

Power factor: The power factor F_p of the circuit is $\cos 53.13° = \mathbf{0.6}$ **lagging,** where $53.13°$ is the phase angle between $\mathbf{E}$ and $\mathbf{I}$.

If we write the basic power equation $P = EI \cos \theta$ as follows:

$$\cos \theta = \frac{P}{EI}$$

where E and I are the input quantities and P is the power delivered to the network, and then perform the following substitutions from the basic series ac circuit:

$$\cos \theta = \frac{P}{EI} = \frac{I^2 R}{EI} = \frac{IR}{E} = \frac{R}{E/I} = \frac{R}{Z_T}$$

we find

$$\boxed{F_p = \cos \theta_T = \frac{R}{Z_T}} \qquad (15.9)$$

Reference to Fig. 15.28 also indicates that θ is the impedance angle θ_T as written in Eq. (15.9), further supporting the fact that the impedance angle θ_T is also the phase angle between the input voltage and current for a series ac circuit. To determine the power factor, it is necessary only to form the ratio of the total resistance to the magnitude of the input impedance. For the case at hand,

$$F_p = \cos \theta = \frac{R}{Z_T} = \frac{3 \text{ } \Omega}{5 \text{ } \Omega} = \mathbf{0.6 \text{ lagging}}$$

as found above.

R-C

Refer to Fig. 15.30.

Phasor Notation

$$i = 7.07 \sin(\omega t + 53.13°) \Rightarrow \mathbf{I} = 5 \text{ A } \angle 53.13°$$

Note Fig. 15.31.

$\mathbf{Z}_T$

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 = 6 \text{ } \Omega \angle 0° + 8 \text{ } \Omega \angle -90° = 6 \text{ } \Omega - j8 \text{ } \Omega$$

and

$$\mathbf{Z}_T = \mathbf{10 \text{ } \Omega \angle -53.13°}$$
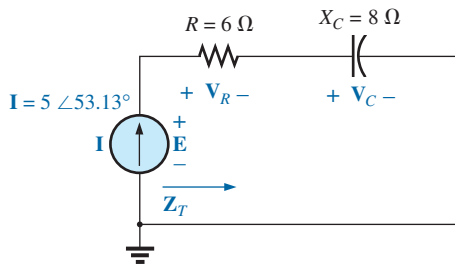
Impedance diagram: As shown in Fig. 15.32.

E
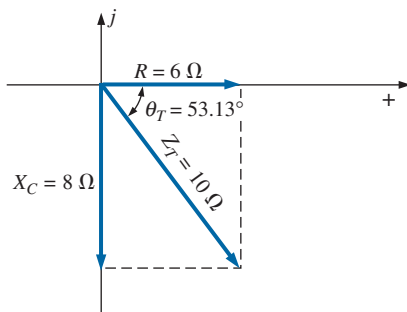
$$\mathbf{E} = \mathbf{I}\mathbf{Z}_T = (5 \text{ A } \angle 53.13°)(10 \text{ } \Omega \angle -53.13°) = \mathbf{50 \text{ V } \angle 0°}$$
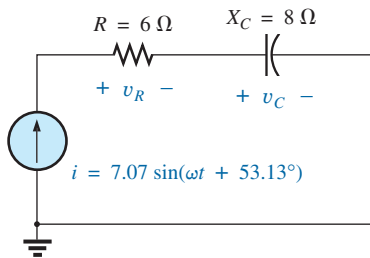
FIG. 15.30

Series R-C ac circuit.

FIG. 15.31

Applying phasor notation to the circuit in Fig. 15.30.

FIG. 15.32

Impedance diagram for the series R-C circuit in Fig. 15.30.

V_R and V_C

$$\mathbf{V}_R = \mathbf{IZ}_R = (I \angle\theta)(R \angle 0°) = (5\,A \angle 53.13°)(6\,\Omega \angle 0°)$$
$$= \mathbf{30\,V} \angle \mathbf{53.13°}$$

$$\mathbf{V}_C = \mathbf{IZ}_C = (I \angle\theta)(X_C \angle -90°) = (5\,A \angle 53.13°)(8\,\Omega \angle -90°)$$
$$= \mathbf{40\,V} \angle \mathbf{-36.87°}$$

Kirchhoff's voltage law:

$$\Sigma_C \mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_C = 0$$

or
$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_C$$

which can be verified by vector algebra as demonstrated for the *R-L* circuit.

Phasor diagram: Note on the phasor diagram in Fig. 15.33 that the current **I** is in phase with the voltage across the resistor and leads the voltage across the capacitor by 90°.

Time domain: In the time domain,

$$e = \sqrt{2}(50) \sin \omega t = \mathbf{70.70 \sin \omega t}$$
$$v_R = \sqrt{2}(30) \sin(\omega t + 53.13°) = \mathbf{42.42 \sin(\omega t + 53.13°)}$$
$$v_C = \sqrt{2}(40) \sin(\omega t - 36.87°) = \mathbf{56.56\ \sin(\omega t - 36.87°)}$$

A plot of all of the voltages and the current of the circuit appears in Fig. 15.34. Note again that i and v_R are in phase and that v_C lags i by 90°.
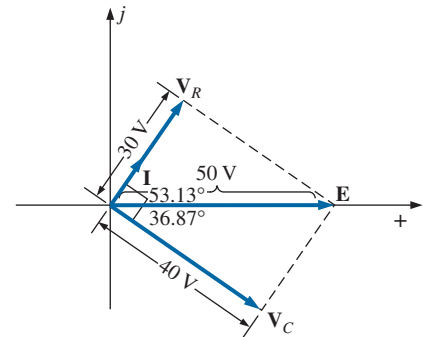
FIG. 15.33
Phasor diagram for the series R-C circuit in Fig. 15.30.
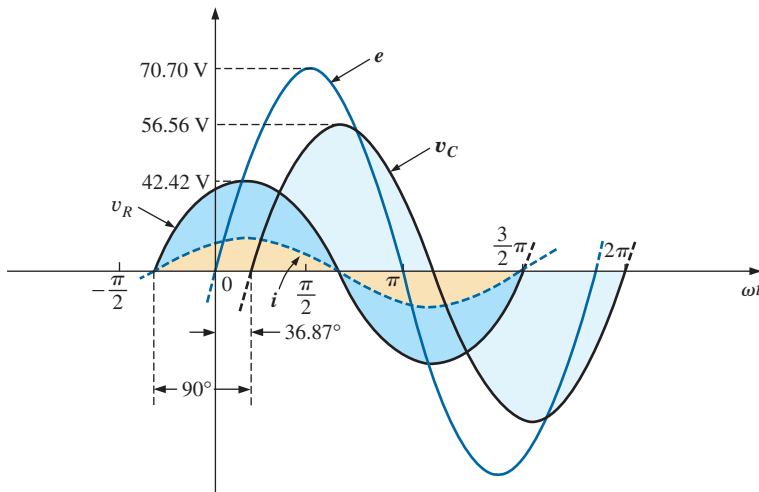
FIG. 15.34
Waveforms for the series R-C circuit in Fig. 15.30.

Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta_T = (50\,V)(5\,A) \cos 53.13°$$
$$= (250)(0.6) = \mathbf{150\,W}$$

or
$$P_T = I^2 R = (5\,A)^2(6\,\Omega) = (25)(6)$$
$$= \mathbf{150\,W}$$

or, finally,

$$P_T = P_R + P_C = V_R I \cos \theta_R + V_C I \cos \theta_C$$
$$= (30\,V)(5\,A) \cos 0° + (40\,V)(5\,A) \cos 90°$$
$$= 150\,W + 0$$
$$= \mathbf{150\,W}$$

Power factor: The power factor of the circuit is

$$F_p = \cos\theta = \cos 53.13° = \textbf{0.6 leading}$$

Using Eq. (15.9), we obtain

$$F_p = \cos\theta = \frac{R}{Z_T} = \frac{6\,\Omega}{10\,\Omega}$$
$$= \textbf{0.6 leading}$$

as determined above.

R-L-C

Refer to Fig. 15.35.
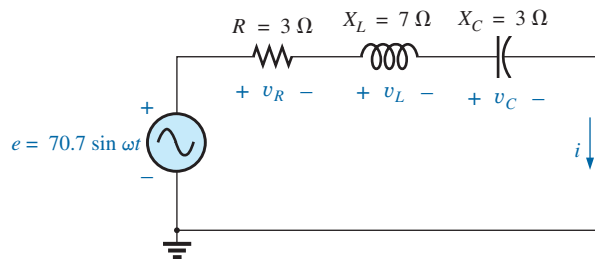
FIG. 15.35
Series R-L-C ac circuit.

Phasor Notation   As shown in Fig. 15.36.
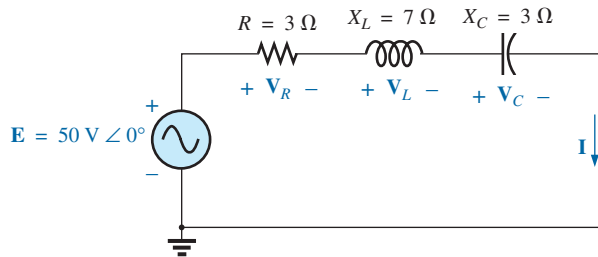
FIG. 15.36
Applying phasor notation to the circuit in Fig. 15.35.

Z_T

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 = R\angle 0° + X_L\angle 90° + X_C\angle -90°$$
$$= 3\,\Omega + j7\,\Omega - j3\,\Omega = 3\,\Omega + j4\,\Omega$$

and
$$\mathbf{Z}_T = \textbf{5 }\Omega\,\angle\textbf{53.13°}$$

Impedance diagram: As shown in Fig. 15.37.
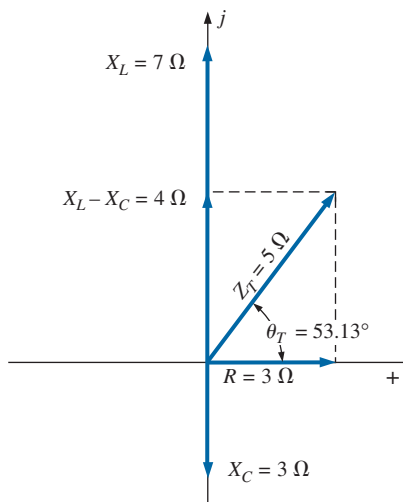
FIG. 15.37
Impedance diagram for the series R-L-C circuit in Fig. 15.35.

I

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{50\text{ V}\angle 0°}{5\,\Omega\angle 53.13°} = \textbf{10 A}\,\angle\,\textbf{−53.13°}$$

V_R, V_L, and V_C

$$\mathbf{V}_R = \mathbf{IZ}_R = (I \angle\theta)(R \angle0°) = (10 \text{ A} \angle-53.13°)(3 \text{ } \Omega \angle0°)$$
$$= \mathbf{30 \text{ V} \angle-53.13°}$$

$$\mathbf{V}_L = \mathbf{IZ}_L = (I \angle\theta)(X_L \angle90°) = (10 \text{ A} \angle-53.13°)(7 \text{ } \Omega \angle90°)$$
$$= \mathbf{70 \text{ V} \angle36.87°}$$

$$\mathbf{V}_C = \mathbf{IZ}_C = (I \angle\theta)(X_C \angle-90°) = (10 \text{ A} \angle-53.13°)(3 \text{ } \Omega \angle-90°)$$
$$= \mathbf{30 \text{ V} \angle-143.13°}$$

Kirchhoff's voltage law:

$$\Sigma_\circlearrowleft \mathbf{V} = \mathbf{E} - \mathbf{V}_R - \mathbf{V}_L - \mathbf{V}_C = 0$$

or
$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

which can also be verified through vector algebra.

Phasor diagram: The phasor diagram in Fig. 15.38 indicates that the current **I** is in phase with the voltage across the resistor, lags the voltage across the inductor by 90°, and leads the voltage across the capacitor by 90°.

Time domain:

$$i = \sqrt{2}(10) \sin(\omega t - 53.13°) = \mathbf{14.14 \sin(\omega t - 53.13°)}$$
$$v_R = \sqrt{2}(30) \sin(\omega t - 53.13°) = \mathbf{42.42 \sin(\omega t - 53.13°)}$$
$$v_L = \sqrt{2}(70) \sin(\omega t + 36.87°) = \mathbf{98.98 \sin(\omega t + 36.87°)}$$
$$v_C = \sqrt{2}(30) \sin(\omega t - 143.13°) = \mathbf{42.42 \ \sin(\omega t - 143.13°)}$$

A plot of all the voltages and the current of the circuit appears in Fig. 15.39.
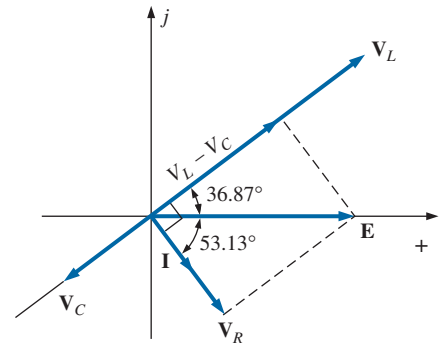
FIG. 15.38

Phasor diagram for the series R-L-C circuit in Fig. 15.35.
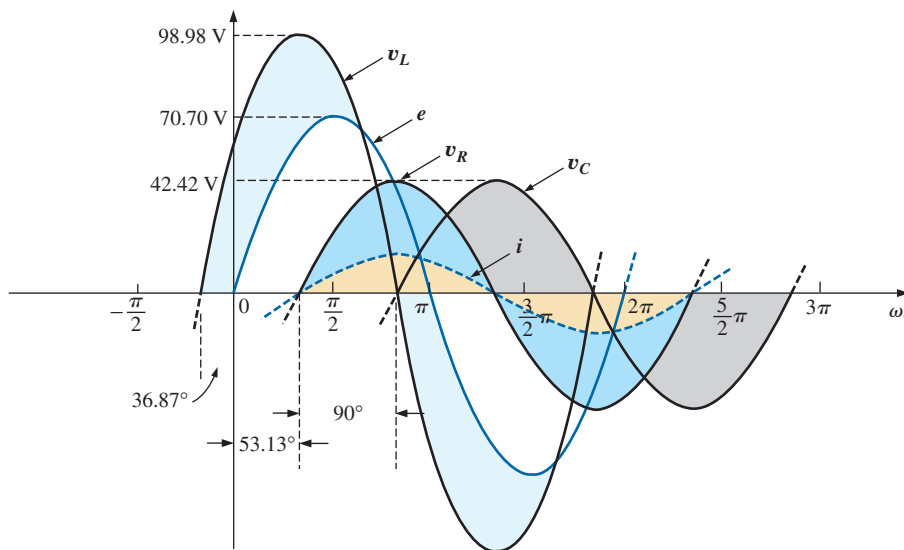
FIG. 15.39

Waveforms for the series R-L-C circuit in Fig. 15.35.

Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos\theta_T = (50 \text{ V})(10 \text{ A}) \cos 53.13° = (500)(0.6) = \mathbf{300 \text{ W}}$$

or
$$P_T = I^2R = (10 \text{ A})^2(3 \text{ } \Omega) = (100)(3) = \mathbf{300 \text{ W}}$$

or

$$P_T = P_R + P_L + P_C$$
$$= V_R I \cos \theta_R + V_L I \cos \theta_L + V_C I \cos \theta_C$$
$$= (30 \text{ V})(10 \text{ A}) \cos 0° + (70 \text{ V})(10 \text{ A}) \cos 90° + (30 \text{ V})(10 \text{ A}) \cos 90°$$
$$= (30 \text{ V})(10 \text{ A}) + 0 + 0 = \textbf{300 W}$$

Power factor: The power factor of the circuit is

$$F_p = \cos \theta_T = \cos 53.13° = \textbf{0.6 lagging}$$

Using Eq. (15.9), we obtain

$$F_p = \cos \theta = \frac{R}{Z_T} = \frac{3 \, \Omega}{5 \, \Omega} = \textbf{0.6 lagging}$$

15.4 VOLTAGE DIVIDER RULE

The basic format for the **voltage divider rule** in ac circuits is exactly the same as that for dc circuits:

$$\boxed{\mathbf{V}_x = \frac{\mathbf{Z}_x \mathbf{E}}{\mathbf{Z}_T}} \tag{15.10}$$

where $\mathbf{V}_x$ is the voltage across one or more elements in a series that have total impedance $\mathbf{Z}_x$, $\mathbf{E}$ is the total voltage appearing across the series circuit, and $\mathbf{Z}_T$ is the total impedance of the series circuit.
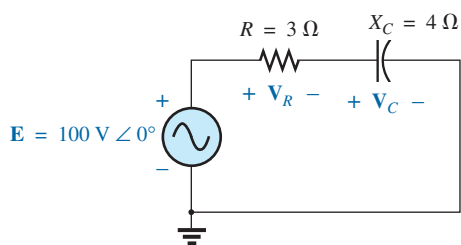
FIG. 15.40
Example 15.9.

EXAMPLE 15.9 Using the voltage divider rule, find the voltage across each element of the circuit in Fig. 15.40.

Solution:

$$\mathbf{V}_C = \frac{\mathbf{Z}_C \mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(4 \, \Omega \angle -90°)(100 \text{ V} \angle 0°)}{4 \, \Omega \angle -90° + 3 \, \Omega \angle 0°} = \frac{400 \angle -90°}{3 - j4}$$

$$= \frac{400 \angle -90°}{5 \angle -53.13°} = \textbf{80 V} \angle \textbf{−36.87°}$$

$$\mathbf{V}_R = \frac{\mathbf{Z}_R \mathbf{E}}{\mathbf{Z}_C + \mathbf{Z}_R} = \frac{(3 \, \Omega \angle 0°)(100 \text{ V} \angle 0°)}{5 \, \Omega \angle -53.13°} = \frac{300 \angle 0°}{5 \angle -53.13°}$$

$$= \textbf{60 V} \angle \textbf{+53.13°}$$

EXAMPLE 15.10 Using the voltage divider rule, find the unknown voltages $\mathbf{V}_R$, $\mathbf{V}_L$, $\mathbf{V}_C$, and $\mathbf{V}_1$ for the circuit in Fig. 15.41.
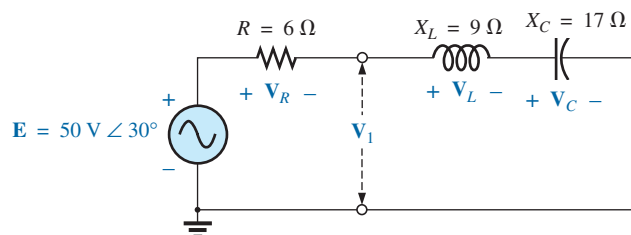
FIG. 15.41
Example 15.10.

Solution:

$$\mathbf{V}_R = \frac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C} = \frac{(6\ \Omega\ \angle 0°)(50\ \text{V}\ \angle 30°)}{6\ \Omega\ \angle 0° + 9\ \Omega\ \angle 90° + 17\ \Omega\ \angle -90°}$$

$$= \frac{300\ \angle 30°}{6 + j\,9 - j\,17} = \frac{300\ \angle 30°}{6 - j\,8}$$

$$= \frac{300\ \angle 30°}{10\ \angle -53.13°} = \textbf{30 V}\ \angle \textbf{83.13°}$$

Calculator The above calculation provides an excellent opportunity to demonstrate the power of today's calculators. Using the TI-89 calculator, the sequence of steps to calculate $\mathbf{V}_R$ are shown in Fig. 15.42.

(6 ∠ 0 °) × (5 0 ∠ 3 0 °) ÷ ((
6 ∠ 0 °) + (9 ∠ 9 0 °) + (1
7 ∠ (−) 9 0 °)) ▶ Polar ENTER ENTER 30.00E0 ∠ 83.13E0
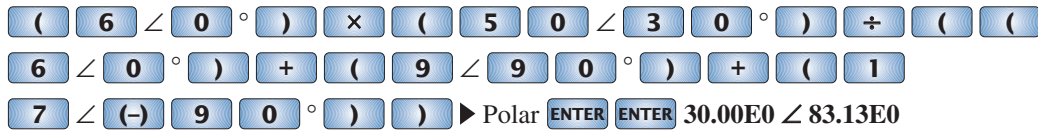
FIG. 15.42
Using the TI-89 calculator to determine V_R in Example 15.10.

$$\mathbf{V}_L = \frac{\mathbf{Z}_L\mathbf{E}}{\mathbf{Z}_T} = \frac{(9\ \Omega\ \angle 90°)(50\ \text{V}\ \angle 30°)}{10\ \Omega\ \angle -53.13°} = \frac{450\ \text{V}\ \angle 120°}{10\ \angle -53.13°}$$

$$= \textbf{45 V} \angle \textbf{173.13°}$$

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_T} = \frac{(17\ \Omega\ \angle -90°)(50\ \text{V}\ \angle 30°)}{10\ \Omega\ \angle -53.13°} = \frac{850\ \text{V}\ \angle -60°}{10\ \angle -53°}$$

$$= \textbf{85 V} \angle \textbf{−6.87°}$$

$$\mathbf{V}_1 = \frac{(\mathbf{Z}_L + \mathbf{Z}_C)\mathbf{E}}{\mathbf{Z}_T} = \frac{(9\ \Omega\ \angle 90° + 17\ \Omega\ \angle -90°)(50\ \text{V}\ \angle 30°)}{10\ \Omega\ \angle -53.13°}$$

$$= \frac{(8\ \angle -90°)(50\ \angle 30°)}{10\ \angle -53.13°}$$

$$= \frac{400\ \angle -60°}{10\ \angle -53.13°} = \textbf{40 V}\ \angle \textbf{−6.87°}$$

EXAMPLE 15.11 For the circuit in Fig. 15.43:

a. Calculate $\mathbf{I}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$ in phasor form.
b. Calculate the total power factor.
c. Calculate the average power delivered to the circuit.
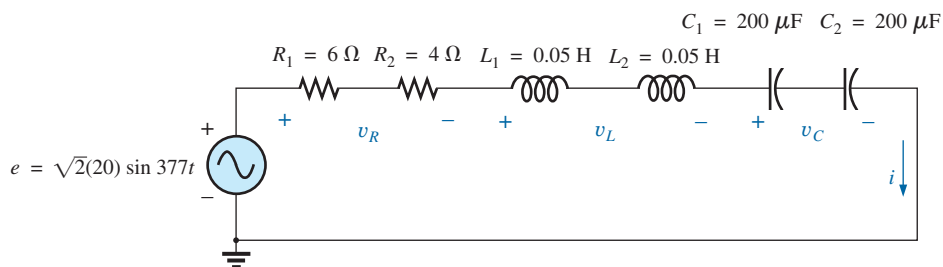
FIG. 15.43
Example 15.11.

d. Draw the phasor diagram.
e. Obtain the phasor sum of V_R, V_L, and V_C, and show that it equals the input voltage **E.**
f. Find V_R and V_C using the voltage divider rule.

Solutions:

a. Combining common elements and finding the reactance of the inductor and capacitor, we obtain

$$R_T = 6\,\Omega + 4\,\Omega = 10\,\Omega$$
$$L_T = 0.05\,\text{H} + 0.05\,\text{H} = 0.1\,\text{H}$$
$$C_T = \frac{200\,\mu\text{F}}{2} = 100\,\mu\text{F}$$
$$X_L = \omega L = (377\,\text{rad/s})(0.1\,\text{H}) = 37.70\,\Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{(377\,\text{rad/s})(100 \times 10^{-6}\,\text{F})} = \frac{10^6\,\Omega}{37{,}700} = 26.53\,\Omega$$

Redrawing the circuit using phasor notation results in Fig. 15.44.
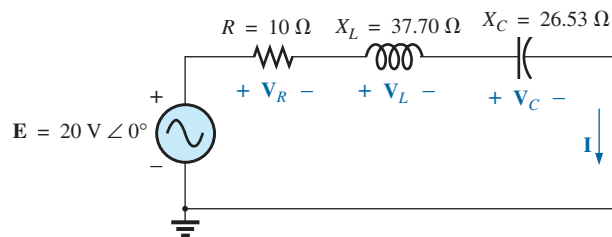
FIG. 15.44
Applying phasor notation to the circuit in Fig. 15.43.

For the circuit in Fig. 15.44,

$$\mathbf{Z}_T = R\angle 0° + X_L\angle 90° + X_C\angle -90°$$
$$= 10\,\Omega + j\,37.70\,\Omega - j\,26.53\,\Omega$$
$$= 10\,\Omega + j\,11.17\,\Omega = \mathbf{15\,\Omega\,\angle 48.16°}$$

The current **I** is

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{20\,\text{V}\angle 0°}{15\,\Omega\,\angle 48.16°} = \mathbf{1.33\,A\,\angle -48.16°}$$

The voltage across the resistor, inductor, and capacitor can be found using Ohm's law:

$$\mathbf{V}_R = \mathbf{IZ}_R = (I\angle\theta)(R\angle 0°) = (1.33\,\text{A}\angle -48.16°)(10\,\Omega\angle 0°)$$
$$= \mathbf{13.30\,V\,\angle -48.16°}$$
$$\mathbf{V}_L = \mathbf{IZ}_L = (I\angle\theta)(X_L\angle 90°) = (1.33\,\text{A}\angle -48.16°)(37.70\,\Omega\angle 90°)$$
$$= \mathbf{50.14\,V\,\angle 41.84°}$$
$$\mathbf{V}_C = \mathbf{IZ}_C = (I\angle\theta)(X_C\angle -90°) = (1.33\,\text{A}\angle -48.16°)(26.53\,\Omega\angle -90°)$$
$$= \mathbf{35.28\,V\,\angle -138.16°}$$

b. The total power factor, determined by the angle between the applied voltage **E** and the resulting current **I,** is 48.16°:

$$F_p = \cos\theta = \cos 48.16° = \mathbf{0.667\ lagging}$$

or $$F_p = \cos\theta = \frac{R}{Z_T} = \frac{10\,\Omega}{15\,\Omega} = \mathbf{0.667\ lagging}$$

c. The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta = (20 \text{ V})(1.33 \text{ A})(0.667) = \textbf{17.74 W}$$

d. The phasor diagram appears in Fig. 15.45.

e. The phasor sum of $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$ is

$$\mathbf{E} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$
$$= 13.30 \text{ V} \angle{-48.16°} + 50.14 \text{ V} \angle{41.84°} + 35.28 \text{ V} \angle{-138.16°}$$
$$\mathbf{E} = 13.30 \text{ V} \angle{-48.16°} + 14.86 \text{ V} \angle{41.84°}$$

Therefore,

$$E = \sqrt{(13.30 \text{ V})^2 + (14.86 \text{ V})^2} = \textbf{20 V}$$

and $\qquad \theta_E = \textbf{0°} \qquad$ (from phasor diagram)

and $\qquad\qquad \mathbf{E} = 20 \angle{0°}$

f. $\mathbf{V}_R = \dfrac{\mathbf{Z}_R\mathbf{E}}{\mathbf{Z}_T} = \dfrac{(10 \text{ } \Omega \text{ } \angle{0°})(20 \text{ V} \angle{0°})}{15 \text{ } \Omega \text{ } \angle{48.16°}} = \dfrac{200 \text{ V} \angle{0°}}{15 \angle{48.16°}}$

$\qquad = \textbf{13.3 V} \angle{\textbf{−48.16°}}$

$\mathbf{V}_C = \dfrac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_T} = \dfrac{(26.5 \text{ } \Omega \text{ } \angle{-90°})(20 \text{ V}\angle{0°})}{15 \text{ } \Omega \text{ } \angle{48.16°}} = \dfrac{530.6 \text{ V} \angle{-90°}}{15 \angle{48.16°}}$

$\qquad = \textbf{35.37 V}\angle{\textbf{−138.16°}}$
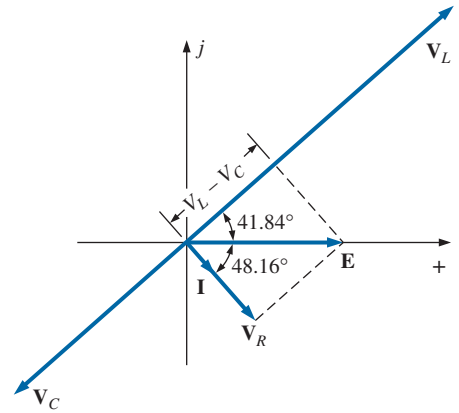
FIG. 15.45

Phasor diagram for the circuit in Fig. 15.43.

15.5 FREQUENCY RESPONSE FOR SERIES ac CIRCUITS

Thus far, the analysis has been for a fixed frequency, resulting in a fixed value for the reactance of an inductor or a capacitor. We now examine how the response of a series circuit changes as the frequency changes. We assume ideal elements throughout the discussion so that the response of each element will be as shown in Fig. 15.46. Each response in Fig. 15.46 was discussed in detail in Chapter 14.
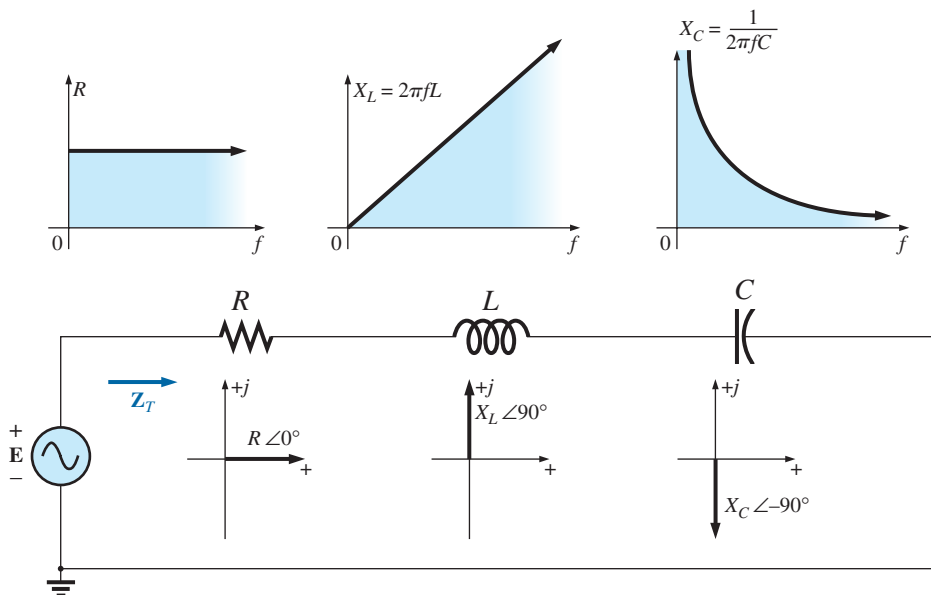
FIG. 15.46

Reviewing the frequency response of the basic elements.

When considering elements in series, remember that the total imped-ance is the sum of the individual elements and that the reactance of an in-ductor is in direct opposition to that of a capacitor. For Fig. 15.46, we are first aware that the resistance will remain fixed for the full range of fre-quencies: It will always be there, but, more importantly, its magnitude will not change. The inductor, however, will provide increasing levels of impedance as the frequency increases, while the capacitor will provide lower levels of impedance.

We are also aware from Chapter 14 that the inductor has a short-circuit equivalence at $f = 0$ Hz or very low frequencies, while the capacitor is nearly an open circuit for the same frequency range. For very high fre-quencies, the capacitor approaches the short-circuit equivalence, and the inductor approaches the open-circuit equivalence.

In general, therefore, if we encounter a series R-L-C circuit at very low frequencies, we can assume that the capacitor, with its very large imped-ance, will be the predominant factor. If the circuit is just an R-L series cir-cuit, the impedance may be determined primarily by the resistive element since the reactance of the inductor is so small. As the frequency in-creases, the reactance of the coil increases to the point where it totally outshadows the impedance of the resistor. For an R-L-C combination, as the frequency increases, the reactance of the capacitor begins to approach a short-circuit equivalence, and the total impedance will be determined primarily by the inductive element. At very high frequencies, for an R-C series circuit, the total impedance eventually approaches that of the re-sistor since the impedance of the capacitor is dropping off so quickly.

In total, therefore,

when encountering a series ac circuit of any combination of elements, always use the idealized response of each element to establish some feeling for how the circuit will respond as the frequency changes.

Once you have a logical, overall sense for what the response will be, you can concentrate on working out the details.

Series *R-C* ac Circuit

As an example of establishing the frequency response of a circuit, con-sider the series R-C circuit in Fig. 15.47. As noted next to the source, the frequency range of interest is from 0 to 20 kHz. A great deal of detail is provided for this particular combination so that obtaining the response of a series R-L or R-L-C combination will be quite straightforward.

Since the resistance remains fixed at 5 kΩ for the full frequency range, and the total impedance is the sum of the impedances, it is immediately obvious that the lowest possible impedance is 5 kΩ. The highest imped-
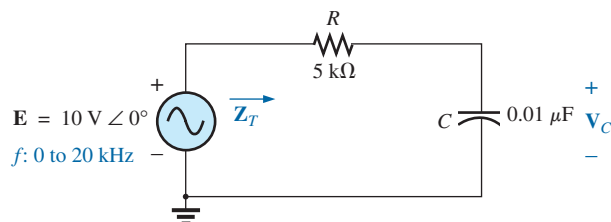
FIG. 15.47

Determining the frequency response of a series R-C circuit.

ance, however, is dependent on the capacitive element since its imped-
ance at very low frequencies is extremely high. At very low frequencies
we can conclude, without a single calculation, that the impedance is de-
termined primarily by the impedance of the capacitor. At the highest fre-
quencies, we can assume that the reactance of the capacitor has dropped
to such low levels that the impedance of the combination will approach
that of the resistance.

The frequency at which the reactance of the capacitor drops to that of
the resistor can be determined by setting the reactance of the capacitor
equal to that of the resistor as follows:

$$X_C = \frac{1}{2\pi f_1 C} = R$$

Solving for the frequency yields

$$f_1 = \frac{1}{2\pi RC} \qquad \textbf{(15.11)}$$

This significant point appears in the frequency plots in Fig. 15.48.
Substituting values, we find that it occurs at

$$f_1 = \frac{1}{2\pi RC} = \frac{1}{2\pi(5 \text{ k}\Omega)(0.01 \ \mu\text{F})} \cong 3.18 \text{ kHz}$$

We now know that for frequencies greater than f_1, $R > X_C$ and that for fre-
quencies less than f_1, $X_C > R$, as shown in Fig. 15.48.

Now for the details. The total impedance is determined by the follow-
ing equation:

$$\mathbf{Z}_T = R - jX_C$$

and $$\mathbf{Z}_T = Z_T \angle\theta_T = \sqrt{R^2 + X_C^2} \ \angle -\tan^{-1}\frac{X_C}{R} \qquad \textbf{(15.12)}$$

The magnitude and angle of the total impedance can now be found at any
frequency of interest by simply substituting into Eq. (15.12). The presence
of the capacitor suggests that we start from a low frequency (100 Hz) and
then open the spacing until we reach the upper limit of interest (20 kHz).
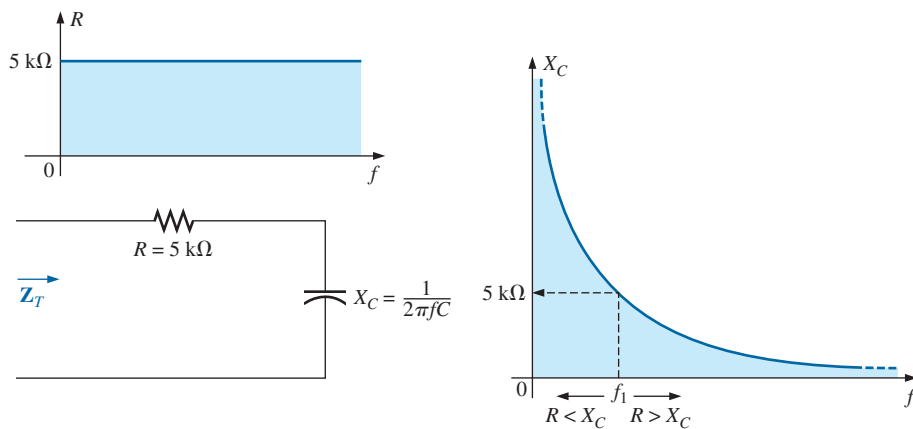

FIG. 15.48
The frequency response for the individual elements of a series R-C circuit.

f = 100 Hz

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(100 \text{ Hz})(0.01 \text{ } \mu\text{F})} = 159.16 \text{ k}\Omega$$

and

$$Z_T = \sqrt{R^2 + X_C^2} = \sqrt{(5 \text{ k}\Omega)^2 + (159.16 \text{ k}\Omega)^2} = 159.24 \text{ k}\Omega$$

with

$$\theta_T = -\tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{159.16 \text{ k}\Omega}{5 \text{ k}\Omega} = -\tan^{-1} 31.83$$

$$= -88.2°$$

and

$$\mathbf{Z}_T = \mathbf{159.24 \text{ k}\Omega} \angle \mathbf{-88.2°}$$

which compares very closely with $\mathbf{Z}_C = 159.16 \text{ k}\Omega \angle -90°$ if the circuit were purely capacitive ($R = 0 \text{ } \Omega$). Our assumption that the circuit is primarily capacitive at low frequencies is therefore confirmed.

f = 1 kHz

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(1 \text{ kHz})(0.01 \text{ } \mu\text{F})} = 15.92 \text{ k}\Omega$$

and

$$Z_T = \sqrt{R^2 + X_C^2} = \sqrt{(5 \text{ k}\Omega)^2 + (15.92 \text{ k}\Omega)^2} = 16.69 \text{ k}\Omega$$

with

$$\theta_T = -\tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{15.92 \text{ k}\Omega}{5 \text{ k}\Omega}$$

$$= -\tan^{-1} 3.18 = -72.54°$$

and

$$\mathbf{Z}_T = \mathbf{16.69 \text{ k}\Omega} \angle \mathbf{-72.54°}$$

A noticeable drop in the magnitude has occurred, and the impedance angle has dropped almost 17° from the purely capacitive level.

Continuing:

$$f = 5 \text{ kHz:} \quad \mathbf{Z}_T = \mathbf{5.93 \text{ k}\Omega} \angle \mathbf{-32.48°}$$
$$f = 10 \text{ kHz:} \quad \mathbf{Z}_T = \mathbf{5.25 \text{ k}\Omega} \angle \mathbf{-17.66°}$$
$$f = 15 \text{ kHz:} \quad \mathbf{Z}_T = \mathbf{5.11 \text{ k}\Omega} \angle \mathbf{-11.98°}$$
$$f = 20 \text{ kHz:} \quad \mathbf{Z}_T = \mathbf{5.06 \text{ k}\Omega} \angle \mathbf{-9.04°}$$

Note how close the magnitude of Z_T at $f = 20$ kHz is to the resistance level of 5 kΩ. In addition, note how the phase angle is approaching that associated with a pure resistive network (0°).

A plot of Z_T versus frequency in Fig. 15.49 completely supports our assumption based on the curves in Fig. 15.48. The plot of θ_T versus frequency in Fig. 15.50 further suggests that the total impedance made a transition from one of a capacitive nature ($\theta_T = -90°$) to one with resistive characteristics ($\theta_T = 0°$).

Applying the voltage divider rule to determine the voltage across the capacitor in phasor form yields

$$\mathbf{V}_C = \frac{\mathbf{Z}_C\mathbf{E}}{\mathbf{Z}_R + \mathbf{Z}_C}$$

$$= \frac{(X_C\angle -90°)(E \angle 0°)}{R - jX_C} = \frac{X_C E \angle -90°}{R - jX_C}$$

$$= \frac{X_C E \angle -90°}{\sqrt{R^2 + X_C^2} \angle -\tan^{-1} X_C/R}$$

or

$$\mathbf{V}_C = V_C \angle \theta_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} \angle -90° + \tan^{-1}(X_C/R)$$
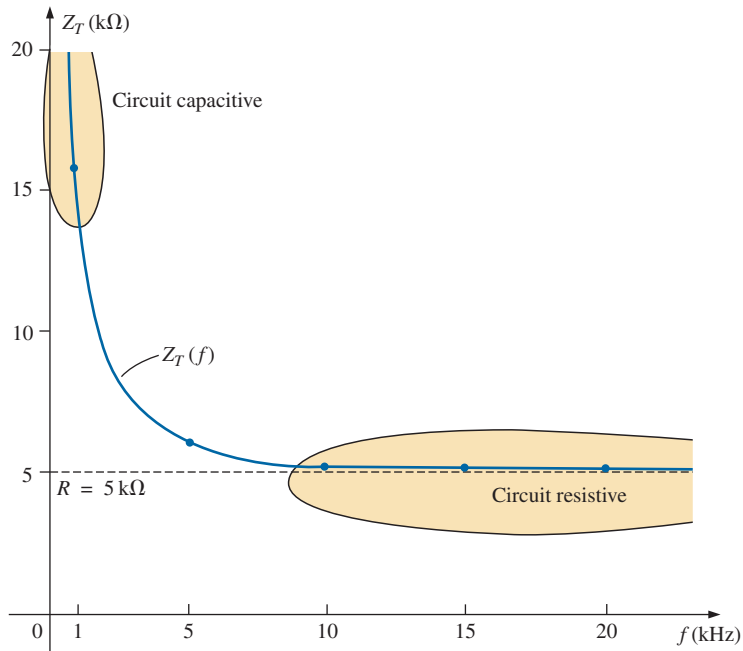
FIG. 15.49

The magnitude of the input impedance versus frequency for the circuit in Fig. 15.47.
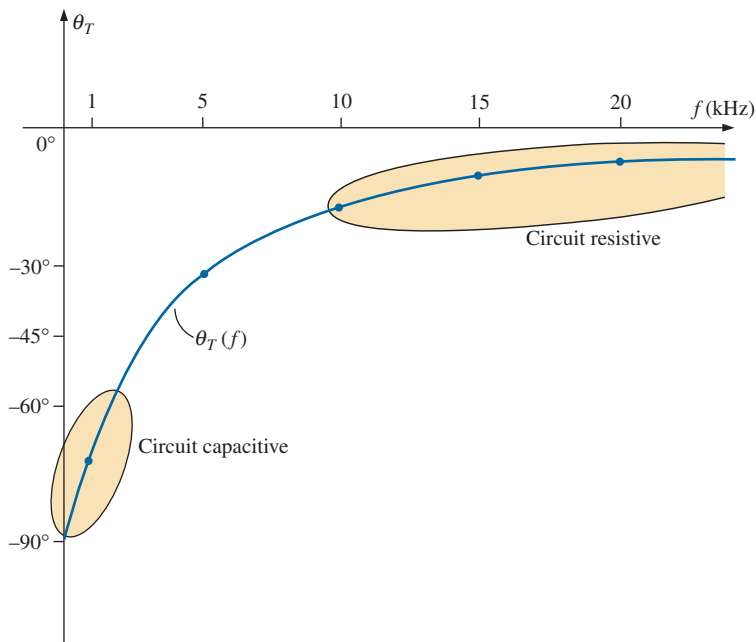
FIG. 15.50

The phase angle of the input impedance versus frequency for the circuit in Fig. 15.47.

The magnitude of $\mathbf{V}_C$ is therefore determined by

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} \qquad (15.13)$$

and the phase angle θ_C by which $\mathbf{V}_C$ leads $\mathbf{E}$ is given by

$$\theta_C = -90° + \tan^{-1}\frac{X_C}{R} = -\tan^{-1}\frac{R}{X_C} \qquad \textbf{(15.14)}$$

To determine the frequency response, X_C must be calculated for each frequency of interest and inserted into Eqs. (15.13) and (15.14).

To begin our analysis, we should consider the case of $f = 0$ Hz (dc conditions).

$f = 0$ Hz

$$X_C = \frac{1}{2\pi(0)C} = \frac{1}{0} \Rightarrow \text{very large value}$$

Applying the open-circuit equivalent for the capacitor based on the above calculation results in the following:

$$\mathbf{V}_C = \mathbf{E} = 10 \text{ V} \angle 0°$$

If we apply Eq. (15.13), we find

$$X_C^2 \gg R^2$$

and

$$\sqrt{R^2 + X_C^2} \cong \sqrt{X_C^2} = X_C$$

and

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{X_C E}{X_C} = E$$

with

$$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1} 0 = 0°$$

verifying the above conclusions.

$f = 1$ kHz   Applying Eq. (15.13):

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(2\pi)(1 \times 10^3 \text{ Hz})(0.01 \times 10^{-6} \text{ F})} \cong \textbf{15.92 k}\Omega$$

$$\sqrt{R^2 + X_C^2} = \sqrt{(5 \text{ k}\Omega)^2 + (15.92 \text{ k}\Omega)^2} \cong 16.69 \text{ k}\Omega$$

and

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{(15.92 \text{ k}\Omega)(10)}{16.69 \text{ k}\Omega} = \textbf{9.54 V}$$

Applying Eq. (15.14):

$$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1}\frac{5 \text{ k}\Omega}{15.9 \text{ k}\Omega}$$

$$= -\tan^{-1} 0.314 = \textbf{-17.46}°$$

and

$$\mathbf{V}_C = \textbf{9.53 V} \angle \textbf{-17.46}°$$

As expected, the high reactance of the capacitor at low frequencies has resulted in the major part of the applied voltage appearing across the capacitor.

If we plot the phasor diagrams for $f = 0$ Hz and $f = 1$ kHz, as shown in Fig. 15.51, we find that $\mathbf{V}_C$ is beginning a clockwise rotation with an increase in frequency that will increase the angle θ_C and decrease the
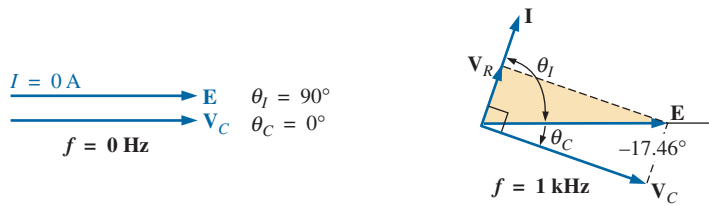
FIG. 15.51
The phasor diagram for the circuit in Fig. 15.47 for f = 0 Hz and 1 kHz.

phase angle between **I** and **E.** Recall that for a purely capacitive network, **I** leads **E** by 90°. As the frequency increases, therefore, the capacitive reactance is decreasing, and eventually $R \gg X_C$ with $\theta_C = -90°$, and the angle between **I** and **E** will approach 0°. Keep in mind as we proceed through the other frequencies that θ_C is the phase angle between $\mathbf{V}_C$ and **E** and that the magnitude of the angle by which **I** leads **E** is determined by

$$\boxed{|\theta_I| = 90° - |\theta_C|} \qquad (15.15)$$

f = 5 kHz   Applying Eq. (15.13):

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(2\pi)(5 \times 10^3 \text{ Hz})(0.01 \times 10^{-6} \text{ F})} \cong \mathbf{3.18 \text{ k}\Omega}$$

Note the dramatic drop in X_C from 1 kHz to 5 kHz. In fact, X_C is now less than the resistance R of the network, and the phase angle determined by $\tan^{-1}(X_C/R)$ must be less than 45°. Here,

$$V_C = \frac{X_C E}{\sqrt{R^2 + X_C^2}} = \frac{(3.18 \text{ k}\Omega)(10 \text{ V})}{\sqrt{(5 \text{ k}\Omega)^2 + (3.18 \text{ k}\Omega)^2}} = \mathbf{5.37 \text{ V}}$$

with

$$\theta_C = -\tan^{-1}\frac{R}{X_C} = -\tan^{-1}\frac{5 \text{ k}\Omega}{3.2 \text{ k}\Omega}$$

$$= -\tan^{-1} 1.56 = \mathbf{-57.38°}$$

f = 10 kHz

$$X_C \cong \mathbf{1.59 \text{ k}\Omega} \qquad V_C = \mathbf{3.03 \text{ V}} \qquad \theta_C = \mathbf{-72.34°}$$

f = 15 kHz

$$X_C \cong \mathbf{1.06 \text{ k}\Omega} \qquad V_C = \mathbf{2.07 \text{ V}} \qquad \theta_C = \mathbf{-78.02°}$$

f = 20 kHz

$$X_C \cong \mathbf{795.78 \text{ }\Omega} \qquad V_C = \mathbf{1.57 \text{ V}} \qquad \theta_C = \mathbf{-80.96°}$$

The phasor diagrams for $f = 5$ kHz and $f = 20$ kHz appear in Fig. 15.52 to show the continuing rotation of the $\mathbf{V}_C$ vector.

Note also from Figs. 15.51 and 15.52 that the vector $\mathbf{V}_R$ and the current **I** have grown in magnitude with the reduction in the capacitive reactance.
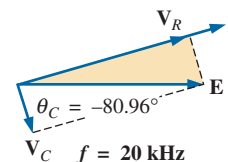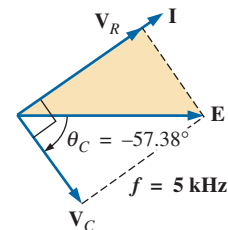
FIG. 15.52
The phasor diagram for the circuit in Fig. 15.47 for f = 5 kHz and 20 kHz.

Eventually, at very high frequencies X_C will approach zero ohms and the short-circuit equivalent can be applied, resulting in $V_C \cong 0$ V and $\theta_C \cong -90°$, and producing the phasor diagram in Fig. 15.53. The network is then resistive, the phase angle between **I** and **E** is essentially zero degrees, and V_R and I are their maximum values.

$$V_R \quad \theta_I \cong 0°$$
$$V_C \cong 0\text{ V} \qquad \mathbf{E} \quad \theta_C \cong -90°$$

$$f = \textbf{very high frequencies}$$
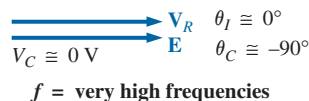
FIG. 15.53

The phasor diagram for the circuit in Fig. 15.47 at very high frequencies.

A plot of V_C versus frequency appears in Fig. 15.54. At low frequencies, $X_C \gg R$, and V_C is very close to E in magnitude. As the applied frequency increases, X_C decreases in magnitude along with V_C as V_R captures more of the applied voltage. A plot of θ_C versus frequency is provided in Fig. 15.55. At low frequencies, the phase angle between $\mathbf{V}_C$ and **E** is very small since $\mathbf{V}_C \cong \mathbf{E}$. Recall that if two phasors are equal, they must have the same angle. As the applied frequency increases, the network becomes more resistive, and the phase angle between $\mathbf{V}_C$ and **E** approaches 90°. Keep in mind that, at high frequencies, **I** and **E** are approaching an in-phase situation and the angle between $\mathbf{V}_C$ and **E** will approach that between $\mathbf{V}_C$ and **I**, which we know must be 90° ($\mathbf{I}_C$ leading $\mathbf{V}_C$).

A plot of V_R versus frequency approaches E volts from zero volts with an increase in frequency, but remember $V_R \neq E - V_C$ due to the vector relationship. The phase angle between **I** and **E** could be plotted directly from Fig. 15.55 using Eq. (15.15).

In Chapter 21, the analysis of this section is extended to a much wider frequency range using a log axis for frequency. It will be demonstrated
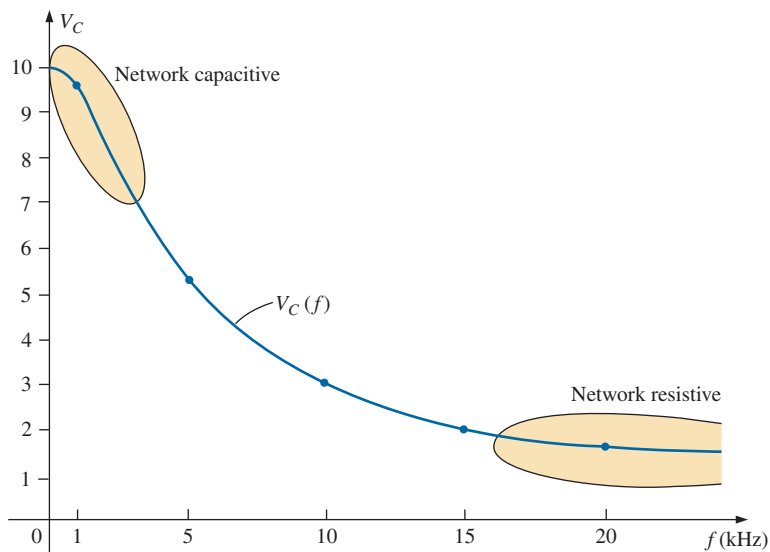
FIG. 15.54

The magnitude of the voltage V_C versus frequency for the circuit in Fig. 15.47.
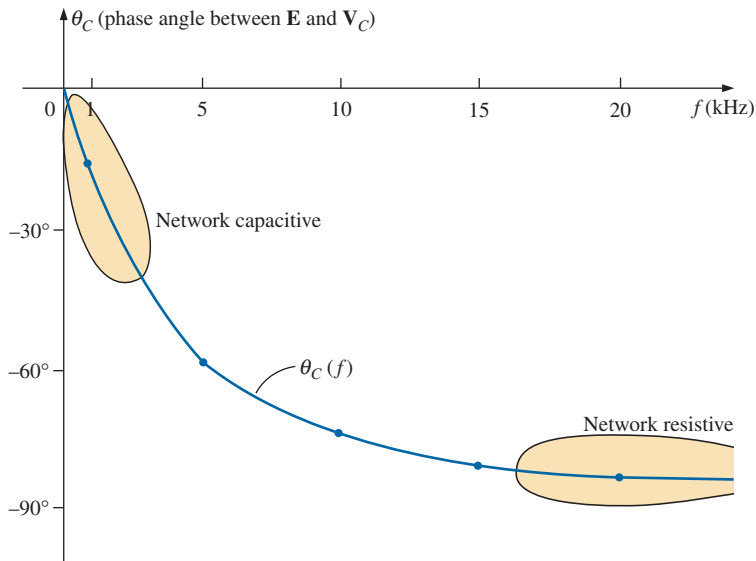
FIG. 15.55

*The phase angle between **E** and **V**$_C$ versus frequency for the circuit in Fig. 15.47.*

that an *R-C* circuit such as that in Fig. 15.47 can be used as a filter to determine which frequencies will have the greatest impact on the stage to follow. From our current analysis, it is obvious that any network connected across the capacitor will receive the greatest potential level at low frequencies and be effectively "shorted out" at very high frequencies.

The analysis of a series *R-L* circuit proceeds in much the same manner, except that X_L and V_L increases with frequency and the angle between **I** and **E** approaches 90° (voltage leading the current) rather than 0°. If **V**$_L$ is plotted versus frequency, **V**$_L$ will approach **E,** as demonstrated in Example 15.12, and X_L will eventually attain a level at which the open-circuit equivalent is appropriate.

EXAMPLE 15.12 For the series *R-L* circuit in Fig. 15.56:

a. Determine the frequency at which $X_L = R$.
b. Develop a mental image of the change in total impedance with frequency without doing any calculations.
c. Find the total impedance at $f = 100$ Hz and 40 kHz, and compare your answer with the assumptions of part (b).
d. Plot the curve of V_L versus frequency.
e. Find the phase angle of the total impedance at $f = 40$ kHz. Can the circuit be considered inductive at this frequency? Why?
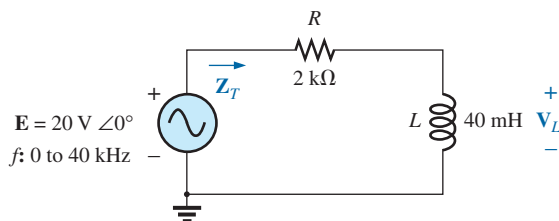
FIG. 15.56

Circuit for Example 15.12.

Solutions:

a. $X_L = 2\pi f_1 L = R$

and $\qquad f_1 = \dfrac{R}{2\pi L} = \dfrac{2\text{ k}\Omega}{2\pi(40\text{ mH})} = \textbf{7957.7 Hz}$

b. At low frequencies, $R > X_L$ and the impedance will be very close to that of the resistor, or 2 kΩ. As the frequency increases, X_L increases to a point where it is the predominant factor. The result is that the curve starts almost horizontal at 2 kΩ and then increases linearly to very high levels.

c. $Z_T = R + j\,X_L = Z_T \angle \theta_T = \sqrt{R^2 + X_L^2}\,\angle\tan^{-1}\dfrac{X_L}{R}$

At $f = 100$ Hz:

$$X_L = 2\pi f L = 2\pi(100\text{ Hz})(40\text{ mH}) = 25.13\ \Omega$$

and $\quad Z_T = \sqrt{R^2 + X_L^2} = \sqrt{(2\text{ k}\Omega)^2 + (25.13\ \Omega)^2}$
$$= 2000.16\ \Omega \cong R$$

At $f = 40$ kHz:

$$X_L = 2\pi f L = 2\pi(40\text{ kHz})(40\text{ mH}) \cong 10.05\text{ k}\Omega$$

and $\quad Z_T = \sqrt{R^2 + X_L^2} = \sqrt{(2\text{ k}\Omega)^2 + (10.05\text{ k}\Omega)^2}$
$$= 10.25\text{ k}\Omega \cong X_L$$

Both calculations support the conclusions of part (b).

d. Applying the voltage divider rule:

$$\mathbf{V}_L = \dfrac{\mathbf{Z}_L \mathbf{E}}{\mathbf{Z}_T}$$

From part (c), we know that at 100 Hz, $Z_T \cong R$ so that $V_R \cong 20$ V and $V_L \cong 0$ V. Part (c) revealed that at 40 kHz, $Z_T \cong X_L$ so that $V_L \cong 20$ V and $V_R \cong 0$ V. The result is two plot points for the curve in Fig. 15.57.
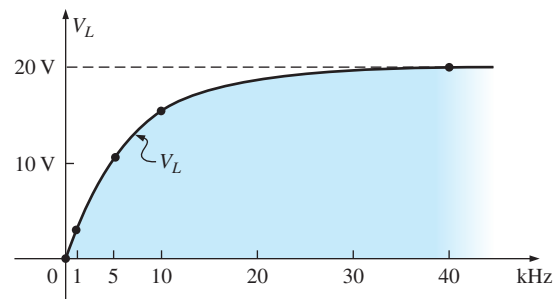
FIG. 15.57
Plotting V_L versus f for the series R-L circuit in Fig. 15.56.

At 1 kHz: $\qquad X_L = 2\pi f L \cong 0.25\text{ k}\Omega$

and $\qquad \mathbf{V}_L = \dfrac{(0.25\text{ k}\Omega\ \angle 90°)(20\text{ V }\angle 0°)}{2\text{ k}\Omega\ +\ j\,0.25\text{ k}\Omega} = \textbf{2.48 V}\angle\textbf{82.87°}$

At 5 kHz: $\qquad X_L = 2\pi f L \cong 1.26\text{ k}\Omega$

and $\qquad \mathbf{V}_L = \dfrac{(1.26\text{ k}\Omega\ \angle 90°)(20\text{ V }\angle 0°)}{2\text{ k}\Omega\ +\ j\,1.26\text{ k}\Omega} = \textbf{10.68 V}\angle\textbf{57.79°}$

At 10 kHz: $\qquad X_L = 2\pi fL \cong 2.5 \text{ k}\Omega$

and $\qquad \mathbf{V}_L = \dfrac{(2.5 \text{ k}\Omega \angle 90°)(20 \text{ V} \angle 0°)}{2.5 \text{ k}\Omega + j\,2.5 \text{ k}\Omega} = \mathbf{15.63 \text{ V} \angle 38.66°}$

The complete plot appears in Fig. 15.57.

e. $\quad \theta_T = \tan^{-1}\dfrac{X_L}{R} = \tan^{-1}\dfrac{10.05 \text{ k}\Omega}{2 \text{ k}\Omega} = \mathbf{78.75°}$

The angle θ_T is closing in on the 90° of a purely inductive network. Therefore, the network can be considered quite inductive at a frequency of 40 kHz.

15.6 SUMMARY: SERIES ac CIRCUITS

The following is a review of important conclusions that can be derived from the discussion and examples of the previous sections. The list is not all-inclusive, but it does emphasize some of the conclusions that should be carried forward in the future analysis of ac systems.

For series ac circuits with reactive elements:

1. *The total impedance will be frequency dependent.*
2. *The impedance of any one element can be greater than the total impedance of the network.*
3. *The inductive and capacitive reactances are always in direct opposition on an impedance diagram.*
4. *Depending on the frequency applied, the same circuit can be either predominantly inductive or predominantly capacitive.*
5. *At lower frequencies, the capacitive elements will usually have the most impact on the total impedance, while at high frequencies the inductive elements will usually have the most impact.*
6. *The magnitude of the voltage across any one element can be greater than the applied voltage.*
7. *The magnitude of the voltage across an element compared to the other elements of the circuit is directly related to the magnitude of its impedance; that is, the larger the impedance of an element, the larger the magnitude of the voltage across the element.*
8. *The voltages across a coil or capacitor are always in direct opposition on a phasor diagram.*
9. *The current is always in phase with the voltage across the resistive elements, lags the voltage across all the inductive elements by 90°, and leads the voltage across all the capacitive elements by 90°.*
10. *The larger the resistive element of a circuit compared to the net reactive impedance, the closer the power factor is to unity.*

PARALLEL ac CIRCUITS

15.7 ADMITTANCE AND SUSCEPTANCE

The discussion for **parallel ac circuits** is very similar to that for dc circuits. In dc circuits, *conductance* (*G*) was defined as being equal to $1/R$. The total conductance of a parallel circuit was then found by adding the conductance of each branch. The total resistance R_T is simply $1/G_T$.

In ac circuits, we define **admittance** (**Y**) as being equal to $1/\mathbf{Z}$. The unit of measure for admittance as defined by the SI system is *siemens,*

which has the symbol S. Admittance is a measure of how well an ac circuit will *admit,* or allow, current to flow in the circuit. The larger its value, therefore, the heavier the current flow for the same applied potential. The total admittance of a circuit can also be found by finding the sum of the parallel admittances. The total impedance Z_T of the circuit is then $1/Y_T$; that is, for the network in Fig. 15.58:

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \cdots + \mathbf{Y}_N \tag{15.16}$$
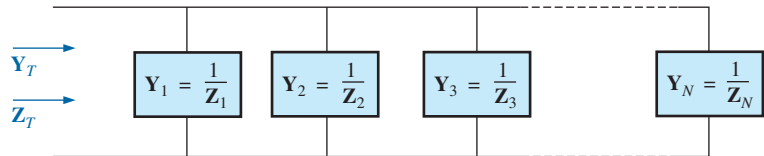
FIG. 15.58
Parallel ac network.

or, since $\mathbf{Z} = 1/\mathbf{Y}$,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots + \frac{1}{\mathbf{Z}_N} \tag{15.17}$$

and

$$\mathbf{Z}_T = \cfrac{1}{\cfrac{1}{\mathbf{Z}_1} + \cfrac{1}{\mathbf{Z}_2} + \cfrac{1}{\mathbf{Z}_3} + \cdots + \cfrac{1}{\mathbf{Z}_N}} \tag{15.18}$$

matching Eq. (6.3) for dc networks.

For two impedances in parallel,

$$\frac{1}{\mathbf{Z}_T} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

If the manipulations used in Chapter 6 to find the total resistance of two parallel resistors are now applied, the following similar equation results:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{15.19}$$

For N parallel equal impedances ($\mathbf{Z}_1$), the total impedance is determined by

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1}{N} \tag{15.20}$$

For three parallel impedances,

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1 \mathbf{Z}_2 \mathbf{Z}_3}{\mathbf{Z}_1 \mathbf{Z}_2 + \mathbf{Z}_2 \mathbf{Z}_3 + \mathbf{Z}_1 \mathbf{Z}_3} \tag{15.21}$$

As pointed out in the introduction to this section, conductance is the reciprocal of resistance, and

$$\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{R \angle 0°} = G \angle 0° \qquad \text{(15.22)}$$

The reciprocal of reactance ($1/X$) is called **susceptance** and is a measure of how *susceptible* an element is to the passage of current through it. Susceptance is also measured in *siemens* and is represented by the capital letter B.

For the inductor,

$$\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \frac{1}{X_L \angle 90°} = \frac{1}{X_L} \angle -90° \qquad \text{(15.23)}$$

Defining

$$B_L = \frac{1}{X_L} \quad \text{(siemens, S)} \qquad \text{(15.24)}$$

we have

$$\mathbf{Y}_L = B_L \angle -90° \qquad \text{(15.25)}$$

Note that for inductance, an increase in frequency or inductance will result in a decrease in susceptance or, correspondingly, in admittance.

For the capacitor,

$$\mathbf{Y}_C = \frac{1}{\mathbf{Z}_C} = \frac{1}{X_C \angle -90°} = \frac{1}{X_C} \angle 90° \qquad \text{(15.26)}$$

Defining

$$B_C = \frac{1}{X_C} \quad \text{(siemens, S)} \qquad \text{(15.27)}$$

we have

$$\mathbf{Y}_C = B_C \angle 90° \qquad \text{(15.28)}$$

For the capacitor, therefore, an increase in frequency or capacitance will result in an increase in its susceptibility.

For parallel ac circuits, the **admittance diagram** is used with the three admittances, represented as shown in Fig. 15.59.

Note in Fig. 15.59 that the conductance (like resistance) is on the positive real axis, whereas inductive and capacitive susceptances are in direct opposition on the imaginary axis.

For any configuration (series, parallel, series-parallel, and so on), the angle associated with the total admittance is the angle by which the source current leads the applied voltage. For inductive networks, θ_T is negative, whereas for capacitive networks, θ_T is positive.

For parallel ac networks, the components of the configuration and the desired quantities determine whether to use an impedance or admittance approach. If the total impedance is requested, the most direct route may be to use impedance parameters. However, sometimes using admittance parameters can also be very efficient, as demonstrated in some of the examples in the rest of the text. In general, use the approach with which you are more comfortable. Naturally, if the format of the desired quantity is spelled out, it is usually best to work with those parameters.
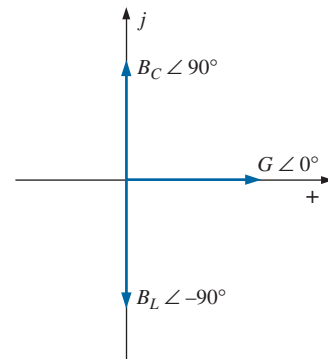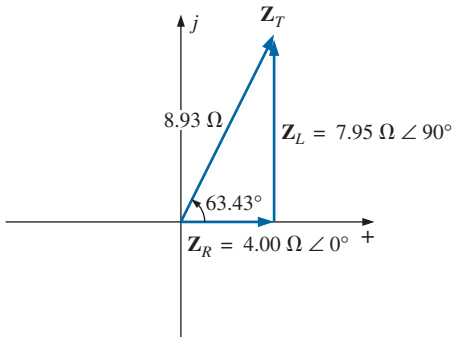

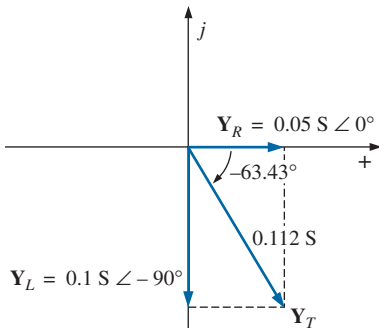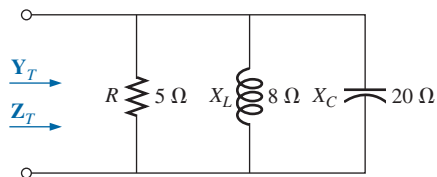
FIG. 15.59
Admittance diagram.
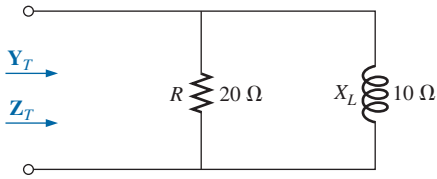
FIG. 15.60
Example 15.12.

FIG. 15.61
Impedance diagram for the network in Fig. 15.60.

FIG. 15.62
Admittance diagram for the network in Fig. 15.60.

FIG. 15.63
Example 15.14.

EXAMPLE 15.13 For the network in Fig. 15.60:

a. Calculate the input impedance.
b. Draw the impedance diagram.
c. Find the admittance of each parallel branch.
d. Determine the input admittance and draw the admittance diagram.

Solutions:

a. $\mathbf{Z}_T = \dfrac{\mathbf{Z}_R\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \dfrac{(20\ \Omega\ \angle 0°)(10\ \Omega\ \angle 90°)}{20\ \Omega + j\ 10\ \Omega}$

$= \dfrac{200\ \Omega\ \angle 90°}{22.361\ \angle 26.57°} = \mathbf{8.93\ \Omega\ \angle 63.43°}$

$= \mathbf{4.00\ \Omega + j\ 7.95\ \Omega} = R_T + j\ X_{L_T}$

b. The impedance diagram appears in Fig. 15.61.

c. $\mathbf{Y}_R = G\angle 0° = \dfrac{1}{R}\angle 0° = \dfrac{1}{20\ \Omega}\angle 0° = \mathbf{0.05\ S\ \angle 0°}$

$= \mathbf{0.05\ S + j\ 0}$

$\mathbf{Y}_L = B_L\angle -90° = \dfrac{1}{X_L}\angle -90° = \dfrac{1}{10\ \Omega}\angle -90°$

$= \mathbf{0.1\ S\ \angle -90°} = \mathbf{0 - j\ 0.1\ S}$

d. $\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L = (0.05\ S + j\ 0) + (0 - j\ 0.1\ S)$
$= \mathbf{0.05\ S - j\ 0.1\ S} = G - jB_L$

The admittance diagram appears in Fig. 15.62.

EXAMPLE 15.14 Repeat Example 15.13 for the parallel network in Fig. 15.63.

Solutions:

a. $\mathbf{Z}_T = \dfrac{1}{\dfrac{1}{\mathbf{Z}_R} + \dfrac{1}{\mathbf{Z}_L} + \dfrac{1}{\mathbf{Z}_C}}$

$= \dfrac{1}{\dfrac{1}{5\ \Omega\ \angle 0°} + \dfrac{1}{8\ \Omega\ \angle 90°} + \dfrac{1}{20\ \Omega\ \angle -90°}}$

$= \dfrac{1}{0.2\ S\ \angle 0° + 0.125\ S\ \angle -90° + 0.05\ S\ \angle 90°}$

$= \dfrac{1}{0.2\ S - j\ 0.075\ S} = \dfrac{1}{0.2136\ S\ \angle -20.56°}$

$= \mathbf{4.68\ \Omega\ \angle 20.56°}$

or

$\mathbf{Z}_T = \dfrac{\mathbf{Z}_R\mathbf{Z}_L\mathbf{Z}_C}{\mathbf{Z}_R\mathbf{Z}_L + \mathbf{Z}_L\mathbf{Z}_C + \mathbf{Z}_R\mathbf{Z}_C}$

$= \dfrac{(5\ \Omega\ \angle 0°)(8\ \Omega\ \angle 90°)(20\ \Omega\ \angle -90°)}{(5\ \Omega\ \angle 0°)(8\ \Omega\ \angle 90°) + (8\ \Omega\ \angle 90°)(20\ \Omega\ \angle -90°) + (5\ \Omega\ \angle 0°)(20\ \Omega\ \angle -90°)}$

$= \dfrac{800\ \Omega\ \angle 0°}{40\ \angle 90° + 160\ \angle 0° + 100\ \angle -90°}$

$$= \frac{800 \ \Omega}{160 + j\,40 - j\,100} = \frac{800 \ \Omega}{160 - j\,60}$$

$$= \frac{800 \ \Omega}{170.88 \ \angle -20.56°}$$

$$= \textbf{4.68} \ \Omega \ \angle \textbf{20.56°} = \textbf{4.38} \ \Omega \ + \ j\,\textbf{1.64} \ \Omega$$

b. The impedance diagram appears in Fig. 15.64.

c. $\mathbf{Y}_R = G \ \angle 0° = \dfrac{1}{R} \ \angle 0° = \dfrac{1}{5 \ \Omega} \ \angle 0°$

$\quad = \textbf{0.2 S} \ \angle \textbf{0°} = \textbf{0.2 S} + \textbf{j\,0}$

$\mathbf{Y}_L = B_L \angle -90° = \dfrac{1}{X_L} \ \angle -90° = \dfrac{1}{8 \ \Omega} \ \angle -90°$

$\quad = \textbf{0.125 S} \ \angle -\textbf{90°} = \textbf{0} - \textbf{j\,0.125 S}$

$\mathbf{Y}_C = B_C \angle 90° = \dfrac{1}{X_C} \ \angle 90° = \dfrac{1}{20 \ \Omega} \ \angle 90°$

$\quad = \textbf{0.050 S} \ \angle \ +\textbf{90°} = \textbf{0} + \textbf{j\,0.050 S}$

d. $\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}_C$

$\quad = (0.2 \ \text{S} + j\,0) + (0 - j\,0.125 \ \text{S}) + (0 + j\,0.050 \ \text{S})$

$\quad = 0.2 \ \text{S} - j\,0.075 \ \text{S} = \textbf{0.214 S} \ \angle -\textbf{20.56°}$

The admittance diagram appears in Fig. 15.65.
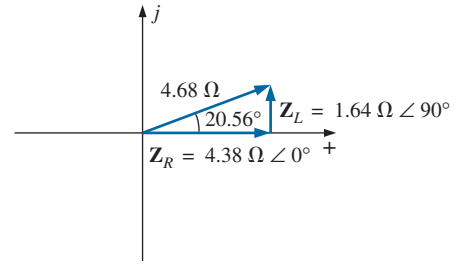
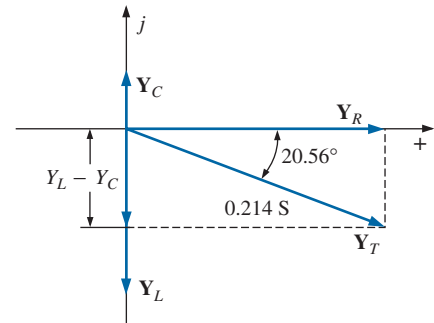On many occasions, the inverse relationship $\mathbf{Y}_T = 1/\mathbf{Z}_T$ or $\mathbf{Z}_T = 1/\mathbf{Y}_T$ will require that we divide the number 1 by a complex number having a real and an imaginary part. This division, if not performed in the polar form, requires that we multiply the numerator and denominator by the conjugate of the denominator, as follows:

$$\mathbf{Y}_T = \frac{1}{\mathbf{Z}_T} = \frac{1}{4 \ \Omega + j\,6 \ \Omega} = \left(\frac{1}{4 \ \Omega + j\,6 \ \Omega} \right) \left(\frac{(4 \ \Omega - j\,6 \ \Omega)}{(4 \ \Omega - j\,6 \ \Omega)} \right)$$

$$= \frac{4 - j\,6}{4^2 + 6^2}$$

and
$$\mathbf{Y}_T = \frac{4}{52} \ \text{S} - j\,\frac{6}{52} \ \text{S}$$

To avoid this laborious task each time we want to find the reciprocal of a complex number in rectangular form, a format can be developed using the following complex number, which is symbolic of any impedance or admittance in the first or fourth quadrant:

$$\frac{1}{a_1 \pm j\,b_1} = \left(\frac{1}{a_1 \pm j\,b_1} \right) \left(\frac{a_1 \mp j\,b_1}{a_1 \mp j\,b_1} \right) = \frac{a_1 \mp j\,b_1}{a_1^2 + b_1^2}$$

or
$$\boxed{\frac{1}{a_1 \pm j\,b_1} = \frac{a_1}{a_1^2 + b_1^2} \mp j\,\frac{b_1}{a_1^2 + b_1^2}} \qquad \textbf{(15.29)}$$

Note that the denominator is simply the sum of the squares of each term. The sign is inverted between the real and imaginary parts. A few examples will develop some familiarity with the use of this equation.

EXAMPLE 15.15 Find the admittance of each set of series elements in Fig. 15.66.
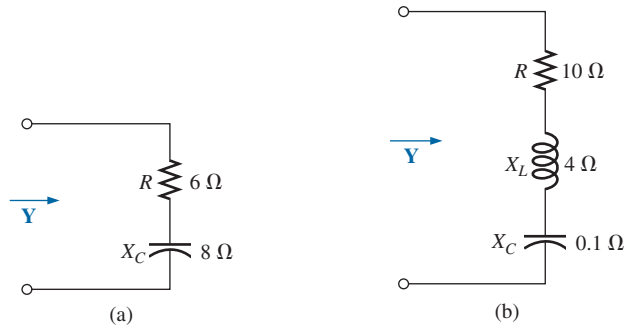
FIG. 15.66

Example 15.15.

Solutions:

a. $\mathbf{Z} = R - jX_C = 6\ \Omega - j\ 8\ \Omega$

Eq. (15.29):

$$\mathbf{Y} = \frac{1}{6\ \Omega - j\ 8\ \Omega} = \frac{6}{(6)^2 + (8)^2} + j\frac{8}{(6)^2 + (8)^2}$$
$$= \frac{6}{100}\ \mathbf{S} + j\frac{8}{100}\ \mathbf{S}$$

b. $\mathbf{Z} = 10\ \Omega + j\ 4\ \Omega + (-j\ 0.1\ \Omega) = 10\ \Omega + j\ 3.9\ \Omega$
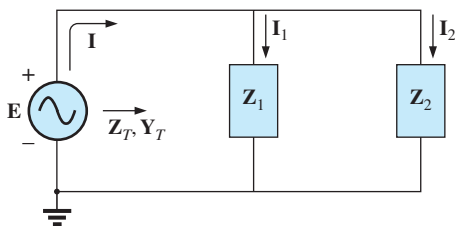
Eq. (15.29):

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{10\ \Omega + j\ 3.9\ \Omega} = \frac{10}{(10)^2 + (3.9)^2} - j\frac{3.9}{(10)^2 + (3.9)^2}$$
$$= \frac{10}{115.21} - j\frac{3.9}{115.21} = \mathbf{0.087\ S} - j\ \mathbf{0.034\ S}$$

15.8 PARALLEL ac NETWORKS

For the representative parallel ac network in Fig. 15.67, the total impedance or admittance is determined as described in the previous section, and the source current is determined by Ohm's law as follows:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \mathbf{E}\mathbf{Y}_T \qquad (15.30)$$

Since the voltage is the same across parallel elements, the current through each branch can then be found through another application of Ohm's law:

$$\mathbf{I}_1 = \frac{\mathbf{E}}{\mathbf{Z}_1} = \mathbf{E}\mathbf{Y}_1 \qquad (15.31a)$$

FIG. 15.67

Parallel ac network.

$$I_2 = \frac{E}{Z_2} = EY_2 \qquad\qquad (15.31b)$$

Kirchhoff's current law can then be applied in the same manner as used for dc networks. However, keep in mind that we are now dealing with the algebraic manipulation of quantities that have both magnitude and direction.

$$I - I_1 - I_2 = 0$$

or $\qquad\qquad$ $I = I_1 + I_2$ $\qquad\qquad$ (15.32)
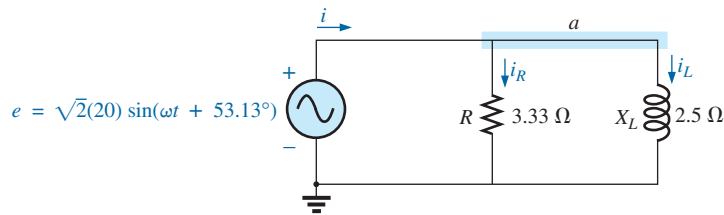
The power to the network can be determined by

$$P = EI \cos \theta_T \qquad\qquad (15.33)$$

where θ_T is the phase angle between **E** and **I.**

Let us now look at a few examples carried out in great detail for the first exposure.
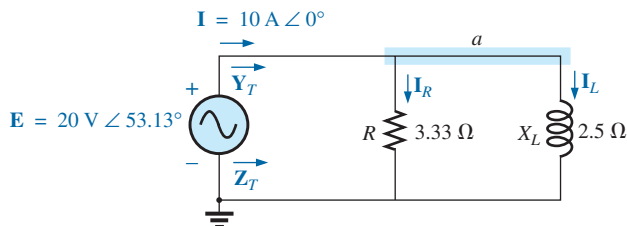
R-L

Refer to Fig. 15.68.

FIG. 15.68
Parallel R-L network.

Phasor Notation As shown in Fig. 15.69.

FIG. 15.69
Applying phasor notation to the network in Fig. 15.68.

$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L$

$\qquad = G \angle 0° + B_L \angle -90° = \dfrac{1}{3.33\ \Omega} \angle 0° + \dfrac{1}{2.5\ \Omega} \angle -90°$

$\qquad = 0.3\ \text{S} \angle 0° + 0.4\ \text{S} \angle -90° = 0.3\ \text{S} - j\,0.4\ \text{S}$

$\qquad = \mathbf{0.5\ S} \angle -\mathbf{53.13°}$

$\mathbf{Z}_T = \dfrac{1}{\mathbf{Y}_T} = \dfrac{1}{0.5\ \text{S} \angle -53.13°} = \mathbf{2\ \Omega} \angle \mathbf{53.13°}$
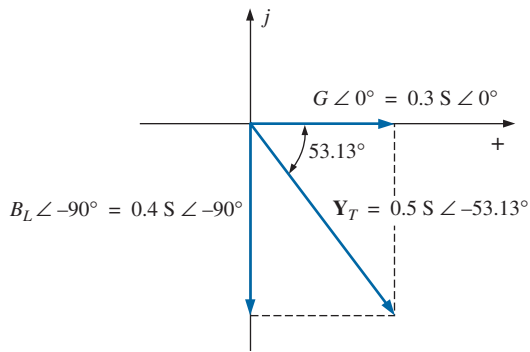
FIG. 15.70

Admittance diagram for the parallel R-L network in Fig. 15.68.
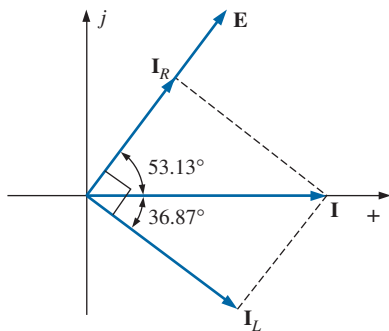

FIG. 15.71

Phasor diagram for the parallel R-L network in Fig. 15.68.

Admittance diagram: As shown in Fig. 15.70.

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \mathbf{E}\mathbf{Y}_T = (20 \text{ V} \angle 53.13°)(0.5 \text{ S} \angle -53.13°) = \mathbf{10 \text{ A} \angle 0°}$$

$$\mathbf{I}_R = \frac{E \angle \theta}{R \angle 0°} = (E \angle \theta)(G \angle 0°)$$
$$= (20 \text{ V} \angle 53.13°)(0.3 \text{ S} \angle 0°) = \mathbf{6 \text{ A} \angle 53.13°}$$

$$\mathbf{I}_L = \frac{E \angle \theta}{X_L \angle 90°} = (E \angle \theta)(B_L \angle -90°)$$
$$= (20 \text{ V} \angle 53.13°)(0.4 \text{ S} \angle -90°)$$
$$= \mathbf{8 \text{ A} \angle -36.87°}$$

Kirchhoff's current law: At node a,

$$\mathbf{I} - \mathbf{I}_R - \mathbf{I}_L = 0$$

or $$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L$$

$10 \text{ A} \angle 0° = 6 \text{ A} \angle 53.13° + 8 \text{ A} \angle -36.87°$
$10 \text{ A} \angle 0° = (3.60 \text{ A} + j\,4.80 \text{ A}) + (6.40 \text{ A} - j\,4.80 \text{ A}) = 10 \text{ A} + j\,0$

and $$\mathbf{10 \text{ A} \angle 0°} = \mathbf{10 \text{ A} \angle 0°} \quad \text{(checks)}$$

Phasor diagram: The phasor diagram in Fig. 15.71 indicates that the applied voltage $\mathbf{E}$ is in phase with the current $\mathbf{I}_R$ and leads the current $\mathbf{I}_L$ by 90°.

Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta_T$$
$$= (20 \text{ V})(10 \text{ A}) \cos 53.13° = (200 \text{ W})(0.6)$$
$$= \mathbf{120 \text{ W}}$$

or $$P_T = I^2 R = \frac{V_R^2}{R} = V_R^2 G = (20 \text{ V})^2(0.3 \text{ S}) = \mathbf{120 \text{ W}}$$

or, finally,

$$P_T = P_R + P_L = EI_R \cos \theta_R + EI_L \cos \theta_L$$
$$= (20 \text{ V})(6 \text{ A}) \cos 0° + (20 \text{ V})(8 \text{ A}) \cos 90° = 120 \text{ W} + 0$$
$$= \mathbf{120 \text{ W}}$$

Power factor: The power factor of the circuit is

$$F_p = \cos \theta_T = \cos 53.13° = \mathbf{0.6 \text{ lagging}}$$

or, through an analysis similar to that used for a series ac circuit,

$$\cos \theta_T = \frac{P}{EI} = \frac{E^2/R}{EI} = \frac{EG}{I} = \frac{G}{I/V} = \frac{G}{Y_T}$$

and $$\boxed{F_p = \cos \theta_T = \frac{G}{Y_T}} \qquad \textbf{(15.34)}$$

where G and Y_T are the magnitudes of the total conductance and admittance of the parallel network. For this case,

$$F_p = \cos \theta_T = \frac{0.3 \text{ S}}{0.5 \text{ S}} = \mathbf{0.6 \text{ lagging}}$$

Impedance approach: The current $\mathbf{I}$ can also be found by first finding the total impedance of the network:
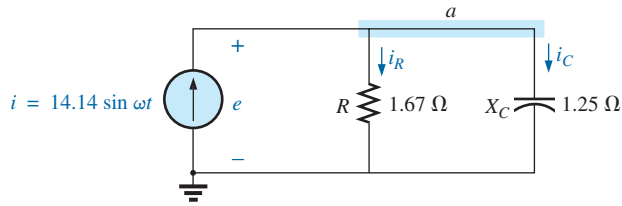
$$\mathbf{Z}_T = \frac{\mathbf{Z}_R\mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(3.33\ \Omega\ \angle 0°)(2.5\ \Omega\ \angle 90°)}{3.33\ \Omega\ \angle 0° + 2.5\ \Omega\ \angle 90°}$$

$$= \frac{8.325\ \angle 90°}{4.164\ \angle 36.87°} = \mathbf{2\ \Omega\ \angle 53.13°}$$

And then, using Ohm's law, we obtain

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{20\ V\ \angle 53.13°}{2\ \Omega\ \angle 53.13°} = \mathbf{10\ A\ \angle 0°}$$
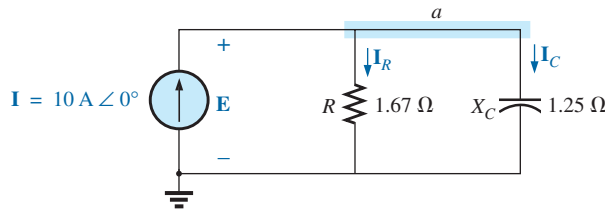
R-C

Refer to Fig. 15.72.

FIG. 15.72
Parallel R-C network.

Phasor Notation   As shown in Fig. 15.73.

FIG. 15.73
Applying phasor notation to the network in Fig. 15.72.

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_C = G\ \angle 0° + B_C\ \angle 90° = \frac{1}{1.67\ \Omega}\ \angle 0° + \frac{1}{1.25\ \Omega}\ \angle 90°$$

$$= 0.6\ S\ \angle 0° + 0.8\ S\ \angle 90° = 0.6\ S + j\,0.8\ S = \mathbf{1.0\ S\ \angle 53.13°}$$

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{1.0\ S\ \angle 53.13°} = \mathbf{1\ \Omega\ \angle -53.13°}$$

Admittance diagram: As shown in Fig. 15.74.

$$\mathbf{E} = \mathbf{IZ}_T = \frac{\mathbf{I}}{\mathbf{Y}_T} = \frac{10\ A\ \angle 0°}{1\ S\ \angle 53.13°} = \mathbf{10\ V\ \angle -53.13°}$$

$$\mathbf{I}_R = (E\ \angle\theta)(G\ \angle 0°)$$
$$= (10\ V\ \angle -53.13°)(0.6\ S\ \angle 0°) = \mathbf{6\ A\ \angle -53.13°}$$

$$\mathbf{I}_C = (E\ \angle\theta)(B_C\ \angle 90°)$$
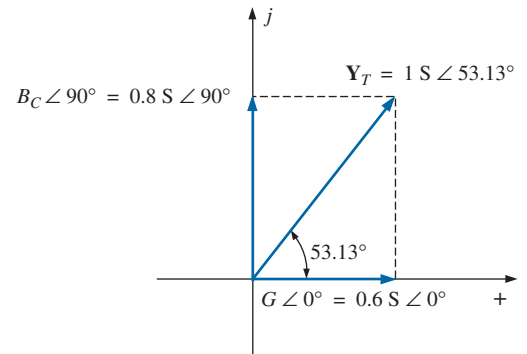$$= (10\ V\ \angle -53.13°)(0.8\ S\ \angle 90°) = \mathbf{8\ A\ \angle 36.87°}$$

Kirchhoff's current law: At node *a*,

$$\mathbf{I} - \mathbf{I}_R - \mathbf{I}_C = 0$$

or
$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_C$$

which can also be verified (as for the *R-L* network) through vector algebra.

FIG. 15.74
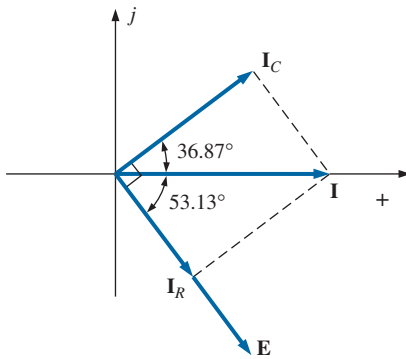*Admittance diagram for the parallel R-C network in Fig. 15.72.*

FIG. 15.75

Phasor diagram for the parallel R-C network in Fig. 15.74.

Phasor diagram: The phasor diagram in Fig. 15.75 indicates that $\mathbf{E}$ is in phase with the current through the resistor $\mathbf{I}_R$ and lags the capacitive current $\mathbf{I}_C$ by 90°.

Time domain:

$$e = \sqrt{2}(10) \sin(\omega t - 53.13°) = \mathbf{14.14 \sin(\omega t - 53.13°)}$$
$$i_R = \sqrt{2}(6) \sin(\omega t - 53.13°) = \mathbf{8.48 \sin(\omega t - 53.13°)}$$
$$i_C = \sqrt{2}(8) \sin(\omega t + 36.87°) = \mathbf{11.31 \sin(\omega t + 36.87°)}$$

A plot of all of the currents and the voltage appears in Fig. 15.76. Note that e and i_R are in phase and e lags i_C by 90°.
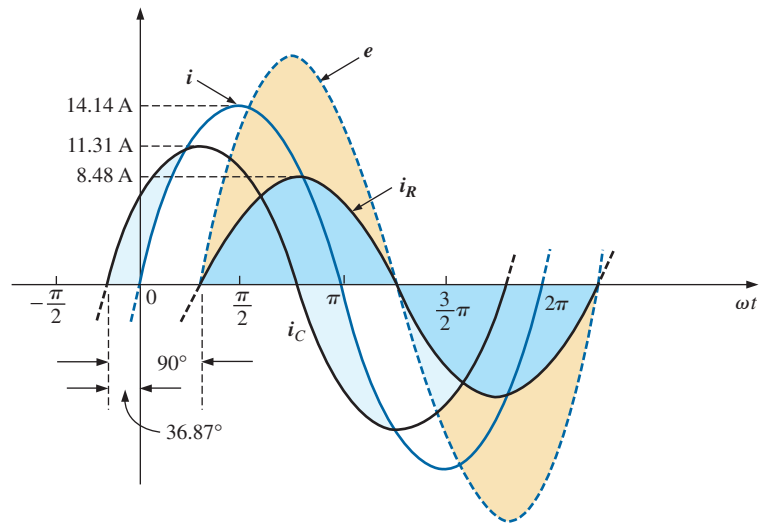

FIG. 15.76

Waveforms for the parallel R-C network in Fig. 15.72.

Power:

$$P_T = EI \cos \theta = (10 \text{ V})(10 \text{ A}) \cos 53.13° = (10)^2(0.6)$$
$$= \mathbf{60 \ W}$$

or $\qquad\qquad P_T = E^2 G = (10 \text{ V})^2(0.6 \text{ S}) = \mathbf{60 \ W}$

or, finally,

$$P_T = P_R + P_C = EI_R \cos \theta_R + EI_C \cos \theta_C$$
$$= (10 \text{ V})(6 \text{ A}) \cos 0° + (10 \text{ V})(8 \text{ A}) \cos 90°$$
$$= \mathbf{60 \ W}$$

Power factor: The power factor of the circuit is

$$F_p = \cos 53.13° = \mathbf{0.6 \ leading}$$

Using Eq. (15.34), we have

$$F_p = \cos \theta_T = \frac{G}{Y_T} = \frac{0.6 \text{ S}}{1.0 \text{ S}} = \mathbf{0.6 \ leading}$$

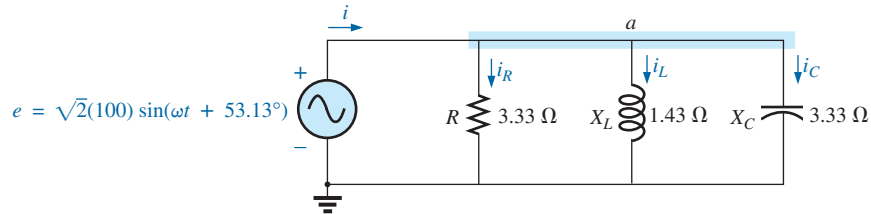*Impedance approach:* The voltage $\mathbf{E}$ can also be found by first finding the total impedance of the circuit:

$$\mathbf{Z}_T = \frac{\mathbf{Z}_R \mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{(1.67 \ \Omega \ \angle 0°)(1.25 \ \Omega \ \angle -90°)}{1.67 \ \Omega \ \angle 0° + 1.25 \ \Omega \ \angle -90°}$$
$$= \frac{2.09 \ \angle -90°}{2.09 \ \angle -36.81°} = \mathbf{1 \ \Omega \ \angle -53.19°}$$

and then, using Ohm's law, we find

$$\mathbf{E} = \mathbf{IZ}_T = (10 \text{ A } \angle 0°)(1 \text{ } \Omega \angle -53.19°) = \mathbf{10 \text{ V } \angle -53.19°}$$
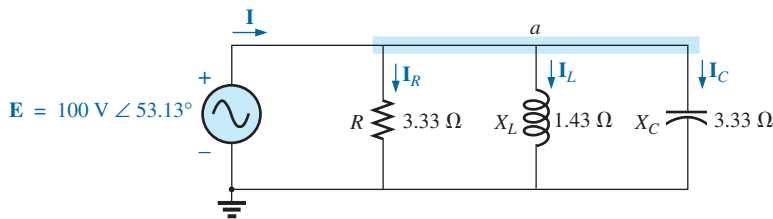
R-L-C

Refer to Fig. 15.77.

FIG. 15.77
Parallel R-L-C ac network.

Phasor notation: As shown in Fig. 15.78.

FIG. 15.78
Applying phasor notation to the network in Fig. 15.77.

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}_C = G \angle 0° + B_L \angle -90° + B_C \angle 90°$$
$$= \frac{1}{3.33 \text{ } \Omega} \angle 0° + \frac{1}{1.43 \text{ } \Omega} \angle -90° + \frac{1}{3.33 \text{ } \Omega} \angle 90°$$
$$= 0.3 \text{ S } \angle 0° + 0.7 \text{ S } \angle -90° + 0.3 \text{ S } \angle 90°$$
$$= 0.3 \text{ S } - j\, 0.7 \text{ S } + j\, 0.3 \text{ S}$$
$$= 0.3 \text{ S } - j\, 0.4 \text{ S } = \mathbf{0.5 \text{ S } \angle -53.13°}$$

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{0.5 \text{ S } \angle -53.13°} = \mathbf{2 \text{ } \Omega \angle 53.13°}$$

Admittance diagram: As shown in Fig. 15.79.

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \mathbf{EY}_T = (100 \text{ V } \angle 53.13°)(0.5 \text{ S } \angle -53.13°) = \mathbf{50 \text{ A } \angle 0°}$$

$$\mathbf{I}_R = (E \angle \theta)(G \angle 0°)$$
$$= (100 \text{ V } \angle 53.13°)(0.3 \text{ S } \angle 0°) = \mathbf{30 \text{ A } \angle 53.13°}$$
$$\mathbf{I}_L = (E \angle \theta)(B_L \angle -90°)$$
$$= (100 \text{ V } \angle 53.13°)(0.7 \text{ S } \angle -90°) = \mathbf{70 \text{ A } \angle -36.87°}$$
$$\mathbf{I}_C = (E \angle \theta)(B_C \angle 90°)$$
$$= (100 \text{ V } \angle 53.13°)(0.3 \text{ S } \angle +90°) = \mathbf{30 \text{ A } \angle 143.13°}$$

Kirchhoff's current law: At node *a*,

$$\mathbf{I} - \mathbf{I}_R - \mathbf{I}_L - \mathbf{I}_C = 0$$

or

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L + \mathbf{I}_C$$
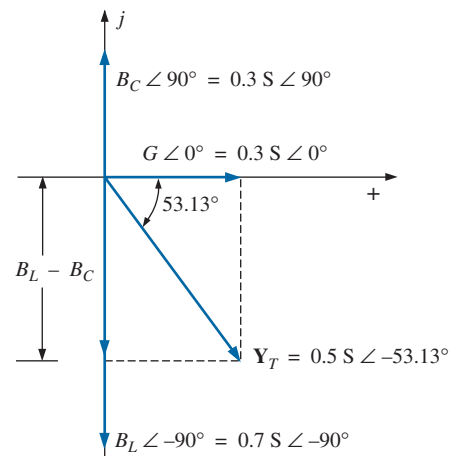
FIG. 15.79
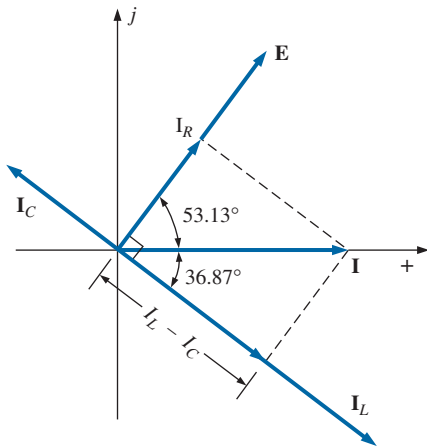*Admittance diagram for the parallel R-L-C network in Fig. 15.77.*

FIG. 15.80
Phasor diagram for the parallel R-L-C network in Fig. 15.77.

Phasor diagram: The phasor diagram in Fig. 15.80 indicates that the impressed voltage **E** is in phase with the current $\mathbf{I}_R$ through the resistor, leads the current $\mathbf{I}_L$ through the inductor by 90°, and lags the current $\mathbf{I}_C$ of the capacitor by 90°.

Time domain:

$$i = \sqrt{2}(50) \sin \omega t = \mathbf{70.70 \sin \omega t}$$
$$i_R = \sqrt{2}(30) \sin(\omega t + 53.13°) = \mathbf{42.42 \sin(\omega t + 53.13°)}$$
$$i_L = \sqrt{2}(70) \sin(\omega t - 36.87°) = \mathbf{98.98 \sin(\omega t - 36.87°)}$$
$$i_C = \sqrt{2}(30) \sin(\omega t + 143.13°) = \mathbf{42.42 \sin(\omega t + 143.13°)}$$

A plot of all of the currents and the impressed voltage appears in Fig. 15.81.

Power: The total power in watts delivered to the circuit is

$$P_T = EI \cos \theta = (100 \text{ V})(50 \text{ A}) \cos 53.13° = (5000)(0.6)$$
$$= \mathbf{3000 \ W}$$
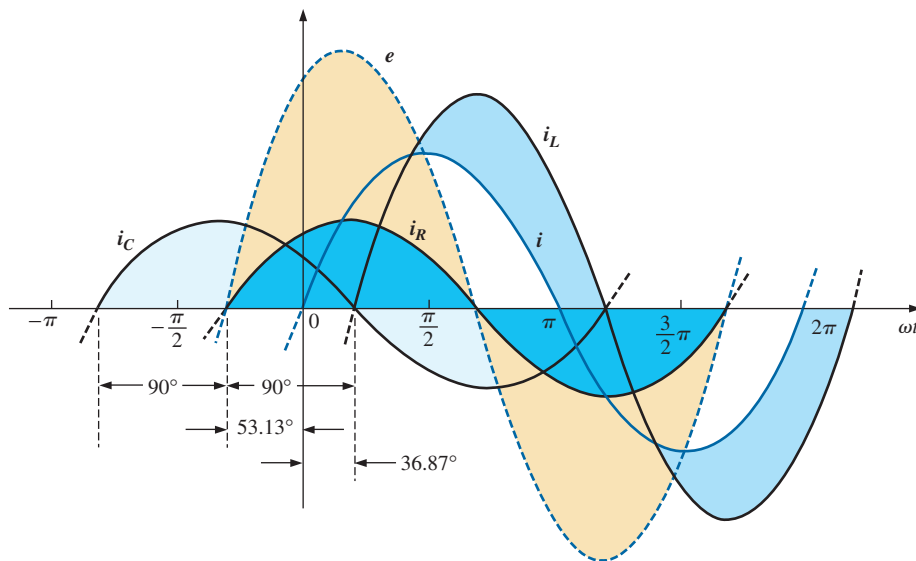
FIG. 15.81
Waveforms for the parallel R-L-C network in Fig. 15.77.

or $$P_T = E^2 G = (100 \text{ V})^2 (0.3 \text{ S}) = \mathbf{3000 \ W}$$

or, finally,

$$P_T = P_R + P_L + P_C$$
$$= EI_R \cos \theta_R + EI_L \cos \theta_L + EI_C \cos \theta_C$$
$$= (100 \text{ V})(30 \text{ A}) \cos 0° + (100 \text{ V})(70 \text{ A}) \cos 90°$$
$$+ (100 \text{ V})(30 \text{ A}) \cos 90°$$
$$= 3000 \text{ W} + 0 + 0$$
$$= \mathbf{3000 \ W}$$

Power factor: The power factor of the circuit is

$$F_p = \cos \theta_T = \cos 53.13° = \mathbf{0.6 \ lagging}$$

Using Eq. (15.34), we obtain

$$F_p = \cos \theta_T = \frac{G}{Y_T} = \frac{0.3 \text{ S}}{0.5 \text{ S}} = \mathbf{0.6 \ lagging}$$

Impedance approach: The input current **I** can also be determined by first finding the total impedance in the following manner:

$$Z_T = \frac{Z_R Z_L Z_C}{Z_R Z_L + Z_L Z_C + Z_R Z_C} = 2\ \Omega\ \angle 53.13°$$

and, applying Ohm's law, we obtain

$$I = \frac{E}{Z_T} = \frac{100\ V\ \angle 53.13°}{2\ \Omega\ \angle 53.13°} = 50\ A\ \angle 0°$$

15.9 CURRENT DIVIDER RULE

The basic format for the **current divider rule** in ac circuits is exactly the same as that for dc circuits; that is, for two parallel branches with impedances Z_1 and Z_2 as shown in Fig. 15.82,

$$I_1 = \frac{Z_2 I_T}{Z_1 + Z_2} \quad \text{or} \quad I_2 = \frac{Z_1 I_T}{Z_1 + Z_2} \qquad \text{(15.35)}$$
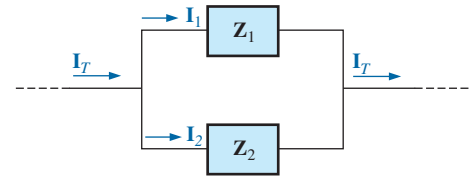

FIG. 15.82

Applying the current divider rule.

EXAMPLE 15.16 Using the current divider rule, find the current through each impedance in Fig. 15.83.

Solution:

$$I_R = \frac{Z_L I_T}{Z_R + Z_L} = \frac{(4\ \Omega\ \angle 90°)(20\ A\ \angle 0°)}{3\ \Omega\ \angle 0° + 4\ \Omega\ \angle 90°} = \frac{80\ A\ \angle 90°}{5\ \angle 53.13°}$$

$$= 16\ A\ \angle 36.87°$$

$$I_L = \frac{Z_R I_T}{Z_R + Z_L} = \frac{(3\ \Omega\ \angle 0°)(20\ A\ \angle 0°)}{5\ \Omega\ \angle 53.13°} = \frac{60\ A\ \angle 0°}{5\ \angle 53.13°}$$

$$= 12\ A\ \angle -53.13°$$
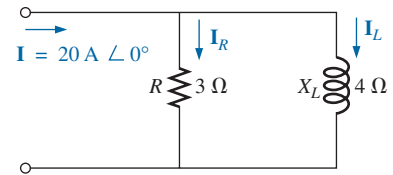
FIG. 15.83

Example 15.16.

EXAMPLE 15.17 Using the current divider rule, find the current through each parallel branch in Fig. 15.84.

Solution:

$$I_{R-L} = \frac{Z_C I_T}{Z_C + Z_{R-L}} = \frac{(2\ \Omega\ \angle -90°)(5\ A\ \angle 30°)}{-j\,2\ \Omega + 1\ \Omega + j\,8\ \Omega} = \frac{10\ A\ \angle -60°}{1 + j6}$$

$$= \frac{10\ A\ \angle -60°}{6.083\ \angle 80.54°} \cong 1.64\ A\ \angle -140.54°$$

$$I_C = \frac{Z_{R-L} I_T}{Z_{R-L} + Z_C} = \frac{(1\ \Omega + j\,8\ \Omega)(5\ A\ \angle 30°)}{6.08\ \Omega\ \angle 80.54°}$$

$$= \frac{(8.06\ \angle 82.87°)(5\ A\ \angle 30°)}{6.08\ \angle 80.54°} = \frac{40.30\ A\ \angle 112.87°}{6.083\ \angle 80.54°}$$

$$= 6.63\ A\ \angle 32.33°$$
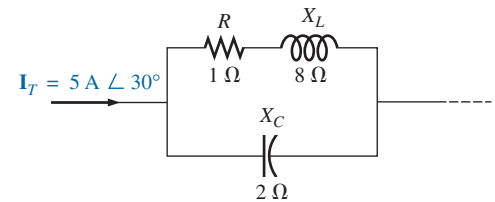
FIG. 15.84

Example 15.17.

15.10 FREQUENCY RESPONSE OF PARALLEL ELEMENTS

Recall that for elements in series, the total impedance is the direct sum of the impedances of each element, and the largest real or imaginary

component has the most impact on the total impedance. For parallel elements, it is important to remember that *the smallest parallel resistor or the smallest parallel reactance will have the most impact on the real or imaginary component, respectively, of the total impedance.*

In Fig. 15.85, the frequency response has been included for each element of a parallel *R-L-C* combination. At very low frequencies, the impedance of the coil will be less than that of the resistor or capacitor, resulting in an inductive network in which the reactance of the inductor will have the most impact on the total impedance. As the frequency increases, the impedance of the inductor will increase while the impedance of the capacitor will decrease. Depending on the components chosen, it is possible that the reactance of the capacitor will drop to a point where it will equal the impedance of the coil before either one reaches the resistance level.
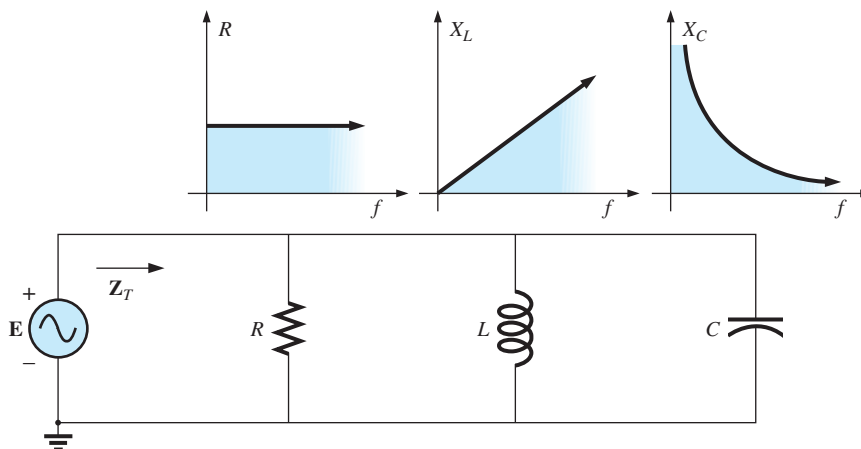
FIG. 15.85
Frequency response for parallel R-L-C elements.

Therefore, it is impossible to make too many broad statements about the effect of each element as the frequency increases. In general, however, for very low frequencies, we can assume that a parallel *R-L-C* network will be inductive as described above, and at very high frequencies it will be capacitive since X_C will drop to very low levels. In between, a point will result at which X_L will equal X_C and where X_L or X_C will equal *R*. The frequencies at which these events occur, however, depend on the elements chosen and the frequency range of interest. In general, however, keep in mind that the smaller the resistance or reactance, the greater its impact on the total impedance of a parallel system.

Let us now note the impact of frequency on the total impedance and inductive current for the parallel *R-L* network in Fig. 15.86 for a frequency range of zero through 40 kHz.
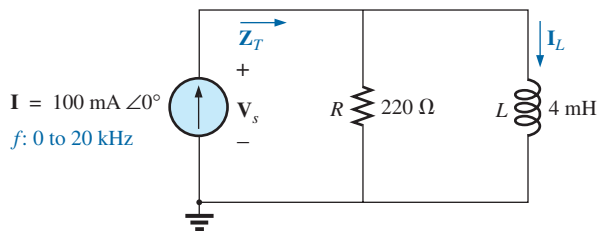
FIG. 15.86
Determining the frequency response of a parallel R-L network.
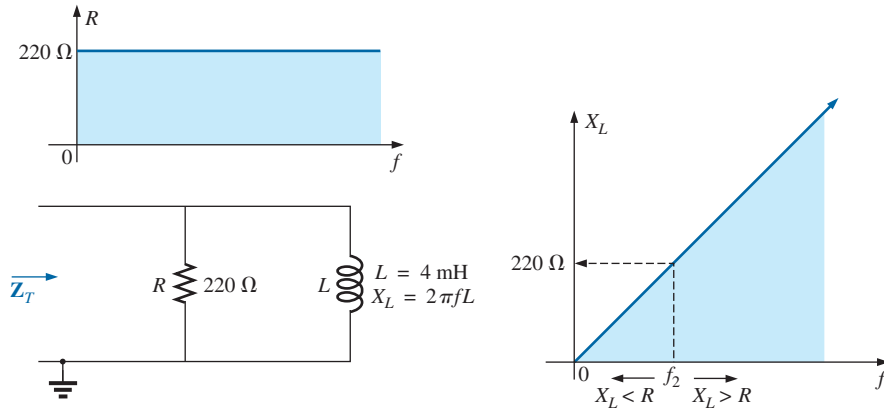
FIG. 15.87

The frequency response of the individual elements of a parallel R-L network.

$\mathbf{Z_T}$ Before getting into specifics, let us first develop a "sense" for the impact of frequency on the network in Fig. 15.86 by noting the impedance-versus-frequency curves of the individual elements, as shown in Fig. 15.87. The fact that the elements are now in parallel requires that we consider their characteristics in a different manner than occurred for the series R-C circuit in Section 15.5. Recall that for parallel elements, the element with the smallest impedance will have the greatest impact on the total impedance at that frequency. In Fig. 15.87, for example, X_L is very small at low frequencies compared to R, establishing X_L as the predominant factor in this frequency range. In other words, at low frequencies the network will be primarily inductive, and the angle associated with the total impedance will be close to 90°, as with a pure inductor. As the frequency increases, X_L increases until it equals the impedance of the resistor (220 Ω). The frequency at which this situation occurs can be determined in the following manner:

$$X_L = 2\pi f_2 L = R$$

and
$$f_2 = \frac{R}{2\pi L} \qquad \textbf{(15.36)}$$

which for the network in Fig. 15.86 is

$$f_2 = \frac{R}{2\pi L} = \frac{220\ \Omega}{2\pi(4 \times 10^{-3}\ \text{H})}$$

$$\cong \textbf{8.75 kHz}$$

which falls within the frequency range of interest.

For frequencies less than f_2, $X_L < R$, and for frequencies greater than f_2, $X_L > R$, as shown in Fig. 15.87. A general equation for the total impedance in vector form can be developed in the following manner:

$$\mathbf{Z_T} = \frac{\mathbf{Z_R Z_L}}{\mathbf{Z_R} + \mathbf{Z_L}}$$

$$= \frac{(R\ \angle 0°)(X_L\ \angle 90°)}{R + j X_L} = \frac{R X_L\ \angle 90°}{\sqrt{R^2 + X_L^2}\ \angle \tan^{-1} X_L/R}$$

and
$$\mathbf{Z_T} = \frac{R X_L}{\sqrt{R^2 + X_L^2}}\ \underline{/90° - \tan^{-1} X_L/R}$$

so that

$$Z_T = \frac{RX_L}{\sqrt{R^2 + X_L^2}} \qquad \text{(15.37)}$$

and

$$\theta_T = 90° - \tan^{-1}\frac{X_L}{R} = \tan^{-1}\frac{R}{X_L} \qquad \text{(15.38)}$$

The magnitude and angle of the total impedance can now be found at any frequency of interest simply by substituting Eqs. (15.37) and (15.38).

f = 1 kHz

$$X_L = 2\pi fL = 2\pi(1 \text{ kHz})(4 \times 10^{-3} \text{ H}) = 25.12 \text{ } \Omega$$

and

$$Z_T = \frac{RX_L}{\sqrt{R^2 + X_L^2}} = \frac{(220 \text{ } \Omega)(25.12 \text{ } \Omega)}{\sqrt{(220 \text{ } \Omega)^2 + (25.12 \text{ } \Omega)^2}} = \textbf{24.96 } \Omega$$

with

$$\theta_T = \tan^{-1}\frac{R}{X_L} = \tan^{-1}\frac{220 \text{ } \Omega}{25.12 \text{ } \Omega}$$

$$= \tan^{-1} 8.76 = 83.49°$$

and

$$\mathbf{Z_T = 24.96 \text{ } \Omega \text{ } \angle 83.49°}$$

This value compares very closely with $X_L = 25.12 \text{ } \Omega \text{ } \angle 90°$, which it would be if the network were purely inductive ($R = \infty \text{ } \Omega$). Our assumption that the network is primarily inductive at low frequencies is therefore confirmed.

Continuing:

$$
\begin{array}{lll}
f = & 5 \text{ kHz}: & \mathbf{Z_T = 109.1 \text{ } \Omega \text{ } \angle 60.23°} \\
f = & 10 \text{ kHz}: & \mathbf{Z_T = 165.5 \text{ } \Omega \text{ } \angle 41.21°} \\
f = & 15 \text{ kHz}: & \mathbf{Z_T = 189.99 \text{ } \Omega \text{ } \angle 30.28°} \\
f = & 20 \text{ kHz}: & \mathbf{Z_T = 201.53 \text{ } \Omega \text{ } \angle 23.65°} \\
f = & 30 \text{ kHz}: & \mathbf{Z_T = 211.19 \text{ } \Omega \text{ } \angle 16.27°} \\
f = & 40 \text{ kHz}: & \mathbf{Z_T = 214.91 \text{ } \Omega \text{ } \angle 12.35°}
\end{array}
$$

At $f = 40$ kHz, note how closely the magnitude of Z_T has approached the resistance level of 220 Ω and how the associated angle with the total impedance is approaching zero degrees. The result is a network with terminal characteristics that are becoming more and more resistive as the frequency increases, which further confirms the earlier conclusions developed by the curves in Fig. 15.87.

Plots of Z_T versus frequency in Fig. 15.88 and θ_T in Fig. 15.89 clearly reveal the transition from an inductive network to one that has resistive characteristics. Note that the transition frequency of 8.75 kHz occurs right in the middle of the "knee" of the curves for both Z_T and θ_T.

A review of Figs. 15.49 and 15.88 reveals that a series R-C and a parallel R-L network will have an impedance level that approaches the resistance of the network at high frequencies. The capacitive circuit approaches the level from above, whereas the inductive network does the same from below. For the series R-L circuit and the parallel R-C network, the total impedance will begin at the resistance level and then display the characteristics of the reactive elements at high frequencies.
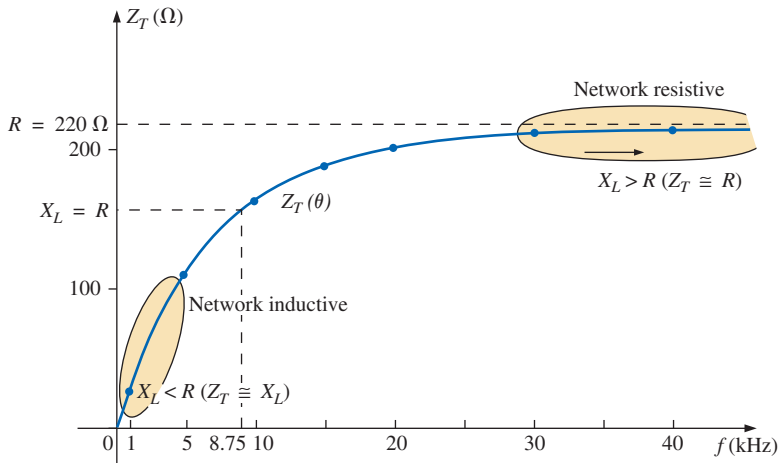
FIG. 15.88

The magnitude of the input impedance versus frequency for the network in Fig. 15.86.
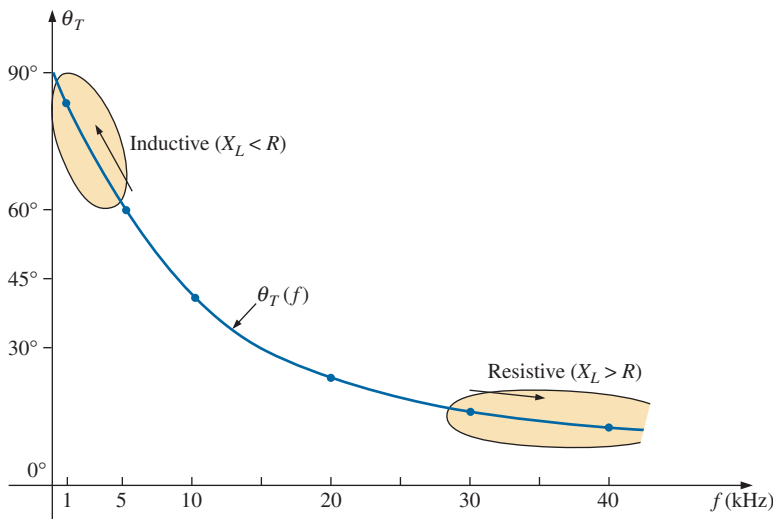
FIG. 15.89

The phase angle of the input impedance versus frequency for the network in Fig. 15.86.

I_L Applying the current divider rule to the network in Fig. 15.86 results in the following:

$$\mathbf{I}_L = \frac{\mathbf{Z}_R\mathbf{I}}{\mathbf{Z}_R + \mathbf{Z}_L}$$

$$= \frac{(R\angle 0°)(I\angle 0°)}{R + jX_L} = \frac{RI\angle 0°}{\sqrt{R^2 + X_L^2}\;\underline{/\tan^{-1}X_L/R}}$$

and

$$\mathbf{I}_L = I_L\angle\theta_L = \frac{RI}{\sqrt{R^2 + X_L^2}}\;\underline{/-\tan^{-1}X_L/R}$$

The magnitude of I_L is therefore determined by

$$I_L = \frac{RI}{\sqrt{R^2 + X_L^2}} \tag{15.39}$$

and the phase angle θ_L, by which $\mathbf{I}_L$ leads $\mathbf{I}$, is given by

$$\theta_L = -\tan^{-1}\frac{X_L}{R} \tag{15.40}$$

Because θ_L is always negative, the magnitude of θ_L is, in actuality, the angle by which $\mathbf{I}_L$ lags $\mathbf{I}$.

To begin our analysis, let us first consider the case of $f = 0$ Hz (dc conditions).

$f = 0$ Hz

$$X_L = 2\pi f L = 2\pi(0 \text{ Hz})L = 0 \ \Omega$$

Applying the short-circuit equivalent for the inductor in Fig. 15.86 results in

$$\mathbf{I}_L = \mathbf{I} = 100 \text{ mA } \angle 0°$$

as appearing in Figs. 15.90 and 15.91.

$f = 1$ kHz   Applying Eq. (15.39):

$$X_L = 2\pi f L = 2\pi(1 \text{ kHz})(4 \text{ mH}) = 25.12 \ \Omega$$

and $\quad \sqrt{R^2 + X_L^2} = \sqrt{(220 \ \Omega)^2 + (25.12 \ \Omega)^2} = 221.43 \ \Omega$

and $\quad I_L = \dfrac{RI}{\sqrt{R^2 + X_L^2}} = \dfrac{(220 \ \Omega)(100 \text{ mA})}{221.43 \ \Omega} = \mathbf{99.35 \text{ mA}}$
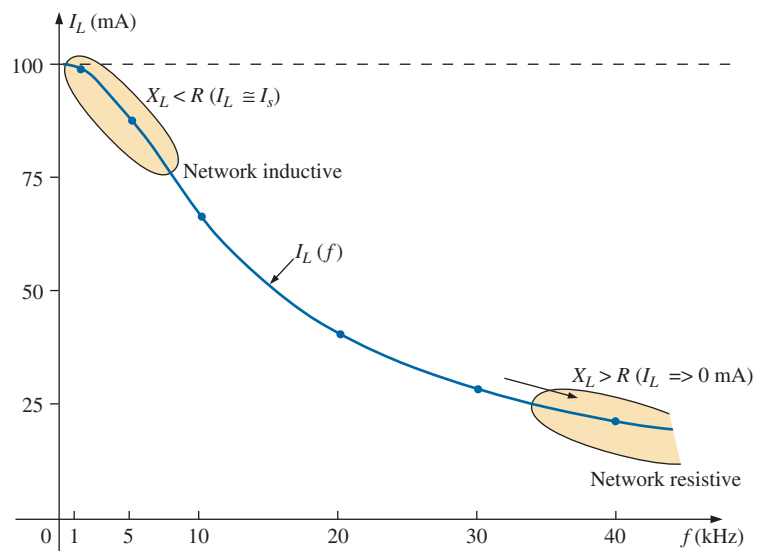
FIG. 15.90
The magnitude of the current $\mathbf{I}_L$ versus frequency for the parallel R-L network in Fig. 15.86.

FIG. 15.91

The phase angle of the current $\mathbf{I}_L$ *versus frequency for the parallel R-L network in Fig. 15.86.*

with

$$\theta_L = \tan^{-1}\frac{X_L}{R} = -\tan^{-1}\frac{25.12\ \Omega}{220\ \Omega} = -\tan^{-1}0.114 = -6.51°$$

and
$$\mathbf{I}_L = \mathbf{99.35\ mA}\ \angle -6.51°$$

The result is a current $\mathbf{I}_L$ that is still very close to the source current $\mathbf{I}$ in both magnitude and phase.

Continuing:

$$
\begin{array}{lll}
f = & 5\ \text{kHz:} & \mathbf{I}_L = \mathbf{86.84\ mA}\ \angle -\mathbf{29.72°} \\
f = 10\ \text{kHz:} & & \mathbf{I}_L = \mathbf{65.88\ mA}\ \angle -\mathbf{48.79°} \\
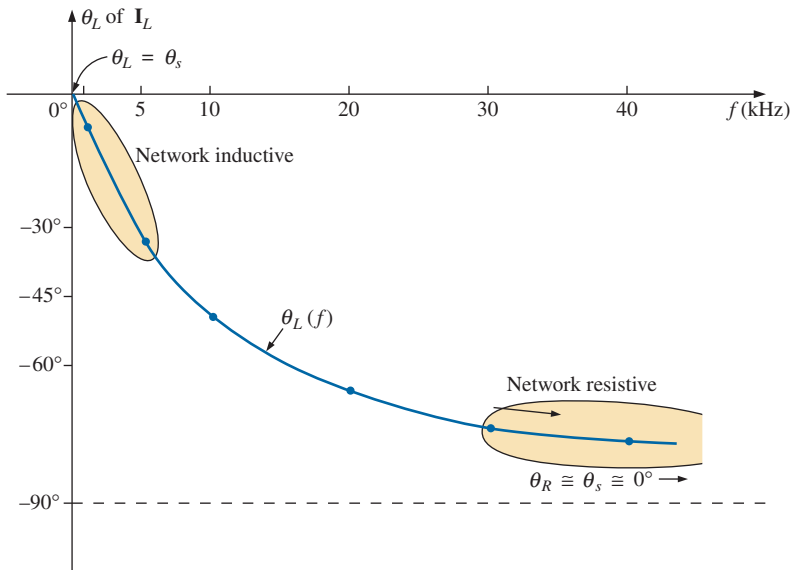f = 15\ \text{kHz:} & & \mathbf{I}_L = \mathbf{50.43\ mA}\ \angle -\mathbf{59.72°} \\
f = 20\ \text{kHz:} & & \mathbf{I}_L = \mathbf{40.11\ mA}\ \angle -\mathbf{66.35°} \\
f = 30\ \text{kHz:} & & \mathbf{I}_L = \mathbf{28.02\ mA}\ \angle -\mathbf{73.73°} \\
f = 40\ \text{kHz:} & & \mathbf{I}_L = \mathbf{21.38\ mA}\ \angle -\mathbf{77.65°}
\end{array}
$$

The plot of the magnitude of I_L versus frequency is provided in Fig. 15.90 and reveals that the current through the coil dropped from its maximum of 100 mA to almost 20 mA at 40 kHz. As the reactance of the coil increased with frequency, more of the source current chose the lower-resistance path of the resistor. The magnitude of the phase angle between $\mathbf{I}_L$ and $\mathbf{I}$ is approaching 90° with an increase in frequency, as shown in Fig. 15.91, leaving its initial value of zero degrees at $f = 0$ Hz far behind.

At $f = 1$ kHz, the phasor diagram of the network appears as shown in Fig. 15.92. First note that the magnitude and the phase angle of $\mathbf{I}_L$ are very close to those of $\mathbf{I}$. Since the voltage across a coil must lead the current through a coil by 90°, the voltage $\mathbf{V}_s$ appears as shown. The voltage across a resistor is in phase with the current through the resistor, resulting in the direction of $\mathbf{I}_R$ shown in Fig. 15.92. Of course, at this frequency $R > X_L$, and the current I_R is relatively small in magnitude.
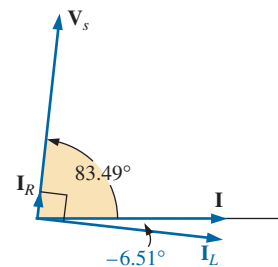
FIG. 15.92

The phasor diagram for the parallel R-L network in Fig. 15.86 at $f = 1$ kHz.
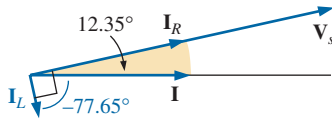
FIG. 15.93

The phasor diagram for the parallel R-L network in Fig. 15.86 at f = 40 kHz.

At $f = 40$ kHz, the phasor diagram changes to that appearing in Fig. 15.93. Note that now $\mathbf{I}_R$ and $\mathbf{I}$ are close in magnitude and phase because $X_L > R$. The magnitude of $\mathbf{I}_L$ has dropped to very low levels, and the phase angle associated with $\mathbf{I}_L$ is approaching $-90°$. The network is now more "resistive" compared to its "inductive" characteristics at low frequencies.

The analysis of a parallel R-C or R-L-C network would proceed in much the same manner, with the inductive impedance predominating at low frequencies and the capacitive reactance predominating at high frequencies.

15.11 SUMMARY: PARALLEL ac NETWORKS

The following is a review of important conclusions that can be derived from the discussion and examples of the previous sections. The list is not all-inclusive, but it does emphasize some of the conclusions that should be carried forward in the future analysis of ac systems.

For parallel ac networks with reactive elements:

1. *The total admittance or impedance will be frequency dependent.*
2. *Depending on the frequency applied, the same network can be either predominantly inductive or predominantly capacitive.*
3. *The magnitude of the current through any one branch can be greater than the source current.*
4. *The inductive and capacitive susceptances are in direct opposition on an admittance diagram.*
5. *At lower frequencies, the inductive elements will usually have the most impact on the total impedance, while at high frequencies, the capacitive elements will usually have the most impact.*
6. *The impedance of any one element can be less than the total impedance (recall that for dc circuits, the total resistance must always be less than the smallest parallel resistor).*
7. *The magnitude of the current through an element, compared to the other elements of the network, is directly related to the magnitude of its impedance; that is, the smaller the impedance of an element, the larger the magnitude of the current through the element.*
8. *The current through a coil is always in direct opposition with the current through a capacitor on a phasor diagram.*
9. *The applied voltage is always in phase with the current through the resistive elements, leads the voltage across all the inductive elements by 90°, and lags the current through all capacitive elements by 90°.*
10. *The smaller the resistive element of a network compared to the net reactive susceptance, the closer the power factor is to unity.*

15.12 EQUIVALENT CIRCUITS

In a series ac circuit, the total impedance of two or more elements in series is often equivalent to an impedance that can be achieved with fewer elements of different values, the elements and their values being determined by the frequency applied. This is also true for parallel circuits. For the circuit in Fig. 15.94(a),

$$\mathbf{Z}_T = \frac{\mathbf{Z}_C \mathbf{Z}_L}{\mathbf{Z}_C + \mathbf{Z}_L} = \frac{(5\ \Omega\ \angle{-90°})(10\ \Omega\ \angle 90°)}{5\ \Omega\ \angle{-90°}\ +\ 10\ \Omega\ \angle 90°} = \frac{50\ \angle 0°}{5\ \angle 90°}$$

$$= 10\ \Omega\ \angle{-90°}$$
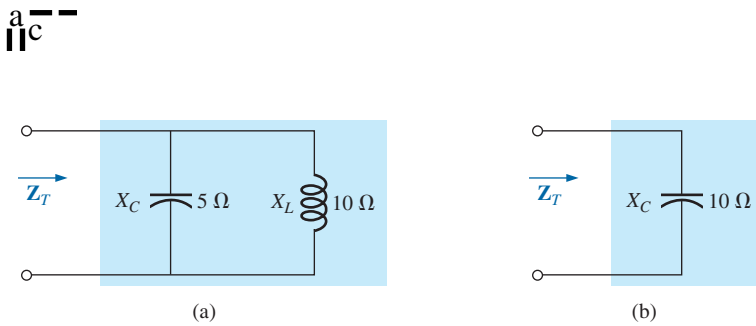
(a)　　　　　　　　　(b)

FIG. 15.94

Defining the equivalence between two networks at a specific frequency.

The total impedance at the frequency applied is equivalent to a capacitor with a reactance of 10 Ω, as shown in Fig. 15.94(b). Always keep in mind that this equivalence is true only at the applied frequency. If the frequency changes, the reactance of each element changes, and the equivalent circuit changes—perhaps from capacitive to inductive in the above example.

Another interesting development appears if the impedance of a parallel circuit, such as the one in Fig. 15.95(a), is found in rectangular form. In this case,

(a)

$$\mathbf{Z}_T = \frac{\mathbf{Z}_L\mathbf{Z}_R}{\mathbf{Z}_L + \mathbf{Z}_R} = \frac{(4 \ \Omega \ \angle 90°)(3 \ \Omega \ \angle 0°)}{4 \ \Omega \ \angle 90° + 3 \ \Omega \ \angle 0°}$$

$$= \frac{12 \ \angle 90°}{5 \ \angle 53.13°} = 2.40 \ \Omega \ \angle 36.87°$$

$$= 1.92 \ \Omega + j \ 1.44 \ \Omega$$

(b)

which is the impedance of a series circuit with a resistor of 1.92 Ω and an inductive reactance of 1.44 Ω, as shown in Fig. 15.95(b).

The current **I** will be the same in each circuit in Fig. 15.94 or Fig. 15.95 if the same input voltage **E** is applied. For a parallel circuit of one resistive element and one reactive element, the series circuit with the same input impedance will always be composed of one resistive and one reactive element. The impedance of each element of the series circuit will be different from that of the parallel circuit, but the reactive elements will always be of the same type; that is, an *R-L* circuit and an *R-C* parallel circuit will have an equivalent *R-L* and *R-C* series circuit, respectively. The same is true when converting from a series to a parallel circuit. In the discussion to follow, keep in mind that

FIG. 15.95

Finding the series equivalent circuit for a parallel R-L network.

the term equivalent refers only to the fact that for the same applied potential, the same impedance and input current will result.

To formulate the equivalence between the series and parallel circuits, the equivalent series circuit for a resistor and reactance in parallel can be found by determining the total impedance of the circuit in rectangular form; that is, for the circuit in Fig. 15.96(a),

$$\mathbf{Y}_p = \frac{1}{R_p} + \frac{1}{\pm jX_p} = \frac{1}{R_p} \mp j \frac{1}{X_p}$$

and

$$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = \frac{1}{(1/R_p) \mp j(1/X_p)}$$

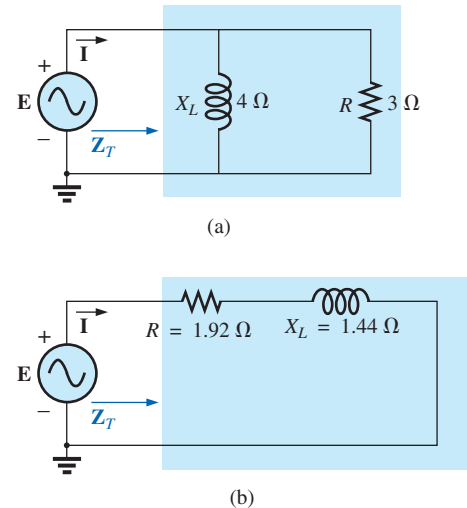$$= \frac{1/R_p}{(1/R_p)^2 + (1/X_p)^2} \pm j \frac{1/X_p}{(1/R_p)^2 + (1/X_p)^2}$$
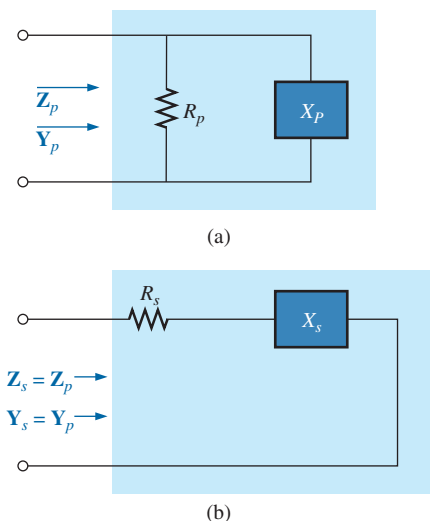

(a)

(b)

FIG. 15.96

Defining the parameters of equivalent series and parallel networks.

Multiplying the numerator and denominator of each term by $R_p^2 X_p^2$ results in

$$\mathbf{Z}_p = \frac{R_p X_p^2}{X_p^2 + R_p^2} \pm j \frac{R_p^2 X_p}{X_p^2 + R_p^2}$$
$$= R_s \pm j X_s \qquad \text{[Fig. 15.96(b)]}$$

and

$$R_s = \frac{R_p X_p^2}{X_p^2 + R_p^2} \qquad (15.41)$$

with

$$X_s = \frac{R_p^2 X_p}{X_p^2 + R_p^2} \qquad (15.42)$$

For the network in Fig. 15.95,

$$R_s = \frac{R_p X_p^2}{X_p^2 + R_p^2} = \frac{(3\ \Omega)(4\ \Omega)^2}{(4\ \Omega)^2 + (3\ \Omega)^2} = \frac{48\ \Omega}{25} = \mathbf{1.92\ \Omega}$$

and

$$X_s = \frac{R_p^2 X_p}{X_p^2 + R_p^2} = \frac{(3\ \Omega)^2(4\ \Omega)}{(4\ \Omega)^2 + (3\ \Omega)^2} = \frac{36\ \Omega}{25} = \mathbf{1.44\ \Omega}$$

which agrees with the previous result.

The equivalent parallel circuit for a circuit with a resistor and reactance in series can be found by finding the total admittance of the system in rectangular form; that is, for the circuit in Fig. 15.96(b),

$$\mathbf{Z}_s = R_s \pm j X_s$$
$$\mathbf{Y}_s = \frac{1}{\mathbf{Z}_s} = \frac{1}{R_s \pm j X_s} = \frac{R_s}{R_s^2 + X_s^2} \mp j \frac{X_s}{R_s^2 + X_s^2}$$
$$= G_p \mp j B_p = \frac{1}{R_p} \mp j \frac{1}{X_p} \qquad \text{[Fig. 15.96(a)]}$$

or

$$R_p = \frac{R_s^2 + X_s^2}{R_s} \qquad (15.43)$$

with

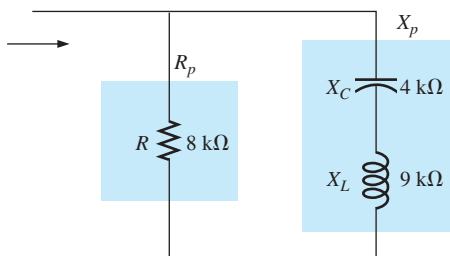$$X_p = \frac{R_s^2 + X_s^2}{X_s} \qquad (15.44)$$

For the above example,

$$R_p = \frac{R_s^2 + X_s^2}{R_s} = \frac{(1.92\ \Omega)^2 + (1.44\ \Omega)^2}{1.92\ \Omega} = \frac{5.76\ \Omega}{1.92} = \mathbf{3.0\ \Omega}$$

and

$$X_p = \frac{R_s^2 + X_s^2}{X_s} = \frac{5.76\ \Omega}{1.44} = \mathbf{4.0\ \Omega}$$

as shown in Fig. 15.95(a).

FIG. 15.97

Example 15.18.

EXAMPLE 15.18 Determine the series equivalent circuit for the network in Fig. 15.97.

Solution:

$$R_p = 8 \text{ k}\Omega$$
$$X_p \text{ (resultant)} = |X_L - X_C| = |9 \text{ k}\Omega - 4 \text{ k}\Omega|$$
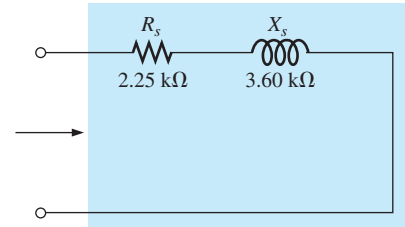$$= 5 \text{ k}\Omega$$

and

$$R_s = \frac{R_p X_p^2}{X_p^2 + R_p^2} = \frac{(8 \text{ k}\Omega)(5 \text{ k}\Omega)^2}{(5 \text{ k}\Omega)^2 + (8 \text{ k}\Omega)^2} = \frac{200 \text{ k}\Omega}{89} = \mathbf{2.25 \text{ k}\Omega}$$

with

$$X_s = \frac{R_p^2 X_p}{X_p^2 + R_p^2} = \frac{(8 \text{ k}\Omega)^2(5 \text{ k}\Omega)}{(5 \text{ k}\Omega)^2 + (8 \text{ k}\Omega)^2} = \frac{320 \text{ k}\Omega}{89}$$
$$= \mathbf{3.60 \text{ k}\Omega} \quad \textbf{(inductive)}$$

The equivalent series circuit appears in Fig. 15.98.

FIG. 15.98

The equivalent series circuit for the parallel network in Fig. 15.97.

EXAMPLE 15.19 For the network in Fig. 15.99:

a. Determine $\mathbf{Y}_T$ and $\mathbf{Z}_T$.
b. Sketch the admittance diagram.
c. Find $\mathbf{E}$ and $\mathbf{I}_L$.
d. Compute the power factor of the network and the power delivered to the network.
e. Determine the equivalent series circuit as far as the terminal characteristics of the network are concerned.
f. Using the equivalent circuit developed in part (e), calculate $\mathbf{E}$, and compare it with the result of part (c).
g. Determine the power delivered to the network, and compare it with the solution of part (d).
h. Determine the equivalent parallel network from the equivalent series circuit, and calculate the total admittance $\mathbf{Y}_T$. Compare the result with the solution of part (a).
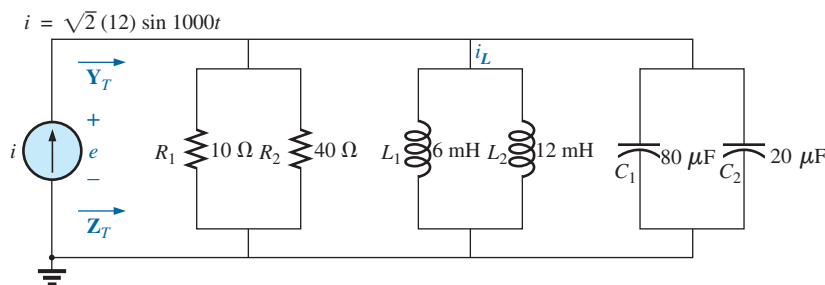
FIG. 15.99

Example 15.19.

Solutions:

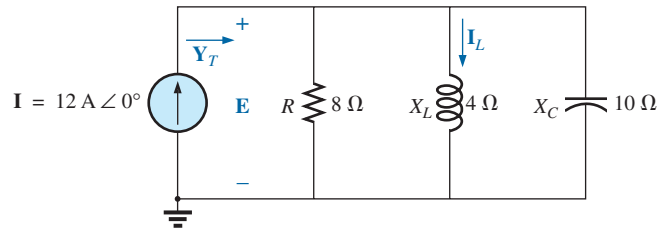a. Combining common elements and finding the reactance of the inductor and capacitor, we obtain

$$R_T = 10 \ \Omega \,\|\, 40 \ \Omega = 8 \ \Omega$$
$$L_T = 6 \text{ mH} \,\|\, 12 \text{ mH} = 4 \text{ mH}$$
$$C_T = 80 \ \mu\text{F} + 20 \ \mu\text{F} = 100 \ \mu\text{F}$$
$$X_L = \omega L = (1000 \text{ rad/s})(4 \text{ mH}) = 4 \ \Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{(1000 \text{ rad/s})(100 \ \mu\text{F})} = 10 \ \Omega$$
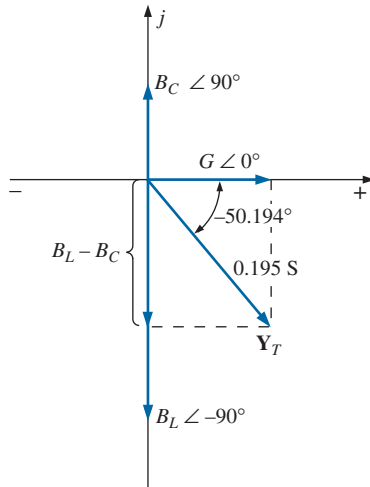
FIG. 15.100

Applying phasor notation to the network in Fig. 15.99.

The network is redrawn in Fig. 15.100 with phasor notation. The total admittance is

$$\mathbf{Y}_T = \mathbf{Y}_R + \mathbf{Y}_L + \mathbf{Y}_C$$
$$= G \angle 0° + B_L \angle -90° + B_C \angle +90°$$
$$= \frac{1}{8\,\Omega} \angle 0° + \frac{1}{4\,\Omega} \angle -90° + \frac{1}{10\,\Omega} \angle +90°$$
$$= 0.125\text{ S} \angle 0° + 0.25\text{ S} \angle -90° + 0.1\text{ S} \angle +90°$$
$$= 0.125\text{ S} - j\,0.25\text{ S} + j\,0.1\text{ S}$$
$$= 0.125\text{ S} - j\,0.15\text{ S} = \mathbf{0.195\text{ S} \angle -50.194°}$$

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{0.195\text{ S}} \angle -50.194° = \mathbf{5.13\ \Omega \angle 50.19°}$$

b. See Fig. 15.101.

FIG. 15.101

Admittance diagram for the parallel R-L-C network in Fig. 15.99.

c. $\mathbf{E} = \mathbf{IZ}_T = \dfrac{\mathbf{I}}{\mathbf{Y}_T} = \dfrac{12\text{ A} \angle 0°}{0.195\text{ S} \angle -50.194°} = \mathbf{61.54\text{ V} \angle 50.19°}$

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{Z}_L} = \frac{\mathbf{E}}{\mathbf{Z}_L} = \frac{61.538\text{ V} \angle 50.194°}{4\,\Omega \angle 90°} = \mathbf{15.39\text{ A} \angle -39.81°}$$
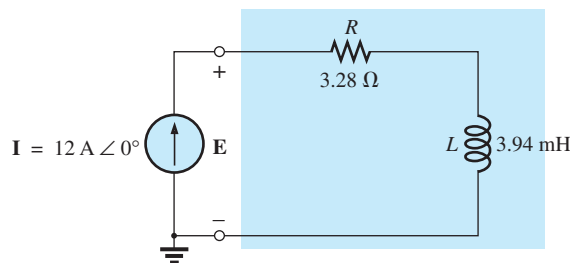
d. $F_p = \cos\theta = \dfrac{G}{Y_T} = \dfrac{0.125\text{ S}}{0.195\text{ S}} = \mathbf{0.641\text{ lagging}}$ (**E** leads **I**)

$$P = EI\cos\theta = (61.538\text{ V})(12\text{ A})\cos 50.194°$$
$$= \mathbf{472.75\text{ W}}$$

e. $\mathbf{Z}_T = \dfrac{1}{\mathbf{Y}_T} = \dfrac{1}{0.195\text{ S} \angle -50.194°} = 5.128\ \Omega \angle +50.194°$

$$= 3.28\ \Omega + j\,3.94\ \Omega$$
$$= R + jX_L$$

$$X_L = 3.94\ \Omega = \omega L$$

$$L = \frac{3.94\ \Omega}{\omega} = \frac{3.94\ \Omega}{1000\text{ rad/s}} = \mathbf{3.94\text{ mH}}$$

The series equivalent circuit appears in Fig. 15.102.

FIG. 15.102

Series equivalent circuit for the parallel R-L-C network in Fig. 15.99 with $\omega = 1000$ rad/s.

f. $\mathbf{E} = \mathbf{IZ}_T = (12\ \text{A}\ \angle 0°)(5.128\ \Omega\ \angle 50.194°)$

   $\qquad = \mathbf{61.54\ V}\ \angle \mathbf{50.194°}$ (as above)

g. $P = I^2 R = (12\ \text{A})^2(3.28\ \Omega) = \mathbf{472.32\ W}$ (as above)

h. $R_p = \dfrac{R_s^2 + X_s^2}{R_s} = \dfrac{(3.28\ \Omega)^2 + (3.94\ \Omega)^2}{3.28\ \Omega} = \mathbf{8\ \Omega}$

   $X_p = \dfrac{R_s^2 + X_s^2}{X_s} = \dfrac{(3.28\ \Omega)^2 + (3.94\ \Omega)^2}{3.94\ \Omega} = \mathbf{6.67\ \Omega}$

The parallel equivalent circuit appears in Fig. 15.103.

$$\mathbf{Y}_T = G\ \angle 0° + B_L\ \angle -90° = \frac{1}{8\ \Omega}\ \angle 0° + \frac{1}{6.675\ \Omega}\ \angle -90°$$

$$= 0.125\ \text{S}\ \angle 0° + 0.15\ \text{S}\ \angle -90°$$

$$= 0.125\ \text{S} - j\,0.15\ \text{S} = \mathbf{0.195\ S}\ \angle -\mathbf{50.194°}\quad\text{(as above)}$$
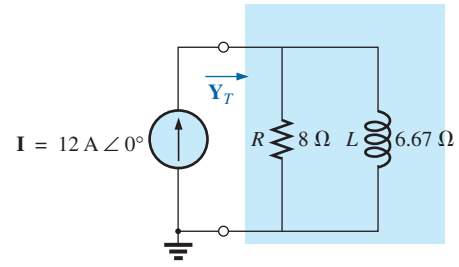

FIG. 15.103
Parallel equivalent of the circuit in Fig. 15.102.

15.13 PHASE MEASUREMENTS

Measuring the phase angle between quantities is one of the most important functions that an oscilloscope can perform. It is an operation that must be performed carefully, however, or you may obtain the incorrect result or damage the equipment. Whenever you are using the dual-trace capability of an oscilloscope, the most important thing to remember is that

both channels of a dual-trace oscilloscope must be connected to the same ground.

Measuring Z_T and θ_T

For ac parallel networks restricted to resistive loads, the total impedance can be found in the same manner as described for dc circuits: Simply remove the source and place an ohmmeter across the network terminals. However,

for parallel ac networks with reactive elements, the total impedance cannot be measured with an ohmmeter.

An experimental procedure must be defined that permits determining the magnitude and the angle of the terminal impedance.

The phase angle between the applied voltage and the resulting source current is one of the most important because (a) it is also the phase angle associated with the total impedance; (b) it provides an instant indication of whether a network is resistive or reactive; (c) it reveals whether a network is inductive or capacitive; and (d) it can be used to find the power delivered to the network.

In Fig. 15.104, a resistor has been added to the configuration between the source and the network to permit measuring the current and finding the phase angle between the applied voltage and the source current.

At the frequency of interest, the applied voltage establishes a voltage across the sensing resistor which can be displayed by one channel of the dual-trace oscilloscope. In Fig. 15.104, channel 1 is displaying the applied voltage, and channel 2 the voltage across the sensing resistor. Sensitivities for each channel are chosen to establish the waveforms appearing on the screen in Fig. 15.105. As emphasized above, note that both channels have the same ground connection. In fact, the need for a common ground connection is the only reason that the sensing resistor
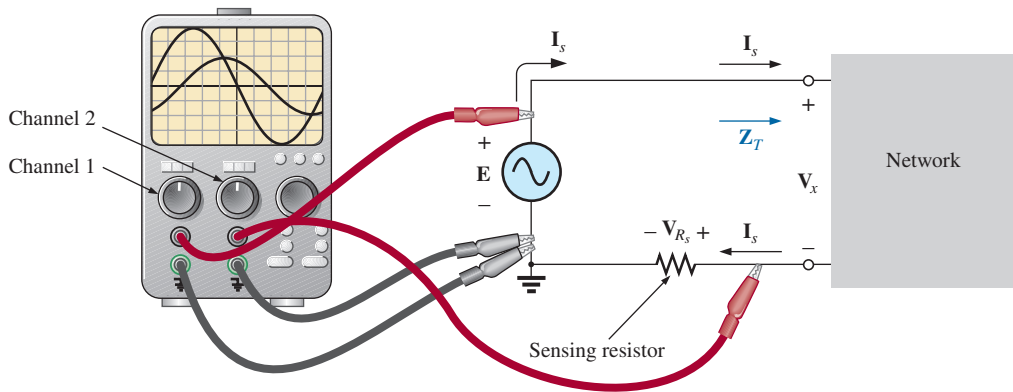
FIG. 15.104
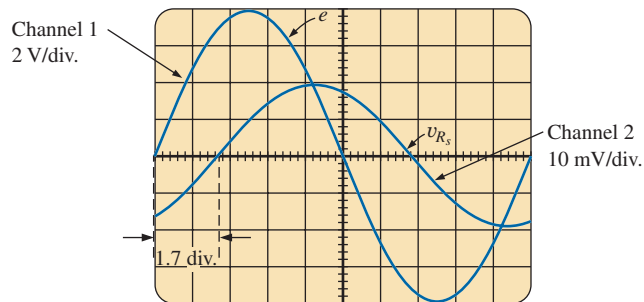*Using an oscilloscope to measure Z_T and θ_T.*

FIG. 15.105
e and v_{R_s}, for the configuration in Fig. 15.104.

was not connected to the positive side of the supply. Since oscilloscopes display only voltages versus time, the peak value of the source current must be found using Ohm's law. *Since the voltage across a resistor and the current through the resistor are in phase, the phase angle between the two voltages will be the same as that between the applied voltage and the resulting source current.*

Using the sensitivities, the peak value of the applied voltage is

$$E_m = (4 \text{ div.})(2 \text{ V/div.}) = 8 \text{ V}$$

while the peak value of the voltage across the sensing resistor is

$$V_{R_s(\text{peak})} = (2 \text{ div.})(10 \text{ mV/div.}) = 20 \text{ mV}$$

Using Ohm's law, the peak value of the current is

$$I_{s(\text{peak})} = \frac{V_{R_s(\text{peak})}}{R_s} = \frac{20 \text{ mV}}{10 \text{ }\Omega} = 2 \text{ mA}$$

The sensing resistor is chosen small enough so that the voltage across the sensing resistor is small enough to permit the approximation $\mathbf{V}_x = \mathbf{E} - \mathbf{V}_{R_s} \cong \mathbf{E}$. The magnitude of the input impedance is then

$$Z_T = \frac{V_x}{I_s} \cong \frac{E}{I_s} = \frac{8 \text{ V}}{2 \text{ mA}} = \mathbf{4 \text{ k}\Omega}$$

For the chosen horizontal sensitivity, each waveform in Fig. 15.105 has a period T defined by ten horizontal divisions, and the phase angle between the two waveforms is 1.7 divisions. Using the fact that each pe-

riod of a sinusoidal waveform encompasses 360°, the following ratios can be set up to determine the phase angle θ:

$$\frac{10 \text{ div.}}{360°} = \frac{1.7 \text{ div.}}{\theta}$$

and

$$\theta = \left(\frac{1.7}{10}\right)360° = \textbf{61.2}°$$

In general,

$$\boxed{\theta = \frac{(\text{div. for } \theta)}{(\text{div. for } T)} \times 360°} \qquad \textbf{(15.45)}$$

Therefore, the total impedance is

$$\mathbf{Z}_T = \textbf{4 k}\Omega \angle \textbf{61.2}° = \textbf{1.93 k}\Omega + j \textbf{3.51 k}\Omega = R + j X_L$$

which is equivalent to the series combination of a 1.93 kΩ resistor and an inductor with a reactance of 3.51 kΩ (at the frequency of interest).

Measuring the Phase Angle between Various Voltages

In Fig. 15.106, an oscilloscope is being used to find the phase relationship between the applied voltage and the voltage across the inductor. Note again that each channel shares the same ground connection. The resulting pattern appears in Fig. 15.107 with the chosen sensitivities. This time, both channels have the same sensitivity, resulting in the following peak values for the voltages:
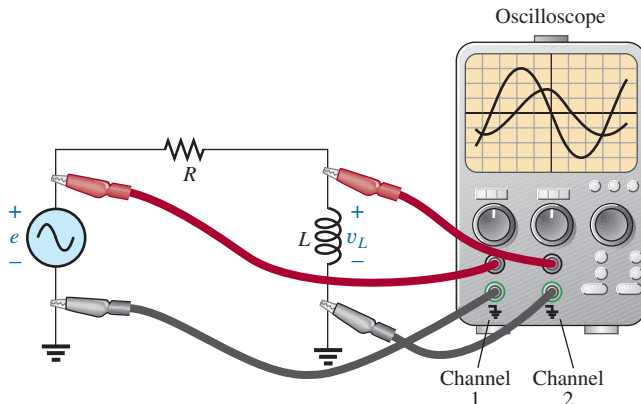
$$E_m = (3 \text{ div.})(2 \text{ V/div.}) = \textbf{6 V}$$
$$V_{L(\text{peak})} = (1.6 \text{ div.})(2 \text{ V/div.}) = \textbf{3.2 V}$$

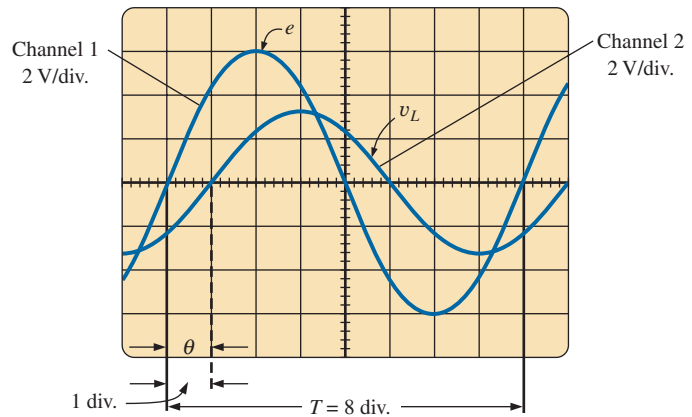The phase angle is determined using Eq. 15.45:

$$\theta = \frac{(1 \text{ div.})}{(8 \text{ div.})} \times 360°$$
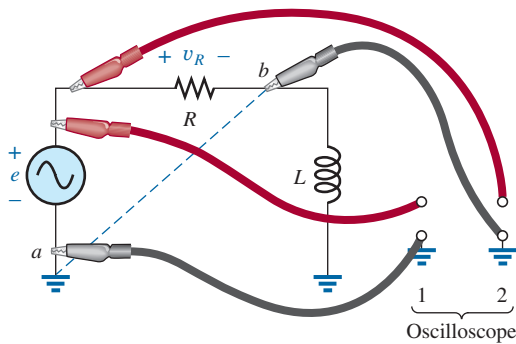$$\theta = \textbf{45}°$$

FIG. 15.106

Determining the phase relationship between e and v_L.

FIG. 15.107

*Determining the phase angle between e and v_L for the configuration in Fig. 15.106.*

FIG. 15.108

An improper phase-measurement connection.

If the phase relationship between e and v_R is desired, the oscilloscope *cannot be connected* as shown in Fig. 15.108. *The grounds of each channel are internally connected in the oscilloscope,* forcing point b to have the same potential as point a. The result would be a direct connection between points a and b that would short out the inductive element. If the inductive element is the predominant factor in controlling the level of the current, the current in the circuit could rise to dangerous levels and damage the oscilloscope or supply. The easiest way to find the phase relationship between e and v_R would be to simply interchange the positions of the resistor and the inductor and proceed as before.

For the parallel network in Fig. 15.109, the phase relationship between two of the branch currents, i_R and i_L, can be determined using a sensing resistor, as shown in the figure. The value of the sensing resistor is chosen small enough in comparison to the value of the series inductive reactance to ensure that it will not affect the general response of the network. Channel 1 displays the voltage v_R, and channel 2 the voltage v_{R_s}. Since v_R is in phase with i_R, and v_{R_s} is in phase with i_L, the phase relationship between v_R and v_{R_s} will be the same as between i_R and i_L. The peak value of each current can be found through a simple application of Ohm's law.
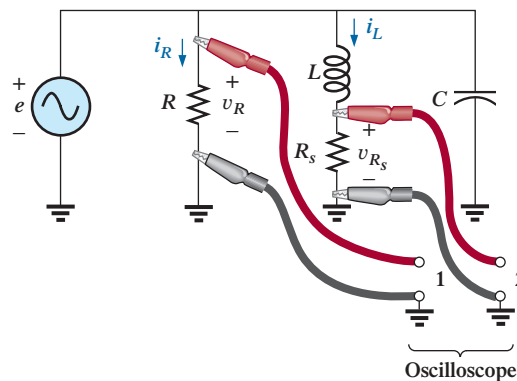
FIG. 15.109

Determining the phase relationship between i_R and i_L.

15.14 APPLICATIONS

Home Wiring

An expanded view of house wiring is provided in Fig. 15.110 to permit a discussion of the entire system. The house panel has been included with the "feed" and the important grounding mechanism. In addition, a number of typical circuits found in the home have been included to provide a sense for the manner in which the total power is distributed.

First note how the copper bars in the panel are laid out to provide both 120 V and 240 V. Between any one bar and ground is the single-phase 120 V supply. However, the bars have been arranged so that 240 V can be obtained between two vertical adjacent bars using a double-gang circuit breaker. When time permits, examine your own panel (but do not remove the cover), and note the dual circuit breaker arrangement for the 240 V supply.

For appliances such as fixtures and heaters that have a metal casing, the ground wire is connected to the metal casing to provide a direct path to ground path for a "shorting" or errant current as described in Section 6.8. For outlets that do not have a conductive casing, the ground lead is connected to a point on the outlet that distributes to all important points of the outlet.

Note the series arrangement between the thermostat and the heater but the parallel arrangement between heaters on the same circuit. In addition,
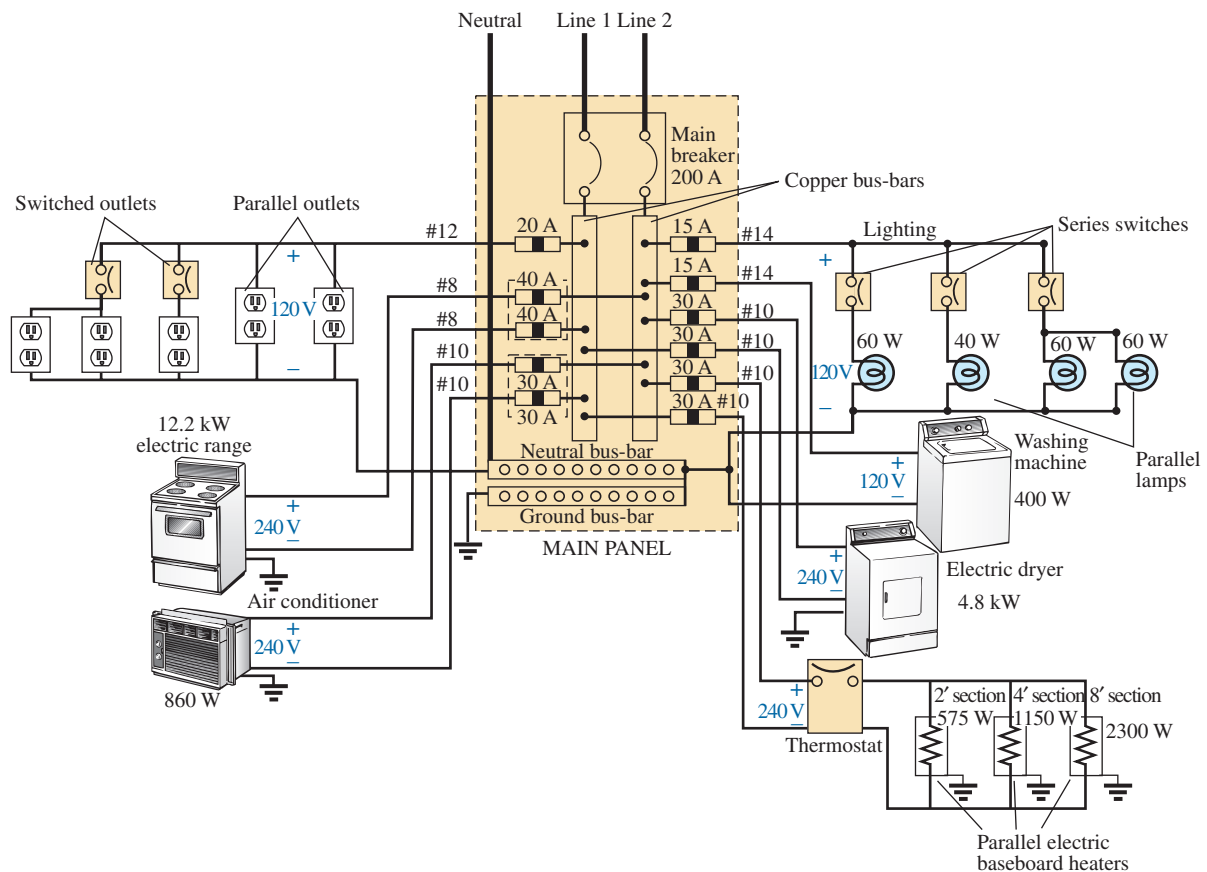
FIG. 15.110
Home wiring diagram.

note the series connection of switches to lights in the upper-right corner but the parallel connection of lights and outlets. Due to high current demand, the air conditioner, heaters, and electric stove have 30 A breakers. Keep in mind that the total current does not equal the product of the two (or 60 A) since each breaker is in a line and the same current will flow through each breaker.

In general, you now have a surface understanding of the general wiring in your home. You may not be a qualified, licensed electrician, but at least you should now be able to converse with some intelligence about the system.

Speaker Systems

The best reproduction of sound is obtained using a different speaker for the low-, mid-, and high-frequency regions. Although the typical audio range for the human ear is from about 100 Hz to 20 kHz, speakers are available from 20 Hz to 40 kHz. For the low-frequency range usually extending from about 20 Hz to 300 Hz, a speaker referred to as a *woofer* is used. Of the three speakers, it is normally the largest. The mid-range speaker is typically smaller in size and covers the range from about 100 Hz to 5 kHz. The *tweeter,* as it is normally called, is usually the smallest of the three speakers and typically covers the range from about 2 kHz to 25 kHz. There is an overlap of frequencies to ensure that frequencies aren't lost in those regions where the response of one drops off and the other takes over. A great deal more about the range of each speaker and their dB response (a term you may have heard when discussing speaker response) is covered in detail in Chapter 21.

One popular method for hooking up the three speakers is the *cross-over* configuration in Fig. 15.111. Note that it is nothing more than a parallel network with a speaker in each branch and full applied voltage across each branch. The added elements (inductors and capacitors) were carefully chosen to set the range of response for each speaker. Note that each speaker is labeled with an impedance level and associated frequency. This type of information is typical when purchasing a quality speaker. It immediately identifies the type of speaker and reveals at which frequency it will have its maximum response. A detailed analysis of the same network will be included in Section 21.15. For now, however, it should prove interesting to determine the total impedance of each branch at specific frequencies to see if indeed the response of one will far outweigh the response of the other two. Since an amplifier with an output impedance of 8 Ω is to be used, maximum transfer of power (see Section 18.5 for ac networks) to the speaker results when the impedance of the branch is equal to or very close to 8 Ω.

Let us begin by examining the response of the frequencies to be carried primarily by the mid-range speaker since it represents the greatest portion of the human hearing range. Since the mid-range speaker branch is rated at 8 Ω at 1.4 kHz, let us test the effect of applying 1.4 kHz to all branches of the crossover network.

For the mid-range speaker:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(1.4 \text{ kHz})(47 \ \mu\text{F})} = 2.42 \ \Omega$$

$$X_L = 2\pi fL = 2\pi(1.4 \text{ kHz})(270 \ \mu\text{H}) = 2.78 \ \Omega$$

$$R = 8 \ \Omega$$
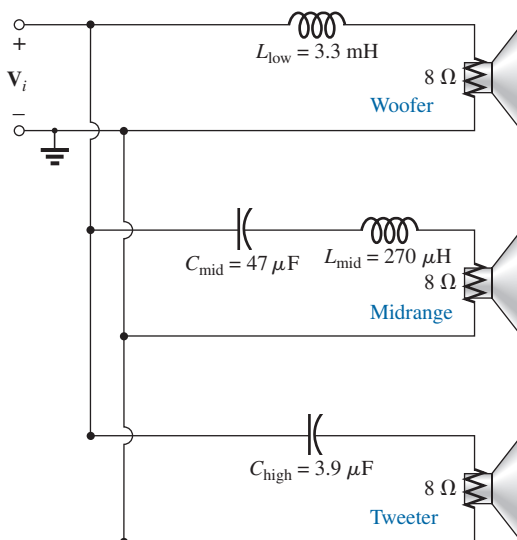
FIG. 15.111

Crossover speaker system.

and $Z_{\text{mid-range}} = R + j (X_L - X_C) = 8 \ \Omega + j (2.78 \ \Omega - 2.42 \ \Omega)$
$= 8 \ \Omega + j \ 0.36 \ \Omega$
$= 8.008 \ \Omega \ \angle -2.58° \cong 8 \ \Omega \ \angle \ 0° = R$

In Fig. 15.112(a), the amplifier with the output impedance of 8 Ω has been applied across the mid-range speaker at a frequency of 1.4 kHz. Since the total reactance offered by the two series reactive elements is so small compared to the 8 Ω resistance of the speaker, we can essentially replace the series combination of the coil and capacitor by a short circuit of 0 Ω. We are then left with a situation where the load impedance is an exact match with the output impedance of the amplifier, and maximum power will be delivered to the speaker. Because of the equal series impedances, each will capture half the applied voltage or 6 V. The power to the speaker is then $V^2/R = (6 \text{ V})^2/8 \ \Omega = 4.5$ W.

At a frequency of 1.4 kHz, we would expect the woofer and tweeter to have minimum impact on the generated sound. We will now test the validity of this statement by determining the impedance of each branch at 1.4 kHz.
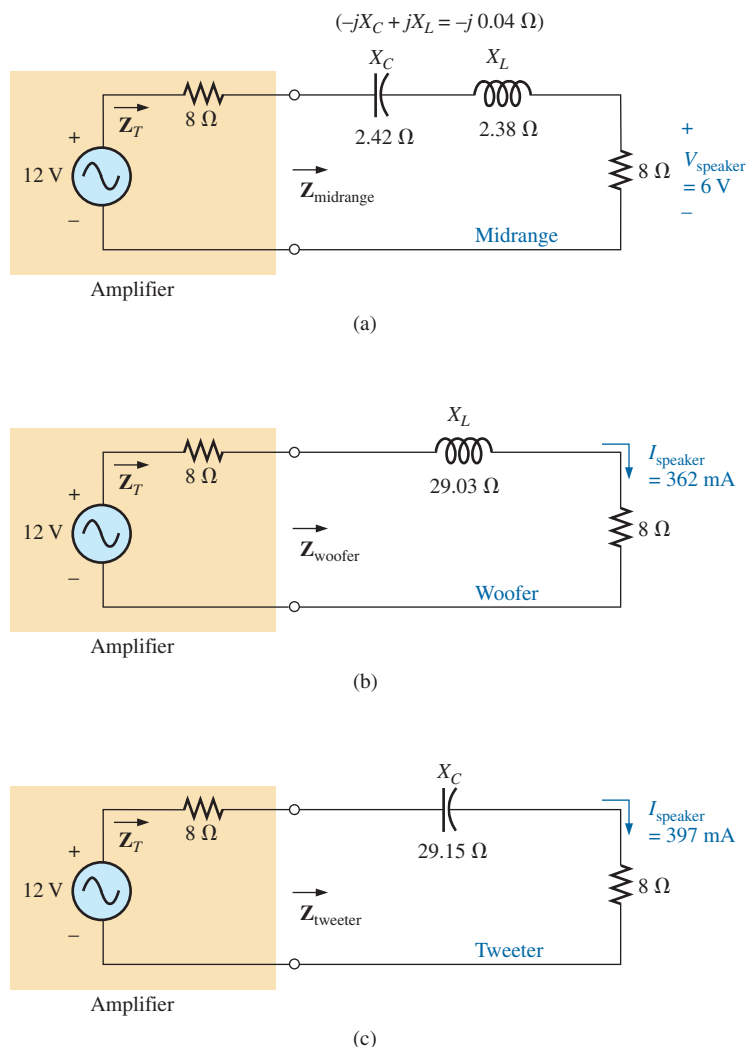
FIG. 15.112
Crossover network: (a) mid-range speaker at 1.4 kHz; (b) woofer at 1.4 kHz;
(c) tweeter.

For the woofer:

$$X_L = 2\pi fL = 2\pi(1.4 \text{ kHz})(3.3 \text{ mH}) = 29.03 \ \Omega$$

and
$$Z_{woofer} = R + j X_L = 8 \ \Omega + j \ 29.03 \ \Omega$$
$$= 30.11 \ \Omega \angle 74.59°$$

which is a poor match with the output impedance of the amplifier. The resulting network is shown in Fig. 15.112(b).

The total load on the source of 12 V is

$$Z_T = 8 \ \Omega + 8 \ \Omega + j \ 29.03 \ \Omega = 16 \ \Omega + j \ 29.03 \ \Omega$$
$$= 33.15 \ \Omega \angle 61.14°$$

and the current is

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{12 \text{ V} \angle 0°}{33.15 \ \Omega \angle 61.14°}$$
$$= 362 \text{ mA} \angle -61.14°$$

The power to the 8 Ω speaker is then

$$P_{woofer} = I^2R = (362 \text{ mA})^2(8 \ \Omega) = \textbf{1.05 W}$$

or about 1 W.

Consequently, the sound generated by the mid-range speaker far outweighs the response of the woofer (as it should).

For the tweeter in Fig. 15.112,

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(1.4 \text{ kHz})(3.9 \ \mu\text{F})} = 29.15 \ \Omega$$

and
$$Z_{tweeter} = R - j X_C = 8 \ \Omega - j \ 29.15 \ \Omega$$
$$= 30.23 \ \Omega \angle -74.65°$$

which, as for the woofer, is a poor match with the output impedance of the amplifier. The current

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{12 \text{ V} \angle 0°}{30.23 \ \Omega \angle -74.65°}$$
$$= 397 \text{ mA} \angle 74.65°$$

The power to the 8 Ω speaker is then

$$P_{tweeter} = I^2R = (397 \text{ mA})^2(8 \ \Omega) = \textbf{1.26 W}$$

or about 1.3 W.

Consequently, the sound generated by the mid-range speaker far outweighs the response of the tweeter also.

All in all, the mid-range speaker predominates at a frequency of 1.4 kHz for the crossover network in Fig. 15.111.

Let us now determine the impedance of the tweeter at 20 kHz and the impact of the woofer at this frequency.

For the tweeter:

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(20 \text{ kHz})(3.9 \ \mu\text{F})} = 2.04 \ \Omega$$

with
$$Z_{tweeter} = 8 \ \Omega - j \ 2.04 \ \Omega = 8.26 \ \Omega \angle -14.31°$$

Even though the magnitude of the impedance of the branch is not exactly 8 Ω, it is very close, and the speaker will receive a high level of power (actually 4.43 W).

For the woofer:

$$X_L = 2\pi f L = 2\pi(20 \text{ kHz})(3.3 \text{ mH}) = 414.69 \text{ } \Omega$$

with $\qquad Z_{\text{woofer}} = 8 \text{ } \Omega - j \text{ } 414.69 \text{ } \Omega = 414.77 \text{ } \Omega \angle 88.9°$

which is a terrible match with the output impedance of the amplifier. Therefore, the speaker will receive a very low level of power (6.69 mW $\cong 0.007$ W).

For all the calculations, note that the capacitive elements predominate at low frequencies, and the inductive elements at high frequencies. For the low frequencies, the reactance of the coil is quite small, permitting a full transfer of power to the speaker. For the high-frequency tweeter, the reactance of the capacitor is quite small, providing a direct path for power flow to the speaker.

Phase-Shift Power Control

In Chapter 11, the internal structure of a light dimmer was examined and its basic operation described. We can now turn our attention to how the power flow to the bulb is controlled.

If the dimmer were composed of simply resistive elements, all the voltages of the network would be in phase as shown in Fig. 15.113(a). If we assume that 20 V are required to turn on the triac in Fig. 11.68, the power will be distributed to the bulb for the period highlighted by the blue area of Fig. 15.113(a). For this situation, the bulb is close to full brightness since the applied voltage is available to the bulb for almost the entire cycle. To reduce the power to the bulb (and therefore reduce its brightness), the controlling voltage would need a lower peak voltage as shown in Fig. 15.113(b). In fact, the waveform in Fig. 15.113(b) is such that the turn-on voltage is not reached until the peak value occurs. In this case, power is delivered to the bulb for only half the cycle, and the brightness of the bulb is reduced. The problem with using only resistive elements in a dimmer now becomes apparent: The bulb can be made no dimmer than the situation depicted by Fig. 15.113(b). Any further reduction in the controlling voltage would reduce its peak value below the trigger level, and the bulb would never turn on.

This dilemma can be resolved by using a series combination of elements such as shown in Fig. 15.114(a) from the dimmer in Fig. 11.68. Note that the controlling voltage is the voltage across the capacitor, while the full line voltage of 120 V rms, 170 V peak, is across the entire branch. To describe the behavior of the network, let us examine the case defined by

(a)

(b)
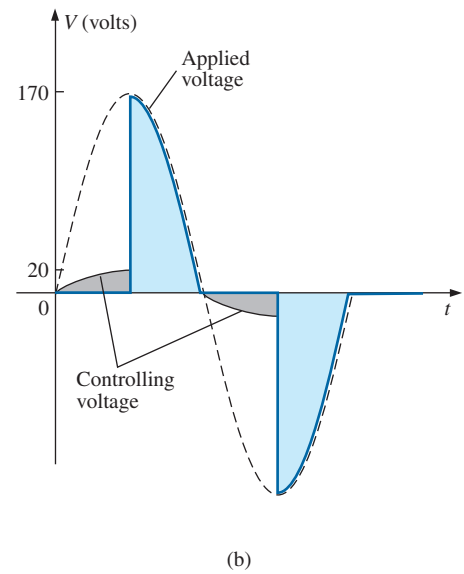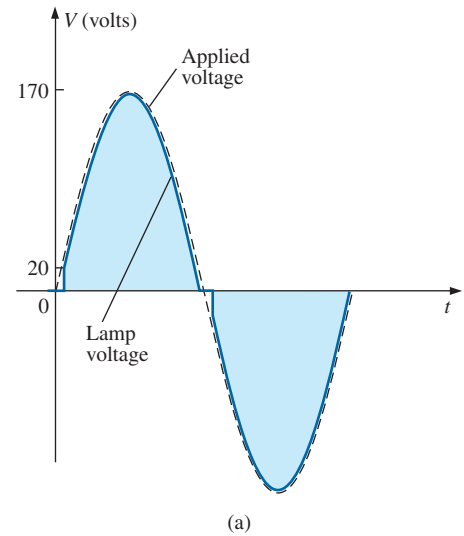
FIG. 15.113

Light dimmer: (a) with purely resistive elements; (b) half-cycle power distribution.

(a)

(b)
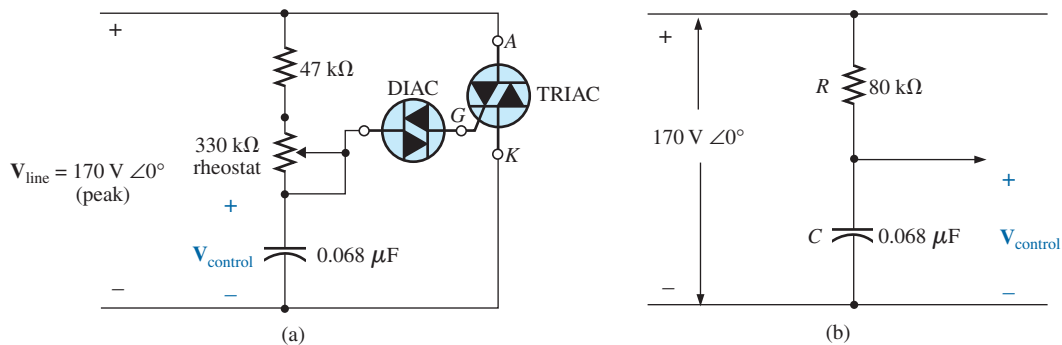
FIG. 15.114

Light dimmer: (a) from Fig. 11.68; (b) with rheostat set at 33 kΩ.

setting the potentiometer (used as a rheostat) to 1/10 its maximum value, or 33 kΩ. Combining the 33 kΩ with the fixed resistance of 47 kΩ results in a total resistance of 80 kΩ and the equivalent network in Fig. 15.114(b).

At 60 Hz, the reactance of the capacitor is

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(60 \text{ Hz})(62 \text{ } \mu\text{F})} = 42.78 \text{ k}\Omega$$

Applying the voltage divider rule:

$$\mathbf{V}_{control} = \frac{\mathbf{Z}_C \mathbf{V}_s}{\mathbf{Z}_R + \mathbf{Z}_C}$$
$$= \frac{(42.78 \text{ k}\Omega \angle -90°)(V_s \angle 0°)}{80 \text{ k}\Omega - j \, 42.78 \text{ k}\Omega} = \frac{42.78 \text{ k}\Omega \, V_s \angle -90°}{90.72 \text{ k}\Omega \angle -28.14°}$$
$$= 0.472 \, V_s \angle -61.86°$$

Using a peak value of 170 V:

$$\mathbf{V}_{control} = 0.472(170 \text{ V}) \angle -61.86°$$
$$= 80.24 \text{ V} \angle -61.86°$$

producing the waveform in Fig. 15.115(a). The result is a waveform with a phase shift of 61.86° (lagging the applied line voltage) and a relatively high peak value. The high peak value results in a quick transition to the 20 V turn-on level, and power is distributed to the bulb for the major portion of the applied signal. Recall from the discussion in Chapter 11 that the response in the negative region is a replica of that achieved in the positive region. If we reduced the potentiometer resistance further, the phase angle would be reduced, and the bulb would burn brighter. The situation is now very similar to that described for the response in Fig. 15.113(a). In other words, nothing has been gained thus far by using the capacitive element in the control network. However, let us now increase the potentiometer resistance to 200 kΩ and note the effect on the controlling voltage.

That is,

$$R_T = 200 \text{ k}\Omega + 47 \text{ k}\Omega = 247 \text{ k}\Omega$$
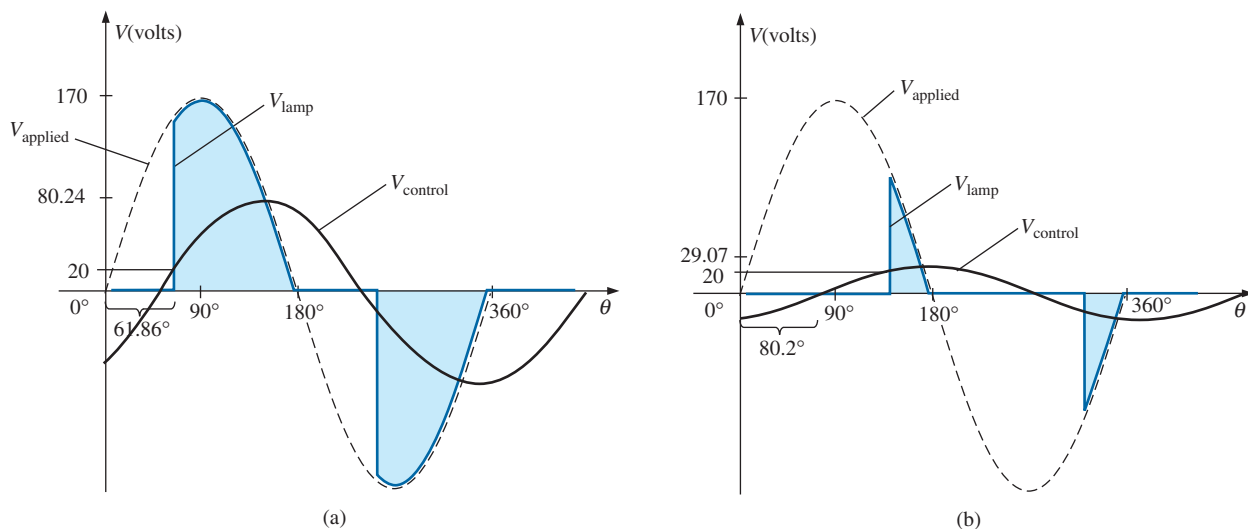
FIG. 15.115

Light dimmer in Fig. 11.68; (a) rheostat set at 33 kΩ; (b) rheostat set at 200 kΩ.

$$\mathbf{V}_{control} = \frac{\mathbf{Z}_C \mathbf{V}_s}{\mathbf{Z}_R + \mathbf{Z}_C}$$

$$= \frac{(42.78 \text{ k}\Omega \angle -90°)(V_s \angle 0°)}{247 \text{ k}\Omega - j42.78 \text{ k}\Omega} = \frac{42.78 \text{ k}\Omega \ V_s \angle -90°}{250.78 \text{ k}\Omega \angle -9.8°}$$

$$= 0.171 \ V_s \angle -80.2°$$

and using a peak value of 170 V, we have

$$\mathbf{V}_{control} = 0.171(170 \text{ V}) \angle -80.2°$$

$$= 29.07 \text{ V} \angle -80.2°$$

The peak value has been substantially reduced to only 29.07 V, and the phase-shift angle has increased to 80.2°. The result, as depicted by Fig. 15.115(b), is that the firing potential of 20 V is not reached until near the end of the positive region of the applied voltage. Power is delivered to the bulb for only a very short period of time, causing the bulb to be quite dim, significantly dimmer than obtained from the response in Fig. 15.113(b).

A conduction angle less than 90° is therefore possible due only to the phase shift introduced by the series *R-C* combination. Thus, it is possible to construct a network of some significance with a rather simple pair of elements.

15.15 COMPUTER ANALYSIS

PSpice

Series *R-L-C* Circuit The *R-L-C* network in Fig. 15.35 is now analyzed using OrCAD Capture. Since the inductive and capacitive reactances cannot be entered onto the screen, the associated inductive and capacitive levels were first determined as follows:

$$X_L = 2\pi f L \Rightarrow L = \frac{X_L}{2\pi f} = \frac{7 \ \Omega}{2\pi(1 \text{ kHz})} = 1.114 \text{ mH}$$

$$X_C = \frac{1}{2\pi f C} \Rightarrow C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(1 \text{ kHz})3 \ \Omega} = 53.05 \ \mu\text{F}$$

Enter the values into the schematic as shown in Fig. 15.116. For the ac source, the sequence is **Place part** icon-**SOURCE-VSIN-OK** with **VOFF** set at 0 V, **VAMPL** set at 70.7 V (the peak value of the applied sinusoidal source in Fig. 15.35), and **FREQ** = 1 kHz. Double-click on the source symbol and the **Property Editor** appears, confirming the above choices and showing that **DF** = 0 s, **PHASE** = 0°, and **TD** = 0 s as set by the default levels. You are now ready to do an analysis of the circuit for the fixed frequency of 1 kHz.

The simulation process is initiated by first selecting the **New Simulation Profile** icon and inserting **PSpice 15-1** as the **Name** followed by **Create.** The **Simulation Settings** dialog appears and since you are continuing to plot the results against time, select the **Time Domain (Transient)** option under **Analysis type.** Since the period of each **cycle** of the applied source is 1 ms, set the **Run to time** at 5 ms so that five cycles appear. Leave the **Start saving data after** at 0 s even though there will be an oscillatory period for the reactive elements before the circuit settles down. Set the **Maximum step size** at 5 ms/1000 = 5 μs. Finally, select **OK** followed by the **Run PSpice** key. The result is a blank screen with an *x*-axis extending from 0 s to 5 ms.
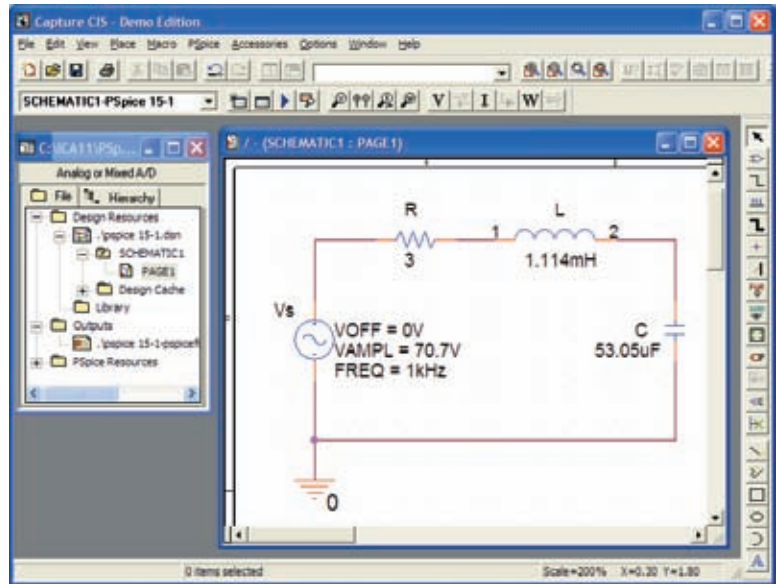
FIG. 15.116
Using PSpice to analyze a series R-L-C ac circuit.

The first quantity of interest is the current through the circuit, so select **Trace-Add-Trace** followed by **I(R)** and **OK.** The resulting plot in Fig. 15.117 clearly shows that there is a period of storing and discharging of the reactive elements before a steady-state level is established. It would appear that after 3 ms, steady-state conditions have been essentially established. Select the **Toggle cursor** key, and left-click; a cursor appears that can be moved along the axis near the maximum value around 1.4 ms.
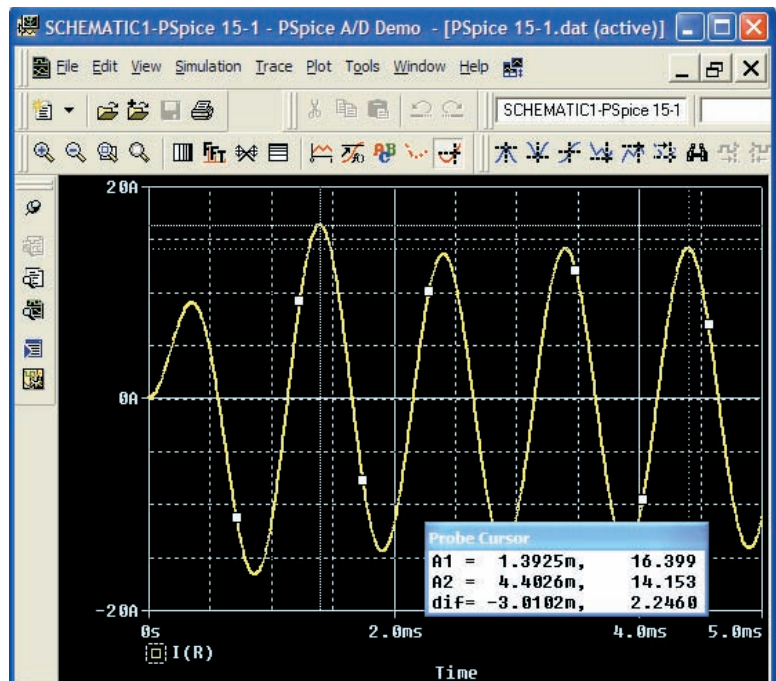
FIG. 15.117
A plot of the current for the circuit in Fig. 15.116 showing the transition from the transient state to the steady-state response.

In fact, the cursor reveals a maximum value of 16.4 A which exceeds the steady-state solution by over 2 A. Right-click to establish a second cursor on the screen that can be placed near the steady-state peak around 4.4 ms. The resulting peak value is about 14.15 A which is a match with the longhand solution for Fig. 15.35. We will therefore assume that steady-state conditions have been established for the circuit after 4 ms.

Now add the source voltage through **Trace-Add Trace-V(Vs:+)-OK** to obtain the multiple plot at the bottom of Fig. 15.118. For the voltage across the coil, the sequence **Plot-Add Plot to Window-Trace-Add Trace-V(L:1)-V(L:2)** results in the plot appearing at the top of Fig. 15.118. Take special note that the **Trace Expression** is **V(L:1)–V(L:2)** rather than just **V(L:1)** because **V(L:1)** would be the voltage from that point to ground which would include the voltage across the capacitor. In addition, the − sign between the two comes from the **Functions or Macros** list at the right of the **Add Traces** dialog box. Finally, since we know that the waveforms are fairly steady after 3 ms, cut away the waveforms before 3 ms with **Plot-Axis Settings-X axis-User Defined-3ms to 5ms-OK** to obtain the two cycles of Fig. 15.118. Now you can clearly see that the peak value of the voltage across the coil is 100 V to match the analysis of Fig. 15.35. It is also clear that the applied voltage leads the input current by an angle that can be determined using the cursors. First activate the cursor option by selecting the cursor key (a red plot through the origin) in the second toolbar down from the menu bar. Then select **V(Vs:+)** at the bottom left of the screen with a left click, and set it at that point where the applied voltage passes through the horizontal axis with a positive slope. The result is **A1** = 4 ms at −4.243 μV ≅ 0 V. Then select **I(R)** at the bottom left of the screen with a right click, and place it at the point where the current waveform passes through the horizontal axis with a positive slope. The result is **A2** = 4.15 ms at −55.15 mA = 0.55 A ≅
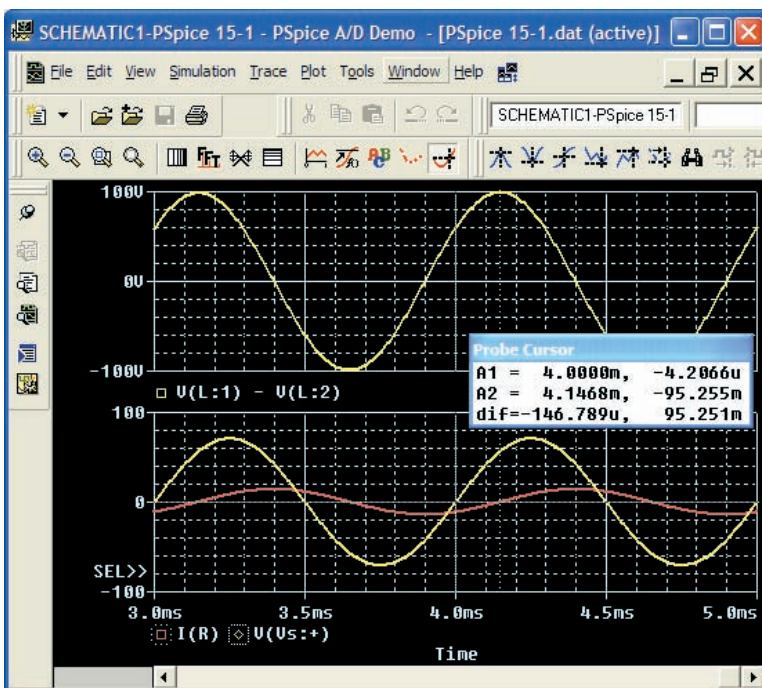

FIG. 15.118

A plot of the steady-state response (t > 3 ms) for v_L, v_s, and i for the circuit in Fig. 15.116.

0 A (compared to a peak value of 14.14 A). At the bottom of the **Probe Cursor** dialog box, the time difference is 147.24 μs.

Now set up the ratio

$$\frac{147.24\ \mu s}{1000\ \mu s} = \frac{\theta}{360°}$$

$$\theta = 52.99°$$

The phase angle by which the applied voltage leads the source is 52.99° which is very close to the theoretical solution of 53.13° obtained in Fig. 15.39. Increasing the number of data points for the plot would have increased the accuracy level and brought the results closer to 53.13°.

Multisim

We now examine the response of a network versus frequency rather than time using the network in Fig. 15.86 which now appears on the schematic in Fig. 15.119. The ac current source appears as **AC_CURRENT_ SOURCE** under the **SIGNAL_CURRENT_SOURCES Family** listing. Note that the current source was given an amplitude of 1 A to establish a magnitude match between the response of the voltage across the network and the impedance of the network. That is,

$$|Z_T| = \left|\frac{V_s}{I_s}\right| = \left|\frac{V_s}{1\ A}\right| = |V_s|$$

Before applying computer methods, we should develop a rough idea of what to expect so that we have something to which to compare the computer solution. At very high frequencies such as 1 MHz, the impedance of the inductive element will be about 25 kΩ which when placed in parallel with the 220 Ω will look like an open circuit. The result is that as the fre-
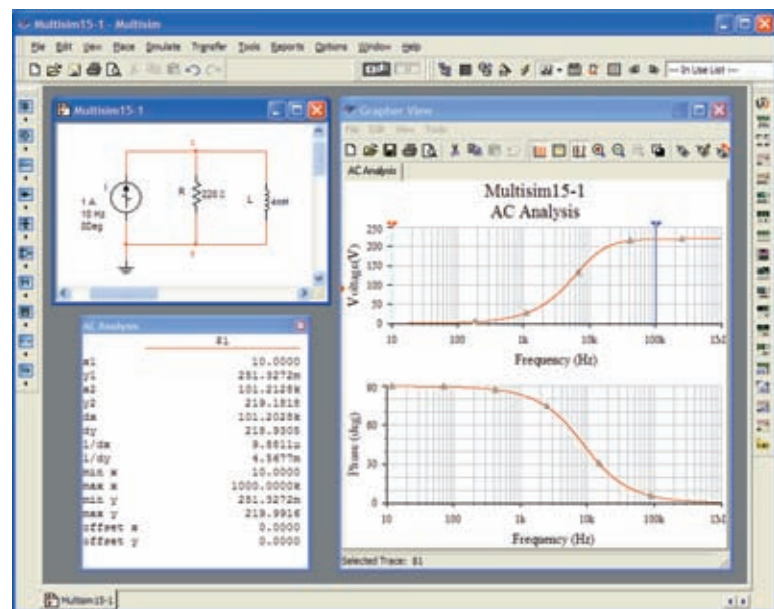
FIG. 15.119

Obtaining an impedance plot for a parallel R-L network using Multisim.

quency gets very high, we should expect the impedance of the network to approach the 220 Ω level of the resistor. In addition, since the network will take on resistive characteristics at very high frequencies, the angle associated with the input impedance should also approach 0 Ω. At very low frequencies the reactance of the inductive element will be much less than the 220 Ω of the resistor, and the network will take on inductive characteristics. In fact, at, say, 10 Hz, the reactance of the inductor is only about 0.25 Ω which is very close to a short-circuit equivalent compared to the parallel 220 Ω resistor. The result is that the impedance of the network is very close to 0 Ω at very low frequencies. Again, since the inductive effects are so strong at low frequencies, the phase angle associated with the input impedance should be very close to 90°.

Now for the computer analysis. The current source, the resistor element, and the inductor are all placed and connected using procedures described in detail in earlier chapters. However, there is one big difference this time: Since the output will be plotted versus frequency, the **AC Analysis Magnitude** in the **SIGNAL_CURRENT_SOURCES** dialog box for the source must be set to 1 A. In this case, the default level of **1A** matches that of the applied source, so you were set even if you failed to check the setting. In the future, however, a voltage or current source may be used that does not have an amplitude of 1, and proper entries must be made to this listing.

For the simulation, first apply the sequence **Simulate-Analyses-AC Analysis** to obtain the **AC Analysis** dialog box. Set the **Start frequency** at **10 Hz** so that you have entries at very low frequencies, and set the **Stop frequency** at **1MHz** so that you have data points at the other end of the spectrum. The **Sweep type** can remain **Decade,** but the number of points per decade will be 1000 so that you obtain a detailed plot. Set the **Vertical scale** on **Linear.** Within **Output variables,** only one node, **1,** is defined. Shifting it over to the **Selected variables for analysis** column using the **Add** keypad and then hitting the **Simulate** key results in the two plots in Fig. 15.119. Select the **Show/Hide Grid** key to place the grid on the graph, and select the **Show/Hide Cursors** key to place the **AC Analysis** dialog box appearing in Fig. 15.119. Since two graphs are present, define the one you are working on by clicking on the **Voltage** or **Phase** heading on the left side of each plot. A small red arrow appears when selected so you know which is the active plot. When setting up the cursors, be sure that you have activated the correct plot. When the red cursor is moved to 10 Hz (**x1**), you find that the voltage across the network is only 0.251 V (**y1**), resulting in an input impedance of only 0.25 Ω—quite small and matching your theoretical prediction. In addition, note that the phase angle is essentially at 90° in the other plot, confirming your other assumption above—a totally inductive network. If you set the blue cursor near 100 kHz (**x2** = 102.3 kHz), you find that the impedance at 219.2 Ω (**y2**) is closing in on the resistance of the parallel resistor of 220 Ω, again confirming the preliminary analysis above. As noted in the bottom of the **AC Analysis** box, the maximum value of the voltage is 219.99 Ω or essentially 220 Ω at 1 MHz. Before leaving the plot, note the advantages of using a log axis when you want a response over a wide frequency range.

PROBLEMS

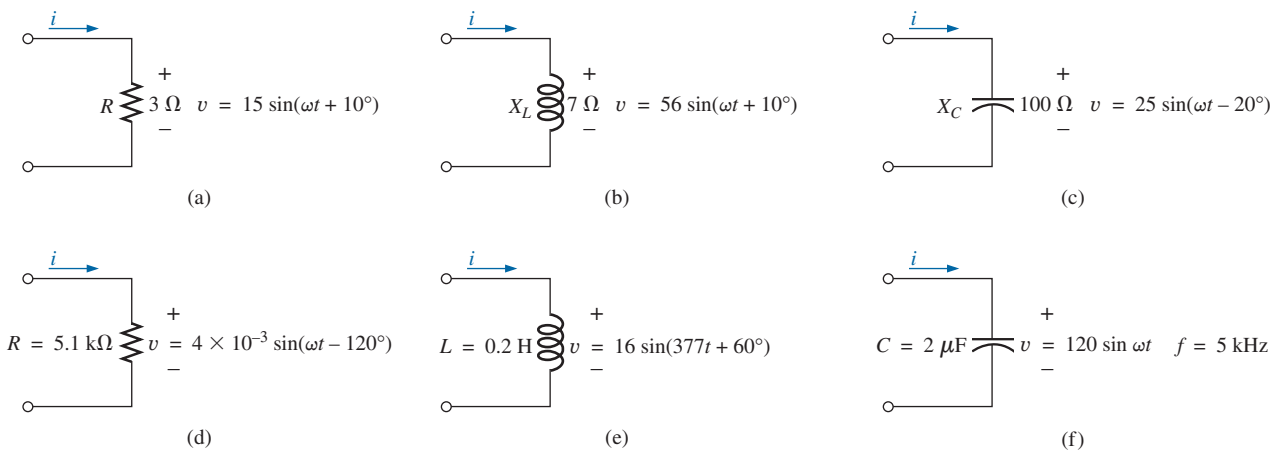## SECTION 15.2 Impedance and the Phasor Diagram

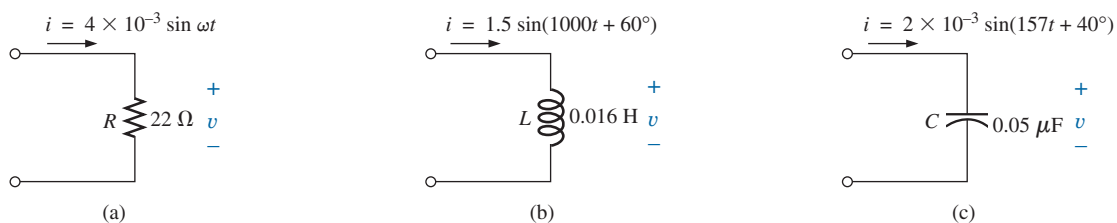1. Express the impedances in Fig. 15.120 in both polar and rectangular forms.

$R = 6.8\ \Omega$
(a)

$L = 1.2\ \text{H}$
$\omega = 377\ \text{rad/s}$
(b)

$L = 0.05\ \text{H}$
$f = 50\ \text{Hz}$
(c)

$C = 10\ \mu\text{F}$
$\omega = 100\ \text{rad/s}$
(d)

$C = 0.05\ \mu\text{F}$
$f = 10\ \text{kHz}$
(e)

$R = 220\ \Omega$
$\omega = 157\ \text{rad/s}$
(f)
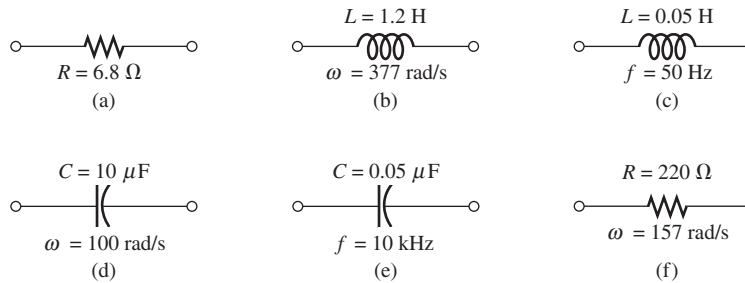
FIG. 15.120
Problem 1.

2. Find the current i for the elements in Fig. 15.121 using complex algebra. Sketch the waveforms for v and i on the same set of axes.

$R \;\; 3\ \Omega$ $v = 15\sin(\omega t + 10°)$
(a)

$X_L \;\; 7\ \Omega$ $v = 56\sin(\omega t + 10°)$
(b)

$X_C \;\; 100\ \Omega$ $v = 25\sin(\omega t - 20°)$
(c)

$R = 5.1\ \text{k}\Omega$ $v = 4 \times 10^{-3}\sin(\omega t - 120°)$
(d)

$L = 0.2\ \text{H}$ $v = 16\sin(377t + 60°)$
(e)

$C = 2\ \mu\text{F}$ $v = 120\sin\omega t$ $f = 5\ \text{kHz}$
(f)

FIG. 15.121
Problem 2.

3. Find the voltage v for the elements in Fig. 15.122 using complex algebra. Sketch the waveforms of v and i on the same set of axes.

$i = 4 \times 10^{-3}\sin\omega t$

$R \;\; 22\ \Omega$ v
(a)

$i = 1.5\sin(1000t + 60°)$

$L \;\; 0.016\ \text{H}$ v
(b)

$i = 2 \times 10^{-3}\sin(157t + 40°)$

$C \;\; 0.05\ \mu\text{F}$ v
(c)

FIG. 15.122
Problem 3.

SECTION 15.3   Series Configuration

4. Calculate the total impedance of the circuits in Fig. 15.123. Express your answer in rectangular and polar forms, and draw the impedance diagram.
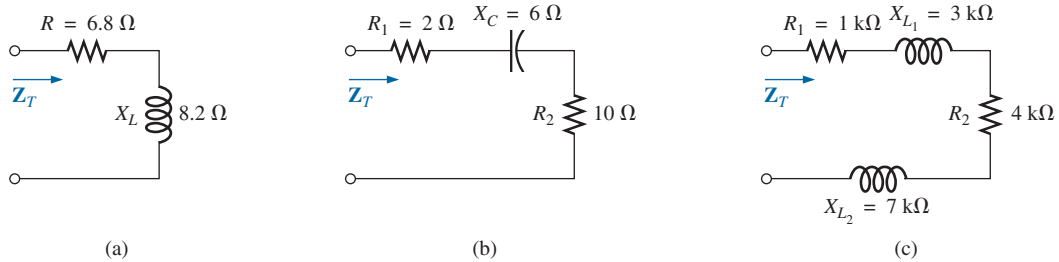
FIG. 15.123
Problem 4.

5. Calculate the total impedance of the circuits in Fig. 15.124. Express your answer in rectangular and polar forms, and draw the impedance diagram.
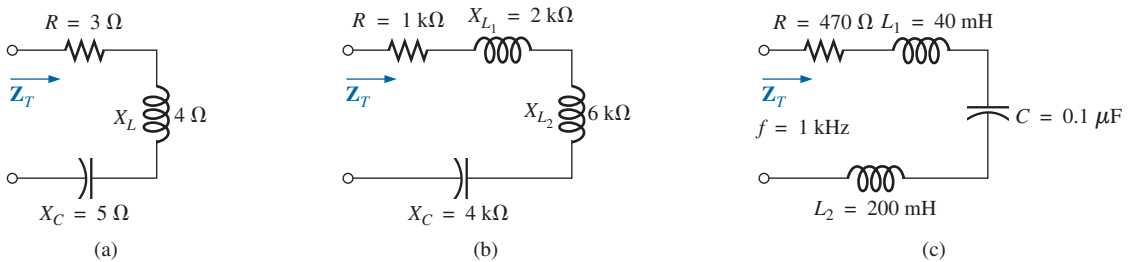
FIG. 15.124
Problem 5.

6. Find the type and impedance in ohms of the series circuit elements that must be in the closed container in Fig. 15.125 for the indicated voltages and currents to exist at the input terminals. (Find the simplest series circuit that will satisfy the indicated conditions.)
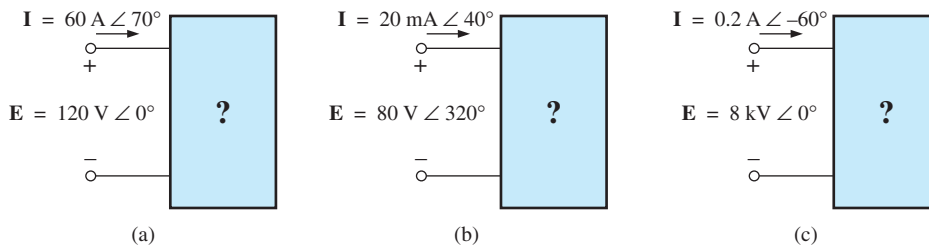
FIG. 15.125
Problems 6 and 27.

7. For the circuit in Fig. 15.126:
   a. Find the total impedance $\mathbf{Z}_T$ in polar form.
   b. Draw the impedance diagram.
   c. Find the current $\mathbf{I}$ and the voltages $\mathbf{V}_R$ and $\mathbf{V}_L$ in phasor form.
   d. Draw the phasor diagram of the voltages $\mathbf{E}$, $\mathbf{V}_R$, and $\mathbf{V}_L$, and the current $\mathbf{I}$.
   e. Verify Kirchhoff's voltage law around the closed loop.
   f. Find the average power delivered to the circuit.
   g. Find the power factor of the circuit, and indicate whether it is leading or lagging.
   h. Find the sinusoidal expressions for the voltages and current if the frequency is 60 Hz.
   i. Plot the waveforms for the voltages and current on the same set of axes.
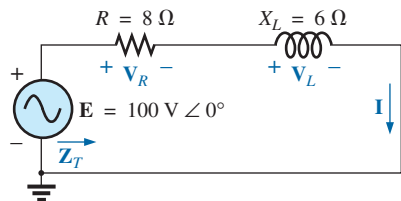
FIG. 15.126
Problems 7 and 48.

8. Repeat Problem 7 for the circuit in Fig. 15.127, replacing $\mathbf{V}_L$ with $\mathbf{V}_C$ in parts (c) and (d).
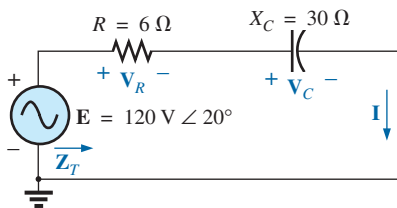
FIG. 15.127
Problem 8.

9. Given the network in Fig. 15.128:
   a. Determine $\mathbf{Z}_T$.
   b. Find $\mathbf{I}$.
   c. Calculate $\mathbf{V}_R$ and $\mathbf{V}_C$.
   d. Find P and F_p.
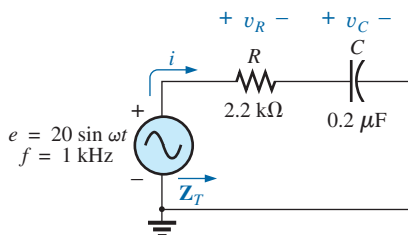
FIG. 15.128
Problems 9 and 50.

10. For the circuit in Fig. 15.129:
   a. Find the total impedance $\mathbf{Z}_T$ in polar form.
   b. Draw the impedance diagram.
   c. Find the value of C in microfarads and L in henries.
   d. Find the current $\mathbf{I}$ and the voltages $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$ in phasor form.
   e. Draw the phasor diagram of the voltages $\mathbf{E}$, $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$, and the current $\mathbf{I}$.
   f. Verify Kirchhoff's voltage law around the closed loop.
   g. Find the average power delivered to the circuit.
   h. Find the power factor of the circuit, and indicate whether it is leading or lagging.
   i. Find the sinusoidal expressions for the voltages and current.
   j. Plot the waveforms for the voltages and current on the same set of axes.
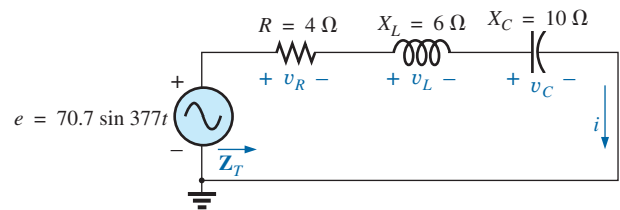
FIG. 15.129
Problem 10.

11. Repeat Problem 10 for the circuit in Fig. 15.130.
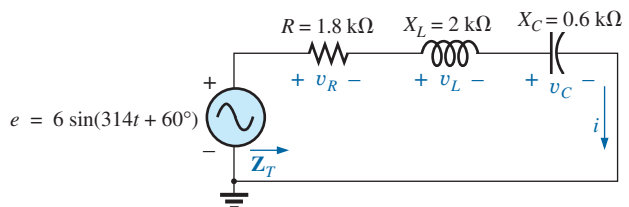
FIG. 15.130
Problem 11.

12. Using the oscilloscope reading in Fig. 15.131, determine the resistance R.
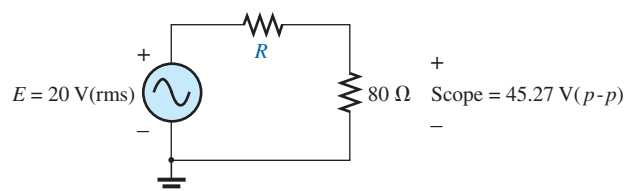

FIG. 15.131
Problem 12.

*13. Using the DMM current reading and the oscilloscope mea-
surement in Fig. 15.132:
  a. Determine the inductance L.
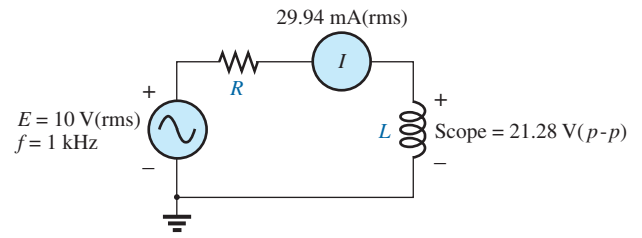 b. Find the resistance R.

FIG. 15.132
Problem 13.

*14. Using the oscilloscope reading in Fig. 15.133, determine the
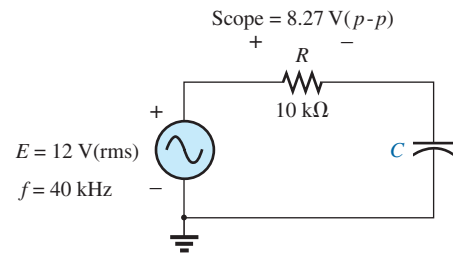capacitance C.

FIG. 15.133
Problem 14.

SECTION 15.4 Voltage Divider Rule

15. Calculate the voltages V_1 and V_2 for the circuits in Fig.
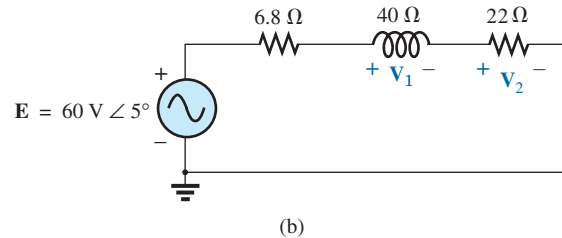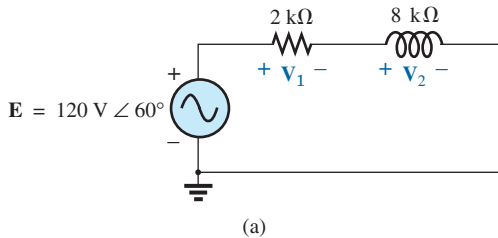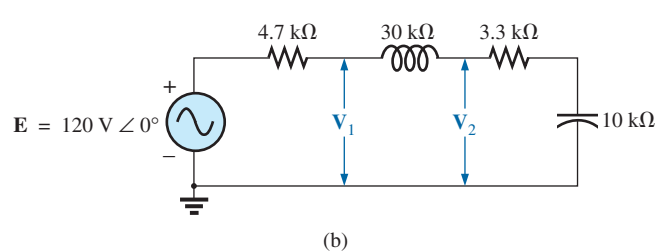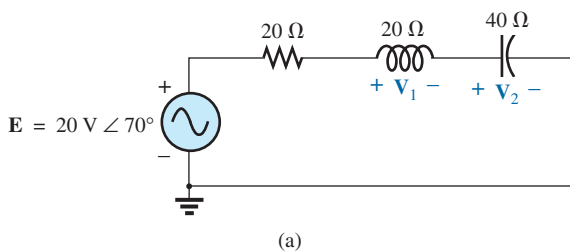15.134 in phasor form using the voltage divider rule.

(a)

(b)

FIG. 15.134
Problem 15.

16. Calculate the voltages V_1 and V_2 for the circuits in Fig.
15.135 in phasor form using the voltage divider rule.

(a)

(b)

FIG. 15.135
Problem 16.

*17. For the circuit in Fig. 15.136:
   a. Determine **I**, $\mathbf{V}_R$, and $\mathbf{V}_C$ in phasor form.
   b. Calculate the total power factor, and indicate whether it is leading or lagging.
   c. Calculate the average power delivered to the circuit.
   d. Draw the impedance diagram.
   e. Draw the phasor diagram of the voltages **E**, $\mathbf{V}_R$, and $\mathbf{V}_C$, and the current **I**.
   f. Find the voltages $\mathbf{V}_R$ and $\mathbf{V}_C$ using the voltage divider rule, and compare them with the results of part (a) above.
   g. Draw the equivalent series circuit of the above as far as the total impedance and the current i are concerned.
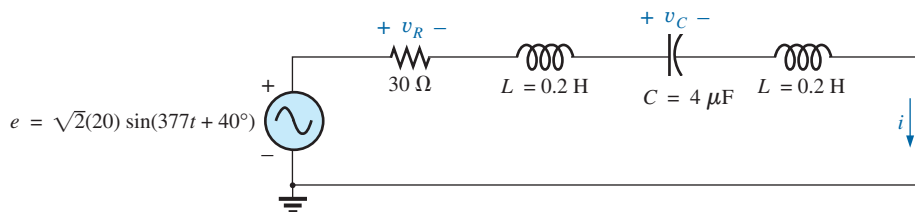
FIG. 15.136
Problems 17, 18, and 51.

*18. Repeat Problem 17 if the capacitance is changed to 220 μF.

19. An electrical load has a power factor of 0.8 lagging. It dissipates 8 kW at a voltage of 200 V. Calculate the impedance of this load in rectangular coordinates.

*20. Find the series element or elements that must be in the enclosed container in Fig. 15.137 to satisfy the following conditions:
   a. Average power to circuit = 300 W.
   b. Circuit has a lagging power factor.
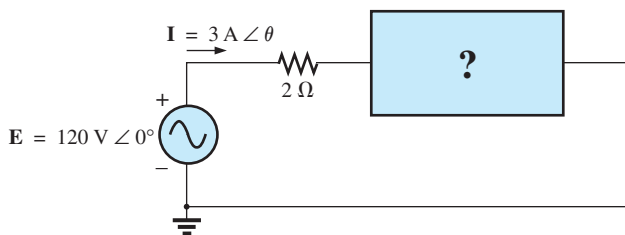
FIG. 15.137
Problem 20.

SECTION 15.5   Frequency Response for Series ac Circuits

*21. For the circuit in Fig. 15.138:
   a. Plot Z_T and θ_T versus frequency for a frequency range of zero to 20 kHz.
   b. Plot V_L versus frequency for the frequency range of part (a).
   c. Plot θ_L versus frequency for the frequency range of part (a).
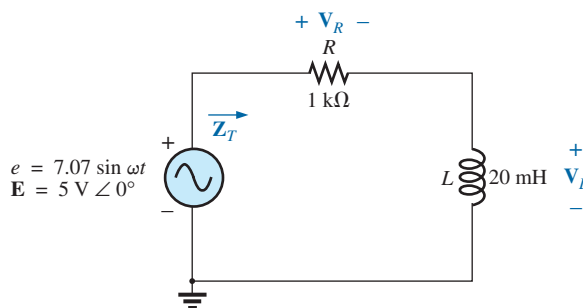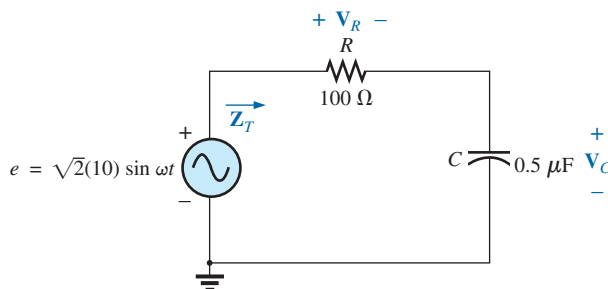   d. Plot V_R versus frequency for the frequency range of part (a).

FIG. 15.138
Problem 21.

*22. For the circuit in Fig. 15.139:
   a. Plot Z_T and θ_T versus frequency for a frequency range of zero to 10 kHz.
   b. Plot V_C versus frequency for the frequency range of part (a).

FIG. 15.139
Problem 22.

c. Plot θ_C versus frequency for the frequency range of part (a).

d. Plot V_R versus frequency for the frequency range of part (a).

*23. For the series *R-L-C* circuit in Fig. 15.140:

a. Plot Z_T and θ_T versus frequency for a frequency range of zero to 20 kHz in increments of 1 kHz.

b. Plot V_C (magnitude only) versus frequency for the same frequency range of part (a).

c. Plot I (magnitude only) versus frequency for the same frequency range of part (a).
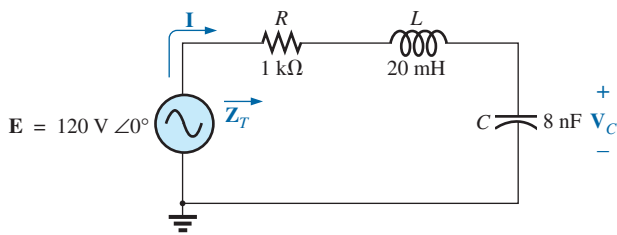
FIG. 15.140

Problem 23.

24. For the series *R-C* circuit in Fig. 15.141:

a. Determine the frequency at which $X_C = R$.

b. Develop a mental image of the change in total impedance with frequency without resorting to a single calculation.

c. Find the total impedance at 100 Hz and 10 kHz, and compare your answer with the assumptions of part (b).

d. Plot the curve of V_C versus frequency.

e. Find the phase angle of the total impedance at $f = 40$ kHz. Is the network resistive or capacitive at this frequency?
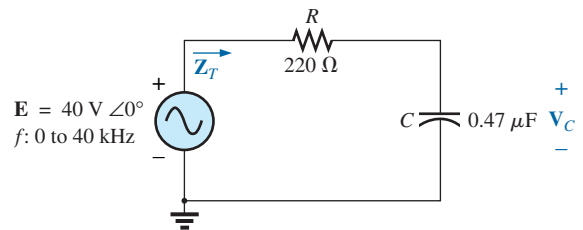
FIG. 15.141

Problem 24.

SECTION 15.7   Admittance and Susceptance

25. Find the total admittance and impedance of the circuits in Fig. 15.142. Identify the values of conductance and susceptance, and draw the admittance diagram.

26. Find the total admittance and impedance of the circuits in Fig. 15.143. Identify the values of conductance and susceptance, and draw the admittance diagram.
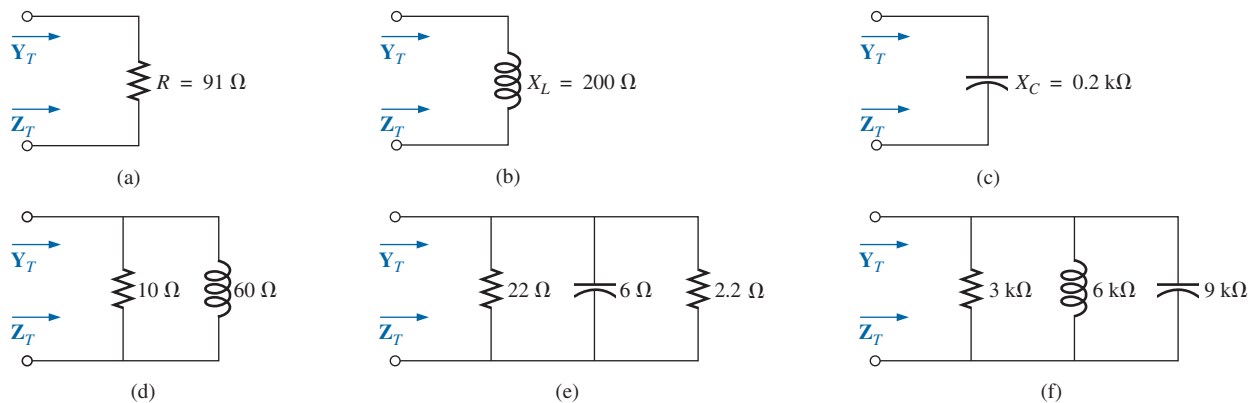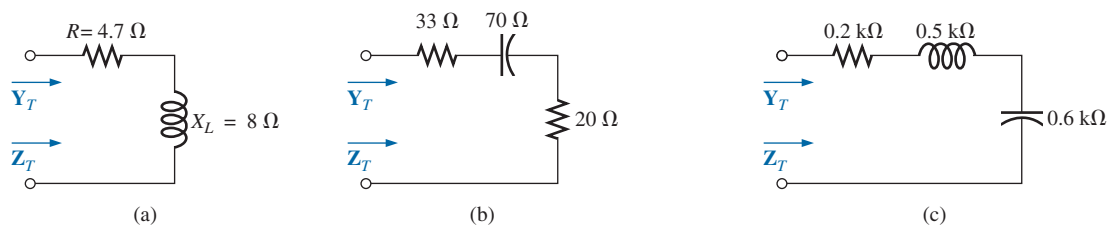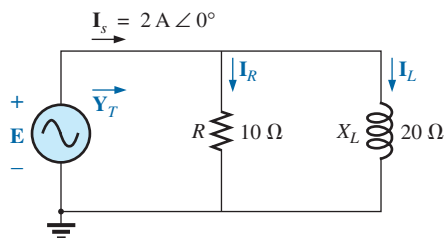

FIG. 15.142

Problem 25.

FIG. 15.143

Problem 26.

27. Repeat Problem 6 for the parallel circuit elements that must be in the closed container for the same voltage and current to exist at the input terminals. (Find the simplest parallel circuit that will satisfy the conditions indicated.)

SECTION 15.8   Parallel ac Networks

28. For the circuit in Fig. 15.144:
    a. Find the total admittance Y_T in polar form.
    b. Draw the admittance diagram.
    c. Find the voltage E and the currents I_R and I_L in phasor form.
    d. Draw the phasor diagram of the currents I_s, I_R, and I_L, and the voltage E.
    e. Verify Kirchhoff's current law at one node.
    f. Find the average power delivered to the circuit.
    g. Find the power factor of the circuit, and indicate whether it is leading or lagging.
    h. Find the sinusoidal expressions for the currents and voltage if the frequency is 60 Hz.
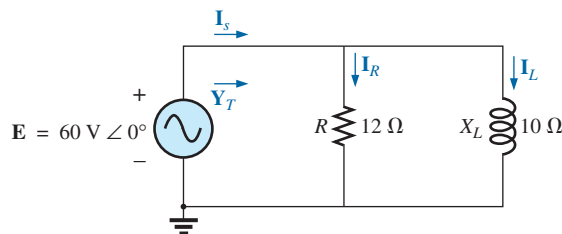   i. Plot the waveforms for the currents and voltage on the same set of axes.

FIG. 15.144
Problem 28.

29. Repeat Problem 28 for the circuit in Fig. 15.145, replacing I_L with I_C in parts (c) and (d).
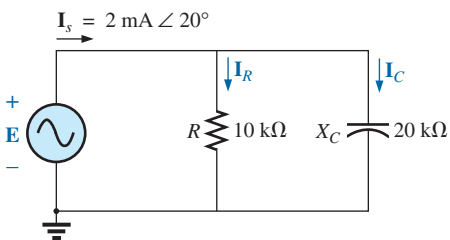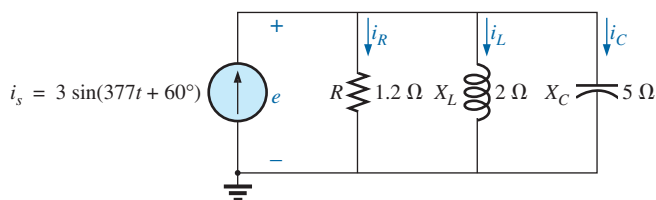

FIG. 15.145
Problem 29.

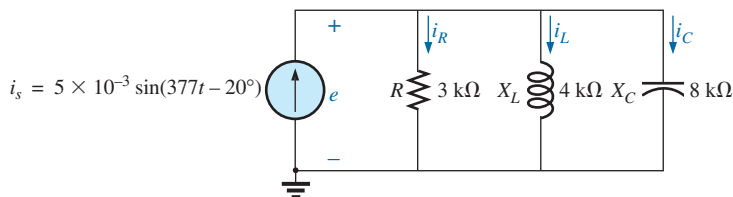30. Repeat Problem 28 for the circuit in Fig. 15.146, replacing E with I_s in part (c).

FIG. 15.146
Problems 30 and 49.

31. For the circuit in Fig. 15.147:
    a. Find the total admittance and impedance in polar form.
    b. Draw the admittance and impedance diagrams.
    c. Find the value of C in microfarads and L in henries.
    d. Find the voltage E and currents I_R, I_L, and I_C in phasor form.
    e. Draw the phasor diagram of the currents I_s, I_R, I_L, and I_C, and the voltage E.
    f. Verify Kirchhoff's current law at one node.
    g. Find the average power delivered to the circuit.
    h. Find the power factor of the circuit, and indicate whether it is leading or lagging.
    i. Find the sinusoidal expressions for the currents and voltage.
    j. Plot the waveforms for the currents and voltage on the same set of axes.

FIG. 15.147
Problem 31.

32. Repeat Problem 31 for the circuit in Fig. 15.148.

FIG. 15.148
Problem 32.

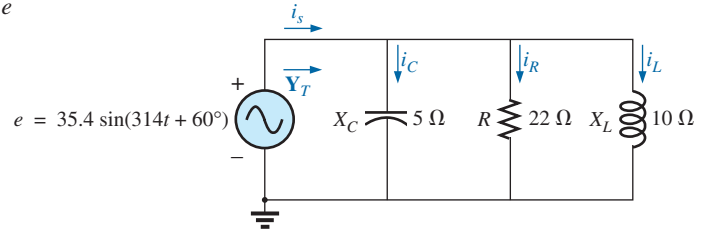**33.** Repeat Problem 31 for the circuit in Fig. 15.149, replacing e with i_s in part (d).

FIG. 15.149
Problem 33.

SECTION 15.9 Current Divider Rule

34. Calculate the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in Fig. 15.150 in phasor form using the current divider rule.

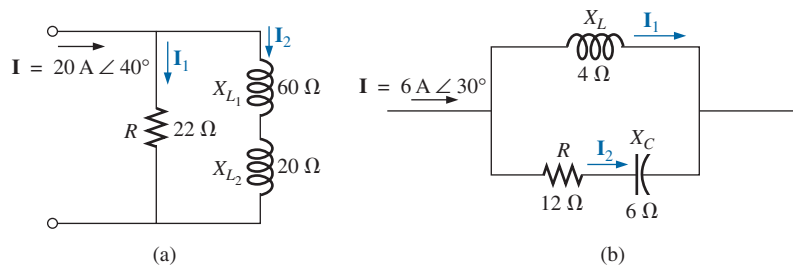(a)   (b)

FIG. 15.150
Problem 34.

SECTION 15.10 Frequency Response of Parallel Elements

*35. For the parallel *R-C* network in Fig. 15.151:
  a. Plot Z_T and θ_T versus frequency for a frequency range of zero to 20 kHz.
  b. Plot V_C versus frequency for the frequency range of part (a).
  c. Plot I_R versus frequency for the frequency range of part (a).

FIG. 15.152
Problems 36 and 38.

37. Plot Y_T and θ_T (of $\mathbf{Y}_T = Y_T \angle \theta_T$) for a frequency range of zero to 20 kHz for the network in Fig. 15.151.

38. Plot Y_T and θ_T (of $\mathbf{Y}_T = Y_T \angle \theta_T$) for a frequency range of zero to 10 kHz for the network in Fig. 15.152.

39. For the parallel *R-L-C* network in Fig. 15.153:
  a. Plot Y_T and θ_T (of $\mathbf{Y}_T = Y_T \angle \theta_T$) for a frequency range of zero to 20 kHz.
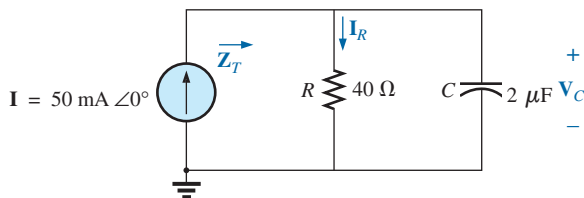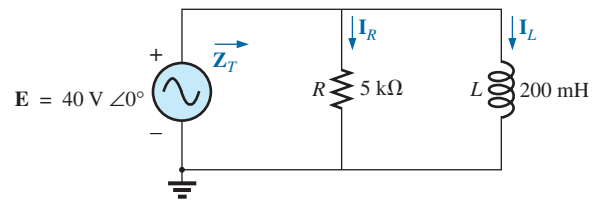

FIG. 15.151
Problems 35 and 37.

*36. For the parallel *R-L* network in Fig. 15.152:
  a. Plot Z_T and θ_T versus frequency for a frequency range of zero to 10 kHz.
  b. Plot I_L versus frequency for the frequency range of part (a).
  c. Plot I_R versus frequency for the frequency range of part (a).
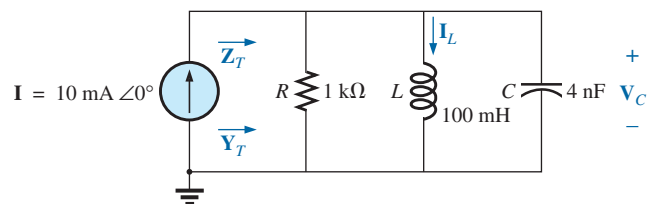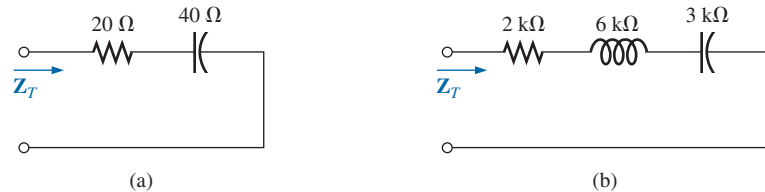

FIG. 15.153
Problem 39.

b. Repeat part (a) for Z_T and θ_T (of $\mathbf{Z}_T = Z_T \angle \theta_T$).
c. Plot V_C versus frequency for the frequency range of part (a).
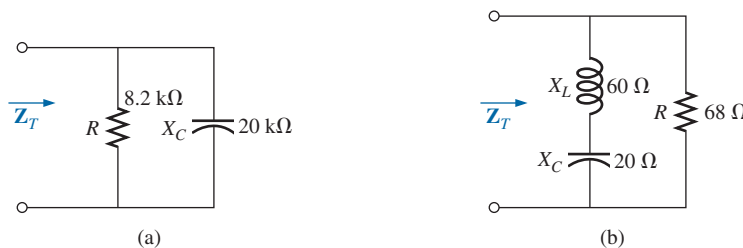d. Plot I_L versus frequency for the frequency range of part (a).

SECTION 15.12 Equivalent Circuits

40. For the series circuits in Fig. 15.154, find a parallel circuit that will have the same total impedance ($\mathbf{Z}_T$).
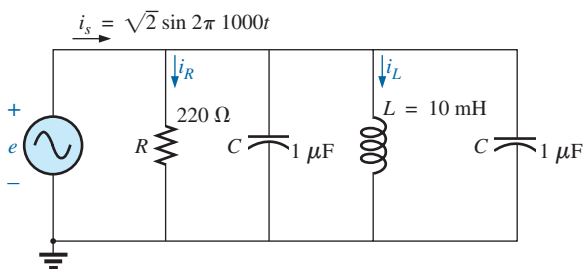

FIG. 15.154
Problem 40.

41. For the parallel circuits in Fig. 15.155, find a series circuit that will have the same total impedance.

FIG. 15.155
Problem 41.

42. For the network in Fig. 15.156:
    a. Calculate $\mathbf{E}$, $\mathbf{I}_R$, and $\mathbf{I}_L$ in phasor form.
    b. Calculate the total power factor, and indicate whether it is leading or lagging.
    c. Calculate the average power delivered to the circuit.
    d. Draw the admittance diagram.
    e. Draw the phasor diagram of the currents $\mathbf{I}_s$, $\mathbf{I}_R$, and $\mathbf{I}_L$, and the voltage $\mathbf{E}$.

f. Find the current $\mathbf{I}_C$ for each capacitor using only Kirchhoff's current law.
g. Find the series circuit of one resistive and reactive element that will have the same impedance as the original circuit.

*43. Repeat Problem 42 if the inductance is changed to 1 H.

44. Find the element or elements that must be in the closed container in Fig. 15.157 to satisfy the following conditions. (Find the simplest parallel circuit that will satisfy the indicated conditions.)
   a. Average power to the circuit = 3000 W.
    b. Circuit has a lagging power factor.

FIG. 15.156
Problems 42 and 43.
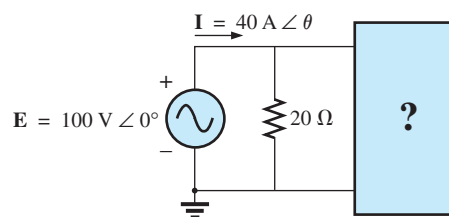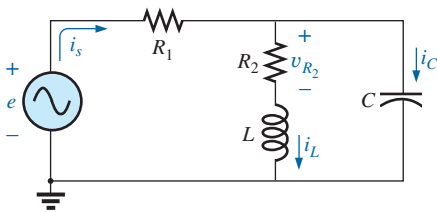

FIG. 15.157
Problem 44.

SECTION 15.13   Phase Measurements

45. For the circuit in Fig. 15.158, determine the phase relationship between the following using a dual-trace oscilloscope. The circuit can be reconstructed differently for each part, but do not use sensing resistors. Show all connections on a redrawn diagram.
   a. e and v_C
   b. e and i_s
   c. e and v_L
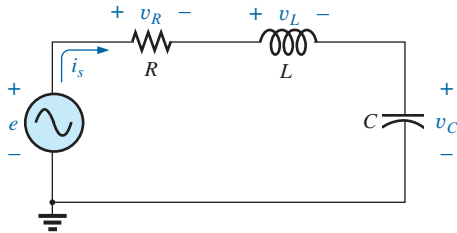
FIG. 15.158
Problem 45.

46. For the network in Fig. 15.159, determine the phase relationship between the following using a dual-trace oscilloscope. The network must remain as constructed in Fig. 15.159, but sensing resistors can be introduced. Show all connections on a redrawn diagram.
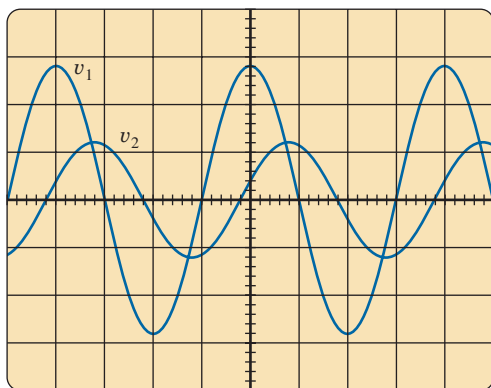

FIG. 15.159
Problem 46.

   a. e and v_{R_2}
   b. e and i_s
   c. i_L and i_C

47. For the oscilloscope traces in Fig. 15.160:
   a. Determine the phase relationship between the waveforms, and indicate which one leads or lags.
   b. Determine the peak-to-peak and rms values of each waveform.
   c. Find the frequency of each waveform.

SECTION 15.15   Computer Analysis

PSpice or Multisim

48. For the network in Fig. 15.126 (use $f = 1$ kHz):
   a. Determine the rms values of the voltages $\mathbf{V}_R$ and $\mathbf{V}_L$ and the current $\mathbf{I}$.
   b. Plot v_R, v_L, and i versus time on separate plots.
   c. Place e, v_R, v_L, and i on the same plot, and label accordingly.

49. For the network in Fig. 15.146:
   a. Determine the rms values of the currents $\mathbf{I}_s$, $\mathbf{I}_R$, and $\mathbf{I}_L$.
   b. Plot i_s, i_R, and i_L versus time on separate plots.
   c. Place e, i_s, i_R, and i_L on the same plot, and label accordingly.

50. For the network in Fig. 15.128:
   a. Plot the impedance of the network versus frequency from 0 to 10 kHz.
   b. Plot the current i versus frequency for the frequency range zero to 10 kHz.

*51. For the network in Fig. 15.136:
   a. Find the rms values of the voltages v_R and v_C at a frequency of 1 kHz.
   b. Plot v_C versus frequency for the frequency range zero to 10 kHz.
   c. Plot the phase angle between e and i for the frequency range zero to 10 kHz.

Vertical sensitivity = 0.5 V/div.
Horizontal sensitivity = 0.2 ms/div.

(I)
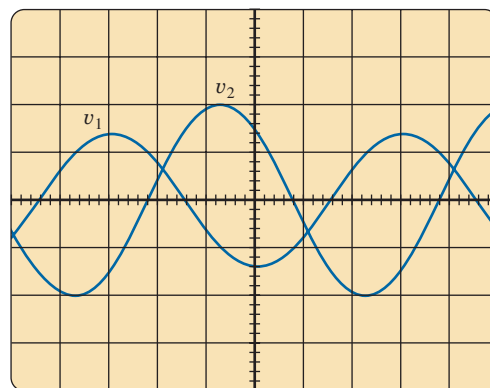
Vertical sensitivity = 2 V/div.
Horizontal sensitivity = 10 μs/div.

(II)

FIG. 15.160
Problem 47.

GLOSSARY

Admittance A measure of how easily a network will "admit" the passage of current through that system. It is measured in siemens, abbreviated S, and is represented by the capital letter *Y*.

Admittance diagram A vector display that clearly depicts the magnitude of the admittance of the conductance, capacitive susceptance, and inductive susceptance, and the magnitude and angle of the total admittance of the system.

Current divider rule A method by which the current through either of two parallel branches can be determined in an ac network without first finding the voltage across the parallel branches.

Equivalent circuits For every series ac network, there is a parallel ac network (and vice versa) that will be "equivalent" in the sense that the input current and impedance are the same.

Impedance diagram A vector display that clearly depicts the magnitude of the impedance of the resistive, reactive, and capacitive components of a network, and the magnitude and angle of the total impedance of the system.

Parallel ac circuits A connection of elements in an ac network in which all the elements have two points in common. The voltage is the same across each element.

Phasor diagram A vector display that provides at a glance the magnitude and phase relationships among the various voltages and currents of a network.

Series ac configuration A connection of elements in an ac network in which no two impedances have more than one terminal in common and the current is the same through each element.

Susceptance A measure of how "susceptible" an element is to the passage of current through it. It is measured in siemens, abbreviated S, and is represented by the capital letter *B*.

Voltage divider rule A method through which the voltage across one element of a series of elements in an ac network can be determined without first having to find the current through the elements.