Linear Algebra and Its Applications

THIRD EDITION UPDATE CHAPTER 8

David C. Lay
University of Maryland - College Park



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Answers to Exercises A1

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Answers to Exercises A1

The Geometry of Vector Spaces



INTRODUCTORY EXAMPLE

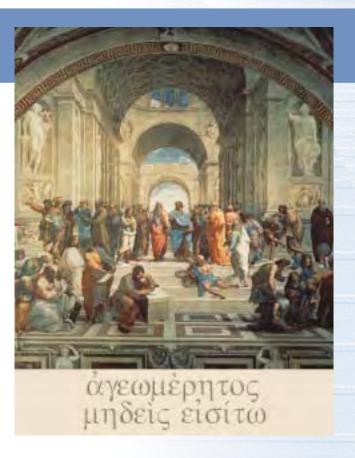
The Platonic Solids

In the city of Athens in 387 B.C., the Greek philosopher Plato founded an Academy, sometimes referred to as the world's first university. While the curriculum included astronomy, biology, political theory, and philosophy, the subject closest to his heart was geometry. Indeed, inscribed over the doors of his academy were these words: "Let no one destitute of geometry enter my doors."

The Greeks were greatly impressed by geometric patterns such as the regular solids. A polyhedron is called regular if its faces are congruent regular polygons and all the angles at the vertices are equal. As early as 150 years before Euclid, the Pythagoreans knew at least three of the regular solids: the tetrahedron (4 triangular faces), the cube (6 square faces), and the octahedron (8 triangular faces). (See Fig. 1.) These shapes occur naturally as crystals of common minerals. There are only five such regular solids, the remaining two being the dodecahedron (12 pentagonal faces) and the icosahedron (20 triangular faces).

Plato discussed the basic theory of these five solids in Book XIII of his *Elements*, and since then they have carried his name: the Platonic solids.

For centuries there was no need to envision geometric objects in more than three dimensions. But nowadays mathematicians regularly deal with objects in vector



spaces having four, five, or even hundreds of dimensions. It is not necessarily clear what geometrical properties one might ascribe to these objects in higher dimensions.

For example, what properties do lines have in 2-space and planes have in 3-space that would be useful in higher dimensions? How can one characterize such objects? Sections 8.1 and 8.4 provide some answers. The hyperplanes of Section 8.4 will be important for understanding the multi-dimensional nature of the linear programming problems in Chapter 9.

What would the analogue of a polyhedron "look like" in more than three dimensions? A partial answer is provided by two-dimensional projections of the four-dimensional object, created in a manner analogous to two-dimensional projections of a three-dimensional object. Section 8.5 illustrates this idea for the

four-dimensional "cube" and the four-dimensional "simplex."

The study of geometry in higher dimensions not only provides new ways of visualizing abstract algebraic concepts, but also creates tools that may be applied in \mathbb{R}^3 .

For instance, Sections 8.2 and 8.6 include applications to computer graphics, and Section 8.5 outlines a proof (in Exercise 13) that there are only five regular polyhedra in \mathbb{R}^3 .

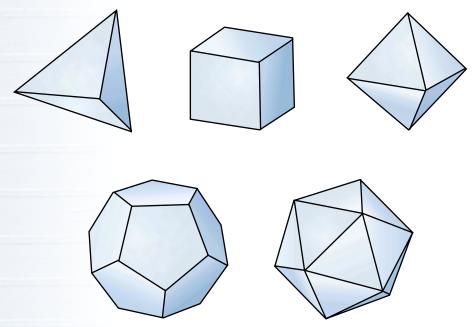


FIGURE 1 The five Platonic solids.

ost applications in earlier chapters involved algebraic calculations with subspaces and linear combinations of vectors. This chapter studies sets of vectors that can be visualized as geometric objects such as line segments, polygons, and solid objects. Individual vectors are viewed as points. The concepts introduced here

¹I thank my brother, Dr. Steven R. Lay, for designing and writing most of this chapter and for class testing it at Lee University. Several colleagues and I have also class tested the chapter, and I have made a few changes and additions. Comments from faculty and students who use it would be appreciated. Write to: lay@math.umd.edu.

are used in computer graphics, linear programming (in Chapter 9), and other areas of mathematics.²

Throughout the chapter, sets of vectors are described by linear combinations, but with various restrictions on the weights used in the combinations. For instance, in Section 8.1, the sum of the weights is one, while in Section 8.2, the weights are positive and sum to one. The visualizations are in \mathbb{R}^2 or \mathbb{R}^3 , of course, but the concepts also apply to \mathbb{R}^n and other vector spaces.

8.1 AFFINE COMBINATIONS

An affine combination of vectors is a special kind of linear combination. Given vectors (or "points") $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and scalars c_1, \dots, c_p , an **affine combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ is a linear combination

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

such that the weights satisfy $c_1 + \cdots + c_p = 1$.

DEFINITION

The set of all affine combinations of points in a set S is called the **affine hull** (or **affine span**) of S, denoted by aff S.

The affine hull of a single point \mathbf{v}_1 is just the set $\{\mathbf{v}_1\}$, since it has the form $c_1\mathbf{v}_1$ where $c_1=1$. The affine hull of two distinct points is often written in a special way. Suppose $\mathbf{y}=c_1\mathbf{v}_1+c_2\mathbf{v}_2$ with $c_1+c_2=1$. Write t in place of c_2 , so that $c_1=1-c_2=1-t$. Then the affine hull of $\{\mathbf{v}_1,\mathbf{v}_2\}$ is the set

$$\mathbf{y} = (1 - t)\mathbf{v}_1 + t\mathbf{v}_2, \quad \text{with } t \text{ in } \mathbb{R}$$
 (1)

This set of points includes \mathbf{v}_1 (when t=0) and \mathbf{v}_2 (when t=1). If $\mathbf{v}_2=\mathbf{v}_1$, then (1) again describes just one point. Otherwise, (1) describes the *line* through \mathbf{v}_1 and \mathbf{v}_2 . To see this, rewrite (1) in the form

$$\mathbf{y} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{p} + t\mathbf{u}$$
, with t in \mathbb{R}

where \mathbf{p} is \mathbf{v}_1 and \mathbf{u} is $\mathbf{v}_2 - \mathbf{v}_1$. The set of all multiples of \mathbf{u} is Span $\{\mathbf{u}\}$, the line through \mathbf{u} and the origin. Adding \mathbf{p} to each point on this line translates Span $\{\mathbf{u}\}$ into the line through \mathbf{p} parallel to the line through \mathbf{u} and the origin. See Fig. 1. (Compare this figure with Fig. 5 in Section 1.5.)

Figure 2 uses the original points \mathbf{v}_1 and \mathbf{v}_2 , and displays aff $\{\mathbf{v}_1, \mathbf{v}_2\}$ as the line through \mathbf{v}_1 and \mathbf{v}_2 .

p tu p + tu

FIGURE 1

²See Foley, van Dam, Feiner, and Hughes, *Computer Graphics—Principles and Practice*, 2nd edition (Boston: Addison Wesley, 1996), pp. 1083–1112. That material also discusses coordinate-free "affine spaces."

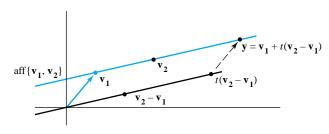


FIGURE 2

Notice that while the point \mathbf{y} in Fig. 2 is an affine combination of \mathbf{v}_1 and \mathbf{v}_2 , the point $\mathbf{y} - \mathbf{v}_1$ equals $t(\mathbf{v}_2 - \mathbf{v}_1)$, which is a linear combination (in fact, a multiple) of $\mathbf{v}_2 - \mathbf{v}_1$. This relation between \mathbf{y} and $\mathbf{y} - \mathbf{v}_1$ holds for any affine combination of points, as the following theorem shows.

THEOREM 1

A point \mathbf{y} in \mathbb{R}^n is an affine combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ in \mathbb{R}^n if and only if $\mathbf{y} - \mathbf{v}_1$ is a linear combination of the translated points $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1$.

PROOF If $\mathbf{y} - \mathbf{v}_1$ is a linear combination of $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1$, there exist weights c_2, \dots, c_p such that

$$\mathbf{y} - \mathbf{v}_1 = c_2(\mathbf{v}_2 - \mathbf{v}_1) + \dots + c_n(\mathbf{v}_n - \mathbf{v}_1)$$
 (2)

Then

$$\mathbf{y} = (1 - c_2 - \dots - c_p)\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$
(3)

and the weights in this linear combination sum to one. So \mathbf{y} is an affine combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$. Conversely, suppose

$$\mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p \tag{4}$$

where $c_1 + \cdots + c_p = 1$. Since $c_1 = 1 - c_2 - \cdots - c_p$, equation (4) may be written as in (3), and this leads to (2), which shows that $\mathbf{y} - \mathbf{v}_1$ is a linear combination of $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1$.

In the statement of Theorem 1, the point \mathbf{v}_1 could be replaced by any of the other points in the list $\mathbf{v}_1, \dots, \mathbf{v}_p$. Only the notation in the proof would change.

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. If possible, write \mathbf{y} as an affine combination of \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 .

Solution Compute the translated points

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \mathbf{y} - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

To find scalars c_2 , c_3 , and c_4 such that

$$c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) + c_4(\mathbf{v}_4 - \mathbf{v}_1) = \mathbf{y} - \mathbf{v}_1$$
 (5)

row reduce the augmented matrix having these points as columns:

$$\begin{bmatrix} 1 & 0 & -3 & 3 \\ 3 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & 9 & -10 \end{bmatrix}$$

This shows that equation (5) is consistent, and the general solution is $c_2 = 3c_4 + 3$, $c_3 = -9c_4 - 10$, with c_4 free. When $c_4 = 0$,

$$\mathbf{v} - \mathbf{v}_1 = 3(\mathbf{v}_2 - \mathbf{v}_1) - 10(\mathbf{v}_3 - \mathbf{v}_1) + 0(\mathbf{v}_4 - \mathbf{v}_1)$$

and

$$\mathbf{y} = 8\mathbf{v}_1 + 3\mathbf{v}_2 - 10\mathbf{v}_3$$

As another example, take $c_4 = 1$. Then $c_2 = 6$ and $c_3 = -19$, so

$$\mathbf{y} - \mathbf{v}_1 = 6(\mathbf{v}_2 - \mathbf{v}_1) - 19(\mathbf{v}_3 - \mathbf{v}_1) + 1(\mathbf{v}_4 - \mathbf{v}_1)$$

and

$$y = 13v_1 + 6v_2 - 19v_3 + v_4$$

While the procedure in Example 1 works for arbitrary points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n , the question can be answered more directly if the chosen points \mathbf{v}_i are a basis for \mathbb{R}^n . For example, let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be such a basis. Then any \mathbf{y} in \mathbb{R}^n is a unique *linear* combination of $\mathbf{b}_1, \dots, \mathbf{b}_n$. This combination is an affine combination of the \mathbf{b} 's if and only if the weights sum to one. (These weights are just the \mathcal{B} -coordinates of \mathbf{y} , as in Section 4.4.)

EXAMPLE 2 Let
$$\mathbf{b}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$
, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

The set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for \mathbb{R}^3 . Determine whether the points \mathbf{p}_1 and \mathbf{p}_2 are affine combinations of the points in \mathcal{B} .

Solution Find the \mathcal{B} -coordinates of \mathbf{p}_1 and \mathbf{p}_2 . These two calculations can be combined by row reducing the matrix $[\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \ \mathbf{p}_1 \ \mathbf{p}_2]$, with two augmented columns:

$$\begin{bmatrix} 4 & 0 & 5 & 2 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 3 & 2 & 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & \frac{2}{3} \\ 0 & 1 & 0 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & 2 & -\frac{1}{3} \end{bmatrix}$$

Read column 4 to build \mathbf{p}_1 , and read column 5 to build \mathbf{p}_2 :

$$\mathbf{p}_1 = -2\mathbf{b}_1 - \mathbf{b}_2 + 2\mathbf{b}_3$$
 and $\mathbf{p}_2 = \frac{2}{3}\mathbf{b}_1 + \frac{2}{3}\mathbf{b}_2 - \frac{1}{3}\mathbf{b}_3$

The sum of the weights in the linear combination for \mathbf{p}_1 is -1, not 1, so \mathbf{p}_1 is *not* an affine combination of the **b**'s. However, \mathbf{p}_2 is an affine combination of the **b**'s, because the sum of the weights for \mathbf{p}_2 is 1.

DEFINITION

A set S is **affine** if \mathbf{p} , $\mathbf{q} \in S$ implies that $(1-t)\mathbf{p} + t\mathbf{q} \in S$ for each real number t.

Geometrically, a set is affine if whenever two points are in the set, the entire line through these points is in the set. (If S contains only one point, \mathbf{p} , then the line through \mathbf{p} and \mathbf{p} is just a point, a "degenerate" line.) Algebraically, for a set S to be affine, the definition requires that every affine combination of two points of S belong to S. Remarkably, this is equivalent to requiring that S contain every affine combination of an arbitrary number of points of S.

THEOREM 2

A set S is affine if and only if every affine combination of points of S lies in S. That is, S is affine if and only if S = aff S.

PROOF Suppose that S is affine and use induction on the number m of points of S occurring in an affine combination. When m is 1 or 2, an affine combination of m points of S lies in S, by the definition of an affine set. Now, assume that every affine combination of k or fewer points of S yields a point in S, and consider a combination of k+1 points. Take \mathbf{v}_i in S for $i=1,\ldots,k+1$, and let $\mathbf{y}=c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k+c_{k+1}\mathbf{v}_{k+1}$, where $c_1+\cdots+c_{k+1}=1$. Since the c_i 's sum to one, at least one of them must not be equal to one. By re-indexing the \mathbf{v}_i and c_i , if necessary, we may assume that $c_{k+1}\neq 1$. Let $t=c_1+\cdots+c_k$. Then $t=1-c_{k+1}\neq 0$, and

$$\mathbf{y} = (1 - c_{k+1}) \left(\frac{c_1}{t} \mathbf{v}_1 + \dots + \frac{c_k}{t} \mathbf{v}_k \right) + c_{k+1} \mathbf{v}_{k+1}$$
 (6)

By the induction hypothesis, the point $\mathbf{z} = (c_1/t)\mathbf{v}_1 + \cdots + (c_k/t)\mathbf{v}_k$ is in S, since the coefficients sum to one. Thus (6) displays \mathbf{y} as an affine combination of two points in S, and so $\mathbf{y} \in S$. By the principle of induction, every affine combination of such points lies in S. That is, aff $S \subset S$. But the reverse inclusion, $S \subset A$ aff S, always applies. Thus, when S is affine, S = A aff S. Conversely, if S = A aff S, then affine combinations of two (or more) points of S lie in S, so S is affine.

The next definition provides terminology for affine sets that emphasizes their close connection with subspaces of \mathbb{R}^n .

DEFINITION

A translate of a set S in \mathbb{R}^n by a vector \mathbf{p} is the set $S + \mathbf{p} = \{\mathbf{s} + \mathbf{p} : \mathbf{s} \in S\}$. A **flat** in \mathbb{R}^n is a translate of a subspace of \mathbb{R}^n . Two flats are **parallel** if one is a translate of the other. The **dimension of a flat** is the dimension of the corresponding parallel subspace. The **dimension of a set** S, written as dim S, is the dimension of the smallest flat containing S. A **line** in \mathbb{R}^n is a flat of dimension one. A **hyperplane** in \mathbb{R}^n is a flat of dimension n-1.

In \mathbb{R}^3 , the proper subspaces consist of the origin $\mathbf{0}$, the set of all lines through $\mathbf{0}$, and the set of all planes through $\mathbf{0}$. Thus the proper flats in \mathbb{R}^3 are points (zero-dimensional), lines (one-dimensional), and planes (two-dimensional), which may or may not pass through the origin.

The next theorem shows that these geometric descriptions of lines and planes in \mathbb{R}^3 (as translates of subspaces) actually coincide with their earlier algebraic descriptions as sets of all affine combinations of two or three points, respectively.

THEOREM 3 A nonempty set S is affine if and only if it is a flat.

PROOF Suppose that S is affine. Let \mathbf{p} be any fixed point in S and let $W = S + (-\mathbf{p})$, so that $S = W + \mathbf{p}$. To show that S is a flat, it suffices to show that S is a subspace of \mathbb{R}^n . Since \mathbf{p} is in S, the zero vector is in S. To show that S is closed under sums and scalar multiples, it suffices to show that if \mathbf{u}_1 and \mathbf{u}_2 are elements of S, then S is in S for every real S in S such that S in S such that S in S and S in S such that S in S and S in S such that S in S and S in S such that S in S such that S is affine and S in S such that S in S such that S is affine and S in S such that S is a flat, it suffices to show that S is a flat, it suffices to sho

$$\mathbf{u}_1 + t\mathbf{u}_2 = (\mathbf{s}_1 - \mathbf{p}) + t(\mathbf{s}_2 - \mathbf{p})$$
$$= (1 - t)\mathbf{s}_1 + t(\mathbf{s}_1 + \mathbf{s}_2 - \mathbf{p}) - \mathbf{p}$$

Let $\mathbf{y} = \mathbf{s}_1 + \mathbf{s}_2 - \mathbf{p}$. Then \mathbf{y} is an affine combination of points in S. Since S is affine, \mathbf{y} is in S (by Theorem 2). But then $(1 - t)\mathbf{s}_1 + t\mathbf{y}$ is also in S. So $\mathbf{u}_1 + t\mathbf{u}_2$ is in $-\mathbf{p} + S = W$. This shows that W is a subspace of \mathbb{R}^n . Thus S is a flat, because $S = W + \mathbf{p}$.

Conversely, suppose S is a flat. That is, $S = W + \mathbf{p}$ for some $\mathbf{p} \in \mathbb{R}^n$ and some subspace W. To show that S is affine, it suffices to show that for any pair \mathbf{s}_1 and \mathbf{s}_2 of points in S, the line through \mathbf{s}_1 and \mathbf{s}_2 lies in S. By definition of W, there exist \mathbf{u}_1 and \mathbf{u}_2 in W such that $\mathbf{s}_1 = \mathbf{u}_1 + \mathbf{p}$ and $\mathbf{s}_2 = \mathbf{u}_2 + \mathbf{p}$. So, for each real t,

$$(1-t)\mathbf{s}_1 + t\mathbf{s}_2 = (1-t)(\mathbf{u}_1 + \mathbf{p}) + t(\mathbf{u}_2 + \mathbf{p})$$
$$= (1-t)\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{p}$$

Since W is a subspace, $(1 - t)\mathbf{u}_1 + t\mathbf{u}_2 \in W$ and so $(1 - t)\mathbf{s}_1 + t\mathbf{s}_2 \in W + \mathbf{p} = S$. Thus S is affine.

³If $\mathbf{p} = \mathbf{0}$, then the translate is just S itself. See Fig. 4 in Section 1.5.

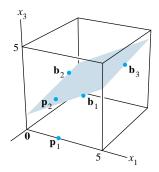


FIGURE 3

Theorem 3 provides a geometric way to view the affine hull of a set: it is the flat that consists of all the affine combinations of points in the set. For instance, Fig. 3 shows the points studied in Example 2. Although the set of all *linear* combinations of \mathbf{b}_1 , \mathbf{b}_2 , and \mathbf{b}_3 is all of \mathbb{R}^3 , the set of all *affine* combinations is only the plane through \mathbf{b}_1 , \mathbf{b}_2 , and \mathbf{b}_3 . Note that \mathbf{p}_2 (from Example 2) is in the plane through \mathbf{b}_1 , \mathbf{b}_2 , and \mathbf{b}_3 , while \mathbf{p}_1 is not in that plane. Also, see Exercise 12.

Earlier, Theorem 1 displayed an important connection between affine combinations and linear combinations. The next theorem provides another view of affine combinations, which for \mathbb{R}^2 and \mathbb{R}^3 is closely connected to applications in computer graphics, discussed in the next section (and in Section 2.7).

DEFINITION

For \mathbf{v} in \mathbb{R}^n , the standard **homogeneous form** of \mathbf{v} is the point $\tilde{\mathbf{v}} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$ in \mathbb{R}^{n+1} .

THEOREM 4

A point \mathbf{y} in \mathbb{R}^n is an affine combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in \mathbb{R}^n if and only if the homogeneous form of \mathbf{y} is in Span $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_p\}$. In fact, $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, with $c_1 + \cdots + c_p = 1$, if and only if $\tilde{\mathbf{y}} = c_1\tilde{\mathbf{v}}_1 + \cdots + c_p\tilde{\mathbf{v}}_p$.

PROOF A point **y** is in aff $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ if and only if there exist weights c_1, \dots, c_p such that

$$\begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \dots + c_p \begin{bmatrix} \mathbf{v}_p \\ 1 \end{bmatrix}$$

This happens if and only if $\tilde{\mathbf{y}}$ is in Span $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_p\}$.

EXAMPLE 3 Let
$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. Use Theorem 4

to write **p** as an affine combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 , if possible

Solution Row reduce the augmented matrix for the equation

$$x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$$

To simplify the arithmetic, move the fourth row of 1's to the top (equivalent to three row interchanges). After this, the number of arithmetic operations here is basically the same as the number needed for the method using Theorem 1.

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 4 \\ 1 & 2 & 7 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 6 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & .5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4, $1.5\mathbf{v}_1 - \mathbf{v}_2 + .5\mathbf{v}_3 = \mathbf{p}$. See Fig. 4, which shows the plane that contains $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and \mathbf{p} (together with points on the coordinate axes).

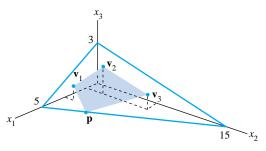


FIGURE 4

PRACTICE PROBLEM

Plot the points $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ on graph paper, and explain why \mathbf{p} must be an affine combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . Then find the affine combination for \mathbf{p} . [Hint: What is the dimension of aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?]

8.1 EXERCISES

In Exercises 1–5, write y as an affine combination of the other points listed, if possible.

1.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

2.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

3.
$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 17 \\ 1 \\ 5 \end{bmatrix}$

4.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}$

5.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 7 \\ -2 \\ -8 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -1 \\ 3 \\ -9 \\ 2 \end{bmatrix}$$

6. Let
$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$
, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, and

 $S = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Write each point below as an affine combination of the points in the set S, if possible. Note that S is an orthogonal basis for \mathbb{R}^3 . [*Hint:* Use Theorem 5 in Section 6.2 instead of row reduction to find the weights.]

a.
$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ -19 \\ -5 \end{bmatrix}$$
 b. $\mathbf{p}_2 = \begin{bmatrix} 1.5 \\ -1.3 \\ -.5 \end{bmatrix}$ c. $\mathbf{p}_3 = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$

7. Let
$$\mathbf{v}_{1} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$
, $\mathbf{v}_{2} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_{3} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{p}_{1} = \begin{bmatrix} 5 \\ -3 \\ 5 \\ 3 \end{bmatrix}$, $\mathbf{p}_{2} = \begin{bmatrix} -9 \\ 10 \\ 9 \\ -13 \end{bmatrix}$, $\mathbf{p}_{3} = \begin{bmatrix} 4 \\ 2 \\ 8 \\ 5 \end{bmatrix}$, and $S = \{\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\}$. It can be

shown that *S* is linearly independent.

- a. Is \mathbf{p}_1 in Span S? Is \mathbf{p}_1 in aff S?
- b. Is \mathbf{p}_2 in Span S? Is \mathbf{p}_2 in aff S?
- c. Is \mathbf{p}_3 in Span S? Is \mathbf{p}_3 in aff S?

8. Repeat Exercise 7 when
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 6 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 12 \\ -6 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 4 \\ -1 \\ 15 \\ -7 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} -5 \\ 3 \\ -8 \\ 6 \end{bmatrix}$, and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 6 \\ -6 \\ -8 \end{bmatrix}$.

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

- **9.** a. The set of all affine combinations of points in a set *S* is called the affine hull of *S*.
 - b. If $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a linearly independent subset of \mathbb{R}^n and if \mathbf{p} is a linear combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$, then \mathbf{p} is an affine combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$.
 - c. The affine hull of two distinct points is called a line.
 - d. A flat is a subspace.
 - e. A plane in \mathbb{R}^3 is a hyperplane.
- **10.** a. If $S = \{x\}$, then aff S is the empty set.
 - b. A set is affine if and only if it contains its affine hull.
 - c. A flat of dimension 1 is called a line.
 - d. A flat of dimension 2 is called a hyperplane.
 - e. A flat through the origin is a subspace.

- **11.** Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 . Show that Span $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane in \mathbb{R}^3 . [*Hint:* What can you say about \mathbf{u} and \mathbf{v} when Span $\{\mathbf{u}, \mathbf{v}\}$ is a plane?]
- 12. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 , then aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the plane through $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .
- **13.** Let *A* be an $m \times n$ matrix and, given **b** in \mathbb{R}^m , show that the set *S* of all solutions of $A\mathbf{x} = \mathbf{b}$ is an affine subset of \mathbb{R}^n .
- **14.** Let $\mathbf{v} \in \mathbb{R}^n$ and let $k \in \mathbb{R}$. Prove that $S = {\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = k}$ is an affine subset of \mathbb{R}^n .
- **15.** Choose a set *S* of three points such that aff *S* is the plane in \mathbb{R}^3 whose equation is $x_3 = 5$. Justify your work.
- **16.** Choose a set *S* of four distinct points in \mathbb{R}^3 such that aff *S* is the plane $2x_1 + x_2 3x_3 = 12$. Justify your work.
- **17.** Let *S* be an affine subset of \mathbb{R}^n , suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and let f(S) denote the set of images $\{f(\mathbf{x}) : \mathbf{x} \in S\}$. Prove that f(S) is an affine subset of \mathbb{R}^m .
- **18.** Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, let T be an affine subset of \mathbb{R}^m , and let $S = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T\}$. Show that S is an affine subset of \mathbb{R}^n .

In Exercises 19–22, prove the given statement about subsets A and B of \mathbb{R}^n . A proof for an exercise may use results from earlier exercises.

- **19.** If $A \subset B$ and B is affine, then aff $A \subset B$.
- **20.** If $A \subset B$, then aff $A \subset \text{aff } B$.
- **21.** $[(aff A) \cup (aff B)] \subset aff (A \cup B)$. *[Hint:* To show that $D \cup E \subset F$, show that $D \subset F$ and $E \subset F$.]
- **22.** aff $(A \cap B) \subset (\text{aff } A \cap \text{aff } B)$. [*Hint:* To show that $D \subset E \cap F$, show that $D \subset E$ and $D \subset F$.]
- **23.** Find an example in \mathbb{R}^2 to show that equality need not hold in the statement of Exercise 21. [*Hint:* Consider sets *A* and *B* that each contain only one or two points.]
- **24.** Find an example in \mathbb{R}^2 to show that equality need not hold in the statement of Exercise 22.

\mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_1

SOLUTION TO PRACTICE PROBLEM

Since the points \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are not collinear (that is, not on a single line), aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cannot be one-dimensional. Thus, aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must equal \mathbb{R}^2 . To find the actual weights used to express \mathbf{p} as an affine combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 , first compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \ \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{p} - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

To write $\mathbf{p} - \mathbf{v}_1$ as a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$, row reduce the matrix having these points as columns:

8.2

$$\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix}$$

Thus $\mathbf{p} - \mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) + 2(\mathbf{v}_3 - \mathbf{v}_1)$, which shows that

$$\mathbf{p} = (1 - \frac{1}{2} - 2)\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3 = -\frac{3}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3$$

This expresses \mathbf{p} as an affine combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 , because the coefficients sum to one.

Alternatively, use the method of Example 3 and row reduce:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{p} \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & 4 \\ 0 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This shows that $\mathbf{p} = -\frac{3}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3$.

8.2 AFFINE INDEPENDENCE

Because affine sets are simply translates of subspaces, some important concepts for subspaces "translate" into related concepts for affine sets.

DEFINITION

An indexed set of points $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is **affinely dependent** if there exist real numbers c_1, \dots, c_p , not all zero, such that

$$c_1 + \dots + c_p = 0$$
 and $c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p = \mathbf{0}$ (1)

Otherwise, the set is **affinely independent**.

A set $\{\mathbf{v}_1\}$ of only one point (even the zero vector) must be affinely independent because the required properties of the coefficients c_i cannot be satisfied when there is only one coefficient. For $\{\mathbf{v}_1\}$, the first equation in (1) is just $c_1 = 0$, and yet at least one (the only one) coefficient must be nonzero.

Exercise 13 asks you to show that an indexed set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent if and only if $\mathbf{v}_1 = \mathbf{v}_2$. The following theorem handles the general case and shows how the concept of affine dependence is analogous to that of linear dependence. Parts (c) and (d) give useful methods for determining whether a set is affinely dependent. Recall from Section 8.1 that if \mathbf{v} is in \mathbb{R}^n , then the vector $\tilde{\mathbf{v}}$ in \mathbb{R}^{n+1} denotes the homogeneous form of \mathbf{v} .

THEOREM 5

Given an indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n , with $p \ge 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

- a. *S* is affinely dependent.
- b. One of the points in S is an affine combination of the other points in S.
- c. The set $\{\mathbf{v}_2 \mathbf{v}_1, \dots, \mathbf{v}_p \mathbf{v}_1\}$ in \mathbb{R}^n is linearly dependent.
- d. The set $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_p\}$ in \mathbb{R}^{n+1} is linearly dependent.

PROOF Suppose (a) is true, and let c_1, \ldots, c_p satisfy (1). By renaming the points if necessary, one may assume that $c_1 \neq 0$ and divide both equations in (1) by c_1 , so that $1 + (c_2/c_1) + \cdots + (c_p/c_1) = 0$ and

$$\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2 + \dots + (-c_p/c_1)\mathbf{v}_p \tag{2}$$

Note that the coefficients on the right of (2) sum to one. Thus (a) implies (b). Now, suppose that (b) is true. By renaming the points if necessary, one may assume that $\mathbf{v}_1 = c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p$, where $c_2 + \cdots + c_p = 1$. Then

$$(c_2 + \dots + c_p)\mathbf{v}_1 = c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p \tag{3}$$

and

$$c_2(\mathbf{v}_2 - \mathbf{v}_1) + \dots + c_n(\mathbf{v}_n - \mathbf{v}_1) = 0 \tag{4}$$

Not all of c_2, \ldots, c_p can be zero because they sum to one. So (b) implies (c).

Next, if (c) is true, then there exist weights c_2, \ldots, c_p , not all zero, such that (4) holds. Rewrite (4) as (3) and set $c_1 = -(c_2 + \cdots + c_p)$. Then $c_1 + \cdots + c_p = 0$. Thus (3) shows that (1) is true. So (c) implies (a), which proves that (a), (b), and (c) are logically equivalent. Finally, (d) is equivalent to (a) because the two equations in (1) are equivalent to the following equation involving the homogeneous forms of the points in S:

$$c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \dots + c_p \begin{bmatrix} \mathbf{v}_p \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}$$

In statement (c) of Theorem 5, \mathbf{v}_1 could be replaced by any of the other points in the list $\mathbf{v}_1, \dots, \mathbf{v}_p$. Only the notation in the proof would change. So, to test whether a set is affinely dependent, subtract one point in the set from the other points, and check whether the translated set of p-1 points is linearly dependent.

EXAMPLE 1 The affine hull of two distinct points \mathbf{p} and \mathbf{q} is a line. If a third point \mathbf{r} is on the line, then $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is an affinely dependent set. If a point \mathbf{s} is not on the line through \mathbf{p} and \mathbf{q} , then these three points are not collinear and $\{\mathbf{p}, \mathbf{q}, \mathbf{s}\}$ is an affinely independent set. See Fig. 1.



FIGURE 1 $\{p, q, r\}$ is affinely dependent.

EXAMPLE 2 Let
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3}$. Deter-

mine whether S is affinely independent.

Solution Compute
$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -.5 \end{bmatrix}$$
 and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. These two points are

not multiples and hence form a linearly independent set, S'. So all statements in Theorem 5 are false, and S is affinely independent. Figure 2 shows S and the translated set S'. Notice that Span S' is a plane through the origin and aff S is a parallel plane through \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . (Only a portion of each plane is shown here, of course.)

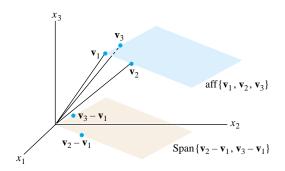


FIGURE 2 An affinely independent set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

EXAMPLE 3 Let
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 14 \\ 6 \end{bmatrix}$, and let

 $S = \{\mathbf{v}_1, \dots, \mathbf{v}_4\}$. Is S affinely dependent?

Solution Compute
$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -.5 \end{bmatrix}$$
, $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 11 \\ -1 \end{bmatrix}$,

and row reduce the matrix:

$$\begin{bmatrix} 1 & -1 & -1 \\ 4 & 1 & 11 \\ -.5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 5 & 15 \\ 0 & -.5 & -1.5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 5 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

Recall from Section 4.6 (or Section 2.8) that the columns are linearly dependent because not every column is a pivot column; so $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_4 - \mathbf{v}_1$ are linearly dependent. By (c) in Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely dependent. This dependence can also be established using (d) in Theorem 5 instead of (c).

The calculations in Example 3 show that $\mathbf{v}_4 - \mathbf{v}_1$ is a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$, which means that $\mathbf{v}_4 - \mathbf{v}_1$ is in Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$. By Theorem 1 in Section 8.1, \mathbf{v}_4 is in aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In fact, complete row reduction of the matrix in Example 3 would show that

$$\mathbf{v}_4 - \mathbf{v}_1 = 2(\mathbf{v}_2 - \mathbf{v}_1) + 3(\mathbf{v}_3 - \mathbf{v}_1)$$
 (5)

$$\mathbf{v}_4 = -4\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 \tag{6}$$

See Fig. 3.

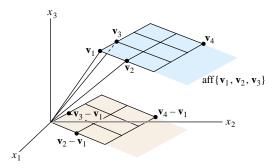


FIGURE 3 \mathbf{v}_4 is in the plane aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Figure 3 shows grids on both Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ and aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The grid on aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is based on (5). Another "coordinate system" can be based on (6), in which the coefficients -4, 3, 2 are called *affine* or *barycentric* coordinates of \mathbf{v}_4 .

Barycentric Coordinates

The definition of barycentric coordinates depends on the following affine version of the Unique Representation Theorem in Section 4.4. See Exercise 17.

THEOREM 6

Let $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be an affinely independent set in \mathbb{R}^n . Then each \mathbf{p} in aff S has a unique representation as an affine combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$. That is, for each \mathbf{p} there exists a unique set of scalars c_1, \dots, c_k such that

$$\mathbf{p} = c_1 \mathbf{b}_1 + \dots + c_k \mathbf{b}_k \quad \text{and} \quad c_1 + \dots + c_k = 1$$
 (7)

DEFINITION

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be an affinely independent set. Then for each point \mathbf{p} in aff S, the coefficients c_1, \dots, c_p in the unique representation (7) of \mathbf{p} are called the **barycentric** (or, sometimes, **affine**) **coordinates** of \mathbf{p} .

Observe that (7) is equivalent to the single equation

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{b}_1 \\ 1 \end{bmatrix} + \dots + c_k \begin{bmatrix} \mathbf{b}_k \\ 1 \end{bmatrix}$$
 (8)

involving the homogeneous forms of the points. Row reduction of the augmented matrix $\begin{bmatrix} \tilde{\mathbf{b}}_1 & \cdots & \tilde{\mathbf{b}}_k & \tilde{\mathbf{p}} \end{bmatrix}$ for (8) produces the barycentric coordinates of \mathbf{p} .

EXAMPLE 4 Let $\mathbf{a} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the barycentric coordinates of \mathbf{p} determined by the affinely independent set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

Solution Row reduce the augmented matrix of points in homogeneous form, moving the last row of ones to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} & \tilde{\mathbf{c}} & \tilde{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{12} \end{bmatrix}$$

The coordinates are $\frac{1}{4}$, $\frac{1}{3}$, $\frac{5}{12}$, so $\mathbf{p} = \frac{1}{4}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{5}{12}\mathbf{c}$.

Barycentric coordinates have both physical and geometric interpretations. They were originally defined by A. F. Moebius in 1827 for a point $\bf p$ inside a triangular region with vertices $\bf a$, $\bf b$, and $\bf c$. He wrote that the barycentric coordinates of $\bf p$ are three nonnegative numbers $m_{\bf a}$, $m_{\bf b}$, and $m_{\bf c}$ such that $\bf p$ is the center of mass of a system consisting of the triangle (with no mass) and masses $m_{\bf a}$, $m_{\bf b}$, and $m_{\bf c}$ at the corresponding vertices. The masses are uniquely determined by requiring that their sum be one. This view is still useful in physics today.¹

Figure 4 gives a geometric interpretation to the barycentric coordinates in Example 4, showing the triangle Δabc and three small triangles Δpbc , Δapc , and Δabp . The areas of the small triangles are proportional to the barycentric coordinates of **p**. In fact,

$$\operatorname{area}(\Delta \mathbf{pbc}) = \frac{1}{4} \cdot \operatorname{area}(\Delta \mathbf{abc})$$

$$\operatorname{area}(\Delta \mathbf{apc}) = \frac{1}{3} \cdot \operatorname{area}(\Delta \mathbf{abc})$$

$$\operatorname{area}(\Delta \mathbf{abp}) = \frac{5}{12} \cdot \operatorname{area}(\Delta \mathbf{abc})$$
(9)

 $^{^1}$ See Exercise 29 in Section 1.3. In astronomy, however, "barycentric coordinates" usually refer to ordinary \mathbb{R}^3 coordinates of points in what is now called the *International Celestial Reference System*, a Cartesian coordinate system for outer space, with the origin at the center of mass (the barycenter) of the solar system.

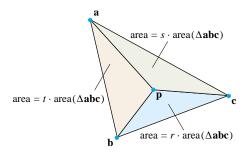


FIGURE 4
$$\mathbf{p} = r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$$
. Here, $r = \frac{1}{4}$, $s = \frac{1}{3}$, $t = \frac{5}{12}$.

The formulas in Fig. 4 are verified in Exercises 19–21. Analogous equalities for volumes of tetrahedrons hold for the case when \mathbf{p} is a point inside a tetrahedron in \mathbb{R}^3 , with vertices \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{d} .

When a point is not inside the triangle (or tetrahedron), some or all of the barycentric coordinates will be negative. The case of a triangle is illustrated in Fig. 5, for vertices $\bf a$, $\bf b$, $\bf c$, and coordinate values r, s, t, as above. The points on the line through $\bf b$ and $\bf c$, for instance, have r=0 because they are affine combinations of only $\bf b$ and $\bf c$. The parallel line through $\bf a$ identifies points with r=1.

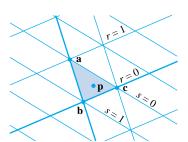


FIGURE 5 Barycentric coordinates for points in aff $\{a, b, c\}$.

Barycentric Coordinates in Computer Graphics

When working with geometric objects in a computer graphics program, a designer may use a "wire-frame" approximation to an object at certain key points in the process of creating a realistic final image.² For instance, if the surface of part of an object consists of small flat triangular surfaces, then a graphics program can easily add color, lighting, and shading to each small surface when that information is known only at the vertices.

²The Introductory Example for Chapter 2 shows a wire-frame model of a Boeing 777 airplane, used to visualize the flow of air over the surface of the plane.

Barycentric coordinates provide the tool for smoothly interpolating the vertex information over the interior of a triangle. The interpolation at a point is simply the linear combination of the vertex values using the barycentric coordinates as weights.

8.2

Colors on a computer screen are often described by RGB coordinates. A triple (r, g, b) indicates the amount of each color–red, green, and blue–with the parameters varying from 0 to 1. For example, pure red is (1, 0, 0), white is (1, 1, 1), and black is (0, 0, 0).

EXAMPLE 5 Let
$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 3 \\ 3 \\ 3.5 \end{bmatrix}$. The colors at

the vertices \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 of a triangle are magenta (1, 0, 1), light magenta (1, .4, 1), and purple (.6, 0, 1), respectively. Find the interpolated color at \mathbf{p} . See Fig. 6.



FIGURE 6 Interpolated colors.

Solution First, find the barycentric coordinates of **p**. Here is the calculation using homogeneous forms of the points, with the first step moving row 4 to row 1:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 1 & 3 \\ 1 & 3 & 5 & 3 \\ 5 & 4 & 1 & 3.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .25 \\ 0 & 1 & 0 & .50 \\ 0 & 0 & 1 & .25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathbf{p} = .25\mathbf{v}_1 + .5\mathbf{v}_2 + .25\mathbf{v}_3$. Use the barycentric coordinates of \mathbf{p} to make a linear combination of the color data. The RGB values for \mathbf{p} are

$$.25\begin{bmatrix} 1\\0\\1 \end{bmatrix} + .50\begin{bmatrix} 1\\.4\\1 \end{bmatrix} + .25\begin{bmatrix} .6\\0\\1 \end{bmatrix} = \begin{bmatrix} .9\\.2\\1 \end{bmatrix} \text{ red green blue}$$

One of the last steps in preparing a graphics scene for display on a computer screen is to remove "hidden surfaces" that should not be visible on the screen. Imagine the viewing screen as consisting of, say, a million pixels, and consider a ray or "line of sight" from the viewer's eye through a pixel and into the collection of objects that make up the 3D display. The color and other information displayed in the pixel on the screen

³Applets for color experiments are currently on the Web at www.nacs.uci.edu/~wiedeman/cspace/. Click on rgb.

should come from the object that the ray first intersects. See Fig. 7. When the objects in the graphics scene are approximated by wire frames with triangular patches, the hidden surface problem can be solved using barycentric coordinates.

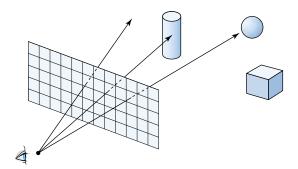


FIGURE 7 A ray from the eye through the screen to the nearest object.

The mathematics for finding the ray-triangle intersections can also be used to perform extremely realistic shading of objects. Currently, this *ray-tracing* method is too slow for real-time rendering, but recent advances in hardware implementation may change that in the future.⁴

EXAMPLE 6 Let
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 11 \\ -2 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} .7 \\ .4 \\ -3 \end{bmatrix}$,

and $\mathbf{x}(t) = \mathbf{a} + t\mathbf{b}$ for $t \ge 0$. Find the point where the ray $\mathbf{x}(t)$ intersects the plane that contains the triangle with vertices \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . Is this point inside the triangle?

Solution The plane is aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. A typical point in this plane may be written as $(1 - c_2 - c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some c_2 and c_3 . (The weights in this combination sum to one.) The ray $\mathbf{x}(t)$ intersects the plane when c_2 , c_3 , and t satisfy

$$(1 - c_2 - c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{a} + t\mathbf{b}$$

Rearrange this as $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) + t(-\mathbf{b}) = \mathbf{a} - \mathbf{v}_1$. In matrix form,

$$\begin{bmatrix} \mathbf{v}_2 - \mathbf{v}_1 & \mathbf{v}_3 - \mathbf{v}_1 & -\mathbf{b} \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ t \end{bmatrix} = \mathbf{a} - \mathbf{v}_1$$

For the specific points given here,

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}, \mathbf{a} - \mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 16 \end{bmatrix}$$

⁴See Joshua Fender and Jonathan Rose, "A High-Speed Ray Tracing Engine Built on a Field-Programmable System," currently online at www.eecg.toronto.edu/~fender/pdfs/raytrace_fender.pdf. (A single processor can calculate 600 million ray-triangle intersections per second.)

Row reduction of the augmented matrix above produces

$$\begin{bmatrix} 7 & 4 & -.7 & -1 \\ 0 & 10 & -.4 & -1 \\ 2 & 4 & 3 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .3 \\ 0 & 1 & 0 & .1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Thus $c_2 = .3$, $c_3 = .1$, and t = 5. Therefore, the intersection point is

$$\mathbf{x}(5) = \mathbf{a} + 5\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} + 5 \begin{bmatrix} .7 \\ .4 \\ -3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.0 \\ -5.0 \end{bmatrix}$$

Also,

$$\mathbf{x}(5) = (1 - .3 - .1)\mathbf{v}_1 + .3\mathbf{v}_2 + .1\mathbf{v}_3 = .6 \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix} + .3 \begin{bmatrix} 8 \\ 1 \\ -4 \end{bmatrix} + .1 \begin{bmatrix} 5 \\ 11 \\ -2 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.0 \\ -5.0 \end{bmatrix}$$

The intersection point is inside the triangle because the barycentric weights for $\mathbf{x}(5)$ are all positive.

PRACTICE PROBLEMS

- 1. Describe a fast way to determine when three points are collinear.
- **2.** The points $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, form an affinely dependent set. Find weights c_1, \ldots, c_4 that produce an affine dependence relation $c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4 = \mathbf{0}$, where $c_1 + \cdots + c_4 = 0$ and not all c_i are zero. [Hint: See the end of the proof of Theorem 5.]

8.2 EXERCISES

In Exercises 1-6, determine if the set of points is affinely dependent. (See Practice Problem 2.) If so, construct an affine dependence relation for the points.

1.
$$\begin{bmatrix} 3 \\ -3 \end{bmatrix}$$
, $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
2. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

2.
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
, $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

3.
$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 15 \\ -9 \end{bmatrix}$$

4.
$$\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$
, $\begin{bmatrix} 0 \\ -3 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 7 \\ -3 \end{bmatrix}$

5.
$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$
, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}$

6.
$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$$

In Exercises 7 and 8, find the barycentric coordinates of **p** with respect to the affinely independent set of points that precedes it.

7.
$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 5 \\ 4 \\ -2 \\ 2 \end{bmatrix}$$

8.
$$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$
, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ -6 \\ 5 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} -1 \\ 1 \\ -4 \\ 0 \end{bmatrix}$

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

- **9.** a. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbb{R}^n and if the set $\{\mathbf{v}_1 \mathbf{v}_2, \mathbf{v}_3 \mathbf{v}_2, \dots, \mathbf{v}_p \mathbf{v}_2\}$ is linearly dependent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is affinely dependent. (Read this carefully.)
 - b. If $\mathbf{v}_1, \dots \mathbf{v}_p$ are in \mathbb{R}^n and if the set of homogeneous forms $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_p\}$ in \mathbb{R}^{n+1} is linearly independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is affinely dependent.
 - c. A finite set of points $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is affinely dependent if there exist real numbers c_1, \dots, c_k , not all zero, such that $c_1 + \dots + c_k = 1$ and $c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}$.
 - d. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an affinely independent set in \mathbb{R}^n and if \mathbf{p} in \mathbb{R}^n has a negative barycentric coordinate determined by S, then \mathbf{p} is not in aff S.
 - e. If \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , \mathbf{a} , and \mathbf{b} are in \mathbb{R}^3 and if a ray $\mathbf{a} + t\mathbf{b}$ for $t \ge 0$ intersects the triangle with vertices \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 , then the barycentric coordinates of the intersection point are all nonnegative.
- **10.** a. If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an affinely dependent set in \mathbb{R}^n , then the set $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_p\}$ in \mathbb{R}^{n+1} of homogeneous forms may be linearly independent.
 - b. If \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 are in \mathbb{R}^3 and if the set $\{\mathbf{v}_2-\mathbf{v}_1,\mathbf{v}_3-\mathbf{v}_1,\mathbf{v}_4-\mathbf{v}_1\}$ is linearly independent, then $\{\mathbf{v}_1,\ldots,\mathbf{v}_4\}$ is affinely independent.
 - c. Given $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ in \mathbb{R}^n , each \mathbf{p} in aff S has a unique representation as an affine combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$.
 - d. When color information is specified at each vertex \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 of a triangle in \mathbb{R}^3 , then the color may be interpolated at a point \mathbf{p} in aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ using the barycentric coordinates of \mathbf{p} .
 - e. If T is a triangle in \mathbb{R}^2 and if a point \mathbf{p} is on an edge of the triangle, then the barycentric coordinates of \mathbf{p} (for this triangle) are not all positive.
- 11. Explain why any set of five or more points in \mathbb{R}^3 must be affinely dependent.
- **12.** Show that a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^n is affinely dependent when p > n + 2.
- 13. Use only the definition of affine dependence to show that an indexed set $\{\mathbf{v}_1, \mathbf{v}_2\}$ in \mathbb{R}^n is affinely dependent if and only if $\mathbf{v}_1 = \mathbf{v}_2$.
- **14.** The conditions for affine dependence are stronger than those for linear dependence, so an affinely dependent set is automatically linearly dependent. Also, a linearly independent set

cannot be affinely dependent and therefore must be affinely independent. Construct two linearly dependent indexed sets S_1 and S_2 in \mathbb{R}^2 such that S_1 is affinely dependent and S_2 is affinely independent. In each case, the set should contain either one, two, or three nonzero points.

15. Let
$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3}$.

- a. Show that the set S is affinely independent.
- b. Find the barycentric coordinates of $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{p}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{p}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with respect
- c. Let T be the triangle with vertices \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . When the sides of T are extended, the lines divide \mathbb{R}^2 into seven regions. See Fig. 8. Note the signs of the barycentric coordinates of the points in each region. For example, \mathbf{p}_5 is inside the triangle T and all its barycentric coordinates are positive. Point \mathbf{p}_1 has coordinates (-, +, +). Its third coordinate is positive because \mathbf{p}_1 is on the \mathbf{v}_3 side of the line through \mathbf{v}_1 and \mathbf{v}_2 . Its first coordinate is negative because \mathbf{p}_1 is opposite the \mathbf{v}_1 side of the line through \mathbf{v}_2 and \mathbf{v}_3 . Point \mathbf{p}_2 is on the $\mathbf{v}_2\mathbf{v}_3$ edge of T. Its coordinates are (0, +, +). Without calculating the actual values, determine the signs of the barycentric coordinates of points \mathbf{p}_6 , \mathbf{p}_7 , and \mathbf{p}_8 as shown in Fig. 8.

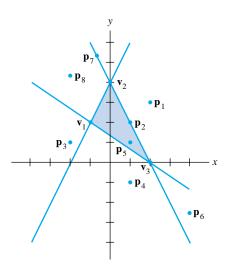


FIGURE 8

16. Let
$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{p}_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{p}_5 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\mathbf{p}_6 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{p}_7 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3}$.

- a. Show that the set S is affinely independent.
- b. Find the barycentric coordinates of \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 with respect to S.
- c. On graph paper, sketch the triangle T with vertices $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 , extend the sides as in Fig. 5, and plot the points \mathbf{p}_4 , \mathbf{p}_5 , \mathbf{p}_6 , and \mathbf{p}_7 . Without calculating the actual values, determine the signs of the barycentric coordinates of points \mathbf{p}_4 , \mathbf{p}_5 , \mathbf{p}_6 , and \mathbf{p}_7 .
- **17.** Prove Theorem 6 for an affinely independent set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in \mathbb{R}^n . [*Hint:* One method is to mimic the proof of Theorem 7 in Section 4.4.]
- **18.** Let T be a tetrahedron in "standard" position, with three edges along the three positive coordinate axes in \mathbb{R}^3 , and suppose the vertices are $a\mathbf{e}_1$, $b\mathbf{e}_2$, $c\mathbf{e}_3$, and $\mathbf{0}$, where $[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] = I_3$. Find formulas for the barycentric coordinates of an arbitrary point \mathbf{p} in \mathbb{R}^3 .

In Exercises 19–22, **a**, **b**, and **c** are noncollinear points in \mathbb{R}^2 and **p** is any other point in \mathbb{R}^2 . Let $\Delta \mathbf{abc}$ denote the closed triangular region determined by **a**, **b**, and **c**, and let $\Delta \mathbf{pbc}$ be the region determined by **p**, **b**, and **c**. For convenience, assume that **a**, **b**, and **c** are arranged so that det $[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$ is positive, where $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$, and $\tilde{\mathbf{c}}$ are the standard homogeneous forms for the points.

- **19.** Show that the area of Δabc is det $[\tilde{a} \quad \tilde{b} \quad \tilde{c}]/2$. [*Hint:* Consult Sections 3.2 and 3.3, including the Exercises.]
- **20.** Let **p** be a point on the line through **a** and **b**. Show that $\det \begin{bmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} & \tilde{\mathbf{p}} \end{bmatrix} = 0$.
- **21.** Let **p** be any point in the interior of Δ **abc**, with barycentric coordinates (r, s, t), so that

$$\begin{bmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} & \tilde{\mathbf{c}} \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$$

Use Exercise 19 and a fact about determinants (Chapter 3) to show that

 $r = (\text{area of } \Delta \mathbf{pbc}) / (\text{area of } \Delta \mathbf{abc})$

 $s = (\text{area of } \Delta \mathbf{apc}) / (\text{area of } \Delta \mathbf{abc})$

 $t = (\text{area of } \Delta \mathbf{abp}) / (\text{area of } \Delta \mathbf{abc})$

22. Take \mathbf{q} on the line segment from \mathbf{b} to \mathbf{c} and consider the line through \mathbf{q} and \mathbf{a} , which may be written as $\mathbf{p} = (1-x)\mathbf{q} + x\mathbf{a}$ for all real x. Show that, for each x, $\det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = x \cdot \det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$. From this and earlier work, conclude that the parameter x is the first barycentric coordinate of \mathbf{p} . However, by construction, the parameter x also determines the relative distance between \mathbf{p} and \mathbf{q} along the segment from \mathbf{q} to \mathbf{a} . (When x = 1, $\mathbf{p} = \mathbf{a}$.) When this fact is applied to Example 5, it shows that the colors at vertex \mathbf{a} and the point \mathbf{q} are smoothly interpolated as \mathbf{p} moves along the line between \mathbf{a} and \mathbf{q} .

SOLUTIONS TO PRACTICE PROBLEMS

- 1. From Example 1, the problem is to determine if the points are affinely dependent. Use the method of Example 2 and subtract one point from the other two. If one of these two new points is a multiple of the other, the original three points lie on a line.
- **2.** The proof of Theorem 5 essentially points out that an affine dependence relation among points corresponds to a linear dependence relation among the homogeneous forms of the points, using the *same* weights. So, row reduce:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 5 & 1 \\ 1 & 0 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 5 & 1 \\ 1 & 0 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1.25 \\ 0 & 0 & 1 & .75 \end{bmatrix}$$

View this matrix as the coefficient matrix for $A\mathbf{x} = \mathbf{0}$ with four variables. Then x_4 is free, $x_1 = x_4$, $x_2 = -1.25x_4$, and $x_3 = -.75x_4$. One solution is $x_1 = x_4 = 4$, $x_2 = -5$, and $x_3 = -3$. A linear dependence among the homogeneous forms is $4\tilde{\mathbf{v}}_1 - 5\tilde{\mathbf{v}}_2 - 3\tilde{\mathbf{v}}_3 + 4\tilde{\mathbf{v}}_4 = \mathbf{0}$. So $4\mathbf{v}_1 - 5\mathbf{v}_2 - 3\mathbf{v}_3 + 4\mathbf{v}_4 = \mathbf{0}$.

Another solution method is to translate the problem to the origin by subtracting \mathbf{v}_1 from the other points, find a linear dependence relation among the translated points, and then rearrange the terms. The amount of arithmetic involved is about the same as in the approach shown above.

8.3 CONVEX COMBINATIONS

Section 8.1 considered special linear combinations of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$
, where $c_1 + c_2 + \cdots + c_k = 1$

This section further restricts the weights to be nonnegative.

DEFINITION

A **convex combination** of points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in \mathbb{R}^n is a linear combination of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

such that $c_1 + c_2 + \cdots + c_k = 1$ and $c_i \ge 0$ for all i. The set of all convex combinations of points in a set S is called the **convex hull** of S, denoted by conv S.

The convex hull of a single point \mathbf{v}_1 is just the set $\{\mathbf{v}_1\}$, the same as the affine hull. In other cases, the convex hull is properly contained in the affine hull. Recall that the affine hull of distinct points \mathbf{v}_1 and \mathbf{v}_2 is the line

$$\mathbf{y} = (1-t)\mathbf{v}_1 + t\mathbf{v}_2$$
, with t in \mathbb{R}

Because the weights in a convex combination are nonnegative, the points in conv $\{v_1, v_2\}$ may be written as

$$y = (1 - t)v_1 + tv_2$$
, with $0 < t < 1$

which is the **line segment** between \mathbf{v}_1 and \mathbf{v}_2 , hereafter denoted by $\overline{\mathbf{v}_1\mathbf{v}_2}$.

If a set *S* is affinely independent and if $\mathbf{p} \in \text{aff } S$, then $\mathbf{p} \in \text{conv } S$ if and only if the barycentric coordinates of \mathbf{p} are nonnegative. Example 1 shows a special situation in which *S* is much more than just affinely independent.

EXAMPLE 1 Let
$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} -10 \\ 5 \\ 11 \\ -4 \end{bmatrix}$,

and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that S is an orthogonal set. Determine whether \mathbf{p}_1 is in Span S, aff S, and conv S. Then do the same for \mathbf{p}_2 .

Solution If \mathbf{p}_1 is at least a *linear* combination of the points in S, then the weights are easily found, because S is an orthogonal set. Let W be the subspace spanned by S. A calculation as in Section 6.3 shows that the orthogonal projection of \mathbf{p}_1 onto W is \mathbf{p}_1 itself:

$$\operatorname{proj}_{W} \mathbf{p}_{1} = \frac{\mathbf{p}_{1} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}} \mathbf{v}_{1} + \frac{\mathbf{p}_{1} \cdot \mathbf{v}_{2}}{\mathbf{v}_{2} \cdot \mathbf{v}_{2}} \mathbf{v}_{2} + \frac{\mathbf{p}_{1} \cdot \mathbf{v}_{3}}{\mathbf{v}_{3} \cdot \mathbf{v}_{3}} \mathbf{v}_{3}$$

$$= \frac{18}{54} \mathbf{v}_{1} + \frac{18}{54} \mathbf{v}_{2} + \frac{18}{54} \mathbf{v}_{3}$$

$$= \frac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \mathbf{p}_{1}$$

This shows that \mathbf{p}_1 is in Span S. Also, since the coefficients sum to one, \mathbf{p}_1 is in aff S. In fact, \mathbf{p}_1 is in conv S, because the coefficients are also nonnegative.

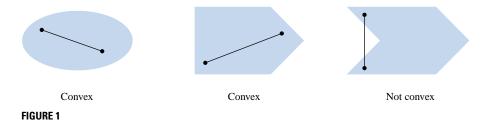
For \mathbf{p}_2 , a similar calculation shows that $\operatorname{proj}_W \mathbf{p}_2 \neq \mathbf{p}_2$. Since $\operatorname{proj}_W \mathbf{p}_2$ is the closest point in Span S to \mathbf{p}_2 , the point \mathbf{p}_2 is not in Span S. In particular, \mathbf{p}_2 cannot be in aff S or conv S.

Recall that a set *S* is affine if it contains all lines determined by pairs of points in *S*. When attention is restricted to convex combinations, the appropriate condition involves line segments rather than lines.

DEFINITION

A set S is **convex** if for each \mathbf{p} , $\mathbf{q} \in S$, the line segment $\overline{\mathbf{pq}}$ is contained in S.

Intuitively, a set S is convex if every two points in the set can "see" each other without the line of sight leaving the set. Figure 1 illustrates this idea.



The next result is analogous to Theorem 2 for affine sets.

THEOREM 7

A set *S* is convex if and only if every convex combination of points of *S* lies in *S*. That is, *S* is convex if and only if S = conv S.

PROOF The argument is similar to the proof of Theorem 2. The only difference is in the induction step. When taking a convex combination of k + 1 points, consider

 $\mathbf{y} = c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k + c_{k+1} \mathbf{v}_{k+1}$, where $c_1 + \dots + c_{k+1} = 1$ and $0 \le c_i \le 1$ for all i. If $c_{k+1} = 1$, then $\mathbf{y} = \mathbf{v}_{k+1}$, which belongs to S, and there is nothing further to prove. If $c_{k+1} < 1$, let $t = c_1 + \dots + c_k$. Then $t = 1 - c_{k+1} > 0$ and

$$\mathbf{y} = (1 - c_{k+1}) \left(\frac{c_1}{t} \mathbf{v}_1 + \dots + \frac{c_k}{t} \mathbf{v}_k \right) + c_{k+1} \mathbf{v}_{k+1}$$
 (1)

By the induction hypothesis, the point $\mathbf{z} = (c_1/t)\mathbf{v}_1 + \cdots + (c_k/t)\mathbf{v}_k$ is in S, since the nonnegative coefficients sum to one. Thus (1) displays \mathbf{y} as a convex combination of two points in S. By the principle of induction, every convex combination of such points lies in S.

Theorem 9 below provides a more geometric characterization of the convex hull of a set. It requires a preliminary result on intersections of sets. Recall from Section 4.1 (Exercise 32) that the intersection of two subspaces is itself a subspace. In fact, the intersection of any collection of subspaces is itself a subspace. A similar result holds for affine sets and convex sets.

THEOREM 8

Let $\{S_{\alpha}: \alpha \in \mathcal{A}\}$ be any collection of convex sets. Then $\bigcap_{\alpha \in \mathcal{A}} S_{\alpha}$ is convex. If $\{T_{\beta}: \beta \in \mathcal{B}\}$ is any collection of affine sets, then $\bigcap_{\beta \in \mathcal{B}} T_{\beta}$ is affine.

PROOF If **p** and **q** are in $\cap S_{\alpha}$, then **p** and **q** are in each S_{α} . Since each S_{α} is convex, the line segment between **p** and **q** is in S_{α} for all α and hence that segment is contained in $\cap S_{\alpha}$. The proof of the affine case is similar.

THEOREM 9

For any set S, the convex hull of S is the intersection of all the convex sets that contain S.

PROOF Let T denote the intersection of all the convex sets containing S. Since conv S is a convex set containing S, it follows that $T \subset \text{conv } S$. On the other hand, let C be any convex set containing S. Then C contains every convex combination of points of C (Theorem 7), and hence also contains every convex combination of points of the subset S. That is, conv $S \subset C$. Since this is true for every convex set C containing S, it is also true for the intersection of them all. That is, conv $S \subset T$.

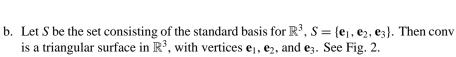
Theorem 9 shows that conv S is in a natural sense the "smallest" convex set containing S. For example, consider a set S that lies inside some large rectangle in \mathbb{R}^2 , and imagine stretching a rubber band around the outside of S. As the rubber band contracts around S, it outlines the boundary of the convex hull of S. Or to use another analogy, the convex hull of S fills in all the holes in the inside of S and fills out all the dents in the boundary of S.

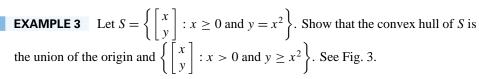
EXAMPLE 2

a. The convex hulls of sets *S* and *T* in \mathbb{R}^2 are shown below.



b. Let S be the set consisting of the standard basis for \mathbb{R}^3 , $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then conv S is a triangular surface in \mathbb{R}^3 , with vertices \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 . See Fig. 2.





Solution Every point in conv S must lie on a line segment that connects two points of S. The dashed line in Fig. 3 indicates that, except for the origin, the positive y-axis is not in conv S, because the origin is the only point of S on the y-axis. It may seem reasonable that Fig. 3 does show conv S, but how can you be sure that the point $(10^{-2}, 10^4)$, for example, is on a line segment from the origin to a point on the curve in S?

Consider any point **p** in the shaded region of Fig. 3, say

$$\mathbf{p} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \text{with } a > 0 \text{ and } b \ge a^2$$

The line through **0** and **p** has the equation y = (b/a)t for t real. That line intersects S where t satisfies $(b/a)t = t^2$, that is, when t = b/a. Thus, **p** is on the line segment from **0** to $\begin{bmatrix} b/a \\ b^2/a^2 \end{bmatrix}$, which shows that Fig. 3 is correct.

The following theorem is basic in the study of convex sets. It was first proved by Constantin Caratheodory in 1907. If **p** is in the convex hull of S, then, by definition, **p** must be a convex combination of points of S. But the definition makes no stipulation as to how many points of S are required to make the combination. Caratheodory's remarkable theorem says that in an n-dimensional space, the number of points of S in the convex combination never has to be more than n + 1.

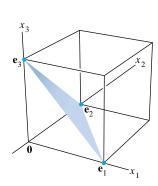


FIGURE 2

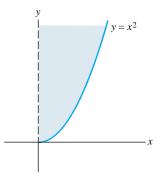


FIGURE 3

THEOREM 10

(Caratheodory) If S is a nonempty subset of \mathbb{R}^n , then every point in conv S can be expressed as a convex combination of n + 1 or fewer points of S.

PROOF Given **p** in conv *S*, one may write $\mathbf{p} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$, where $\mathbf{v}_i \in S$, $c_1 + \cdots + c_k = 1$, and $c_i \geq 0$, for some *k* and $i = 1, \dots, k$. The goal is to show that such an expression exists for **p** with $k \leq n + 1$.

If k > n + 1, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is affinely dependent, by Exercise 12 in Section 8.2. Thus there exist scalars d_1, \dots, d_k , not all zero, such that

$$\sum_{i=1}^k d_i \mathbf{v}_i = \mathbf{0} \quad \text{and} \quad \sum_{i=1}^k d_i = 0$$

Consider the two equations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{p}$$

and

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_k\mathbf{v}_k = \mathbf{0}$$

By subtracting an appropriate multiple of the second equation from the first, we now eliminate one of the \mathbf{v}_i terms and obtain a convex combination of fewer than k elements of S that is equal to \mathbf{p} .

Since not all of the d_i coefficients are zero, we may assume (by reordering subscripts if necessary) that $d_k > 0$ and that $c_k/d_k \le c_i/d_i$ for all those i for which $d_i > 0$. For $i = 1, \ldots, k$, let $b_i = c_i - (c_k/d_k)d_i$. Then $b_k = 0$ and

$$\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} c_i - \frac{c_k}{d_k} \sum_{i=1}^{k} d_i = 1 - 0 = 1$$

Furthermore, each $b_i \ge 0$. Indeed, if $d_i \le 0$, then $b_i \ge c_i \ge 0$. If $d_i > 0$, then $b_i = d_i(c_i/d_i - c_k/d_k) \ge 0$. By construction,

$$\sum_{i=1}^{k-1} b_i \mathbf{v}_i = \sum_{i=1}^k b_i \mathbf{v}_i = \sum_{i=1}^k \left(c_i - \frac{c_k}{d_k} d_i \right) \mathbf{v}_i$$
$$= \sum_{i=1}^k c_i \mathbf{v}_i - \frac{c_k}{d_k} \sum_{i=1}^k d_i \mathbf{v}_i = \sum_{i=1}^k c_i \mathbf{v}_i = \mathbf{p}$$

Thus **p** is now a convex combination of k-1 of the points $\mathbf{v}_1, \dots, \mathbf{v}_k$. This process may be repeated until **p** is expressed as a convex combination of at most n+1 of the points of S.

The following example illustrates the calculations in the proof above.

EXAMPLE 4 Let
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{10}{3} \\ \frac{5}{2} \end{bmatrix}$, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Then

$$\frac{1}{4}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \frac{1}{12}\mathbf{v}_4 = \mathbf{p}$$
 (2)

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Use the procedure in the proof of Caratheodory's Theorem to express \mathbf{p} as a convex combination of three points of S.

Solution The set *S* is affinely dependent. Use the technique of Section 8.2 to obtain an affine dependence relation

$$-5\mathbf{v}_1 + 4\mathbf{v}_2 - 3\mathbf{v}_3 + 4\mathbf{v}_4 = \mathbf{0} \tag{3}$$

Next, choose the points \mathbf{v}_2 and \mathbf{v}_4 in (3), whose coefficients are positive. For each point, compute the ratio of the quotients in equations (2) and (3). The ratio for \mathbf{v}_2 is $\frac{1}{6} \div 4 = \frac{1}{24}$, and that for \mathbf{v}_4 is $\frac{1}{12} \div 4 = \frac{1}{48}$. The ratio for \mathbf{v}_4 is smaller, so subtract $\frac{1}{48}$ times equation (3) from equation (2) to eliminate \mathbf{v}_4 :

$$(\frac{1}{4} + \frac{5}{48})\mathbf{v}_1 + (\frac{1}{6} - \frac{4}{48})\mathbf{v}_2 + (\frac{1}{2} + \frac{3}{48})\mathbf{v}_3 + (\frac{1}{12} - \frac{4}{48})\mathbf{v}_4 = \mathbf{p}$$

$$\frac{17}{48}\mathbf{v}_1 + \frac{4}{48}\mathbf{v}_2 + \frac{27}{48}\mathbf{v}_3 = \mathbf{p}$$

This result cannot, in general, be improved by decreasing the required number of points. Indeed, given any three non-collinear points in \mathbb{R}^2 , the centroid of the triangle formed by them is in the convex hull of all three, but is not in the convex hull of any two.

PRACTICE PROBLEMS

- 1. Let $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Determine whether \mathbf{p}_1 and \mathbf{p}_2 are in conv S.
- **2.** Let *S* be the set of points on the curve y = 1/x for x > 0. Explain geometrically why conv *S* consists of all points on and above the curve *S*.

8.3 EXERCISES

- **1.** In \mathbb{R}^2 , let $S = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : 0 \le y < 1 \right\} \bigcup \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$. Describe (or sketch) the convex hull of S.
- **2.** Describe the convex hull of the set *S* of points $\begin{bmatrix} x \\ y \end{bmatrix}$ in \mathbb{R}^2 that satisfy the given conditions. Justify your answers. (Show that an arbitrary point **p** in *S* belongs to conv *S*.)

a.
$$y = 1/x \text{ and } x \ge 1/2$$

b.
$$y = \sin x$$

c.
$$y = x^{1/2} \text{ and } x \ge 0$$

- 3. Consider the points in Exercise 7 of Section 8.1. Which of p₁, p₂, and p₃ are in conv S?
- **4.** Consider the points in Exercise 6 of Section 8.1. Which of **p**₁, **p**₂, and **p**₃ are in conv *S*?

5. Let
$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Determine whether \mathbf{p}_1 and \mathbf{p}_2 are in conv. S .

6. Let
$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 2 \\ -\frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$,

$$\mathbf{p}_2 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{4} \\ \frac{7}{4} \end{bmatrix}, \ \mathbf{p}_3 = \begin{bmatrix} 6 \\ -4 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{p}_4 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \text{ and let } S \text{ be}$$

the orthogonal set $\{v_1, v_2, v_3\}$. Determine whether each \mathbf{p}_i is in Span S, aff S, or conv S.

- a. \mathbf{p}_1
- b. **p**₂
- c. **p**₃
- d. **p**₄

Exercises 7 and 8 use the terminology from Section 8.2.

- 7. a. Let $T = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and let $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$, and $\mathbf{p}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find the barycentric coordinates of \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_3 , and \mathbf{p}_4 with respect to T.
 - b. Use your answers in part (a) to determine whether each of
 - $\mathbf{p}_1, \dots, \mathbf{p}_4$ in part (a) is inside, outside, or on the edge of conv T, a triangle.
- **8.** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be an affinely independent set. Consider the points $\mathbf{p}_1, \dots, \mathbf{p}_5$ whose barycentric coordinates with respect to S are given by (2,0,0,-1), $\left(0,\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$, $(\frac{1}{2}, 0, \frac{3}{2}, -1), (\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}), \text{ and } (\frac{1}{3}, 0, \frac{2}{3}, 0), \text{ respectively.}$ Determine whether each of $\mathbf{p}_1, \dots, \mathbf{p}_5$ is inside, outside, or on the surface of conv S, a tetrahedron. Are any of these points on an edge of conv S?
- **9.** Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Confirm that

$$\mathbf{p} = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$$
 and $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}$

Use the procedure in the proof of Caratheodory's Theorem to express \mathbf{p} as a convex combination of three of the \mathbf{v}_i 's. Do this in two ways.

10. Repeat Exercise 9 for points
$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Confirm that
$$\mathbf{p} = \frac{1}{121}\mathbf{v}_1 + \frac{72}{121}\mathbf{v}_2 + \frac{37}{121}\mathbf{v}_3 + \frac{1}{11}\mathbf{v}_4$$

and

$$10\mathbf{v}_1 - 6\mathbf{v}_2 + 7\mathbf{v}_3 - 11\mathbf{v}_4 = \mathbf{0}$$

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

- **11.** a. If $\mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$ and $c_1 + c_2 + c_3 = 1$, then \mathbf{y} is a convex combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 .
 - b. If S is a nonempty set, then conv S contains some points that are not in S.
 - c. If S and T are convex sets, then $S \cup T$ is also convex.
- 12. a. A set is convex if $\mathbf{x}, \mathbf{y} \in S$ implies that the line segment between \mathbf{x} and \mathbf{y} is contained in S.
 - b. If *S* and *T* are convex sets, then $S \cap T$ is also convex.
 - c. If S is a nonempty subset of \mathbb{R}^5 and $\mathbf{y} \in \text{conv } S$, then there exist distinct points $\mathbf{v}_1, \ldots, \mathbf{v}_6$ in S such that \mathbf{y} is a convex combination of $\mathbf{v}_1, \ldots, \mathbf{v}_6$.

Exercises 13–16 use the following definition:

A point y is called a **positive combination** of the points $\mathbf{v}_1, \ldots, \mathbf{v}_m$ if $\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_m \mathbf{v}_m$, where all $c_i \geq 0$. The set of all positive combinations of points of a set S is called the **positive hull** of S and is denoted by pos S.

- **13.** In \mathbb{R}^2 , find pos S when $S = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.
- **14.** Observe that in Exercise 13, (pos S) \cap (aff S) = conv S.
 - a. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Verify that $\mathbf{p} \in \text{pos}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathbf{p} \in \text{aff}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, but $\mathbf{p} \notin \text{conv} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. This shows that the result in Exercise 13 is not true in general.
 - b. What special property does the set S in Exercise 13 have so that (pos S) \cap (aff S) = conv S?
- **15.** Prove that pos S = pos (conv S) for any set S.
- **16.** Let S be a convex set. Prove that $\mathbf{x} \in \text{pos } S$ if and only if $\mathbf{x} = \lambda \mathbf{s}$ for some $\lambda \geq 0$ and $\mathbf{s} \in S$.
- **17.** Let S be a convex subset of \mathbb{R}^n and suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Prove that the set $f(S) = \{ f(\mathbf{x}) : \mathbf{x} \in S \}$ is a convex subset of \mathbb{R}^m .
- **18.** Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let T be a convex subset of \mathbb{R}^m . Prove that the set $S = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T\}$ is a convex subset of \mathbb{R}^n .

In Exercises 19–22, prove the given statement about subsets A and B of \mathbb{R}^n . A proof for an exercise may use results of earlier exercises.

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20. If $A \subset B$, then conv $A \subset \text{conv } B$.

21. $[(\operatorname{conv} A) \cup (\operatorname{conv} B)] \subset \operatorname{conv} (A \cup B)$

22. conv $(A \cap B) \subset [(\operatorname{conv} A) \cap (\operatorname{conv} B)]$

23. Find an example in \mathbb{R}^2 to show that equality need not hold in the statement of Exercise 21.

24. Find an example in \mathbb{R}^2 to show that equality need not hold in the statement of Exercise 22.

25. a. Let F and G be subspaces of \mathbb{R}^n . Prove that $F \cup G$ is convex if and only if $F \subset G$ or $G \subset F$.

b. Show by an example that the union of two convex sets may be convex without either of the sets being a subset of the other

26. Let \mathbf{p}_0 , \mathbf{p}_1 , and \mathbf{p}_2 be points in \mathbb{R}^n , and define $\mathbf{f}_0(t) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$, $\mathbf{f}_1(t) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$, and $\mathbf{g}(t) = (1-t)\mathbf{f}_0(t) + t\mathbf{f}_1(t)$ for $0 \le t \le 1$.

a. For the points shown in the figure, draw a picture that shows $\mathbf{g}(\frac{1}{4})$, $\mathbf{g}(\frac{1}{2})$, and $\mathbf{g}(\frac{3}{4})$.



b. The graph of $\mathbf{g}(t)$ is called a *quadratic Bézier curve*, and it is used in some computer graphics designs. The points \mathbf{p}_0 , \mathbf{p}_1 , and \mathbf{p}_2 are called the *control points* for the curve. Compute a formula for $\mathbf{g}(t)$ that involves only \mathbf{p}_0 , \mathbf{p}_1 , and \mathbf{p}_2 . Then show that $\mathbf{g}(t)$ is in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ for $0 \le t \le 1$.

27. Given control points \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 in \mathbb{R}^n , let $\mathbf{g}_1(t)$ for $0 \le t \le 1$ be the quadratic Bézier curve from Exercise 26 determined by \mathbf{p}_0 , \mathbf{p}_1 , and \mathbf{p}_2 , and let $\mathbf{g}_2(t)$ be defined similarly for \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 . For $0 \le t \le 1$, define $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$. Show that the graph of $\mathbf{h}(t)$ lies in the convex hull of the four control points. This curve is called a *cubic Bézier curve*, and its definition here is one step of the de Casteljau algorithm for constructing Bézier curves. A Bézier curve of degree k is determined by k+1 control points, and its graph lies in the convex hull of these control points.

28. Given $\mathbf{p}_0, \dots, \mathbf{p}_{k+1}$ in \mathbb{R}^n , let $S_0 = \operatorname{conv} \{\mathbf{p}_0, \dots, \mathbf{p}_k\}$ and $S_1 = \operatorname{conv} \{\mathbf{p}_1, \dots, \mathbf{p}_{k+1}\}$. For $0 \le t \le 1$, let $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$ be curves that lie in S_0 and S_1 , respectively, and let $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$.

Show that $\mathbf{h}(t)$ lies in conv $\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{k+1}\}$. This fact provides the induction step in the proof that for $k \geq 2$, a Bézier curve of degree k lies in the convex hull of its control points.

SOLUTIONS TO PRACTICE PROBLEMS

1. The points v_1 , v_2 , and v_3 are not orthogonal, so compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -8 \\ 2 \\ -3 \end{bmatrix}, \mathbf{p}_1 - \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \mathbf{p}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$$

Augment the matrix $[\mathbf{v}_2 - \mathbf{v}_1 \quad \mathbf{v}_3 - \mathbf{v}_1]$ with both $\mathbf{p}_1 - \mathbf{v}_1$ and $\mathbf{p}_2 - \mathbf{v}_1$, and row reduce:

$$\begin{bmatrix} 1 & -8 & -5 & -3 \\ -1 & 2 & 1 & 0 \\ 3 & -3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix}$$

The third column shows that $\mathbf{p}_1 - \mathbf{v}_1 = \frac{1}{3}(\mathbf{v}_2 - \mathbf{v}_1) + \frac{2}{3}(\mathbf{v}_3 - \mathbf{v}_1)$, which leads to $\mathbf{p}_1 = 0\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3$. Thus \mathbf{p}_1 is in conv S. In fact, \mathbf{p}_1 is in conv $\{\mathbf{v}_2, \mathbf{v}_3\}$.

The last column of the matrix shows that $\mathbf{p}_2 - \mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$. Thus \mathbf{p}_2 is not an affine combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 , so \mathbf{p}_2 cannot possibly be in conv S.

An alternative method of solution is to row reduce the augmented matrix of homogeneous forms:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}}_1 & \tilde{\mathbf{p}}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. If **p** is a point above S, then the line through **p** with slope -1 will intersect S at two points before it reaches the positive x- and y-axes.

8.4 HYPERPLANES

Hyperplanes play a special role in the geometry of \mathbb{R}^n because they divide the space into two disjoint pieces, just as a plane separates \mathbb{R}^3 into two parts and a line cuts through \mathbb{R}^2 . The key to working with hyperplanes is to use simple *implicit* descriptions, rather than the *explicit* or parametric representations of lines and planes used in the earlier work with affine sets.¹

An implicit equation of a line in \mathbb{R}^2 has the form ax + by = d. An implicit equation of a plane in \mathbb{R}^3 has the form ax + by + cz = d. Both equations describe the line or plane as the set of all points at which a linear expression, or linear functional, such as ax + by has a fixed value, d.

DEFINITION

A linear functional on \mathbb{R}^n is a linear transformation f from \mathbb{R}^n into \mathbb{R} . For each scalar d in \mathbb{R} , the symbol [f:d] denotes the set of all \mathbf{x} in \mathbb{R}^n at which the value of f is d. That is,

[
$$f:d$$
] is the set $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = d\}$

The **zero** *functional* is the transformation such that $f(\mathbf{x}) = 0$ for all \mathbf{x} in \mathbb{R}^n . All other linear functionals on \mathbb{R}^n are said to be **nonzero**.

- **EXAMPLE 1** In \mathbb{R}^2 , the line x 4y = 13 is a hyperplane in \mathbb{R}^2 , and it is the set of points at which the linear functional f(x, y) = x 4y has the value 13. That is, the line is the set [f:13].
- **EXAMPLE 2** In \mathbb{R}^3 , the plane 5x 2y + 3z = 21 is a hyperplane, the set of points at which the linear functional g(x, y, z) = 5x 2y + 3z has the value 21. This hyperplane is the set [g:21].

¹Parametric representations were introduced in Section 1.5.

If f is a linear functional on \mathbb{R}^n , then the standard matrix of this linear transformation f is a $1 \times n$ matrix A, say $A = [a_1 \ a_2 \ \cdots \ a_n]$. So

[
$$f:0$$
] is the same as $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\} = \text{Nul } A$ (1)

If f is a nonzero functional, then rank A = 1, and dim Nul A = n - 1, by the Rank Theorem.² Thus, the subspace [f:0] has dimension n - 1 and so is a hyperplane. Also, if d is any number in \mathbb{R} , then

[
$$f:d$$
] is the same as $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = d\}$ (2)

Recall from Theorem 6 in Section 1.5 that the set of solutions of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution \mathbf{p} of $A\mathbf{x} = \mathbf{b}$. When A is the standard matrix of the transformation f, this theorem says that

$$[f:d] = [f:0] + \mathbf{p} \quad \text{for any } \mathbf{p} \text{ in } [f:d]$$
(3)

Thus the sets [f:d] are hyperplanes parallel to [f:0]. See Fig. 1.

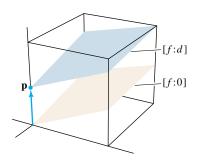


FIGURE 1 Parallel hyperplanes, with $f(\mathbf{p}) = d$.

When *A* is a $1 \times n$ matrix, the equation $A\mathbf{x} = d$ may be written with an inner product $\mathbf{n} \cdot \mathbf{x}$, using \mathbf{n} in \mathbb{R}^n with the same entries as *A*. Thus, from (2),

[
$$f:d$$
] is the same as $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = d\}$ (4)

Then $[f:0] = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = 0 \}$, which shows that [f:0] is the orthogonal complement of the subspace spanned by \mathbf{n} . In the terminology of calculus and geometry for \mathbb{R}^3 , \mathbf{n} is called a **normal** vector to [f:0]. (A "normal" vector in this sense need not have unit length.) Also, \mathbf{n} is said to be **normal** to each parallel hyperplane [f:d], even though $\mathbf{n} \cdot \mathbf{x}$ is not zero when $d \neq 0$.

Another name for [f:d] is a *level set* of f, and \mathbf{n} is sometimes called the *gradient* of f when $f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x}$ for each \mathbf{x} .

²See Theorem 14 in Section 2.9 or Theorem 14 in Section 4.6.

EXAMPLE 3 Let $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$, and let $H = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 12\}$, so H = [f:12], where f(x, y) = 3x + 4y. Thus H is the line 3x + 4y = 12. Find an implicit description of the parallel hyperplane (line) $H_1 = H + \mathbf{v}$.

Solution First, find a point \mathbf{p} in H_1 . To do this, find a point in H and add \mathbf{v} to it. For instance, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ is in H, so $\mathbf{p} = \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is in H_1 . Now, compute $\mathbf{n} \cdot \mathbf{p} = -9$. This shows that $H_1 = [f: -9]$. See Fig. 2, which also shows the subspace $H_0 = {\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 0}$.

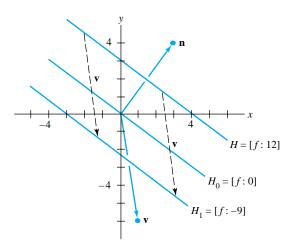


FIGURE 2

The next three examples show connections between implicit and explicit descriptions of hyperplanes. Example 4 begins with an implicit form.

EXAMPLE 4 In \mathbb{R}^2 , write the line x - 4y = 13 in parametric vector form.

Solution This amounts to solving a nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -4 \end{bmatrix}$ and \mathbf{b} is the number 13 in \mathbb{R} . Write x = 13 + 4y, where y is a free variable. In parametric form, the solution is

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 + 4y \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \end{bmatrix} + y \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{p} + y\mathbf{q}, \quad y \in \mathbb{R}$$

Converting an explicit description of a line into implicit form is more involved. The basic idea is to construct [f:0] and then find d for [f:d].

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and let L_1 be the line through \mathbf{v}_1 and \mathbf{v}_2 . Find a linear functional f and a constant d such that $L_1 = [f:d]$.

Solution The line L_1 is parallel to the translated line L_0 through $\mathbf{v}_2 - \mathbf{v}_1$ and the origin. The defining equation for L_0 has the form

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = 0, \quad \text{where} \quad \mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$$
 (5)

Since **n** is orthogonal to the subspace L_0 , which contains $\mathbf{v}_2 - \mathbf{v}_1$, compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and solve

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 0$$

By inspection, a solution is $[a \ b] = [2 \ 5]$. Let f(x, y) = 2x + 5y. From (5), $L_0 = [f:0]$, and $L_1 = [f:d]$ for some d. Since \mathbf{v}_1 is on line $L_1, d = f(\mathbf{v}_1) = 2(1) + 5(2) = 12$. Thus, the equation for L_1 is 2x + 5y = 12. As a check, note that $f(\mathbf{v}_2) = f(6, 0) = 2(6) + 5(0) = 12$, so \mathbf{v}_2 is on L_1 , too.

EXAMPLE 6 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Find an implicit description [f:d] of the plane H_1 that passes through \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 .

Solution H_1 is parallel to a plane H_0 through the origin that contains the translated points

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$
 and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Since these two points are linearly independent, $H_0 = \text{Span} \{ \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1 \}$. Let

 $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the normal to H_0 . Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are each orthogonal to \mathbf{n} . That

is, $(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n} = 0$ and $(\mathbf{v}_3 - \mathbf{v}_1) \cdot \mathbf{n} = 0$. These two equations form a system whose augmented matrix can be row reduced:

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \quad \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \quad \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

Row operations yield $a = (-\frac{2}{4})c$, $b = (\frac{5}{4})c$, with c free. Set c = 4, for instance. Then

$$\mathbf{n} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$
 and $H_0 = [f:0]$, where $f(\mathbf{x}) = -2x_1 + 5x_2 + 4x_3$.

The parallel hyperplane H_1 is [f:d]. To find d, use the fact that \mathbf{v}_1 is in H_1 , and compute $d = f(\mathbf{v}_1) = f(1, 1, 1) = -2(1) + 5(1) + 4(1) = 7$. As a check, compute $f(\mathbf{v}_2) = f(2, -1, 4) = -2(2) + 5(-1) + 4(4) = 16 - 9 = 7$.

The procedure in Example 6 generalizes to higher dimensions. However, for the special case of \mathbb{R}^3 , one can also use the **cross product** formula to compute **n**, using a symbolic determinant as a mnemonic device:

$$\mathbf{n} = (\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_1)$$

$$= \begin{vmatrix} 1 & 2 & \mathbf{i} \\ -2 & 0 & \mathbf{j} \\ 3 & 1 & \mathbf{k} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} \mathbf{k}$$

$$= -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

If only the formula for f is needed, the cross product calculation may be written as an ordinary determinant:

$$f(x_1, x_2, x_3) = \begin{vmatrix} 1 & 2 & x_1 \\ -2 & 0 & x_2 \\ 3 & 1 & x_3 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} x_1 - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} x_2 + \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} x_3$$
$$= -2x_1 + 5x_2 + 4x_3$$

So far, every hyperplane examined has been described as [f:d] for some linear functional f and some d in \mathbb{R} , or equivalently as $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = d\}$ for some \mathbf{n} in \mathbb{R}^n . The following theorem shows that *every* hyperplane has these equivalent descriptions.

THEOREM 11

A subset H of \mathbb{R}^n is a hyperplane if and only if H = [f:d] for some nonzero linear functional f and some scalar d in \mathbb{R} . Thus, if H is a hyperplane, there exist a nonzero vector \mathbf{n} and a real number d such that $H = {\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = d}$.

PROOF Suppose that H is a hyperplane, take $\mathbf{p} \in H$, and let $H_0 = H - \mathbf{p}$. Then H_0 is an (n-1)-dimensional subspace. Next, take any point \mathbf{y} that is not in H_0 . By the Orthogonal Decomposition Theorem in Section 6.3,

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{n}$$

where \mathbf{y}_1 is a vector in H_0 and \mathbf{n} is orthogonal to every vector in H_0 . The function f defined by

$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x}$$
 for $\mathbf{x} \in \mathbb{R}^n$

is a linear functional, by properties of the inner product. Now, [f:0] is a hyperplane that contains H_0 , by construction of **n**. It follows that

$$H_0 = [f:0]$$

[Argument: H_0 contains a basis S of n-1 vectors, and since S is in the (n-1)-dimensional subspace [f:0], S must also be a basis for [f:0], by the Basis Theorem.]

Finally, let $d = f(\mathbf{p}) = \mathbf{n} \cdot \mathbf{p}$. Then, as in (3) shown earlier,

$$[f:d] = [f:0] + \mathbf{p} = H_0 + \mathbf{p} = H$$

The converse statement that [f:d] is a hyperplane follows from (1) and (3).

Many important applications of hyperplanes depend on the possibility of "separating" two sets by a hyperplane. Intuitively, this means that one of the sets is on one side of the hyperplane and the other set is on the other side. The following terminology and notation will help to make this idea more precise.

TOPOLOGY IN \mathbb{R}^n : TERMS AND FACTS

For any point **p** in \mathbb{R}^n and any real $\delta > 0$, the **open ball** $B(\mathbf{p}, \delta)$ with center **p** and radius δ is given by

$$B(\mathbf{p}, \delta) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{p}\| < \delta\}$$

Given a set S, a point \mathbf{p} is an **interior point** of S if there exists a $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset S$. If every open ball centered at \mathbf{p} intersects both S and the complement of S, then \mathbf{p} is called a **boundary point** of S. A set is **open** if it contains none of its boundary points. (This is equivalent to saying that all of its points are interior points.) A set is **closed** if it contains all of its boundary points. (If S contains some but not all of its boundary points, then S is neither open nor closed.) A set S is **bounded** if there exists a S of such that $S \subset B(\mathbf{0}, S)$. A set is **compact** if it is closed and bounded.

Theorem: The convex hull of an open set is open, and the convex hull of a compact set is compact. (The convex hull of a closed set need not be closed. See Exercise 17.)

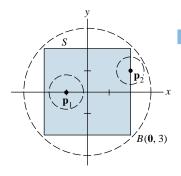


FIGURE 3 The set *S* is closed and bounded.

EXAMPLE 7 Let $S = \text{conv}\left\{ \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}, \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

 $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, as shown in Fig. 3. Then \mathbf{p}_1 is an interior point since $B\left(\mathbf{p}, \frac{3}{4}\right) \subset S$. The point \mathbf{p}_2 is a boundary point since every open ball centered at \mathbf{p}_2 intersects both S and the complement of S. The set S is closed since it contains all its boundary points. The set S is bounded since $S \subset B(\mathbf{0}, 3)$. Thus S is also compact.

Notation: If f is a linear functional, then $f(A) \le d$ means $f(\mathbf{x}) \le d$ for each $\mathbf{x} \in A$. Corresponding notations will be used when the inequalities are reversed or when they are strict.

DEFINITION

The hyperplane H = [f:d] separates two sets A and B if one of the following holds:

(i)
$$f(A) \le d$$
 and $f(B) \ge d$, or

(ii)
$$f(A) \ge d$$
 and $f(B) \le d$.

If in the conditions above all the weak inequalities are replaced by strict inequalities, then H is said to **strictly separate** A and B.

Notice that strict separation requires that the two sets be disjoint, while mere separation does not. Indeed, if two circles in the plane are externally tangent, then their common tangent line separates them (but does not separate them strictly).

Although it is necessary that two sets be disjoint in order to strictly separate them, this condition is not sufficient, even for closed convex sets. Indeed, in \mathbb{R}^2 , let

$$A = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge \frac{1}{2} \quad \text{and} \quad \frac{1}{x} \le y \le 2 \right\} \quad \text{and} \quad B = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge 0 \text{ and } y = 0 \right\}$$

Then A and B are disjoint closed convex sets, but they cannot be strictly separated by a hyperplane (line in \mathbb{R}^2). See Fig. 4. Thus the problem of separating (or strictly separating) two sets by a hyperplane is more complex than it might at first appear.

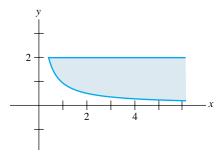


FIGURE 4 Disjoint closed convex sets.

There are many interesting conditions on the sets A and B that imply the existence of a separating hyperplane, but the following two theorems are sufficient for this section. The proof of the first theorem requires quite a bit of preliminary material, but the second theorem follows easily from the first.

THEOREM 12

Suppose A and B are nonempty convex sets such that A is compact and B is closed. Then there exists a hyperplane B that strictly separates A and B if and only if $A \cap B = \emptyset$.

³A proof of Theorem 12 is given in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Melbourne, FL: Krieger Pub., 1992), pp. 34–39.

THEOREM 13

Suppose *A* and *B* are nonempty compact sets. Then there exists a hyperplane that strictly separates *A* and *B* if and only if $(conv A) \cap (conv B) = \emptyset$.

PROOF Suppose that $(\operatorname{conv} A) \cap (\operatorname{conv} B) = \emptyset$. Since the convex hull of a compact set is compact, Theorem 12 ensures that there is a hyperplane H that strictly separates conv A and conv B. Clearly, H also strictly separates the smaller sets A and B.

Conversely, suppose the hyperplane H = [f:d] strictly separates A and B. Without loss of generality, assume that f(A) < d and f(B) > d. Let $\mathbf{x} = c_1 \mathbf{x}_1 + \cdots + c_k \mathbf{x}_k$ be any convex combination of elements of A. Then

$$f(\mathbf{x}) = c_1 f(\mathbf{x}_1) + \dots + c_k f(\mathbf{x}_k) < c_1 d + \dots + c_k d = d$$

since $c_1 + \cdots + c_k = 1$. Thus f(conv A) < d. Likewise, f(conv B) > d, so H = [f:d] strictly separates conv A and conv B. By Theorem 12, conv A and conv B must be disjoint.

EXAMPLE 8 Let $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$,

and let $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Show that the hyperplane H = [f:5], where $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + x_3$, does not separate A and B. Is there a hyperplane parallel to H that does separate A and B? Do the convex hulls of A and B intersect?

Solution Evaluate the linear functional f at each of the points in A and B:

$$f(\mathbf{a}_1) = 2$$
, $f(\mathbf{a}_2) = -11$, $f(\mathbf{a}_3) = -6$, $f(\mathbf{b}_1) = 4$, and $f(\mathbf{b}_2) = 12$

Since $f(\mathbf{b}_1) = 4$ is less than 5 and $f(\mathbf{b}_2) = 12$ is greater than 5, points of B lie on both sides of H = [f:5] and so H does not separate A and B.

Since f(A) < 3 and f(B) > 3, the parallel hyperplane [f:3] strictly separates A and B. By Theorem 13, $(\operatorname{conv} A) \cap (\operatorname{conv} B) = \emptyset$.

Caution: If there were no hyperplane parallel to *H* that strictly separated *A* and *B*, this would *not* necessarily imply that their convex hulls intersect. It might be that some other hyperplane not parallel to *H* would strictly separate them.

PRACTICE PROBLEM

Let
$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$
, $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{n}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{n}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$, let H_1 be the hyperplane

(plane) in \mathbb{R}^3 passing through the point \mathbf{p}_1 and having normal vector \mathbf{n}_1 , and let H_2 be the hyperplane passing through the point \mathbf{p}_2 and having normal vector \mathbf{n}_2 . Describe $H_1 \cap H_2$.

8.4 EXERCISES

- **1.** Let L be the line in \mathbb{R}^2 through the points $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find a linear functional f and a real number d such that L = [f:d].
- **2.** Let L be the line in \mathbb{R}^2 through the points $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$. Find a linear functional f and a real number d such that L = [f:d].

In Exercises 3–6, let H be the hyperplane through the listed points. (a) Find a normal vector **n** to the hyperplane. (b) Find a linear functional f and a real number d such that H = [f:d].

$$\mathbf{3.} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

$$, \begin{bmatrix} 2\\4\\1 \end{bmatrix}, \begin{bmatrix} -1\\-2\\5 \end{bmatrix} \qquad \qquad \mathbf{4.} \begin{bmatrix} 1\\-2\\1 \end{bmatrix}, \begin{bmatrix} 4\\-2\\3 \end{bmatrix}, \begin{bmatrix} 7\\-4\\4 \end{bmatrix}$$

5.
$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
, $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

6.
$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

7. Let
$$\mathbf{p} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 2 \end{bmatrix}$$
, $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$, and

 $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, and let H be the hyperplane in \mathbb{R}^4 with normal \mathbf{n}

and passing through **p**. Which of the points \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are on the same side of H as the origin, and which are not?

8. Let
$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$
, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{n} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, and let

 $A = {\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3}$ and $B = {\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3}$. Find a hyperplane Hwith normal \mathbf{n} that separates A and B. Is there a hyperplane parallel to H that strictly separates A and B?

9. Let
$$\mathbf{p}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 2 \end{bmatrix}$$
, $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{n}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}$, and $\mathbf{n}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}$,

let H_1 be the hyperplane in \mathbb{R}^4 through \mathbf{p}_1 with normal \mathbf{n}_1 , and let H_2 be the hyperplane through \mathbf{p}_2 with normal \mathbf{n}_2 . Find a point **p** in $H_1 \cap H_2$ and two linearly independent vectors \mathbf{v}_1 and \mathbf{v}_2 that span a subspace parallel to the 2-dimensional flat $H_1 \cap H_2$.

10. Let F_1 and F_2 be 4-dimensional flats in \mathbb{R}^6 , and suppose that $F_1 \cap F_2 \neq \emptyset$. What are the possible dimensions of $F_1 \cap F_2$?

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

- 11. a. A linear transformation from \mathbb{R} to \mathbb{R}^n is called a linear functional.
 - b. If f is a linear functional defined on \mathbb{R}^n , then there exists a real number k such that $f(\mathbf{x}) = k\mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n .
 - c. If a hyperplane strictly separates sets A and B, then $A \cap B = \emptyset$.
 - d. If A and B are closed convex sets and $A \cap B = \emptyset$, then there exists a hyperplane that strictly separates A and B.
- **12.** a. If d is a real number and f is a nonzero linear functional defined on \mathbb{R}^n , then [f:d] is a hyperplane in \mathbb{R}^n .
 - b. Given any vector \mathbf{n} and any real number d, the set $\{\mathbf{x}: \mathbf{n} \cdot \mathbf{x} = d\}$ is a hyperplane.
 - c. If A and B are nonempty disjoint sets such that A is compact and B is closed, then there exists a hyperplane that strictly separates A and B.
 - d. If there exists a hyperplane H such that H does not strictly separate two sets A and B, then $(\operatorname{conv} A) \cap (\operatorname{conv} B) \neq \emptyset$.
- **13.** Prove that the open ball $B(\mathbf{p}, \delta) = {\mathbf{x} : ||\mathbf{x} \mathbf{p}|| < \delta}$ is a convex set.
- 14. Prove that the convex hull of a bounded set is bounded.
- **15.** Let $\mathbf{p} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Find a hyperplane [f:d] that strictly separates $B(\overline{\mathbf{0}}, \overline{\mathbf{3}})$ and $B(\mathbf{p}, 1)$. [Hint: After finding f, show that the point $\mathbf{v} = (1 - .75)\mathbf{0} + .75\mathbf{p}$ is neither in $B(\mathbf{0}, 3)$ nor in $B(\mathbf{p}, 1).$
- **16.** Let $\mathbf{q} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$. Find a hyperplane [f:d] that strictly separates $B(\mathbf{q}, 3)$ and $B(\mathbf{p}, 1)$.

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Exercises 19–30 use the following notations: Let S be an n-dimensional subset of \mathbb{R}^n . Denote the set of all interior points of S by int S, and the set of all boundary points of S by bd S. Also, define the **closure** of S, denoted cl S, by cl $S = S \cup$ bd S. For any $\lambda \in \mathbb{R}$, let $\lambda S = \{\lambda \mathbf{s} : \mathbf{s} \in S\}$. Prove the following. You may use the fact that if S is an n-dimensional convex subset of \mathbb{R}^n , then int $S \neq \emptyset$. You may also use the results in the earlier exercises in the proofs of the later exercises. (It may be helpful to draw a diagram in doing the proofs, but the proof needs to be more than just a diagram.)

19. Let $\mathbf{p} \in \mathbb{R}^n$ and let $\delta > 0$.

a. If $\lambda > 0$, then $\lambda B(\mathbf{p}, \delta) = B(\lambda \mathbf{p}, \lambda \delta)$.

b. If $\lambda < 0$, then $\lambda B(\mathbf{p}, \delta) = B(\lambda \mathbf{p}, |\lambda| \delta)$.

20. A point **p** is in cl *S* if and only if, for every $\delta > 0$, the open ball $B(\mathbf{p}, \delta)$ contains at least one point of *S*.

21. cl (cl S) = cl S

22. If $\mathbf{x} \in \text{int } S$ and $\mathbf{y} \in S$, then for every λ such that $0 < \lambda < 1$, the point $\mathbf{z} = \lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in \text{int } S$.

23. If $\mathbf{x} \in \text{int } S$ and $\mathbf{y} \in \text{cl } S$, then for every λ such that $0 < \lambda < 1$, the point $\mathbf{z} = \lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in \text{int } S$.

24. If *S* is convex, then int *S* is convex.

25. If *S* is convex, then cl *S* is convex.

26. a. If S is convex, then cl (int S) = cl S.

b. Find an example to show that part (a) may not hold if *S* is not convex.

27. a. If S is convex, then int (cl S) = int S.

b. Find an example to show that part (a) may not hold if *S* is not convex.

28. a. If *S* is convex, then bd (cl S) = bd S.

b. Find an example to show that part (a) may not hold if *S* is not convex.

29. If *S* is convex, then bd *S* may be a convex set.

30. If *S* is convex, then bd *S* may be a non-convex set.

SOLUTION TO PRACTICE PROBLEM

First, compute $\mathbf{n}_1 \cdot \mathbf{p}_1 = -3$ and $\mathbf{n}_2 \cdot \mathbf{p}_2 = 7$. The hyperplane H_1 is the solution set of the equation $x_1 + x_2 - 2x_3 = -3$, and H_2 is the solution set of the equation $-2x_1 + x_2 + 3x_3 = 7$. Next, solve this system of equations by row reduction:

$$\begin{bmatrix} 1 & 1 & -2 & -3 \\ -2 & 1 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{10}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Thus
$$x_1 = -\frac{10}{3} + \frac{5}{3}x_3$$
, $x_2 = \frac{1}{3} + \frac{1}{3}x_3$, $x_3 = x_3$. Let $\mathbf{p} = \begin{bmatrix} -\frac{10}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$. The

general solution can be written as $\mathbf{x} = \mathbf{p} + x_3 \mathbf{v}$. This is the line through \mathbf{p} in the direction of \mathbf{v} . Note that \mathbf{v} is orthogonal to both \mathbf{n}_1 and \mathbf{n}_2 .

8.5 POLYTOPES

This section studies geometric properties of an important class of compact convex sets called polytopes. These sets arise in all sorts of applications, including game theory (Section 9.1), linear programming (Sections 9.2 to 9.4), and more general optimization problems, such as the design of feedback controls for engineering systems.

A **polytope** in \mathbb{R}^n is the convex hull of a finite set of points. In \mathbb{R}^2 , a polytope is simply a polygon. In \mathbb{R}^3 , a polytope is called a polyhedron. Important features of a polyhedron are its faces, edges, and vertices. For example, the cube has 6 square faces, 12 edges, and 8 vertices. The following definitions provide terminology for higher dimensions as well as \mathbb{R}^2 and \mathbb{R}^3 . Recall that the dimension of a set in \mathbb{R}^n is the dimension of the smallest flat that contains it. Also, note that a polytope is a special type of compact convex set, because a finite set in \mathbb{R}^n is compact and the convex hull of this set is compact, by the theorem in the topology facts box in Section 8.4.

DEFINITION

Let *S* be a compact convex subset of \mathbb{R}^n . A nonempty subset *F* of *S* is called a (proper) **face** of *S* if $F \neq S$ and there exists a hyperplane H = [f:d] such that $F = S \cap H$ and either $f(S) \leq d$ or $f(S) \geq d$. The hyperplane *H* is called a **supporting hyperplane** to *S*. If the dimension of *F* is *k*, then *F* is called a *k*-face of *S*

If P is a polytope of dimension k, then P is called a k-polytope. A 0-face of P is called a **vertex** (plural: **vertices**), a 1-face is an **edge**, and a (k-1)-dimensional face is a **facet** of S.

EXAMPLE 1 Suppose S is a cube in \mathbb{R}^3 . When a plane H is translated through \mathbb{R}^3 until it just touches (supports) the cube but does not cut through the interior of the cube, there are three possibilities for $H \cap S$, depending on the orientation of H. (See Figure 1.)

 $H \cap S$ may be a 2-dimensional square face (facet) of the cube.

 $H \cap S$ may be a 1-dimensional edge of the cube.

 $H \cap S$ may be a 0-dimensional vertex of the cube.

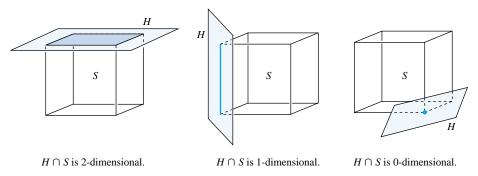


FIGURE 1

Most applications of polytopes involve the vertices in some way, because they have a special property that is identified in the following definition.

DEFINITION

Let *S* be a convex set. A point **p** in *S* is called an **extreme point** of *S* if **p** is not in the interior of any line segment that lies in *S*. More precisely, if $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{p} \in \overline{\mathbf{xy}}$, then $\mathbf{p} = \mathbf{x}$ or $\mathbf{p} = \mathbf{y}$. The set of all extreme points of *S* is called the **profile** of *S*.

A vertex of any compact convex set S is automatically an extreme point of S. This fact is proved during the proof of Theorem 14, below. In working with a polytope, say $P = \text{conv } \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for $\mathbf{v}_1, \dots, \mathbf{v}_k$ in \mathbb{R}^n , it is usually helpful to know that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are the extreme points of P. However, such a list might contain extraneous points. For example, some vector \mathbf{v}_i could be the midpoint of an edge of the polytope. Of course, in this case \mathbf{v}_i is not really needed to generate the convex hull. The following definition describes the property of the vertices that will make them all extreme points.

DEFINITION

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a **minimal representation** of the polytope P if $P = \text{conv } \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and for each $i = 1, \dots, k, \mathbf{v}_i \notin \text{conv } \{\mathbf{v}_j : j \neq i\}$.

Every polytope has a minimal representation. For if $P = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and if some \mathbf{v}_i is a convex combination of the other points, then \mathbf{v}_i may be deleted from the set of points without changing the convex hull. This process may be repeated until the minimal representation is left. It can be shown that the minimal representation is unique.

THEOREM 14

Suppose $M = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is the minimal representation of the polytope P. Then the following three statements are equivalent:

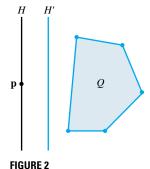
- a. $\mathbf{p} \in M$.
- b. \mathbf{p} is a vertex of P.
- c. \mathbf{p} is an extreme point of P.

PROOF (a) \Rightarrow (b) Suppose $\mathbf{p} \in M$ and let $Q = \operatorname{conv} \{ \mathbf{v} : \mathbf{v} \in M \text{ and } \mathbf{v} \neq \mathbf{p} \}$. It follows from the definition of M that $\mathbf{p} \notin Q$, and since Q is compact, Theorem 13 implies the existence of a hyperplane H' that strictly separates $\{\mathbf{p}\}$ and Q. Let H be the hyperplane through \mathbf{p} parallel to H'. See Fig. 2.

Then Q lies in one of the closed half-spaces H^+ bounded by H and so $P \subseteq H^+$. Thus H supports P at \mathbf{p} . Furthermore, \mathbf{p} is the only point of P that can lie on H, so $H \cap P = \{\mathbf{p}\}$ and \mathbf{p} is a vertex of P.

(b) \Rightarrow (c) Let **p** be a vertex of P. Then there exists a hyperplane H = [f:d] such that $H \cap P = \{\mathbf{p}\}$ and $f(P) \geq d$. If **p** were not an extreme point, then there would exist points **x** and **y** in P such that $\mathbf{p} = (1 - c)\mathbf{x} + c\mathbf{y}$ with 0 < c < 1. That is,

$$c\mathbf{y} = \mathbf{p} - (1 - c)\mathbf{x}$$
 and $\mathbf{y} = \left(\frac{1}{c}\right)(\mathbf{p}) - \left(\frac{1}{c} - 1\right)(\mathbf{x})$



It follows that $f(\mathbf{y}) = \frac{1}{c} f(\mathbf{p}) - \left(\frac{1}{c} - 1\right) f(\mathbf{x})$. But $f(\mathbf{p}) = d$ and $f(\mathbf{x}) \ge d$, so $f(\mathbf{y}) \le \left(\frac{1}{c}\right) (d) - \left(\frac{1}{c} - 1\right) (d) = d$

On the other hand, $\mathbf{y} \in P$, so $f(\mathbf{y}) \ge d$. It follows that $f(\mathbf{y}) = d$ and that $\mathbf{y} \in H \cap P$. This contradicts the fact that \mathbf{p} is a vertex. So \mathbf{p} must be an extreme point. (Note that this part of the proof does not depend on P being a polytope. It holds for any compact convex set.)

(c) \Rightarrow (a) It is clear that any extreme point of P must be a member of M.

EXAMPLE 2 Recall that the profile of a set S is the set of extreme points of S. Theorem 14 shows that the profile of a polygon in \mathbb{R}^2 is the set of vertices. (See Fig. 3.) The profile of a closed ball is its boundary. An open set has no extreme points, so its profile is empty. A closed half-space has no extreme points, so its profile is empty.



FIGURE 3

Exercise 10 asks you to show that a point \mathbf{p} in a convex set S is an extreme point of S if and only if, when \mathbf{p} is removed from S, the remaining points still form a convex set. It follows that if S^* is any subset of S such that conv S^* is equal to S, then S^* must contain the profile of S. The sets in Example 2 show that in general S^* may have to be larger than the profile of S. It is true, however, that when S is compact we may actually take S^* to be the profile of S, as Theorem 15 will show. Thus every nonempty compact set S has an extreme point, and the set of all extreme points is the smallest subset of S whose convex hull is equal to S.

THEOREM 15

Let S be a nonempty compact convex set. Then S is the convex hull of its profile (the set of extreme points of S).

PROOF The proof is by induction on the dimension of the set S.¹

One important application of Theorem 15 is the following theorem. It is one of the key theoretical results in the development of linear programming. Linear functionals are continuous, and continuous functions always attain their maximum and minimum on a compact set. The significance of Theorem 16 is that for compact convex sets, the maximum (and minimum) is actually attained at an extreme point of *S*.

¹The details may be found in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Melbourne, FL: Krieger Pub., 1992), p. 43.

THEOREM 16

Let f be a linear functional defined on a nonempty compact convex set S. Then there exist extreme points $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ of S such that

$$f(\hat{\mathbf{v}}) = \max_{\mathbf{v} \in S} f(\mathbf{v})$$
 and $f(\hat{\mathbf{w}}) = \min_{\mathbf{v} \in S} f(\mathbf{v})$

PROOF We assume that f attains its maximum m on S at some point \mathbf{v}' in S. That is, $f(\mathbf{v}') = m$. We wish to show that there exists an extreme point in S with the same property. By Theorem 15, \mathbf{v}' is a convex combination of the extreme points of S. That is, there exist extreme points $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of S and nonnegative c_1, \ldots, c_k such that

$$\mathbf{v}' = c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k$$
 with $c_1 + \dots + c_k = 1$

If none of the extreme points of S satisfy $f(\mathbf{v}) = m$, then

$$f(\mathbf{v}_i) < m$$
 for $i = 1, \dots, k$

since m is the maximum of f on S. But then, because f is linear,

$$m = f(\mathbf{v}') = f(c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k)$$

= $c_1 f(\mathbf{v}_1) + \dots + c_k f(\mathbf{v}_k)$
< $c_1 m + \dots + c_k m = m(c_1 + \dots + c_k) = m$

This contradiction implies that some extreme point $\hat{\mathbf{v}}$ of S must satisfy $f(\hat{\mathbf{v}}) = m$. The proof for $\hat{\mathbf{w}}$ is similar.

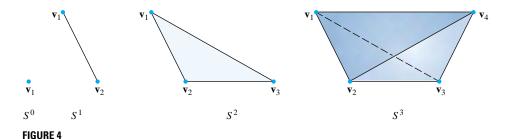
The remainder of this section discusses the construction of two basic polytopes in \mathbb{R}^3 (and higher dimensions). The first appears in linear programming problems, the subject of Chapter 9. Both polytopes provide opportunities to visualize \mathbb{R}^4 in a remarkable way.

Simplex

A **simplex** is the convex hull of an affinely independent finite set of vectors. To construct a k-dimensional simplex (or k-simplex), proceed as follows:

```
0-simplex S^0: a single point \{\mathbf{v}_1\}
1-simplex S^1: conv (S^0 \cup \{\mathbf{v}_2\}), with \mathbf{v}_2 not in aff S^0
2-simplex S^2: conv (S^1 \cup \{\mathbf{v}_3\}), with \mathbf{v}_3 not in aff S^1
\vdots
k-simplex S^k: conv (S^{k-1} \cup \{\mathbf{v}_{k+1}\}), with \mathbf{v}_{k+1} not in aff S^{k-1}
```

The simplex S^1 is a line segment. The triangle S^2 comes from choosing a point \mathbf{v}_3 that is not in the line containing S_1 and then forming the convex hull with S_2 . (See Fig. 4.) The tetrahedron S_3 is produced by choosing a point \mathbf{v}_4 not in the plane of S^2 and then forming the convex hull with S^2 .



Before continuing, consider some of the patterns that are being followed. The triangle S^2 has three edges. Each of these edges is a line segment like S^1 . Where do these three line segments come from? One of them is S^1 . One of them comes by joining the endpoint \mathbf{v}_2 to the new point \mathbf{v}_3 . The third comes from joining the other endpoint \mathbf{v}_1 to \mathbf{v}_3 . You might say that each endpoint in S^1 is stretched out into a line segment in S^2 .

The tetrahedron S^3 has four triangular faces. One of these is the original triangle S^2 , and the other three come from stretching the edges of S^2 out to the new point \mathbf{v}_4 . Notice too that the vertices of S^2 get stretched out into edges in S^3 . The other edges in S^3 come from the edges in S^2 . This suggests how to "visualize" the four-dimensional S^4 .

The construction of S^4 , called a pentatope, involves forming the convex hull of S^3 with a point \mathbf{v}_5 not in the 3-space of S^3 . A complete picture is impossible, of course, but Fig. 5 is suggestive: S^4 has five vertices, and any four of the vertices determine a

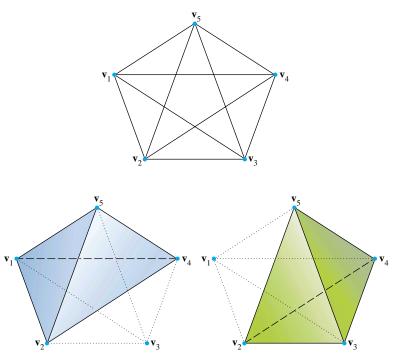


FIGURE 5 The 4-dimensional simplex S^4 projected onto \mathbb{R}^2 , with two tetrahedral facets emphasized.

facet in the shape of a tetrahedron. For example, the figure emphasizes the facet with vertices \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_4 , and \mathbf{v}_5 and the facet with vertices \mathbf{v}_2 , \mathbf{v}_3 , \mathbf{v}_4 , and \mathbf{v}_5 . There are five such facets. Figure 5 identifies all ten edges of S^4 , and these can be used to visualize the ten triangular faces.

Figure 6 shows another representation of the 4-dimensional simplex S^4 . This time the fifth vertex appears "inside" the tetrahedron S^3 . The highlighted tetrahedral facets also appear to be "inside" S^3 .

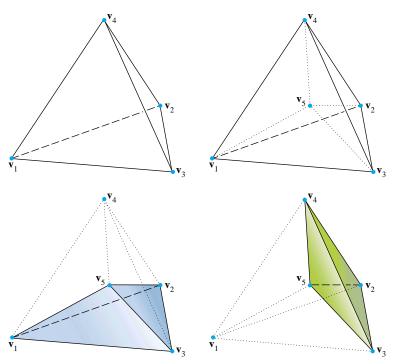


FIGURE 6 The fifth vertex of S^4 is "inside" S^3 .

Hypercube

Let $I_i = \overline{\mathbf{0}\mathbf{e}_i}$ be the line segment from the origin $\mathbf{0}$ to the standard basis vector \mathbf{e}_i in \mathbb{R}^n . Then for k such that $1 \le k \le n$, the vector sum²

$$C^k = I_1 + I_2 + \cdots + I_k$$

is called a *k*-dimensional **hypercube**.

To visualize the construction of C^k , start with the simple cases. The hypercube C^1 is the line segment I_1 . If C^1 is translated by \mathbf{e}_2 , the convex hull of its initial and final

²The vector sum of two sets A and B is defined by $A + B = \{ \mathbf{a} + \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B \}.$

positions describes a square C^2 . (See Fig. 7.) Translating C^2 by \mathbf{e}_3 creates the cube C^3 . A similar translation of C^3 by the vector \mathbf{e}_4 yields the 4-dimensional hypercube C^4 .

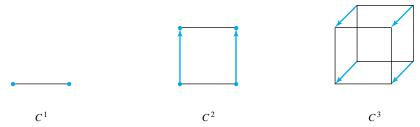


FIGURE 7 Constructing the cube C^3 .

Again, this is hard to visualize, but Fig. 8 shows a 2-dimensional projection of C^4 . Each of the edges of C^3 is stretched into a square face of C^4 . And each of the square faces of C^3 is stretched into a cubic face of C^4 . Figure 9 shows three facets of C^4 . Part (a) highlights the cube that comes from the left square face of C^3 . Part (b) shows the cube that comes from the front square face of C^3 . And part (c) emphasizes the cube that comes from the top square face of C^3 .

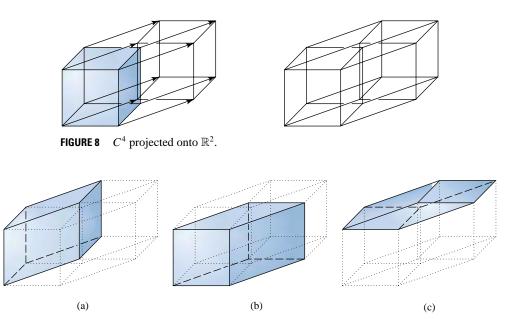


FIGURE 9 Three of the cubic facets of C^4 .

Figure 10 shows another representation of C^4 in which the translated cube is placed "inside" C^3 . This makes it easier to visualize the cubic facets of C^4 , since there is less distortion.

Altogether, the 4-dimensional cube C^4 has 8 cubic faces. Two come from the original and translated images of C^3 , and six come from the square faces of C^3 that are

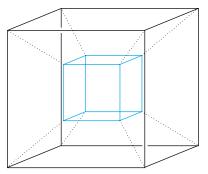


FIGURE 10 The translated image of C^3 is placed "inside" C^3 to obtain C^4 .

stretched into cubes. The square 2-dimensional faces of C^4 come from the square faces of C^3 and its translate, and the edges of C^3 that are stretched into squares. Thus there are $2 \times 6 + 12 = 24$ square faces. To count the edges, take 2 times the number of edges in C^3 and add the number of vertices in C^3 . This makes $2 \times 12 + 8 = 32$ edges in C^4 . The vertices in C^4 all come from C^3 and its translate, so there are $2 \times 8 = 16$ vertices.

One of the truly remarkable results in the study of polytopes is the following formula, first proved by Leonard Euler (1707–1783). It establishes a simple relationship between the number of faces of different dimensions in a polytope. To simplify the statement of the formula, let $f_k(P)$ denote the number of k-dimensional faces of an n-dimensional polytope P.³

Euler's formula:
$$\sum_{k=0}^{n-1} (-1)^k f_k(P) = 1 + (-1)^{n-1}$$

In particular, when n = 3, v - e + f = 2, where v, e, and f denote the number of vertices, edges, and facets (respectively) of P.

8.5 EXERCISES

- **1.** Let $S = \{(x, y) : x^2 + (y 1)^2 \le 1\} \cup \{(3, 0)\}$. Is the origin an extreme point of conv S? Is the origin a vertex of conv S?
- **2.** Find an example of a closed convex set S in \mathbb{R}^2 such that its profile P is nonempty but conv $P \neq S$.
- **3.** Find an example of a bounded convex set *S* in \mathbb{R}^2 such that its profile *P* is nonempty but conv $P \neq S$.
- **4.** a. Determine the number of k-faces of the 5-dimensional simplex S^5 for k = 0, 1, ..., 4. Verify that your answer satisfies Euler's formula.
 - b. Make a chart of the values of $f_k(S^n)$ for n = 1, ..., 5 and k = 0, 1, ..., 4. Can you see a pattern? Guess at a general formula for $f_k(S^n)$.

³A proof is in Steven R. Lay, *Convex Sets and Their Applications* (New York: John Wiley & Sons, 1982; Melbourne, FL: Krieger Pub., 1992), p. 131.

- **5.** a. Determine the number of k-faces of the 5-dimensional hypercube C^5 for $k = 0, 1, \ldots, 4$. Verify that your answer satisfies Euler's formula.
 - b. Make a chart of the values of $f_k(\mathbb{C}^n)$ for n = 1, ..., 5 and k = 0, 1, ..., 4. Can you see a pattern? Guess at a general formula for $f_k(\mathbb{C}^n)$.
- **6.** Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent vectors in \mathbb{R}^n $(1 \le k \le n)$. Then the set $X^k = \text{conv } \{\pm \mathbf{v}_1, \ldots, \pm \mathbf{v}_k\}$ is called a k-crosspolytope.
 - a. Sketch X^1 and X^2 .
 - b. Determine the number of k-faces of the 3-dimensional crosspolytope X^3 for k = 0, 1, 2. What is another name for X^3 ?
 - c. Determine the number of k-faces of the 4-dimensional crosspolytope X^4 for k=0,1,2,3. Verify that your answer satisfies Euler's formula.
 - d. Find a formula for $f_k(X^n)$, the number of k-faces of X^n , for 0 < k < n 1.
- 7. A *k*-pyramid P^k is the convex hull of a (k-1)-polytope Q and a point $\mathbf{x} \notin \text{aff } Q$. Find a formula for each of the following in terms of $f_i(Q)$, $j=1,\ldots,k-1$.
 - a. The number of vertices of P^n : $f_0(P^n)$.
 - b. The number of k-faces of P^n : $f_k(P^n)$ for $1 \le k \le n-2$.
 - c. The number of (n-1)-dimensional facets of P^n : $f_{n-1}(P^n)$.

In Exercises 8 and 9, mark each statement True or False. Justify each answer.

- **8.** a. A polytope is the convex hull of a finite set of points.
 - b. Let **p** be an extreme point of a convex set *S*. If $\mathbf{u}, \mathbf{v} \in S$, $\mathbf{p} \in \overline{\mathbf{u}}\overline{\mathbf{v}}$, and $\mathbf{p} \neq \mathbf{u}$, then $\mathbf{p} = \mathbf{v}$.
 - c. If *S* is a nonempty convex subset of \mathbb{R}^n , then *S* is the convex hull of its profile.
 - d. The 4-dimensional simplex S⁴ has exactly five facets, each of which is a 3-dimensional tetrahedron.

- **9.** a. A cube in \mathbb{R}^3 has five facets.
 - b. A point **p** is an extreme point of a polytope *P* if and only if **p** is a vertex of *P*.
 - c. If *S* is a nonempty compact convex set and a linear functional attains its maximum at a point **p**, then **p** is an extreme point of *S*.
 - d. A 2-dimensional polytope always has the same number of vertices and edges.
- 10. Let **v** be an element of the convex set *S*. Prove that **v** is an extreme point of *S* if and only if the set $\{\mathbf{x} \in S : \mathbf{x} \neq \mathbf{v}\}$ is convex.
- **11.** If A and B are convex sets, prove that A + B is convex.
- **12.** If $c \in \mathbb{R}$ and S is a set, define $cS = \{c\mathbf{x} : \mathbf{x} \in S\}$.
 - a. Let *S* be a convex set and suppose c > 0 and d > 0. Prove that cS + dS = (c + d)S.
 - b. Find an example to show that the convexity of *S* is necessary in part (a).
- 13. A polyhedron (3-polytope) is called regular if all its facets are congruent regular polygons and all the angles at the vertices are equal. Supply the details in the following proof that there are only five regular polyhedra.
 - a. Suppose that a regular polyhedron has r facets, each of which is a k-sided regular polygon, and that s edges meet at each vertex. Letting v and e denote the numbers of vertices and edges in the polyhedron, explain why kr = 2e and sv = 2e.
 - b. Use Euler's formula to show that 1/s + 1/k = 1/2 + 1/e.
 - c. Find all the integral solutions of the equation in part (b) that satisfy the geometric constraints of the problem. (How small can k and s be?)

For your information, the five regular polyhedra are the tetrahedron (4, 6, 4), the cube (8, 12, 6), the octahedron (6, 12, 8), the dodecahedron (20, 30, 12), and the icosahedron (12, 30, 20). (The numbers in parentheses indicate the numbers of vertices, edges, and faces, respectively.)

8.6 CURVES AND SURFACES

For thousands of years, builders used long thin strips of wood to create the hull of a boat. In more recent times, designers used long flexible metal strips to lay out the surfaces of cars and airplanes. Weights and pegs shaped the strips into smooth curves called *natural cubic splines*. The curve between two successive control points (pegs or weights) has a parametric representation using cubic polynomials. Unfortunately, such curves have

the property that moving one control point affects the shape of the entire curve, because of physical forces that the pegs and weights exert on the strip. Design engineers had long wanted local control of the curve—in which movement of one control point would affect only a small portion of the curve. In 1962, a French automotive engineer, Pierre Bézier, solved this problem by adding extra control points and using a class of curves now called by his name.

Bézier Curves

The curves described below play an important role in computer graphics as well as engineering. For example, they are used in Adobe Illustrator and Macromedia Freehand, and in application programming languages such as OpenGL. These curves permit a program to store exact information about curved segments and surfaces in a relatively small number of control points. All graphics commands for the segments and surfaces have only to be computed for the control points. The special structure of these curves also speeds up other calculations in the "graphics pipeline" that creates the final display on the viewing screen.

Exercises in Section 8.3 introduced quadratic Bézier curves and showed one method for constructing Bézier curves of higher degree. The discussion here focuses on quadratic and cubic Bézier curves, which are determined by three or four control points, denoted by \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 . These points can be in \mathbb{R}^2 or \mathbb{R}^3 , or they can be represented by homogeneous forms in \mathbb{R}^3 or \mathbb{R}^4 . The standard parametric descriptions of these curves, for $0 \le t \le 1$, are

$$\mathbf{w}(t) = (1-t)^2 \mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2 \mathbf{p}_2$$
 (1)

$$\mathbf{x}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2 (1-t) \mathbf{p}_2 + t^3 \mathbf{p}_3$$
 (2)

Figure 1 shows two typical curves. Usually, the curves pass through only the initial and terminal control points, but a Bézier curve is always in the convex hull of its control points. (See Exercises 26–28 in Section 8.3.)

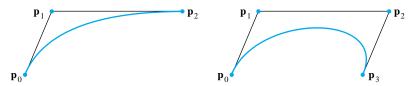


FIGURE 1 Quadratic and cubic Bézier curves.

Bézier curves are useful in computer graphics because their essential properties are preserved under the action of linear transformations and translations. For instance, if A is a matrix of appropriate size, then from the linearity of matrix multiplication, for $0 \le t \le 1$,

$$A\mathbf{x}(t) = A[(1-t)^3\mathbf{p}_0 + 3t(1-t)^2\mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3\mathbf{p}_3]$$

= $(1-t)^3A\mathbf{p}_0 + 3t(1-t)^2A\mathbf{p}_1 + 3t^2(1-t)A\mathbf{p}_2 + t^3A\mathbf{p}_3$

The new control points are $A\mathbf{p}_0, \ldots, A\mathbf{p}_3$. Translations of Bézier curves are considered in Exercise 1.

The curves in Fig. 1 suggest that the control points determine the tangent lines to the curves at the initial and terminal control points. Recall from calculus that for any parametric curve, say $\mathbf{y}(t)$, the direction of the tangent line to the curve at a point $\mathbf{y}(t)$ is given by the derivative $\mathbf{y}'(t)$, called the **tangent vector** of the curve. (This derivative is computed entry by entry.)

EXAMPLE 1 Determine how the tangent vector of the quadratic Bézier curve $\mathbf{w}(t)$ is related to the control points of the curve, at t = 0 and t = 1.

Solution Write the weights in (1) as simple polynomials

$$\mathbf{w}(t) = (1 - 2t + t^2)\mathbf{p}_0 + (2t - 2t^2)\mathbf{p}_1 + t^2\mathbf{p}_2$$

Then, because differentiation is a linear transformation on functions,

$$\mathbf{w}'(t) = (-2 + 2t)\mathbf{p}_0 + (2 - 4t)\mathbf{p}_1 + 2t\mathbf{p}_2$$

So

$$\mathbf{w}'(0) = -2\mathbf{p}_0 + 2\mathbf{p}_1 = 2(\mathbf{p}_1 - \mathbf{p}_0)$$

$$\mathbf{w}'(1) = -2\mathbf{p}_1 + 2\mathbf{p}_2 = 2(\mathbf{p}_2 - \mathbf{p}_1)$$

The tangent vector at \mathbf{p}_0 , for instance, points from \mathbf{p}_0 to \mathbf{p}_1 , but it is twice as long as the segment from \mathbf{p}_0 to \mathbf{p}_1 . Notice that $\mathbf{w}'(0) = \mathbf{0}$ when $\mathbf{p}_1 = \mathbf{p}_0$. In this case, $\mathbf{w}(t) = (1 - t^2)\mathbf{p}_1 + t^2\mathbf{p}_2$, and the graph of $\mathbf{w}(t)$ is the line segment from \mathbf{p}_1 to \mathbf{p}_2 .

Connecting Two Bézier Curves

Two basic Bézier curves can be joined end to end, with the terminal point of the first curve $\mathbf{x}(t)$ being the initial point \mathbf{p}_2 of the second curve $\mathbf{y}(t)$. The combined curve is said to have G^0 geometric continuity (at \mathbf{p}_2) because the two segments join at \mathbf{p}_2 . If the tangent line to curve 1 at \mathbf{p}_2 has a different direction than the tangent line to curve 2, then a "corner," or abrupt change of direction, may be apparent at \mathbf{p}_2 . See Fig. 2.

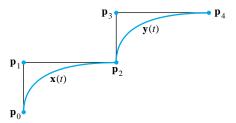


FIGURE 2 G^0 continuity at \mathbf{p}_2 .

To avoid a sharp bend, it usually suffices to adjust the curves to have what is called G^1 geometric continuity, where both tangent vectors at \mathbf{p}_2 point in the same direction. That is, the derivatives $\mathbf{x}'(1)$ and $\mathbf{y}'(0)$ point in the same direction, even though their magnitudes may be different. When the tangent vectors are actually equal at \mathbf{p}_2 , the tangent vector is continuous at \mathbf{p}_2 , and the combined curve is said to have C^1 continuity, or C^1 parametric continuity. Figure 3 shows G^1 continuity in (a) and C^1 continuity in (b).

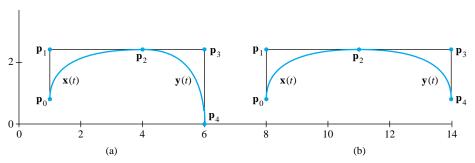


FIGURE 3 (a) G^1 continuity and (b) C^1 continuity.

- **EXAMPLE 2** Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ determine two quadratic Bézier curves, with control points $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$, respectively. The curves are joined at $\mathbf{p}_2 = \mathbf{x}(1) = \mathbf{y}(0)$.
 - a. Suppose the combined curve has G^1 continuity (at \mathbf{p}_2). What algebraic restriction does this condition impose on the control points? Express this restriction in geometric language.
 - b. Repeat part (a) for C^1 continuity.

Solution

a. From Example 1, $\mathbf{x}'(1) = 2(\mathbf{p}_2 - \mathbf{p}_1)$. Also, using the control points for $\mathbf{y}(t)$ in place of $\mathbf{w}(t)$, Example 1 shows that $\mathbf{y}'(0) = 2(\mathbf{p}_3 - \mathbf{p}_2)$. G^1 continuity means that $\mathbf{y}'(0) = k\mathbf{x}'(1)$ for some positive constant k. Equivalently,

$$\mathbf{p}_3 - \mathbf{p}_2 = k(\mathbf{p}_2 - \mathbf{p}_1), \text{ with } k > 0$$
 (3)

Geometrically, (3) implies that \mathbf{p}_2 lies on the line segment from \mathbf{p}_1 to \mathbf{p}_3 . To prove this, let $t = (k+1)^{-1}$, and note that 0 < t < 1. Solve for k to obtain k = (1-t)/t. When this expression is used for k in (3), a rearrangement shows that $\mathbf{p}_2 = (1-t)\mathbf{p}_1 + t\mathbf{p}_3$, which verifies the assertion about \mathbf{p}_2 .

b. C^1 continuity means that $\mathbf{y}'(0) = \mathbf{x}'(1)$. Thus $2(\mathbf{p}_3 - \mathbf{p}_2) = 2(\mathbf{p}_2 - \mathbf{p}_1)$, so $\mathbf{p}_3 - \mathbf{p}_2 = \mathbf{p}_2 - \mathbf{p}_1$, and $\mathbf{p}_2 = (\mathbf{p}_1 + \mathbf{p}_3)/2$. Geometrically, \mathbf{p}_2 is the midpoint of the line segment from \mathbf{p}_1 to \mathbf{p}_3 . See Fig. 3.

Figure 4 shows C^1 continuity for two cubic Bézier curves. Notice how the point joining the two segments lies in the middle of the line segment between the adjacent control points.

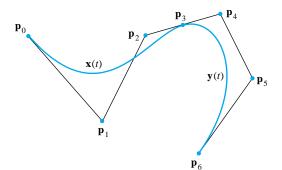


FIGURE 4 Two cubic Bézier curves.

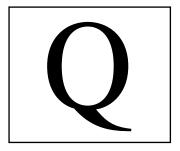
Two curves have C^2 (parametric) continuity when they have C^1 continuity and the *second* derivatives $\mathbf{x}''(1)$ and $\mathbf{y}''(0)$ are equal. This is possible for cubic Bézier curves, but it severely limits the positions of the control points. Another class of cubic curves, called *B-splines*, always have C^2 continuity because each pair of curves share three control points rather than one. Graphics figures using B-splines have more control points and consequently require more computations. Some exercises for this section examine these curves.

Surprisingly, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ join at \mathbf{p}_3 , the apparent smoothness of the curve at \mathbf{p}_3 is usually the same for both G^1 continuity and C^1 continuity. This is because the magnitude of $\mathbf{x}'(t)$ is not related to the physical shape of the curve. The magnitude reflects only the mathematical parameterization of the curve. For instance, if a new vector function $\mathbf{z}(t)$ equals $\mathbf{x}(2t)$, then the point $\mathbf{z}(t)$ traverses the curve from \mathbf{p}_0 to \mathbf{p}_3 twice as fast as the original version, because 2t reaches 1 when t is .5. But, by the chain rule of calculus, $\mathbf{z}'(t) = 2 \cdot \mathbf{x}'(2t)$, so the tangent vector to $\mathbf{z}(t)$ at \mathbf{p}_3 is twice the tangent vector to $\mathbf{x}(t)$ at \mathbf{p}_3 .

In practice, many simple Bézier curves are joined to create graphics objects. Type-setting programs provide one important application, because many letters in a type font involve curved segments. Each letter in a PostScript[®] font, for example, is stored as a set of control points, along with information on how to construct the "outline" of the letter using line segments and Bézier curves. Enlarging such a letter basically requires multiplying the coordinates of each control point by one constant scale factor. Once the outline of the letter has been computed, the appropriate solid parts of the letter are filled in. Figure 5 illustrates this for a character in a PostScript font. Note the control points.

Matrix Equations for Bézier Curves

Since a Bézier curve is a linear combination of control points using polynomials as weights, the formula for $\mathbf{x}(t)$ may be written as



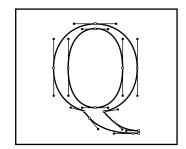


FIGURE 5 A PostScript character.

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} 1 - 3t + 3t^2 - t^3 \\ 3t - 6t^2 + 3t^3 \\ 3t^2 - 3t^3 \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} 1 - 3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

The matrix whose columns are the four control points is called a **geometry matrix**, G. The 4×4 matrix of polynomial coefficients is the **Bézier basis matrix**, M_B . If $\mathbf{u}(t)$ is the column vector of powers of t, then the Bézier curve is given by

$$\mathbf{x}(t) = GM_B\mathbf{u}(t) \tag{4}$$

Other parametric cubic curves in computer graphics are written in this form, too. For instance, if the entries in the matrix M_B are changed appropriately, the resulting curves are B-splines. They are "smoother" than Bézier curves, but they do not pass through any of the control points. A **Hermite** cubic curve arises when the matrix M_B is replaced by a Hermite basis matrix. In this case, the columns of the geometry matrix consist of the starting and ending points of the curves and the tangent vectors to the curves at those points.¹

The Bézier curve in (4) can also be "factored" in another way, to be used in the discussion of Bézier surfaces. For convenience later, the parameter t is replaced by a parameter s:

¹The term *basis matrix* comes from the rows of the matrix that list the coefficients of the *blending* polynomials used to define the curve. For a cubic Bézier curve, the four polynomials are $(1-t)^3$, $3t(1-t)^2$, $3t^2(1-t)$, and t^3 . They form a basis for the space \mathbb{P}_3 of polynomials of degree 3 or less. Each entry in the vector $\mathbf{x}(t)$ is a linear combination of these polynomials. The weights come from the rows of the geometry matrix G in (4).

$$\mathbf{x}(s) = \mathbf{u}(s)^{T} M_{B}^{T} \begin{bmatrix} \mathbf{p}_{0} \\ \mathbf{p}_{1} \\ \mathbf{p}_{2} \\ \mathbf{p}_{3} \end{bmatrix} = \begin{bmatrix} 1 & s & s^{2} & s^{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{0} \\ \mathbf{p}_{2} \\ \mathbf{p}_{3} \end{bmatrix}$$
$$= \begin{bmatrix} (1-s)^{3} & 3s(1-s)^{2} & 3s^{2}(1-s) & s^{3} \end{bmatrix} \begin{bmatrix} \mathbf{p}_{0} \\ \mathbf{p}_{1} \\ \mathbf{p}_{2} \\ \mathbf{p}_{3} \end{bmatrix}$$
(5)

This formula is not quite the same as the transpose of the product on the right of (4), because $\mathbf{x}(s)$ and the control points appear in (5) without transpose symbols. The matrix of control points in (5) is called a **geometry vector**. This should be viewed as a 4×1 block (partitioned) matrix whose entries are column vectors. The matrix to the left of the geometry vector, in the second part of (5), can be viewed as a block matrix, too, with a scalar in each block. The partitioned matrix multiplication makes sense, because each (vector) entry in the geometry vector can be left-multiplied by a scalar as well as by a matrix. Thus, the column vector $\mathbf{x}(s)$ is represented by (5).

Bézier Surfaces

A 3D bicubic surface patch can be constructed from a set of four Bézier curves. Consider the four geometry matrices

$$\begin{bmatrix} \textbf{p}_{11} & \textbf{p}_{12} & \textbf{p}_{13} & \textbf{p}_{14} \\ \textbf{p}_{21} & \textbf{p}_{22} & \textbf{p}_{23} & \textbf{p}_{24} \\ \textbf{p}_{31} & \textbf{p}_{32} & \textbf{p}_{33} & \textbf{p}_{34} \\ \textbf{p}_{41} & \textbf{p}_{42} & \textbf{p}_{43} & \textbf{p}_{44} \end{bmatrix}$$

and recall from (4) that a Bézier curve is produced when any one of these matrices is multiplied on the right by the following vector of weights:

$$M_B \mathbf{u}(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}$$

Let *G* be the block (partitioned) 4×4 matrix whose entries are the control points \mathbf{p}_{ij} displayed above. Then the following product is a block 4×1 matrix, and each entry is a Bézier curve:

$$GM_{B}\mathbf{u}(t) = \begin{bmatrix} \mathbf{p}_{11} & \mathbf{p}_{12} & \mathbf{p}_{13} & \mathbf{p}_{14} \\ \mathbf{p}_{21} & \mathbf{p}_{22} & \mathbf{p}_{23} & \mathbf{p}_{24} \\ \mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} & \mathbf{p}_{34} \\ \mathbf{p}_{41} & \mathbf{p}_{42} & \mathbf{p}_{43} & \mathbf{p}_{44} \end{bmatrix} \begin{bmatrix} (1-t)^{3} \\ 3t(1-t)^{2} \\ 3t^{2}(1-t) \\ t^{3} \end{bmatrix}$$

In fact,

$$GM_{B}\mathbf{u}(t) = \begin{bmatrix} (1-t)^{3}\mathbf{p}_{11} + 3t(1-t)^{2}\mathbf{p}_{12} + 3t^{2}(1-t)\mathbf{p}_{13} + t^{3}\mathbf{p}_{14} \\ (1-t)^{3}\mathbf{p}_{21} + 3t(1-t)^{2}\mathbf{p}_{22} + 3t^{2}(1-t)\mathbf{p}_{23} + t^{3}\mathbf{p}_{24} \\ (1-t)^{3}\mathbf{p}_{31} + 3t(1-t)^{2}\mathbf{p}_{32} + 3t^{2}(1-t)\mathbf{p}_{33} + t^{3}\mathbf{p}_{34} \\ (1-t)^{3}\mathbf{p}_{41} + 3t(1-t)^{2}\mathbf{p}_{42} + 3t^{2}(1-t)\mathbf{p}_{43} + t^{3}\mathbf{p}_{44} \end{bmatrix}$$

Now fix t. Then $GM_B\mathbf{u}(t)$ is a column vector that can be used as a geometry vector in (5) for a Bézier curve in another variable s. This observation produces the **Bézier bicubic surface**:

$$\mathbf{x}(s,t) = \mathbf{u}(s)^T M_R^T G M_R \mathbf{u}(t), \quad \text{where } 0 \le s, t \le 1$$
 (6)

The formula for $\mathbf{x}(s,t)$ is a linear combination of the sixteen control points. If one imagines that these control points are arranged in a fairly uniform rectangular array, as in Fig. 6, then the Bézier surface is controlled by a web of eight Bézier curves, four in the "s-direction" and four in the "t-direction." The surface actually passes through the four control points at its "corners." When it is in the middle of a larger surface, the sixteen-point surface shares its twelve boundary control points with its neighbors.

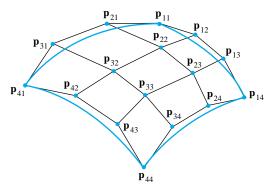


FIGURE 6 Sixteen control points for a Bézier bicubic surface patch.

Approximations to Curves and Surfaces

In CAD programs and in programs used to create realistic computer games, the designer often works at a graphics workstation to compose a "scene" involving various geometric structures. This process requires interaction between the designer and the geometric objects. Each slight repositioning of an object requires new mathematical computations by the graphics program. Bézier curves and surfaces can be useful in this process because they involve fewer control points than objects approximated by many polygons. This dramatically reduces the computation time and speeds up the designer's work.

After the scene composition, however, the final image preparation has different computational demands that are more easily met by objects consisting of flat surfaces and straight edges, such as polyhedra. The designer needs to *render* the scene, by introducing light sources, adding color and texture to surfaces, and simulating reflections from the surfaces.

Computing the direction of a reflected light at a point $\bf p$ on a surface, for instance, requires knowing the direction of both the incoming light and the *surface normal*—the vector perpendicular to the tangent plane at $\bf p$. Computing such normal vectors is much easier on a surface composed of, say, tiny flat polygons than on a curved surface whose normal vector changes continuously as $\bf p$ moves. If $\bf p_1$, $\bf p_2$, and $\bf p_3$ are adjacent

vertices of a flat polygon, then the surface normal is just plus or minus the cross product $(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_2 - \mathbf{p}_3)$. When the polygon is small, only one normal vector is needed for rendering the entire polygon. Also, two widely used shading routines, Gouraud shading and Phong shading, both require a surface to be defined by polygons.

As a result of these needs for flat surfaces, the Bézier curves and surfaces from the scene composition stage now are usually approximated by straight line segments and polyhedral surfaces. The basic idea for approximating a Bézier curve or surface is to divide the curve or surface into smaller pieces, with more and more control points.

Recursive Subdivision of Bézier Curves and Surfaces

Figure 7 shows the four control points $\mathbf{p}_0, \ldots, \mathbf{p}_3$ for a Bézier curve, along with control points for two new curves, each coinciding with half of the original curve. The "left" curve begins at $\mathbf{q}_0 = \mathbf{p}_0$ and ends at \mathbf{q}_3 , at the midpoint of the original curve. The "right" curve begins at $\mathbf{r}_0 = \mathbf{q}_3$ and ends at $\mathbf{r}_3 = \mathbf{p}_3$.

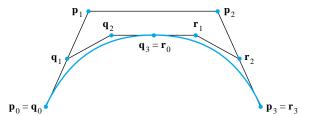


FIGURE 7 Subdivision of a Bézier curve.

Figure 8 shows how the new control points enclose regions that are "thinner" than the region enclosed by the original control points. As the distances between the control points decrease, the control points of each curve segment also move closer to a line segment. This *variation-diminishing property* of Bézier curves depends on the fact that a Bézier curve always lies in the convex hull of the control points.

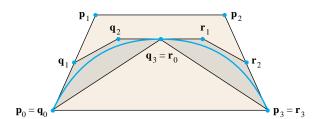


FIGURE 8 Convex hulls of the control points.

The new control points are related to the original control points by simple formulas. Of course, $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. The midpoint of the original curve $\mathbf{x}(t)$ occurs at $\mathbf{x}(.5)$ when $\mathbf{x}(t)$ has the standard parameterization,

$$\mathbf{x}(t) = (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 + (3t - 6t^2 + 3t^3)\mathbf{p}_1 + (3t^2 - 3t^3)\mathbf{p}_2 + t^3\mathbf{p}_3$$
 (7)

for $0 \le t \le 1$. Thus, the new control points \mathbf{q}_3 and \mathbf{r}_0 are given by

$$\mathbf{q}_3 = \mathbf{r}_0 = \mathbf{x}(.5) = \frac{1}{9}(\mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3)$$
 (8)

The formulas for the remaining "interior" control points are also simple, but the derivation of the formulas requires some work involving the tangent vectors of the curves. By definition, the tangent vector to a parameterized curve $\mathbf{x}(t)$ is the derivative $\mathbf{x}'(t)$. This vector shows the direction of the line tangent to the curve at $\mathbf{x}(t)$. For the Bézier curve in (7),

$$\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$$

for $0 \le t \le 1$. In particular,

$$\mathbf{x}'(0) = 3(\mathbf{p}_1 - \mathbf{p}_0)$$
 and $\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$ (9)

Geometrically, \mathbf{p}_1 is on the line tangent to the curve at \mathbf{p}_0 , and \mathbf{p}_2 is on the line tangent to the curve at \mathbf{p}_3 . See Fig. 8. Also, from $\mathbf{x}'(t)$, compute

$$\mathbf{x}'(.5) = \frac{3}{4}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{10}$$

Let $\mathbf{y}(t)$ be the Bézier curve determined by $\mathbf{q}_0, \ldots, \mathbf{q}_3$, and let $\mathbf{z}(t)$ be the Bézier curve determined by $\mathbf{r}_0, \ldots, \mathbf{r}_3$. Since $\mathbf{y}(t)$ traverses the same path as $\mathbf{x}(t)$ but only gets to $\mathbf{x}(.5)$ as t goes from 0 to 1, $\mathbf{y}(t) = \mathbf{x}(.5t)$ for $0 \le t \le 1$. Similarly, since $\mathbf{z}(t)$ starts at $\mathbf{x}(.5)$ when t = 0, $\mathbf{z}(t) = \mathbf{x}(.5 + .5t)$ for $0 \le t \le 1$. By the chain rule for derivatives,

$$\mathbf{y}'(t) = .5\mathbf{x}'(.5t)$$
 and $\mathbf{z}'(t) = .5\mathbf{x}'(.5 + .5t)$ for $0 \le t \le 1$ (11)

From (9) with $\mathbf{y}'(0)$ in place of $\mathbf{x}'(0)$, from (11) with t = 0, and from (9), the control points for $\mathbf{y}(t)$ satisfy

$$3(\mathbf{q}_1 - \mathbf{q}_0) = \mathbf{y}'(0) = .5\mathbf{x}'(0) = \frac{3}{2}(\mathbf{p}_1 - \mathbf{p}_0)$$
 (12)

From (9) with $\mathbf{v}'(1)$ in place of $\mathbf{x}'(1)$, from (11) with t = 1, and from (10),

$$3(\mathbf{q}_3 - \mathbf{q}_2) = \mathbf{y}'(1) = .5\mathbf{x}'(.5) = \frac{3}{8}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$
(13)

Equations (8), (9), (10), (12), and (13) can be solved to produce the formulas for $\mathbf{q}_0, \ldots, \mathbf{q}_3$ shown in Exercise 13. Geometrically, the formulas are displayed in Fig. 9. The interior control points \mathbf{q}_1 and \mathbf{r}_2 are the midpoints, respectively, of the segment from \mathbf{p}_0 to \mathbf{p}_1 and the segment from \mathbf{p}_2 to \mathbf{p}_3 . When the midpoint of the segment from \mathbf{p}_1 to \mathbf{p}_2 is connected to \mathbf{q}_1 , the resulting line segment has \mathbf{q}_2 in the middle!

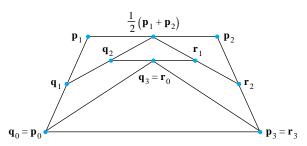


FIGURE 9 Geometric structure of new control points.

This completes one step of the subdivision process. The "recursion" begins, and both new curves are subdivided. The recursion continues to a depth at which all curves are sufficiently straight. Alternatively, at each step the recursion can be "adaptive" and not subdivide one of the two new curves if that curve is sufficiently straight. Once the subdivision completely stops, the endpoints of each curve are joined by line segments, and the scene is ready for the next step in the final image preparation.

A Bézier bicubic surface has the same variation-diminishing property as the Bézier curves that make up each cross-section of the surface, so the process described above can be applied in each cross-section. With the details omitted, here is the basic strategy. Consider the four "parallel" Bézier curves whose parameter is s, and apply the subdivision process to each of them. This produces four sets of eight control points; each set determines a curve as s varies from 0 to 1. As t varies, however, there are eight curves, each with four control points. Apply the subdivision process to each of these sets of four points, creating a total of 64 control points. Adaptive recursion is possible in this setting, too, but there are some subtleties involved. See the following Web site for a color/interactive demo: <a href="http://id.mind.net/~zona/mmts/curveFitting/bezierCurves/bezie

8.6 EXERCISES

1. Suppose a Bézier curve is translated to $\mathbf{x}(t) + \mathbf{b}$. That is, for 0 < t < 1, the new curve is

$$\mathbf{x}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3 + \mathbf{b}$$

Show that this new curve is again a Bézier curve. [*Hint:* Where are the new control points?]

 A spline usually refers to a curve that passes through specified points. A B-spline, however, usually does not pass through its control points. A single segment has the parametric vector form

$$\mathbf{x}(t) = \frac{1}{6} [(1-t)^3 \mathbf{p}_0 + (3t^3 - 6t^2 + 4)\mathbf{p}_1 + (-3t^3 + 3t^2 + 3t + 1)\mathbf{p}_2 + t^3 \mathbf{p}_3]$$
(14)

for $0 \le t \le 1$, where \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 are the control points. When t varies from 0 to 1, $\mathbf{x}(t)$ creates a short curve that lies close to \mathbf{p}_1 and \mathbf{p}_2 . Figure 10 compares a B-spline with a Bézier curve that has the same control points.

 a. Compare the B-spline with the cubic Bézier curve by showing that the B-spline has the form

$$\mathbf{x}(t) = \frac{1}{6} [(1-t)^3 \mathbf{p}_0 + (3t(t-1)^2 + 4 - 3t) \mathbf{p}_1 + (3t^2(1-t) + 3t + 1) \mathbf{p}_2 + t^3 \mathbf{p}_3]$$
(15)



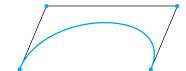


FIGURE 10 A B-spline segment and a Bézier curve.

- b. Show that for $0 \le t \le 1$, $\mathbf{x}(t)$ lies in the convex hull of the control points.
- c. Suppose that a B-spline curve $\mathbf{x}(t)$ is translated to $\mathbf{x}(t) + \mathbf{b}$ (as in Exercise 1). Show that this new curve is again a B-spline.

²See Foley, van Dam, Feiner, and Hughes, *Computer Graphics—Principles and Practice*, 2nd Ed. (Boston: Addison-Wesley, 1996), pp. 527–528.

- Let x(t) be a cubic Bézier curve determined by points p₀, p₁, p₂, and p₃.
 - a. Compute the *tangent* vector $\mathbf{x}'(t)$. Determine how $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are related to the control points, and give geometric descriptions of the *directions* of these tangent vectors. Is it possible to have $\mathbf{x}'(1) = \mathbf{0}$?
 - b. Compute the second derivative $\mathbf{x}''(t)$ and determine how $\mathbf{x}''(0)$ and $\mathbf{x}''(1)$ are related to the control points. Draw a figure based on Fig. 10, and construct a line segment that points in the direction of $\mathbf{x}''(0)$. [*Hint:* Use \mathbf{p}_1 as the origin of the coordinate system.]
- Let x(t) be the B-spline in Exercise 2, with control points p₀, p₁, p₂, and p₃.
 - a. Compute the tangent vector $\mathbf{x}'(t)$ and determine how the derivatives $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are related to the control points. Give geometric descriptions of the *directions* of these tangent vectors. Explore what happens when both $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ equal $\mathbf{0}$. Justify your assertions.
 - b. Compute the second derivative $\mathbf{x}''(t)$ and determine how $\mathbf{x}''(0)$ and $\mathbf{x}''(1)$ are related to the control points. Draw a figure based on Fig. 10, and construct a line segment that points in the direction of $\mathbf{x}''(1)$. [*Hint:* Use \mathbf{p}_2 as the origin of the coordinate system.]
- **5.** Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be cubic Bézier curves with control points $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and $\{\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6\}$, respectively, so that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are joined at \mathbf{p}_3 . The following questions refer to the curve consisting of $\mathbf{x}(t)$ followed by $\mathbf{y}(t)$. For simplicity, assume that the curve is in \mathbb{R}^2 .
 - a. What condition on the control points will guarantee that the curve has C^1 continuity at \mathbf{p}_3 ? Justify your answer.
 - b. What happens when $\mathbf{x}'(1)$ and $\mathbf{y}'(0)$ are both the zero vector?
- 6. A B-spline is built out of B-spline segments, described in Exercise 2. Let p₀,..., p₄ be control points. For 0 ≤ t ≤ 1, let x(t) and y(t) be determined by the geometry matrices [p₀ p₁ p₂ p₃] and [p₁ p₂ p₃ p₄], respectively. Notice how the two segments share three control points. The two segments do not overlap, however—they join at a common endpoint, close to p₂.
 - a. Show that the combined curve has G^0 continuity—that is, $\mathbf{x}(1) = \mathbf{y}(0)$.
 - b. Show that the curve has C^1 continuity at the join point, $\mathbf{x}(1)$. That is, show that $\mathbf{x}'(1) = \mathbf{y}'(0)$.
- 7. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be Bézier curves from Exercise 5, and suppose the combined curve has C^2 continuity (which includes

- C^1 continuity) at \mathbf{p}_3 . Set $\mathbf{x}''(1) = \mathbf{y}''(0)$ and show that \mathbf{p}_5 is completely determined by \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 . Thus, the points $\mathbf{p}_0, \ldots, \mathbf{p}_3$ and the C^2 condition determine all but one of the control points for $\mathbf{y}(t)$.
- 8. Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be segments of a B-spline as in Exercise 6. Show that the curve has C^2 continuity (as well as C^1 continuity) at $\mathbf{x}(1)$. That is, show that $\mathbf{x}''(1) = \mathbf{y}''(0)$. This higher-order continuity is desirable in CAD applications such as automotive body design, since the curves and surfaces appear much smoother. However, B-splines require three times the computation of Bézier curves, for curves of comparable length. For surfaces, B-splines require nine times the computation of Bézier surfaces. Programmers often choose Bézier surfaces for applications (such as an airplane cockpit simulator) that require real-time rendering.
- A quartic Bézier curve is determined by five control points, p₀, p₁, p₂, p₃, and p₄:

$$\mathbf{x}(t) = (1-t)^4 \mathbf{p}_0 + 4t(1-t)^3 \mathbf{p}_1 + 6t^2(1-t)^2 \mathbf{p}_2 + 4t^3(1-t)\mathbf{p}_3 + t^4 \mathbf{p}_4 \quad \text{for } 0 \le t \le 1$$

Construct the quartic basis matrix M_B for $\mathbf{x}(t)$.

10. The "B" in B-spline refers to the fact that a segment $\mathbf{x}(t)$ may be written in terms of a basis matrix, M_S , in a form similar to a Bézier curve. That is,

$$\mathbf{x}(t) = GM_S\mathbf{u}(t)$$
 for $0 \le t \le 1$

where G is the geometry matrix [$\mathbf{p}_0 \quad \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3$] and $\mathbf{u}(t)$ is the column vector $(1, t, t^2, t^3)$. In a *uniform* B-spline, each segment uses the same basis matrix, but the geometry matrix changes. Construct the basis matrix M_S for $\mathbf{x}(t)$.

- 11. [M] The MATLAB program bezier.m creates Bézier curves of degrees 2, 3, and 4. This exercise involves only cubic curves.
 - a. Begin by entering the 2×4 geometry matrix for the control points $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$. Notice how the symmetry of the control points is reflected in the symmetry of the curve.
 - b. Rerun the program to create a Bézier curve $\mathbf{x}(t)$ for the points $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1.8 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 4.2 \end{bmatrix}$. Print the output, and keep the MATLAB Figure window open. (The program includes the command hold on, so MATLAB will display any new graphs in the same window.)
 - c. Design a new Bézier curve $\mathbf{y}(t)$ that joins $\mathbf{x}(t)$ at $\mathbf{p}_3 = (5, 4.2)$ such that the combined curve has G^1

continuity but *not* C^1 continuity (at \mathbf{p}_3). Also, design the new curve so that the combined curve has an inflection point at \mathbf{p}_3 . (The combined curve should cross the tangent line at \mathbf{p}_3 .) Print the new curve in the same window so that the combined curve is shown. [If you wish to start a new figure, close the existing figure before you run the program again. In this case you will need to do part (a) again.] By hand (or in your MATLAB printout), identify the coordinates you choose for the second set of control points.

- 12. [M] Suppose that as part of an animated sequence, a person or animal needs to move along a curve $\mathbf{x}(t)$ (in the xy-plane, for simplicity). Then the curve must have C^2 continuity. Otherwise, the movement of the person may appear jerky at the join points of the segments of the curve, because $\mathbf{x}''(t)$ reflects the *acceleration* (rate of change of the velocity or tangent vector) along the curve. In general, cubic Bézier curves are not suitable for this task. However, from Exercise 8, every B-spline has C^2 continuity. This exercise asks you to watch as a program computes three segments of a uniform B-spline.
 - a. The MATLAB program bspline.m creates a uniform B-spline consisting of three segments that form a continuous curve. The first three lines of the program define the geometry matrices. Currently, lines 2 and 3 begin with a % "comment" symbol. Run the program, and print the output. Close the MATLAB Figure window.
 - b. Generate the second curve segment by moving the comment symbol from line 2 to line 1. Print the result and close the Figure window. Repeat this for the third curve segment (moving the comment symbol from line 3 to line 2). Parts (a) and (b) should generate three figures.
 - Print all three B-spline segments onto one figure. [That
 is, repeat parts (a) and (b) without closing the Figure window.]
 - d. Modify the program so that it asks for a 2×6 matrix such that the program uses the first four columns for one segment, the middle four columns for the second segment, and the last four columns for the third segment. The program should print all three segments, pausing between segments for the user to see how the graph is being constructed. Use the pause command and a disp('') command that asks the user to check the graph and then press any key to continue. (See bezier.m for ideas, if you need them.) You may modify the input step if you think that the data entry would be easier if the points were entered in order as the rows of a matrix rather than the columns: [poi poz; pii piz; ...; psi psz].

Exercises 13–15 concern the subdivision of a Bézier curve shown in Fig. 7. Let $\mathbf{x}(t)$ be the Bézier curve, with control points $\mathbf{p}_0, \ldots, \mathbf{p}_3$, and let $\mathbf{y}(t)$ and $\mathbf{z}(t)$ be the subdividing Bézier curves as in the text, with control points $\mathbf{q}_0, \ldots, \mathbf{q}_3$ and $\mathbf{r}_0, \ldots, \mathbf{r}_3$, respectively.

- 13. a. Use equation (12) to show that \mathbf{q}_1 is the midpoint of the segment from \mathbf{p}_0 to \mathbf{p}_1 .
 - b. Use equation (13) to show that $8\mathbf{q}_2 = 8\mathbf{q}_3 + \mathbf{p}_0 + \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3$.
 - c. Use (b), equation (8), and (a) to show that \mathbf{q}_2 is the midpoint of the segment from \mathbf{q}_1 to the midpoint of the segment from \mathbf{p}_1 to \mathbf{p}_2 . That is, $\mathbf{q}_2 = \frac{1}{2}[\mathbf{q}_1 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)]$.
- **14.** a. Justify each equals sign: $3(\mathbf{r}_3 \mathbf{r}_2) = \mathbf{z}'(1) = .5\mathbf{x}'(1) = \frac{3}{2}(\mathbf{p}_3 \mathbf{p}_2).$
 - b. Show that \mathbf{r}_2 is the midpoint of the segment from \mathbf{p}_2 to \mathbf{p}_3 .
 - c. Justify each equals sign: $3(\mathbf{r}_1 \mathbf{r}_0) = \mathbf{z}'(0) = .5\mathbf{x}'(.5)$.
 - d. Use (c) to show that $8\mathbf{r}_1 = -\mathbf{p}_0 \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + 8\mathbf{r}_0$.
 - e. Use (d), equation (8), and (b) to show that \mathbf{r}_1 is the midpoint of the segment from \mathbf{r}_2 to the midpoint of the segment from \mathbf{p}_2 to \mathbf{p}_3 . That is, $\mathbf{r}_1 = \frac{1}{2}[\mathbf{r}_2 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)]$.
- 15. Sometimes only one half of a Bézier curve needs further subdividing. For example, subdivision of the "left" side is accomplished with parts (a) and (c) of Exercise 13 and equation (8). When both halves of the curve $\mathbf{x}(t)$ are divided, it is possible to organize calculations efficiently to calculate both left and right control points concurrently, without using (8) directly.
 - a. Show that the tangent vectors $\mathbf{y}'(1)$ and $\mathbf{z}'(0)$ are equal.
 - b. Use (a) to show that \mathbf{q}_3 (which equals \mathbf{r}_0) is the midpoint of the segment from \mathbf{q}_2 to \mathbf{r}_1 .
 - c. Using (b) and the results of Exercises 13 and 14, write an algorithm that computes the control points for both $\mathbf{y}(t)$ and $\mathbf{z}(t)$ in an efficient manner. The only operations needed are sums and division by 2.
- **16.** Explain why a cubic Bézier curve is completely determined by $\mathbf{x}(0)$, $\mathbf{x}'(0)$, $\mathbf{x}(1)$, and $\mathbf{x}'(1)$.
- 17. TrueType[®] fonts created by Apple Computer and Adobe Systems use quadratic Bézier curves; PostScript fonts created by Microsoft use cubic Bézier curves. The cubic curves provide more flexibility for typeface design, but it is important to Microsoft that every typeface using quadratic curves can be transformed into one that uses cubic curves. Suppose that $\mathbf{w}(t)$ is a quadratic curve, with control points \mathbf{p}_0 , \mathbf{p}_1 , and \mathbf{p}_2 .
 - a. Find control points \mathbf{r}_0 , \mathbf{r}_1 , \mathbf{r}_2 , and \mathbf{r}_3 such that the cubic Bézier curve $\mathbf{x}(t)$ with these control points has the property that $\mathbf{x}(t)$ and $\mathbf{w}(t)$ have the same initial and terminal

points and the same tangent vectors at t = 0 and t = 1. (See Exercise 16.)

- b. Show that if $\mathbf{x}(t)$ is constructed as in part (a), then $\mathbf{x}(t) = \mathbf{w}(t)$ for $0 \le t \le 1$.
- **18.** Use partitioned matrix multiplication to compute the following matrix product, which appears in the alternative formula

(5) for a Bézier curve:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Answers to Exercises Chapter 8

Section 8.1, page 9

1. Some possible answers: $\mathbf{y} = 2\mathbf{v}_1 - 1.5\mathbf{v}_2 + .5\mathbf{v}_3$, $\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$, $\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 7\mathbf{v}_3 + 3\mathbf{v}_4$

STUDY GUIDE SOLUTION:

1.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$

Solve $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) + c_4(\mathbf{v}_4 - \mathbf{v}_1) =$

 $\mathbf{y} - \mathbf{v}_1$ by row reducing the augmented matrix:

$$\begin{bmatrix} -3 & -1 & 2 & 4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & -1 & 2 & 4 \\ 0 & 1 & 2.5 & .5 \end{bmatrix}$$
$$\sim \begin{bmatrix} -3 & 0 & 4.5 & 4.5 \\ 0 & 1 & 2.5 & .5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -1.5 & -1.5 \\ 0 & 1 & 2.5 & .5 \end{bmatrix}$$

The general solution is $c_2 = 1.5c_4 - 1.5$, $c_3 = -2.5c_4 + .5$, with c_4 free. When $c_4 = 0$,

$$\mathbf{y} - \mathbf{v}_1 = -1.5(\mathbf{v}_2 - \mathbf{v}_1) + .5(\mathbf{v}_3 - \mathbf{v}_1)$$

and

$$y = 2v_1 - 1.5v_2 + .5v_3$$

If
$$c_4 = 1$$
, then $c_2 = 0$ and

$$\mathbf{y} - \mathbf{v}_1 = -2(\mathbf{v}_3 - \mathbf{v}_1) + 1(\mathbf{v}_4 - \mathbf{v}_1)$$

and

$$\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$$

If $c_4 = 3$, then

$$\begin{aligned} \bm{y}-\bm{v}_1 &= 3(\bm{v}_2-\bm{v}_1) - 7(\bm{v}_3-\bm{v}_1) + 3(\bm{v}_4-\bm{v}_1) \quad \text{and} \\ \bm{y} &= 2\bm{v}_1 + 3\bm{v}_2 - 7\bm{v}_3 + 3\bm{v}_4 \end{aligned}$$

Of course, many other answers are possible. Note that in all cases, the weights in the linear combination sum to one

- **2.** $\mathbf{y} = -5\mathbf{v}_1 + 2\mathbf{v}_2 + 4\mathbf{v}_3$. The weights sum to one, so this is an affine sum.
- 3. $\mathbf{y} = -3\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3$. The weights sum to one, so this is an affine sum.

4.
$$\mathbf{y} = 2.6\mathbf{v}_1 - .4\mathbf{v}_2 - 1.2\mathbf{v}_3$$

5.
$$\mathbf{y} = -1.4\mathbf{v}_1 + 2.3\mathbf{v}_2 + .1\mathbf{v}_3$$

6. a.
$$\mathbf{p}_1 = -4\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$$
, so $\mathbf{p}_1 \in \text{aff } S$

b.
$$\mathbf{p}_2 = 0.2\mathbf{b}_1 + 0.5\mathbf{b}_2 + 0.3\mathbf{b}_3$$
, so $\mathbf{p}_2 \in \text{aff } S$

c.
$$\mathbf{p}_3 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$$
, so $\mathbf{p}_3 \notin \text{aff } S$

7. **a.**
$$\mathbf{p}_1 \in \operatorname{Span} S$$
, but $\mathbf{p}_1 \notin \operatorname{aff} S$

b.
$$\mathbf{p}_2 \in \operatorname{Span} S$$
, and $\mathbf{p}_2 \in \operatorname{aff} S$

c.
$$\mathbf{p}_3 \notin \operatorname{Span} S$$
, so $\mathbf{p}_3 \notin \operatorname{aff} S$

STUDY GUIDE SOLUTION:

7. The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}.$$

Parts (a), (b), and (c) use columns 4, 5, and 6, respectively, as the "augmented" column.

- **a.** $\mathbf{p}_1 = 2\mathbf{v}_1 + \mathbf{v}_2 \mathbf{v}_3$, so \mathbf{p}_1 is in Span S. The weights do not sum to one, so $\mathbf{p}_1 \notin \text{aff } S$.
- **b.** $\mathbf{p}_2 = 2\mathbf{v}_1 4\mathbf{v}_2 + 3\mathbf{v}_3$, so \mathbf{p}_2 is in Span S. The weights sum to one, so $\mathbf{p}_2 \in \text{aff } S$.

- **c.** $\mathbf{p}_3 \notin \text{Span } S \text{ because } 0 \neq -5, \text{ so } \mathbf{p}_3 \text{ cannot possibly be in aff } S.$
- **8.** The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Parts (a), (b), and (c) use columns 4, 5, and 6, respectively, as the "augmented" column.

- **a.** $\mathbf{p}_1 = 3\mathbf{v}_1 \mathbf{v}_2 + \mathbf{v}_3$, so \mathbf{p}_1 is in Span S. The weights do not sum to one, so $\mathbf{p}_1 \notin \text{aff } S$.
- **b.** $\mathbf{p}_2 \notin Span \ S$ because $0 \neq 1$ (column 5 is the augmented column), so \mathbf{p}_2 cannot possibly be in aff S.
- **c.** $\mathbf{p}_3 = -2\mathbf{v}_1 + 6\mathbf{v}_2 3\mathbf{v}_3$, so \mathbf{p}_3 is in Span S. The weights sum to one, so $\mathbf{p}_3 \in \text{aff } S$.
- **9. a.** True. This is the definition.
 - **b.** False. The weights in the linear combination must sum to one.
 - c. True. See equation (1).
 - **d.** False. A flat is a translate of a subspace.
 - **e.** True. A hyperplane in \mathbb{R}^3 has dimension 2, so it is a plane.
- **10. a.** False. If $S = \{x\}$, then aff $S = \{x\}$.
 - **b.** True. Theorem 2.
 - c. True. This is the definition.
 - **d.** False. A flat of dimension 2 is called a hyperplane only if the flat is considered a subset of \mathbb{R}^3 . In general, a hyperplane is a flat of dimension n-1.
 - e. True. A flat through the origin is a subspace translated by the ${\bf 0}$ vector.
- 11. Span $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane if and only if $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is linearly independent. Suppose c_2 and c_3 satisfy $c_2(\mathbf{v}_2 \mathbf{v}_1) + c_3(\mathbf{v}_3 \mathbf{v}_1) = \mathbf{0}$. Then $c_2\mathbf{v}_2 + c_3\mathbf{v}_3 (c_2 + c_3)\mathbf{v}_1 = \mathbf{0}$. Then $c_2 = c_3 = 0$, because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. This shows that $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a linearly independent set. Thus, Span $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane in \mathbb{R}^3 .
- 12. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 , the set $W = \operatorname{Span} \{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane in \mathbb{R}^3 , by Exercise 11. Thus, $W + \mathbf{v}_1$ is a plane parallel to W that contains \mathbf{v}_1 . Since $\mathbf{v}_2 = (\mathbf{v}_2 \mathbf{v}_1) + \mathbf{v}_1$, $W + \mathbf{v}_1$ contains \mathbf{v}_2 . Similarly, $W + \mathbf{v}_1$ contains \mathbf{v}_3 . Finally, Theorem 1 shows that aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the plane $W + \mathbf{v}_1$ that contains $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .
- **13.** Let $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$. To show that S is affine, it suffices to show that S is a flat, by Theorem 3. Let $W = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$. Then W is a subspace of \mathbb{R}^n , by Theorem 2 in Section 4.2

- (or Theorem 12 in Section 2.8). Since $S = W + \mathbf{p}$, where \mathbf{p} satisfies $A\mathbf{p} = \mathbf{b}$, by Theorem 6 in Section 1.5, S is a translate of W, and hence S is a flat.
- **14.** Suppose $\mathbf{p}, \mathbf{q} \in S$ and $t \in \mathbb{R}$. Then, by properties of the dot product (Theorem 1 in Section 6.1),

$$[(1-t)\mathbf{p} + t\mathbf{q}] \cdot \mathbf{v} = (1-t)(\mathbf{p} \cdot \mathbf{v}) + t(\mathbf{q} \cdot \mathbf{v})$$

= $(1-t)k + tk = k$

Thus, $[(1 - t)\mathbf{p} + t\mathbf{q}] \in S$, by definition of S. This shows that S is an affine set.

- **15.** A suitable set consists of any three vectors that are not collinear and have 5 as their third entry. If 5 is their third entry, they lie in the plane $x_3 = 5$. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.
- **16.** A suitable set consists of any four vectors that lie in the plane $2x_1 + x_2 3x_3 = 12$ and are not collinear. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.
- 17. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. Given any $t \in \mathbb{R}$, we must show that $\mathbf{z} = (1 t)\mathbf{p} + t\mathbf{q}$ is in f(S). Since f is linear,

$$\mathbf{z} = (1-t)\mathbf{p} + t\mathbf{q} = (1-t)f(\mathbf{r}) + tf(\mathbf{s}) = f((1-t)\mathbf{r} + t\mathbf{s})$$

Since *S* is affine, $(1 - t)\mathbf{r} + t\mathbf{s} \in S$. Thus, $\mathbf{z} \in f(S)$ and f(S) is affine.

18. Given an affine set T, let $S = \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T \}$. Consider $\mathbf{x}, \mathbf{y} \in S$ and $t \in \mathbb{R}$. Then

$$f((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)f(\mathbf{x}) + tf(\mathbf{y})$$

But $f(\mathbf{x}) \in T$ and $f(\mathbf{y}) \in T$, so $(1-t)f(\mathbf{x}) + tf(\mathbf{y}) \in T$ because T is an affine set. It follows that $(1-t)\mathbf{x} + t\mathbf{y} \in S$. This is true for all $\mathbf{x}, \mathbf{y} \in S$ and $t \in \mathbb{R}$, so S is an affine set.

- **19.** Since *B* is affine, Theorem 2 implies that *B* contains all affine combinations of points of *B*. Hence *B* contains all affine combinations of points of *A*. That is, aff $A \subset B$.
- **20.** Since aff *B* is an affine set, this result follows from Exercise 19, with *B* replaced by aff *B*.
- **21.** Since $A \subset (A \cup B)$, it follows from Exercise 20 that aff $A \subset$ aff $(A \cup B)$. Similarly, aff $B \subset$ aff $(A \cup B)$, so [aff $A \cup$ aff B] \subset aff $(A \cup B)$.
- **22.** Since $(A \cap B) \subset A$, it follows from Exercise 20 that aff $(A \cap B) \subset$ aff A. Similarly, aff $(A \cap B) \subset$ aff B, so aff $(A \cap B) \subset$ (aff $A \cap$ aff B).

- **23.** One possibility is to let $A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then (aff A) \cup (aff B) consists of the two coordinate axes, but aff $(A \cup B) = \mathbb{R}^2$.
- **24.** One possibility is to let $A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$. Then both aff A and aff B are equal to the x-axis. But $A \cap B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, so aff $(A \cap B) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Section 8.2, page 19

- 1. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_3 \mathbf{v}_1$ is a multiple of $\mathbf{v}_2 \mathbf{v}_1$, these two points are linearly dependent. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely dependent. Note that $(\mathbf{v}_2 \mathbf{v}_1) 3(\mathbf{v}_3 \mathbf{v}_1) = \mathbf{0}$. A rearrangement produces the affine dependence relation $2\mathbf{v}_1 + \mathbf{v}_2 3\mathbf{v}_3 = \mathbf{0}$. (Note that the weights sum to one.) Geometrically, \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are collinear.
- **2.** $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$; $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. Since $\mathbf{v}_3 \mathbf{v}_1$ and $\mathbf{v}_2 \mathbf{v}_1$ are not multiples, they are linearly independent. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely independent.
- **3.** The set is affinely independent. If the points are called \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 , then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 and $\mathbf{v}_4 = 16\mathbf{v}_1 + 5\mathbf{v}_2 3\mathbf{v}_3$, but the weights in the linear combination do not sum to one.

Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$,

$$\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix}$$
, $\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 13 \\ -8 \end{bmatrix}$. To study linear

independence of these points, row reduce the matrix

$$\begin{bmatrix} -3 & 1 & -1 \\ -6 & -3 & 13 \\ 9 & 12 & -8 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & -1 \\ 0 & -5 & 15 \\ 0 & 15 & -11 \end{bmatrix} \sim$$

$$\begin{bmatrix} -3 & 1 & -1 \\ 0 & -5 & 15 \\ 0 & 0 & 34 \end{bmatrix}.$$
 The points are linearly independent

because there is a pivot in each column. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely independent, by Theorem 5.

Alternative Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Use Theorem 5(d) and study the homogeneous forms of the points. The first step is to move the bottom row of ones (in the augmented matrix) to the top to simplify the arithmetic:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & -1 & 15 \\ -1 & 8 & 11 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & -6 & -3 & 13 \\ 0 & 9 & 12 & -8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & 15 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & 34 \end{bmatrix}$$

The homogeneous points are linearly independent because there is a pivot in each column. So the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely independent, by Theorem 5.

4.
$$-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$$

Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 4 \end{bmatrix}$,

$$\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -7 \\ -9 \end{bmatrix}, \mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}$$
. To study linear

independence of the translated points, row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ -8 & -7 & 2 & 0 \\ 4 & -9 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -15 & -6 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.6 & 0 \\ 0 & 1 & .4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns are linearly dependent, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely dependent, by Theorem 5. To find the affine dependence relation, write the general solution of this system: $x_1 = .6x_3$, $x_2 = -.4x_3$, with x_3 free. Set $x_3 = 5$, for instance. Then $x_1 = 3$, $x_2 = -2$, and $x_3 = 5$. Thus, $3(\mathbf{v}_2 - \mathbf{v}_1) - 2(\mathbf{v}_3 - \mathbf{v}_1) + 5(\mathbf{v}_4 - \mathbf{v}_1) = \mathbf{0}$. Rearranging gives $-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$.

Alternative Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Use Theorem 5(d) and study the homogeneous forms of the points. The first step is to move the bottom row (in the augmented matrix) to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 & \tilde{\mathbf{0}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ -2 & 0 & 1 & -2 & 0 \\ 5 & -3 & -2 & 7 & 0 \\ 3 & 7 & -6 & -3 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 1.2 & 0 \\ 0 & 1 & 0 & -.6 & 0 \\ 0 & 0 & 1 & .4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 + 1.2x_4 = 0$, $x_2 - .6x_4 = 0$, and $x_3 + .4x_4 = 0$, with x_4 free. Take $x_4 = 5$, for example, and get $x_1 = -6$, $x_2 = 3$, and $x_3 = -2$. An affine dependence relation is $-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$.

5.
$$-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$$

Solution:

One solution method is to translate the points to the origin. The following solution uses homogeneous forms. The first step is to move the bottom row (in the augmented matrix) to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 & \tilde{\mathbf{0}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 5 & 5 & 0 \\ -2 & 1 & 1 & -3 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -\frac{4}{3}x_4$, $x_2 = \frac{5}{3}x_4$, $x_3 = -\frac{4}{3}x_4$. An affine dependence relation is $-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$.

6. The set is affinely independent, as the following calculation with homogeneous forms shows:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 3 & -1 & 5 & 5 \\ 1 & -2 & 2 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row reduction of [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4] shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 and $\mathbf{v}_4 = -2\mathbf{v}_1 + 1.5\mathbf{v}_2 + 2.5\mathbf{v}_3$, but the weights in the linear combination do not sum to one. *Instructor*: (Possible exam question)

If the last entry of \mathbf{v}_4 is changed from 0 to 1, then row reduction of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$ shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 and $\mathbf{v}_4 = -3\mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3$.

7. The barycentric coordinates are (-2, 4, -1).

Solution:

Denote the given points as \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{p} . Row reduce the augmented matrix for the equation $x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$. Remember to move the bottom row of ones to the top as the first step, to simplify the arithmetic by hand.

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 5 \\ -1 & 1 & 2 & 4 \\ 2 & 0 & -2 & -2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -2$, $x_2 = 4$, $x_3 = -1$, and $\tilde{\mathbf{p}} = -2\tilde{\mathbf{v}}_1 + 4\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_3$, so $\mathbf{p} = -2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$, and the barycentric coordinates are (-2, 4, -1).

8. The barycentric coordinates are (2, -1, 0).

Solution:

Denote the given points as \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{p} . Row reduce the augmented matrix for the equation $x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$.

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 4 & 1 \\ -2 & 0 & -6 & -4 \\ 1 & 2 & 5 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

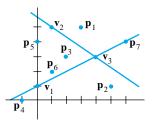
Thus, $\tilde{\mathbf{p}} = 2\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2 + 0\tilde{\mathbf{v}}_3$, so $\mathbf{p} = 2\mathbf{v}_1 - \mathbf{v}_2$. The barycentric coordinates are (2, -1, 0).

Instructor: $\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$

A5

- **9.** a. True. Theorem 5 uses the point \mathbf{v}_1 for the translation, but the paragraph after the theorem points out that any one of the points in the set can be used for the translation.
 - **b.** False, by (d) of Theorem 5.
 - c. False. The weights in the linear combination must sum to zero, not one.
 - d. False. The only points that have barycentric coordinates determined by S belong to aff S.
 - e. True. The barycentric coordinates are zero on the edges of the triangle and positive for interior points.
- **10. a.** False. By Theorem 5, the set of homogeneous forms must be linearly dependent, too.
 - **b.** True. If one statement in Theorem 5 is false, the other statements are false, too.
 - **c.** False. Theorem 6 applies only when S is affinely independent.
 - **d.** False. The color interpolation applies only to points whose barycentric coordinates are nonnegative, since the colors are formed by nonnegative combinations of red, green, and blue.
 - e. True. See the discussion of Fig. 5.
- 11. When a set of five points is translated by subtracting, say, the first point, the new set of four points must be linearly dependent, by Theorem 8 in Section 1.7, because the four points are in \mathbb{R}^3 . By Theorem 5, the original set of five points is affinely dependent.
- **12.** Suppose $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbb{R}^n and $p \ge n + 2$. Since $p-1 \ge n+1$, the points $\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1$ are linearly dependent, by Theorem 8 in Section 1.7. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is affinely dependent.
- **13.** If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent, then there exist c_1 and c_2 , not both zero, such that $c_1 + c_2 = 0$, and $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = 0$. Then $c_1 = -c_2 \neq 0$ and $c_1 \mathbf{v}_1 = -c_2 \mathbf{v}_2 = c_1 \mathbf{v}_2$, which implies that $\mathbf{v}_1 = \mathbf{v}_2$. Conversely, if $\mathbf{v}_1 = \mathbf{v}_2$, let $c_1 = 1$ and $c_2 = -1$. Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{v}_1 + (-1)\mathbf{v}_1 = \mathbf{0}$ and $c_1 + c_2 = 0$, which shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent.
- **14.** Let S_1 consist of three (distinct) points on a line through the origin. The set is affinely dependent because the third point is on the line determined by the first two points. Let S_2 consist of two (distinct) points on a line through the origin. By Exercise 13, the set is affinely independent because the two points are distinct. (A correct solution should include a justification for the sets presented.)
- **15.** a. The vectors $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are not multiples and hence are linearly independent. By Theorem 5, S is affinely independent.

- **b.** $\mathbf{p}_1 \leftrightarrow \left(-\frac{6}{8}, \frac{9}{8}, \frac{5}{8}\right), \mathbf{p}_2 \leftrightarrow \left(0, \frac{1}{2}, \frac{1}{2}\right), \mathbf{p}_3 \leftrightarrow \left(\frac{14}{8}, -\frac{5}{8}, -\frac{1}{8}\right),$ $\begin{array}{l} \textbf{p}_{4} \leftrightarrow \left(\frac{6}{8}, -\frac{5}{8}, \frac{7}{8}\right), \, \textbf{p}_{5} \leftrightarrow \left(\frac{2}{8}, \frac{1}{8}, \frac{5}{8}\right) \\ \textbf{c.} \ \ \textbf{p}_{6} \text{ is } (-, -, +), \textbf{p}_{7} \text{ is } (0, +, -), \text{ and } \textbf{p}_{8} \text{ is } (+, +, -). \end{array}$
- **16.** a. The vectors $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are not Theorem 5, S is affinely independent.
 - **b.** $\mathbf{p}_1 \leftrightarrow (-\frac{2}{7}, \frac{5}{7}, \frac{4}{7}), \mathbf{p}_2 \leftrightarrow (\frac{2}{7}, -\frac{5}{7}, \frac{10}{7}), \mathbf{p}_3 \leftrightarrow (\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$
 - c. $\mathbf{p}_4 \leftrightarrow (+,-,-), \mathbf{p}_5 \leftrightarrow (+,+,-), \mathbf{p}_6 \leftrightarrow (+,+,+),$ $\mathbf{p}_7 \leftrightarrow (-,0,+)$. See the figure below. Actually,
 - $\mathbf{p}_4 \leftrightarrow (\frac{19}{14}, -\frac{2}{14}, -\frac{3}{14}), \mathbf{p}_5 \leftrightarrow (\frac{5}{14}, \frac{12}{14}, -\frac{3}{14}),$
 - $\mathbf{p}_6 \leftrightarrow (\frac{9}{14}, \frac{2}{14}, \frac{3}{14}), \text{ and } \mathbf{p}_7 \leftrightarrow (-\frac{1}{2}, 0, \frac{3}{2}).$



17. Suppose $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set. Then (7) has a solution, because \mathbf{p} is in aff S. Hence (8) has a solution. By Theorem 5, the homogeneous forms of the points in S are linearly independent. Thus, (8) has a unique solution. Then (7) also has a unique solution, because (8) encodes both equations that appear in (7).

The following argument mimics the proof of Theorem 7 in Section 4.4. If $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set, then scalars c_1, \ldots, c_k exist that satisfy (7), by definition of aff S. Suppose \mathbf{p} also has the representation

$$\mathbf{p} = d_1 \mathbf{b}_1 + \dots + d_k \mathbf{b}_k \quad \text{and} \quad d_1 + \dots + d_k = 1$$
 (7a)

for scalars d_1, \ldots, d_k . Then subtraction produces the equation

$$\mathbf{0} = \mathbf{p} - \mathbf{p} = (c_1 - d_1)\mathbf{b}_1 + \dots + (c_k - d_k)\mathbf{b}_k$$
 (7b)

The weights in (7b) sum to zero because the c's and the d's separately sum to one. This is impossible, unless each weight in (8) is zero, because S is an affinely independent set. This proves that $c_i = d_i$ for i = 1, ..., k.

18. Let $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{x}{a} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \frac{y}{b} \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \frac{z}{b} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \left(1 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So the barycentric

coordinates are x/a, y/b, z/c, and 1 - x/a - y/b - z/c. This holds for any nonzero choices of a, b, and c.

19. Let
$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$
, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then $\det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} =$

$$\det\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}$$
 by using the transpose property of the

determinant (Theorem 5 in Section 3.2). By Exercise 30 in Section 3.3, this determinant equals 2 times the area of the triangle with vertices at **a**, **b**, and **c**.

- 20. If p is on the line through a and b, then p is an affine combination of a and b, so \tilde{p} is a linear combination of \tilde{a} and \tilde{b} . Thus, the columns of $[\tilde{a} \quad \tilde{b} \quad \tilde{p}]$ are linearly dependent. So the determinant of this matrix is zero.
- **21.** If $[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$, then Cramer's rule gives

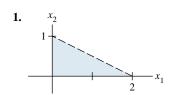
 $r = \det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}} \]/\det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}} \].$ By Exercise 19, the numerator of this quotient is twice the area of $\Delta \mathbf{pbc}$, and the denominator is twice the area of $\Delta \mathbf{abc}$. This proves the formula for r. The other formulas are proved using Cramer's rule for s and t.

22. Let $\mathbf{p} = (1 - x)\mathbf{q} + x\mathbf{a}$, where \mathbf{q} is on the line segment from \mathbf{b} to \mathbf{c} . Then, because the determinant is a linear function of the first column when the other columns are fixed (Section 3.2),

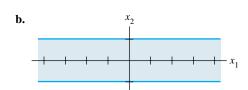
$$\det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = \det[(1-x)\tilde{\mathbf{q}} + x\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$$
$$= (1-x)\cdot \det[\tilde{\mathbf{q}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] + x\cdot \det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$$

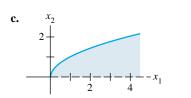
Now, $[\tilde{\mathbf{q}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$ is a singular matrix because $\tilde{\mathbf{q}}$ is a linear combination of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$. So $\det[\tilde{\mathbf{q}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = 0$ and $\det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = x \cdot \det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$.

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2. a. x_2





3. None are in conv *S*.

Solution:

From Exercise 7 in Section 8.1, \mathbf{p}_1 and \mathbf{p}_3 are not in aff S, so they certainly are not in conv S. Since $\mathbf{p}_2 = 2\mathbf{v}_1 - 4\mathbf{v}_2 + 3\mathbf{v}_3$, and the weights sum to one, \mathbf{p}_2 is in aff S. However, S is affinely independent (because S is linearly independent), so the barycentric coordinates of \mathbf{p}_2 are 2, -4 and 3. Since one coordinate is negative, \mathbf{p}_2 is not in conv S.

4. $\mathbf{p}_2 \in \text{conv } S$

Solution:

From Exercise 6 in Section 8.1, \mathbf{p}_3 is not in aff S, so it certainly is not in conv S. Since $\mathbf{p}_1 = -4\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$ and $\mathbf{p}_2 = 0.2\mathbf{b}_1 + 0.5\mathbf{b}_2 + .3\mathbf{b}_3$, and in each case the weights sum to one, both \mathbf{p}_1 and \mathbf{p}_2 are in aff S. However, S is affinely independent (because S is linearly independent), so the weights in these combinations are barycentric coordinates. Thus, \mathbf{p}_2 is in conv S, because its barycentric coordinates are nonnegative. This is not the case for \mathbf{p}_1 , so $\mathbf{p}_1 \notin \text{conv } S$.

A7

5.
$$\mathbf{p}_1 = -\frac{1}{6}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$$
, so $\mathbf{p}_1 \notin \text{conv } S$. $\mathbf{p}_2 = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_2 \in \text{conv } S$.

6. a.
$$\mathbf{p}_1 = \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 + \mathbf{v}_3 \in \text{aff } S$$

b.
$$\mathbf{p}_2 = \frac{1}{4}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \in \text{conv } S$$

c.
$$\mathbf{p}_3 = \mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 \in \text{Span } S$$

b.
$$\mathbf{p}_2 = \frac{1}{4}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \in \text{conv } S$$

c. $\mathbf{p}_3 = \mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 \in \text{Span } S$
d. $\text{proj}_{\text{Span } S} \mathbf{p}_4 = \frac{2}{3}\mathbf{v}_1 + \frac{8}{9}\mathbf{v}_2 + \frac{8}{9}\mathbf{v}_3 = \left(-\frac{4}{9}, -\frac{8}{9}, \frac{10}{9}, 4\right) \neq \mathbf{p}_4,$
so $\mathbf{p}_4 \notin \text{Span } S$

7. a. The barycentric coordinates of
$$\mathbf{p}_1$$
, \mathbf{p}_2 , \mathbf{p}_3 , and \mathbf{p}_4 are, respectively, $\left(\frac{12}{13}, \frac{3}{13}, -\frac{2}{13}\right)$, $\left(\frac{8}{13}, \frac{2}{13}, \frac{3}{13}\right)$, $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$, and $\left(\frac{9}{13}, -\frac{1}{13}, \frac{5}{13}\right)$.

b.
$$\mathbf{p}_1$$
 and \mathbf{p}_4 are outside conv T . \mathbf{p}_2 is inside conv T . \mathbf{p}_3 is on the edge $\overline{\mathbf{v}_1\mathbf{v}_3}$ of conv T .

9.
$$\mathbf{p} = \frac{1}{6}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_4$$
 and $\mathbf{p} = \frac{1}{2}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3$.

10.
$$\mathbf{p} = \frac{3}{5}\mathbf{v}_2 + \frac{3}{10}\mathbf{v}_3 + \frac{1}{10}\mathbf{v}_4$$
 and $\mathbf{p} = \frac{1}{11}\mathbf{v}_1 + \frac{6}{11}\mathbf{v}_2 + \frac{4}{11}\mathbf{v}_3$

b. True. Theorem 9.

c. False. The points do not have to be distinct. For example, S might consist of two points in \mathbb{R}^5 . A point in conv S would be a convex combination of these two points. Caratheodory's Theorem requires n + 1 or fewer points.

13. pos S is a cone with vertex at (0, 0) containing the positive *x*-axis, with sides on the lines $y = \pm x$.

14. a. There are many possibilities. For example, $p = v_1 + v_2 + 2v_3$, so $p \in pos S$.

> $\mathbf{p} = -2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$, so $\mathbf{p} \in \text{aff } S$. It is clear from a graph that $\mathbf{p} \notin \text{conv } S$.

b. The set in Exercise 13 consists of exactly two points, say \mathbf{v}_1 and \mathbf{v}_2 . These points form a basis for \mathbb{R}^2 . Any other point **p** in \mathbb{R}^2 has a *unique* representation as a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . If this combination is both affine and positive, it must be convex. In part (a) above, there were two different linear combinations giving **p**, one affine and one positive, but no single linear combination with both properties.

15. Let $\mathbf{y} \in \text{pos (conv } S)$. Then $\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$, where $\mathbf{v}_i \in \text{conv } S \text{ and } c_i \geq 0.$ But by Caratheodory's Theorem,

each \mathbf{v}_i is a convex combination of n+1 (or fewer) points of S, so

$$\mathbf{v}_i = (d_{i1}\mathbf{w}_{i1} + \dots + d_{i,n+1}\mathbf{w}_{i,n+1}),$$

where

$$d_{ij} \geq 0$$
, $\Sigma d_{ij} = 1$, and $\mathbf{w}_{ij} \in S$

$$\mathbf{y} = c_1(d_{11}\mathbf{w}_{11} + \dots + d_{1,n+1}\mathbf{w}_{1,n+1}) + \dots + c_k(d_{k1}\mathbf{w}_{k1} + \dots + d_{k,n+1}\mathbf{w}_{k,n+1})$$

Since all these coefficients are nonnegative, this shows that $\mathbf{y} \in \text{pos } S$.

The converse is immediate since $S \subset \text{conv } S$.

16. Suppose S is convex and let $y \in pos S$. Then $\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$, where $\mathbf{v}_i \in S$ and $c_i \geq 0$. Let $\alpha = c_1 + \cdots + c_k$. If $\alpha = 0$, then all $c_i = 0$. It follows that y = 0, and $y = \lambda s$ for $\lambda = 0$. If $\alpha \neq 0$, then

$$\frac{\mathbf{y}}{\alpha} = \frac{c_1}{\alpha} \mathbf{v}_1 + \dots + \frac{c_k}{\alpha} \mathbf{v}_k \in S$$
, since S is convex

That is, $\mathbf{y}/\alpha = \mathbf{s}$ for some $\mathbf{s} \in S$, and $\mathbf{y} = \alpha \mathbf{s}$.

17. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. If $0 \le t \le 1$, we must show that $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$ is in f(S). Since f is linear, $\mathbf{y} = (1-t)\mathbf{p} + t\mathbf{q} = (1-t)f(\mathbf{r}) + tf(\mathbf{s}) = f((1-t)\mathbf{r} + t\mathbf{s})$

Since S is convex, $(1-t)\mathbf{r} + t\mathbf{s} \in S$ whenever 0 < t < 1. Thus, $y \in S$ and f(S) is convex.

18. Suppose $\mathbf{r}, \mathbf{s} \in S$ and $0 \le t \le 1$. Then, since f is a linear transformation,

$$f((1-t)\mathbf{r} + t\mathbf{s}) = (1-t)f(\mathbf{r}) + tf(\mathbf{s})$$

But $f(\mathbf{r}) \in T$ and $f(\mathbf{s}) \in T$, so $(1-t)f(\mathbf{r}) + tf(\mathbf{s}) \in T$ since T is convex. It follows that $(1 - t)\mathbf{r} + t\mathbf{s} \in S$. This shows that S is convex.

19. Suppose $A \subset B$, where B is convex. Then, since B is convex, Theorem 7 implies that B contains all convex combinations of points of B. Hence B contains all convex combinations of points of A. That is, conv $A \subset B$.

20. Suppose $A \subset B$. Then $A \subset B \subset \text{conv } B$. Since conv B is convex, Exercise 19 shows that conv $A \subset \text{conv } B$.

21. Since $A \subset (A \cup B)$, Exercise 20 shows that conv $A \subset \text{conv}(A \cup B)$. Similarly, conv $B \subset \text{conv } (A \cup B)$. Thus, $[(\operatorname{conv} A) \cup (\operatorname{conv} B)] \subset \operatorname{conv} (A \cup B).$

22. Since $(A \cap B) \subset A$, Exercise 20 shows that conv $(A \cap B) \subset \text{conv } A$. Similarly,

c. False. For example, the union of two distinct points is not convex, but the individual points form convex sets.

conv $(A \cap B) \subset \text{conv } B$. Thus, conv $(A \cap B) \subset [(\text{conv } A) \cap (\text{conv } B)]$.

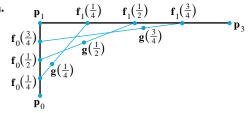
- **23.** One possibility is to let A be two adjacent corners of a square and B be the other two corners. Then $(\operatorname{conv} A) \cup (\operatorname{conv} B)$ consists of two opposite sides of the square, but $\operatorname{conv} (A \cup B)$ is the whole square.
- **24.** One possibility is to let A be a pair of opposite vertices of a square and B be the other pair of opposite vertices. Then conv A and conv B are intersecting diagonals of the square. $A \cap B$ is the empty set, so conv $(A \cap B)$ must be empty, too. But conv $A \cap \text{conv } B$ contains the single point where the diagonals intersect. So conv $(A \cap B)$ is a proper subset of conv $A \cap \text{conv } B$.
- **25. a.** *Hint*: Suppose F is not a subset of G and G is not a subset of F. Then there exist points \mathbf{p} and \mathbf{q} such that $\mathbf{p} \in F$, $\mathbf{p} \notin G$, $\mathbf{q} \in G$, and $\mathbf{q} \notin F$. Consider the point $\mathbf{r} = (\frac{1}{2})(\mathbf{p} + \mathbf{q})$. Since $F \cup G$ is convex, \mathbf{r} belongs to the line segment between \mathbf{p} and \mathbf{q} , which is contained in $F \cup G$. Is $\mathbf{r} \in F$? Is $\mathbf{r} \in G$?

STUDY GUIDE SOLUTION:

If $\mathbf{r} \in F$, then $\mathbf{q} \in F$ since $\mathbf{q} = 2\mathbf{r} - \mathbf{p}$ and F is a subspace. This contradicts $\mathbf{q} \notin F$. Similarly, if $\mathbf{r} \in G$, we reach the contradiction $\mathbf{p} \in G$. Thus, the points \mathbf{p} and \mathbf{q} cannot both exist as described, and either $F \subset G$ or $G \subset F$. The converse is immediate.

b. Here is one possibility in \mathbb{R}^1 : let A = [1, 4] and B = [2, 5].

26. a.



- **b.** $\mathbf{g}(t) = (1-t)\mathbf{f}_0(t) + t\mathbf{f}_1(t) = (1-t)[(1-t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1-t)\mathbf{p}_1 + t\mathbf{p}_2] = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2.$ The sum of the weights in the linear combination for \mathbf{g} is $(1-t)^2 + 2t(1-t) + t^2$, which equals $(1-2t+t^2) + (2t-2t^2) + t^2 = 1$. The weights are each between 0 and 1 when $0 \le t \le 1$, so $\mathbf{g}(t)$ is in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.
- **27.** $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$. Use the representation for $\mathbf{g}_1(t)$ from Exercise 26, and the analogous representation for $\mathbf{g}_2(t)$, based on the control points \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 , and obtain

$$\begin{aligned} \mathbf{h}(t) &= (1-t)[(1-t)^2 \mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2 \mathbf{p}_2] \\ &+ t[(1-t)^2 \mathbf{p}_1 + 2t(1-t)\mathbf{p}_2 + t^2 \mathbf{p}_3] \\ &= (1-t)^3 \mathbf{p}_0 + 2t(1-2t+t^2)\mathbf{p}_1 + (t^2-t^3)\mathbf{p}_2 \\ &+ t(1-2t+t^2)\mathbf{p}_1 + 2t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3 \\ &= (1-3t+3t^2-t^3)\mathbf{p}_0 + (2t-4t^2+2t^3)\mathbf{p}_1 \\ &+ (t^2-t^3)\mathbf{p}_2 + (t-2t^2+t^3)\mathbf{p}_1 + (2t^2-2t^3)\mathbf{p}_2 + t^3 \mathbf{p}_3 \\ &= (1-3t+3t^2-t^3)\mathbf{p}_0 + (3t-6t^2+3t^3)\mathbf{p}_1 \\ &+ (3t^2-3t^3)\mathbf{p}_2 + t^3 \mathbf{p}_3 \end{aligned}$$

By inspection, the sum of the weights in this linear combination is 1, for all t. To show that the weights are nonnegative for $0 \le t \le 1$, factor the coefficients and write

$$\mathbf{h}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2 (1-t) \mathbf{p}_2 + t^3 \mathbf{p}_3 \text{ for } \le t \le 1$$

Thus, $\mathbf{h}(t)$ is in the convex hull of the control points \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 .

28. Fix t, with $0 \le t \le 1$. If $\mathbf{g}_1(t)$ is in S_0 and if $\mathbf{g}_2(t)$ is in S_1 , then there exist nonnegative weights c_0, \ldots, c_k that sum to one, and nonnegative weights d_1, \ldots, d_{k+1} that also sum to one, such that

$$\mathbf{g}_1(t) = c_0 \mathbf{p}_0 + c_1 \mathbf{p}_1 + \dots + c_k \mathbf{p}_k$$

an

$$\mathbf{g}_2(t) = d_1 \mathbf{p}_1 + \dots + d_k \mathbf{p}_k + d_{k+1} \mathbf{p}_{k+1}$$

If $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t \cdot \mathbf{g}_2(t)$, then the weights on $\mathbf{p}_0, \ldots, \mathbf{p}_{k+1}$ in the expression for $\mathbf{h}(t)$ are nonnegative because they are nonnegative c_i and d_i multiplied by (1-t) and t. Further, the sum of these weights is $(1-t)[c_0 + \cdots + c_k] + t[d_1 + \cdots + d_{k+1}] = (1-t) \cdot 1 + t \cdot 1 = 1$. Thus, $\mathbf{h}(t) \in \text{conv } \{\mathbf{p}_0, \ldots, \mathbf{p}_{k+1}\}$ for each t.

(More detailed solution)

Fix t, with $0 \le t \le 1$. By hypothesis, $\mathbf{g}_1(t) = \sum_{i=0}^k c_i \mathbf{p}_i$ and $\mathbf{g}_2(t) = \sum_{j=1}^{k+1} d_j \mathbf{p}_j$ for some nonnegative constants c_i and d_j , respectively, such that $\sum c_i = 1$ and $\sum d_j = 1$. Let

$$\mathbf{y} = (1 - t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$$

$$= (1 - t)\sum_{i=0}^k c_i \mathbf{p}_i + t\sum_{j=1}^{k+1} d_j \mathbf{p}_j$$

$$= \sum_{i=0}^k (1 - t)c_i \mathbf{p}_i + \sum_{j=1}^{k+1} td_j \mathbf{p}_j$$

The weights in this linear combination are all nonnegative, so to show $\mathbf{y} \in \text{conv} \{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{k+1}\}$, it suffices to show that the weights sum to one. In fact, this sum is

$$\sum_{i=0}^{k} (1-t)c_i + \sum_{j=1}^{k+1} t d_j = (1-t) \sum_{i=0}^{k} c_i + t \sum_{j=1}^{k+1} d_j$$
$$= (1-t) \cdot 1 + t \cdot 1 = 1$$

Possible Test Question. Let \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 be points in \mathbb{R}^n . Show that a line segment from a point in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ to a point in conv $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is contained in conv $\{{\bf p}_0, {\bf p}_1, {\bf p}_2, {\bf p}_3\}.$

Solution:

Given $\mathbf{p} \in \text{conv} \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ and $\mathbf{q} \in \text{conv} \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, let $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$ for some t such that $0 \le t \le 1$. Write $\mathbf{p} = c_0 \mathbf{p}_0 + c_1 \mathbf{p}_1 + c_2 \mathbf{p}_2$ and $\mathbf{q} = d_1 \mathbf{p}_1 + d_2 \mathbf{p}_2 + d_3 \mathbf{p}_3$, where the weights in each case are nonnegative and sum to one. Etc.

Section 8.4, page 38

1. $f(x_1, x_2) = 3x_1 + 4x_2$ and d = 13

$$\mathbf{v}_{2} - \mathbf{v}_{1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

$$f(x_{1}, x_{2}) = 3x_{1} + 4x_{2}. \quad f(\mathbf{v}_{1}) = 3(3) + 4(1) = 13.$$

2. $f(x_1, x_2) = 5x_1 - 3x_2$ and d = -7

solution:

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

 $f(x_1, x_2) = 5x_1 - 3x_2. \quad f(\mathbf{v}_1) = 5(1) - 3(4) = -7$

- **3. a.** $\mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ or a multiple **b.** $f(\mathbf{x}) = 2x_2 + 3x_3, d = 11$
- **4. a.** $\mathbf{n} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$ or a multiple
 - **b.** $f(\mathbf{x}) = 4x_1 + 3x_2 6x_3, d = -8$

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{vmatrix} 3 & 6 & x_1 \\ 0 & -2 & x_2 \\ 2 & 3 & x_3 \end{vmatrix} = 4x_1 + 3x_2 - 6x_3 = f(x_1, x_2, x_3)$$

So
$$\mathbf{n} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$$
 and $d = \mathbf{n} \cdot \mathbf{v}_1 = -8$.

5. **a.**
$$\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$
 or a multiple

b.
$$f(\mathbf{x}) = 3x_1 - x_2 + 2x_3 + x_4, d = 5$$

6. a.
$$\mathbf{n} = \begin{bmatrix} -2 \\ 3 \\ -5 \\ 1 \end{bmatrix}$$
 or a multiple

b.
$$f(\mathbf{x}) = -2x_1 + 3x_2 - 5x_3 + x_4, d = 4$$

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1\\2\\0\\0 \end{bmatrix}, \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1\\0\\-1\\-3 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 0\\1\\2\\7 \end{bmatrix},$$

$$\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}. \text{ Solve the equations } (\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n} = 0,$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 2 & 0 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}.$$

Thus, $x_1 = -2x_4$, $x_2 = 3x_4$, $x_3 = -5x_4$, with x_4 free. Take

$$x_4 = 1$$
, for example, to get $\mathbf{n} = \begin{bmatrix} -2 \\ 3 \\ -5 \\ 1 \end{bmatrix}$.

b. Let
$$f(x_1, x_2, x_3, x_4) = -2x_1 + 3x_2 - 5x_3 + x_4$$
. Let $d = f(\mathbf{v}_1) = -2(1) + 3(2) + 0 + 0 = 4$.

7. \mathbf{v}_2 is on the same side as $\mathbf{0}$, \mathbf{v}_1 is on the other side, and \mathbf{v}_3 is in H.

Solution:

Compute $\mathbf{n} \cdot \mathbf{p} = 2$. Then $H = {\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 2}$. The origin is in $H_0 = \{ \mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 0 \}$. Compute $\mathbf{n} \cdot \mathbf{v}_1 = 5$. This is more than 2, so \mathbf{v}_1 and $\mathbf{0}$ are on the opposite sides of H. Compute $\mathbf{n} \cdot \mathbf{v}_2 = -2$, which is less than 2, so \mathbf{v}_2 and $\mathbf{0}$ are on the same side of H. Compute $\mathbf{n} \cdot \mathbf{v}_3 = 2$, so \mathbf{v}_3 is in H.

- **8.** Let H = [f : d], where $f(x_1, x_2, x_3) = 3x_1 + x_2 2x_3$ and d = 4. There is no hyperplane parallel to H that strictly separates A and B.
- 9. One possibility is $\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Solution:

 $H_1 = \{ \mathbf{x} : \mathbf{n}_1 \cdot \mathbf{x} = d_1 \}$ and $H_2 = \{ \mathbf{x} : \mathbf{n}_2 \cdot \mathbf{x} = d_2 \}$. Since $\mathbf{p}_1 \in H_1, d_1 = \mathbf{n}_1 \cdot \mathbf{p}_1 = 4$. Similarly, $d_2 = \mathbf{n}_2 \cdot \mathbf{p}_2 = 22$. Solve the simultaneous system $\begin{bmatrix} 1 & 2 & 4 & 2 \end{bmatrix} \mathbf{x} = 4$ and $[2 \ 3 \ 1 \ 5]$ **x** = 22:

$$\begin{bmatrix} 1 & 2 & 4 & 2 & 4 \\ 2 & 3 & 1 & 5 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -10 & 4 & 32 \\ 0 & 1 & 7 & -1 & -14 \end{bmatrix}$$

The general solution provides one set of vectors, \mathbf{p} , \mathbf{v}_1 , and \mathbf{v}_2 . Other choices are possible.

$$\mathbf{x} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + x_3 \mathbf{v}_1 + x_4 \mathbf{v}_2, \text{ where}$$

$$\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- **10.** 2, 3, or 4
- **11. a.** False. A linear functional goes from \mathbb{R}^n to \mathbb{R} .
 - **b.** False. See the discussion of (1) and (4). There is a $1 \times n$ matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Equivalently, there is a point \mathbf{n} in \mathbb{R}^n such that $f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n .
 - c. True. See the comments after the definition of strictly separate.
 - d. False. See the sets in Fig. 4.
- 12. a. True. See the statement after (3).
 - **b.** False. The vector **n** must be nonzero. If $\mathbf{n} = \mathbf{0}$, then the given set is empty if $d \neq 0$ and the set is all of \mathbb{R}^n if d = 0.
 - **c.** False. Theorem 12 requires that the sets *A* and *B* be convex. For example, *A* could be the boundary of a circle and *B* could be the center of the circle.
 - **d.** False. Some other hyperplane might strictly separate them. See the caution at the end of Example 8.
- **13.** Let $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$ and suppose $\mathbf{z} = (1 t)\mathbf{x} + t\mathbf{y}$, where $0 \le t \le 1$. Then

$$\|\mathbf{z} - \mathbf{p}\| = \|[(1-t)\mathbf{x} + t\mathbf{y}] - \mathbf{p}\|$$

$$= \|(1-t)(\mathbf{x} - \mathbf{p}) + t(\mathbf{y} - \mathbf{p})\|$$

$$\leq (1-t)\|\mathbf{x} - \mathbf{p}\| + t\|\mathbf{y} - \mathbf{p}\|$$

$$< (1-t)\delta + t\delta = \delta$$

where the first inequality comes from the Triangle Inequality (Theorem 17 in Section 6.7) and the second inequality follows from $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$. It follows that $\mathbf{z} \in B(\mathbf{p}, \delta)$ and $B(\mathbf{p}, \delta)$ is convex.

- **14.** Let *S* be a bounded set. Then there exists a $\delta > 0$ such that $S \subset B(\mathbf{0}, \delta)$. Since $B(\mathbf{0}, \delta)$ is convex by Exercise 13, Theorem 9 in Section 8.3 implies that conv $S \subset B(\mathbf{0}, \delta)$ and conv *S* is bounded.
- **15.** f(x, y) = 4x + y. A natural choice for d is 12.75, which equals f(3, .75). The point (3, .75) is three-fourths of the distance between the center of $B(\mathbf{0}, 3)$ and the center of $B(\mathbf{p}, 1)$.

Solution:

Let L be the line segment from the center of $B(\mathbf{0},3)$ to the center of $B(\mathbf{p},1)$. This is on the line through the origin in the direction of \mathbf{p} . The length of L is $(4^2+1^2)^{1/2}\approx 4.1231$. This exceeds the sum of the radii of the two disks, so the disks do not touch. If the disks did touch, the separating hyperplane (line) would have to be orthogonal to \mathbf{p} , the vector that in this case determines the direction of L. Since the disks are separated slightly, the hyperplane need not be exactly perpendicular, but the easiest one to find is a hyperplane H whose normal vector is \mathbf{p} . So define f by $f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x}$.

To find d, evaluate f at any point on L that is between the two disks. If the disks were tangent, that point would be three-fourths of the distance between their centers, since the radii are 3 and 1. Since the disks are slightly separated, the distance is 4.1231. Three-fourths of this distance is greater than 3, and one-fourth of this distance is greater than 1. A suitable value of d is $f(\mathbf{q})$, where

$$\mathbf{q} = (\frac{1}{4})\mathbf{0} + (\frac{3}{4})\mathbf{p} = (3, .75)$$
. So $d = 4(3) + 1(.75) = 12.75$.

16. Note:
$$\mathbf{p} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$
. $f(x, y) = 4x - 2y$. A natural choice for *d* is $f(5, 1.5) = 17$.

Solution:

The normal to the separating hyperplane has the direction of the line segment between ${\bf p}$ and ${\bf q}$. So, let

$$\mathbf{n} = \mathbf{p} - \mathbf{q} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$
. The distance between \mathbf{p} and \mathbf{q} is $\sqrt{20}$,

which is more than the sum of the radii of the two balls. The large ball has center \mathbf{q} . A point three-fourths of the distance from \mathbf{q} to \mathbf{p} will be greater than 3 units from \mathbf{q} and greater than 1 unit from \mathbf{p} . This point is

$$\mathbf{x} = .75\mathbf{p} + .25\mathbf{q} = .75\begin{bmatrix} 6\\1 \end{bmatrix} + .25\begin{bmatrix} 2\\3 \end{bmatrix} = \begin{bmatrix} 5.0\\1.5 \end{bmatrix}$$

Compute $\mathbf{n} \cdot \mathbf{x} = 17$. The desired hyperplane is

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 4x - 2y = 17 \right\}.$$

- 17. Exercise 2(a) in Section 8.3 gives one possibility. Or let $S = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}$. Then conv *S* is the upper (open) half-plane.
- **18.** One possibility is $A = \{(x, y) : |x| \le 1 \text{ and } y = 0\}$ and $B = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}.$
- 19. Let $\mathbf{y} \in B(\mathbf{p}, \delta)$ so that $\lambda \mathbf{y} \in \lambda B(\mathbf{p}, \delta)$. Then $\|\mathbf{y} \mathbf{p}\| < \delta$, so $\|\lambda \mathbf{y} \lambda \mathbf{p}\| = \lambda \|\mathbf{y} \mathbf{p}\| < \lambda \delta$, and $\lambda \mathbf{y} \in B(\lambda \mathbf{p}, \lambda \delta)$. Conversely, suppose $\mathbf{z} \in B(\lambda \mathbf{p}, \lambda \delta)$. Then $\|\mathbf{z} \lambda \mathbf{p}\| < \lambda \delta$, so

$$\left\|\frac{\mathbf{z}}{1} - \mathbf{p}\right\| = \frac{1}{1} \left\|\mathbf{z} - \lambda \mathbf{p}\right\| < \frac{1}{1} (\lambda \delta) = \delta$$

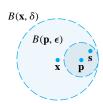
Thus, $\frac{\mathbf{z}}{\lambda} \in B(\mathbf{p}, \delta)$ and $\mathbf{z} \in \lambda B(\mathbf{p}, \delta)$. The proof of the second part is similar.

20. If $\mathbf{p} \in \operatorname{cl} S$, then $\mathbf{p} \in S$ or $\mathbf{p} \in \operatorname{bd} S$. If $\mathbf{p} \in S$, then every open ball centered at \mathbf{p} contains the point \mathbf{p} , which is in S. If $\mathbf{p} \in \operatorname{bd} S$, then, by definition, every open ball centered at \mathbf{p} intersects S.

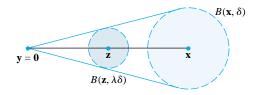
On the other hand, suppose every open ball centered at **p** contains a point of *S*. If **p** is not in *S*, then every open ball centered at **p** intersects both *S* and its complement. Thus, $\mathbf{p} \in \text{bd } S$.

21. Let $\mathbf{x} \in \operatorname{cl}$ (cl S). Then, given any $\delta > 0$, there exists a point $\mathbf{p} \in B(\mathbf{x}, \delta) \cap \operatorname{cl} S$. We must show that some point of $B(\mathbf{x}, \delta)$ is in S. Since $\mathbf{p} \in \operatorname{cl} S$, either $\mathbf{p} \in S$ or $\mathbf{p} \in \operatorname{bd} S$. If $\mathbf{p} \in S$, we are done. Otherwise, $\mathbf{p} \in \operatorname{bd} S$. Let $\varepsilon = \delta - \|\mathbf{x} - \mathbf{p}\|$. Then $\varepsilon > 0$ and the open ball $B(\mathbf{p}, \varepsilon) \subset B(\mathbf{x}, \delta)$. See the figure below. Since $\mathbf{p} \in \operatorname{bd} S$, there exists a point \mathbf{s} in $B(\mathbf{p}, \varepsilon) \cap S$. But then $\mathbf{s} \in B(\mathbf{x}, \delta) \cap S$ and $\mathbf{x} \in \operatorname{cl} S$.

Conversely, let $\mathbf{x} \in \operatorname{cl} S$. Then, given any $\delta > 0$, there exists a point $\mathbf{q} \in B(\mathbf{x}, \delta) \cap S$. But $S \subset \operatorname{cl} S$, so $\mathbf{q} \in B(\mathbf{x}, \delta) \cap \operatorname{cl} S$ and $\mathbf{x} \in \operatorname{cl} (\operatorname{cl} S)$.

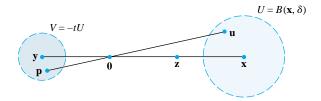


22. Assume without loss of generality that $\mathbf{y} = \mathbf{0}$. Then, for any λ such that $0 < \lambda < 1$, we have $\mathbf{z} = \lambda \mathbf{x}$. See the figure below. If $\mathbf{x} \in \text{int } S$, then there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset S$. Now $B(\lambda \mathbf{x}, \lambda \delta) = \lambda B(\mathbf{x}, \delta)$, by Exercise 19. Since S is convex and $\mathbf{0} \in S$, $\lambda B(\mathbf{x}, \delta) \subset S$. Thus, $B(\mathbf{z}, \lambda \delta) \subset S$ and $\mathbf{z} \in \text{int } S$.



23. Let $\mathbf{z} = \lambda \mathbf{x} + (1 - \lambda)\mathbf{y}$, with $0 < \lambda < 1$. We may assume without loss of generality that $\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z} = \mathbf{0}$. Then $\mathbf{y} = -t\mathbf{x}$ for some t > 0. See the figure below. Since $\mathbf{x} \in \text{int } S$, there exists an open ball $U = B(\mathbf{x}, \delta)$ contained in S. Let V = -tU. Then V is an open ball of radius $t\delta$ about \mathbf{y} .

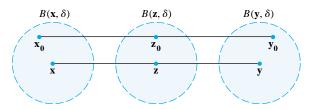
Since $\mathbf{y} \in \operatorname{cl} S$, there exists a point \mathbf{p} in $V \cap S$. That is, $\mathbf{p} = -t\mathbf{u}$ for some $\mathbf{u} \in U$. Since \mathbf{u} and $-t\mathbf{u}$ are in the convex set S, it follows that $\mathbf{0} \in S$. But then Exercise 22 implies $\mathbf{z} \in \operatorname{int} S$.



- **24.** Let $\mathbf{x}, \mathbf{y} \in \text{int } S$. By Exercise 22, all the points $(1 t)\mathbf{x} + t\mathbf{y}$, where 0 < t < 1, are in the interior of S. Thus, $\overline{\mathbf{xy}} \subset \text{int } S$, and int S is convex.
- **25.** Let $\mathbf{x}, \mathbf{y} \in \operatorname{cl} S$ and let $\mathbf{z} = \alpha \mathbf{x} + \beta \mathbf{y}$, where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$. Furthermore, let $B(\mathbf{z}, \delta)$ be an open ball centered at \mathbf{z} . See the figure below. Since \mathbf{x} and \mathbf{y} are in $\operatorname{cl} S$, Exercise 20 implies that there exist points $\mathbf{x}_0 \in B(\mathbf{x}, \delta) \cap S$ and $\mathbf{y}_0 \in B(\mathbf{y}, \delta) \cap S$. We claim that $\mathbf{z}_0 = \alpha \mathbf{x}_0 + \beta \mathbf{y}_0 \in B(z, \delta)$. Indeed,

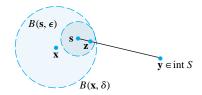
$$\begin{aligned} \|\mathbf{z} - \mathbf{z}_0\| &= \|(\alpha \mathbf{x} + \beta \mathbf{y}) - (\alpha \mathbf{x}_0 + \beta \mathbf{y}_0)\| \\ &= \|\alpha (\mathbf{x} - \mathbf{x}_0) + \beta (\mathbf{y} - \mathbf{y}_0)\| \\ &\leq \|\alpha (\mathbf{x} - \mathbf{x}_0)\| + \|\beta (\mathbf{y} - \mathbf{y}_0)\| \\ &= \alpha \|\mathbf{x} - \mathbf{x}_0\| + \beta \|\mathbf{y} - \mathbf{y}_0\| < \alpha \delta + \beta \delta = (\alpha + \beta)\delta = \delta \end{aligned}$$

Since $\mathbf{z}_0 \in \overline{\mathbf{x}_0 \mathbf{y}_0}$, we have $\mathbf{z}_0 \in S$. Thus, $\mathbf{z} \in \operatorname{cl} S$ and $\operatorname{cl} S$ is convex.

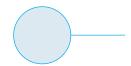


26. a. Suppose $\mathbf{x} \in \operatorname{cl}$ (int S). Then, for every $\delta > 0$, $B(\mathbf{x}, \delta)$ contains at least one point of int S, by Exercise 20. Since int $S \subset S$, this means $B(\mathbf{x}, \delta)$ contains at least one point of S. Thus, $\mathbf{x} \in \operatorname{cl} S$.

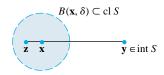
Conversely, suppose $\mathbf{x} \in \operatorname{cl} S$. Given any $\delta > 0$, there exists a point \mathbf{s} in $S \cap B(\mathbf{x}, \delta)$. Since $\|\mathbf{x} - \mathbf{s}\| < \delta$, we have $\varepsilon = \delta - \|\mathbf{x} - \mathbf{s}\| > 0$. It follows that $B(\mathbf{s}, \varepsilon) \subset B(\mathbf{x}, \delta)$. See the figure below. Let $\mathbf{y} \in \operatorname{int} S$ and consider the line segment $\overline{\mathbf{s}}\overline{\mathbf{y}}$. Since $\varepsilon > 0$, there exists $\mathbf{z} \in \overline{\mathbf{s}}\overline{\mathbf{y}}$ such that $\mathbf{z} \in B(\mathbf{s}, \varepsilon)$. But Exercise 22 implies that $\mathbf{z} \in \operatorname{int} S$, so $\mathbf{z} \in B(\mathbf{x}, \delta) \cap (\operatorname{int} S)$, and $\mathbf{x} \in \operatorname{cl} (\operatorname{int} S)$.



b. Let *S* be a "lollipop" shape in \mathbb{R}^2 :



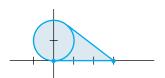
27. a. Since $S \subset \operatorname{cl} S$, int $S \subset \operatorname{int} (\operatorname{cl} S)$. Conversely, suppose $\mathbf{x} \in \operatorname{int} (\operatorname{cl} S)$. Then there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset \operatorname{cl} S$. Let $\mathbf{y} \in \operatorname{int} S$. Then there exists a point \mathbf{z} in $\operatorname{cl} S$ such that $\mathbf{x} \in \overline{\mathbf{z}}\mathbf{y}$, with $\mathbf{x} \neq \mathbf{z}$. See the figure below. It follows from Exercise 23 that $\mathbf{x} \in \operatorname{int} S$.



- **b.** Let $S = [0, 1) \cup (1, 2]$ in \mathbb{R}^2 . Then int $S = (0, 1) \cup (1, 2)$, but int (cl S) = (0, 2). Or let S be a solid square in \mathbb{R}^2 with one diagonal removed. Then int S consists of two triangular pieces, but int (cl S) is the whole square (without the boundary).
- 28. a. If x ∈ bd S, then x ∈ cl S and x ∉ int S. But cl S = cl (cl S), by applying Exercise 21 to cl S in place of S, and int S = int (cl S), by Exercise 27. Thus, x ∈ cl (cl S) and x ∉ int (cl S), so x ∈ bd (cl S). For the converse, reverse the argument above.
 - **b.** Let $S = [0, 1) \cup (1, 2]$ in \mathbb{R}^1 . Then bd $S = \{0, 1, 2\}$, but bd (cl S) = $\{0, 2\}$. Or let S be a solid square in \mathbb{R}^2 with one diagonal removed. Then bd S consists of the diagonal and the four edges of the square, but bd (cl S) does not include the diagonal.
- **29.** Let H = [f : d] be a hyperplane and let $S = \{\mathbf{x} : f(\mathbf{x}) \ge d\}$. Then S is a closed half-space, and its boundary is the convex set H.
- **30.** Let *S* be the convex hull of a circle in \mathbb{R}^2 . Then bd *S* is the circle itself, which is not convex.

Section 8.5, page 47

1. The origin is an extreme point, but it is not a vertex.

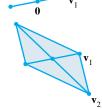


- 2. One possibility is a ray. It has an extreme point at one end.
- **3.** One possibility is to let *S* be a square that includes part of the boundary but not all of it. For example, include just two adjacent edges. The convex hull of the profile is a triangle.



- **4. a.** $f_0(S^5) = 6$, $f_1(S^5) = 15$, $f_2(S^5) = 20$, $f_3(S^5) = 15$, $f_4(S^5) = 6$, and 6 15 + 20 15 + 6 = 2.
- **5. a.** $f_0(C^5) = 32$, $f_1(C^5) = 80$, $f_2(C^5) = 80$, $f_3(C^5) = 40$, $f_4(C^5) = 10$, and 32 80 + 80 40 + 10 = 2.

6. a. X^1 is a line segment:



 X^2 is a parallelogram:

- **b.** $f_0(X^3) = 6$, $f_1(X^3) = 12$, $f_2(X^3) = 8$. X^3 is an octahedron.
- **c.** $f_0(X^4) = 8$, $f_1(X^4) = 24$, $f_2(X^4) = 32$, $f_3(X^4) = 16$, 8 24 + 32 16 = 0
- **d.** $f_k(X^n) = 2^{k+1} \binom{n}{k+1}$, $0 \le k \le n-1$, where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is the binomial coefficient.
- 7. **a.** $f_0(P^n) = f_0(Q) + 1$
 - **b.** $f_k(P^n) = f_k(Q) + f_{k-1}(Q)$
 - **c.** $f_{n-1}(P^n) = f_{n-2}(Q) + 1$
- 8. a. True. Definition.
 - **b.** True. Definition.
 - **c.** False. *S* must be compact. See Theorem 15.
 - **d.** True. See the comment after Fig. 4.
- **9.** a. False. It has six facets (faces).
 - **b.** True. Theorem 14.
 - **c.** False. The maximum is always attained at some extreme point, but there may be other points that are not extreme points at which the maximum is attained.
 - **d.** True. Follows from Euler's formula with n = 2.
- 10. Let \mathbf{x} be an extreme point of the convex set S and let $T = {\mathbf{y} \in S : \mathbf{y} \neq \mathbf{x}}$. If \mathbf{y} and \mathbf{z} are in T, then $\overline{\mathbf{y}}\overline{\mathbf{z}} \subset S$ since S is convex. But since \mathbf{x} is an extreme point of S, $\mathbf{x} \notin \overline{\mathbf{y}}\overline{\mathbf{z}}$, so $\overline{\mathbf{y}}\overline{\mathbf{z}} \subset T$. Thus, T is convex.

Conversely, suppose $\mathbf{x} \in S$, but \mathbf{x} is not an extreme point of S. Then there exist \mathbf{y} and \mathbf{z} in S such that $\mathbf{x} \in \overline{\mathbf{y}}\overline{\mathbf{z}}$, with $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{z}$. It follows that \mathbf{y} and \mathbf{z} are in T, but $\overline{\mathbf{y}}\overline{\mathbf{z}} \not\subset T$. Hence T is not convex.

11. Suppose A and B are convex. Let $\mathbf{x}, \mathbf{y} \in A + B$. Then there exist $\mathbf{a}, \mathbf{c} \in A$ and $\mathbf{b}, \mathbf{d} \in B$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\mathbf{y} = \mathbf{c} + \mathbf{d}$. For any t such that $0 \le t \le 1$, we have

$$(1-t)\mathbf{x} + t\mathbf{y} = (1-t)(\mathbf{a} + \mathbf{b}) + t(\mathbf{c} + \mathbf{d})$$

= $[(1-t)\mathbf{a} + t\mathbf{c}] + [(1-t)\mathbf{b} + t\mathbf{d}]$

But $(1 - t)\mathbf{a} + t\mathbf{c} \in A$ since A is convex, and $(1 - t)\mathbf{b} + t\mathbf{d} \in B$ since B is convex. Thus, $(1 - t)\mathbf{x} + t\mathbf{y}$ is in A + B, which shows that A + B is convex.

12. a. Let *S* be convex and let $\mathbf{x} \in cS + dS$, where c > 0 and d > 0. Then there exist \mathbf{s}_1 and \mathbf{s}_2 in *S* such that $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2$. But then

$$\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2 = (c+d) \left[\frac{c}{c+d} \mathbf{s}_1 + \frac{d}{c+d} \mathbf{s}_2 \right]$$

Now $\frac{c}{c+d}$ and $\frac{d}{c+d}$ are both nonnegative and sum to one. Since *S* is convex, $\frac{c}{c+d}\mathbf{s}_1 + \frac{d}{c+d}\mathbf{s}_2 \in S$. Thus, $\mathbf{x} \in (c+d)S$

Conversely, let $\mathbf{x} \in (c+d)S$. Then $\mathbf{x} = (c+d)\mathbf{s}$ for some $\mathbf{s} \in S$. But then $\mathbf{x} = c\mathbf{s} + d\mathbf{s} \in cS + dS$, as desired.

- 13. a. Since each edge belongs to two facets, kr is twice the number of edges: kr = 2e. Since each edge has two vertices, sv = 2e.
 - **b.** v e + r = 2, so $\frac{2e}{s} e + \frac{2e}{k} = 2 \Rightarrow \frac{1}{s} + \frac{1}{k} = \frac{1}{2} + \frac{1}{e}$
 - **c.** A polygon must have at least three sides, so $k \ge 3$. At least three edges meet at each vertex, so $s \ge 3$. But k and s cannot both be greater than 3, for then the left side of the equation in (b) could not exceed $\frac{1}{2}$.

When k = 3, we get $\frac{1}{s} - \frac{1}{6} = \frac{1}{e}$, so s = 3, 4, or 5. For these values, we get e = 6, 12, or 30, corresponding to the tetrahedron, the octahedron, and the icosahedron, respectively.

When s = 3, we get $\frac{1}{k} - \frac{1}{6} = \frac{1}{e}$, so k = 2, 3, or 5 and e = 6, 12, or 30, respectively. These values correspond to the tetrahedron, the cube, and the dodecahedron.

Section 8.6, page 58

1. Given

 $\mathbf{x}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3,$ $0 \le t \le 1$, let **b** be any fixed vector. Observe (by algebraic computation) that

$$(1-t)^3 \mathbf{b} + 3t(1-t)^2 \mathbf{b} + 3t^2 (1-t) \mathbf{b} + t^3 \mathbf{b} = \mathbf{b}$$

$$\mathbf{x}(t) + \mathbf{b} = (1 - t)^3 (\mathbf{p}_0 + \mathbf{b}) + 3t(1 - t)^2 (\mathbf{p}_1 + \mathbf{b}) + 3t^2 (1 - t)(\mathbf{p}_2 + \mathbf{b}) + t^3 (\mathbf{p}_3 + \mathbf{b})$$

This representation of $\mathbf{x}(t) + \mathbf{b}$ has the form of a Bézier curve, whose control points are translates by \mathbf{b} of the original control points. So translation by \mathbf{b} maps a Bézier curve into a Bézier curve.

2. a. The coefficients of \mathbf{p}_0 and \mathbf{p}_3 in (15) match those in (14). For \mathbf{p}_1 .

$$3t(t-1)^{2} + 4 - 3t = 3t(t^{2} - 2t + 1) + 4 - 3t$$
$$= 3t^{3} - 6t^{2} + 3t + 4 - 3t$$
$$= 3t^{3} - 6t^{2} + 4$$

The verification for \mathbf{p}_2 is trivial. So (15) is an equivalent description for $\mathbf{x}(t)$.

b. Equation (15) reveals that each polynomial weight is nonnegative for $0 \le t \le 1$, since 4 - 3t > 0. For the

sum of the coefficients, use (15) with the first term expanded: $1 - 3t + 3t^2 - t^3$. The 1 here plus the 4 and 1 in the coefficients of \mathbf{p}_1 and \mathbf{p}_2 , respectively, sum to 6, while the other terms sum to 0. This explains the $\frac{1}{6}$ in the formula for $\mathbf{x}(t)$, which makes the coefficients sum to 1. Thus, $\mathbf{x}(t)$ is a convex combination of the control points for 0 < t < 1.

c. Since the coefficients inside the brackets in equation (14) sum to 6, it follows that

$$\mathbf{b} = \frac{1}{6} [6\mathbf{b}]$$

$$= \frac{1}{6} [(1-t)^3 \mathbf{b} + (3t^3 - 6t^2 + 4)\mathbf{b} + (-3t^3 + 3t^2 + 3t + 1)\mathbf{b} + t^3 \mathbf{b}]$$

and hence $\mathbf{x}(t) + \mathbf{b}$ may be written in a similar form, with \mathbf{p}_i replaced by $\mathbf{p}_i + \mathbf{b}$ for each i. This shows that $\mathbf{x}(t) + \mathbf{b}$ is a cubic B-spline with control points $\mathbf{p}_i + \mathbf{b}$ for $i = 0, \dots, 3$.

3. a. Start with

$$\mathbf{x}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3,$$

and compute

$$\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$$

Then $\mathbf{x}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0)$, and $\mathbf{x}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2)$. This shows that the tangent vector $\mathbf{x}'(0)$ points in the direction from \mathbf{p}_0 to \mathbf{p}_1 and is three times the length of $\mathbf{p}_1 - \mathbf{p}_0$. Likewise, $\mathbf{x}'(1)$ points in the direction from \mathbf{p}_2 to \mathbf{p}_3 and is three times the length of $\mathbf{p}_3 - \mathbf{p}_2$. In particular, $\mathbf{x}'(1) = 0$ if and only if $\mathbf{p}_3 = \mathbf{p}_2$.

b. From part (a), differentiate $\mathbf{x}'(t)$ to get

$$\mathbf{x}''(t) = (6 - 6t)\mathbf{p}_0 + (-12 + 18t)\mathbf{p}_1 + (6 - 18t)\mathbf{p}_2 + 6t\mathbf{p}_3$$

Then

$$\mathbf{x}''(0) = 6\mathbf{p}_0 - 12\mathbf{p}_1 + 6\mathbf{p}_2 = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$$

$$\mathbf{x}''(1) = 6\mathbf{p}_1 - 12\mathbf{p}_2 + 6\mathbf{p}_3 = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at \mathbf{p}_1 , temporarily, label \mathbf{p}_0 as $\mathbf{p}_0 - \mathbf{p}_1$ and label \mathbf{p}_2 as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin through the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$, extended out a bit. That line points in the direction of $\mathbf{x}''(0)$.

4. a. $\mathbf{x}'(t) = \frac{1}{6} \left[\left(-3t^2 + 6t - 3 \right) \mathbf{p}_0 + \left(9t^2 - 12t \right) \mathbf{p}_1 + \left(-9t^2 + 6t + 3 \right) \mathbf{p}_2 + 3t^2 \mathbf{p}_3 \right]$ $\mathbf{x}'(0) = \frac{1}{2} \left(\mathbf{p}_2 - \mathbf{p}_0 \right)$ and $\mathbf{x}'(1) = \frac{1}{2} \left(\mathbf{p}_3 - \mathbf{p}_1 \right)$ (Verify that, in the first part of Fig. 10, a line drawn through \mathbf{p}_0 and \mathbf{p}_2 is parallel to the tangent line at the beginning of the B-spline.) When $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are both zero, the figure collapses and the convex hull of the set of control points is the line segment between \mathbf{p}_0 and \mathbf{p}_3 , in which case $\mathbf{x}(t)$ is a straight line. Where does $\mathbf{x}(t)$ start? In this case,

$$\mathbf{x}(t) = \frac{1}{6} \left[(-4t^3 + 6t^2 + 2)\mathbf{p}_0 + (4t^3 - 6t^2 + 4)\mathbf{p}_3 \right]$$

$$\mathbf{x}(0) = \frac{1}{3}\mathbf{p}_0 + \frac{2}{3}\mathbf{p}_3$$
 and $\mathbf{x}(1) = \frac{2}{3}\mathbf{p}_0 + \frac{1}{3}\mathbf{p}_3$

The curve begins closer to \mathbf{p}_3 and finishes closer to \mathbf{p}_0 . Could it turn around during its travel? Since $\mathbf{x}'(t) = 2t(1-t)(\mathbf{p}_0 - \mathbf{p}_3)$, the curve travels in the direction $\mathbf{p}_0 - \mathbf{p}_3$, so when $\mathbf{x}'(0) = \mathbf{x}'(1) = 0$, the curve always moves away from \mathbf{p}_3 toward \mathbf{p}_0 for $0 \le t \le 1$.

b.
$$\mathbf{x}''(t) = (1-t)\mathbf{p}_0 + (-2+3t)\mathbf{p}_1 + (1-3t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{x}''(0) = \mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2 = (\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1)$$

and

$$\mathbf{x}''(1) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = (\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2)$$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at \mathbf{p}_1 , temporarily, label \mathbf{p}_0 as $\mathbf{p}_0 - \mathbf{p}_1$, and label \mathbf{p}_2 as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin to the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$. That segment represents $\mathbf{x}''(0)$.

For a picture of $\mathbf{x}''(1)$, construct a coordinate system with the origin at \mathbf{p}_2 , temporarily, label \mathbf{p}_1 as $\mathbf{p}_1 - \mathbf{p}_2$, and label \mathbf{p}_3 as $\mathbf{p}_3 - \mathbf{p}_2$. Finally, construct a line from this new origin to the sum of $\mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{p}_3 - \mathbf{p}_2$. That segment represents $\mathbf{x}''(1)$.

5. a. From Exercise 3(a) or equation (9) in the text,

$$\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for $\mathbf{x}'(0)$, with the control points from $\mathbf{y}(t)$, and obtain

$$\mathbf{y}'(0) = -3\mathbf{p}_3 + 3\mathbf{p}_4 = 3(\mathbf{p}_4 - \mathbf{p}_3)$$

For C^1 continuity, $3(\mathbf{p}_3 - \mathbf{p}_2) = 3(\mathbf{p}_4 - \mathbf{p}_3)$, so $\mathbf{p}_3 = (\mathbf{p}_4 + \mathbf{p}_2)/2$, and \mathbf{p}_3 is the midpoint of the line segment from \mathbf{p}_2 to \mathbf{p}_4 .

- **b.** If $\mathbf{x}'(1) = \mathbf{y}'(0) = \mathbf{0}$, then $\mathbf{p}_2 = \mathbf{p}_3$ and $\mathbf{p}_3 = \mathbf{p}_4$. Thus, the "line segment" from \mathbf{p}_2 to \mathbf{p}_4 is just the point \mathbf{p}_3 . [*Note*: In this case, the combined curve is still C^1 continuous, by definition. However, some choices of the other control points, \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_5 , and \mathbf{p}_6 , can produce a curve with a visible "corner" at \mathbf{p}_3 , in which case the curve is not G^1 continuous at \mathbf{p}_3 .]
- **6. a.** With $\mathbf{x}(t)$ as in Exercise 2,

$$\mathbf{x}(0) = (\mathbf{p}_0 + 4\mathbf{p}_1 + \mathbf{p}_2)/6$$

and

$$\mathbf{x}(1) = (\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3)/6$$

Use the formula for $\mathbf{x}(0)$, but with the shifted control points for $\mathbf{y}(t)$, and obtain

$$\mathbf{y}(0) = (\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3)/6$$

This equals $\mathbf{x}(1)$, so the B-spline is G^0 continuous at the join point.

b. From Exercise 4(a),

$$\mathbf{x}'(1) = (\mathbf{p}_3 - \mathbf{p}_1)/2$$
 and $\mathbf{x}'(0) = (\mathbf{p}_2 - \mathbf{p}_0)/2$

Use the formula for $\mathbf{x}'(0)$, with the control points for $\mathbf{y}(t)$, and obtain

$$\mathbf{y}'(0) = (\mathbf{p}_3 - \mathbf{p}_1)/2 = \mathbf{x}'(1)$$

Thus, the B-spline is C^1 continuous at the join point.

7. From Exercise 3(b),

$$\mathbf{x}''(0) = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$$

and

$$\mathbf{x}''(1) = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for $\mathbf{x}''(0)$, but with the control points for $\mathbf{y}(t)$, to get

$$\mathbf{y}''(0) = 6(\mathbf{p}_3 - \mathbf{p}_4) + 6(\mathbf{p}_5 - \mathbf{p}_4)$$

Set $\mathbf{x}''(1) = \mathbf{y}''(0)$ and divide by 6, to get

$$(\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{p}_3 - \mathbf{p}_4) + (\mathbf{p}_5 - \mathbf{p}_4)$$

Since the curve is C^1 continuous at \mathbf{p}_3 , the point \mathbf{p}_3 is the midpoint of the segment from \mathbf{p}_2 to \mathbf{p}_4 , by Exercise 5(a).

Thus, $\mathbf{p}_4 - \mathbf{p}_3 = \mathbf{p}_3 - \mathbf{p}_2$. Substituting gives

$$(\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = -(\mathbf{p}_3 - \mathbf{p}_2) + \mathbf{p}_5 - \mathbf{p}_4$$

$$(\mathbf{p}_1 - \mathbf{p}_2) + 2(\mathbf{p}_3 - \mathbf{p}_2) + \mathbf{p}_4 = \mathbf{p}_5$$

Finally, again from C^1 continuity, $\mathbf{p}_4 = \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_2$. Thus,

$$\mathbf{p}_5 = \mathbf{p}_3 + (\mathbf{p}_1 - \mathbf{p}_2) + 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Only \mathbf{p}_6 can be chosen arbitrarily.

8. From Exercise 4(b), $\mathbf{x}''(0) = \mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{x}''(1) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3$. Use the formula for $\mathbf{x}''(0)$, with the shifted control points for $\mathbf{y}(t)$, to get

$$\mathbf{y}''(0) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = \mathbf{x}''(1)$$

Thus, the curve has C^2 continuity at $\mathbf{x}(1)$.

9. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, and factor the vector as $M_B \mathbf{u}(t)$:

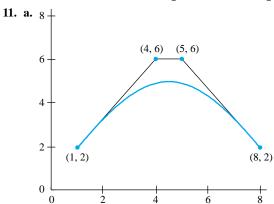
$$\begin{bmatrix} 1 - 4t + 6t^2 - 4t^3 + t^4 \\ 4t - 12t^2 + 12t^3 - 4t^4 \\ 6t^2 - 12t^3 + 6t^4 \\ 4t^3 - 4t^4 \\ t^4 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix}$$

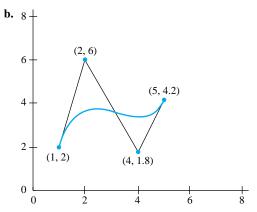
$$M_B = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, taking care to write the terms in ascending powers of t, and factor the vector as $M_S \mathbf{u}(t)$:

$$\frac{1}{6} \begin{bmatrix} 1 - 3t + 3t^2 - t^3 \\ 4 - 6t^2 + 3t^3 \\ 1 + 3t + 3t^2 - 3t^3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t^2 \\ t^3 \end{bmatrix}$$
$$= M_{S}\mathbf{u}(t),$$

$$M_S = \frac{1}{6} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

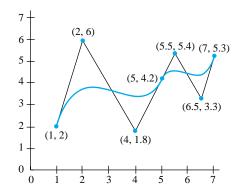




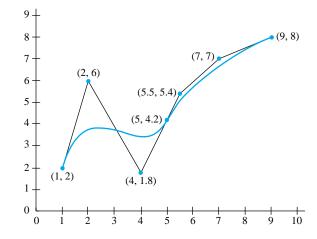
c. The graph for this part must show an inflection point. The first new control point \mathbf{p}_4 must be on the line determined by \mathbf{p}_2 and \mathbf{p}_3 , placed so that \mathbf{p}_3 is **not** the

A16 Answers to Exercises

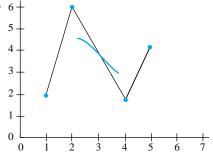
midpoint of the line segment from \mathbf{p}_2 to \mathbf{p}_4 . So \mathbf{p}_4 cannot be $\begin{bmatrix} 6.0 \\ 6.6 \end{bmatrix}$, but two natural choices are $\begin{bmatrix} 5.5 \\ 5.4 \end{bmatrix}$ and $\begin{bmatrix} 6.5 \\ 7.8 \end{bmatrix}$. The first figure below was produced from $\begin{bmatrix} 5.0 \\ 4.2 \end{bmatrix}$, $\begin{bmatrix} 5.5 \\ 5.4 \end{bmatrix}$, $\begin{bmatrix} 6.5 \\ 3.3 \end{bmatrix}$, and $\begin{bmatrix} 7.0 \\ 5.3 \end{bmatrix}$. The MATLAB input is [5.5.5 6.5.7; 4.2.5.4.3.3.5.3]



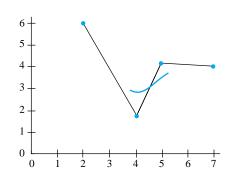
The second graph, from a student, uses $\begin{bmatrix} 5 \\ 4.2 \end{bmatrix}$, $\begin{bmatrix} 5.5 \\ 5.4 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$.



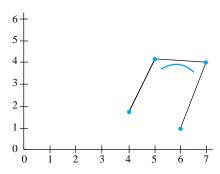
12. a. ₆₊



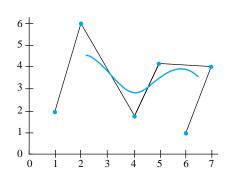
b. Second segment (one new control point):



Third segment:



c. Three segments combined into one curve:



d. Sample bspline.m modification

```
%B-spline curve
```

disp('Sample B-spline Curve')

%Input data points

A = input('Enter 2x6 Geometry Matrix:');

\$Set up the axes and keep them fixed.

axis([0 10 0 10])

%Simple default setting hold on

%Create the parameter t and the vector u
that contains the powers of t.

t = linspace(0,1);

 $u = [t.^0; t; t.^2; t.^3];$

%plot each segment of the curve

for i = 1:3

 $G = [A(:,i) \ A(:,i+1) \ A(:,i+2) \ A(:,i+3)]$

%Current geometry matrix

M = [1 -3 3 -1; 4 0 -6 3; 1 3 3 -3;

0 0 0 1]/6

%B-spline basis curve

x = G*M*u;

%The B-spline curve

plot(G(1,:),G(2,:),'-ob',

`MarkerFacecolor', `b', `MarkerSize',2)

This comment plots the control points with connecting lines.

%'-ob' draws lines and sets a circle for each point, in blue.

%Use 'ob' instead, if you want no lines between the circles.

%MarkerFaceColor fills in each circle
(marker) in blue.

%MarkerSize specifies the size of the circle, in points.

plot(x(1,:), x(2,:), r') %Plot the B-sline curve in red.

if(i < 3)

disp('Press any key to continue')
 pause
end

end

13. a. From (12), $\mathbf{q}_1 - \mathbf{q}_0 = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_0) = \frac{1}{2}\mathbf{p}_1 - \frac{1}{2}\mathbf{p}_0$. Since $\mathbf{q}_0 = \mathbf{p}_0$, $\mathbf{q}_1 = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_0)$.

b. From (13), $(\mathbf{q}_3 - \mathbf{q}_2) = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$. So $8\mathbf{q}_3 + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = 8\mathbf{q}_2$.

c. Use (8) to substitute for $8\mathbf{q}_3$, and obtain

$$8\mathbf{q}_2 = (\mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3) + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3$$

= $2\mathbf{p}_0 + 4\mathbf{p}_1 + 2\mathbf{p}_2$

Then dividing by 8 and using part (a) gives

$$\mathbf{q}_{2} = \frac{1}{4}\mathbf{p}_{0} + \frac{1}{2}\mathbf{p}_{1} + \frac{1}{4}\mathbf{p}_{2}$$

$$= (\frac{1}{4}\mathbf{p}_{0} + \frac{1}{4}\mathbf{p}_{1}) + (\frac{1}{4}\mathbf{p}_{1} + \frac{1}{4}\mathbf{p}_{2})$$

$$= \frac{1}{2}\mathbf{q}_{1} + \frac{1}{4}(\mathbf{p}_{1} + \mathbf{p}_{2})$$

$$= \frac{1}{2}[\mathbf{q}_{1} + \frac{1}{2}(\mathbf{p}_{1} + \mathbf{p}_{2})]$$

14. a. $3(\mathbf{r}_3 - \mathbf{r}_2) = \mathbf{z}'(1)$, by (9) with $\mathbf{z}'(1)$ and \mathbf{r}_i in place of $\mathbf{x}'(1)$ and \mathbf{p}_i .

 $\mathbf{z}'(1) = .5\mathbf{x}'(1)$, by (11) with t = 1.

 $.5\mathbf{x}'(1) = (.5)3(\mathbf{p}_3 - \mathbf{p}_2), \text{ by } (9).$

b. From part (a), $6(\mathbf{r}_3 - \mathbf{r}_2) = 3(\mathbf{p}_3 - \mathbf{p}_2)$, $\mathbf{r}_3 - \mathbf{r}_2 = \frac{1}{2}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_2$, and $\mathbf{r}_3 - \frac{1}{2}\mathbf{p}_3 + \frac{1}{2}\mathbf{p}_2 = \mathbf{r}_2$.

 $\mathbf{r}_3 - \mathbf{r}_2 = \frac{1}{2}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_2$, and $\mathbf{r}_3 - \frac{1}{2}\mathbf{p}_3 + \frac{1}{2}\mathbf{p}_2 = \mathbf{r}_2$. Since $\mathbf{r}_3 = \mathbf{p}_3$, this equation becomes $\mathbf{r}_2 = \frac{1}{2}(\mathbf{p}_3 + \mathbf{p}_2)$.

c. $3(\mathbf{r}_1 - \mathbf{r}_0) = \mathbf{z}'(0)$, by (9) with $\mathbf{z}'(0)$ and \mathbf{r}_i in place of $\mathbf{x}'(0)$ and \mathbf{p}_i .

 $\mathbf{z}'(0) = .5\mathbf{x}'(.5)$, by (11) with t = 0.

d. Part (c) and (10) show that $3(\mathbf{r}_1 - \mathbf{r}_0) = \frac{3}{8}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$. Multiply by $\frac{8}{3}$ and rearrange to obtain $8\mathbf{r}_1 = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + 8\mathbf{r}_0$.

e. From (8), $8\mathbf{r}_0 = \mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3$. Substitute into the equation from part (d), and obtain

 $8\mathbf{r}_1 = 2\mathbf{p}_1 + 4\mathbf{p}_2 + 2\mathbf{p}_3$. Divide by 8 and use part (b) to obtain

$$\mathbf{r}_{1} = \frac{1}{4}\mathbf{p}_{1} + \frac{1}{2}\mathbf{p}_{2} + \frac{1}{4}\mathbf{p}_{3} = (\frac{1}{4}\mathbf{p}_{1} + \frac{1}{4}\mathbf{p}_{2}) + \frac{1}{4}(\mathbf{p}_{2} + \mathbf{p}_{3})$$
$$= \frac{1}{2} \cdot \frac{1}{2}(\mathbf{p}_{1} + \mathbf{p}_{2}) + \frac{1}{2}\mathbf{r}_{2}$$

Interchange the terms on the right, and obtain $\mathbf{r}_1 = \frac{1}{2}[\mathbf{r}_2 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)].$

- **15.** a. From (11), $\mathbf{y}'(1) = .5\mathbf{x}'(.5) = \mathbf{z}'(0)$.
 - **b.** Observe that $\mathbf{y}'(1) = 3(\mathbf{q}_3 \mathbf{q}_2)$. This follows from (9), with $\mathbf{y}(t)$ and its control points in place of $\mathbf{x}(t)$ and its control points. Similarly, for $\mathbf{z}(t)$ and its control points,

- $\mathbf{z}'(0) = 3(\mathbf{r}_1 \mathbf{r}_0)$. By part (a), $3(\mathbf{q}_3 \mathbf{q}_2) = 3(\mathbf{r}_1 \mathbf{r}_0)$. Replace \mathbf{r}_0 by \mathbf{q}_3 , and obtain $\mathbf{q}_3 \mathbf{q}_2 = \mathbf{r}_1 \mathbf{q}_3$, and hence $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$.
- **c.** Set $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. Compute $\mathbf{q}_1 = (\mathbf{p}_0 + \mathbf{p}_1)/2$ and $\mathbf{r}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$. Compute $\mathbf{m} = (\mathbf{p}_1 + \mathbf{p}_2)/2$. Compute $\mathbf{q}_2 = (\mathbf{l}_1 + \mathbf{m})/2$ and $\mathbf{r}_1 = (\mathbf{m} + \mathbf{r}_2)/2$. Compute $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$ and set $\mathbf{r}_0 = \mathbf{q}_3$.
- **16.** A Bézier curve is completely determined by its four control points. Two are given directly: $\mathbf{p}_0 = \mathbf{x}(0)$ and $\mathbf{p}_3 = \mathbf{x}(1)$. From equation (9), $\mathbf{x}'(0) = 3(\mathbf{p}_1 \mathbf{p}_0)$ and $\mathbf{x}'(1) = 3(\mathbf{p}_3 \mathbf{p}_2)$. Solving gives

$$\mathbf{p}_1 = \frac{1}{3}\mathbf{x}'(0) + \mathbf{p}_0$$
 and $\mathbf{p}_2 = \mathbf{p}_3 - \frac{1}{3}\mathbf{x}'(1)$

17. a. The quadratic curve is $\mathbf{w}(t) = (1-t)^2 \mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2 \mathbf{p}_2$. From Example 1, the tangent vectors at the endpoints are $\mathbf{w}'(0) = 2\mathbf{p}_1 - 2\mathbf{p}_0$ and $\mathbf{w}'(1) = 2\mathbf{p}_2 - 2\mathbf{p}_1$. Denote the control points of $\mathbf{x}(t)$ by \mathbf{r}_0 , \mathbf{r}_1 , \mathbf{r}_2 , and \mathbf{r}_3 . Then

$$\mathbf{r}_0 = \mathbf{x}(0) = \mathbf{w}(0) = \mathbf{p}_0$$
 and $\mathbf{r}_3 = \mathbf{x}(1) = \mathbf{w}(1) = \mathbf{p}_2$

From equation (9) or Exercise 3(a) (using \mathbf{r}_i in place of \mathbf{p}_i) and Example 1,

$$-3\mathbf{r}_0 + 3\mathbf{r}_1 = \mathbf{x}'(0) = \mathbf{w}'(0) = 2\mathbf{p}_1 - 2\mathbf{p}_0$$

so

$$-\mathbf{p}_0 + \mathbf{r}_1 = \frac{2\mathbf{p}_1 - 2\mathbf{p}_0}{3}$$

and

$$\mathbf{r}_1 = \frac{2\mathbf{p}_1 + \mathbf{p}_0}{3}$$

Similarly, from the tangent data at t = 1, along with equation (9) and Example 1,

$$-3\mathbf{r}_2 + 3\mathbf{r}_3 = \mathbf{x}'(1) = \mathbf{w}'(1) = 2\mathbf{p}_2 - 2\mathbf{p}_1,$$

$$-\mathbf{r}_2 + \mathbf{p}_2 = \frac{2\mathbf{p}_2 - 2\mathbf{p}_1}{3}, \mathbf{r}_2 = \mathbf{p}_2 - \frac{2\mathbf{p}_2 - 2\mathbf{p}_1}{3}, \text{ and }$$

$$\mathbf{r}_2 = \frac{2\mathbf{p}_1 + \mathbf{p}_2}{3}$$
b. Write the standard formula (7), with \mathbf{r}_i in place of \mathbf{p}_i for

b. Write the standard formula (7), with \mathbf{r}_i in place of \mathbf{p}_i for i = 0, ..., 3, and then replace \mathbf{r}_0 by \mathbf{p}_0 , and replace \mathbf{r}_3 by \mathbf{p}_2 :

$$\mathbf{x}(t) = (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 + (3t - 6t^2 + 3t^3)\mathbf{r}_1 + (3t^2 - 3t^3)\mathbf{r}_2 + t^3\mathbf{p}_2$$
 (*)

Use the formulas for \mathbf{r}_1 and \mathbf{r}_2 to examine the second and third terms in (*):

$$(3t - 6t^2 + 3t^3)\mathbf{r}_1 = \frac{1}{3}(3t - 6t^2 + 3t^3)\mathbf{p}_0 + \frac{2}{3}(3t - 6t^2 + 3t^3)\mathbf{p}_1$$

= $(t - 2t^2 + t^3)\mathbf{p}_0 + (2t - 4t^2 + 2t^3)\mathbf{p}_1$

$$(3t^2 - 3t^3)\mathbf{r}_2 = \frac{2}{3}(3t^2 - 3t^3)\mathbf{p}_1 + \frac{1}{3}(3t^2 - 3t^3)\mathbf{p}_2$$

= $(2t^2 - 2t^3)\mathbf{p}_1 + (t^2 - t^3)\mathbf{p}_2$

When these two results are substituted in (*), the coefficient of \mathbf{p}_0 is

$$(1-3t+3t^2-t^3)+(t-2t^2+t^3)=1-2t+t^2=(1-t)^2$$

The coefficient of \mathbf{p}_1 is

$$(2t - 4t^2 + 2t^3) + (2t^2 - 2t^3) = 2t - 2t^2 = 2t(1 - t)$$

The coefficient of \mathbf{p}_2 is $(t^2 - t^3) + t^3 = t^2$. So $\mathbf{x}(t) = (1 - t)^2 \mathbf{p}_0 + 2t(1 - t)\mathbf{p}_1 + t^2 \mathbf{p}_2$, which shows that $\mathbf{x}(t)$ is the quadratic Bézier curve $\mathbf{w}(t)$.

18.
$$\begin{bmatrix} \mathbf{p}_0 \\ -3\mathbf{p}_0 + 3\mathbf{p}_1 \\ 3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2 \\ -\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3 \end{bmatrix}$$

Answers to Exercises Chapter 8

Section 8.1, page 9

1. Some possible answers: $\mathbf{y} = 2\mathbf{v}_1 - 1.5\mathbf{v}_2 + .5\mathbf{v}_3$, $\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$, $\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 7\mathbf{v}_3 + 3\mathbf{v}_4$

STUDY GUIDE SOLUTION:

1.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$

Solve $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) + c_4(\mathbf{v}_4 - \mathbf{v}_1) =$

 $\mathbf{y} - \mathbf{v}_1$ by row reducing the augmented matrix:

$$\begin{bmatrix} -3 & -1 & 2 & 4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & -1 & 2 & 4 \\ 0 & 1 & 2.5 & .5 \end{bmatrix}$$
$$\sim \begin{bmatrix} -3 & 0 & 4.5 & 4.5 \\ 0 & 1 & 2.5 & .5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -1.5 & -1.5 \\ 0 & 1 & 2.5 & .5 \end{bmatrix}$$

The general solution is $c_2 = 1.5c_4 - 1.5$, $c_3 = -2.5c_4 + .5$, with c_4 free. When $c_4 = 0$,

$$\mathbf{y} - \mathbf{v}_1 = -1.5(\mathbf{v}_2 - \mathbf{v}_1) + .5(\mathbf{v}_3 - \mathbf{v}_1)$$

and

$$y = 2v_1 - 1.5v_2 + .5v_3$$

If
$$c_4 = 1$$
, then $c_2 = 0$ and

$$\mathbf{y} - \mathbf{v}_1 = -2(\mathbf{v}_3 - \mathbf{v}_1) + 1(\mathbf{v}_4 - \mathbf{v}_1)$$

and

$$\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$$

If $c_4 = 3$, then

$$\begin{aligned} \bm{y}-\bm{v}_1 &= 3(\bm{v}_2-\bm{v}_1) - 7(\bm{v}_3-\bm{v}_1) + 3(\bm{v}_4-\bm{v}_1) \quad \text{and} \\ \bm{y} &= 2\bm{v}_1 + 3\bm{v}_2 - 7\bm{v}_3 + 3\bm{v}_4 \end{aligned}$$

Of course, many other answers are possible. Note that in all cases, the weights in the linear combination sum to one

- **2.** $\mathbf{y} = -5\mathbf{v}_1 + 2\mathbf{v}_2 + 4\mathbf{v}_3$. The weights sum to one, so this is an affine sum.
- 3. $\mathbf{y} = -3\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3$. The weights sum to one, so this is an affine sum.

4.
$$\mathbf{y} = 2.6\mathbf{v}_1 - .4\mathbf{v}_2 - 1.2\mathbf{v}_3$$

5.
$$\mathbf{y} = -1.4\mathbf{v}_1 + 2.3\mathbf{v}_2 + .1\mathbf{v}_3$$

6. a.
$$\mathbf{p}_1 = -4\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$$
, so $\mathbf{p}_1 \in \text{aff } S$

b.
$$\mathbf{p}_2 = 0.2\mathbf{b}_1 + 0.5\mathbf{b}_2 + 0.3\mathbf{b}_3$$
, so $\mathbf{p}_2 \in \text{aff } S$

c.
$$\mathbf{p}_3 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$$
, so $\mathbf{p}_3 \notin \text{aff } S$

7. **a.**
$$\mathbf{p}_1 \in \operatorname{Span} S$$
, but $\mathbf{p}_1 \notin \operatorname{aff} S$

b.
$$\mathbf{p}_2 \in \operatorname{Span} S$$
, and $\mathbf{p}_2 \in \operatorname{aff} S$

c.
$$\mathbf{p}_3 \notin \operatorname{Span} S$$
, so $\mathbf{p}_3 \notin \operatorname{aff} S$

STUDY GUIDE SOLUTION:

7. The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}.$$

Parts (a), (b), and (c) use columns 4, 5, and 6, respectively, as the "augmented" column.

- **a.** $\mathbf{p}_1 = 2\mathbf{v}_1 + \mathbf{v}_2 \mathbf{v}_3$, so \mathbf{p}_1 is in Span S. The weights do not sum to one, so $\mathbf{p}_1 \notin \text{aff } S$.
- **b.** $\mathbf{p}_2 = 2\mathbf{v}_1 4\mathbf{v}_2 + 3\mathbf{v}_3$, so \mathbf{p}_2 is in Span S. The weights sum to one, so $\mathbf{p}_2 \in \text{aff } S$.

- **c.** $\mathbf{p}_3 \notin \text{Span } S \text{ because } 0 \neq -5, \text{ so } \mathbf{p}_3 \text{ cannot possibly be in aff } S.$
- **8.** The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Parts (a), (b), and (c) use columns 4, 5, and 6, respectively, as the "augmented" column.

- **a.** $\mathbf{p}_1 = 3\mathbf{v}_1 \mathbf{v}_2 + \mathbf{v}_3$, so \mathbf{p}_1 is in Span S. The weights do not sum to one, so $\mathbf{p}_1 \notin \text{aff } S$.
- **b.** $\mathbf{p}_2 \notin \text{Span } S \text{ because } 0 \neq 1 \text{ (column 5 is the augmented column), so <math>\mathbf{p}_2$ cannot possibly be in aff S.
- **c.** $\mathbf{p}_3 = -2\mathbf{v}_1 + 6\mathbf{v}_2 3\mathbf{v}_3$, so \mathbf{p}_3 is in Span S. The weights sum to one, so $\mathbf{p}_3 \in \text{aff } S$.
- **9. a.** True. This is the definition.
 - **b.** False. The weights in the linear combination must sum to one.
 - c. True. See equation (1).
 - **d.** False. A flat is a translate of a subspace.
 - **e.** True. A hyperplane in \mathbb{R}^3 has dimension 2, so it is a plane.
- **10. a.** False. If $S = \{x\}$, then aff $S = \{x\}$.
 - **b.** True. Theorem 2.
 - c. True. This is the definition.
 - **d.** False. A flat of dimension 2 is called a hyperplane only if the flat is considered a subset of \mathbb{R}^3 . In general, a hyperplane is a flat of dimension n-1.
 - e. True. A flat through the origin is a subspace translated by the ${\bf 0}$ vector.
- 11. Span $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane if and only if $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is linearly independent. Suppose c_2 and c_3 satisfy $c_2(\mathbf{v}_2 \mathbf{v}_1) + c_3(\mathbf{v}_3 \mathbf{v}_1) = \mathbf{0}$. Then $c_2\mathbf{v}_2 + c_3\mathbf{v}_3 (c_2 + c_3)\mathbf{v}_1 = \mathbf{0}$. Then $c_2 = c_3 = 0$, because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. This shows that $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a linearly independent set. Thus, Span $\{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane in \mathbb{R}^3 .
- 12. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 , the set $W = \operatorname{Span} \{\mathbf{v}_2 \mathbf{v}_1, \mathbf{v}_3 \mathbf{v}_1\}$ is a plane in \mathbb{R}^3 , by Exercise 11. Thus, $W + \mathbf{v}_1$ is a plane parallel to W that contains \mathbf{v}_1 . Since $\mathbf{v}_2 = (\mathbf{v}_2 \mathbf{v}_1) + \mathbf{v}_1$, $W + \mathbf{v}_1$ contains \mathbf{v}_2 . Similarly, $W + \mathbf{v}_1$ contains \mathbf{v}_3 . Finally, Theorem 1 shows that aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the plane $W + \mathbf{v}_1$ that contains $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .
- **13.** Let $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$. To show that S is affine, it suffices to show that S is a flat, by Theorem 3. Let $W = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$. Then W is a subspace of \mathbb{R}^n , by Theorem 2 in Section 4.2

- (or Theorem 12 in Section 2.8). Since $S = W + \mathbf{p}$, where \mathbf{p} satisfies $A\mathbf{p} = \mathbf{b}$, by Theorem 6 in Section 1.5, S is a translate of W, and hence S is a flat.
- **14.** Suppose $\mathbf{p}, \mathbf{q} \in S$ and $t \in \mathbb{R}$. Then, by properties of the dot product (Theorem 1 in Section 6.1),

$$[(1-t)\mathbf{p} + t\mathbf{q}] \cdot \mathbf{v} = (1-t)(\mathbf{p} \cdot \mathbf{v}) + t(\mathbf{q} \cdot \mathbf{v})$$

= $(1-t)k + tk = k$

Thus, $[(1 - t)\mathbf{p} + t\mathbf{q}] \in S$, by definition of S. This shows that S is an affine set.

- **15.** A suitable set consists of any three vectors that are not collinear and have 5 as their third entry. If 5 is their third entry, they lie in the plane $x_3 = 5$. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.
- **16.** A suitable set consists of any four vectors that lie in the plane $2x_1 + x_2 3x_3 = 12$ and are not collinear. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.
- 17. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. Given any $t \in \mathbb{R}$, we must show that $\mathbf{z} = (1 t)\mathbf{p} + t\mathbf{q}$ is in f(S). Since f is linear,

$$\mathbf{z} = (1-t)\mathbf{p} + t\mathbf{q} = (1-t)f(\mathbf{r}) + tf(\mathbf{s}) = f((1-t)\mathbf{r} + t\mathbf{s})$$

Since *S* is affine, $(1 - t)\mathbf{r} + t\mathbf{s} \in S$. Thus, $\mathbf{z} \in f(S)$ and f(S) is affine.

18. Given an affine set T, let $S = \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T \}$. Consider $\mathbf{x}, \mathbf{y} \in S$ and $t \in \mathbb{R}$. Then

$$f((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)f(\mathbf{x}) + tf(\mathbf{y})$$

But $f(\mathbf{x}) \in T$ and $f(\mathbf{y}) \in T$, so $(1-t)f(\mathbf{x}) + tf(\mathbf{y}) \in T$ because T is an affine set. It follows that $(1-t)\mathbf{x} + t\mathbf{y} \in S$. This is true for all $\mathbf{x}, \mathbf{y} \in S$ and $t \in \mathbb{R}$, so S is an affine set.

- **19.** Since *B* is affine, Theorem 2 implies that *B* contains all affine combinations of points of *B*. Hence *B* contains all affine combinations of points of *A*. That is, aff $A \subset B$.
- **20.** Since aff *B* is an affine set, this result follows from Exercise 19, with *B* replaced by aff *B*.
- **21.** Since $A \subset (A \cup B)$, it follows from Exercise 20 that aff $A \subset$ aff $(A \cup B)$. Similarly, aff $B \subset$ aff $(A \cup B)$, so [aff $A \cup$ aff B] \subset aff $(A \cup B)$.
- **22.** Since $(A \cap B) \subset A$, it follows from Exercise 20 that aff $(A \cap B) \subset$ aff A. Similarly, aff $(A \cap B) \subset$ aff B, so aff $(A \cap B) \subset$ (aff $A \cap$ aff B).

- **23.** One possibility is to let $A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $B = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then (aff A) \cup (aff B) consists of the two coordinate axes, but aff $(A \cup B) = \mathbb{R}^2$.
- **24.** One possibility is to let $A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$. Then both aff A and aff B are equal to the x-axis. But $A \cap B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, so aff $(A \cap B) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Section 8.2, page 19

- 1. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_3 \mathbf{v}_1$ is a multiple of $\mathbf{v}_2 \mathbf{v}_1$, these two points are linearly dependent. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely dependent. Note that $(\mathbf{v}_2 \mathbf{v}_1) 3(\mathbf{v}_3 \mathbf{v}_1) = \mathbf{0}$. A rearrangement produces the affine dependence relation $2\mathbf{v}_1 + \mathbf{v}_2 3\mathbf{v}_3 = \mathbf{0}$. (Note that the weights sum to one.) Geometrically, \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are collinear.
- **2.** $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$; $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. Since $\mathbf{v}_3 \mathbf{v}_1$ and $\mathbf{v}_2 \mathbf{v}_1$ are not multiples, they are linearly independent. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely independent.
- **3.** The set is affinely independent. If the points are called \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 , then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 and $\mathbf{v}_4 = 16\mathbf{v}_1 + 5\mathbf{v}_2 3\mathbf{v}_3$, but the weights in the linear combination do not sum to one.

Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$,

$$\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix}$$
, $\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 13 \\ -8 \end{bmatrix}$. To study linear

independence of these points, row reduce the matrix

$$\begin{bmatrix} -3 & 1 & -1 \\ -6 & -3 & 13 \\ 9 & 12 & -8 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & -1 \\ 0 & -5 & 15 \\ 0 & 15 & -11 \end{bmatrix} \sim$$

$$\begin{bmatrix} -3 & 1 & -1 \\ 0 & -5 & 15 \\ 0 & 0 & 34 \end{bmatrix}.$$
 The points are linearly independent

because there is a pivot in each column. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely independent, by Theorem 5.

Alternative Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Use Theorem 5(d) and study the homogeneous forms of the points. The first step is to move the bottom row of ones (in the augmented matrix) to the top to simplify the arithmetic:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & -1 & 15 \\ -1 & 8 & 11 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & -6 & -3 & 13 \\ 0 & 9 & 12 & -8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & 15 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & 34 \end{bmatrix}$$

The homogeneous points are linearly independent because there is a pivot in each column. So the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely independent, by Theorem 5.

4.
$$-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$$

Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 4 \end{bmatrix}$,

$$\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -7 \\ -9 \end{bmatrix}, \mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}$$
. To study linear

independence of the translated points, row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ -8 & -7 & 2 & 0 \\ 4 & -9 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -15 & -6 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.6 & 0 \\ 0 & 1 & .4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns are linearly dependent, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely dependent, by Theorem 5. To find the affine dependence relation, write the general solution of this system: $x_1 = .6x_3$, $x_2 = -.4x_3$, with x_3 free. Set $x_3 = 5$, for instance. Then $x_1 = 3$, $x_2 = -2$, and $x_3 = 5$. Thus, $3(\mathbf{v}_2 - \mathbf{v}_1) - 2(\mathbf{v}_3 - \mathbf{v}_1) + 5(\mathbf{v}_4 - \mathbf{v}_1) = \mathbf{0}$. Rearranging gives $-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$.

Alternative Solution:

Name the points \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{v}_4 . Use Theorem 5(d) and study the homogeneous forms of the points. The first step is to move the bottom row (in the augmented matrix) to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 & \tilde{\mathbf{0}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ -2 & 0 & 1 & -2 & 0 \\ 5 & -3 & -2 & 7 & 0 \\ 3 & 7 & -6 & -3 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 1.2 & 0 \\ 0 & 1 & 0 & -.6 & 0 \\ 0 & 0 & 1 & .4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 + 1.2x_4 = 0$, $x_2 - .6x_4 = 0$, and $x_3 + .4x_4 = 0$, with x_4 free. Take $x_4 = 5$, for example, and get $x_1 = -6$, $x_2 = 3$, and $x_3 = -2$. An affine dependence relation is $-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$.

5.
$$-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$$

Solution:

One solution method is to translate the points to the origin. The following solution uses homogeneous forms. The first step is to move the bottom row (in the augmented matrix) to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 & \tilde{\mathbf{0}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 5 & 5 & 0 \\ -2 & 1 & 1 & -3 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -\frac{4}{3}x_4$, $x_2 = \frac{5}{3}x_4$, $x_3 = -\frac{4}{3}x_4$. An affine dependence relation is $-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$.

6. The set is affinely independent, as the following calculation with homogeneous forms shows:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 3 & -1 & 5 & 5 \\ 1 & -2 & 2 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row reduction of [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4] shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 and $\mathbf{v}_4 = -2\mathbf{v}_1 + 1.5\mathbf{v}_2 + 2.5\mathbf{v}_3$, but the weights in the linear combination do not sum to one. *Instructor*: (Possible exam question)

If the last entry of \mathbf{v}_4 is changed from 0 to 1, then row reduction of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$ shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 and $\mathbf{v}_4 = -3\mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3$.

7. The barycentric coordinates are (-2, 4, -1).

Solution:

Denote the given points as \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{p} . Row reduce the augmented matrix for the equation $x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$. Remember to move the bottom row of ones to the top as the first step, to simplify the arithmetic by hand.

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 5 \\ -1 & 1 & 2 & 4 \\ 2 & 0 & -2 & -2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -2$, $x_2 = 4$, $x_3 = -1$, and $\tilde{\mathbf{p}} = -2\tilde{\mathbf{v}}_1 + 4\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_3$, so $\mathbf{p} = -2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$, and the barycentric coordinates are (-2, 4, -1).

8. The barycentric coordinates are (2, -1, 0).

Solution:

Denote the given points as \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{p} . Row reduce the augmented matrix for the equation $x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$.

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 4 & 1 \\ -2 & 0 & -6 & -4 \\ 1 & 2 & 5 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

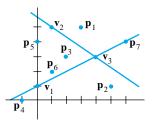
Thus, $\tilde{\mathbf{p}} = 2\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2 + 0\tilde{\mathbf{v}}_3$, so $\mathbf{p} = 2\mathbf{v}_1 - \mathbf{v}_2$. The barycentric coordinates are (2, -1, 0).

Instructor: $\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$

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- **9.** a. True. Theorem 5 uses the point \mathbf{v}_1 for the translation, but the paragraph after the theorem points out that any one of the points in the set can be used for the translation.
 - **b.** False, by (d) of Theorem 5.
 - c. False. The weights in the linear combination must sum to zero, not one.
 - d. False. The only points that have barycentric coordinates determined by S belong to aff S.
 - e. True. The barycentric coordinates are zero on the edges of the triangle and positive for interior points.
- **10. a.** False. By Theorem 5, the set of homogeneous forms must be linearly dependent, too.
 - **b.** True. If one statement in Theorem 5 is false, the other statements are false, too.
 - **c.** False. Theorem 6 applies only when S is affinely independent.
 - **d.** False. The color interpolation applies only to points whose barycentric coordinates are nonnegative, since the colors are formed by nonnegative combinations of red, green, and blue.
 - e. True. See the discussion of Fig. 5.
- 11. When a set of five points is translated by subtracting, say, the first point, the new set of four points must be linearly dependent, by Theorem 8 in Section 1.7, because the four points are in \mathbb{R}^3 . By Theorem 5, the original set of five points is affinely dependent.
- **12.** Suppose $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbb{R}^n and $p \ge n + 2$. Since $p-1 \ge n+1$, the points $\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \dots, \mathbf{v}_p - \mathbf{v}_1$ are linearly dependent, by Theorem 8 in Section 1.7. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is affinely dependent.
- **13.** If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent, then there exist c_1 and c_2 , not both zero, such that $c_1 + c_2 = 0$, and $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = 0$. Then $c_1 = -c_2 \neq 0$ and $c_1 \mathbf{v}_1 = -c_2 \mathbf{v}_2 = c_1 \mathbf{v}_2$, which implies that $\mathbf{v}_1 = \mathbf{v}_2$. Conversely, if $\mathbf{v}_1 = \mathbf{v}_2$, let $c_1 = 1$ and $c_2 = -1$. Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{v}_1 + (-1)\mathbf{v}_1 = \mathbf{0}$ and $c_1 + c_2 = 0$, which shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent.
- **14.** Let S_1 consist of three (distinct) points on a line through the origin. The set is affinely dependent because the third point is on the line determined by the first two points. Let S_2 consist of two (distinct) points on a line through the origin. By Exercise 13, the set is affinely independent because the two points are distinct. (A correct solution should include a justification for the sets presented.)
- **15.** a. The vectors $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are not multiples and hence are linearly independent. By Theorem 5, S is affinely independent.

- **b.** $\mathbf{p}_1 \leftrightarrow \left(-\frac{6}{8}, \frac{9}{8}, \frac{5}{8}\right), \mathbf{p}_2 \leftrightarrow \left(0, \frac{1}{2}, \frac{1}{2}\right), \mathbf{p}_3 \leftrightarrow \left(\frac{14}{8}, -\frac{5}{8}, -\frac{1}{8}\right),$ $\begin{array}{l} \textbf{p}_{4} \leftrightarrow \left(\frac{6}{8}, -\frac{5}{8}, \frac{7}{8}\right), \, \textbf{p}_{5} \leftrightarrow \left(\frac{2}{8}, \frac{1}{8}, \frac{5}{8}\right) \\ \textbf{c.} \ \ \textbf{p}_{6} \text{ is } (-, -, +), \textbf{p}_{7} \text{ is } (0, +, -), \text{ and } \textbf{p}_{8} \text{ is } (+, +, -). \end{array}$
- **16.** a. The vectors $\mathbf{v}_2 \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{v}_3 \mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are not Theorem 5, S is affinely independent.
 - **b.** $\mathbf{p}_1 \leftrightarrow (-\frac{2}{7}, \frac{5}{7}, \frac{4}{7}), \mathbf{p}_2 \leftrightarrow (\frac{2}{7}, -\frac{5}{7}, \frac{10}{7}), \mathbf{p}_3 \leftrightarrow (\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$
 - c. $\mathbf{p}_4 \leftrightarrow (+,-,-), \mathbf{p}_5 \leftrightarrow (+,+,-), \mathbf{p}_6 \leftrightarrow (+,+,+),$ $\mathbf{p}_7 \leftrightarrow (-,0,+)$. See the figure below. Actually,
 - $\mathbf{p}_4 \leftrightarrow (\frac{19}{14}, -\frac{2}{14}, -\frac{3}{14}), \mathbf{p}_5 \leftrightarrow (\frac{5}{14}, \frac{12}{14}, -\frac{3}{14}),$
 - $\mathbf{p}_6 \leftrightarrow (\frac{9}{14}, \frac{2}{14}, \frac{3}{14}), \text{ and } \mathbf{p}_7 \leftrightarrow (-\frac{1}{2}, 0, \frac{3}{2}).$



17. Suppose $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set. Then (7) has a solution, because \mathbf{p} is in aff S. Hence (8) has a solution. By Theorem 5, the homogeneous forms of the points in S are linearly independent. Thus, (8) has a unique solution. Then (7) also has a unique solution, because (8) encodes both equations that appear in (7).

The following argument mimics the proof of Theorem 7 in Section 4.4. If $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set, then scalars c_1, \ldots, c_k exist that satisfy (7), by definition of aff S. Suppose \mathbf{p} also has the representation

$$\mathbf{p} = d_1 \mathbf{b}_1 + \dots + d_k \mathbf{b}_k \quad \text{and} \quad d_1 + \dots + d_k = 1$$
 (7a)

for scalars d_1, \ldots, d_k . Then subtraction produces the equation

$$\mathbf{0} = \mathbf{p} - \mathbf{p} = (c_1 - d_1)\mathbf{b}_1 + \dots + (c_k - d_k)\mathbf{b}_k$$
 (7b)

The weights in (7b) sum to zero because the c's and the d's separately sum to one. This is impossible, unless each weight in (8) is zero, because S is an affinely independent set. This proves that $c_i = d_i$ for i = 1, ..., k.

18. Let $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{x}{a} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \frac{y}{b} \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \frac{z}{c} \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} + \left(1 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So the barycentric

coordinates are x/a, y/b, z/c, and 1 - x/a - y/b - z/c. This holds for any nonzero choices of a, b, and c.

19. Let
$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$
, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then $\det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} =$

$$\det\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}$$
 by using the transpose property of the

determinant (Theorem 5 in Section 3.2). By Exercise 30 in Section 3.3, this determinant equals 2 times the area of the triangle with vertices at **a**, **b**, and **c**.

- 20. If p is on the line through a and b, then p is an affine combination of a and b, so \tilde{p} is a linear combination of \tilde{a} and \tilde{b} . Thus, the columns of $[\tilde{a} \quad \tilde{b} \quad \tilde{p}]$ are linearly dependent. So the determinant of this matrix is zero.
- **21.** If $[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$, then Cramer's rule gives

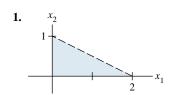
 $r = \det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}} \]/\det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}} \].$ By Exercise 19, the numerator of this quotient is twice the area of $\Delta \mathbf{pbc}$, and the denominator is twice the area of $\Delta \mathbf{abc}$. This proves the formula for r. The other formulas are proved using Cramer's rule for s and t.

22. Let $\mathbf{p} = (1 - x)\mathbf{q} + x\mathbf{a}$, where \mathbf{q} is on the line segment from \mathbf{b} to \mathbf{c} . Then, because the determinant is a linear function of the first column when the other columns are fixed (Section 3.2),

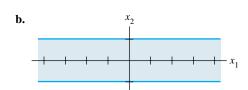
$$\det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = \det[(1-x)\tilde{\mathbf{q}} + x\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$$
$$= (1-x)\cdot \det[\tilde{\mathbf{q}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] + x\cdot \det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$$

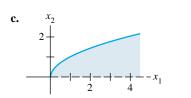
Now, $[\tilde{\mathbf{q}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$ is a singular matrix because $\tilde{\mathbf{q}}$ is a linear combination of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$. So $\det[\tilde{\mathbf{q}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = 0$ and $\det[\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = x \cdot \det[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$.

Section 8.3, page 27



2. a. x_2





3. None are in conv *S*.

Solution:

From Exercise 7 in Section 8.1, \mathbf{p}_1 and \mathbf{p}_3 are not in aff S, so they certainly are not in conv S. Since $\mathbf{p}_2 = 2\mathbf{v}_1 - 4\mathbf{v}_2 + 3\mathbf{v}_3$, and the weights sum to one, \mathbf{p}_2 is in aff S. However, S is affinely independent (because S is linearly independent), so the barycentric coordinates of \mathbf{p}_2 are 2, -4 and 3. Since one coordinate is negative, \mathbf{p}_2 is not in conv S.

4. $\mathbf{p}_2 \in \text{conv } S$

Solution:

From Exercise 6 in Section 8.1, \mathbf{p}_3 is not in aff S, so it certainly is not in conv S. Since $\mathbf{p}_1 = -4\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$ and $\mathbf{p}_2 = 0.2\mathbf{b}_1 + 0.5\mathbf{b}_2 + .3\mathbf{b}_3$, and in each case the weights sum to one, both \mathbf{p}_1 and \mathbf{p}_2 are in aff S. However, S is affinely independent (because S is linearly independent), so the weights in these combinations are barycentric coordinates. Thus, \mathbf{p}_2 is in conv S, because its barycentric coordinates are nonnegative. This is not the case for \mathbf{p}_1 , so $\mathbf{p}_1 \notin \text{conv } S$.

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5.
$$\mathbf{p}_1 = -\frac{1}{6}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$$
, so $\mathbf{p}_1 \notin \text{conv } S$. $\mathbf{p}_2 = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_2 \in \text{conv } S$.

6. a.
$$\mathbf{p}_1 = \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 + \mathbf{v}_3 \in \text{aff } S$$

b.
$$\mathbf{p}_2 = \frac{1}{4}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \in \text{conv } S$$

c.
$$\mathbf{p}_3 = \mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 \in \text{Span } S$$

b.
$$\mathbf{p}_2 = \frac{1}{4}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 \in \text{conv } S$$

c. $\mathbf{p}_3 = \mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 \in \text{Span } S$
d. $\text{proj}_{\text{Span } S} \mathbf{p}_4 = \frac{2}{3}\mathbf{v}_1 + \frac{8}{9}\mathbf{v}_2 + \frac{8}{9}\mathbf{v}_3 = \left(-\frac{4}{9}, -\frac{8}{9}, \frac{10}{9}, 4\right) \neq \mathbf{p}_4,$
so $\mathbf{p}_4 \notin \text{Span } S$

7. a. The barycentric coordinates of
$$\mathbf{p}_1$$
, \mathbf{p}_2 , \mathbf{p}_3 , and \mathbf{p}_4 are, respectively, $\left(\frac{12}{13}, \frac{3}{13}, -\frac{2}{13}\right)$, $\left(\frac{8}{13}, \frac{2}{13}, \frac{3}{13}\right)$, $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$, and $\left(\frac{9}{3}, -\frac{1}{13}, \frac{5}{13}\right)$.

b.
$$\mathbf{p}_1$$
 and \mathbf{p}_4 are outside conv T . \mathbf{p}_2 is inside conv T . \mathbf{p}_3 is on the edge $\overline{\mathbf{v}_1\mathbf{v}_3}$ of conv T .

9.
$$\mathbf{p} = \frac{1}{6}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_4$$
 and $\mathbf{p} = \frac{1}{2}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3$.

10.
$$\mathbf{p} = \frac{3}{5}\mathbf{v}_2 + \frac{3}{10}\mathbf{v}_3 + \frac{1}{10}\mathbf{v}_4$$
 and $\mathbf{p} = \frac{1}{11}\mathbf{v}_1 + \frac{6}{11}\mathbf{v}_2 + \frac{4}{11}\mathbf{v}_3$

b. True. Theorem 9.

c. False. The points do not have to be distinct. For example, S might consist of two points in \mathbb{R}^5 . A point in conv S would be a convex combination of these two points. Caratheodory's Theorem requires n + 1 or fewer points.

13. pos S is a cone with vertex at (0, 0) containing the positive *x*-axis, with sides on the lines $y = \pm x$.

14. a. There are many possibilities. For example, $p = v_1 + v_2 + 2v_3$, so $p \in pos S$.

> $\mathbf{p} = -2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$, so $\mathbf{p} \in \text{aff } S$. It is clear from a graph that $\mathbf{p} \notin \text{conv } S$.

b. The set in Exercise 13 consists of exactly two points, say \mathbf{v}_1 and \mathbf{v}_2 . These points form a basis for \mathbb{R}^2 . Any other point **p** in \mathbb{R}^2 has a *unique* representation as a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . If this combination is both affine and positive, it must be convex. In part (a) above, there were two different linear combinations giving **p**, one affine and one positive, but no single linear combination with both properties.

15. Let $\mathbf{y} \in \text{pos (conv } S)$. Then $\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$, where $\mathbf{v}_i \in \text{conv } S \text{ and } c_i \geq 0.$ But by Caratheodory's Theorem,

each \mathbf{v}_i is a convex combination of n+1 (or fewer) points of S, so

$$\mathbf{v}_i = (d_{i1}\mathbf{w}_{i1} + \dots + d_{i,n+1}\mathbf{w}_{i,n+1}),$$

where

$$d_{ij} \geq 0$$
, $\Sigma d_{ij} = 1$, and $\mathbf{w}_{ij} \in S$

$$\mathbf{y} = c_1(d_{11}\mathbf{w}_{11} + \dots + d_{1,n+1}\mathbf{w}_{1,n+1}) + \dots + c_k(d_{k1}\mathbf{w}_{k1} + \dots + d_{k,n+1}\mathbf{w}_{k,n+1})$$

Since all these coefficients are nonnegative, this shows that $\mathbf{y} \in \text{pos } S$.

The converse is immediate since $S \subset \text{conv } S$.

16. Suppose S is convex and let $y \in pos S$. Then $\mathbf{y} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$, where $\mathbf{v}_i \in S$ and $c_i \geq 0$. Let $\alpha = c_1 + \cdots + c_k$. If $\alpha = 0$, then all $c_i = 0$. It follows that y = 0, and $y = \lambda s$ for $\lambda = 0$. If $\alpha \neq 0$, then

$$\frac{\mathbf{y}}{\alpha} = \frac{c_1}{\alpha} \mathbf{v}_1 + \dots + \frac{c_k}{\alpha} \mathbf{v}_k \in S$$
, since S is convex

That is, $\mathbf{y}/\alpha = \mathbf{s}$ for some $\mathbf{s} \in S$, and $\mathbf{y} = \alpha \mathbf{s}$.

17. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. If $0 \le t \le 1$, we must show that $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$ is in f(S). Since f is linear, $\mathbf{y} = (1-t)\mathbf{p} + t\mathbf{q} = (1-t)f(\mathbf{r}) + tf(\mathbf{s}) = f((1-t)\mathbf{r} + t\mathbf{s})$ Since S is convex, $(1-t)\mathbf{r} + t\mathbf{s} \in S$ whenever 0 < t < 1.

Thus, $y \in S$ and f(S) is convex.

18. Suppose $\mathbf{r}, \mathbf{s} \in S$ and $0 \le t \le 1$. Then, since f is a linear transformation,

$$f((1-t)\mathbf{r} + t\mathbf{s}) = (1-t)f(\mathbf{r}) + tf(\mathbf{s})$$

But $f(\mathbf{r}) \in T$ and $f(\mathbf{s}) \in T$, so $(1-t)f(\mathbf{r}) + tf(\mathbf{s}) \in T$ since T is convex. It follows that $(1 - t)\mathbf{r} + t\mathbf{s} \in S$. This shows that S is convex.

19. Suppose $A \subset B$, where B is convex. Then, since B is convex, Theorem 7 implies that B contains all convex combinations of points of B. Hence B contains all convex combinations of points of A. That is, conv $A \subset B$.

20. Suppose $A \subset B$. Then $A \subset B \subset \text{conv } B$. Since conv B is convex, Exercise 19 shows that conv $A \subset \text{conv } B$.

21. Since $A \subset (A \cup B)$, Exercise 20 shows that conv $A \subset \text{conv}(A \cup B)$. Similarly, conv $B \subset \text{conv } (A \cup B)$. Thus, $[(\operatorname{conv} A) \cup (\operatorname{conv} B)] \subset \operatorname{conv} (A \cup B).$

22. Since $(A \cap B) \subset A$, Exercise 20 shows that conv $(A \cap B) \subset \text{conv } A$. Similarly,

c. False. For example, the union of two distinct points is not convex, but the individual points form convex sets.

conv $(A \cap B) \subset \text{conv } B$. Thus, conv $(A \cap B) \subset [(\text{conv } A) \cap (\text{conv } B)]$.

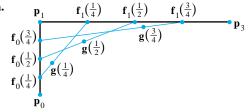
- **23.** One possibility is to let A be two adjacent corners of a square and B be the other two corners. Then $(\operatorname{conv} A) \cup (\operatorname{conv} B)$ consists of two opposite sides of the square, but $\operatorname{conv} (A \cup B)$ is the whole square.
- **24.** One possibility is to let A be a pair of opposite vertices of a square and B be the other pair of opposite vertices. Then conv A and conv B are intersecting diagonals of the square. $A \cap B$ is the empty set, so conv $(A \cap B)$ must be empty, too. But conv $A \cap \text{conv } B$ contains the single point where the diagonals intersect. So conv $(A \cap B)$ is a proper subset of conv $A \cap \text{conv } B$.
- **25. a.** *Hint*: Suppose F is not a subset of G and G is not a subset of F. Then there exist points \mathbf{p} and \mathbf{q} such that $\mathbf{p} \in F$, $\mathbf{p} \notin G$, $\mathbf{q} \in G$, and $\mathbf{q} \notin F$. Consider the point $\mathbf{r} = (\frac{1}{2})(\mathbf{p} + \mathbf{q})$. Since $F \cup G$ is convex, \mathbf{r} belongs to the line segment between \mathbf{p} and \mathbf{q} , which is contained in $F \cup G$. Is $\mathbf{r} \in F$? Is $\mathbf{r} \in G$?

STUDY GUIDE SOLUTION:

If $\mathbf{r} \in F$, then $\mathbf{q} \in F$ since $\mathbf{q} = 2\mathbf{r} - \mathbf{p}$ and F is a subspace. This contradicts $\mathbf{q} \notin F$. Similarly, if $\mathbf{r} \in G$, we reach the contradiction $\mathbf{p} \in G$. Thus, the points \mathbf{p} and \mathbf{q} cannot both exist as described, and either $F \subset G$ or $G \subset F$. The converse is immediate.

b. Here is one possibility in \mathbb{R}^1 : let A = [1, 4] and B = [2, 5].

26. a.



- **b.** $\mathbf{g}(t) = (1-t)\mathbf{f}_0(t) + t\mathbf{f}_1(t) = (1-t)[(1-t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1-t)\mathbf{p}_1 + t\mathbf{p}_2] = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2.$ The sum of the weights in the linear combination for \mathbf{g} is $(1-t)^2 + 2t(1-t) + t^2$, which equals $(1-2t+t^2) + (2t-2t^2) + t^2 = 1$. The weights are each between 0 and 1 when $0 \le t \le 1$, so $\mathbf{g}(t)$ is in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.
- **27.** $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$. Use the representation for $\mathbf{g}_1(t)$ from Exercise 26, and the analogous representation for $\mathbf{g}_2(t)$, based on the control points \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 , and obtain

$$\begin{aligned} \mathbf{h}(t) &= (1-t)[(1-t)^2 \mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2 \mathbf{p}_2] \\ &+ t[(1-t)^2 \mathbf{p}_1 + 2t(1-t)\mathbf{p}_2 + t^2 \mathbf{p}_3] \\ &= (1-t)^3 \mathbf{p}_0 + 2t(1-2t+t^2)\mathbf{p}_1 + (t^2-t^3)\mathbf{p}_2 \\ &+ t(1-2t+t^2)\mathbf{p}_1 + 2t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3 \\ &= (1-3t+3t^2-t^3)\mathbf{p}_0 + (2t-4t^2+2t^3)\mathbf{p}_1 \\ &+ (t^2-t^3)\mathbf{p}_2 + (t-2t^2+t^3)\mathbf{p}_1 + (2t^2-2t^3)\mathbf{p}_2 + t^3 \mathbf{p}_3 \\ &= (1-3t+3t^2-t^3)\mathbf{p}_0 + (3t-6t^2+3t^3)\mathbf{p}_1 \\ &+ (3t^2-3t^3)\mathbf{p}_2 + t^3 \mathbf{p}_3 \end{aligned}$$

By inspection, the sum of the weights in this linear combination is 1, for all t. To show that the weights are nonnegative for $0 \le t \le 1$, factor the coefficients and write

$$\mathbf{h}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2 (1-t) \mathbf{p}_2 + t^3 \mathbf{p}_3 \text{ for } \le t \le 1$$

Thus, $\mathbf{h}(t)$ is in the convex hull of the control points \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 .

28. Fix t, with $0 \le t \le 1$. If $\mathbf{g}_1(t)$ is in S_0 and if $\mathbf{g}_2(t)$ is in S_1 , then there exist nonnegative weights c_0, \ldots, c_k that sum to one, and nonnegative weights d_1, \ldots, d_{k+1} that also sum to one, such that

$$\mathbf{g}_1(t) = c_0 \mathbf{p}_0 + c_1 \mathbf{p}_1 + \dots + c_k \mathbf{p}_k$$

an

$$\mathbf{g}_2(t) = d_1 \mathbf{p}_1 + \dots + d_k \mathbf{p}_k + d_{k+1} \mathbf{p}_{k+1}$$

If $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t \cdot \mathbf{g}_2(t)$, then the weights on $\mathbf{p}_0, \ldots, \mathbf{p}_{k+1}$ in the expression for $\mathbf{h}(t)$ are nonnegative because they are nonnegative c_i and d_i multiplied by (1-t) and t. Further, the sum of these weights is $(1-t)[c_0 + \cdots + c_k] + t[d_1 + \cdots + d_{k+1}] = (1-t) \cdot 1 + t \cdot 1 = 1$. Thus, $\mathbf{h}(t) \in \text{conv } \{\mathbf{p}_0, \ldots, \mathbf{p}_{k+1}\}$ for each t.

(More detailed solution)

Fix t, with $0 \le t \le 1$. By hypothesis, $\mathbf{g}_1(t) = \sum_{i=0}^k c_i \mathbf{p}_i$ and $\mathbf{g}_2(t) = \sum_{j=1}^{k+1} d_j \mathbf{p}_j$ for some nonnegative constants c_i and d_j , respectively, such that $\sum c_i = 1$ and $\sum d_j = 1$. Let

$$\mathbf{y} = (1 - t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$$

$$= (1 - t)\sum_{i=0}^k c_i \mathbf{p}_i + t\sum_{j=1}^{k+1} d_j \mathbf{p}_j$$

$$= \sum_{i=0}^k (1 - t)c_i \mathbf{p}_i + \sum_{j=1}^{k+1} td_j \mathbf{p}_j$$

The weights in this linear combination are all nonnegative, so to show $y \in \text{conv} \{p_0, p_1, \dots, p_{k+1}\}$, it suffices to show that the weights sum to one. In fact, this sum is

$$\sum_{i=0}^{k} (1-t)c_i + \sum_{j=1}^{k+1} t d_j = (1-t) \sum_{i=0}^{k} c_i + t \sum_{j=1}^{k+1} d_j$$
$$= (1-t) \cdot 1 + t \cdot 1 = 1$$

Possible Test Question. Let \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 be points in \mathbb{R}^n . Show that a line segment from a point in conv $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ to a point in conv $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is contained in conv $\{{\bf p}_0, {\bf p}_1, {\bf p}_2, {\bf p}_3\}$.

Solution:

Given $\mathbf{p} \in \text{conv} \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ and $\mathbf{q} \in \text{conv} \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, let $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$ for some t such that $0 \le t \le 1$. Write $\mathbf{p} = c_0 \mathbf{p}_0 + c_1 \mathbf{p}_1 + c_2 \mathbf{p}_2$ and $\mathbf{q} = d_1 \mathbf{p}_1 + d_2 \mathbf{p}_2 + d_3 \mathbf{p}_3$, where the weights in each case are nonnegative and sum to one. Etc.

Section 8.4, page 38

1. $f(x_1, x_2) = 3x_1 + 4x_2$ and d = 13

$$\mathbf{v}_{2} - \mathbf{v}_{1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

$$f(x_{1}, x_{2}) = 3x_{1} + 4x_{2}. \quad f(\mathbf{v}_{1}) = 3(3) + 4(1) = 13.$$

2. $f(x_1, x_2) = 5x_1 - 3x_2$ and d = -7

solution:

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

 $f(x_1, x_2) = 5x_1 - 3x_2. \quad f(\mathbf{v}_1) = 5(1) - 3(4) = -7$

- **3. a.** $\mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ or a multiple **b.** $f(\mathbf{x}) = 2x_2 + 3x_3, d = 11$
- **4. a.** $\mathbf{n} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$ or a multiple
 - **b.** $f(\mathbf{x}) = 4x_1 + 3x_2 6x_3, d = -8$

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{vmatrix} 3 & 6 & x_1 \\ 0 & -2 & x_2 \\ 2 & 3 & x_3 \end{vmatrix} = 4x_1 + 3x_2 - 6x_3 = f(x_1, x_2, x_3)$$

So
$$\mathbf{n} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$$
 and $d = \mathbf{n} \cdot \mathbf{v}_1 = -8$.

5. **a.**
$$\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$
 or a multiple

b.
$$f(\mathbf{x}) = 3x_1 - x_2 + 2x_3 + x_4, d = 5$$

6. a.
$$\mathbf{n} = \begin{bmatrix} -2 \\ 3 \\ -5 \\ 1 \end{bmatrix}$$
 or a multiple

b.
$$f(\mathbf{x}) = -2x_1 + 3x_2 - 5x_3 + x_4, d = 4$$

Solution:

$$\mathbf{v}_1 = \begin{bmatrix} 1\\2\\0\\0 \end{bmatrix}, \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1\\0\\-1\\-3 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 0\\1\\2\\7 \end{bmatrix},$$

$$\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}. \text{ Solve the equations } (\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n} = 0,$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 2 & 0 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}.$$

Thus, $x_1 = -2x_4$, $x_2 = 3x_4$, $x_3 = -5x_4$, with x_4 free. Take

$$x_4 = 1$$
, for example, to get $\mathbf{n} = \begin{bmatrix} -2 \\ 3 \\ -5 \\ 1 \end{bmatrix}$.

b. Let
$$f(x_1, x_2, x_3, x_4) = -2x_1 + 3x_2 - 5x_3 + x_4$$
. Let $d = f(\mathbf{v}_1) = -2(1) + 3(2) + 0 + 0 = 4$.

7. \mathbf{v}_2 is on the same side as $\mathbf{0}$, \mathbf{v}_1 is on the other side, and \mathbf{v}_3 is in H.

Solution:

Compute $\mathbf{n} \cdot \mathbf{p} = 2$. Then $H = {\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 2}$. The origin is in $H_0 = \{ \mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 0 \}$. Compute $\mathbf{n} \cdot \mathbf{v}_1 = 5$. This is more than 2, so \mathbf{v}_1 and $\mathbf{0}$ are on the opposite sides of H. Compute $\mathbf{n} \cdot \mathbf{v}_2 = -2$, which is less than 2, so \mathbf{v}_2 and $\mathbf{0}$ are on the same side of H. Compute $\mathbf{n} \cdot \mathbf{v}_3 = 2$, so \mathbf{v}_3 is in H.

- **8.** Let H = [f : d], where $f(x_1, x_2, x_3) = 3x_1 + x_2 2x_3$ and d = 4. There is no hyperplane parallel to H that strictly separates A and B.
- 9. One possibility is $\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Solution:

 $H_1 = \{ \mathbf{x} : \mathbf{n}_1 \cdot \mathbf{x} = d_1 \}$ and $H_2 = \{ \mathbf{x} : \mathbf{n}_2 \cdot \mathbf{x} = d_2 \}$. Since $\mathbf{p}_1 \in H_1, d_1 = \mathbf{n}_1 \cdot \mathbf{p}_1 = 4$. Similarly, $d_2 = \mathbf{n}_2 \cdot \mathbf{p}_2 = 22$. Solve the simultaneous system $\begin{bmatrix} 1 & 2 & 4 & 2 \end{bmatrix} \mathbf{x} = 4$ and $[2 \ 3 \ 1 \ 5]$ **x** = 22:

$$\begin{bmatrix} 1 & 2 & 4 & 2 & 4 \\ 2 & 3 & 1 & 5 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -10 & 4 & 32 \\ 0 & 1 & 7 & -1 & -14 \end{bmatrix}$$

The general solution provides one set of vectors, \mathbf{p} , \mathbf{v}_1 , and \mathbf{v}_2 . Other choices are possible.

$$\mathbf{x} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + x_3 \mathbf{v}_1 + x_4 \mathbf{v}_2, \text{ where}$$

$$\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- **10.** 2, 3, or 4
- **11. a.** False. A linear functional goes from \mathbb{R}^n to \mathbb{R} .
 - **b.** False. See the discussion of (1) and (4). There is a $1 \times n$ matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Equivalently, there is a point \mathbf{n} in \mathbb{R}^n such that $f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n .
 - c. True. See the comments after the definition of strictly separate.
 - d. False. See the sets in Fig. 4.
- 12. a. True. See the statement after (3).
 - **b.** False. The vector **n** must be nonzero. If $\mathbf{n} = \mathbf{0}$, then the given set is empty if $d \neq 0$ and the set is all of \mathbb{R}^n if d = 0.
 - **c.** False. Theorem 12 requires that the sets *A* and *B* be convex. For example, *A* could be the boundary of a circle and *B* could be the center of the circle.
 - **d.** False. Some other hyperplane might strictly separate them. See the caution at the end of Example 8.
- **13.** Let $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$ and suppose $\mathbf{z} = (1 t)\mathbf{x} + t\mathbf{y}$, where $0 \le t \le 1$. Then

$$\|\mathbf{z} - \mathbf{p}\| = \|[(1-t)\mathbf{x} + t\mathbf{y}] - \mathbf{p}\|$$

$$= \|(1-t)(\mathbf{x} - \mathbf{p}) + t(\mathbf{y} - \mathbf{p})\|$$

$$\leq (1-t)\|\mathbf{x} - \mathbf{p}\| + t\|\mathbf{y} - \mathbf{p}\|$$

$$< (1-t)\delta + t\delta = \delta$$

where the first inequality comes from the Triangle Inequality (Theorem 17 in Section 6.7) and the second inequality follows from $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$. It follows that $\mathbf{z} \in B(\mathbf{p}, \delta)$ and $B(\mathbf{p}, \delta)$ is convex.

- **14.** Let *S* be a bounded set. Then there exists a $\delta > 0$ such that $S \subset B(\mathbf{0}, \delta)$. Since $B(\mathbf{0}, \delta)$ is convex by Exercise 13, Theorem 9 in Section 8.3 implies that conv $S \subset B(\mathbf{0}, \delta)$ and conv *S* is bounded.
- **15.** f(x, y) = 4x + y. A natural choice for d is 12.75, which equals f(3, .75). The point (3, .75) is three-fourths of the distance between the center of $B(\mathbf{0}, 3)$ and the center of $B(\mathbf{p}, 1)$.

Solution:

Let L be the line segment from the center of $B(\mathbf{0},3)$ to the center of $B(\mathbf{p},1)$. This is on the line through the origin in the direction of \mathbf{p} . The length of L is $(4^2+1^2)^{1/2}\approx 4.1231$. This exceeds the sum of the radii of the two disks, so the disks do not touch. If the disks did touch, the separating hyperplane (line) would have to be orthogonal to \mathbf{p} , the vector that in this case determines the direction of L. Since the disks are separated slightly, the hyperplane need not be exactly perpendicular, but the easiest one to find is a hyperplane H whose normal vector is \mathbf{p} . So define f by $f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x}$.

To find d, evaluate f at any point on L that is between the two disks. If the disks were tangent, that point would be three-fourths of the distance between their centers, since the radii are 3 and 1. Since the disks are slightly separated, the distance is 4.1231. Three-fourths of this distance is greater than 3, and one-fourth of this distance is greater than 1. A suitable value of d is $f(\mathbf{q})$, where

$$\mathbf{q} = (\frac{1}{4})\mathbf{0} + (\frac{3}{4})\mathbf{p} = (3, .75)$$
. So $d = 4(3) + 1(.75) = 12.75$.

16. Note:
$$\mathbf{p} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$
. $f(x, y) = 4x - 2y$. A natural choice for *d* is $f(5, 1.5) = 17$.

Solution:

The normal to the separating hyperplane has the direction of the line segment between ${\bf p}$ and ${\bf q}$. So, let

$$\mathbf{n} = \mathbf{p} - \mathbf{q} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$
. The distance between \mathbf{p} and \mathbf{q} is $\sqrt{20}$,

which is more than the sum of the radii of the two balls. The large ball has center \mathbf{q} . A point three-fourths of the distance from \mathbf{q} to \mathbf{p} will be greater than 3 units from \mathbf{q} and greater than 1 unit from \mathbf{p} . This point is

$$\mathbf{x} = .75\mathbf{p} + .25\mathbf{q} = .75\begin{bmatrix} 6\\1 \end{bmatrix} + .25\begin{bmatrix} 2\\3 \end{bmatrix} = \begin{bmatrix} 5.0\\1.5 \end{bmatrix}$$

Compute $\mathbf{n} \cdot \mathbf{x} = 17$. The desired hyperplane is

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 4x - 2y = 17 \right\}.$$

- 17. Exercise 2(a) in Section 8.3 gives one possibility. Or let $S = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}$. Then conv *S* is the upper (open) half-plane.
- **18.** One possibility is $A = \{(x, y) : |x| \le 1 \text{ and } y = 0\}$ and $B = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}.$
- 19. Let $\mathbf{y} \in B(\mathbf{p}, \delta)$ so that $\lambda \mathbf{y} \in \lambda B(\mathbf{p}, \delta)$. Then $\|\mathbf{y} \mathbf{p}\| < \delta$, so $\|\lambda \mathbf{y} \lambda \mathbf{p}\| = \lambda \|\mathbf{y} \mathbf{p}\| < \lambda \delta$, and $\lambda \mathbf{y} \in B(\lambda \mathbf{p}, \lambda \delta)$. Conversely, suppose $\mathbf{z} \in B(\lambda \mathbf{p}, \lambda \delta)$. Then $\|\mathbf{z} \lambda \mathbf{p}\| < \lambda \delta$, so

$$\left\|\frac{\mathbf{z}}{1} - \mathbf{p}\right\| = \frac{1}{1} \left\|\mathbf{z} - \lambda \mathbf{p}\right\| < \frac{1}{1} (\lambda \delta) = \delta$$

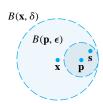
Thus, $\frac{\mathbf{z}}{\lambda} \in B(\mathbf{p}, \delta)$ and $\mathbf{z} \in \lambda B(\mathbf{p}, \delta)$. The proof of the second part is similar.

20. If $\mathbf{p} \in \operatorname{cl} S$, then $\mathbf{p} \in S$ or $\mathbf{p} \in \operatorname{bd} S$. If $\mathbf{p} \in S$, then every open ball centered at \mathbf{p} contains the point \mathbf{p} , which is in S. If $\mathbf{p} \in \operatorname{bd} S$, then, by definition, every open ball centered at \mathbf{p} intersects S.

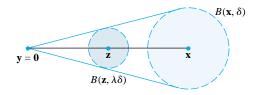
On the other hand, suppose every open ball centered at **p** contains a point of *S*. If **p** is not in *S*, then every open ball centered at **p** intersects both *S* and its complement. Thus, $\mathbf{p} \in \text{bd } S$.

21. Let $\mathbf{x} \in \operatorname{cl}$ (cl S). Then, given any $\delta > 0$, there exists a point $\mathbf{p} \in B(\mathbf{x}, \delta) \cap \operatorname{cl} S$. We must show that some point of $B(\mathbf{x}, \delta)$ is in S. Since $\mathbf{p} \in \operatorname{cl} S$, either $\mathbf{p} \in S$ or $\mathbf{p} \in \operatorname{bd} S$. If $\mathbf{p} \in S$, we are done. Otherwise, $\mathbf{p} \in \operatorname{bd} S$. Let $\varepsilon = \delta - \|\mathbf{x} - \mathbf{p}\|$. Then $\varepsilon > 0$ and the open ball $B(\mathbf{p}, \varepsilon) \subset B(\mathbf{x}, \delta)$. See the figure below. Since $\mathbf{p} \in \operatorname{bd} S$, there exists a point \mathbf{s} in $B(\mathbf{p}, \varepsilon) \cap S$. But then $\mathbf{s} \in B(\mathbf{x}, \delta) \cap S$ and $\mathbf{x} \in \operatorname{cl} S$.

Conversely, let $\mathbf{x} \in \operatorname{cl} S$. Then, given any $\delta > 0$, there exists a point $\mathbf{q} \in B(\mathbf{x}, \delta) \cap S$. But $S \subset \operatorname{cl} S$, so $\mathbf{q} \in B(\mathbf{x}, \delta) \cap \operatorname{cl} S$ and $\mathbf{x} \in \operatorname{cl} (\operatorname{cl} S)$.

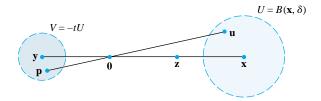


22. Assume without loss of generality that $\mathbf{y} = \mathbf{0}$. Then, for any λ such that $0 < \lambda < 1$, we have $\mathbf{z} = \lambda \mathbf{x}$. See the figure below. If $\mathbf{x} \in \text{int } S$, then there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset S$. Now $B(\lambda \mathbf{x}, \lambda \delta) = \lambda B(\mathbf{x}, \delta)$, by Exercise 19. Since S is convex and $\mathbf{0} \in S$, $\lambda B(\mathbf{x}, \delta) \subset S$. Thus, $B(\mathbf{z}, \lambda \delta) \subset S$ and $\mathbf{z} \in \text{int } S$.



23. Let $\mathbf{z} = \lambda \mathbf{x} + (1 - \lambda)\mathbf{y}$, with $0 < \lambda < 1$. We may assume without loss of generality that $\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z} = \mathbf{0}$. Then $\mathbf{y} = -t\mathbf{x}$ for some t > 0. See the figure below. Since $\mathbf{x} \in \text{int } S$, there exists an open ball $U = B(\mathbf{x}, \delta)$ contained in S. Let V = -tU. Then V is an open ball of radius $t\delta$ about \mathbf{y} .

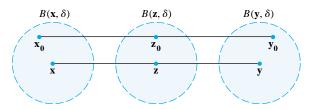
Since $\mathbf{y} \in \operatorname{cl} S$, there exists a point \mathbf{p} in $V \cap S$. That is, $\mathbf{p} = -t\mathbf{u}$ for some $\mathbf{u} \in U$. Since \mathbf{u} and $-t\mathbf{u}$ are in the convex set S, it follows that $\mathbf{0} \in S$. But then Exercise 22 implies $\mathbf{z} \in \operatorname{int} S$.



- **24.** Let $\mathbf{x}, \mathbf{y} \in \text{int } S$. By Exercise 22, all the points $(1 t)\mathbf{x} + t\mathbf{y}$, where 0 < t < 1, are in the interior of S. Thus, $\overline{\mathbf{xy}} \subset \text{int } S$, and int S is convex.
- **25.** Let $\mathbf{x}, \mathbf{y} \in \operatorname{cl} S$ and let $\mathbf{z} = \alpha \mathbf{x} + \beta \mathbf{y}$, where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$. Furthermore, let $B(\mathbf{z}, \delta)$ be an open ball centered at \mathbf{z} . See the figure below. Since \mathbf{x} and \mathbf{y} are in $\operatorname{cl} S$, Exercise 20 implies that there exist points $\mathbf{x}_0 \in B(\mathbf{x}, \delta) \cap S$ and $\mathbf{y}_0 \in B(\mathbf{y}, \delta) \cap S$. We claim that $\mathbf{z}_0 = \alpha \mathbf{x}_0 + \beta \mathbf{y}_0 \in B(z, \delta)$. Indeed,

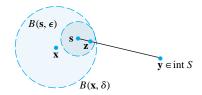
$$\begin{aligned} \|\mathbf{z} - \mathbf{z}_0\| &= \|(\alpha \mathbf{x} + \beta \mathbf{y}) - (\alpha \mathbf{x}_0 + \beta \mathbf{y}_0)\| \\ &= \|\alpha (\mathbf{x} - \mathbf{x}_0) + \beta (\mathbf{y} - \mathbf{y}_0)\| \\ &\leq \|\alpha (\mathbf{x} - \mathbf{x}_0)\| + \|\beta (\mathbf{y} - \mathbf{y}_0)\| \\ &= \alpha \|\mathbf{x} - \mathbf{x}_0\| + \beta \|\mathbf{y} - \mathbf{y}_0\| < \alpha \delta + \beta \delta = (\alpha + \beta)\delta = \delta \end{aligned}$$

Since $\mathbf{z}_0 \in \overline{\mathbf{x}_0 \mathbf{y}_0}$, we have $\mathbf{z}_0 \in S$. Thus, $\mathbf{z} \in \operatorname{cl} S$ and $\operatorname{cl} S$ is convex.

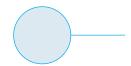


26. a. Suppose $\mathbf{x} \in \operatorname{cl}$ (int S). Then, for every $\delta > 0$, $B(\mathbf{x}, \delta)$ contains at least one point of int S, by Exercise 20. Since int $S \subset S$, this means $B(\mathbf{x}, \delta)$ contains at least one point of S. Thus, $\mathbf{x} \in \operatorname{cl} S$.

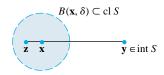
Conversely, suppose $\mathbf{x} \in \operatorname{cl} S$. Given any $\delta > 0$, there exists a point \mathbf{s} in $S \cap B(\mathbf{x}, \delta)$. Since $\|\mathbf{x} - \mathbf{s}\| < \delta$, we have $\varepsilon = \delta - \|\mathbf{x} - \mathbf{s}\| > 0$. It follows that $B(\mathbf{s}, \varepsilon) \subset B(\mathbf{x}, \delta)$. See the figure below. Let $\mathbf{y} \in \operatorname{int} S$ and consider the line segment $\overline{\mathbf{s}}\overline{\mathbf{y}}$. Since $\varepsilon > 0$, there exists $\mathbf{z} \in \overline{\mathbf{s}}\overline{\mathbf{y}}$ such that $\mathbf{z} \in B(\mathbf{s}, \varepsilon)$. But Exercise 22 implies that $\mathbf{z} \in \operatorname{int} S$, so $\mathbf{z} \in B(\mathbf{x}, \delta) \cap (\operatorname{int} S)$, and $\mathbf{x} \in \operatorname{cl} (\operatorname{int} S)$.



b. Let *S* be a "lollipop" shape in \mathbb{R}^2 :



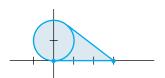
27. a. Since $S \subset \operatorname{cl} S$, int $S \subset \operatorname{int} (\operatorname{cl} S)$. Conversely, suppose $\mathbf{x} \in \operatorname{int} (\operatorname{cl} S)$. Then there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset \operatorname{cl} S$. Let $\mathbf{y} \in \operatorname{int} S$. Then there exists a point \mathbf{z} in $\operatorname{cl} S$ such that $\mathbf{x} \in \overline{\mathbf{z}}\mathbf{y}$, with $\mathbf{x} \neq \mathbf{z}$. See the figure below. It follows from Exercise 23 that $\mathbf{x} \in \operatorname{int} S$.



- **b.** Let $S = [0, 1) \cup (1, 2]$ in \mathbb{R}^2 . Then int $S = (0, 1) \cup (1, 2)$, but int (cl S) = (0, 2). Or let S be a solid square in \mathbb{R}^2 with one diagonal removed. Then int S consists of two triangular pieces, but int (cl S) is the whole square (without the boundary).
- 28. a. If x ∈ bd S, then x ∈ cl S and x ∉ int S. But cl S = cl (cl S), by applying Exercise 21 to cl S in place of S, and int S = int (cl S), by Exercise 27. Thus, x ∈ cl (cl S) and x ∉ int (cl S), so x ∈ bd (cl S). For the converse, reverse the argument above.
 - **b.** Let $S = [0, 1) \cup (1, 2]$ in \mathbb{R}^1 . Then bd $S = \{0, 1, 2\}$, but bd (cl S) = $\{0, 2\}$. Or let S be a solid square in \mathbb{R}^2 with one diagonal removed. Then bd S consists of the diagonal and the four edges of the square, but bd (cl S) does not include the diagonal.
- **29.** Let H = [f : d] be a hyperplane and let $S = \{\mathbf{x} : f(\mathbf{x}) \ge d\}$. Then S is a closed half-space, and its boundary is the convex set H.
- **30.** Let *S* be the convex hull of a circle in \mathbb{R}^2 . Then bd *S* is the circle itself, which is not convex.

Section 8.5, page 47

1. The origin is an extreme point, but it is not a vertex.

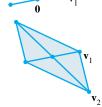


- 2. One possibility is a ray. It has an extreme point at one end.
- **3.** One possibility is to let *S* be a square that includes part of the boundary but not all of it. For example, include just two adjacent edges. The convex hull of the profile is a triangle.



- **4. a.** $f_0(S^5) = 6$, $f_1(S^5) = 15$, $f_2(S^5) = 20$, $f_3(S^5) = 15$, $f_4(S^5) = 6$, and 6 15 + 20 15 + 6 = 2.
- **5. a.** $f_0(C^5) = 32$, $f_1(C^5) = 80$, $f_2(C^5) = 80$, $f_3(C^5) = 40$, $f_4(C^5) = 10$, and 32 80 + 80 40 + 10 = 2.

6. a. X^1 is a line segment:



 X^2 is a parallelogram:

- **b.** $f_0(X^3) = 6$, $f_1(X^3) = 12$, $f_2(X^3) = 8$. X^3 is an octahedron.
- **c.** $f_0(X^4) = 8$, $f_1(X^4) = 24$, $f_2(X^4) = 32$, $f_3(X^4) = 16$, 8 24 + 32 16 = 0
- **d.** $f_k(X^n) = 2^{k+1} \binom{n}{k+1}$, $0 \le k \le n-1$, where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is the binomial coefficient.
- 7. **a.** $f_0(P^n) = f_0(Q) + 1$
 - **b.** $f_k(P^n) = f_k(Q) + f_{k-1}(Q)$
 - **c.** $f_{n-1}(P^n) = f_{n-2}(Q) + 1$
- 8. a. True. Definition.
 - **b.** True. Definition.
 - **c.** False. *S* must be compact. See Theorem 15.
 - **d.** True. See the comment after Fig. 4.
- **9.** a. False. It has six facets (faces).
 - **b.** True. Theorem 14.
 - c. False. The maximum is always attained at some extreme point, but there may be other points that are not extreme points at which the maximum is attained.
 - **d.** True. Follows from Euler's formula with n = 2.
- 10. Let **x** be an extreme point of the convex set *S* and let $T = \{ \mathbf{y} \in S : \mathbf{y} \neq \mathbf{x} \}$. If **y** and **z** are in *T*, then $\overline{\mathbf{yz}} \subset S$ since *S* is convex. But since **x** is an extreme point of *S*, $\mathbf{x} \notin \overline{\mathbf{yz}}$, so $\overline{\mathbf{yz}} \subset T$. Thus, *T* is convex.

Conversely, suppose $\mathbf{x} \in S$, but \mathbf{x} is not an extreme point of S. Then there exist \mathbf{y} and \mathbf{z} in S such that $\mathbf{x} \in \overline{\mathbf{y}}\overline{\mathbf{z}}$, with $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{z}$. It follows that \mathbf{y} and \mathbf{z} are in T, but $\overline{\mathbf{y}}\overline{\mathbf{z}} \not\subset T$. Hence T is not convex.

11. Suppose A and B are convex. Let $\mathbf{x}, \mathbf{y} \in A + B$. Then there exist $\mathbf{a}, \mathbf{c} \in A$ and $\mathbf{b}, \mathbf{d} \in B$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\mathbf{y} = \mathbf{c} + \mathbf{d}$. For any t such that $0 \le t \le 1$, we have

$$(1-t)\mathbf{x} + t\mathbf{y} = (1-t)(\mathbf{a} + \mathbf{b}) + t(\mathbf{c} + \mathbf{d})$$

= $[(1-t)\mathbf{a} + t\mathbf{c}] + [(1-t)\mathbf{b} + t\mathbf{d}]$

But $(1 - t)\mathbf{a} + t\mathbf{c} \in A$ since A is convex, and $(1 - t)\mathbf{b} + t\mathbf{d} \in B$ since B is convex. Thus, $(1 - t)\mathbf{x} + t\mathbf{y}$ is in A + B, which shows that A + B is convex.

12. a. Let *S* be convex and let $\mathbf{x} \in cS + dS$, where c > 0 and d > 0. Then there exist \mathbf{s}_1 and \mathbf{s}_2 in *S* such that $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2$. But then

$$\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2 = (c+d) \left[\frac{c}{c+d} \mathbf{s}_1 + \frac{d}{c+d} \mathbf{s}_2 \right]$$

Now $\frac{c}{c+d}$ and $\frac{d}{c+d}$ are both nonnegative and sum to one. Since *S* is convex, $\frac{c}{c+d}\mathbf{s}_1 + \frac{d}{c+d}\mathbf{s}_2 \in S$. Thus, $\mathbf{x} \in (c+d)S$

Conversely, let $\mathbf{x} \in (c+d)S$. Then $\mathbf{x} = (c+d)\mathbf{s}$ for some $\mathbf{s} \in S$. But then $\mathbf{x} = c\mathbf{s} + d\mathbf{s} \in cS + dS$, as desired.

- 13. a. Since each edge belongs to two facets, kr is twice the number of edges: kr = 2e. Since each edge has two vertices, sv = 2e.
 - **b.** v e + r = 2, so $\frac{2e}{s} e + \frac{2e}{k} = 2 \Rightarrow \frac{1}{s} + \frac{1}{k} = \frac{1}{2} + \frac{1}{e}$
 - **c.** A polygon must have at least three sides, so $k \ge 3$. At least three edges meet at each vertex, so $s \ge 3$. But k and s cannot both be greater than 3, for then the left side of the equation in (b) could not exceed $\frac{1}{2}$.

When k = 3, we get $\frac{1}{s} - \frac{1}{6} = \frac{1}{e}$, so s = 3, 4, or 5. For these values, we get e = 6, 12, or 30, corresponding to the tetrahedron, the octahedron, and the icosahedron, respectively.

When s = 3, we get $\frac{1}{k} - \frac{1}{6} = \frac{1}{e}$, so k = 2, 3, or 5 and e = 6, 12, or 30, respectively. These values correspond to the tetrahedron, the cube, and the dodecahedron.

Section 8.6, page 58

1. Given

 $\mathbf{x}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3,$ $0 \le t \le 1$, let **b** be any fixed vector. Observe (by algebraic computation) that

$$(1-t)^3 \mathbf{b} + 3t(1-t)^2 \mathbf{b} + 3t^2(1-t)\mathbf{b} + t^3 \mathbf{b} = \mathbf{b}$$

$$\mathbf{x}(t) + \mathbf{b} = (1 - t)^3 (\mathbf{p}_0 + \mathbf{b}) + 3t(1 - t)^2 (\mathbf{p}_1 + \mathbf{b}) + 3t^2 (1 - t)(\mathbf{p}_2 + \mathbf{b}) + t^3 (\mathbf{p}_3 + \mathbf{b})$$

This representation of $\mathbf{x}(t) + \mathbf{b}$ has the form of a Bézier curve, whose control points are translates by \mathbf{b} of the original control points. So translation by \mathbf{b} maps a Bézier curve into a Bézier curve.

2. a. The coefficients of \mathbf{p}_0 and \mathbf{p}_3 in (15) match those in (14). For \mathbf{p}_1 .

$$3t(t-1)^{2} + 4 - 3t = 3t(t^{2} - 2t + 1) + 4 - 3t$$
$$= 3t^{3} - 6t^{2} + 3t + 4 - 3t$$
$$= 3t^{3} - 6t^{2} + 4$$

The verification for \mathbf{p}_2 is trivial. So (15) is an equivalent description for $\mathbf{x}(t)$.

b. Equation (15) reveals that each polynomial weight is nonnegative for $0 \le t \le 1$, since 4 - 3t > 0. For the

sum of the coefficients, use (15) with the first term expanded: $1 - 3t + 3t^2 - t^3$. The 1 here plus the 4 and 1 in the coefficients of \mathbf{p}_1 and \mathbf{p}_2 , respectively, sum to 6, while the other terms sum to 0. This explains the $\frac{1}{6}$ in the formula for $\mathbf{x}(t)$, which makes the coefficients sum to 1. Thus, $\mathbf{x}(t)$ is a convex combination of the control points for 0 < t < 1.

c. Since the coefficients inside the brackets in equation (14) sum to 6, it follows that

$$\mathbf{b} = \frac{1}{6} [6\mathbf{b}]$$

$$= \frac{1}{6} [(1-t)^3 \mathbf{b} + (3t^3 - 6t^2 + 4)\mathbf{b} + (-3t^3 + 3t^2 + 3t + 1)\mathbf{b} + t^3 \mathbf{b}]$$

and hence $\mathbf{x}(t) + \mathbf{b}$ may be written in a similar form, with \mathbf{p}_i replaced by $\mathbf{p}_i + \mathbf{b}$ for each i. This shows that $\mathbf{x}(t) + \mathbf{b}$ is a cubic B-spline with control points $\mathbf{p}_i + \mathbf{b}$ for $i = 0, \dots, 3$.

3. a. Start with

$$\mathbf{x}(t) = (1-t)^3 \mathbf{p}_0 + 3t(1-t)^2 \mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3 \mathbf{p}_3,$$

and compute

$$\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$$

Then $\mathbf{x}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0)$, and $\mathbf{x}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2)$. This shows that the tangent vector $\mathbf{x}'(0)$ points in the direction from \mathbf{p}_0 to \mathbf{p}_1 and is three times the length of $\mathbf{p}_1 - \mathbf{p}_0$. Likewise, $\mathbf{x}'(1)$ points in the direction from \mathbf{p}_2 to \mathbf{p}_3 and is three times the length of $\mathbf{p}_3 - \mathbf{p}_2$. In particular, $\mathbf{x}'(1) = 0$ if and only if $\mathbf{p}_3 = \mathbf{p}_2$.

b. From part (a), differentiate $\mathbf{x}'(t)$ to get

$$\mathbf{x}''(t) = (6 - 6t)\mathbf{p}_0 + (-12 + 18t)\mathbf{p}_1 + (6 - 18t)\mathbf{p}_2 + 6t\mathbf{p}_3$$

Then

$$\mathbf{x}''(0) = 6\mathbf{p}_0 - 12\mathbf{p}_1 + 6\mathbf{p}_2 = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$$

$$\mathbf{x}''(1) = 6\mathbf{p}_1 - 12\mathbf{p}_2 + 6\mathbf{p}_3 = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at \mathbf{p}_1 , temporarily, label \mathbf{p}_0 as $\mathbf{p}_0 - \mathbf{p}_1$ and label \mathbf{p}_2 as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin through the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$, extended out a bit. That line points in the direction of $\mathbf{x}''(0)$.

4. a. $\mathbf{x}'(t) = \frac{1}{6} \left[\left(-3t^2 + 6t - 3 \right) \mathbf{p}_0 + \left(9t^2 - 12t \right) \mathbf{p}_1 + \left(-9t^2 + 6t + 3 \right) \mathbf{p}_2 + 3t^2 \mathbf{p}_3 \right]$ $\mathbf{x}'(0) = \frac{1}{2} \left(\mathbf{p}_2 - \mathbf{p}_0 \right) \text{ and } \mathbf{x}'(1) = \frac{1}{2} \left(\mathbf{p}_3 - \mathbf{p}_1 \right) \text{ (Verify that, in the first part of Fig. 10, a line drawn through <math>\mathbf{p}_0$ and \mathbf{p}_2 is parallel to the tangent line at the beginning of the

B-spline.) When $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are both zero, the figure collapses and the convex hull of the set of control points is the line segment between \mathbf{p}_0 and \mathbf{p}_3 , in which case $\mathbf{x}(t)$ is a straight line. Where does $\mathbf{x}(t)$ start? In this case,

$$\mathbf{x}(t) = \frac{1}{6} \left[(-4t^3 + 6t^2 + 2)\mathbf{p}_0 + (4t^3 - 6t^2 + 4)\mathbf{p}_3 \right]$$

$$\mathbf{x}(0) = \frac{1}{3}\mathbf{p}_0 + \frac{2}{3}\mathbf{p}_3$$
 and $\mathbf{x}(1) = \frac{2}{3}\mathbf{p}_0 + \frac{1}{3}\mathbf{p}_3$

The curve begins closer to \mathbf{p}_3 and finishes closer to \mathbf{p}_0 . Could it turn around during its travel? Since $\mathbf{x}'(t) = 2t(1-t)(\mathbf{p}_0 - \mathbf{p}_3)$, the curve travels in the direction $\mathbf{p}_0 - \mathbf{p}_3$, so when $\mathbf{x}'(0) = \mathbf{x}'(1) = 0$, the curve always moves away from \mathbf{p}_3 toward \mathbf{p}_0 for $0 \le t \le 1$.

b.
$$\mathbf{x}''(t) = (1-t)\mathbf{p}_0 + (-2+3t)\mathbf{p}_1 + (1-3t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{x}''(0) = \mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2 = (\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1)$$

and

$$\mathbf{x}''(1) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = (\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2)$$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at \mathbf{p}_1 , temporarily, label \mathbf{p}_0 as $\mathbf{p}_0 - \mathbf{p}_1$, and label \mathbf{p}_2 as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin to the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$. That segment represents $\mathbf{x}''(0)$.

For a picture of $\mathbf{x}''(1)$, construct a coordinate system with the origin at \mathbf{p}_2 , temporarily, label \mathbf{p}_1 as $\mathbf{p}_1 - \mathbf{p}_2$, and label \mathbf{p}_3 as $\mathbf{p}_3 - \mathbf{p}_2$. Finally, construct a line from this new origin to the sum of $\mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{p}_3 - \mathbf{p}_2$. That segment represents $\mathbf{x}''(1)$.

5. a. From Exercise 3(a) or equation (9) in the text,

$$\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for $\mathbf{x}'(0)$, with the control points from $\mathbf{y}(t)$, and obtain

$$\mathbf{y}'(0) = -3\mathbf{p}_3 + 3\mathbf{p}_4 = 3(\mathbf{p}_4 - \mathbf{p}_3)$$

For C^1 continuity, $3(\mathbf{p}_3 - \mathbf{p}_2) = 3(\mathbf{p}_4 - \mathbf{p}_3)$, so $\mathbf{p}_3 = (\mathbf{p}_4 + \mathbf{p}_2)/2$, and \mathbf{p}_3 is the midpoint of the line segment from \mathbf{p}_2 to \mathbf{p}_4 .

- **b.** If $\mathbf{x}'(1) = \mathbf{y}'(0) = \mathbf{0}$, then $\mathbf{p}_2 = \mathbf{p}_3$ and $\mathbf{p}_3 = \mathbf{p}_4$. Thus, the "line segment" from \mathbf{p}_2 to \mathbf{p}_4 is just the point \mathbf{p}_3 . [*Note*: In this case, the combined curve is still C^1 continuous, by definition. However, some choices of the other control points, \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_5 , and \mathbf{p}_6 , can produce a curve with a visible "corner" at \mathbf{p}_3 , in which case the curve is not G^1 continuous at \mathbf{p}_3 .]
- **6.** a. With $\mathbf{x}(t)$ as in Exercise 2,

$$\mathbf{x}(0) = (\mathbf{p}_0 + 4\mathbf{p}_1 + \mathbf{p}_2)/6$$

and

$$\mathbf{x}(1) = (\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3)/6$$

Use the formula for $\mathbf{x}(0)$, but with the shifted control points for y(t), and obtain

$$\mathbf{y}(0) = (\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3)/6$$

This equals $\mathbf{x}(1)$, so the B-spline is G^0 continuous at the join point.

b. From Exercise 4(a),

$$\mathbf{x}'(1) = (\mathbf{p}_3 - \mathbf{p}_1)/2$$
 and $\mathbf{x}'(0) = (\mathbf{p}_2 - \mathbf{p}_0)/2$

Use the formula for $\mathbf{x}'(0)$, with the control points for $\mathbf{y}(t)$, and obtain

$$\mathbf{y}'(0) = (\mathbf{p}_3 - \mathbf{p}_1)/2 = \mathbf{x}'(1)$$

Thus, the B-spline is C^1 continuous at the join point.

7. From Exercise 3(b),

$$\mathbf{x}''(0) = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$$

and

$$\mathbf{x}''(1) = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for $\mathbf{x}''(0)$, but with the control points for $\mathbf{y}(t)$, to get

$$\mathbf{y}''(0) = 6(\mathbf{p}_3 - \mathbf{p}_4) + 6(\mathbf{p}_5 - \mathbf{p}_4)$$

Set $\mathbf{x}''(1) = \mathbf{y}''(0)$ and divide by 6, to get

$$(\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{p}_3 - \mathbf{p}_4) + (\mathbf{p}_5 - \mathbf{p}_4)$$

Since the curve is C^1 continuous at \mathbf{p}_3 , the point \mathbf{p}_3 is the midpoint of the segment from \mathbf{p}_2 to \mathbf{p}_4 , by Exercise 5(a).

Thus, $\mathbf{p}_4 - \mathbf{p}_3 = \mathbf{p}_3 - \mathbf{p}_2$. Substituting gives

$$(\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = -(\mathbf{p}_3 - \mathbf{p}_2) + \mathbf{p}_5 - \mathbf{p}_4$$

 $(\mathbf{p}_1 - \mathbf{p}_2) + 2(\mathbf{p}_3 - \mathbf{p}_2) + \mathbf{p}_4 = \mathbf{p}_5$

Finally, again from C^1 continuity, $\mathbf{p}_4 = \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_2$. Thus,

$$\mathbf{p}_5 = \mathbf{p}_3 + (\mathbf{p}_1 - \mathbf{p}_2) + 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Only \mathbf{p}_6 can be chosen arbitrarily.

8. From Exercise 4(b), $\mathbf{x}''(0) = \mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{x}''(1) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3$. Use the formula for $\mathbf{x}''(0)$, with the shifted control points for y(t), to get

$$\mathbf{y}''(0) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = \mathbf{x}''(1)$$

Thus, the curve has C^2 continuity at $\mathbf{x}(1)$.

9. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, and factor the vector as $M_B \mathbf{u}(t)$:

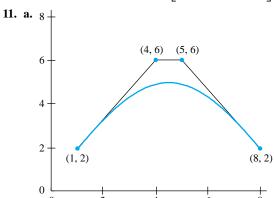
$$\begin{bmatrix} 1 - 4t + 6t^2 - 4t^3 + t^4 \\ 4t - 12t^2 + 12t^3 - 4t^4 \\ 6t^2 - 12t^3 + 6t^4 \\ 4t^3 - 4t^4 \\ t^4 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix}$$

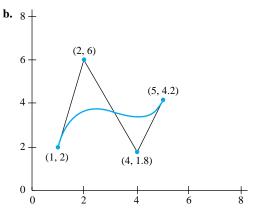
$$M_B = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, taking care to write the terms in ascending powers of t, and factor the vector as M_S **u**(t):

$$\frac{1}{6} \begin{bmatrix} 1 - 3t + 3t^2 - t^3 \\ 4 - 6t^2 + 3t^3 \\ 1 + 3t + 3t^2 - 3t^3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t^2 \\ t^3 \end{bmatrix}$$
$$= M_{S}\mathbf{u}(t),$$

$$M_S = \frac{1}{6} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

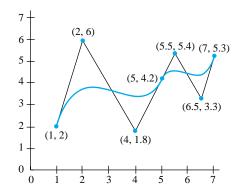




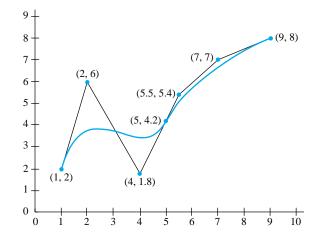
c. The graph for this part must show an inflection point. The first new control point \mathbf{p}_4 must be on the line determined by \mathbf{p}_2 and \mathbf{p}_3 , placed so that \mathbf{p}_3 is **not** the

A16 Answers to Exercises

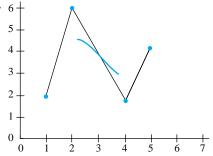
midpoint of the line segment from \mathbf{p}_2 to \mathbf{p}_4 . So \mathbf{p}_4 cannot be $\begin{bmatrix} 6.0 \\ 6.6 \end{bmatrix}$, but two natural choices are $\begin{bmatrix} 5.5 \\ 5.4 \end{bmatrix}$ and $\begin{bmatrix} 6.5 \\ 7.8 \end{bmatrix}$. The first figure below was produced from $\begin{bmatrix} 5.0 \\ 4.2 \end{bmatrix}$, $\begin{bmatrix} 5.5 \\ 5.4 \end{bmatrix}$, $\begin{bmatrix} 6.5 \\ 3.3 \end{bmatrix}$, and $\begin{bmatrix} 7.0 \\ 5.3 \end{bmatrix}$. The MATLAB input is [5.5, 5.6, 5.7; 4.2, 5.4, 3.3, 5.3]



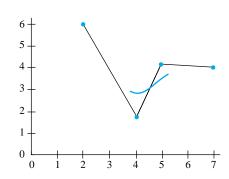
The second graph, from a student, uses $\begin{bmatrix} 5 \\ 4.2 \end{bmatrix}$, $\begin{bmatrix} 5.5 \\ 5.4 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$.



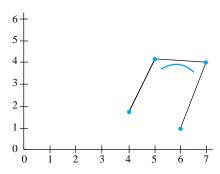
12. a. ₆₊



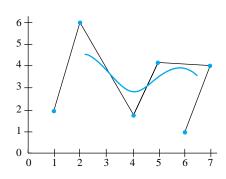
b. Second segment (one new control point):



Third segment:



c. Three segments combined into one curve:



d. Sample bspline.m modification

circle, in points.

```
%B-spline curve
disp('Sample B-spline Curve')
%Input data points
A = input('Enter 2x6 Geometry Matrix:');
%Set up the axes and keep them fixed.
axis([0 10 0 10])
    %Simple default setting
hold on
%Create the parameter t and the vector u
that contains the powers of t.
t = linspace(0,1);
u = [t.^0; t; t.^2; t.^3];
%plot each segment of the curve
for i = 1:3
 G = [A(:,i) \ A(:,i+1) \ A(:,i+2) \ A(:,i+3)]
      %Current geometry matrix
 M = [1 -3 3 -1; 4 0 -6 3; 1 3 3 -3;
 0 0 0 1]/6
      %B-spline basis curve
 x = G*M*u;
      %The B-spline curve
 plot(G(1,:),G(2,:),'-ob',
  `MarkerFacecolor', 'b', 'MarkerSize',2)
   %This comment plots the control points
    with connecting lines.
   %'-ob' draws lines and sets a circle
    for each point, in blue.
   %Use 'ob' instead, if you want no lines
    between the circles.
   %MarkerFaceColor fills in each circle
    (marker) in blue.
   %MarkerSize specifies the size of the
```

```
plot(x(1,:), x(2,:), r')
  %Plot the B-sline curve in red.
 if(i < 3)
   disp('Press any key to continue')
   pause
 end
end
```

13. a. From (12),
$$\mathbf{q}_1 - \mathbf{q}_0 = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_0) = \frac{1}{2}\mathbf{p}_1 - \frac{1}{2}\mathbf{p}_0$$
. Since $\mathbf{q}_0 = \mathbf{p}_0$, $\mathbf{q}_1 = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_0)$.

b. From (13),
$$(\mathbf{q}_3 - \mathbf{q}_2) = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$$
. So $8\mathbf{q}_3 + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = 8\mathbf{q}_2$.

c. Use (8) to substitute for $8q_3$, and obtain

$$8\mathbf{q}_2 = (\mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3) + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3$$

= $2\mathbf{p}_0 + 4\mathbf{p}_1 + 2\mathbf{p}_2$

Then dividing by 8 and using part (a) gives

$$\mathbf{q}_{2} = \frac{1}{4}\mathbf{p}_{0} + \frac{1}{2}\mathbf{p}_{1} + \frac{1}{4}\mathbf{p}_{2}$$

$$= (\frac{1}{4}\mathbf{p}_{0} + \frac{1}{4}\mathbf{p}_{1}) + (\frac{1}{4}\mathbf{p}_{1} + \frac{1}{4}\mathbf{p}_{2})$$

$$= \frac{1}{2}\mathbf{q}_{1} + \frac{1}{4}(\mathbf{p}_{1} + \mathbf{p}_{2})$$

$$= \frac{1}{2}[\mathbf{q}_{1} + \frac{1}{2}(\mathbf{p}_{1} + \mathbf{p}_{2})]$$

14. a.
$$3(\mathbf{r}_3 - \mathbf{r}_2) = \mathbf{z}'(1)$$
, by (9) with $\mathbf{z}'(1)$ and \mathbf{r}_i in place of $\mathbf{x}'(1)$ and \mathbf{p}_i .

$$\mathbf{z}'(1) = .5\mathbf{x}'(1)$$
, by (11) with $t = 1$.
 $.5\mathbf{x}'(1) = (.5)3(\mathbf{p}_3 - \mathbf{p}_2)$, by (9).

b. From part (a),
$$6(\mathbf{r}_3 - \mathbf{r}_2) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$
, $\mathbf{r}_3 - \mathbf{r}_2 = \frac{1}{2}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_2$, and $\mathbf{r}_3 - \frac{1}{2}\mathbf{p}_3 + \frac{1}{2}\mathbf{p}_2 = \mathbf{r}_2$. Since $\mathbf{r}_3 = \mathbf{p}_3$, this equation becomes $\mathbf{r}_2 = \frac{1}{2}(\mathbf{p}_3 + \mathbf{p}_2)$.

c.
$$3(\mathbf{r}_1 - \mathbf{r}_0) = \mathbf{z}'(0)$$
, by (9) with $\mathbf{z}'(0)$ and \mathbf{r}_i in place of $\mathbf{x}'(0)$ and \mathbf{p}_j .

$$\mathbf{z}'(0) = .5\mathbf{x}'(.5)$$
, by (11) with $t = 0$.

d. Part (c) and (10) show that
$$3(\mathbf{r}_1 - \mathbf{r}_0) = \frac{3}{8}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$
. Multiply by $\frac{8}{3}$ and rearrange to obtain $8\mathbf{r}_1 = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + 8\mathbf{r}_0$.

e. From (8), $8\mathbf{r}_0 = \mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3$. Substitute into the equation from part (d), and obtain

 $8\mathbf{r}_1 = 2\mathbf{p}_1 + 4\mathbf{p}_2 + 2\mathbf{p}_3$. Divide by 8 and use part (b) to

$$\begin{aligned} \mathbf{r}_1 &= \frac{1}{4}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2 + \frac{1}{4}\mathbf{p}_3 = (\frac{1}{4}\mathbf{p}_1 + \frac{1}{4}\mathbf{p}_2) + \frac{1}{4}(\mathbf{p}_2 + \mathbf{p}_3) \\ &= \frac{1}{2} \cdot \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) + \frac{1}{2}\mathbf{r}_2 \end{aligned}$$

Interchange the terms on the right, and obtain $\mathbf{r}_1 = \frac{1}{2}[\mathbf{r}_2 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)].$

15. a. From (11),
$$\mathbf{y}'(1) = .5\mathbf{x}'(.5) = \mathbf{z}'(0)$$
.

b. Observe that $\mathbf{y}'(1) = 3(\mathbf{q}_3 - \mathbf{q}_2)$. This follows from (9), with $\mathbf{y}(t)$ and its control points in place of $\mathbf{x}(t)$ and its control points. Similarly, for $\mathbf{z}(t)$ and its control points,

- $\mathbf{z}'(0) = 3(\mathbf{r}_1 \mathbf{r}_0)$. By part (a), $3(\mathbf{q}_3 \mathbf{q}_2) = 3(\mathbf{r}_1 \mathbf{r}_0)$. Replace \mathbf{r}_0 by \mathbf{q}_3 , and obtain $\mathbf{q}_3 \mathbf{q}_2 = \mathbf{r}_1 \mathbf{q}_3$, and hence $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$.
- **c.** Set $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. Compute $\mathbf{q}_1 = (\mathbf{p}_0 + \mathbf{p}_1)/2$ and $\mathbf{r}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$. Compute $\mathbf{m} = (\mathbf{p}_1 + \mathbf{p}_2)/2$. Compute $\mathbf{q}_2 = (\mathbf{l}_1 + \mathbf{m})/2$ and $\mathbf{r}_1 = (\mathbf{m} + \mathbf{r}_2)/2$. Compute $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$ and set $\mathbf{r}_0 = \mathbf{q}_3$.
- **16.** A Bézier curve is completely determined by its four control points. Two are given directly: $\mathbf{p}_0 = \mathbf{x}(0)$ and $\mathbf{p}_3 = \mathbf{x}(1)$. From equation (9), $\mathbf{x}'(0) = 3(\mathbf{p}_1 \mathbf{p}_0)$ and $\mathbf{x}'(1) = 3(\mathbf{p}_3 \mathbf{p}_2)$. Solving gives

$$\mathbf{p}_1 = \frac{1}{3}\mathbf{x}'(0) + \mathbf{p}_0$$
 and $\mathbf{p}_2 = \mathbf{p}_3 - \frac{1}{3}\mathbf{x}'(1)$

17. a. The quadratic curve is $\mathbf{w}(t) = (1-t)^2 \mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2 \mathbf{p}_2$. From Example 1, the tangent vectors at the endpoints are $\mathbf{w}'(0) = 2\mathbf{p}_1 - 2\mathbf{p}_0$ and $\mathbf{w}'(1) = 2\mathbf{p}_2 - 2\mathbf{p}_1$. Denote the control points of $\mathbf{x}(t)$ by \mathbf{r}_0 , \mathbf{r}_1 , \mathbf{r}_2 , and \mathbf{r}_3 . Then

$$\mathbf{r}_0 = \mathbf{x}(0) = \mathbf{w}(0) = \mathbf{p}_0$$
 and $\mathbf{r}_3 = \mathbf{x}(1) = \mathbf{w}(1) = \mathbf{p}_2$

From equation (9) or Exercise 3(a) (using \mathbf{r}_i in place of \mathbf{p}_i) and Example 1,

$$-3\mathbf{r}_0 + 3\mathbf{r}_1 = \mathbf{x}'(0) = \mathbf{w}'(0) = 2\mathbf{p}_1 - 2\mathbf{p}_0$$

so

$$-\mathbf{p}_0 + \mathbf{r}_1 = \frac{2\mathbf{p}_1 - 2\mathbf{p}_0}{3}$$

and

$$\mathbf{r}_1 = \frac{2\mathbf{p}_1 + \mathbf{p}_0}{3}$$

Similarly, from the tangent data at t = 1, along with equation (9) and Example 1,

$$-3\mathbf{r}_2 + 3\mathbf{r}_3 = \mathbf{x}'(1) = \mathbf{w}'(1) = 2\mathbf{p}_2 - 2\mathbf{p}_1,$$

$$-\mathbf{r}_2 + \mathbf{p}_2 = \frac{2\mathbf{p}_2 - 2\mathbf{p}_1}{3}, \mathbf{r}_2 = \mathbf{p}_2 - \frac{2\mathbf{p}_2 - 2\mathbf{p}_1}{3}, \text{ and }$$

$$\mathbf{r}_2 = \frac{2\mathbf{p}_1 + \mathbf{p}_2}{3}$$
b. Write the standard formula (7), with \mathbf{r}_i in place of \mathbf{p}_i for

b. Write the standard formula (7), with \mathbf{r}_i in place of \mathbf{p}_i for i = 0, ..., 3, and then replace \mathbf{r}_0 by \mathbf{p}_0 , and replace \mathbf{r}_3 by \mathbf{p}_2 :

$$\mathbf{x}(t) = (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 + (3t - 6t^2 + 3t^3)\mathbf{r}_1 + (3t^2 - 3t^3)\mathbf{r}_2 + t^3\mathbf{p}_2$$
 (*)

Use the formulas for \mathbf{r}_1 and \mathbf{r}_2 to examine the second and third terms in (*):

$$(3t - 6t^2 + 3t^3)\mathbf{r}_1 = \frac{1}{3}(3t - 6t^2 + 3t^3)\mathbf{p}_0 + \frac{2}{3}(3t - 6t^2 + 3t^3)\mathbf{p}_1$$

= $(t - 2t^2 + t^3)\mathbf{p}_0 + (2t - 4t^2 + 2t^3)\mathbf{p}_1$

$$(3t^2 - 3t^3)\mathbf{r}_2 = \frac{2}{3}(3t^2 - 3t^3)\mathbf{p}_1 + \frac{1}{3}(3t^2 - 3t^3)\mathbf{p}_2$$

= $(2t^2 - 2t^3)\mathbf{p}_1 + (t^2 - t^3)\mathbf{p}_2$

When these two results are substituted in (*), the coefficient of \mathbf{p}_0 is

$$(1-3t+3t^2-t^3)+(t-2t^2+t^3)=1-2t+t^2=(1-t)^2$$

The coefficient of \mathbf{p}_1 is

$$(2t - 4t^2 + 2t^3) + (2t^2 - 2t^3) = 2t - 2t^2 = 2t(1 - t)$$

The coefficient of \mathbf{p}_2 is $(t^2 - t^3) + t^3 = t^2$. So $\mathbf{x}(t) = (1 - t)^2 \mathbf{p}_0 + 2t(1 - t)\mathbf{p}_1 + t^2 \mathbf{p}_2$, which shows that $\mathbf{x}(t)$ is the quadratic Bézier curve $\mathbf{w}(t)$.

18.
$$\begin{bmatrix} \mathbf{p}_0 \\ -3\mathbf{p}_0 + 3\mathbf{p}_1 \\ 3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2 \\ -\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3 \end{bmatrix}$$