Chapter 3 - Wave Properties of Particles

3-1: From Equation (3.1), any particle's wavelength is determined by its momentum, and hence particles with the same wavelength have the same momenta. With a common momentum p, the photon's energy is pc, and the particle's energy is $\sqrt{(pc)^2 + (mc^2)^2}$, which is necessarily greater than pc for a massive particle. The particle's kinetic energy is

$$\text{KE} = E - mc^2 = \sqrt{(pc)^2 + (mc^2)^2} - mc^2.$$

For low values of p ($p \ll mc$ for a nonrelativistic massive particle), the kinetic energy is $\text{KE} \approx \frac{p^2}{2m}$, which is necessarily less than pc. For a realtivistic massive particle, $\text{KE} \approx pc - mc^2$, and KE is less than the photon energy. The kinetic energy of a massive particle will always be less than pc, as can be seen by using $E = \text{KE} + mc^2$, squaring, and subtracting from $E^2 = (pc)^2 + \left(mc^2\right)^2$ to obtain

$$(pc)^2 - \text{KE}^2 = 2\,\text{KE}\,mc^2.$$

3-3: For this nonrelativistic case,

$$\lambda = \frac{h}{m\,v} = \frac{\left(6.626 \times 10^{-34}\ \text{J}\cdot\text{s}\right)}{(1.0 \times 10^{-6}\ \text{kg})\ (20\ \text{m/s})} = 3.3 \times 10^{-29}\ \text{m};$$

quantum effects certainly would not be noticed for such an object.

3-5: Becasue the de Broglie wavelength depends only on the electron's momentum, the percentage error in the wavelength will be the same as the percentage error in the reciprocal of the momentum, with the nonrelativistic calculation giving the higher wavelength due to a lower calculated momentum. The nonrelativistic momentum is

$$\begin{aligned} p_{\text{nr}} = \sqrt{2\,m\,\text{KE}} &= \sqrt{2\ (9.1095 \times 10^{-31}\ \text{kg})\,(100 \times 10^3\ \text{eV})\ (1.602 \times 10^{-19}\ \text{J/eV})} \\ &= 1.708 \times 10^{-22}\ \text{kg}\cdot\text{m/s}, \end{aligned}$$

and the nonrelativistic momentum is

$$\begin{aligned} p_{\text{r}} = \frac{1}{c}\sqrt{\left(\text{KE} + mc^2\right)^2 - \left(mc^2\right)^2} &= \sqrt{(0.100 + 0.511)^2 - (0.511)^2}\ \text{MeV}/c \\ &= 1.790 \times 10^{-22}\ \text{kg}\cdot\text{m/s}, \end{aligned}$$

keeping extra figures in the intermediate calculations. The percentage error in the computed de Broglie wavelength is then

$$\frac{(h/p_{\text{nr}}) - (h/p_{\text{r}})}{(h/p_{\text{r}})} = \frac{p_{\text{r}} - p_{\text{nr}}}{p_{\text{nr}}} = \frac{1.790 - 1.708}{1.708} = 4.8\%.$$

3-7: A nonrelativistic calculation gives

$$\mathrm{KE} = \frac{p^2}{2\,m} = \frac{(hc/\lambda)^2}{2\,mc^2} = \frac{(hc)^2}{2\;(mc^2)\;\lambda^2}$$

$$= \frac{\left(1.240\times10^{-6}\ \mathrm{eV\cdot m}\right)^2}{2\;(939.6\times10^{6}\ \mathrm{eV})\;\left(0.282\times10^{-9}\ \mathrm{m}\right)^2} = 0.00103\ \mathrm{eV}.$$

(Note that in the above caculation, multiplication of numerator and denominator by c^2 and use of the product hc in terms of electronvolts avoided further unit conversion.) This energy is much less than the neutron's rest energy, and so the nonrelativistic calculation is completely valid.

3-9: A nonrelativistic calculation gives

$$\mathrm{KE} = \frac{p^2}{2\,m} = \frac{(hc/\lambda)^2}{2\,mc^2} = \frac{(hc)^2}{2\;(mc^2)\;\lambda^2}$$

$$= \frac{\left(1.240\times10^{-6}\ \mathrm{eV\cdot m}\right)^2}{2\;(511\times10^{3}\ \mathrm{eV})\;\left(550\times10^{-9}\ \mathrm{m}\right)^2} = 5.0\times10^{-6}\ \mathrm{eV},$$

so the electron would have to be accelerated through a potential difference of $5.0\times10^{-6}\ \mathrm{V} = 5.0\ \mu\mathrm{V}$. Note that the kinetic energy is very small compared to the electron rest energy, so the nonrelativistic calculation is valid. (In the above caculation, multiplication of numerator and denominator by c^2 and use of the product hc in terms of electronvolts avoided further unit conversion.)

3-11: If $E^2 = (pc)^2 + \left(mc^2\right)^2 \gg \left(mc^2\right)^2$, then $pc \gg mc^2$ and $E \approx pc$. For a photon with the same energy, $E = pc$, so the momentum of such a particle would be nearly the same as a photon with the same energy, and so the de Broglie wavelengths would be the same.

3-13: For massive particles of the same speed, relativistic or nonrelativistic, the momentum will be proportional to the mass, and so the de Broglie wavelength will be inversely proportional to the mass; the electron will have the longer wavelength by a factor of $(m_p/m_e) = 1838$. From Equation (3.3) the particles have the same phase velocity and from Equation (3.16) they have the same group velocity.

3-15: Suppose that the phase velocity is independent of wavelength, and hence independent of the wave number k; then, from Equation (3.3), the phase velocity $v_p = (\omega/k) = u$, a constant. It follows that because $\omega = u\,k$,

$$v_g = \frac{d\omega}{dk} = u = v_p.$$

3-17: The phase velocity may be expressed in terms of the wave number $k = 2\,\pi/\lambda$ as

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \qquad \text{or} \qquad \omega = \sqrt{g\,k} \qquad \text{or} \qquad \omega^2 = g\,k.$$

Finding the group velocity by differentiating $\omega(k)$ with respect to k,

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{g}\,\frac{1}{\sqrt{k}} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\,\frac{\omega}{k} = \frac{1}{2}\,v_g.$$

Using implicit differentiation in the formula for $\omega^2(k)$,

$$2\,\omega\frac{d\omega}{dk} = 2\,\omega\,v_g = g,$$

So that $$v_g = \frac{g}{2\omega} = \frac{g\,k}{2\,\omega\,k} = \frac{\omega^2}{2\,\omega\,k} = \frac{\omega}{2\,k} = \frac{1}{2}\,v_p,$$

the same result. For those more comfortable with calculus, the dispersion relation may be expressed as

$$2\,\ln(\omega) = \ln(k) + \ln(g),$$

from which $2\,\frac{d\omega}{\omega} = \frac{dk}{k}$, and $v_g = \frac{1}{2}\,\frac{\omega}{k} = \frac{1}{2}\,v_p$.

3-19: For a kinetic energy of 500 keV,

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{\text{KE} + mc^2}{mc^2} = \frac{500 + 511}{511} = 1.978.$$

Solving for v,

$$v = c\sqrt{1 - \frac{1}{\gamma}} = \sqrt{1 - \left(\frac{1}{1.978}\right)^2} = 0.863\,c,$$

and from Equation (3.16), $v_g = v = 0.863\,c$. The phase velocity is then $v_p = c^2/v_g = 1.16\,c$.

3-21: (a) Two equivalent methods will be presented here. Both will assume the validity of Equation (3.16), in that $v_g = v$.

First: Express the wavelength λ in terms of v_g,

$$\lambda = \frac{h}{p} = \frac{h}{m\,v_g\gamma} = \frac{h}{m\,v_g}\sqrt{1-\frac{v_g^2}{c^2}}.$$

Multiplying by $m\,v_g$, squaring and solving for v_g^2 gives

$$v_g^2 = \frac{h^2}{(\lambda\,m)^2 + (h^2/c^2)} = c^2\left[1+\left(\frac{m\,\lambda\,c}{h}\right)^2\right]^{-1}.$$

Taking the square root and using Equation (3.3), $v_p = c^2/v_g$, gives the desired result.

Second: Conisder the particle energy in terms of $v_p = c^2/v_g$;

$$E^2 = (pc)^2 + \left(mc^2\right)^2$$
$$\gamma^2\left(mc^2\right)^2 = \frac{\left(mc^2\right)^2}{1-\dfrac{c^2}{v_p^2}} = \left(\frac{hc}{\lambda}\right)^2 + \left(mc^2\right)^2.$$

Dividing by $\left(mc^2\right)^2$ leads to

$$1-\frac{c^2}{v_p^2} = \frac{1}{1+h^2/\left(m\,c\,\lambda\right)^2}, \qquad \text{so that}$$

$$\frac{c^2}{v_p^2} = 1-\frac{1}{1+h^2/\left(m\,c\,\lambda\right)^2} = \frac{h^2\left(m\,c\,\lambda\right)^2}{h^2\left(m\,c\,\lambda\right)^2+1} = \frac{1}{1+\left(m\,c\,\lambda\right)^2/h^2},$$

which is an equivalent statement of the desired result.

It should be noted that in the first method presented above could be used to find λ in terms of v_p directly, and in the second method the energy could be found in terms of v_g. The final result is, or course, the same.

(b) Using the result of part (a),

$$v_p = c\sqrt{1+\left(\frac{(9.1095\times10^{-31}\ \text{kg})\ (2.998\times10^{8}\ \text{m/s})\ (1.00\times10^{-13}\ \text{m})}{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}\right)^2}$$
$$= 1.00085\,c,$$

and $v_g = c^2/v_g = 0.99915\,c$.

For a calculational shortcut, write the result of part (a) as

$$v_p = c\sqrt{1 + \left(\frac{mc^2\,\lambda}{hc}\right)^2}$$

$$= c\sqrt{1 + \left(\frac{(511\times 10^3\ \text{eV})\ (1.00\times 10^{-13}\ \text{m})}{(1.240\times 10^{-6}\ \text{eV}\cdot\text{m})}\right)^2} = 1.00085\,c.$$

In both of the above answers, the statement that the de Broglie wavelength is "exactly" 10^{-13} m means that the answers can be given to any desired precision.

3-23: Increasing the electron energy increases the electron's momentum, and hence decreases the electron's de Broglie wavelength. From Equation (2.13), a smaller de Broglie wavelength results in a smaller scattering angle.

3-25: (a) For the given energies, a nonrelativistic calculation is sufficient;

$$v = \sqrt{\frac{2\,\text{KE}}{m}} = \sqrt{\frac{2\,(54\ \text{eV})\ (1.602\times 10^{-19}\ \text{J/eV})}{(9.1095\times 10^{-31}\ \text{kg})}} = 4.36\ \text{m/s}$$

outside the crystal, and (from a similar calculation, with KE = 80 eV), $v = 5.30\times 10^6$ m/s inside the crystal (keeping an extra significant figure in both calculations).

(b) With the speeds found in part (a), the de Broglie wavelengths are found from

$$\lambda = \frac{h}{p} = \frac{h}{m\,v} = \frac{(6.626\times 10^{-34}\ \text{J}\cdot\text{s})}{(9.1095\times 10^{-31}\ \text{kg})\ (4.36\times 10^6\ \text{m/s})} = 1.67\times 10^{-10}\ \text{m},$$

or 0.167 nm outside the crystal, with a similar calculation giving 0.137 nm inside the crystal.

3-27: From Equation (3.18),

$$E_n = n^2\,\frac{h^2}{8\,m\,L^2} = n^2\,\frac{\left(6.626\times 10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\ (1.675\times 10^{-27}\ \text{kg})\ (1.00\times 10^{-14}\ \text{m})^2}$$

$$= n^2\,3.28\times 10^{-13}\ \text{J} = n^2\,20.5\ \text{MeV}.$$

The minimum energy, corresponding to $n = 1$, is 20.5 MeV.

3-29: The first excited state corresponds to $n = 2$ in Equation (3.18). Solving for the width L,

$$L = n\sqrt{\frac{h^2}{8\,m\,E_2}} = 2\sqrt{\frac{(6.626\times 10^{-34}\ \text{J}\cdot\text{s})^2}{8\ (1.673\times 10^{-27}\ \text{kg})\ (400\times 10^3\ \text{eV})\ (1.602\times 10^{-19}\ \text{J/eV})}}$$

$$= 4.53\times 10^{-14}\ \text{m} = 45.3\ \text{fm}.$$

3-31: Each atom in a solid is limited to a certain definite region of space – otherwise the assembly of atoms would not be a solid. The uncertainty in position of each atom is therefore finite, and its momentum and hence enrgy cannot be zero. The position of an ideal-gas molecule is not restricted, so the uncertainty in its position is effectively infinite and its momentum and hence enrgy can be zero.

3-33: The percentage uncertainty in the electron's momentum will be at least

$$\begin{aligned}\frac{\Delta p}{p} &= \frac{h}{4\pi\,\Delta x\,p} = \frac{h}{4\pi\,\Delta x\,\sqrt{2\,m\,\mathrm{KE}}} = \frac{hc}{4\pi\,\Delta x\,\sqrt{2\,(mc^2)\,\mathrm{KE}}}\\ &= \frac{(1.240\times10^{-6}\ \mathrm{eV\cdot m})}{4\pi\ (1.00\times10^{-10}\ \mathrm{m})\ \sqrt{2\ (511\times10^3\ \mathrm{eV})\,(1.00\times10^3\ \mathrm{eV})}}\\ &= 3.1\times10^{-2} = 3.1\%.\end{aligned}$$

Note that in the above calculation, conversion of the mass of the electron into its energy equivalent in electronvolts is purely optional; converting the kinetic energy into joules and using $h = 6.626\times10^{-34}$ J·s will of course give the same percentage uncertainty.

3-35: The proton will need to move a minimum distance

$$v\,\Delta t \geq v\,\frac{h}{4\pi\,\Delta E},$$

where v can be taken to be

$$v = \sqrt{\frac{2\,\mathrm{KE}}{m}} = \sqrt{\frac{2\,\Delta E}{m}}, \qquad \text{so that}$$

$$\begin{aligned}v\,\Delta t &= \sqrt{\frac{2\,\mathrm{KE}}{m}}\,\frac{h}{4\pi\,\Delta E} = \frac{h}{2\pi\,\sqrt{2\,m\,\mathrm{KE}}} = \frac{hc}{2\pi\,\sqrt{2\,(mc^2)\ \mathrm{KE}}}\\ &= \frac{1.240\times10^{-6}\ \mathrm{eV\cdot m}}{2\pi\sqrt{2\,(938.28\times10^6\ \mathrm{MeV})\ (1.00\times10^3\ \mathrm{eV})}}\\ &= 1.44\times10^{-13}\ \mathrm{m} = 0.144\ \mathrm{pm}.\end{aligned}$$

(See note to the solution to Problem 3-33 above).

The result for the product $v\,\Delta t$ may be recognized as $v\,\Delta t \geq \dfrac{h}{2\pi\,p}$; this is not inconsistent with Equation (3.21), $\Delta x\,\Delta p \geq \dfrac{h}{4\pi}$. In the current problem, ΔE was taken to be the (maximum) kinetic energy of the proton. In such a situation,

$$\Delta E = \frac{\Delta\left(p^2\right)}{m} = 2\,\frac{p}{m}\,\Delta p = 2\,v\,\Delta p,$$

which is consistent with the previous result.

3-37: (a) The length of each group is

$$c\,\Delta t = \left(2.998\times10^{8}\ \text{m/s}\right)\left(8.00\times10^{-8}\ \text{s}\right) = 24\ \text{m}.$$

The number of waves in each group is the pulse duration divided by the wave period, which is the pulse duration multiplied by the frequency,

$$\left(8.00\times10^{-8}\ \text{s}\right)\left(4900\times10^{6}\ \text{Hz}\right) = 752\ \text{waves}.$$

(b) The bandwidth is the reciprocal of the pulse duration,

$$\left(8.00\times10^{-8}\ \text{s}\right)^{-1} = 12.5\ \text{MHz}.$$

3-39: To use the uncertainty principle, make the identification of p with Δp and x with Δx, so that $p = h/\left(4\pi\,x\right)$, and

$$E = E(x) = \left(\frac{h^2}{8\pi^2\,m}\right)\frac{1}{x^2} + \left(\frac{C}{2}\right)x^2.$$

Differentiating with respect to x and setting $\dfrac{d}{dx}E = 0$,

$$-\left(\frac{h^2}{4\pi^2\,m}\right)\frac{1}{x^3} + C\,x = 0,$$

which is solved for

$$x^2 = \frac{h}{2\pi\,\sqrt{m\,C}}.$$

Substution of this value into $E(x)$ gives

$$E_{\text{min}} = \left(\frac{h^2}{8\pi^2\,m}\right)\left(\frac{2\pi\sqrt{m\,C}}{h}\right) + \left(\frac{C}{2}\right)\left(\frac{h}{2\pi\,\sqrt{m\,C}}\right) = \frac{h}{2\pi}\sqrt{\frac{C}{m}} = \frac{h\,\nu}{2}.$$