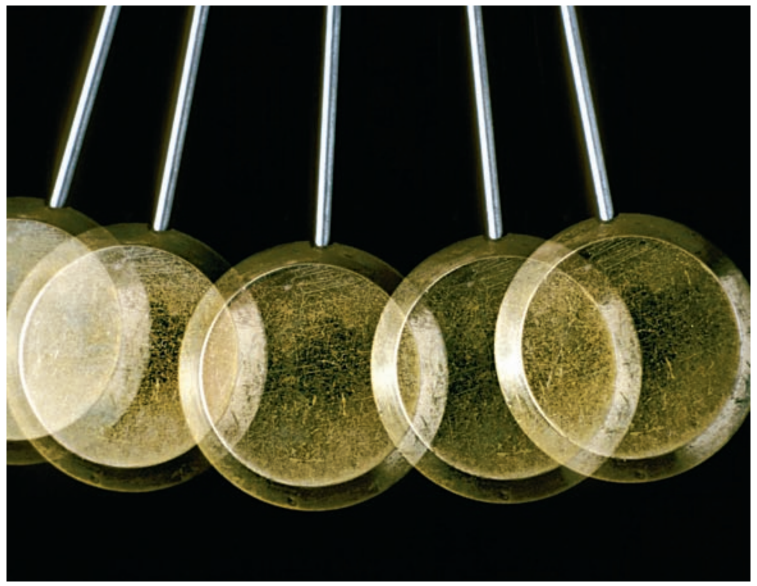
PERIODIC MOTION 13



?Suppose you doubled the mass of a clock's pendulum (including the rod and the weight at the end) while keeping its dimensions the same. Would the clock run fast or slow?

LEARNING GOALS

By studying this chapter, you will learn:

- How to describe oscillations in terms of amplitude, period, frequency, and angular frequency.
- How to do calculations with simple harmonic motion, an important type of oscillation.
- How to use energy concepts to analyze simple harmonic motion.
- How to apply the ideas of simple harmonic motion to different physical situations.
- How to analyze the motions of a simple pendulum.
- What a physical pendulum is, and how to calculate the properties of its motion.
- What determines how rapidly an oscillation dies out.
- How a driving force applied to an oscillator at the right frequency can cause a very large response, or resonance.

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic motion** or **oscillation,** is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

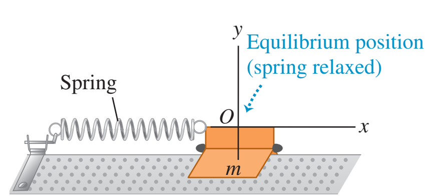
A body that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

In this chapter we will concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We will also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

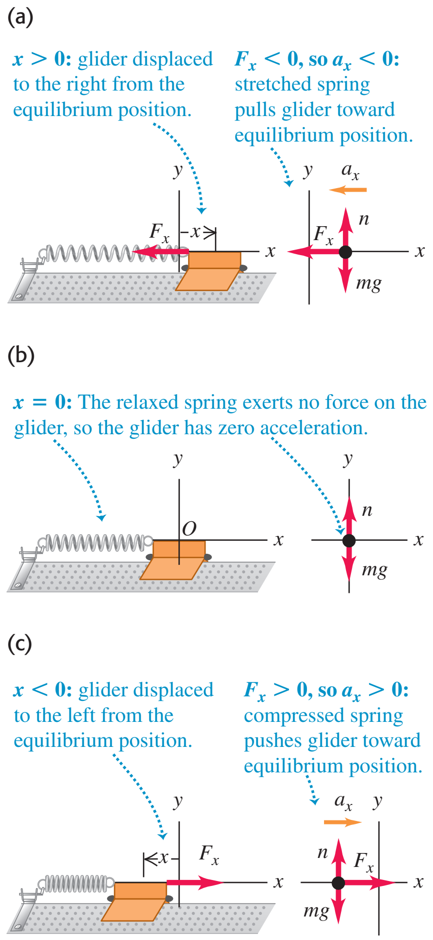
13.1 Describing Oscillation

Figure 13.1 shows one of the simplest systems that can have periodic motion. A body with mass m rests on a frictionless horizontal guide system, such as a linear air track, so it can move only along the x-axis. The body is attached to a spring of negligible mass that can be either stretched or compressed. The left end of the spring is held fixed and the right end is attached to the body. The spring force is the only horizontal force acting on the body; the vertical normal and gravitational forces always add to zero.

13.1 A system that can have periodic motion.



13.2 Model for periodic motion. When the body is displaced from its equilibrium position at $x = 0$, the spring exerts a restoring force back toward the equilibrium position.



It's simplest to define our coordinate system so that the origin O is at the equilibrium position, where the spring is neither stretched nor compressed. Then x is the x-component of the **displacement** of the body from equilibrium and is also the change in the length of the spring. The x-component of the force that the spring exerts on the body is F_x, and the x-component of acceleration a_x is given by $a_x = F_x/m$.

Figure 13.2 shows the body for three different displacements of the spring. Whenever the body is displaced from its equilibrium position, the spring force tends to restore it to the equilibrium position. We call a force with this character a **restoring force.** Oscillation can occur only when there is a restoring force tending to return the system to equilibrium.

Let's analyze how oscillation occurs in this system. If we displace the body to the right to $x = A$ and then let go, the net force and the acceleration are to the left (Fig. 13.2a). The speed increases as the body approaches the equilibrium position O. When the body is at O, the net force acting on it is zero (Fig. 13.2b), but because of its motion it *overshoots* the equilibrium position. On the other side of the equilibrium position the body is still moving to the left, but the net force and the acceleration are to the right (Fig. 13.2c); hence the speed decreases until the body comes to a stop. We will show later that with an ideal spring, the stopping point is at $x = -A$. The body then accelerates to the right, overshoots equilibrium again, and stops at the starting point $x = A$, ready to repeat the whole process. The body is oscillating! If there is no friction or other force to remove mechanical energy from the system, this motion repeats forever; the restoring force perpetually draws the body back toward the equilibrium position, only to have the body overshoot time after time.

In different situations the force may depend on the displacement x from equilibrium in different ways. But oscillation *always* occurs if the force is a *restoring* force that tends to return the system to equilibrium.

Amplitude, Period, Frequency, and Angular Frequency

Here are some terms that we'll use in discussing periodic motions of all kinds:

The **amplitude** of the motion, denoted by A, is the maximum magnitude of displacement from equilibrium—that is, the maximum value of $|x|$. It is always positive. If the spring in Fig. 13.2 is an ideal one, the total overall range of the motion is $2A$. The SI unit of A is the meter. A complete vibration, or **cycle,** is one complete round trip—say, from A to $-A$ and back to A, or from O to A, back through O to $-A$, and back to O. Note that motion from one side to the other (say, $-A$ to A) is a half-cycle, not a whole cycle.

The **period,** T, is the time for one cycle. It is always positive. The SI unit is the second, but it is sometimes expressed as "seconds per cycle."

The **frequency,** f, is the number of cycles in a unit of time. It is always positive. The SI unit of frequency is the hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}$$

This unit is named in honor of the German physicist Heinrich Hertz (1857–1894), a pioneer in investigating electromagnetic waves.

The **angular frequency,** ω, is 2π times the frequency:

$$\omega = 2\pi f$$

We'll learn shortly why ω is a useful quantity. It represents the rate of change of an angular quantity (not necessarily related to a rotational motion) that is always measured in radians, so its units are rad/s. Since f is in cycle/s, we may regard the number 2π as having units rad/cycle.

From the definitions of period T and frequency f we see that each is the reciprocal of the other:

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \qquad \text{(relationships between frequency and period)} \qquad (13.1)$$

Also, from the definition of ω,

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \text{(angular frequency)} \qquad (13.2)$$

Example 13.1 Period, frequency, and angular frequency

An ultrasonic transducer (a kind of loudspeaker) used for medical diagnosis oscillates at a frequency of 6.7 MHz $= 6.7 \times 10^6$ Hz. How much time does each oscillation take, and what is the angular frequency?

SOLUTION

IDENTIFY: Our target variables are the period T and the angular frequency ω.

SET UP: We are given the frequency f, so we can find these variables using Eqs. (13.1) and (13.2).

EXECUTE: From Eqs. (13.1) and (13.2),

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6 \text{ Hz}} = 1.5 \times 10^{-7} \text{ s} = 0.15\ \mu\text{s}$$

$$\begin{aligned}\omega &= 2\pi f = 2\pi(6.7 \times 10^6 \text{ Hz}) \\ &= (2\pi \text{ rad/cycle})(6.7 \times 10^6 \text{ cycle/s}) \\ &= 4.2 \times 10^7 \text{ rad/s}\end{aligned}$$

EVALUATE: This is a very rapid vibration, with large f and ω and small T. A slow vibration has small f and ω and large T.

Test Your Understanding of Section 13.1 A body like that shown in Fig. 13.2 oscillates back and forth. For each of the following values of the body's x-velocity v_x and x-acceleration a_x, state whether its displacement x is positive, negative, or zero. (a) $v_x > 0$ and $a_x > 0$; (b) $v_x > 0$ and $a_x < 0$; (c) $v_x < 0$ and $a_x > 0$; (d) $v_x < 0$ and $a_x < 0$; (e) $v_x = 0$ and $a_x < 0$; (f) $v_x > 0$ and $a_x = 0$.

13.2 Simple Harmonic Motion

The simplest kind of oscillation occurs when the restoring force F_x is *directly proportional* to the displacement from equilibrium x. This happens if the spring in Figs. 13.1 and 13.2 is an ideal one that obeys Hooke's law. The constant of proportionality between F_x and x is the force constant k. (You may want to review Hooke's law and the definition of the force constant in Section 6.3.) On either side of the equilibrium position, F_x and x always have opposite signs. In Section 6.3 we represented the force acting *on* a stretched ideal spring as $F_x = kx$. The x-component of force the spring exerts *on the body* is the negative of this, so the x-component of force F_x on the body is

$$F_x = -kx \qquad \text{(restoring force exerted by an ideal spring)} \qquad (13.3)$$

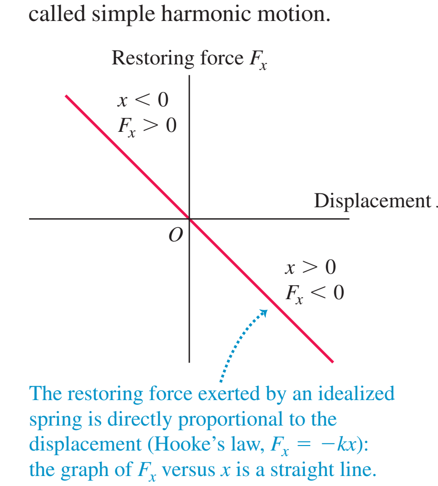
This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero (Fig. 13.3). The force constant k is always positive and has units of N/m (a useful alternative set of units is kg/s^2). We are assuming that there is no friction, so Eq. (13.3) gives the *net* force on the body.

When the restoring force is directly proportional to the displacement from equilibrium, as given by Eq. (13.3), the oscillation is called **simple harmonic motion,** *abbreviated* **SHM.** The acceleration $a_x = d^2x/dt^2 = F_x/m$ of a body in SHM is given by

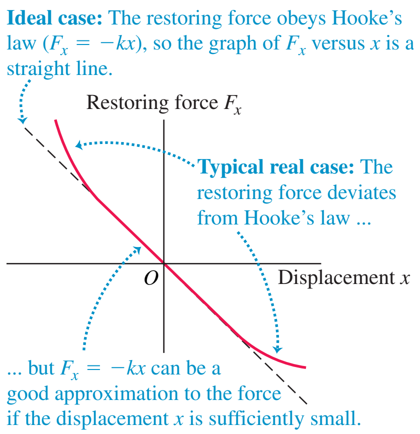
$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \text{(simple harmonic motion)} \qquad (13.4)$$

The minus sign means the acceleration and displacement always have opposite signs. This acceleration is *not* constant, so don't even think of using the constant-acceleration equations from Chapter 2. We'll see shortly how to solve this

13.3 An idealized spring exerts a restoring force that obeys Hooke's law, $F_x = -kx$. Oscillation with such a restoring force is called simple harmonic motion.



13.4 In most real oscillations Hooke's law applies provided the body doesn't move too far from equilibrium. In such a case small-amplitude oscillations are approximately simple harmonic.



equation to find the displacement x as a function of time. A body that undergoes simple harmonic motion is called a **harmonic oscillator.**

Why is simple harmonic motion important? Keep in mind that not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than in Eq. (13.3). But in many systems the restoring force is *approximately* proportional to displacement if the displacement is sufficiently small (Fig. 13.4). That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by Eq. (13.4). Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of the quartz crystal in a watch, the motion of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

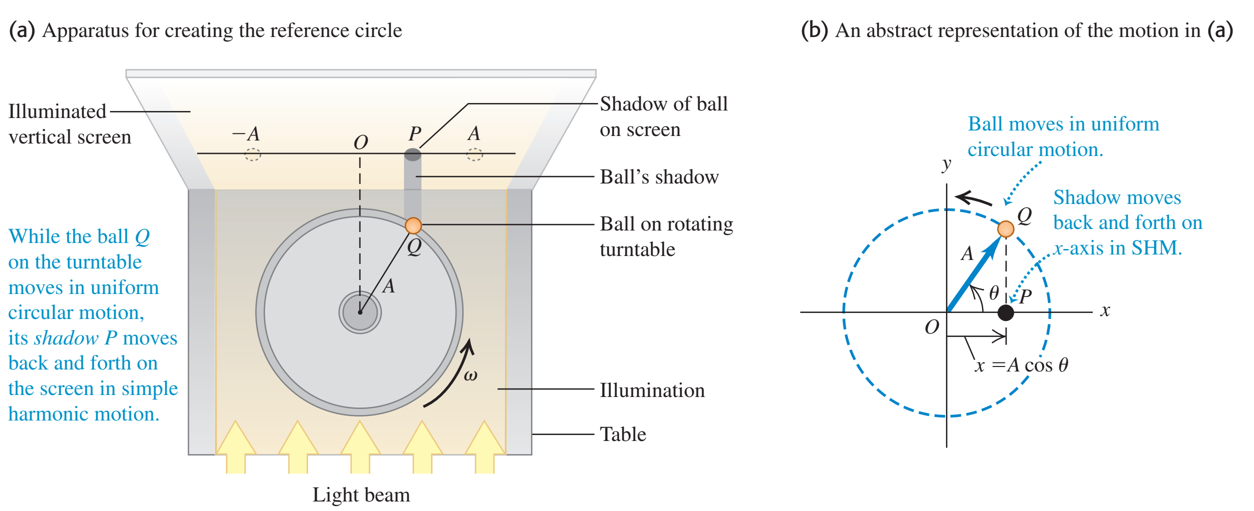
Circular Motion and the Equations of SHM

To explore the properties of simple harmonic motion, we must express the displacement x of the oscillating body as a function of time, $x(t)$. The second derivative of this function, d^2x/dt^2, must be equal to $(-k/m)$ times the function itself, as required by Eq. (13.4). As we mentioned, the formulas for constant acceleration from Section 2.4 are no help because the acceleration changes constantly as the displacement x changes. Instead, we'll find $x(t)$ by noticing a striking similarity between SHM and another form of motion that we've already studied in detail.

Figure 13.5a shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q. The disk rotates with constant angular speed ω (measured in rad/s), so the ball moves in uniform circular motion. A horizontal light beam shines on the rotating disk and casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange a body attached to an ideal spring, like the combination shown in Figs. 13.1 and 13.2, so that the body oscillates parallel to the shadow. We will prove that the motion of the body and the motion of the ball's shadow are *identical* if the amplitude of the body's oscillation is equal to the disk radius A, and if the angular frequency $2\pi f$ of the oscillating body is equal to the angular speed ω of the rotating disk. That is, *simple harmonic motion is the projection of uniform circular motion onto a diameter.*

We can verify this remarkable statement by finding the acceleration of the shadow at P and comparing it to the acceleration of a body undergoing SHM, given by Eq. (13.4). The circle in which the ball moves so that its projection matches the motion of the oscillating body is called the **reference circle;** we will call the point Q the *reference point.* We take the reference circle to lie in the xy-plane, with the origin O at the center of the circle (Fig. 13.5b). At time t the vector OQ from the origin to the reference point Q makes an angle θ with the positive x-axis. As the point Q moves around the reference circle with constant angular speed ω, the vector OQ rotates with the same angular speed. Such a rotating vector is called a **phasor.** (This term was in use long before the invention of the Star Trek stun gun with a similar name. The phasor method for analyzing oscillations is useful in many areas of physics. We'll use phasors when we study alternating-current circuits in Chapter 31 and the interference of light in Chapters 35 and 36.)

13.5 (a) Relating uniform circular motion and simple harmonic motion. (b) The ball's shadow moves exactly like a body oscillating on an ideal spring.



The x-component of the phasor at time t is just the x-coordinate of the point Q:

$$x = A\cos\theta \tag{13.5}$$

This is also the x-coordinate of the shadow P, which is the *projection* of Q onto the x-axis. Hence the x-velocity of the shadow P along the x-axis is equal to the x-component of the velocity vector of the reference point Q (Fig. 13.6a), and the x-acceleration of P is equal to the x-component of the acceleration vector of Q (Fig. 13.6b). Since point Q is in uniform circular motion, its acceleration vector $\vec{a}_Q$ is always directed toward O. Furthermore, the magnitude of $\vec{a}_Q$ is constant and given by the angular speed squared times the radius of the circle (see Section 9.3):

$$a_Q = \omega^2 A \tag{13.6}$$

Figure 13.6b shows that the x-component of $\vec{a}_Q$ is $a_x = -a_Q\cos\theta$. Combining this with Eqs. (13.5) and (13.6), we get that the acceleration of point P is

$$a_x = -a_Q\cos\theta = -\omega^2 A\cos\theta \quad \text{or} \tag{13.7}$$

$$a_x = -\omega^2 x \tag{13.8}$$

The acceleration of the point P is directly proportional to the displacement x and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion.

Equation (13.8) is *exactly* the same as Eq. (13.4) for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass m of the oscillating body by

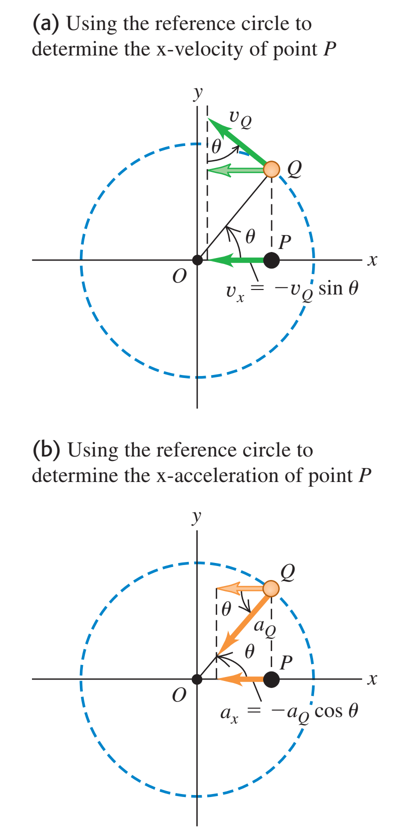
$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}} \tag{13.9}$$

We have been using the same symbol ω for the angular *speed* of the reference point Q and the angular *frequency* of the oscillating point P. The reason is that these quantities are equal! If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through 2π radians, so its angular speed is $\omega = 2\pi/T$. But this is just the same as Eq. (13.2) for the angular frequency of the point P, which verifies our statement about the two interpretations of ω. This is why we introduced angular frequency in Section 13.1; this quantity makes the connection between oscillation and circular motion. So we reinterpret Eq. (13.9) as an expression for the angular frequency of simple harmonic motion for a body of mass m, acted on by a restoring force with force constant k:

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)} \tag{13.10}$$

When you start a body oscillating in SHM, the value of ω is not yours to choose; it is predetermined by the values of k and m. The units of k are N/m or kg/s^2, so

13.6 The (a) x-velocity and (b) x-acceleration of the ball's shadow P (See Fig. 13.5) are the x-components of the velocity and acceleration vectors, respectively, of the ball Q.



k/m is in $(\mathrm{kg/s^2})/\mathrm{kg} = \mathrm{s^{-2}}$. When we take the square root in Eq. (13.10), we get $\mathrm{s^{-1}}$, or more properly rad/s because this is an *angular* frequency (recall that a radian is not a true unit).

According to Eqs. (13.1) and (13.2), the frequency f and period T are

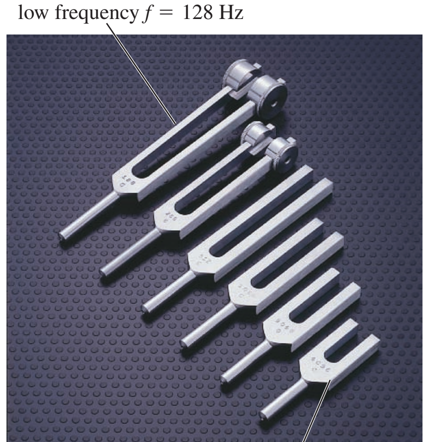
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(simple harmonic motion)} \qquad (13.11)$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \text{(simple harmonic motion)} \qquad (13.12)$$

We see from Eq. (13.12) that a larger mass m, with its greater inertia, will have less acceleration, move more slowly, and take a longer time for a complete cycle (Fig. 13.7). In contrast, a stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x, causing greater acceleration, higher speeds, and a shorter time T per cycle.

13.7 The greater the mass m in a tuning fork's tines, the lower the frequency of oscillation $f = (1/2\pi)\sqrt{k/m}$ and the lower the pitch of the sound that the tuning fork produces.

Tines with large mass m: low frequency $f = 128$ Hz



Tines with small mass m: high frequency $f = 4096$ Hz

CAUTION **Don't confuse frequency and angular frequency** You can run into trouble if you don't make the distinction between frequency f and angular frequency $\omega = 2\pi f$. Frequency tells you how many cycles of oscillation occur per second, while angular frequency tells you how many radians per second this corresponds to on the reference circle. In solving problems, pay careful attention to whether the goal is to find f or ω. ▮

Period and Amplitude in SHM

Equations (13.11) and (13.12) show that the period and frequency of simple harmonic motion are completely determined by the mass m and the force constant k. *In simple harmonic motion the period and frequency do not depend on the amplitude A.* For given values of m and k, the time of one complete oscillation is the same whether the amplitude is large or small. Equation (13.3) shows why we should expect this. Larger A means that the body reaches larger values of $|x|$ and is subjected to larger restoring forces. This increases the average speed of the body over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

The oscillations of a tuning fork are essentially simple harmonic motion, which means that it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to make familiar types of mechanical and electronic clocks run accurately or to play most musical instruments in tune. If you encounter an oscillating body with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

Example 13.2 Angular frequency, frequency, and period in SHM

A spring is mounted horizontally, with its left end held stationary. By attaching a spring balance to the free end and pulling toward the right (Fig. 13.8a), we determine that the stretching force is proportional to the displacement and that a force of 6.0 N causes a displacement of 0.030 m. We remove the spring balance and attach a 0.50-kg glider to the end, pull it a distance of 0.020 m along a frictionless air track, release it, and watch it oscillate (Fig. 13.8b). (a) Find the force constant of the spring. (b) Find the angular frequency, frequency, and period of the oscillation.

SOLUTION

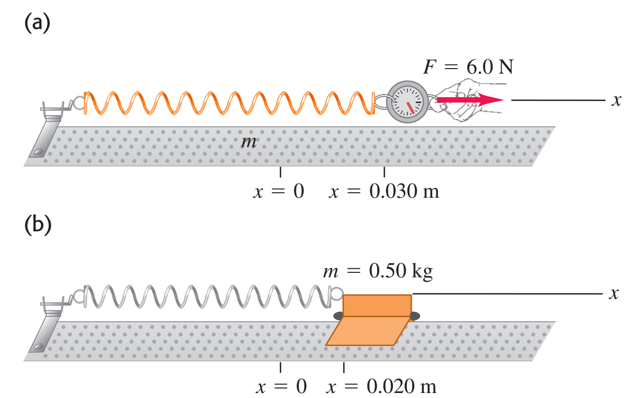
IDENTIFY: Because the spring force (equal in magnitude to the stretching force) is proportional to the displacement, the motion is simple harmonic.

SET UP: We find the value of the force constant k using Hooke's law, Eq. (13.3), and the values of ω, f, and T using Eqs. (13.10), (13.11), and (13.12), respectively.

EXECUTE: (a) When $x = 0.030$ m, the force the spring exerts on the spring balance is $F_x = -6.0$ N. From Eq. (13.3),

$$k = -\frac{F_x}{x} = -\frac{-6.0\ \mathrm{N}}{0.030\ \mathrm{m}} = 200\ \mathrm{N/m} = 200\ \mathrm{kg/s^2}$$

13.8 (a) The force exerted *on* the spring (shown by the vector F) has x-component $F_x = +6.0$ N. The force exerted *by* the spring has x-component $F_x = -6.0$ N. (b) A glider is attached to the same spring and allowed to oscillate.



(b) Using $m = 0.50$ kg in Eq. (13.10), we find

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \mathrm{kg/s^2}}{0.50\ \mathrm{kg}}} = 20\ \mathrm{rad/s}$$

The frequency f is

$$f = \frac{\omega}{2\pi} = \frac{20\ \mathrm{rad/s}}{2\pi\ \mathrm{rad/cycle}} = 3.2\ \mathrm{cycle/s} = 3.2\ \mathrm{Hz}$$

The period T is the reciprocal of the frequency f:

$$T = \frac{1}{f} = \frac{1}{3.2\ \mathrm{cycle/s}} = 0.31\ \mathrm{s}$$

A period is usually stated in "seconds" rather than "seconds per cycle."

EVALUATE: The amplitude of the oscillation is 0.020 m, the distance to the right that we pulled the glider attached to the spring before releasing it. We didn't need to use this information to find the angular frequency, frequency, or period, because in SHM none of these quantities depend on the amplitude.

Displacement, Velocity, and Acceleration in SHM

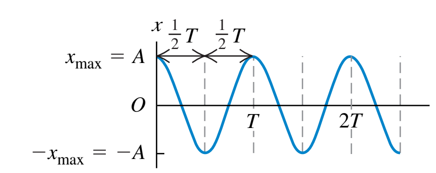
ActivPhysics ONLINE
9.1 Position Graphs and Equations
9.2 Describing Vibrational Motion
9.5 Ape Drops Tarzan

We still need to find the displacement x as a function of time for a harmonic oscillator. Equation (13.4) for a body in simple harmonic motion along the x-axis is identical to Eq. (13.8) for the x-coordinate of the reference point in uniform circular motion with constant angular speed $\omega = \sqrt{k/m}$. It follows that Eq. (13.5), $x = A\cos\theta$, describes the coordinate x for both of these situations. If at $t = 0$ the phasor OQ makes an angle ϕ (the Greck letter phi) with the positive x-axis, then at any later time t this angle is $\theta = \omega t + \phi$. We substitute this into Eq. (13.5) to obtain

$$x = A\cos(\omega t + \phi) \quad \text{(displacement in SHM)} \qquad (13.13)$$

where $\omega = \sqrt{k/m}$. Figure 13.9 shows a graph of Eq. (13.13) for the particular case $\phi = 0$. The displacement x is a periodic function of time, as expected for SHM. We could also have written Eq. (13.13) in terms of a sine function rather than a cosine by using the identity $\cos\alpha = \sin(\alpha + \pi/2)$. *In simple harmonic motion the position is a periodic, sinusoidal function of time.* There are many other periodic functions, but none so smooth and simple as a sine or cosine function.

13.9 Graph of x versus t [See Eq. (13.13)] for simple harmonic motion. The case shown has $\phi = 0$.

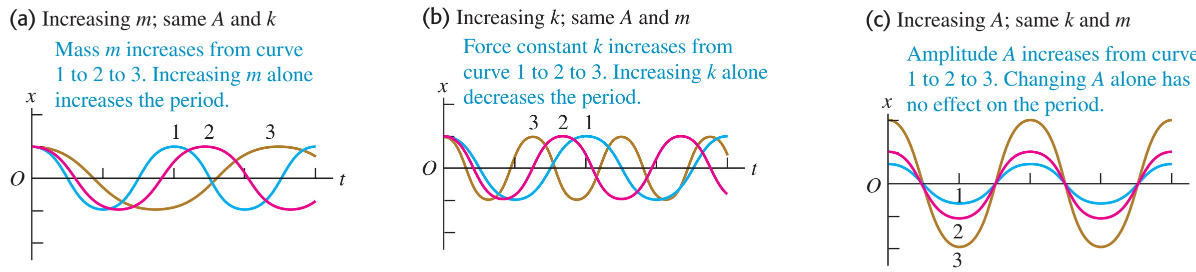


The value of the cosine function is always between -1 and 1, so in Eq. (13.13), x is always between $-A$ and A. This confirms that A is the amplitude of the motion.

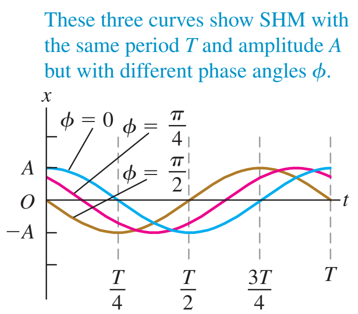
The period T is the time for one complete cycle of oscillation, as Fig. 13.9 shows. The cosine function repeats itself whenever the quantity in parentheses in Eq. (13.13) increases by 2π radians. Thus, if we start at time $t = 0$, the time T to complete one cycle is given by

$$\omega T = \sqrt{\frac{k}{m}}T = 2\pi \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

13.10 Variations of simple harmonic motion. All cases shown have $\phi = 0$ [see Eq. (13.13)].

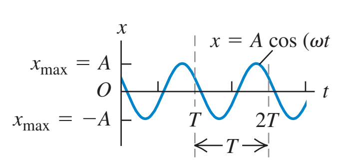


13.11 Variations of SHM: displacement versus time for the same harmonic oscillator with different phase angles ϕ.

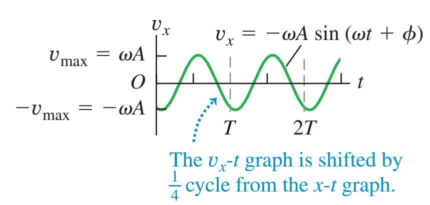


13.12 Graphs of (a) x versus t, (b) v_x versus t, and (c) a_x versus t for a body in SHM. For the motion depicted in these graphs, $\phi = \pi/3$.

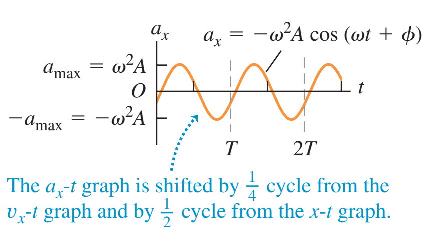
(a) Displacement x as a function of time t



(b) Velocity v_x as a function of time t



(c) Acceleration a_x as a function of time t



which is just Eq. (13.12). Changing either m or k changes the period of oscillation, as shown in Figs 13.10a and 13.10b. The period does not depend on the amplitude A (Fig. 13.10c).

The constant ϕ in Eq. (13.13) is called the **phase angle.** It tells us at what point in the cycle the motion was at $t = 0$ (equivalent to where around the circle the point Q was at $t = 0$). We denote the position at $t = 0$ by x_0. Putting $t = 0$ and $x = x_0$ in Eq. (13.13), we get

$$x_0 = A\cos\phi \tag{13.14}$$

If $\phi = 0$, then $x_0 = A\cos 0 = A$, and the body starts at its maximum positive displacement. If $\phi = \pi$, then $x_0 = A\cos\pi = -A$, and the particle starts at its maximum *negative* displacement. If $\phi = \pi/2$, then $x_0 = A\cos(\pi/2) = 0$, and the particle is initially at the origin. Figure 13.11 shows the displacement x versus time for three different phase angles.

We find the velocity v_x and acceleration a_x as functions of time for a harmonic oscillator by taking derivatives of Eq. (13.13) with respect to time:

$$v_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi) \quad \text{(velocity in SHM)} \tag{13.15}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi) \quad \text{(acceleration in SHM)} \tag{13.16}$$

The velocity v_x oscillates between $v_{max} = +\omega A$ and $-v_{max} = -\omega A$, and the acceleration a_x oscillates between $a_{max} = +\omega^2 A$ and $-a_{max} = -\omega^2 A$ (Fig. 13.12). Comparing Eq. (13.16) with Eq. (13.13) and recalling that $\omega^2 = k/m$ from Eq. (13.9), we see that

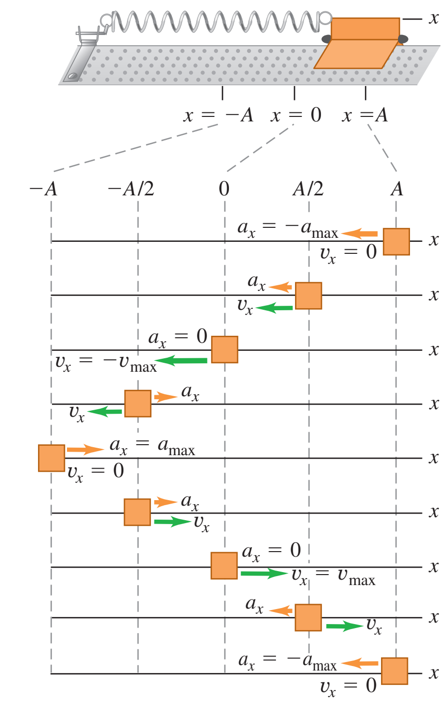
$$a_x = -\omega^2 x = -\frac{k}{m}x$$

which is just Eq. (13.4) for simple harmonic motion. This confirms that Eq. (13.13) for x as a function of time is correct.

We actually derived Eq. (13.16) earlier in a geometrical way by taking the x-component of the acceleration vector of the reference point Q. This was done in Fig. 13.6b and Eq. (13.7) (recall that $\theta = \omega t + \phi$). In the same way, we could have derived Eq. (13.15) by taking the x-component of the velocity vector of Q, as shown in Fig. 13.6b. We'll leave the details for you to work out (see Problem 13.85).

Note that the sinusoidal graph of displacement versus time (Fig. 13.12a) is shifted by one-quarter period from the graph of velocity versus time (Fig. 13.12b) and by one-half period from the graph of acceleration versus time (Fig. 13.12c). Figure 13.13 shows why this is so. When the body is passing through the equilibrium position so that the displacement is zero, the velocity equals either v_{max} or $-v_{max}$ (depending on which way the body is moving) and the acceleration is zero. When the body is at either its maximum positive displacement, $x = +A$, or its maximum negative displacement, $x = -A$, the velocity is zero and the body is instantaneously at rest. At these points, the restoring force $F_x = -kx$ and the acceleration of the body have their maximum magnitudes. At $x = +A$ the acceleration is negative and equal to $-a_{max}$. At $x = -A$ the acceleration is positive: $a_x = +a_{max}$.

13.13 How x-velocity v_x and x-acceleration a_x vary during one cycle of SHM.



If we are given the initial position x_0 and initial velocity v_{0x} for the oscillating body, we can determine the amplitude A and the phase angle ϕ. Here's how to do it. The initial velocity v_{0x} is the velocity at time $t = 0$; putting $v_x = v_{0x}$ and $t = 0$ in Eq. (13.15), we find

$$v_{0x} = -\omega A\sin\phi \tag{13.17}$$

To find ϕ, we divide Eq. (13.17) by Eq. (13.14). This eliminates A and gives an equation that we can solve for ϕ:

$$\frac{v_{0x}}{x_0} = \frac{-\omega A\sin\phi}{A\cos\phi} = -\omega\tan\phi$$

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) \quad \text{(phase angle in SHM)} \tag{13.18}$$

It is also easy to find the amplitude A if we are given x_0 and v_{0x}. We'll sketch the derivation, and you can fill in the details. Square Eq. (13.14); then divide Eq. (13.17) by ω, square it, and add to the square of Eq. (13.14). The right side will be $A^2(\sin^2\phi + \cos^2\phi)$, which is equal to A^2. The final result is

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} \quad \text{(amplitude in SHM)} \tag{13.19}$$

Note that when the body has both an initial displacement x_0 and a nonzero initial velocity v_{0x}, the amplitude A is *not* equal to the initial displacement. That's reasonable; if you start the body at a positive x_0 but give it a positive velocity v_{0x}, it will go *farther* than x_0 before it turns and comes back.

Problem-Solving Strategy 13.1 Simple Harmonic Motion I: Describing Motion

IDENTIFY *the relevant concepts:* An oscillating system undergoes simple harmonic motion (SHM) *only* if the restoring force is directly proportional to the displacement. Be certain that this is the case for the problem at hand before attempting to use any of the results of this section. As always, identify the target variables.

SET UP *the problem* using the following steps:

1. Identify the known and unknown quantities, and determine which are the target variables.
2. It's useful to distinguish between two kinds of quantities. *Basic properties* of the system include the mass m and the force constant k as well as quantities derived from m and k, such as the period T, frequency f, and angular frequency ω. *Properties of the motion* describe how the system behaves when it is set into motion in a particular way. They include the amplitude A, maximum velocity v_{max}, and phase angle ϕ as well as the values of x, v_x, and a_x at a particular time.
3. If necessary, define an x-axis as in Fig. 13.13, with the equilibrium position at $x = 0$.

EXECUTE *the solution* as follows:

1. Use the equations given in Sections 13.1 and 13.2 to solve for the target variables.
2. If you need to calculate the phase angle, be certain to express it in radians. The quantity ωt in Eq. (13.13) is naturally in radians, so ϕ must be as well.
3. If you need to find the values of x, v_x, and a_x at various times, use Eqs. (13.13), (13.15), and (13.16), respectively. If the initial position x_0 and initial velocity v_{0x} are both given, you can determine the phase angle and amplitude from Eqs. (13.18) and (13.19). If the body is given an initial positive displacement x_0 but zero initial velocity $(v_{0x} = 0)$, then the amplitude is $A = x_0$ and the phase angle is $\phi = 0$. If it has an initial positive velocity v_{0x} but no initial displacement $(x_0 = 0)$, the amplitude is $A = v_{0x}/\omega$ and the phase angle is $\phi = -\pi/2$.

EVALUATE *your answer:* Check your results to make sure they're consistent. As an example, suppose you've used the initial position and velocity to find general expressions for x and v_x at time t. If you substitute $t = 0$ into these expressions, you should get back the correct values of x_0 and v_{0x}.

Example 13.3 Describing SHM

Let's return to the system of mass and horizontal spring we considered in Example 13.2, with $k = 200\ \text{N/m}$ and $m = 0.50\ \text{kg}$. This time we give the body an initial displacement of $+0.015\ \text{m}$ and an initial velocity of $+0.40\ \text{m/s}$. (a) Find the period, amplitude, and phase angle of the motion. (b) Write equations for the displacement, velocity, and acceleration as functions of time.

SOLUTION

IDENTIFY: As in Example 13.2, the oscillations are SHM and we may use the expressions developed in this section.

SET UP: We are given the values of k, m, x_0, and v_{0x}. From them, we calculate the target variables T, A, and ϕ and the expressions for x, v_x, and a_x as functions of time.

EXECUTE: (a) The period is the same as in Example 13.2, $T = 0.31\ \text{s}$. In simple harmonic motion the period does not depend on the amplitude, only on the values of k and m. In Example 13.2 we found that $\omega = 20\ \text{rad/s}$. So from Eq. (13.19),

$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}} = \sqrt{(0.015\ \text{m})^2 + \frac{(0.40\ \text{m/s})^2}{(20\ \text{rad/s})^2}} = 0.025\ \text{m}$$

To find the phase angle ϕ, we use Eq. (13.18):

$$\phi = \arctan\left(-\frac{v_{0x}}{\omega x_0}\right) = \arctan\left(-\frac{0.40\ \text{m/s}}{(20\ \text{rad/s})(0.015\ \text{m})}\right) = -53° = -0.93\ \text{rad}$$

(b) The displacement, velocity, and acceleration at any time are given by Eqs. (13.13), (13.15), and (13.16), respectively. Substituting the values, we get

$$x = (0.025\ \text{m})\cos[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$
$$v_x = -(0.50\ \text{m/s})\sin[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$
$$a_x = -(10\ \text{m/s}^2)\cos[(20\ \text{rad/s})t - 0.93\ \text{rad}]$$

The velocity varies sinusoidally between $-0.50\ \text{m/s}$ and $+0.50\ \text{m/s}$, and the acceleration varies sinusoidally between $-10\ \text{m/s}^2$ and $+10\ \text{m/s}^2$.

EVALUATE: You can check the results for x and v_x as functions of time by substituting $t = 0$ and evaluating the result. You should get $x = x_0 = 0.015\ \text{m}$ and $v_x = v_{0x} = 0.40\ \text{m/s}$. Do you?

Test Your Understanding of Section 13.2 A glider is attached to a spring as shown in Fig. 13.13. If the glider is moved to $x = 0.10\ \text{m}$ and released from rest at time $t = 0$, it will oscillate with amplitude $A = 0.10\ \text{m}$ and phase angle $\phi = 0$. (a) Suppose instead that at $t = 0$ the glider is at $x = 0.10\ \text{m}$ and is moving to the right in Fig. 13.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero? (b) Suppose instead that at $t = 0$ the glider is at $x = 0.10\ \text{m}$ and is moving to the left in Fig. 13.13. In this situation is the amplitude greater than, less than, or equal to 0.10 m? Is the phase angle greater than, less than, or equal to zero?

13.3 Energy in Simple Harmonic Motion

ActivPhysics ONLINE

9.3 Vibrational Energy
9.4 Two Ways to Weigh Young Tarzan
9.6 Releasing a Vibrating Skier I
9.7 Releasing a Vibrating Skier II
9.8 One- and Two-Spring Vibrating Systems
9.9 Vibro-Ride

We can learn even more about simple harmonic motion by using energy considerations. Take another look at the body oscillating on the end of a spring in Figs. 13.2 and 13.13. We've already noted that the spring force is the only horizontal force on the body. The force exerted by an ideal spring is a conservative force, and the vertical forces do no work, so the total mechanical energy of the system is *conserved*. We also assume that the mass of the spring itself is negligible.

The kinetic energy of the body is $K = \frac{1}{2}mv^2$ and the potential energy of the spring is $U = \frac{1}{2}kx^2$, just as in Section 7.2. (You'll find it helpful to review that section.) There are no nonconservative forces that do work, so the total mechanical energy $E = K + U$ is conserved:

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \text{constant} \tag{13.20}$$

(Since the motion is one-dimensional, $v^2 = v_x^2$.)

The total mechanical energy E is also directly related to the amplitude A of the motion. When the body reaches the point $x = A$, its maximum displacement from equilibrium, it momentarily stops as it turns back toward the equilibrium position. That is, when $x = A$ (or $-A$), $v_x = 0$. At this point the energy is entirely potential, and $E = \frac{1}{2}kA^2$. Because E is constant, it is equal to $\frac{1}{2}kA^2$ at any other point. Combining this expression with Eq. (13.20), we get

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant} \quad \text{(total mechanical energy in SHM)} \tag{13.21}$$

We can verify this equation by substituting x and v_x from Eqs. (13.13) and (13.15) and using $\omega^2 = k/m$ from Eq. (13.9):

$$\begin{aligned} E &= \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}m[-\omega A\sin(\omega t + \phi)]^2 + \frac{1}{2}k[A\cos(\omega t + \phi)]^2 \\ &= \frac{1}{2}kA^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

(Recall that $\sin^2\alpha + \cos^2\alpha = 1$.) Hence our expressions for displacement and velocity in SHM are consistent with energy conservation, as they must be.

We can use Eq. (13.21) to solve for the velocity v_x of the body at a given displacement x:

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2} \tag{13.22}$$

The $\pm$ sign means that at a given value of x the body can be moving in either direction. For example, when $x = \pm A/2$,

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - \left(\pm\frac{A}{2}\right)^2} = \pm\sqrt{\frac{3}{4}}\sqrt{\frac{k}{m}}A$$

Equation (13.22) also shows that the *maximum* speed v_{max} occurs at $x = 0$. Using Eq. (13.10), $\omega = \sqrt{k/m}$, we find that

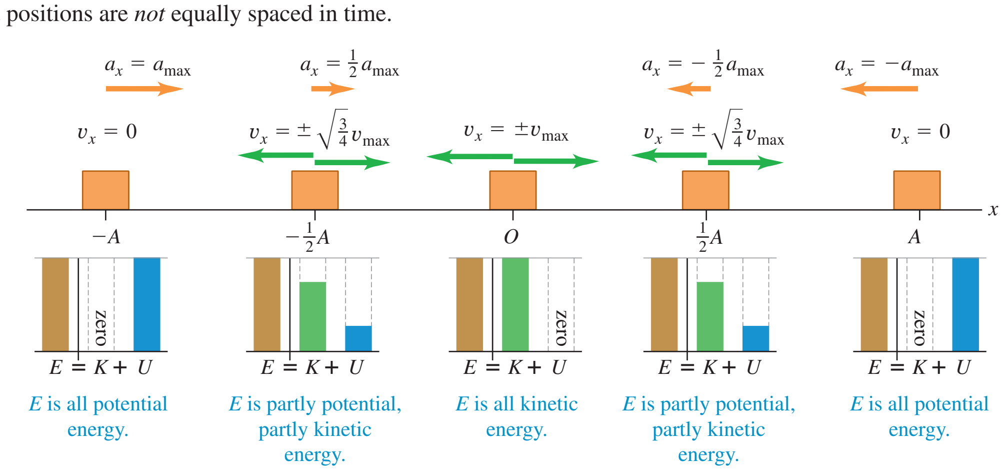
$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A \tag{13.23}$$

This agrees with Eq. (13.15), which showed that v_x oscillates between $-\omega A$ and $+\omega A$.

Interpreting E, K, and U in SHM

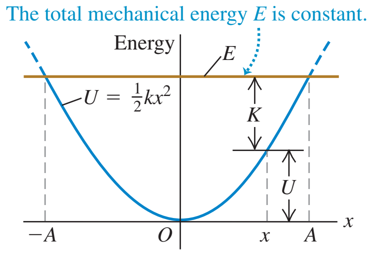
Figure 13.14 shows the energy quantities E, K, and U at $x = 0$, $x = \pm A/2$, and $x = \pm A$. Figure 13.15 is a graphical display of Eq. (13.21); energy (kinetic, potential, and total) is plotted vertically and the coordinate x is plotted horizontally. The

13.14 Graphs of E, K, and U versus displacement in SHM. The velocity of the body is *not* constant, so these images of the body at equally spaced positions are *not* equally spaced in time.

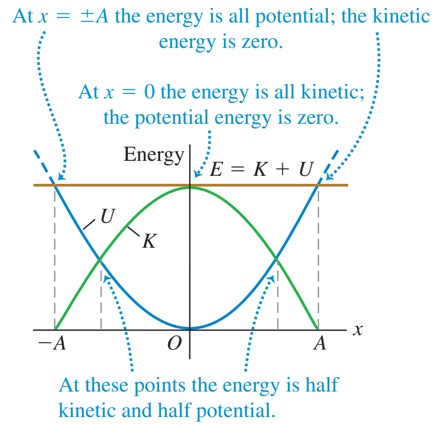


13.15 Kinetic energy K, potential energy U, and total mechanical energy E as functions of position for SHM. At each value of x the sum of the values of K and U equals the constant value of E. Can you show that the energy is half kinetic and half potential at $x = \pm\sqrt{\frac{1}{2}}A$?

(a) The potential energy U and total mechanical energy E for a body in SHM as a function of displacement x



(b) The same graph as in (a), showing kinetic energy K as well



parabolic curve in Fig. 13.15a represents the potential energy $U = \frac{1}{2}kx^2$. The horizontal line represents the total mechanical energy E, which is constant and does not vary with x. At any value of x between $-A$ and A, the vertical distance from the x-axis to the parabola is U; since $E = K + U$, the remaining vertical distance up to the horizontal line is K. Figure 13.15b shows both K and U as functions of x. The horizontal line for E intersects the potential-energy curve at $x = -A$ and $x = A$, so at these points the energy is entirely potential, the kinetic energy is zero, and the body comes momentarily to rest before reversing direction. As the body oscillates between $-A$ and A, the energy is continuously transformed from potential to kinetic and back again.

Figure 13.15a shows the connection between the amplitude A and the corresponding total mechanical energy $E = \frac{1}{2}kA^2$. If we tried to make x greater than A (or less than $-A$), U would be greater than E, and K would have to be negative. But K can never be negative, so x can't be greater than A or less than $-A$.

Problem-Solving Strategy 13.2 Simple Harmonic Motion II: Energy

MP

The energy equation, Eq. (13.21), is a useful alternative relationship between velocity and position, especially when energy quantities are also required. If the problem involves a relationship among position, velocity, and acceleration without reference to time, it is usually easier to use Eq. (13.4) (from Newton's second law) or Eq. (13.21) (from energy conservation) than to use the general expressions for x, v_x, and a_x as functions of time [Eqs. (13.13), (13.15), and (13.16), respectively]. Because the energy equation involves x^2 and v_x^2, it cannot tell you the sign of x or of v_x; you have to infer the sign from the situation. For instance, if the body is moving from the equilibrium position toward the point of greatest positive displacement, then x is positive and v_x is positive.

Example 13.4 Velocity, acceleration, and energy in SHM

In the oscillation described in Example 13.2, $k = 200$ N/m, $m = 0.50$ kg, and the oscillating mass is released from rest at $x = 0.020$ m. (a) Find the maximum and minimum velocities attained by the oscillating body. (b) Compute the maximum acceleration. (c) Determine the velocity and acceleration when the body has moved halfway to the center from its original position. (d) Find the total energy, potential energy, and kinetic energy at this position.

SOLUTION

IDENTIFY: The problem refers to the motion at various *positions* in the motion, not at specified *times.* This is a hint that we can use the energy relationships found in this section to solve for the target variables.

SET UP: Figure 13.13 shows the choice of x-axis. The maximum displacement from equilibrium is $A = 0.020$ m. For any position x we use Eqs. (13.22) and (13.4) to find the velocity v_x and acceleration a_x, respectively. Given the velocity and position, we use Eq. (13.21) to find the energy quantities K, U, and E.

EXECUTE: (a) The velocity v_x at any displacement x is given by Eq. (13.22):

$$v_x = \pm\sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}$$

The maximum velocity occurs when the body is moving to the right through the equilibrium position, where $x = 0$:

$$v_x = v_{max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}(0.020\ \text{m}) = 0.40\ \text{m/s}$$

The minimum (i.e., most negative) velocity occurs when the body is moving to the left through $x = 0$; its value is $-v_{max} = -0.40$ m/s.

(b) From Eq. (13.4),

$$a_x = -\frac{k}{m}x$$

The maximum (most positive) acceleration occurs at the most negative value of x, $x = -A$; therefore

$$a_{max} = -\frac{k}{m}(-A) = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(-0.020\ \text{m}) = 8.0\ \text{m/s}^2$$

The minimum (most negative) acceleration is $-8.0\ \text{m/s}^2$, occurring at $x = +A = +0.020$ m.

(c) At a point halfway to the center from the initial position, $x = A/2 = 0.010$ m. From Eq. (13.22),

$$v_x = -\sqrt{\frac{200\ \text{N/m}}{0.50\ \text{kg}}}\sqrt{(0.020\ \text{m})^2 - (0.010\ \text{m})^2} = -0.35\ \text{m/s}$$

We choose the negative square root because the body is moving from $x = A$ toward $x = 0$. From Eq. (13.4),

$$a_x = -\frac{200\ \text{N/m}}{0.50\ \text{kg}}(0.010\ \text{m}) = -4.0\ \text{m/s}^2$$

At this point the velocity and the acceleration have the same sign, so the speed is increasing. The conditions at $x = 0$, $\pm A/2$, and $\pm A$ are shown in Fig. 13.14.

(d) The total energy has the same value at all points during the motion:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(200\ \text{N/m})(0.020\ \text{m})^2 = 0.040\ \text{J}$$

The potential energy is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(200\ \text{N/m})(0.010\ \text{m})^2 = 0.010\ \text{J}$$

and the kinetic energy is

$$K = \frac{1}{2}mv_x^2 = \frac{1}{2}(0.50\ \text{kg})(-0.35\ \text{m/s})^2 = 0.030\ \text{J}$$

EVALUATE: At the point $x = A/2$, the energy is one-fourth potential energy and three-fourths kinetic energy. You can check this result by inspecting Fig. 13.15b.

Example 13.5 Energy and momentum in SHM

A block with mass M attached to a horizontal spring with force constant k is moving with simple harmonic motion having amplitude A_1. At the instant when the block passes through its equilibrium position, a lump of putty with mass m is dropped vertically onto the block from a very small height and sticks to it. (a) Find the new amplitude and period. (b) Repeat part (a) for the case in which the putty is dropped on the block when it is at one end of its path.

SOLUTION

IDENTIFY: The problem involves the motion at a given position, not a given time, so we can use energy methods. Before the putty lands on the block, the mechanical energy of the oscillating block and spring is constant. When the putty lands on the block, it's a completely inelastic collision (see Section 8.3); the horizontal component of momentum is conserved, but kinetic energy decreases. Once the collision ends, the mechanical energy remains constant at its new value.

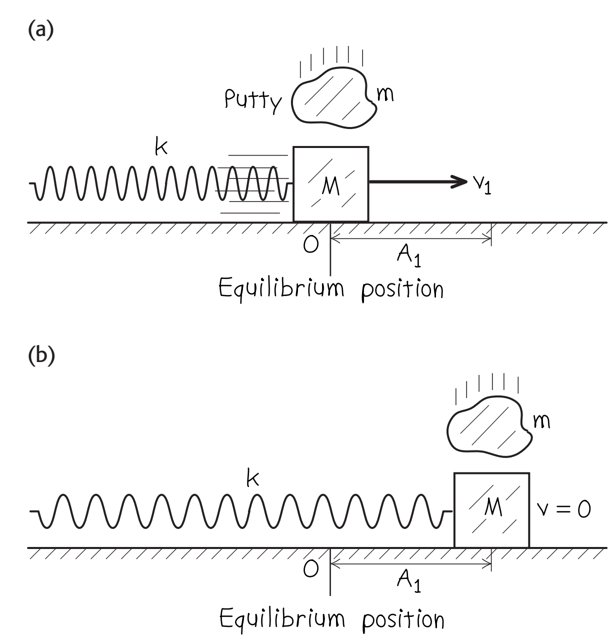
SET UP: Figure 13.16 shows our sketches. In each part we consider what happens before, during, and after the collision. We find the amplitude A_2 after the collision from the final energy of the system, and we find the period T_2 after the collision using the relationship between period and mass.

EXECUTE: (a) Before the collision the total mechanical energy of the block and spring is $E_1 = \frac{1}{2}kA_1^2$. Since the block is at the equilibrium position, $U = 0$, and the energy is purely kinetic (Fig. 13.16a). If we let v_1 be the speed of the block at the equilibrium position, we have

$$E_1 = \frac{1}{2}Mv_1^2 = \frac{1}{2}kA_1^2 \qquad \text{so} \qquad v_1 = \sqrt{\frac{k}{M}}A_1$$

During the collision the x-component of momentum of the system of block and putty is conserved. (Why?) Just before the collision

13.16 Our sketches for this problem.



this component is the sum of Mv_1 (for the block) and zero (for the putty). Just after the collision the block and putty move together with speed v_2, and their combined x-component of momentum is $(M + m)v_2$. From conservation of momentum,

$$Mv_1 + 0 = (M + m)v_2 \qquad \text{so} \qquad v_2 = \frac{M}{M + m}v_1$$

Continued

The collision lasts a very short time, so just after the collision the block and putty are still at the equilibrium position. The energy is still purely kinetic but is *less* than before the collision:

$$E_2 = \frac{1}{2}(M+m)v_2^2 = \frac{1}{2}\frac{M^2}{M+m}v_1^2 = \frac{M}{M+m}\left(\frac{1}{2}Mv_1^2\right)$$
$$= \left(\frac{M}{M+m}\right)E_1$$

Since E_2 equals $\frac{1}{2}kA_2^2$, where A_2 is the amplitude after the collision, we have

$$\frac{1}{2}kA_2^2 = \left(\frac{M}{M+m}\right)\frac{1}{2}kA_1^2$$
$$A_2 = A_1\sqrt{\frac{M}{M+m}}$$

The larger the putty mass m, the smaller the final amplitude.

Finding the period of oscillation after the collision is the easy part. Using Eq. (13.12), we have

$$T_2 = 2\pi\sqrt{\frac{M+m}{k}}$$

(b) When the putty drops, the block is instantaneously at rest (Fig. 13.16b). The x-component of momentum is zero both before and after the collision: The block has zero kinetic energy just before the collision, and the block and putty have zero kinetic energy just after the collision. The energy is all potential energy stored in the spring, so adding the extra mass of the putty has *no effect* on the mechanical energy. That is,

$$E_2 = E_1 = \frac{1}{2}kA_1^2$$

and the amplitude after the collision is unchanged $(A_2 = A_1)$. The period still changes when the putty is added, though; its value doesn't depend on how the mass is added, only on what the total mass is. So T_2 is the same as we found in part (a), $T_2 = 2\pi\sqrt{(M+m)/k}$.

EVALUATE: Why is energy lost in part (a) but not in part (b)? The difference is that in part (a) the putty slides against the moving block during the collision, and energy is dissipated by kinetic friction.

Test Your Understanding of Section 13.3 (a) To double the total energy for a mass-spring system oscillating in SHM, by what factor must the amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$. (b) By what factor will the frequency change due to this amplitude increase? (i) 4; (ii) 2; (iii) $\sqrt{2} = 1.414$; (iv) $\sqrt[4]{2} = 1.189$; (v) it does not change.

13.4 Applications of Simple Harmonic Motion

So far, we've looked at a grand total of *one* situation in which simple harmonic motion (SHM) occurs: a body attached to an ideal horizontal spring. But SHM can occur in any system in which there is a restoring force that is directly proportional to the displacement from equilibrium, as given by Eq. (13.3), $F_x = -kx$. The restoring force will originate in different ways in different situations, so the force constant k has to be found for each case by examining the net force on the system. Once this is done, it's straightforward to find the angular frequency ω, frequency f, and period T; we just substitute the value of k into Eqs. (13.10), (13.11), and (13.12), respectively. Let's use these ideas to examine several examples of simple harmonic motion.

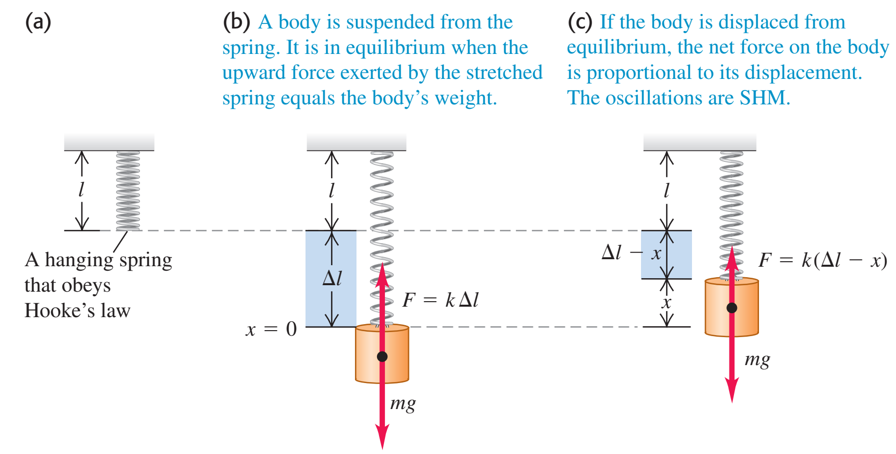
Vertical SHM

Suppose we hang a spring with force constant k (Fig. 13.17a) and suspend from it a body with mass m. Oscillations will now be vertical; will they still be SHM? In Fig. 13.17b the body hangs at rest, in equilibrium. In this position the spring is stretched an amount Δl just great enough that the spring's upward vertical force $k\,\Delta l$ on the body balances its weight mg:

$$k\,\Delta l = mg$$

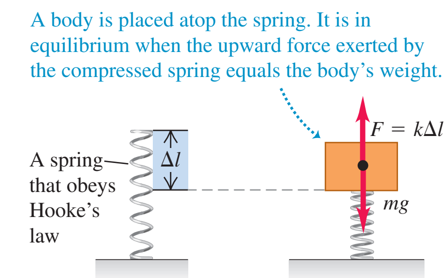
Take $x = 0$ to be this equilibrium position and take the positive x-direction to be upward. When the body is a distance x *above* its equilibrium position (Fig. 13.17c), the extension of the spring is $\Delta l - x$. The upward force it exerts on the body is then $k(\Delta l - x)$, and the net x-component of force on the body is

$$F_{\text{net}} = k(\Delta l - x) + (-mg) = -kx$$



13.17 A body attached to a hanging spring.

13.18 If the weight mg compresses the spring a distance Δl, the force constant is $k = mg/\Delta l$ and the angular frequency for vertical SHM is $\omega = \sqrt{k/m}$—the same as if the body were suspended from the spring (See Fig. 13.17).



that is, a net downward force of magnitude kx. Similarly, when the body is *below* the equilibrium position, there is a net upward force with magnitude kx. In either case there is a restoring force with magnitude kx. If the body is set in vertical motion, it oscillates in SHM with the same angular frequency as though it were horizontal, $\omega = \sqrt{k/m}$. So vertical SHM doesn't differ in any essential way from horizontal SHM. The only real change is that the equilibrium position $x = 0$ no longer corresponds to the point at which the spring is unstretched. The same ideas hold if a body with weight mg is placed atop a compressible spring (Fig. 13.18) and compresses it a distance Δl.

Example 13.6 Vertical SHM in an old car

The shock absorbers in an old car with mass 1000 kg are completely worn out. When a 980-N person climbs slowly into the car to its center of gravity, the car sinks 2.8 cm. When the car, with the person aboard, hits a bump, the car starts oscillating up and down in SHM. Model the car and person as a single body on a single spring, and find the period and frequency of the oscillation.

SOLUTION

IDENTIFY: The situation is like that shown in Fig. 13.18.

SET UP: The compression of the spring when the extra weight is added tells us the force constant, which we can use to find the period and frequency (the target variables).

EXECUTE: When the force increases by 980 N, the spring compresses an additional 0.028 m, and the coordinate x of the car changes by -0.028 m. Hence the effective force constant (including the effect of the entire suspension) is

$$k = -\frac{F_x}{x} = -\frac{980\text{ N}}{-0.028\text{ m}} = 3.5 \times 10^4\text{ kg/s}^2$$

The person's mass is $w/g = (980\text{ N})/(9.8\text{ m/s}^2) = 100$ kg. The *total* oscillating mass is $m = 1000\text{ kg} + 100\text{ kg} = 1100$ kg. The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\text{ kg}}{3.5\times 10^4\text{ kg/s}^2}} = 1.11\text{ s}$$

and the frequency is

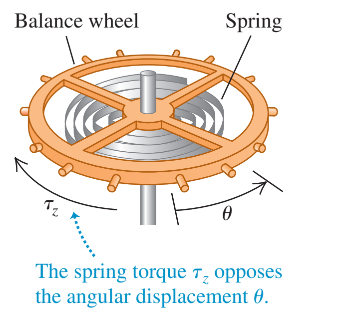
$$f = \frac{1}{T} = \frac{1}{1.11\text{ s}} = 0.90\text{ Hz}$$

EVALUATE: A persistent oscillation with a period of about 1 second makes for a very unpleasant ride. The purpose of shock absorbers is to make such oscillations die out (see Section 13.7).

Angular SHM

A mechanical watch keeps time based on the oscillations of a balance wheel (Fig. 13.19). The wheel has a moment of inertia I about its axis. A coil spring exerts a restoring torque τ_z that is proportional to the angular displacement θ from the equilibrium position. We write $\tau_z = -\kappa\theta$, where κ (the Greek letter kappa) is a constant called the *torsion constant*. Using the rotational analog of

13.19 The balance wheel of a mechanical watch. The spring exerts a restoring torque that is proportional to the angular displacement θ, so the motion is angular SHM.



Newton's second law for a rigid body, $\Sigma\tau_z = I\alpha_z = I\,d^2\theta/dt^2$, we can find the equation of motion:

$$-\kappa\theta = I\alpha \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

The form of this equation is exactly the same as Eq. (13.4) for the acceleration in simple harmonic motion, with x replaced by θ and k/m replaced by κ/I. So we are dealing with a form of *angular* simple harmonic motion. The angular frequency ω and frequency f are given by Eqs. (13.10) and (13.11), respectively, with the same replacement:

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \quad \text{(angular SHM)} \tag{13.24}$$

The motion is described by the function

$$\theta = \Theta\cos(\omega t + \phi)$$

where Θ (the Greek letter theta) plays the role of an angular amplitude.

It's a good thing that the motion of a balance wheel *is* simple harmonic. If it weren't, the frequency might depend on the amplitude, and the watch would run too fast or too slow as the spring ran down.

*Vibrations of Molecules

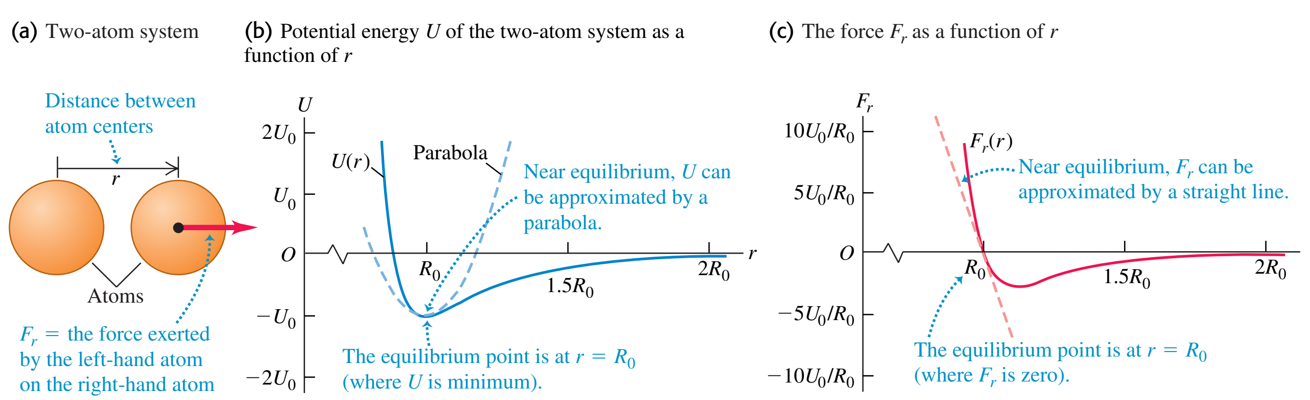
The following discussion of the vibrations of molecules uses the binomial theorem. If you aren't familiar with this theorem, you should read about it in the appropriate section of a math textbook.

When two atoms are separated from each other by a few atomic diameters, they can exert attractive forces on each other. But if the atoms are so close to each other that their electron shells overlap, the forces between the atoms are repulsive. Between these limits, there can be an equilibrium separation distance at which two atoms form a *molecule.* If these atoms are displaced slightly from equilibrium, they will oscillate.

As an example, we'll consider one type of interaction between atoms called the *van der Waals interaction.* Our immediate task here is to study oscillations, so we won't go into the details of how this interaction arises. Let the center of one atom be at the origin and let the center of the other atom be a distance r away (Fig. 13.20a); the equilibrium distance between centers is $r = R_0$. Experiment shows that the van der Waals interaction can be described by the potential-energy function

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right] \tag{13.25}$$

13.20 (a) Two atoms with centers separated by r. (b) Potential energy U in the van der Waals interaction as a function of r. (c) Force F_r on the right-hand atom as a function of r.



where U_0 is a positive constant with units of joules. When the two atoms are very far apart, $U = 0$; when they are separated by the equilibrium distance $r = R_0$, $U = -U_0$. The force on the second atom is the negative derivative of Eq. (13.25):

$$F_r = -\frac{dU}{dr} = U_0\left[\frac{12R_0^{12}}{r^{13}} - 2\frac{6R_0^6}{r^7}\right] = 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7}\right] \tag{13.26}$$

The potential energy and force are plotted in Figs. 13.20b and 13.20c, respectively. The force is positive for $r < R_0$ and negative for $r > R_0$, so it is a *restoring* force.

Let's examine the restoring force F_r in Eq. (13.26). We let x represent the displacement from equilibrium:

$$x = r - R_0 \quad \text{so} \quad r = R_0 + x$$

In terms of x, the force F_r in Eq. (13.26) becomes

$$\begin{aligned} F_r &= 12\frac{U_0}{R_0}\left[\left(\frac{R_0}{R_0+x}\right)^{13} - \left(\frac{R_0}{R_0+x}\right)^{7}\right] \\ &= 12\frac{U_0}{R_0}\left[\frac{1}{(1+x/R_0)^{13}} - \frac{1}{(1+x/R_0)^{7}}\right] \end{aligned} \tag{13.27}$$

This looks nothing like Hooke's law, $F_x = -kx$, so we might be tempted to conclude that molecular oscillations cannot be SHM. But let us restrict ourselves to *small-amplitude* oscillations so that the absolute value of the displacement x is small in comparison to R_0 and the absolute value of the ratio x/R_0 is much less than 1. We can then simplify Eq. (13.27) by using the *binomial theorem:*

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots \tag{13.28}$$

If $|u|$ is much less than 1, each successive term in Eq. (13.28) is much smaller than the one it follows, and we can safely approximate $(1+u)^n$ by just the first two terms. In Eq. (13.27), u is replaced by x/R_0 and n equals -13 or -7, so

$$\frac{1}{(1+x/R_0)^{13}} = (1+x/R_0)^{-13} \approx 1 + (-13)\frac{x}{R_0}$$

$$\frac{1}{(1+x/R_0)^{7}} = (1+x/R_0)^{-7} \approx 1 + (-7)\frac{x}{R_0}$$

$$F_r \approx 12\frac{U_0}{R_0}\left[\left(1 + (-13)\frac{x}{R_0}\right) - \left(1 + (-7)\frac{x}{R_0}\right)\right] = -\left(\frac{72U_0}{R_0^2}\right)x \tag{13.29}$$

This is just Hooke's law, with force constant $k = 72U_0/R_0^2$. (Note that k has the correct units, J/m^2 or N/m.) So oscillations of molecules bound by the van der Waals interaction can be simple harmonic motion, provided that the amplitude is small in comparison to R_0 so that the approximation $|x/R_0| \ll 1$ used in the derivation of Eq. (13.29) is valid.

You can also show that the potential energy U in Eq. (13.25) can be written as $U \approx \frac{1}{2}kx^2 + C$, where $C = -U_0$ and k is again equal to $72U_0/R_0^2$. Adding a constant to the potential energy has no effect on the physics, so the system of two atoms is fundamentally no different from a mass attached to a horizontal spring for which $U = \frac{1}{2}kx^2$. The proof is left to you (see Exercise 13.39).

Example 13.7 Molecular vibration

Two argon atoms can form a weakly bound molecule, Ar_2, held together by a van der Waals interaction with $U_0 = 1.68 \times 10^{-21}$ J and $R_0 = 3.82 \times 10^{-10}$ m. Find the frequency for small oscillations of one of the atoms about its equilibrium position.

SOLUTION

IDENTIFY: This is just the situation shown in Fig. 13.20.

Continued

SET UP: Because the oscillations are small, we can use Eq. (13.11) to obtain the frequency of simple harmonic motion. The force constant is given by Eq. (13.29).

EXECUTE: The force constant is

$$k = \frac{72U_0}{R_0^2} = \frac{72(1.68 \times 10^{-21}\ \text{J})}{(3.82 \times 10^{-10}\ \text{m})^2} = 0.829\ \text{J/m}^2 = 0.829\ \text{N/m}$$

This is comparable to the force constant of a loose, floppy toy spring like a Slinky™.

From the periodic table of the elements (see Appendix D), the average atomic mass of argon is

$$(39.948\ \text{u})(1.66 \times 10^{-27}\ \text{kg/1 u}) = 6.63 \times 10^{-26}\ \text{kg}.$$

If one of the argon atoms is fixed and the other atom oscillates, the frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{0.829\ \text{N/m}}{6.63 \times 10^{-26}\ \text{kg}}} = 5.63 \times 10^{11}\ \text{Hz}$$

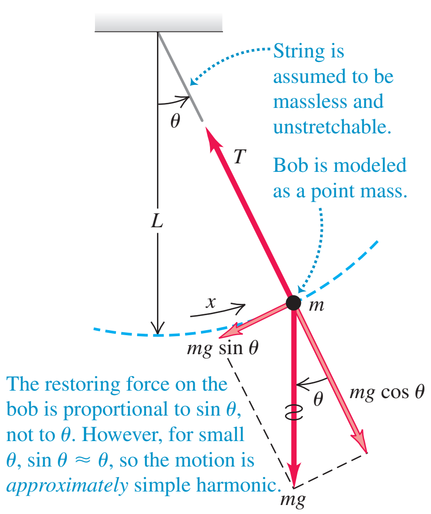
The oscillating mass is very small, so even a floppy spring causes very rapid oscillations.

EVALUATE: Our answer for f isn't quite right. If there is no net external force acting on the molecule, the center of mass of the molecule (located halfway between the two atoms) doesn't accelerate. To ensure this, both atoms must oscillate with the same amplitude in opposite directions. It turns out that we can account for this by replacing m with $m/2$ in the expression for f. (See Problem 13.86.) This makes f larger by a factor of $\sqrt{2}$, so $f = \sqrt{2}(5.63 \times 10^{11}\ \text{Hz}) = 7.96 \times 10^{11}$ Hz. An additional complication is that on the atomic scale we must use quantum mechanics, not Newtonian mechanics, to describe motion; happily, the frequency has the same value in quantum mechanics.

13.21 The dynamics of a simple pendulum.

(a) A real pendulum

(b) An idealized simple pendulum



Test Your Understanding of Section 13.4 A block attached to a hanging ideal spring oscillates up and down with a period of 10 s on earth. If you take the block and spring to Mars, where the acceleration due to gravity is only about 40% as large as on earth, what will be the new period of oscillation? (i) 10 s; (ii) more than 10 s; (iii) less than 10 s.

13.5 The Simple Pendulum

A **simple pendulum** is an idealized model consisting of a point mass suspended by a massless, unstretchable string. When the point mass is pulled to one side of its straight-down equilibrium position and released, it oscillates about the equilibrium position. Familiar situations such as a wrecking ball on a crane's cable or a person on a swing (Fig. 13.21a) can be modeled as simple pendulums.

The path of the point mass (sometimes called a pendulum bob) is not a straight line but the arc of a circle with radius L equal to the length of the string (Fig. 13.21b). We use as our coordinate the distance x measured along the arc. If the motion is simple harmonic, the restoring force must be directly proportional to x or (because $x = L\theta$) to θ. Is it?

In Fig. 13.21b we represent the forces on the mass in terms of tangential and radial components. The restoring force F_θ is the tangential component of the net force:

$$F_\theta = -mg\sin\theta \qquad (13.30)$$

The restoring force is provided by gravity; the tension T merely acts to make the point mass move in an arc. The restoring force is proportional *not* to θ but to $\sin\theta$, so the motion is *not* simple harmonic. However, if the angle θ is *small*, $\sin\theta$ is very nearly equal to θ in radians (Fig. 13.22). For example, when $\theta = 0.1$ rad (about 6°), $\sin\theta = 0.0998$, a difference of only 0.2%. With this approximation, Eq. (13.30) becomes

$$F_\theta = -mg\theta = -mg\frac{x}{L} \quad \text{or}$$

$$F_\theta = -\frac{mg}{L}x \qquad (13.31)$$

The restoring force is then proportional to the coordinate for small displacements, and the force constant is $k = mg/L$. From Eq. (13.10) the angular frequency ω of a simple pendulum with small amplitude is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)} \qquad (13.32)$$

The corresponding frequency and period relationships are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)} \qquad (13.33)$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \quad \text{(simple pendulum, small amplitude)} \qquad (13.34)$$

? Note that these expressions do not involve the *mass* of the particle. This is because the restoring force, a component of the particle's weight, is proportional to m. Thus the mass appears on *both* sides of $\Sigma\vec{F} = m\vec{a}$ and cancels out. (This is the same physics that explains why bodies of different masses fall with the same acceleration in a vacuum.) For small oscillations, the period of a pendulum for a given value of g is determined entirely by its length.

The dependence on L and g in Eqs. (13.32) through (13.34) is just what we should expect. A long pendulum has a longer period than a shorter one. Increasing g increases the restoring force, causing the frequency to increase and the period to decrease.

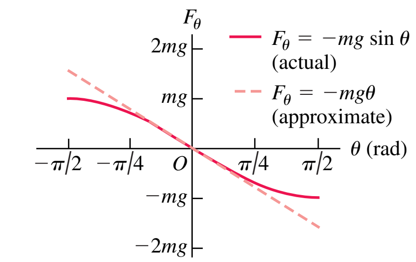
We emphasize again that the motion of a pendulum is only *approximately* simple harmonic. When the amplitude is not small, the departures from simple harmonic motion can be substantial. But how small is "small"? The period can be expressed by an infinite series; when the maximum angular displacement is Θ, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}\sin^4\frac{\Theta}{2} + \cdots\right) \qquad (13.35)$$

We can compute the period to any desired degree of precision by taking enough terms in the series. We invite you to check that when $\Theta = 15°$ (on either side of the central position), the true period is longer than that given by the approximate Eq. (13.34) by less than 0.5%.

The usefulness of the pendulum as a timekeeper depends on the period being *very nearly* independent of amplitude, provided that the amplitude is small. Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps very nearly correct time.

13.22 For small angular displacements θ, the restoring force $F_\theta = -mg\sin\theta$ on a simple pendulum is approximately equal to $-mg\theta$; that is, it is approximately proportional to the displacement θ. Hence for small angles the oscillations are simple harmonic.



ActivPhysics ONLINE
9.10 Pendulum Frequency
9.11 Risky Pendulum Walk
9.12 Physical Pendulum

Example 13.8 A simple pendulum

Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.800\ \text{m/s}^2$.

SOLUTION

IDENTIFY: Since this is a simple pendulum, we can use the ideas of this section.

SET UP: We use Eq. (13.34) to determine the period T of the pendulum from its length, and Eq. (13.1) to find the frequency f from T.

EXECUTE: From Eqs. (13.34) and (13.1),

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000\ \text{m}}{9.800\ \text{m/s}^2}} = 2.007\ \text{s}$$

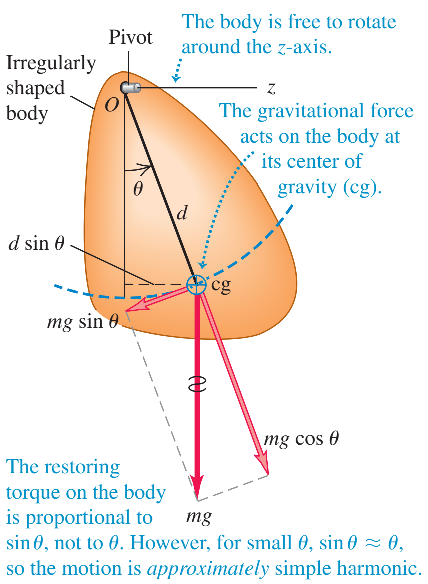
$$f = \frac{1}{T} = \frac{1}{2.007\text{s}} = 0.4983\ \text{Hz}$$

EVALUATE: The period is almost exactly 2 s. In fact, when the metric system was first established, the second was defined as half the period of a 1-meter pendulum. This wasn't a very good standard for time, however, because the value of g varies from place to place. We discussed more modern time standards in Section 1.3.

Test Your Understanding of Section 13.5 When a body oscillating on a horizontal spring passes through its equilibrium position, its acceleration is zero (see Fig. 13.2b). When the bob of an oscillating simple pendulum passes through its equilibrium position, is its acceleration zero?

13.6 The Physical Pendulum

13.23 Dynamics of a physical pendulum.



A **physical pendulum** is any *real* pendulum that uses an extended body, as contrasted to the idealized model of the *simple* pendulum with all the mass concentrated at a single point. For small oscillations, analyzing the motion of a real, physical pendulum is almost as easy as for a simple pendulum. Figure 13.23 shows a body of irregular shape pivoted so that it can turn without friction about an axis through point O. In the equilibrium position the center of gravity is directly below the pivot; in the position shown in the figure, the body is displaced from equilibrium by an angle θ, which we use as a coordinate for the system. The distance from O to the center of gravity is d, the moment of inertia of the body about the axis of rotation through O is I, and the total mass is m. When the body is displaced as shown, the weight mg causes a restoring torque

$$\tau_z = -(mg)(d\sin\theta) \qquad (13.36)$$

The negative sign shows that the restoring torque is clockwise when the displacement is counterclockwise, and vice versa.

When the body is released, it oscillates about its equilibrium position. The motion is not simple harmonic because the torque τ_z is proportional to $\sin\theta$ rather than to θ itself. However, if θ is small, we can approximate $\sin\theta$ by θ in radians, just as we did in analyzing the simple pendulum. Then the motion is *approximately* simple harmonic. With this approximation,

$$\tau_z = -(mgd)\theta$$

The equation of motion is $\sum\tau_z = I\alpha_z$, so

$$-(mgd)\theta = I\alpha_z = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta \qquad (13.37)$$

Comparing this with Eq. (13.4), we see that the role of (k/m) for the spring-mass system is played here by the quantity (mgd/I). Thus the angular frequency is

$$\omega = \sqrt{\frac{mgd}{I}} \quad \text{(physical pendulum, small amplitude)} \qquad (13.38)$$

The frequency f is $1/2\pi$ times this, and the period T is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad \text{(physical pendulum, small amplitude)} \qquad (13.39)$$

Equation (13.39) is the basis of a common method for experimentally determining the moment of inertia of a body with a complicated shape. First locate the center of gravity of the body by balancing. Then suspend the body so that it is free to oscillate about an axis, and measure the period T of small-amplitude oscillations. Finally, use Eq. (13.39) to calculate the moment of inertia I of the body about this axis from T, the body's mass m, and the distance d from the axis to the center of gravity (see Exercise 13.49). Biomechanics researchers use this method to find the moments of inertia of an animal's limbs. This information is important for analyzing how an animal walks, as we'll see in the second of the two following examples.

Example 13.9 Physical pendulum versus simple pendulum

Suppose the body in Fig. 13.23 is a uniform rod with length L, pivoted at one end. Find the period of its motion.

SOLUTION

IDENTIFY: Our target variable is the oscillation period of a rod, which acts as a physical pendulum. We need to know the rod's moment of inertia to do this.

SET UP: We use Table 9.2 (Section 9.4) to find the moment of inertia of the rod, and then substitute this value into Eq. (13.39) to determine the period of oscillation.

EXECUTE: From Table 9.2, the moment of inertia of a uniform rod about an axis through one end is $I = \frac{1}{3}ML^2$. The distance from the pivot to the center of gravity is $d = L/2$. From Eq. (13.39),

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{MgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

EVALUATE: If the rod is a meter stick $(L = 1.00\text{ m})$ and $g = 9.80\text{ m/s}^2$, then

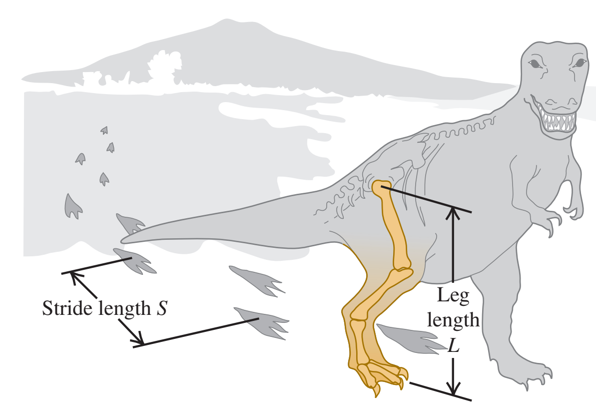
$$T = 2\pi\sqrt{\frac{2(1.00\text{ m})}{3(9.80\text{ m/s}^2)}} = 1.64\text{ s}$$

The period is smaller by a factor of $\sqrt{2/3} = 0.816$ than the period of a simple pendulum with the same length, calculated in Example 13.8. The cg of the rod is half as far from the pivot as the cg of the simple pendulum, which means the torque is half as great. By itself that would give the rod a period $\sqrt{2}$ times greater than the simple pendulum. But the rod's moment of inertia around one end, $I = \frac{1}{3}ML^2$, is one-third that of the simple pendulum, which by itself would make the rod's period $\sqrt{1/3}$ that of the simple pendulum. The moment of inertia factor is more important in this case, which is why the rod has a shorter period than the simple pendulum.

Example 13.10 *Tyrannosaurus rex* and the physical pendulum

All walking animals, including humans, have a natural walking pace—that is, a number of steps per minute that is more comfortable than a faster or slower pace. Suppose this natural pace corresponds to the oscillation of the leg as a physical pendulum. (a) How does the natural walking pace depend on the length L of the leg, measured from hip to foot? Treat the leg as a uniform rod pivoted at the hip joint. (b) Fossil evidence shows that *Tyrannosaurus rex,* a two-legged dinosaur that lived about 65 million years ago at the end of the Cretaceous period, had a leg length $L = 3.1$ m and a stride length $S = 4.0$ m (the distance from one footprint to the next print of the same foot; Fig. 13.24). Estimate the walking speed of *T. rex.*

13.24 The walking speed of *Tyrannosaurus rex* can be estimated from leg length L and stride length S.



SOLUTION

IDENTIFY: Our target variables are (a) the relationship between the walking pace and the leg length and (b) the walking speed of *T. rex.*

SET UP: We treat the leg as a physical pendulum, with a period of oscillation as found in Example 13.9. The shorter the period, the faster the walking pace. We can find the walking speed from the period and the stride length.

EXECUTE: (a) From Example 13.9 the period of oscillation of the leg is $T = 2\pi\sqrt{2L/3g}$, which is proportional to $\sqrt{L}$. Each period (a complete back-and-forth swing of the leg) corresponds to *two* steps, so the walking pace in steps per unit time is just twice the oscillation frequency $f = 1/T$. Hence the walking pace is proportional to $1/\sqrt{L}$. Animals with short legs (small values of L), such as mice or Chihuahuas, have rapid walking paces; humans, giraffes, and other animals with long legs (large values of L) walk at slower paces.

(b) According to our model for the natural walking pace, the elapsed time for one stride of a walking *Tyrannosaurus rex* is

$$T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2(3.1\text{ m})}{3(9.8\text{ m/s}^2)}} = 2.9\text{ s}$$

The distance moved during this time is the stride length S, so the walking speed is

$$v = \frac{S}{T} = \frac{4.0\text{ m}}{2.9\text{ s}} = 1.4\text{ m/s} = 5.0\text{ km/h} = 3.1\text{ mi/h}$$

This is about the same as a typical human walking speed!

Continued

EVALUATE: Our estimate must be somewhat in error because a uniform rod isn't a very good model for a leg. The legs of many animals, including *T. rex* as well as humans, are tapered; there is a lot more mass between the knee and the hip than between the knee and the foot. Thus the center of mass is less than $L/2$ from the hip; a reasonable guess would be about $L/4$. The moment of inertia is therefore *considerably* less than $ML^2/3$, probably somewhere around $ML^2/15$. Try these numbers out with the analysis of Example 13.9; you'll get a shorter oscillation period and an even faster walking speed for *T. rex*.

Test Your Understanding of Section 13.6 The center of gravity of a simple pendulum of mass m and length L is located at the position of the pendulum bob, a distance L from the pivot point. The center of gravity of a uniform rod of the same mass m and length $2L$ pivoted at one end is also a distance L from the pivot point. How does the period of this uniform rod compare to the period of the simple pendulum? (i) The rod has a longer period; (ii) the rod has a shorter period; (iii) the rod has the same period.

13.7 Damped Oscillations

The idealized oscillating systems we have discussed so far are frictionless. There are no nonconservative forces, the total mechanical energy is constant, and a system set into motion continues oscillating forever with no decrease in amplitude.

Real-world systems always have some dissipative forces, however, and oscillations die out with time unless we replace the dissipated mechanical energy (Fig. 13.25). A mechanical pendulum clock continues to run because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction in the pivot and the gears. But eventually the spring runs down or the weights reach the bottom of their travel. Then no more energy is available, and the pendulum swings decrease in amplitude and stop.

13.25 A swinging bell left to itself will eventually stop oscillating due to damping forces (air resistance and friction at the point of suspension).

The decrease in amplitude caused by dissipative forces is called **damping,** and the corresponding motion is called **damped oscillation.** The simplest case to analyze in detail is a simple harmonic oscillator with a frictional damping force that is directly proportional to the *velocity* of the oscillating body. This behavior occurs in friction involving viscous fluid flow, such as in shock absorbers or sliding between oil-lubricated surfaces. We then have an additional force on the body due to friction, $F_x = -bv_x$, where $v_x = dx/dt$ is the velocity and b is a constant that describes the strength of the damping force. The negative sign shows that the force is always opposite in direction to the velocity. The *net* force on the body is then

$$\sum F_x = -kx - bv_x \qquad (13.40)$$

and Newton's second law for the system is

$$-kx - bv_x = ma_x \qquad \text{or} \qquad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad (13.41)$$

Equation (13.41) is a differential equation for x; it would be the same as Eq. (13.4), the equation for the acceleration in SHM, except for the added term $-b\,dx/dt$. Solving this equation is a straightforward problem in differential equations, but we won't go into the details here. If the damping force is relatively small, the motion is described by

$$x = Ae^{-(b/2m)t}\cos(\omega' t + \phi) \quad \text{(oscillator with little damping)} \qquad (13.42)$$

The angular frequency of oscillation ω' is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{(oscillator with little damping)} \qquad (13.43)$$

You can verify that Eq. (13.42) is a solution of Eq. (13.41) by calculating the first and second derivatives of x, substituting them into Eq. (13.41), and checking whether the left and right sides are equal. This is a straightforward but slightly tedious procedure.

The motion described by Eq. (13.42) differs from the undamped case in two ways. First, the amplitude $Ae^{-(b/2m)t}$ is not constant but decreases with time because of the decreasing exponential factor $e^{-(b/2m)t}$. Figure 13.26 is a graph of Eq. (13.42) for the case $\phi = 0$; it shows that the larger the value of b, the more quickly the amplitude decreases.

Second, the angular frequency ω', given by Eq. (13.43), is no longer equal to $\omega = \sqrt{k/m}$ but is somewhat smaller. It becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \qquad \text{or} \qquad b = 2\sqrt{km} \qquad (13.44)$$

When Eq. (13.44) is satisfied, the condition is called **critical damping.** The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

If b is greater than $2\sqrt{km}$, the condition is called **overdamping.** Again there is no oscillation, but the system returns to equilibrium more slowly than with critical damping. For the overdamped case the solutions of Eq. (13.41) have the form

$$x = C_1e^{-a_1t} + C_2e^{-a_2t}$$

where C_1 and C_2 are constants that depend on the initial conditions and a_1 and a_2 are constants determined by m, k, and b.

When b is less than the critical value, as in Eq. (13.42), the condition is called **underdamping.** The system oscillates with steadily decreasing amplitude.

In a vibrating tuning fork or guitar string, it is usually desirable to have as little damping as possible. By contrast, damping plays a beneficial role in the oscillations of an automobile's suspension system. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it doesn't continue bouncing forever (Fig. 13.27). For optimal passenger comfort, the system should be critically damped or slightly underdamped. Too much damping would be counterproductive; if the suspension is overdamped and the car hits a second bump just after the first one, the springs in the suspension will still be compressed somewhat from the first bump and will not be able to fully absorb the impact.

Energy in Damped Oscillations

In damped oscillations the damping force is nonconservative; the mechanical energy of the system is not constant but decreases continuously, approaching zero after a long time. To derive an expression for the rate of change of energy, we first write an expression for the total mechanical energy E at any instant:

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$

To find the rate of change of this quantity, we take its time derivative:

$$\frac{dE}{dt} = mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt}$$

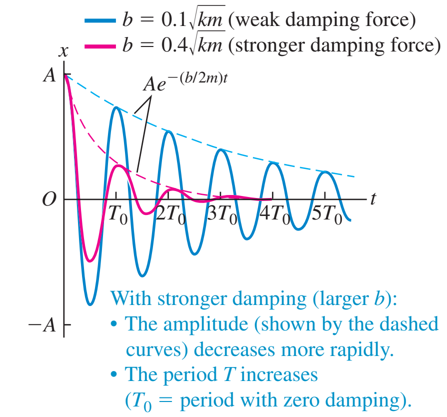
But $dv_x/dt = a_x$ and $dx/dt = v_x$, so

$$\frac{dE}{dt} = v_x(ma_x + kx)$$

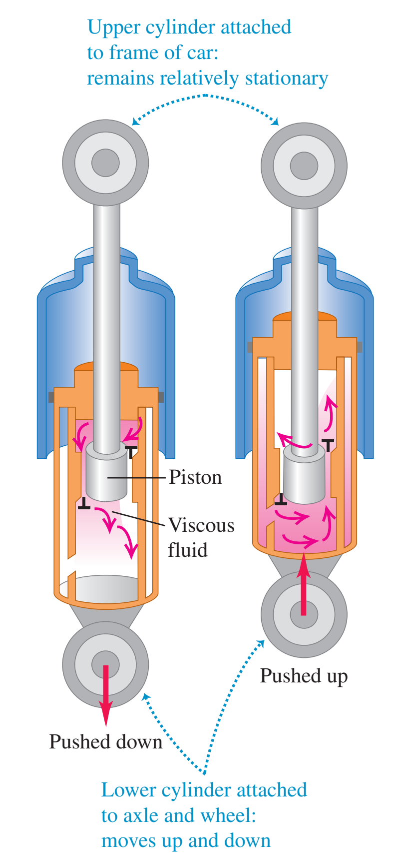
From Eq. (13.41), $ma_x + kx = -b\,dx/dt = -bv_x$, so

$$\frac{dE}{dt} = v_x(-bv_x) = -bv_x^2 \quad \text{(damped oscillations)} \qquad (13.45)$$

13.26 Graph of displacement versus time for an oscillator with little damping [see Eq. (13.42)] and with phase angle $\phi = 0$. The curves are for two values of the damping constant b.



13.27 An automobile shock absorber. The viscous fluid causes a damping force that depends on the relative velocity of the two ends of the unit.



The right side of Eq. (13.45) is **negative** whenever the oscillating body is in motion, whether the x-velocity v_x is positive or negative. This shows that as the body moves, the energy decreases, though not at a uniform rate. The term $-bv_x^2 = (-bv_x)v_x$ (force times velocity) is the rate at which the damping force does (negative) work on the system (that is, the damping *power*). This equals the rate of change of the total mechanical energy of the system.

Similar behavior occurs in electric circuits containing inductance, capacitance, and resistance. There is a natural frequency of oscillation, and the resistance plays the role of the damping constant b. We will study these circuits in detail in Chapters 30 and 31.

Test Your Understanding of Section 13.7 An airplane is flying in a straight line at a constant altitude. If a wind gust strikes and raises the nose of the airplane, the nose will bob up and down until the airplane eventually returns to its original attitude. Are these oscillations (i) undamped, (ii) underdamped, (iii) critically damped, or (iv) overdamped?

13.8 Forced Oscillations and Resonance

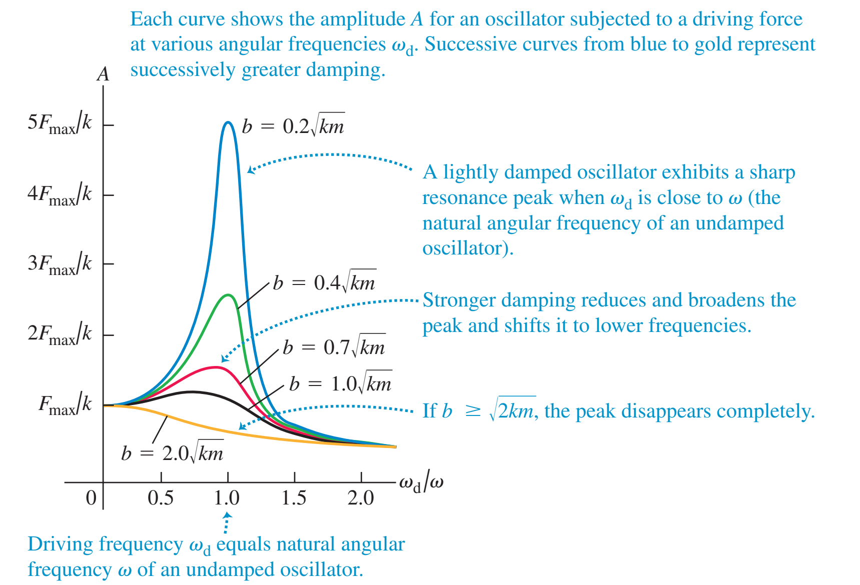
A damped oscillator left to itself will eventually stop moving altogether. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic or cyclic way, with a definite period and frequency. As an example, consider your cousin Throckmorton on a playground swing. You can keep him swinging with constant amplitude by giving him a little push once each cycle. We call this additional force a **driving force.**

Damped Oscillation with a Periodic Driving Force

If we apply a periodically varying driving force with angular frequency ω_d to a damped harmonic oscillator, the motion that results is called a **forced oscillation** or a *driven oscillation.* It is different from the motion that occurs when the system is simply displaced from equilibrium and then left alone, in which case the system oscillates with a **natural angular frequency** ω' determined by m, k, and b, as in Eq. (13.43). In a forced oscillation, however, the angular frequency with which the mass oscillates is equal to the driving angular frequency ω_d. This does *not* have to be equal to the angular frequency ω' with which the system would oscillate without a driving force. If you grab the ropes of Throckmorton's swing, you can force the swing to oscillate with any frequency you like.

Suppose we force the oscillator to vibrate with an angular frequency ω_d that is nearly *equal* to the angular frequency ω' it would have with no driving force. What happens? The oscillator is naturally disposed to oscillate at $\omega = \omega'$, so we expect the amplitude of the resulting oscillation to be larger than when the two frequencies are very different. Detailed analysis and experiment shows that this is just what happens. The easiest case to analyze is a *sinusoidally* varying force—say, $F(t) = F_{max} \cos \omega_d t$. If we vary the frequency ω_d of the driving force, the amplitude of the resulting forced oscillation varies in an interesting way (Fig. 13.28). When there is very little damping (small b), the amplitude goes through a sharp peak as the driving angular frequency ω_d nears the natural oscillation angular frequency ω'. When the damping is increased (larger b), the peak becomes broader and smaller in height and shifts toward lower frequencies.

We could work out an expression that shows how the amplitude A of the forced oscillation depends on the frequency of a sinusoidal driving force, with



13.28 Graph of the amplitude A of forced oscillation as a function of the angular frequency ω_d of the driving force. The horizontal axis shows the ratio of ω_d to the angular frequency $\omega = \sqrt{k/m}$ of an undamped oscillator. Each curve has a different value of the damping constant b.

maximum value F_{max}. That would involve more differential equations than we're ready for, but here is the result:

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad \text{(amplitude of a driven oscillator)} \quad (13.46)$$

When $k - m\omega_d^2 = 0$, the first term under the radical is zero, so A has a maximum near $\omega_d = \sqrt{k/m}$. The height of the curve at this point is proportional to $1/b$; the less damping, the higher the peak. At the low-frequency extreme, when $\omega_d = 0$, we get $A = F_{max}/k$. This corresponds to a *constant* force F_{max} and a constant displacement $A = F_{max}/k$ from equilibrium, as we might expect.

Resonance and Its Consequences

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance.** Physics is full of examples of resonance; building up the oscillations of a child on a swing by pushing with a frequency equal to the swing's natural frequency is one. A vibrating rattle in a car that occurs only at a certain engine speed or wheel-rotation speed is an all-too-familiar example. Inexpensive loudspeakers often have an annoying boom or buzz when a musical note happens to coincide with the resonant frequency of the speaker cone or the speaker housing. In Chapter 16 we will study other examples of resonance that involve sound. Resonance also occurs in electric circuits, as we will see in Chapter 31; a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

Resonance in mechanical systems can be destructive. A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural vibration frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge. Some years ago, vibrations of the engines of a particular airplane had just the right frequency to resonate with the natural frequencies of its wings. Large oscillations built up, and occasionally the wings fell off.

Nearly everyone has seen the film of the collapse of the Tacoma Narrows suspension bridge in 1940 (Fig. 13.29). This is usually cited as an example of resonance driven by the wind, but there's some doubt whether it should be called that.

13.29 The Tacoma Narrows Bridge collapsed four months and six days after it was opened for traffic. The main span was 2800 ft long and 39 ft wide, with 8-ft-high steel stiffening girders on both sides. The maximum amplitude of the torsional vibrations was 35°; the frequency was about 0.2 Hz.

The wind didn't have to vary *periodically* with a frequency close to a natural frequency of the bridge. The airflow past the bridge was turbulent, and vortices were formed in the air with a regular frequency that depended on the flow speed. It is conceivable that this frequency may have coincided with a natural frequency of the bridge. But the cause may well have been something more subtle called a *self-excited oscillation,* in which the aerodynamic forces caused by a *steady* wind blowing on the bridge tended to displace it farther from equilibrium at times when it was already moving away from equilibrium. It is as though we had a damping force such as the $-bv_x$ term in Eq. (13.40) but with the sign reversed. Instead of draining mechanical energy away from the system, this anti-damping force pumps energy into the system, building up the oscillations to destructive amplitudes. The approximate differential equation is Eq. (13.41) with the sign of the b term reversed, and the oscillating solution is Eq. (13.42) with a *positive* sign in the exponent. You can see that we're headed for trouble. Engineers have learned how to stabilize suspension bridges, both structurally and aerodynamically, to prevent such disasters.

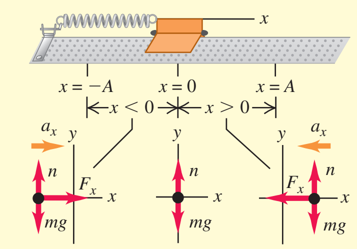
Test Your Understanding of Section 13.8 When driven at a frequency near its natural frequency, an oscillator with very little damping has a much greater response than the same oscillator with more damping. When driven at a frequency that is much higher or lower than the natural frequency, which oscillator will have the greater response: (i) the one with very little damping or (ii) the one with more damping? MP

CHAPTER 13 SUMMARY

Periodic motion: Periodic motion is motion that repeats itself in a definite cycle. It occurs whenever a body has a stable equilibrium position and a restoring force that acts when it is displaced from equilibrium. Period T is the time for one cycle. Frequency f is the number of cycles per unit time. Angular frequency ω is 2π times the frequency. (See Example 13.1.)

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \tag{13.1}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{13.2}$$



Simple harmonic motion: If the restoring force F_x in periodic motion is directly proportional to the displacement x, the motion is called simple harmonic motion (SHM). In many cases this condition is satisfied if the displacement from equilibrium is small. The angular frequency, frequency, and period in SHM do not depend on the amplitude, but only on the mass m and force constant k. The displacement, velocity, and acceleration in SHM are sinusoidal functions of time; the amplitude A and phase angle ϕ of the oscillation are determined by the initial position and velocity of the body. (See Examples 13.2, 13.3, 13.6, and 13.7.)

$$F_x = -kx \tag{13.3}$$

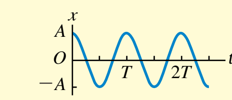
$$a_x = \frac{F_x}{m} = -\frac{k}{m}x \tag{13.4}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{13.10}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{13.11}$$

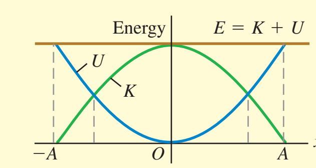
$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{13.12}$$

$$x = A\cos(\omega t + \phi) \tag{13.13}$$



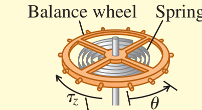
Energy in simple harmonic motion: Energy is conserved in SHM. The total energy can be expressed in terms of the force constant k and amplitude A. (See Examples 13.4 and 13.5.)

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant} \tag{13.21}$$



Angular simple harmonic motion: In angular SHM, the frequency and angular frequency are related to the moment of inertia I and the torsion constant κ.

$$\omega = \sqrt{\frac{\kappa}{I}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \tag{13.24}$$

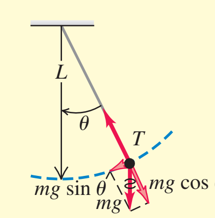


Simple pendulum: A simple pendulum consists of a point mass m at the end of a massless string of length L. Its motion is approximately simple harmonic for sufficiently small amplitude; the angular frequency, frequency, and period then depend only on g and L, not on the mass or amplitude. (See Example 13.8.)

$$\omega = \sqrt{\frac{g}{L}} \tag{13.32}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{13.33}$$

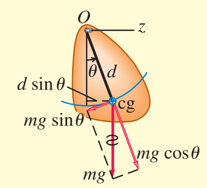
$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \tag{13.34}$$



Physical pendulum: A physical pendulum is any body suspended from an axis of rotation. The angular frequency and period for small-amplitude oscillations are independent of amplitude, but depend on the mass m, distance d from the axis of rotation to the center of gravity, and moment of inertia I about the axis. (See Examples 13.9 and 13.10.)

$$\omega = \sqrt{\frac{mgd}{I}} \quad (13.38)$$

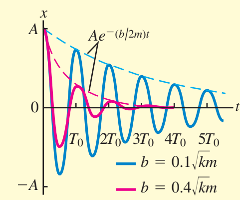
$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad (13.39)$$



Damped oscillations: When a force $F_x = -bv_x$ proportional to velocity is added to a simple harmonic oscillator, the motion is called a damped oscillation. If $b < 2\sqrt{km}$ (called underdamping), the system oscillates with a decaying amplitude and an angular frequency ω' that is lower than it would be without damping. If $b = 2\sqrt{km}$ (called critical damping) or $b > 2\sqrt{km}$ (called overdamping), when the system is displaced it returns to equilibrium without oscillating.

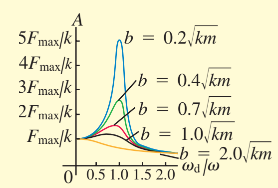
$$x = Ae^{-(b/2m)t}\cos\omega' t \quad (13.42)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad (13.43)$$



Driven oscillations and resonance: When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a forced oscillation. The amplitude is a function of the driving frequency ω_d and reaches a peak at a driving frequency close to the natural frequency of the system. This behavior is called resonance.

$$A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad (13.46)$$



Key Terms

periodic motion (oscillation), *419*
displacement, *420*
restoring force, *420*
amplitude, *420*
cycle, *420*
period, *420*
frequency, *420*
angular frequency, *420*
simple harmonic motion (SHM), *421*
harmonic oscillator, *422*
reference circle, *423*
phasor, *423*
phase angle, *426*
simple pendulum, *436*
physical pendulum, *438*
damping, *440*
damped oscillation, *440*
critical damping, *441*
overdamping, *441*
underdamping, *441*
driving force, *442*
forced oscillation, *442*
natural angular frequency, *442*
resonance, *443*

Answer to Chapter Opening Question ?

Neither—the clock would still keep time correctly. If its rod has negligible mass, then the pendulum is a simple pendulum and its period is independent of the mass [see Eq. (13.34)]. If the rod's mass is included, the pendulum is a physical pendulum. Doubling its mass m also doubles its moment of inertia I, so the ratio I/m is unchanged and the period $T = 2\pi\sqrt{I/mgd}$ [Eq. (13.39)] remains the same.

Answers to Test Your Understanding Questions

13.1 Answers: (a) $x < 0$, (b) $x > 0$, (c) $x < 0$, (d) $x > 0$, (e) $x = 0$, (f) $x > 0$ Figure 13.2 shows that the net x-component of force F_x and the x-acceleration a_x are both positive when $x < 0$ (so the body is displaced to the left and the spring is compressed), while F_x and a_x are both negative when $x > 0$ (so the body is displaced to the right and the spring is stretched). Hence x and a_x always have *opposite* signs. This is true whether the object is moving to the right $(v_x > 0)$, to the left $(v_x < 0)$, or not at all $(v_x = 0)$, since the force exerted by the spring depends only on whether it is compressed or stretched and by what distance. This explains the answers to (a) through (e). If the acceleration is zero as in (f), the net force must also be zero and so the spring must be relaxed; hence $x = 0$.

13.2 Answers: (a) $A > 0.10$ m, $\phi < 0$; (b) $A > 0.10$ m, $\phi > 0$ In both situations the initial $(t = 0)$ x-velocity v_{0x} is nonzero, so from Eq. (13.19) the amplitude $A = \sqrt{x_0^2 + (v_{0x}^2/\omega^2)}$ is greater than the initial x-coordinate $x_0 = 0.10$ m. From Eq. (13.18) the phase angle is $\phi = \arctan(-v_{0x}/\omega x_0)$, which is positive if the quantity $-v_{0x}/\omega x_0$ (the argument of the arctangent function) is positive and negative if $-v_{0x}/\omega x_0$ is negative. In part (a) x_0 and v_{0x} are both positive, so $-v_{0x}/\omega x_0 < 0$ and $\phi < 0$. In part (b) x_0 is positive and v_{0x} is negative, so $-v_{0x}/\omega x_0 > 0$ and $\phi > 0$.

13.3 Answers: (a) (iii), (b) (v) To increase the total energy $E = \frac{1}{2}kA^2$ by a factor of 2, the amplitude A must increase by a factor of $\sqrt{2}$. Because the motion is SHM, changing the amplitude has no effect on the frequency.

13.4 Answer: (i) The oscillation period of a body of mass m attached to a hanging spring of force constant k is given by $T = 2\pi\sqrt{m/k}$, the same expression as for a body attached to a horizontal spring. Neither m nor k changes when the apparatus is taken to Mars, so the period is unchanged. The only difference is that in equilibrium, the spring will stretch a shorter distance on Mars than on earth due to the weaker gravity.

13.5 Answer: no Just as for an object oscillating on a spring, at the equilibrium position the *speed* of the pendulum bob is instantaneously not changing (this is where the speed is maximum, so its derivative at this time is zero). But the *direction* of motion is changing because the pendulum bob follows a circular path. Hence the bob must have a component of acceleration perpendicular to the path and toward the center of the circle (see Section 3.4). To cause this acceleration at the equilibrium position when the string is vertical, the upward tension force at this position must be greater than the weight of the bob. This causes a net upward force on the bob and an upward acceleration toward the center of the circular path.

13.6 Answer: (i) The period of a physical pendulum is given by Eq. (13.39), $T = 2\pi\sqrt{I/mgd}$. The distance $d = L$ from the pivot to the center of gravity is the same for both the rod and the simple pendulum, as is the mass m. This means that for any displacement angle θ the same restoring torque acts on both the rod and the simple pendulum. However, the rod has a greater moment of inertia: $I_{rod} = \frac{1}{3}m(2L)^2 = \frac{4}{3}mL^2$ and $I_{simple} = mL^2$ (all the mass of the pendulum is a distance L from the pivot). Hence the rod has a longer period.

13.7 Answer: (ii) The oscillations are underdamped with a decreasing amplitude on each cycle of oscillation, like those graphed in Fig. 13.26. If the oscillations were undamped, they would continue indefinitely with the same amplitude. If they were critically damped or overdamped, the nose would not bob up and down but would return smoothly to the original equilibrium attitude without overshooting.

13.8 Answer: (i) Figure 13.28 shows that the curve of amplitude versus driving frequency moves upward at *all* frequencies as the value of the damping constant b is decreased. Hence for fixed values of k and m, the oscillator with the least damping (smallest value of b) will have the greatest response at any driving frequency.

PROBLEMS

Discussion Questions

Q13.1. An object is moving with SHM of amplitude A on the end of a spring. If the amplitude is doubled, what happens to the total distance the object travels in one period? What happens to the period? What happens to the maximum speed of the object? Discuss how these answers are related.

Q13.2. Think of several examples in everyday life of motions that are, at least approximately, simple harmonic. In what respects does each differ from SHM?

Q13.3. Does a tuning fork or similar tuning instrument undergo SHM? Why is this a crucial question for musicians?

Q13.4. A box containing a pebble is attached to an ideal horizontal spring and is oscillating on a friction-free air table. When the box has reached its maximum distance from the equilibrium point, the pebble is suddenly lifted out vertically without disturbing the box. Will the following characteristics of the motion increase, decrease, or remain the same in the subsequent motion of the box? Justify each answer. (a) frequency; (b) period; (c) amplitude; (d) the maximum kinetic energy of the box; (e) the maximum speed of the box.

Q13.5. If a uniform spring is cut in half, what is the force constant of each half? Justify your answer. How would the frequency of SHM using a half-spring differ from the frequency using the same mass and the entire spring?

Q13.6. The analysis of SHM in this chapter ignored the mass of the spring. How does the spring's mass change the characteristics of the motion?

Q13.7. Two identical gliders on an air track are connected by an ideal spring. Could such a system undergo SHM? Explain. How would the period compare with that of a single glider attached to a spring whose other end is rigidly attached to a stationary object? Explain.

Q13.8. You are captured by Martians, taken into their ship, and put to sleep. You awake some time later and find yourself locked in a small room with no windows. All the martians have left you with is your digital watch, your school ring, and your long silver-chain necklace. Explain how you can determine whether you are still on earth or have been transported to Mars.

Q13.9. The system shown in Fig. 13.17 is mounted in an elevator. What happens to the period of the motion (does it increase, decrease, or remain the same) if the elevator (a) accelerates upward at 5.0 m/s^2; (b) moves upward at a steady 5.0 m/s; (c) accelerates downward at 5.0 m/s^2? Justify your answers.

Q13.10. If a pendulum has a period of 2.5 s on earth, what would be its period in a space station orbiting the earth? If a mass hung from a vertical spring has a period of 5.0 s on earth, what would its period be in the space station? Justify each of your answers.

Q13.11. A simple pendulum is mounted in an elevator. What happens to the period of the pendulum (does it increase, decrease, or remain the same) if the elevator (a) accelerates upward at 5.0 m/s^2, (b) moves upward at a steady 5.0 m/s, (c) accelerates downward at 5.0 m/s^2, (d) accelerates downward at 9.8 m/s^2? Justify your answers.

Q13.12. What should you do to the length of the string of a simple pendulum to (a) double its frequency; (b) double its period; (c) double its angular frequency?

Q13.13. If a pendulum clock is taken to a mountaintop, does it gain or lose time, assuming it is correct at a lower elevation? Explain your answer.

Q13.14. When the amplitude of a simple pendulum increases, should its period increase or decrease? Give a qualitative argument; do not rely on Eq. (13.35). Is your argument also valid for a physical pendulum?

Q13.15. Why do short dogs (like Chihuahuas) walk with quicker strides than do tall dogs (like Great Danes)?

Q13.16. At what point in the motion of a simple pendulum is the string tension greatest? Least? In each case give the reasoning behind your answer.

Q13.17. Could a standard of time be based on the period of a certain standard pendulum? What advantages and disadvantages would such a standard have compared to the actual present-day standard discussed in Section 1.3?

Q13.18. For a simple pendulum, clearly distinguish between ω (the angular velocity) and ω (the angular frequency). Which is constant and which is variable?

Q13.19. A glider is attached to a fixed ideal spring and oscillates on a horizontal, friction-free air track. A coin is atop the glider and oscillating with it. At what points in the motion is the friction force on the coin greatest? At what points is it least? Justify your answers.

Q13.20. In designing structures in an earthquake-prone region, how should the natural frequencies of oscillation of a structure relate to typical earthquake frequencies? Why? Should the structure have a large or small amount of damping?

Exercises

Section 13.1 Describing Oscillation

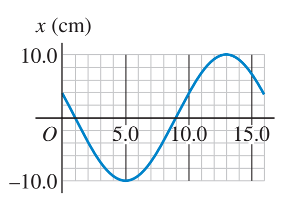
13.1. A piano string sounds a middle A by vibrating primarily at 220 Hz. (a) Calculate the string's period and angular frequency. (b) Calculate the period and angular frequency for a soprano singing an A one octave higher, which is twice the frequency of the piano string.

13.2. If an object on a horizontal, frictionless surface is attached to a spring, displaced, and then released, it will oscillate. If it is displaced 0.120 m from its equilibrium position and released with zero initial speed, then after 0.800 s its displacement is found to be 0.120 m on the opposite side, and it has passed the equilibrium position once during this interval. Find (a) the amplitude; (b) the period; (c) the frequency.

13.3. The tip of a tuning fork goes through 440 complete vibrations in 0.500 s. Find the angular frequency and the period of the motion.

13.4. The displacement of an oscillating object as a function of time is shown in Fig. 13.30. What are (a) the frequency; (b) the amplitude; (c) the period; (d) the angular frequency of this motion?

Figure 13.30 Exercise 13.4.



Section 13.2 Simple Harmonic Motion

13.5. A machine part is undergoing SHM with a frequency of 5.00 Hz and amplitude 1.80 cm. How long does it take the part to go from $x = 0$ to $x = -1.80$ cm?

13.6. In a physics lab, you attach a 0.200-kg air-track glider to the end of an ideal spring of negligible mass and start it oscillating. The elapsed time from when the glider first moves through the equilibrium point to the second time it moves through that point is 2.60 s. Find the spring's force constant.

13.7. When a body of unknown mass is attached to an ideal spring with force constant 120 N/m, it is found to vibrate with a frequency of 6.00 Hz. Find (a) the period of the motion; (b) the angular frequency; (c) the mass of the body.

13.8. When a 0.750-kg mass oscillates on an ideal spring, the frequency is 1.33 Hz. What will the frequency be if 0.220 kg are added to the original mass, and (b) subtracted from the original mass? Try to solve this problem *without* finding the force constant of the spring.

13.9. A harmonic oscillator consists of a 0.500-kg mass attached to an ideal spring with force constant 140 N/m. Find (a) the period; (b) the frequency; (c) the angular frequency of the oscillations.

13.10. Jerk. A guitar string vibrates at a frequency of 440 Hz. A point at its center moves in SHM with an amplitude of 3.0 mm and a phase angle of zero. (a) Write an equation for the position of the center of the string as a function of time. (b) What are the maximum values of the magnitudes of the velocity and acceleration of the center of the string? (c) The derivative of the acceleration with respect to time is a quantity called the *jerk*. Write an equation for the jerk of the center of the string as a function of time, and find the maximum value of the magnitude of the jerk.

13.11. A 2.00-kg, frictionless block is attached to an ideal spring with force constant 300 N/m. At $t = 0$ the spring is neither stretched nor compressed and the block is moving in the negative direction at 12.0 m/s. Find (a) the amplitude and (b) the phase angle. (c) Write an equation for the position as a function of time.

13.12. Repeat Exercise 13.11, but assume that at $t = 0$ the block has velocity -4.00 m/s and displacement $+0.200$ m.

13.13. The point of the needle of a sewing machine moves in SHM along the x-axis with a frequency of 2.5 Hz. At $t = 0$ its position and velocity components are $+1.1$ cm and -15 cm/s, respectively (a) Find the acceleration component of the needle at $t = 0$. (b) Write equations giving the position, velocity, and acceleration components of the point as a function of time.

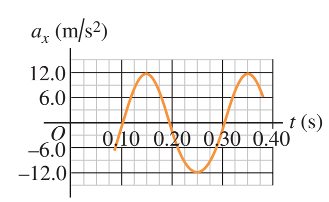
13.14. An object is undergoing SHM with period 1.200 s and amplitude 0.600 m. At $t = 0$ the object is at $x = 0$. How far is the object from the equilibrium position when $t = 0.480$ s?

13.15. Weighing Astronauts. This procedure has actually been used to "weigh" astronauts in space. A 42.5-kg chair is attached to a spring and allowed to oscillate. When it is empty, the chair takes 1.30 s to make one complete vibration. But with an astronaut sitting in it, with her feet off the floor, the chair takes 2.54 s for one cycle. What is the mass of the astronaut?

13.16. A 0.400-kg object undergoing SHM has $a_x = -2.70\ \text{m/s}^2$ when $x = 0.300$ m. What is the time for one oscillation?

13.17. On a frictionless, horizontal air track, a glider oscillates at the end of an ideal spring of force constant 2.50 N/cm. The graph in Fig. 13.31 shows the acceleration of the glider as a function of time. Find (a) the mass of the glider; (b) the maximum displacement of the glider from the equilibrium point; (c) the maximum force the spring exerts on the glider.

Figure 13.31 Exercise 13.17.



13.18. A 0.500-kg mass on a spring has velocity as a function of time given by $v_x(t) = (3.60\ \text{cm/s})\sin[(4.71\ \text{s}^{-1})t - \pi/2]$. What are (a) the period; (b) the amplitude; (c) the maximum acceleration of the mass; (d) the force constant of the spring?

13.19. A 1.50-kg mass on a spring has displacement as a function of time given by the equation

$$x(t) = (7.40\ \text{cm})\cos[(4.16\ \text{s}^{-1})t - 2.42].$$

Find (a) the time for one complete vibration; (b) the force constant of the spring; (c) the maximum speed of the mass; (d) the maximum force on the mass; (e) the position, speed, and acceleration of the mass at $t = 1.00$ s; (f) the force on the mass at that time.

13.20. An object is undergoing SHM with period 0.300 s and amplitude 6.00 cm. At $t = 0$ the object is instantaneously at rest at $x = 6.00$ cm. Calculate the time it takes the object to go from $x = 6.00$ cm to $x = -1.50$ cm.

Section 13.3 Energy in Simple Harmonic Motion

13.21. A tuning fork labeled 392 Hz has the tip of each of its two prongs vibrating with an amplitude of 0.600 mm. (a) What is the maximum speed of the tip of a prong? (b) A housefly (*Musca domestica*) with mass 0.0270 g is holding on to the tip of one of the prongs. As the prong vibrates, what is the fly's maximum kinetic energy? Assume that the fly's mass has a negligible effect on the frequency of oscillation.

13.22. A harmonic oscillator has angular frequency ω and amplitude A. (a) What are the magnitudes of the displacement and velocity when the elastic potential energy is equal to the kinetic energy? (Assume that $U = 0$ at equilibrium.) (b) How often does this occur in each cycle? What is the time between occurrences? (c) At an instant when the displacement is equal to $A/2$, what fraction of the total energy of the system is kinetic and what fraction is potential?

13.23. A 0.500-kg glider, attached to the end of an ideal spring with force constant $k = 450$ N/m, undergoes SHM with an amplitude of 0.040 m. Compute (a) the maximum speed of the glider; (b) the speed of the glider when it is at $x = -0.015$ m; (c) the magnitude of the maximum acceleration of the glider; (d) the acceleration of the glider at $x = -0.015$ m; (e) the total mechanical energy of the glider at any point in its motion.

13.24. A cheerleader waves her pom-pom in SHM with an amplitude of 18.0 cm and a frequency of 0.850 Hz. Find (a) the maximum magnitude of the acceleration and of the velocity; (b) the acceleration and speed when the pom-pom's coordinate is $x = +9.0$ cm; (c) the time required to move from the equilibrium position directly to a point 12.0 cm away. (d) Which of the quantities asked for in parts (a), (b), and (c) can be found using the energy approach used in Section 13.3, and which cannot? Explain.

13.25. For the situation described in part (a) of Example 13.5, what should be the value of the putty mass m so that the amplitude after the collision is one-half the original amplitude? For this value of m, what fraction of the original mechanical energy is converted into heat?

13.26. A 0.150-kg toy is undergoing SHM on the end of a horizontal spring with force constant $k = 300$ N/m. When the object is 0.0120 m from its equilibrium position, it is observed to have a speed of 0.300 m/s. What are (a) the total energy of the object at any point of its motion; (b) the amplitude of the motion; (c) the maximum speed attained by the object during its motion?

13.27. You are watching an object that is moving in SHM. When the object is displaced 0.600 m to the right of its equilibrium position, it has a velocity of 2.20 m/s to the right and an acceleration of 8.40 m/s² to the left. How much farther from this point will the object move before it stops momentarily and then starts to move back to the left?

13.28. On a horizontal, frictionless table, an open-topped 5.20-kg box is attached to an ideal horizontal spring having force constant 375 N/m. Inside the box is a 3.44-kg stone. The system is oscillating with an amplitude of 7.50 cm. When the box has reached its maximum speed, the stone is suddenly plucked vertically out of the box without touching the box. Find (a) the period and (b) the amplitude of the resulting motion of the box. (c) Without doing any calculations, is the new period greater or smaller than the original period? How do you know?

13.29. Inside a NASA test vehicle, a 3.50-kg ball is pulled along by a horizontal ideal spring fixed to a friction-free table. The force constant of the spring is 225 N/m. The vehicle has a steady acceleration of 5.00 m/s², and the ball is not oscillating. Suddenly, when the vehicle's speed has reached 45.0 m/s, its engines turn off, thus eliminating its acceleration but not its velocity. Find (a) the amplitude and (b) the frequency of the resulting oscillations of the ball. (c) What will be the ball's maximum speed relative to the vehicle?

Section 13.4 Applications of Simple Harmonic Motion

13.30. A proud deep-sea fisherman hangs a 65.0-kg fish from an ideal spring having negligible mass. The fish stretches the spring 0.120 m. (a) Find the force constant of the spring. The fish is now pulled down 5.00 cm and released. (b) What is the period of oscillation of the fish? (c) What is the maximum speed it will reach?

13.31. A 175-g glider on a horizontal, frictionless air track is attached to a fixed ideal spring with force constant 155 N/m. At the instant you make measurements on the glider, it is moving at 0.815 m/s and is 3.00 cm from its equilibrium point. Use *energy conservation* to find (a) the amplitude of the motion and (b) the maximum speed of the glider. (c) What is the angular frequency of the oscillations?

13.32. A thrill-seeking cat with mass 4.00 kg is attached by a harness to an ideal spring of negligible mass and oscillates vertically in SHM. The amplitude is 0.050 m, and at the highest point of the motion the spring has its natural unstretched length. Calculate the elastic potential energy of the spring (take it to be zero for the unstretched spring), the kinetic energy of the cat, the gravitational potential energy of the system relative to the lowest point of the motion, and the sum of these three energies when the cat is (a) at its highest point; (b) at its lowest point; (c) at its equilibrium position.

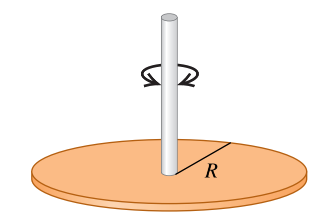
13.33. A 1.50-kg ball and a 2.00-kg ball are glued together with the lighter one below the heavier one. The upper ball is attached to a vertical ideal spring of force constant 165 N/m, and the system is vibrating vertically with amplitude 15.0 cm. The glue connecting the balls is old and weak, and it suddenly comes loose when the balls are at the lowest position in their motion. (a) Why is the glue more likely to fail at the *lowest* point than at any other point in the motion? (b) Find the amplitude and frequency of the vibrations after the lower ball has come loose.

13.34. A uniform, solid metal disk of mass 6.50 kg and diameter 24.0 cm hangs in a horizontal plane, supported at its center by a vertical metal wire. You find that it requires a horizontal force of 4.23 N tangent to the rim of the disk to turn it by 3.34°, thus twisting the wire. You now remove this force and release the disk from rest. (a) What is the torsion constant for the metal wire? (b) What are the frequency and period of the torsional oscillations of the disk? (c) Write the equation of motion for $\theta(t)$ for the disk.

13.35. A certain alarm clock ticks four times each second, with each tick representing half a period. The balance wheel consists of a thin rim with radius 0.55 cm, connected to the balance staff by thin spokes of negligible mass. The total mass of the balance wheel is 0.90 g. (a) What is the moment of inertia of the balance wheel about its shaft? (b) What is the torsion constant of the hairspring?

13.36. A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (Fig. 13.32). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant of the fiber.

Figure 13.32 Exercise 13.36.



13.37. You want to find the moment of inertia of a complicated machine part about an axis through its center of mass. You suspend it from a wire along this axis. The wire has a torsion constant of $0.450\ \text{N} \cdot \text{m/rad}$ You twist the part a small amount about this axis and let it go, timing 125 oscillations in 265 s. What is the moment of inertia you want to find?

13.38. The balance wheel of a watch vibrates with an angular amplitude Θ, angular frequency ω, and phase angle $\phi = 0$. (a) Find expressions for the angular velocity $d\theta/dt$ and angular acceleration $d^2\theta/dt^2$ as functions of time. (b) Find the balance wheel's angular velocity and angular acceleration when its angular displacement is Θ, and when its angular displacement is $\Theta/2$ and θ is decreasing. (*Hint:* Sketch a graph of θ versus t.)

***13.39.** For the van der Waals interaction with potential-energy function given by Eq. (13.25), show that when the magnitude of the displacement x from equilibrium $(r = R_0)$ is small, the potential energy can be written approximately as $U \approx \frac{1}{2}kx^2 - U_0$. [*Hint:* In Eq. (13.25), let $r = R_0 + x$ and $u = x/R_0$. Then approximate $(1 + u)^n$ by the first *three* terms in Eq. (13.28).] How does k in this equation compare with the force constant in Eq. (13.29) for the force?

***13.40.** When displaced from equilibrium, the two hydrogen atoms in an H_2 molecule are acted on by a restoring force $F_x = -kx$ with $k = 580\ \text{N/m}$. Calculate the oscillation frequency of the H_2 molecule. (*Hint:* The mass of a hydrogen atom is 1.008 atomic mass units, or 1 u; see Appendix E. As in Example 13.7 (Section 13.4), use $m/2$ instead of m in the expression for f.)

Section 13.5 The Simple Pendulum

13.41. You pull a simple pendulum 0.240 m long to the side through an angle of 3.50° and release it. (a) How much time does it take the pendulum bob to reach its highest speed? (b) How much time does it take if the pendulum is released at an angle of 1.75° instead of 3.50°?

13.42. An 85.0-kg mountain climber plans to swing down, starting from rest, from a ledge using a light rope 6.50 m long. He holds one end of the rope, and the other end is tied higher up on a rock face. Since the ledge is not very far from the rock face, the rope makes a small angle with the vertical. At the lowest point of his swing, he plans to let go and drop a short distance to the ground. (a) How long after he begins his swing will the climber first reach his lowest point? (b) If he missed the first chance to drop off, how long after first beginning his swing will the climber reach his lowest point for the second time?

13.43. A building in San Francisco has light fixtures consisting of small 2.35-kg bulbs with shades hanging from the ceiling at the end of light thin cords 1.50 m long. If a minor earthquake occurs, how many swings per second will these fixtures make?

13.44. A Pendulum on Mars. A certain simple pendulum has a period on the earth of 1.60 s. What is its period on the surface of Mars, where $g = 3.71\ \text{m/s}^2$?

13.45. An apple weighs 1.00 N. When you hang it from the end of a long spring of force constant 1.50 N/m and negligible mass, it bounces up and down in SHM. If you stop the bouncing and let the apple swing from side to side through a small angle, the frequency of this simple pendulum is half the bounce frequency. (Because the angle is small, the back-and-forth swings do not cause any appreciable change in the length of the spring.) What is the unstretched length of the spring (with the apple removed)?

13.46. A small sphere with mass m is attached to a massless rod of length L that is pivoted at the top, forming a simple pendulum. The pendulum is pulled to one side so that the rod is at an angle Θ from the vertical, and released from rest. (a) In a diagram, show the pendulum just after it is released. Draw vectors representing the *forces* acting on the small sphere and the *acceleration* of the sphere. Accuracy counts! At this point, what is the linear acceleration of the sphere? (b) Repeat part (a) for the instant when the pendulum rod is at an angle $\Theta/2$ from the vertical. (c) Repeat part (a) for the instant when the pendulum rod is vertical. At this point, what is the linear speed of the sphere?

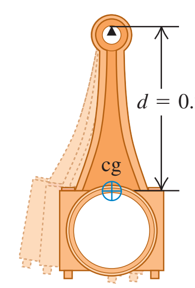
13.47. After landing on an unfamiliar planet, a space explorer constructs a simple pendulum of length 50.0 cm. She finds that the pendulum makes 100 complete swings in 136 s. What is the value of g on this planet?

13.48. A simple pendulum 2.00 m long swings through a maximum angle of 30.0° with the vertical. Calculate its period (a) assuming a small amplitude, and (b) using the first three terms of Eq. (13.35). (c) Which of the answers in parts (a) and (b) is more accurate? For the one that is less accurate, by what percent is it in error from the more accurate answer?

Section 13.6 The Physical Pendulum

13.49. A 1.80-kg connecting rod from a car engine is pivoted about a horizontal knife edge as shown in Fig. 13.33. The center of gravity of the rod was located by balancing and is 0.200 m from the pivot. When the rod is set into small-amplitude oscillation, it makes 100 complete swings in 120 s. Calculate the moment of inertia of the rod about the rotation axis through the pivot.

Figure 13.33 Exercise 13.49.



13.50. We want to hang a thin hoop on a horizontal nail and have the hoop make one complete small-angle oscillation each 2.0 s. What must the hoop's radius be?

13.51. Show that the expression for the period of a physical pendulum reduces to that of a simple pendulum if the physical pendulum consists of a particle with mass m on the end of a massless string of length L.

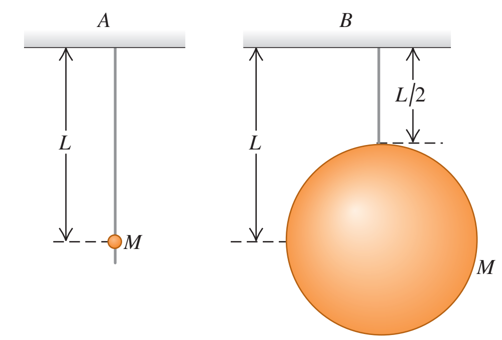
13.52. A 1.80-kg monkey wrench is pivoted 0.250 m from its center of mass and allowed to swing as a physical pendulum. The period for small-angle oscillations is 0.940 s. (a) What is the moment of inertia of the wrench about an axis through the pivot? (b) If the wrench is initially displaced 0.400 rad from its equilibrium position, what is the angular speed of the wrench as it passes through the equilibrium position?

13.53. Two pendulums have the same dimensions (length L) and total mass (m). Pendulum A is a very small ball swinging at the end of a uniform massless bar. In pendulum B, half the mass is in the ball and half is in the uniform bar. Find the period of each pendulum for small oscillations. Which one takes longer for a swing?

13.54. A holiday ornament in the shape of a hollow sphere with mass $M = 0.015\ \text{kg}$ and radius $R = 0.050\ \text{m}$ is hung from a tree limb by a small loop of wire attached to the surface of the sphere. If the ornament is displaced a small distance and released, it swings back and forth as a physical pendulum with negligible friction. Calculate its period. (*Hint:* Use the parallel-axis theorem to find the moment of inertia of the sphere about the pivot at the tree limb.)

13.55. The two pendulums shown in Fig. 13.34 each consist of a uniform solid ball of mass M supported by a massless string, but the ball for pendulum A is very tiny while the ball for pendulum B is much larger. Find the period of each pendulum for small displacements. Which ball takes longer to complete a swing?

Figure 13.34 Exercise 13.55.



Section 13.7 Damped Oscillations

13.56. A 2.20-kg mass oscillates on a spring of force constant 250.0 N/m with a period of 0.615 s. (a) Is this system damped or not? How do you know? If it is damped, find the damping constant b. (b) Is the system undamped, underdamped, critically damped, or overdamped? How do you know?

13.57. An unhappy 0.300-kg rodent, moving on the end of a spring with force constant $k = 2.50\ \text{N/m}$, is acted on by a damping force $F_x = -bv_x$. (a) If the constant b has the value 0.900 kg/s, what is the frequency of oscillation of the rodent? (b) For what value of the constant b will the motion be critically damped?

13.58. A 50.0-g hard-boiled egg moves on the end of a spring with force constant $k = 25.0\ \text{N/m}$. Its initial displacement is 0.300 m. A damping force $F_x = -bv_x$ acts on the egg, and the amplitude of the motion decreases to 0.100 m in 5.00 s. Calculate the magnitude of the damping constant b.

13.59. The motion of an underdamped oscillator is described by Eq. (13.42). Let the phase angle ϕ be zero. (a) According to this equation, what is the value of x at $t = 0$? (b) What are the magnitude and direction of the velocity at $t = 0$? What does the result tell you about the slope of the graph of x versus t near $t = 0$? (c) Obtain an expression for the acceleration a_x at $t = 0$. For what value or range of values of the damping constant b (in terms of k and m) is the acceleration at $t = 0$ negative, zero, and positive? Discuss each case in terms of the shape of the graph of x versus t near $t = 0$.

Section 13.8 Forced Oscillations and Resonance

13.60. A sinusoidally varying driving force is applied to a damped harmonic oscillator of force constant k and mass m. If the damping constant has a value b_1, the amplitude is A_1 when the driving angular frequency equals $\sqrt{k/m}$. In terms of A_1, what is the amplitude for the same driving frequency and the same driving force amplitude F_{max}, if the damping constant is (a) $3b_1$ and (b) $b_1/2$?

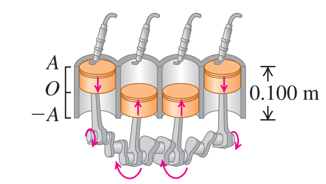
13.61. A sinusoidally varying driving force is applied to a damped harmonic oscillator. (a) What are the units of the damping constant b? (b) Show that the quantity $\sqrt{km}$ has the same units as b. (c) In terms of F_{max} and k, what is the amplitude for $\omega_d = \sqrt{k/m}$ when (i) $b = 0.2\sqrt{km}$ and (ii) $b = 0.4\sqrt{km}$? Compare your results to Fig. 13.28.

13.62. An experimental package and its support structure, which are to be placed on board the International Space Station, act as an underdamped spring–mass system with a force constant of $2.1 \times 10^6\ \text{N/m}$ and mass 108 kg. A NASA requirement is that resonance for forced oscillations not occur for any frequency below 35 Hz. Does this package meet the requirement?

Problems

13.63. SHM in a Car Engine. The motion of the piston of an automobile engine (Fig. 13.35) is approximately simple harmonic. (a) If the stroke of an engine (twice the amplitude) is 0.100 m and the engine runs at 3500 rev/min, compute the acceleration of the piston at the endpoint of its stroke. (b) If the piston has mass 0.450 kg, what net force must be exerted on it at this point? (c) What are the speed and kinetic energy of the piston at the midpoint of its stroke? (d) What average power is required to accelerate the piston from rest to the speed found in part (c)? (e) If the engine runs at 7000 rev/min, what are the answers to parts (b), (c), and (d)?

Figure 13.35 Problem 13.63.



13.64. Four passengers with combined mass 250 kg compress the springs of a car with worn-out shock absorbers by 4.00 cm when they get in. Model the car and passengers as a single body on a single ideal spring. If the loaded car has a period of vibration of 1.08 s, what is the period of vibration of the empty car?

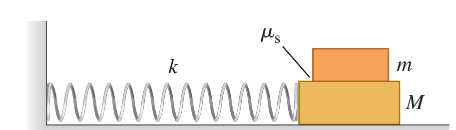
13.65. A glider is oscillating in SHM on an air track with an amplitude A_1. You slow it so that its amplitude is halved. What happens to its (a) period, frequency, and angular frequency; (b) total mechanical energy; (c) maximum speed; (d) speed at $x = \pm A_1/4$; (e) potential and kinetic energies at $x = \pm A_1/4$?

13.66. A child with poor table manners is sliding his 250-g dinner plate back and forth in SHM with an amplitude of 0.100 m on a horizontal surface. At a point 0.060 m away from equilibrium, the speed of the plate is 0.300 m/s. (a) What is the period? (b) What is the displacement when the speed is 0.160 m/s? (c) In the center of the dinner plate is a 10.0-g carrot slice. If the carrot slice is just on the verge of slipping at the endpoint of the path, what is the coefficient of static friction between the carrot slice and the plate?

13.67. A 1.50-kg, horizontal, uniform tray is attached to a vertical ideal spring of force constant 185 N/m and a 275-g metal ball is in the tray. The spring is below the tray, so it can oscillate up-and-down. The tray is then pushed down 15.0 cm below its equilibrium point (call this point A) and released from rest. (a) How high above point A will the tray be when the metal ball leaves the tray? (*Hint:* This does *not* occur when the ball and tray reach their maximum speeds.) (b) How much time elapses between releasing the system at point A and the ball leaving the tray? (c) How fast is the ball moving just as it leaves the tray?

13.68. A block with mass M rests on a frictionless surface and is connected to a horizontal spring of force constant k. The other end of the spring is attached to a wall (Fig. 13.36). A second block with mass m rests on top of the first block. The coefficient of static

Figure 13.36 Problem 13.68.



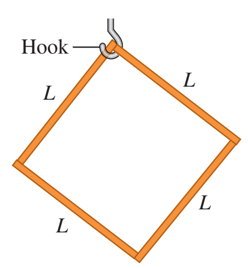
friction between the blocks is μ_s. Find the *maximum* amplitude of oscillation such that the top block will not slip on the bottom block.

13.69. A 10.0-kg mass is traveling to the right with a speed of 2.00 m/s on a smooth horizontal surface when it collides with and sticks to a second 10.0-kg mass that is initially at rest but is attached to a light spring with force constant 80.0 N/m. (a) Find the frequency, amplitude, and period of the subsequent oscillations. (b) How long does it take the system to return the first time to the position it had immediately after the collision?

13.70. A rocket is accelerating upward at 4.00 m/s^2 from the launchpad on the earth. Inside a small 1.50-kg ball hangs from the ceiling by a light 1.10-m wire. If the ball is displaced 8.50° from the vertical and released, find the amplitude and period of the resulting swings of this pendulum.

13.71. A square object of mass m is constructed of four identical uniform thin sticks, each of length L, attached together. This object is hung on a hook at its upper corner (Fig. 13.37). If it is rotated slightly to the left and then released, at what frequency will it swing back and forth?

Figure 13.37 Problem 13.71.



13.72. An object with mass 0.200 kg is acted on by an elastic restoring force with force constant 10.0 N/m. (a) Graph elastic potential energy U as a function of displacement x over a range of x from -0.300 m to $+0.300$ m. On your graph, let 1 cm = 0.05 J vertically and 1 cm = 0.05 m horizontally. The object is set into oscillation with an initial potential energy of 0.140 J and an initial kinetic energy of 0.060 J. Answer the following questions by referring to the graph. (b) What is the amplitude of oscillation? (c) What is the potential energy when the displacement is one-half the amplitude? (d) At what displacement are the kinetic and potential energies equal? (e) What is the value of the phase angle ϕ if the initial velocity is positive and the initial displacement is negative?

13.73. A 2.00-kg bucket containing 10.0 kg of water is hanging from a vertical ideal spring of force constant 125 N/m and oscillating up and down with an amplitude of 3.00 cm. Suddenly the bucket springs a leak in the bottom such that water drops out at a steady rate of 2.00 g/s. When the bucket is half full, find (a) the period of oscillation and (b) the rate at which the period is changing with respect to time. Is the period getting longer or shorter? (c) What is the shortest period this system can have?

13.74. A hanging wire is 1.80 m long. When a 60.0-kg steel ball is suspended from the wire, the wire stretches by 2.00 mm. If the ball is pulled down a small additional distance and released, at what frequency will it vibrate? Assume that the stress on the wire is less than the proportional limit (see Section 11.5).

13.75. A 5.00-kg partridge is suspended from a pear tree by an ideal spring of negligible mass. When the partridge is pulled down 0.100 m below its equilibrium position and released, it vibrates with a period of 4.20 s. (a) What is its speed as it passes through the equilibrium position? (b) What is its acceleration when it is 0.050 m above the equilibrium position? (c) When it is moving upward, how much time is required for it to move from a point 0.050 m below its equilibrium position to a point 0.050 m above it? (d) The motion of the partridge is stopped, and then it is removed from the spring. How much does the spring shorten?

13.76. A 0.0200-kg bolt moves with SHM that has an amplitude of 0.240 m and a period of 1.500 s. The displacement of the bolt is $+0.240$ m when $t = 0$. Compute (a) the displacement of the bolt when $t = 0.500$ s; (b) the magnitude and direction of the force acting on the bolt when $t = 0.500$ s; (c) the minimum time required for the bolt to move from its initial position to the point where $x = -0.180$ m; (d) the speed of the bolt when $x = -0.180$ m.

13.77. SHM of a Butcher's Scale. A spring of negligible mass and force constant $k = 400$ N/m is hung vertically, and a 0.200-kg pan is suspended from its lower end. A butcher drops a 2.2-kg steak onto the pan from a height of 0.40 m. The steak makes a totally inelastic collision with the pan and sets the system into vertical SHM. What are (a) the speed of the pan and steak immediately after the collision; (b) the amplitude of the subsequent motion; (c) the period of that motion?

13.78. A uniform beam is suspended horizontally by two identical vertical springs that are attached between the ceiling and each end of the beam. The beam has mass 225 kg, and a 175-kg sack of gravel sits on the middle of it. The beam is oscillating in SHM, with an amplitude of 40.0 cm and a frequency of 0.600 cycles/s. (a) The sack of gravel falls off the beam when the beam has its maximum upward displacement. What are the frequency and amplitude of the subsequent SHM of the beam? (b) If the gravel instead falls off when the beam has its maximum speed, what are the frequency and amplitude of the subsequent SHM of the beam?

13.79. On the planet Newtonia, a simple pendulum having a bob with mass 1.25 kg and a length of 185.0 cm takes 1.42 s, when released from rest, to swing through an angle of 12.5°, where it again has zero speed. The circumference of Newtonia is measured to be 51,400 km. What is the mass of the planet Newtonia?

13.80. A 40.0-N force stretches a vertical spring 0.250 m. (a) What mass must be suspended from the spring so that the system will oscillate with a period of 1.00 s? (b) If the amplitude of the motion is 0.050 m and the period is that specified in part (a), where is the object and in what direction is it moving 0.35 s after it has passed the equilibrium position, moving downward? (c) What force (magnitude and direction) does the spring exert on the object when it is 0.030 m below the equilibrium position, moving upward?

13.81. Don't Miss the Boat. While on a visit to Minnesota ("Land of 10,000 Lakes"), you sign up to take an excursion around one of the larger lakes. When you go to the dock where the 1500-kg boat is tied, you find that the boat is bobbing up and down in the waves, executing simple harmonic motion with amplitude 20 cm. The boat takes 3.5 s to make one complete up-and-down cycle. When the boat is at its highest point, its deck is at the same height as the stationary dock. As you watch the boat bob up and down, you (mass 60 kg) begin to feel a bit woozy, due in part to the previous night's dinner of lutefisk. As a result, you refuse to board the boat unless the level of the boat's deck is within 10 cm of the dock level. How much time do you have to board the boat comfortably during each cycle of up-and-down motion?

13.82. An interesting, though highly impractical example of oscillation is the motion of an object dropped down a hole that extends from one side of the earth, through its center, to the other side. With the assumption (not realistic) that the earth is a sphere of uniform density, prove that the motion is simple harmonic and find the period. [*Note:* The gravitational force on the object as a function of the object's distance r from the center of the earth was derived in Example 12.10 (Section 12.6). The motion is simple harmonic if the acceleration a_x and the displacement from equilibrium x are related by Eq. (13.8), and the period is then $T = 2\pi/\omega$.]

13.83. Two point masses m are held in place a distance d apart. Another point mass M is midway between them. M is then displaced a small distance x perpendicular to the line connecting the two fixed masses and released. (a) Show that the magnitude of the net gravitational force on M due to the fixed masses is given approximately by $F_{\text{net}} = \dfrac{16\,GmMx}{d^3}$ if $x \ll d$. What is the direction of this force? Is it a restoring force? (b) Show that the mass M will oscillate with an angular frequency of $(4/d)\sqrt{Gm/d}$ and period $\pi d/2\sqrt{d/Gm}$. (c) What would the period be if $m = 100$ kg and $d = 25.0$ cm? Does it seem that you could easily measure this period? What things prevent this experiment from easily being performed in an ordinary physics lab? (d) Will M oscillate if it is displaced from the center a small distance x toward either of the fixed masses? Why?

13.84. For a certain oscillator the net force on the body with mass m is given by $F_x = -cx^3$. (a) What is the potential energy function for this oscillator if we take $U = 0$ at $x = 0$? (b) One-quarter of a period is the time for the body to move from $x = 0$ to $x = A$. Calculate this time and hence the period. [*Hint:* Begin with Eq. (13.20), modified to include the potential-energy function you found in part (a), and solve for the velocity v_x as a function of x. Then replace v_x with dx/dt. Separate the variable by writing all fators containing x on one side and all factors containing t on the other side so that each side can be integrated. In the x-integral make the change of variable $u = x/A$. The resulting integral can be evaluated by numerical methods on a computer and has the value $\int_0^1 du/\sqrt{1 - u^4} = 1.31$.] (c) According to the result you obtained in part (b), does the period depend on the amplitude A of the motion? Are the oscillations simple harmonic?

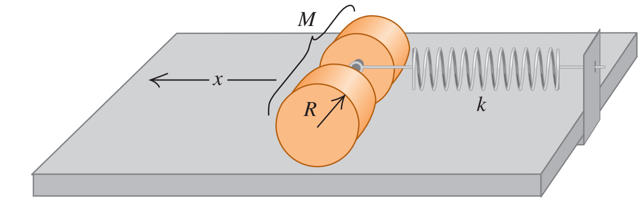
13.85. Consider the circle of reference shown in Fig. 13.6. The x-component of the velocity of Q is the velocity of P. Compute this component, and show that the velocity of P is as given by Eq. (13.15).

***13.86. Diatomic Molecule.** Two identical atoms in a diatomic molecule vibrate as harmonic oscillators. However, their center of mass, midway between them, remains at rest. (a) Show that at any instant, the momenta of the atoms relative to the center of mass are $\vec{p}$ and $-\vec{p}$. (b) Show that the total kinetic energy K of the two atoms at any instant is the same as that of a single object with mass $m/2$ with a momentum of magnitude p. (Use $K = p^2/2m$.) This result shows why $m/2$ should be used in the expression for f in Example 13.7 (Section 13.4). (c) If the atoms are not identical but have masses m_1 and m_2, show that the result of part (a) still holds and the single object's mass in part (b) is $m_1m_2/(m_1 + m_2)$. The quantity $m_1m_2/(m_1 + m_2)$ is called the *reduced mass* of the system.

***13.87.** An approximation for the potential energy of a KCl molecule is $U = A[(R_0^7/8r^8) - 1/r]$, where $R_0 = 2.67 \times 10^{-10}$ m and $A = 2.31 \times 10^{-28}\ \text{J}\cdot\text{m}$. Using this approximation: (a) Show that the radial component of the force on each atom is $F_r = A[(R_0^7/r^9) - 1/r^2]$. (b) Show that R_0 is the equilibrium separation. (c) Find the minimum potential energy. (d) Use $r = R_0 + x$ and the first two terms of the binomial theorem (Eq. 13.28) to show that $F_r \approx -(7A/R_0^3)x$, so that the molecule's force constant is $k = 7A/R_0^3$. (e) With both the K and Cl atoms vibrating in opposite directions on opposite sides of the molecule's center of mass, $m_1m_2/(m_1 + m_2) = 3.06 \times 10^{-26}$ kg is the mass to use in calculating the frequency (see Problem 13.86). Calculate the frequency of small-amplitude vibrations.

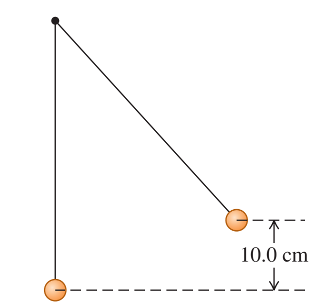
13.88. Two solid cylinders connected along their common axis by a short, light rod have radius R and total mass M and rest on a horizontal tabletop. A spring with force constant k has one end attached to a clamp and the other end attached to a frictionless ring at the center of mass of the cylinders (Fig. 13.38). The cylinders are pulled to the left a distance x, which stretches the spring, and released. There is sufficient friction between the tabletop and the cylinders for the cylinders to roll without slipping as they move back and forth on the end of the spring. Show that the motion of the center of mass of the cylinders is simple harmonic, and calculate its period in terms of M and k. [*Hint:* The motion is simple harmonic if a_x and x are related by Eq. (13.8), and the period then is $T = 2\pi/\omega$. Apply $\sum\tau_z = I_{\text{cm}}\alpha_z$ and $\sum F_x = Ma_{\text{cm-}x}$ to the cylinders in order to relate $a_{\text{cm-}x}$ and the displacement x of the cylinders from their equilibrium position.]

Figure 13.38 Problem 13.88.



13.89. In Fig. 13.39 the upper ball is released from rest, collides with the stationary lower ball, and sticks to it. The strings are both 50.0 cm long. The upper ball has mass 2.00 kg, and it is initially 10.0 cm higher than the lower ball, which has mass 3.00 kg. Find the frequency and maximum angular displacement of the motion after the collision.

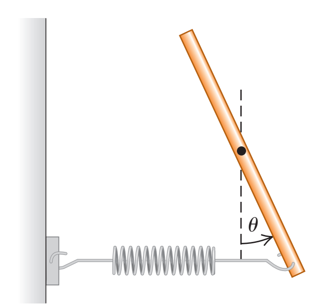
Figure 13.39 Problem 13.89.



13.90. T. rex. Model the leg of the *T. rex* in Example 13.10 (Section 13.6) as two uniform rods, each 1.55 m long, joined rigidly end to end. Let the lower rod have mass M and the upper rod mass $2M$. The composite object is pivoted about the top of the upper rod. Compute the oscillation period of this object for small-amplitude oscillations. Compare your result to that of Example 13.10.

13.91. A slender, uniform, metal rod with mass M is pivoted without friction about an axis through its midpoint and perpendicular to the rod. A horizontal spring with force constant k is attached to the lower end of the rod, with the other end of the spring attached to a rigid support. If the rod is displaced by a small angle Θ from the vertical (Fig. 13.40) and released, show that it moves in angular SHM and calculate the period. (*Hint:* Assume that the angle Θ is small enough for the approximations $\sin\Theta \approx \Theta$ and $\cos\Theta \approx 1$ to be valid. The motion is simple harmonic if $d^2\theta/dt^2 = -\omega^2\theta$, and the period is then $T = 2\pi/\omega$.)

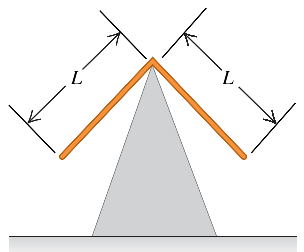
Figure 13.40 Problem 13.91.



13.92. The Silently Ringing Bell Problem. A large bell is hung from a wooden beam so it can swing back and forth with negligible friction. The center of mass of the bell is 0.60 m below the pivot, the bell has mass 34.0 kg, and the moment of inertia of the bell about an axis at the pivot is $18.0\ \text{kg}\cdot\text{m}^2$. The clapper is a small, 1.8-kg mass attached to one end of a slender rod that has length L and negligible mass. The other end of the rod is attached to the inside of the bell so it can swing freely about the same axis as the bell. What should be the length L of the clapper rod for the bell to ring silently—that is, for the period of oscillation for the bell to equal that for the clapper?

13.93. Two identical thin rods, each with mass m and length L, are joined at right angles to form an L-shaped object. This object is balanced on top of a sharp edge (Fig. 13.41). If the L-shaped object is deflected slightly, it oscillates. Find the frequency of oscillation.

Figure 13.41 Problem 13.93.



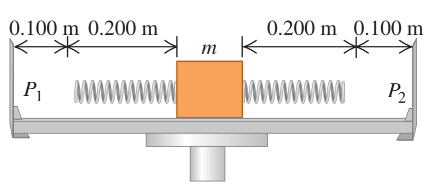
13.94. You want to construct a pendulum with a period of 4.00 s at a location where $g = 9.80\ \text{m/s}^2$. (a) What is the length of a *simple* pendulum having this period? (b) Suppose the pendulum must be mounted in a case that is not more than 0.50 m high. Can you devise a pendulum having a period of 4.00 s that will satisfy this requirement?

13.95. A uniform rod of length L oscillates through small angles about a point a distance x from its center. (a) Prove that its angular frequency is $\sqrt{gx/[(L^2/12) + x^2]}$. (b) Show that its maximum angular frequency occurs when $x = L/\sqrt{12}$. (c) What is the length of the rod if the maximum angular frequency is 2π rad/s?

Challenge Problems

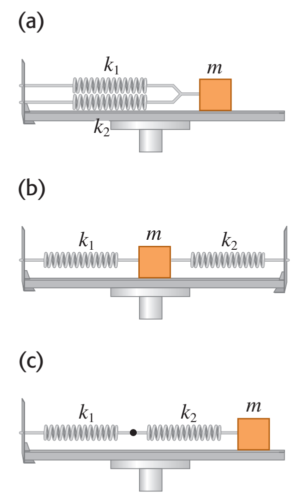
13.96. Two springs, each with unstretched length 0.200 m, but with different force constants k_1 and k_2, are attached to opposite ends of a block with mass m on a level, frictionless surface. The outer ends of the springs are now attached to two pins P_1 and P_2, 0.100 m from the original positions of the ends of the springs (Fig. 13.42). Let $k_1 = 2.00\ \text{N/m}$, $k_2 = 6.00\ \text{N/m}$, and $m = 0.100$ kg. (a) Find the length of each spring when the block is in its new equilibrium position after the springs have been attached to the pins. (b) Find the period of vibration of the block if it is slightly displaced from its new equilibrium position and released.

Figure 13.42 Challenge Problem 13.96.



13.97. The Effective Force Constant of Two Springs. Two springs with the same unstretched length, but different force constants k_1 and k_2 are attached to a block with mass m on a level, frictionless surface. Calculate the effective force constant k_{eff} in each of the three cases (a), (b), and (c) depicted in Fig. 13.43. (The effective force constant is defined by $\sum F_x = -k_{\text{eff}}x$.) (d) An object with mass m, suspended from a uniform spring with a force constant k, vibrates with a frequency f_1. When the spring is cut in half and the same object is suspended from one of the halves, the frequency is f_2. What is the ratio f_2/f_1?

Figure 13.43 Challenge Problem 13.97.



13.98. (a) What is the change ΔT in the period of a simple pendulum when the acceleration of gravity g changes by Δg? (*Hint:* The new period $T + \Delta T$ is obtained by substituting $g + \Delta g$ for g:

$$T + \Delta T = 2\pi\sqrt{\frac{L}{g + \Delta g}}$$

To obtain an approximate expression, expand the factor $(g + \Delta g)^{-1/2}$ using the binomial theorem (Appendix B) and keep only the first two terms:

$$(g + \Delta g)^{-1/2} = g^{-1/2} - \frac{1}{2}g^{-3/2}\Delta g + \cdots$$

The other terms contain higher powers of Δg and are very small if Δg is small.) Express your result as the *fractional* change in period $\Delta T/T$ in terms of the fractional change $\Delta g/g$. (b) A pendulum clock keeps correct time at a point where $g = 9.8000\ \text{m/s}^2$, but is found to lose 4.0 s each day at a higher elevation. Use the result of part (a) to find the approximate value of g at this new location.

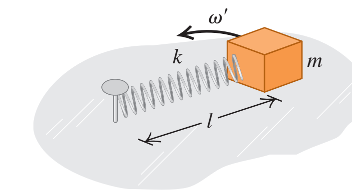
13.99. A Spring with Mass. The preceding problems in this chapter have assumed that the springs had negligible mass. But of course no spring is completely massless. To find the effect of the spring's mass, consider a spring with mass M, equilibrium length L_0, and spring constant k. When stretched or compressed to a length L, the potential energy is $\frac{1}{2}kx^2$, where $x = L - L_0$. (a) Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. Calculate the kinetic energy of the spring in terms of M and v. (*Hint:* Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v, and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) (b) Take the time derivative of the conservation of energy equation, Eq. (13.21), for a mass m moving on the end of a *massless* spring. By comparing your results to Eq. (13.8), which defines ω, show that the angular frequency of oscillation is $\omega = \sqrt{k/m}$. (c) Apply the procedure of part (b) to obtain the angular frequency of oscillation ω of the spring considered in part (a). If the *effective mass* M' of the spring is defined by $\omega = \sqrt{k/M'}$, what is M' in terms of M?

13.100. A uniform, 1.00-m stick hangs from a horizontal axis at one end and oscillates as a physical pendulum. An object of small dimensions and with mass equal to that of the meter stick can be clamped to the stick at a distance y below the axis. Let T represent the period of the system with the body attached and T_0 the period of the meter stick alone. (a) Find the ratio T/T_0. Evaluate your expression for y ranging from 0 to 1.0 m in steps of 0.1 m, and graph T/T_0 versus y. (b) Is there any value of y, other than $y = 0$, for which $T = T_0$? If so, find it and explain why the period is unchanged when y has this value.

13.101. You measure the period of a physical pendulum about one pivot point to be T. Then you find another pivot point on the opposite side of the center of mass that gives the same period. The two points are separated by a distance L. Use the parallel-axis theorem to show that $g = L(2\pi/T)^2$. (This result shows a way that you can measure g without knowing the mass or any moments of inertia of the physical pendulum.)

13.102. Resonance in a Mechanical System. A mass m is attached to one end of a massless spring with a force constant k and an unstretched length l_0. The other end of the spring is free to turn about a nail driven into a frictionless, horizontal surface (Fig. 13.44). The mass is made to revolve in a circle with an angular frequency of revolution ω'. (a) Calculate the length l of the spring as a function of ω'. (b) What happens to the result in part (a) when ω' approaches the natural frequency $\omega = \sqrt{k/m}$ of the mass-spring system? (If your result bothers you, remember that massless springs and frictionless surfaces don't exist as such, but are only approximate descriptions of real springs and surfaces. Also, Hooke's law is only an approximation of the way real springs behave; the greater the elongation of the spring, the greater the deviation from Hooke's law.)

Figure 13.44 Challenge Problem 13.102.



***13.103. Vibration of a Covalently Bonded Molecule.** Many diatomic (two-atom) molecules are bound together by *covalent bonds* that are much stronger than the van der Waals interaction. Examples include H_2, O_2, and N_2. Experiment shows that for many such molecules, the interaction can be described by a force of the form

$$F_r = A\left[e^{-2b(r-R_0)} - e^{-b(r-R_0)}\right]$$

where A and b are positive constants, r is the center-to-center separation of the atoms, and R_0 is the equilibrium separation. For the hydrogen molecule (H_2), $A = 2.97 \times 10^{-8}$ N, $b = 1.95 \times 10^{10}\ \text{m}^{-1}$, and $R_0 = 7.4 \times 10^{-11}$ m. Find the force constant for small oscillations around equilibrium. (*Hint:* Use the expansion for e^x given in Appendix B.) Compare your result to the value given in Exercise 13.40.