MECHANICAL WAVES

15

?When an earthquake strikes, the news of the event travels through the body of the earth in the form of seismic waves. Which aspects of a seismic wave determine how much power is carried by the wave?

LEARNING GOALS

By studying this chapter, you will learn:

- What is meant by a mechanical wave, and the different varieties of mechanical waves.
- How to use the relationship among speed, frequency, and wavelength for a periodic wave.
- How to interpret and use the mathematical expression for a sinusoidal periodic wave.
- How to calculate the speed of waves on a rope or string.
- How to calculate the rate at which a mechanical wave transports energy.
- What happens when mechanical waves overlap and interfere.
- The properties of standing waves on a string, and how to analyze these waves.
- How stringed instruments produce sounds of specific frequencies.

Ripples on a pond, musical sounds, seismic tremors triggered by an earthquake—all these are *wave* phenomena. Waves can occur whenever a system is disturbed from equilibrium and when the disturbance can travel, or *propagate,* from one region of the system to another. As a wave propagates, it carries energy. The energy in light waves from the sun warms the surface of our planet; the energy in seismic waves can crack our planet's crust.

This chapter and the next are about mechanical waves—waves that travel within some material called a *medium.* (Chapter 16 is concerned with sound, an important type of mechanical wave.) We'll begin this chapter by deriving the basic equations for describing waves, including the important special case of *sinusoidal* waves in which the wave pattern is a repeating sine or cosine function. To help us understand waves in general, we'll look at the simple case of waves that travel on a stretched string or rope.

Waves on a string play an important role in music. When a musician strums a guitar or bows a violin, she makes waves that travel in opposite directions along the instrument's strings. What happens when these oppositely directed waves overlap is called *interference.* We'll discover that sinusoidal waves can occur on a guitar or violin string only for certain special frequencies, called *normal-mode frequencies,* determined by the properties of the string. The normal-mode frequencies of a stringed instrument determine the pitch of the musical sounds that the instrument produces. (In the next chapter we'll find that interference also helps explain the pitches of *wind* instruments such as flutes and pipe organs.)

Not all waves are mechanical in nature. *Electromagnetic* waves—including light, radio waves, infrared and ultraviolet radiation, and x rays—can propagate even in empty space, where there is *no* medium. We'll explore these and other nonmechanical waves in later chapters.

ActivPhysics ONLINE
10.1 Properties of Mechanical Waves

15.1 Types of Mechanical Waves

A **mechanical wave** is a disturbance that travels through some material or substance called the **medium** for the wave. As the wave travels through the medium, the particles that make up the medium undergo displacements of various kinds, depending on the nature of the wave.

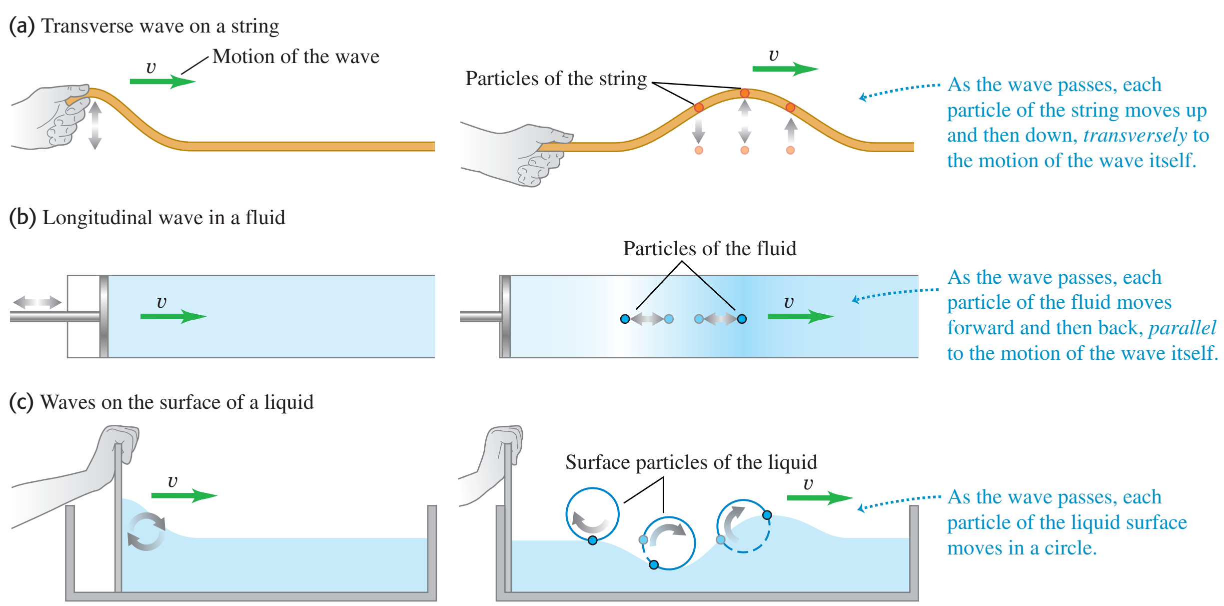
Figure 15.1 shows three varieties of mechanical waves. In Fig. 15.1a the medium is a string or rope under tension. If we give the left end a small upward shake or wiggle, the wiggle travels along the length of the string. Successive sections of string go through the same motion that we gave to the end, but at successively later times. Because the displacements of the medium are perpendicular or *transverse* to the direction of travel of the wave along the medium, this is called a **transverse wave.**

In Fig. 15.1b the medium is a liquid or gas in a tube with a rigid wall at the right end and a movable piston at the left end. If we give the piston a single back-and-forth motion, displacement and pressure fluctuations travel down the length of the medium. This time the motions of the particles of the medium are back and forth along the *same* direction that the wave travels. We call this a **longitudinal wave.**

In Fig. 15.1c the medium is a liquid in a channel, such as water in an irrigation ditch or canal. When we move the flat board at the left end forward and back once, a wave disturbance travels down the length of the channel. In this case the displacements of the water have *both* longitudinal and transverse components.

Each of these systems has an equilibrium state. For the stretched string it is the state in which the system is at rest, stretched out along a straight line. For the fluid in a tube it is a state in which the fluid is at rest with uniform pressure. And for the liquid in a trough it is a smooth, level water surface. In each case the wave motion is a disturbance from the equilibrium state that travels from one region of the medium to another. And in each case there are forces that tend to restore the system to its equilibrium position when it is displaced, just as the force of gravity tends to pull a pendulum toward its straight-down equilibrium position when it is displaced.

15.1 Three ways to make a wave that moves to the right. (a) The hand moves the string up and then returns, producing a transverse wave. (b) The piston moves to the right, compressing the gas or liquid, and then returns, producing a longitudinal wave. (c) The board moves to the right and then returns, producing a combination of longitudinal and transverse waves.



These examples have three things in common. First, in each case the disturbance travels or *propagates* with a definite speed through the medium. This speed is called the speed of propagation, or simply the **wave speed.** Its value is determined in each case by the mechanical properties of the medium. We will use the symbol v for wave speed. (The wave speed is *not* the same as the speed with which particles move when they are disturbed by the wave. We'll return to this point in Section 15.3.) Second, the medium itself does not travel through space; its individual particles undergo back-and-forth or up-and-down motions around their equilibrium positions. The overall pattern of the wave disturbance is what travels. Third, to set any of these systems into motion, we have to put in energy by doing mechanical work on the system. The wave motion transports this energy from one region of the medium to another. *Waves transport energy, but not matter, from one region to another* (Fig. 15.2).

15.2 "Doing the wave" at a sports stadium is an example of a mechanical wave: The disturbance propagates through the crowd, but there is no transport of matter (none of the spectators moves from one seat to another).

Test Your Understanding of Section 15.1 What type of wave is "the wave" shown in Fig. 15.2? (i) transverse; (ii) longitudinal; (iii) a combination of transverse and longitudinal.

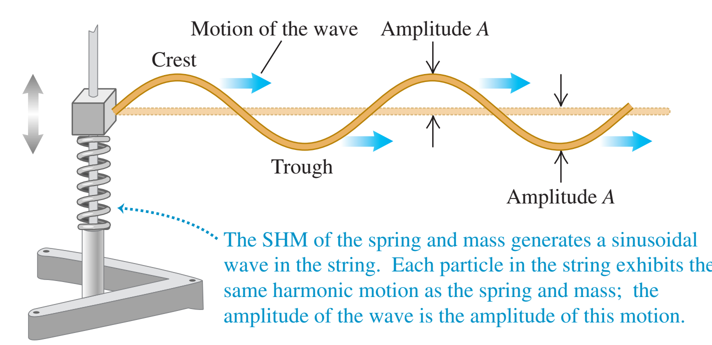
15.2 Periodic Waves

The transverse wave on a stretched string in Fig. 15.1a is an example of a *wave pulse.* The hand shakes the string up and down just once, exerting a transverse force on it as it does so. The result is a single "wiggle," or pulse, that travels along the length of the string. The tension in the string restores its straight-line shape once the pulse has passed.

A more interesting situation develops when we give the free end of the string a repetitive, or *periodic,* motion. (You may want to review the discussion of periodic motion in Chapter 13 before going ahead.) Then each particle in the string also undergoes periodic motion as the wave propagates, and we have a **periodic wave.**

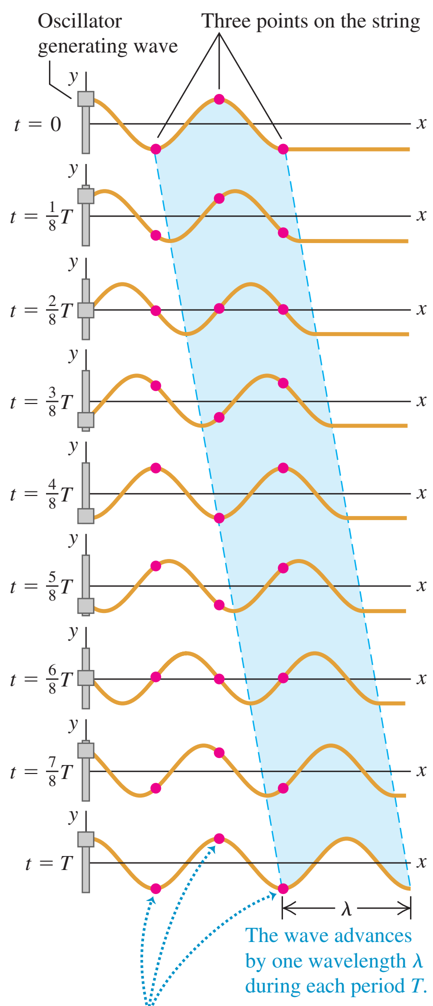
Periodic Transverse Waves

In particular, suppose we move the string up and down with *simple harmonic motion* (SHM) with amplitude A, frequency f, angular frequency $\omega = 2\pi f$, and period $T = 1/f = 2\pi/\omega$. Figure 15.3 shows one way to do this. The wave that results is a symmetrical sequence of *crests* and *troughs.* As we will see, periodic waves with simple harmonic motion are particularly easy to analyze; we call them **sinusoidal waves.** It also turns out that *any* periodic wave can be represented as a combination of sinusoidal waves. So this particular kind of wave motion is worth special attention.



15.3 A block of mass m attached to a spring undergoes simple harmonic motion, producing a sinusoidal wave that travels to the right on the string. (In a real-life system a driving force would have to be applied to the block to replace the energy carried away by the wave.)

15.4 A sinusoidal transverse wave traveling to the right along a string. The vertical scale is exaggerated.



15.5 A series of drops falling into water produces a periodic wave that spreads radially outward. The wave crests and troughs are concentric circles. The wavelength λ is the radial distance between adjacent crests or adjacent troughs.

In Fig. 15.3 the wave that advances along the string is a *continuous succession* of transverse sinusoidal disturbances. Figure 15.4 shows the shape of a part of the string near the left end at time intervals of $\frac{1}{8}$ of a period, for a total time of one period. The wave shape advances steadily toward the right, as indicated by the highlighted area. As the wave moves, any point on the string (any of the red dots, for example) oscillates up and down about its equilibrium position with simple harmonic motion. *When a sinusoidal wave passes through a medium, every particle in the medium undergoes simple harmonic motion with the same frequency.*

CAUTION **Wave motion vs. particle motion** Be very careful to distinguish between the motion of the *transverse wave* along the string and the motion of a *particle* of the string. The wave moves with constant speed v *along* the length of the string, while the motion of the particle is simple harmonic and *transverse* (perpendicular) to the length of the string.

For a periodic wave, the shape of the string at any instant is a repeating pattern. The length of one complete wave pattern is the distance from one crest to the next, or from one trough to the next, or from any point to the corresponding point on the next repetition of the wave shape. We call this distance the **wavelength** of the wave, denoted by λ (the Greek letter lambda). The wave pattern travels with constant speed v and advances a distance of one wavelength λ in a time interval of one period T. So the wave speed v is given by $v = \lambda/T$ or, because $f = 1/T$,

$$v = \lambda f \qquad \text{(periodic wave)} \qquad (15.1)$$

The speed of propagation equals the product of wavelength and frequency. The frequency is a property of the *entire* periodic wave because all points on the string oscillate with the same frequency f.

Waves on a string propagate in just one dimension (in Fig. 15.4, along the x-axis). But the ideas of frequency, wavelength, and amplitude apply equally well to waves that propagate in two or three dimensions. Figure 15.5 shows a wave propagating in two dimensions on the surface of a tank of water. As with waves on a string, the wavelength is the distance from one crest to the next, and the amplitude is the height of a crest above the equilibrium level.

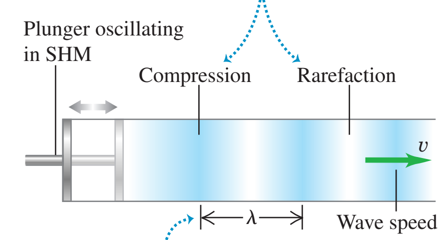
In many important situations including waves on a string, the wave speed v is determined entirely by the mechanical properties of the medium. In this case, increasing f causes λ to decrease so that the product $v = \lambda f$ remains the same, and waves of *all* frequencies propagate with the same wave speed. In this chapter we will consider *only* waves of this kind. (In later chapters we will study the propagation of light waves in matter for which the wave speed depends on frequency; this turns out to be the reason prisms break white light into a spectrum and raindrops create a rainbow.)

Periodic Longitudinal Waves

To understand the mechanics of a periodic *longitudinal* wave, we consider a long tube filled with a fluid, with a piston at the left end as in Fig. 15.1b. If we push the piston in, we compress the fluid near the piston, increasing the pressure in this region. This region then pushes against the neighboring region of fluid, and so on, and a wave pulse moves along the tube.

Now suppose we move the piston back and forth with simple harmonic motion, along a line parallel to the axis of the tube (Fig. 15.6). This motion forms regions in the fluid where the pressure and density are greater or less than the equilibrium values. We call a region of increased density a *compression;* a region of reduced density is a *rarefaction.* Figure 15.6 shows compressions as darkly shaded areas and rarefactions as lightly shaded areas. The wavelength is the distance from one compression to the next or from one rarefaction to the next.

15.6 Using an oscillating piston to make a sinusoidal longitudinal wave in a fluid.



15.7 A sinusoidal longitudinal wave traveling to the right in a fluid. The wave has the same amplitude A and period T as the oscillation of the piston.

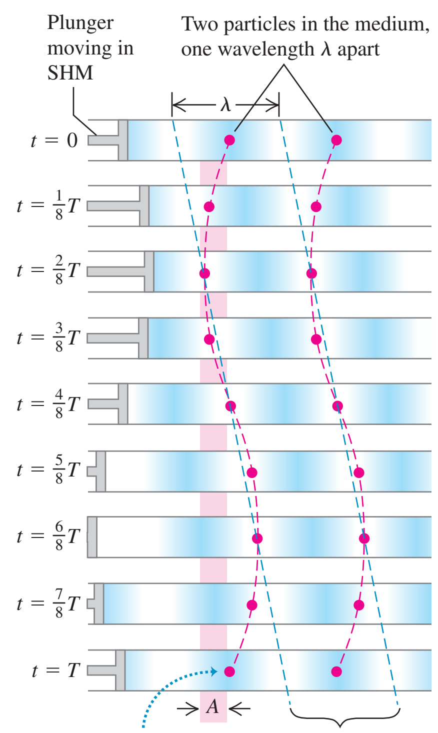


Figure 15.7 shows the wave propagating in the fluid-filled tube at time intervals of $\frac{1}{8}$ of a period, for a total time of one period. The pattern of compressions and rarefactions moves steadily to the right, just like the pattern of crests and troughs in a sinusoidal transverse wave (compare Fig. 15.4). Each particle in the fluid oscillates in SHM parallel to the direction of wave propagation (that is, left and right) with the same amplitude A and period T as the piston. The particles shown by the two red dots in Fig. 15.7 are one wavelength apart, and so oscillate in phase with each other.

Just like the sinusoidal transverse wave shown in Fig. 15.4, in one period T the longitudinal wave in Fig. 15.7 travels one wavelength λ to the right. Hence the fundamental equation $v = \lambda f$ holds for longitudinal waves as well as for transverse waves, and indeed for *all* types of periodic waves. Just as for transverse waves, in this chapter and the next we will consider only situations in which the speed of longitudinal waves does not depend on the frequency.

Example 15.1 Wavelength of a musical sound

Sound waves are longitudinal waves in air. The speed of sound depends on temperature; at 20°C it is 344 m/s (1130 ft/s). What is the wavelength of a sound wave in air at 20°C if the frequency is 262 Hz (the approximate frequency of middle C on a piano)?

SOLUTION

IDENTIFY: This problem involves the relationship among wave speed, wavelength, and frequency for a periodic wave. The target variable is the wavelength λ.

SET UP: The wave speed $v = 344\ \text{m/s}$ and the frequency $f = 262\ \text{Hz}$ are given, so we can use the relationship in Eq. (15.1) among v, λ, and f.

EXECUTE: We solve Eq. (15.1) for the target variable λ:

$$\lambda = \frac{v}{f} = \frac{344\ \text{m/s}}{262\ \text{Hz}} = \frac{344\ \text{m/s}}{262\ \text{s}^{-1}} = 1.31\ \text{m}$$

Note that the units of frequency are either hertz (Hz) or inverse seconds (s^{-1}).

EVALUATE: What happens to the wavelength if the frequency changes? The speed of sound waves is unaffected by changes in frequency, so the relationship $\lambda = v/f$ tells us that wavelength will change in inverse proportion to the frequency. As an example, the high C sung by coloratura sopranos is two octaves above middle C. Each octave corresponds to a factor of 2 in frequency, so the frequency of high C is four times that of middle C: $f = 4(262\ \text{Hz}) = 1048\ \text{Hz}$. Hence the *wavelength* of high C is *one-fourth* as large: $\lambda = (1.31\ \text{m})/4 = 0.328\ \text{m}$.

Test Your Understanding of Section 15.2 If you double the wavelength of a wave on a particular string, what happens to the wave speed v and the frequency f? (i) v doubles and f is unchanged; (ii) v is unchanged and f doubles; (iii) v becomes one-half as great and f is unchanged; (iv) v is unchanged and f becomes one-half as great; (v) none of these.

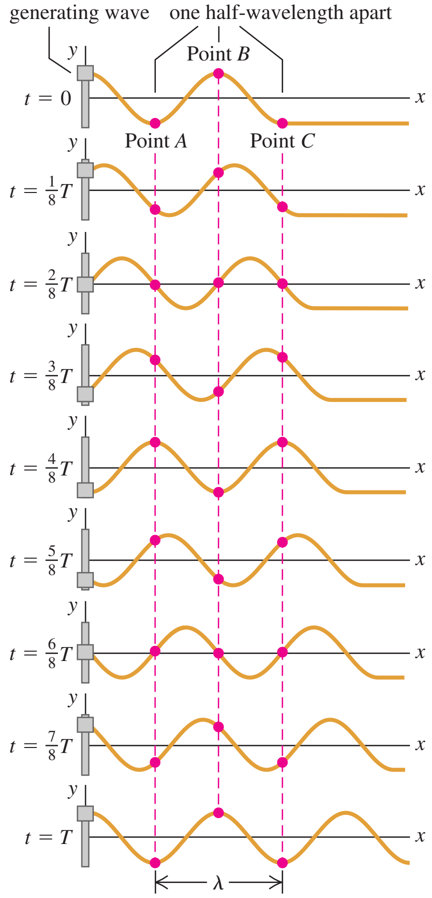
15.3 Mathematical Description of a Wave

Many characteristics of periodic waves can be described by using the concepts of wave speed, amplitude, period, frequency, and wavelength. Often, though, we need a more detailed description of the positions and motions of individual particles of the medium at particular times during wave propagation. For this description we need the concept of a *wave function,* a function that describes the position of any particle in the medium at any time. We will concentrate on *sinusoidal* waves, in which each particle undergoes simple harmonic motion about its equilibrium position.

As a specific example, let's look at waves on a stretched string. If we ignore the sag of the string due to gravity, the equilibrium position of the string is along a straight line. We take this to be the x-axis of a coordinate system. Waves on a string are *transverse;* during wave motion a particle with equilibrium position x is displaced some distance y in the direction perpendicular to the x-axis. The value of y depends on which particle we are talking about (that is, y depends on x) and also on the time t when we look at it. Thus y is a *function* of both x and t; $y = y(x, t)$. We call $y(x, t)$ the **wave function** that describes the wave. If we know this function for a particular wave motion, we can use it to find the displacement (from equilibrium) of any particle at any time. From this we can find the velocity and acceleration of any particle, the shape of the string, and anything else we want to know about the behavior of the string at any time.

15.8 Tracking the oscillations of three points on a string as a sinusoidal wave propagates along it.

The string is shown at time intervals of $\frac{1}{8}$ period for a total of one period T.



Wave Function for a Sinusoidal Wave

Let's see how to determine the form of the wave function for a sinusoidal wave. Suppose a sinusoidal wave travels from left to right (the direction of increasing x) along the string, as in Fig. 15.8. Every particle of the string oscillates with simple harmonic motion with the same amplitude and frequency. But the oscillations of particles at different points on the string are *not* all in step with each other. The particle at point B in Fig. 15.8 is at its maximum positive value of y at $t = 0$ and returns to $y = 0$ at $t = \frac{2}{8}T$; these same events occur for a particle at point A or point C at $t = \frac{4}{8}T$ and $t = \frac{6}{8}T$, exactly one half-period later. For any two particles of the string, the motion of the particle on the right (in terms of the wave, the "downstream" particle) lags behind the motion of the particle on the left by an amount proportional to the distance between the particles.

Hence the cyclic motions of various points on the string are out of step with each other by various fractions of a cycle. We call these differences *phase differences,* and we say that the *phase* of the motion is different for different points. For example, if one point has its maximum positive displacement at the same time that another has its maximum negative displacement, the two are a half-cycle out of phase. (This is the case for points A and B, or points B and C.)

Suppose that the displacement of a particle at the left end of the string $(x = 0)$, where the wave originates, is given by

$$y(x = 0, t) = A\cos\omega t = A\cos 2\pi f t \qquad (15.2)$$

That is, the particle oscillates in simple harmonic motion with amplitude A, frequency f, and angular frequency $\omega = 2\pi f$. The notation $y(x = 0, t)$ reminds us that the motion of this particle is a special case of the wave function $y(x, t)$ that describes the entire wave. At $t = 0$ the particle at $x = 0$ is at its maximum positive displacement $(y = A)$ and is instantaneously at rest (because the value of y is a maximum).

The wave disturbance travels from $x = 0$ to some point x to the right of the origin in an amount of time given by x/v, where v is the wave speed. So the motion of point x at time t is the same as the motion of point $x = 0$ at the earlier time $t - x/v$. Hence we can find the displacement of point x at time t by simply replacing t in Eq. (15.2) by $(t - x/v)$. When we do that, we find the following expression for the wave function:

$$y(x, t) = A\cos\left[\omega\left(t - \frac{x}{v}\right)\right]$$

Because $\cos(-\theta) = \cos\theta$, we can rewrite the wave function as

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right] = A\cos 2\pi f\left(\frac{x}{v} - t\right) \quad \text{(sinusoidal wave moving in } +x\text{-direction)} \qquad (15.3)$$

The displacement $y(x, t)$ is a function of both the location x of the point and the time t. We could make Eq. (15.3) more general by allowing for different values of the phase angle, as we did for simple harmonic motion in Section 13.2, but for now we omit this.

We can rewrite the wave function given by Eq. (15.3) in several different but useful forms. We can express it in terms of the period $T = 1/f$ and the wavelength $\lambda = v/f$:

$$y(x, t) = A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \quad \text{(sinusoidal wave moving in } +x\text{-direction)} \qquad (15.4)$$

We get another convenient form of the wave function if we define a quantity k, called the **wave number:**

$$k = \frac{2\pi}{\lambda} \quad \text{(wave number)} \qquad (15.5)$$

Substituting $\lambda = 2\pi/k$ and $f = \omega/2\pi$ into the wavelength-frequency relationship $v = \lambda f$ gives

$$\omega = vk \quad \text{(periodic wave)} \qquad (15.6)$$

We can then rewrite Eq. (15.4) as

$$y(x, t) = A\cos(kx - \omega t) \quad \text{(sinusoidal wave moving in } +x\text{-direction)} \qquad (15.7)$$

Which of these various forms for the wave function $y(x, t)$ we use in any specific problem is a matter of convenience. Note that ω has units rad/s, so for unit consistency in Eqs. (15.6) and (15.7) the wave number k must have the units rad/m. (Some physicists define the wave number as $1/\lambda$ rather than $2\pi/\lambda$. When reading other texts, be sure to determine how this term is defined.)

Graphing the Wave Function

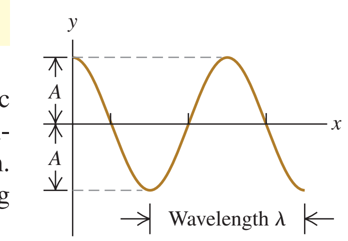
The wave function $y(x, t)$ is graphed as a function of x for a specific time t in Fig. 15.9a. This graph gives the displacement y of a particle from its equilibrium position as a function of the coordinate x of the particle. If the wave is a transverse wave on a string, the graph in Fig. 15.9a represents the shape of the string at that instant, like a flash photograph of the string. In particular, at time $t = 0$,

$$y(x, t = 0) = A\cos kx = A\cos 2\pi\frac{x}{\lambda}$$

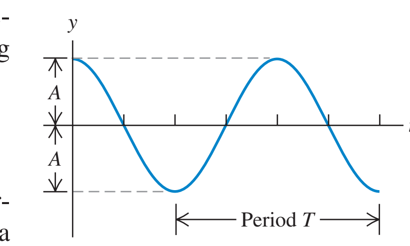
Figure 15.9b is a graph of the wave function versus time t for a specific coordinate x. This graph gives the displacement y of the particle at that coordinate as a

15.9 Two graphs of the wave function $y(x, t)$ in Eq. (15.7). **(a)** Graph of displacement y versus coordinate x at time $t = 0$. **(b)** Graph of displacement y versus time t at coordinate $x = 0$. The vertical scale is exaggerated in both (a) and (b).

(a) If we use Eq. (15.7) to plot y as a function of x for time $t = 0$, the curve shows the *shape* of the string at $t = 0$.



(b) If we use Eq. (15.7) to plot y as a function of t for position $x = 0$, the curve shows the *displacement* y of the particle at $x = 0$ as a function of time.



function of time; that is, it describes the motion of that particle. In particular, at the position $x = 0$,

$$y(x = 0, t) = A\cos(-\omega t) = A\cos\omega t = A\cos 2\pi\frac{t}{T}$$

This is consistent with our original statement about the motion at $x = 0$, Eq. (15.2).

CAUTION **Wave graphs** Although they may look the same at first glance, Figs. 15.9a and 15.9b are *not* identical. Figure 15.9a is a picture of the shape of the string at $t = 0$, while Fig. 15.9b is a graph of the displacement y of a particle at $x = 0$ as a function of time.

More on the Wave Function

We can modify Eqs. (15.3) through (15.7) to represent a wave traveling in the *negative* x-direction. In this case the displacement of point x at time t is the same as the motion of point $x = 0$ at the *later* time $(t + x/v)$, so in Eq. (15.2) we replace t by $(t + x/v)$. For a wave traveling in the negative x-direction,

$$y(x, t) = A\cos 2\pi f\left(\frac{x}{v} + t\right) = A\cos 2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right) = A\cos(kx + \omega t) \quad (15.8)$$

(sinusoidal wave moving in $-x$-direction)

In the expression $y(x, t) = A\cos(kx \pm \omega t)$ for a wave traveling in the $-x$ or $+x$-direction, the quantity $(kx \pm \omega t)$ is called the **phase.** It plays the role of an angular quantity (always measured in radians) in Eq. (15.7) or (15.8), and its value for any values of x and t determines what part of the sinusoidal cycle is occurring at a particular point and time. For a crest (where $y = A$ and the cosine function has the value 1), the phase could be 0, 2π, 4π, and so on; for a trough (where $y = -A$ and the cosine has the value -1), it could be π, 3π, 5π, and so on.

The wave speed is the speed with which we have to move along with the wave to keep alongside a point of a given phase, such as a particular crest of a wave on a string. For a wave traveling in the $+x$-direction, that means $kx - \omega t$ = constant. Taking the derivative with respect to t, we find $k\,dx/dt = \omega$, or

$$\frac{dx}{dt} = \frac{\omega}{k}$$

Comparing this with Eq. (15.6), we see that dx/dt is equal to the speed v of the wave. Because of this relationship, v is sometimes called the *phase velocity* of the wave. (*Phase speed* would be a better term.)

Problem-Solving Strategy 15.1 Mechanical Waves

IDENTIFY *the relevant concepts:* Wave problems fall into two broad categories. *Kinematics* problems are concerned with describing wave motion; they involve wave speed v, wavelength λ (or wave number k), frequency f (or angular frequency ω), and amplitude A. They may also involve the position, velocity, and acceleration of individual particles in the medium. *Dynamics* problems also use concepts from Newton's laws such as force and mass. As an example, later in this chapter we'll encounter problems that involve the relationship of wave speed to the mechanical properties of the wave medium.

As always, make sure that you identify the target variable(s) for the problem. In some cases you'll be asked to find an expression for the wave function.

SET UP *the problem* using the following steps:

1. Make a list of the quantities whose values are given. To help you visualize the situation, you'll find it useful to sketch graphs of y versus x (like Fig. 15.9a) and of y versus t (like Fig. 15.9b). Label your graphs with the values of the known quantities.
2. Decide which equations you'll need to use. If any two of v, f, and λ are given, use Eq. (15.1) $(v = \lambda f)$ to find the third quantity (see Example 15.1). If the problem involves the angular frequency ω and/or the wave number k, use the definitions of those quantities and Eq. (15.6) $(\omega = vk)$. You may also need the various forms of the wave function given in Eqs. (15.3), (15.4), and (15.7).
3. If the wave speed isn't given and you don't have enough information to determine it using $v = \lambda f$, you may be able to use the relationship between v and the mechanical properties of the system. (In the next section we'll develop this relationship for waves on a string.)

EXECUTE *the solution* as follows: Solve for the unknown quantities using the equations you've selected. In some problems all you need to do is find the value of one of the wave variables.

If you're asked to determine the wave function, you need to know A and any two of v, λ, and f (or v, k, and ω). Once you have this information, you can use it in Eq. (15.3), (15.4), or (15.7) to get the specific wave function for the problem at hand. Once you have that, you can find the value of y at any point (value of x) and at any time by substituting into the wave function.

EVALUATE *your answer:* Look at your results with a critical eye. Check to see whether the values of v, f, and λ (or v, ω, and k) agree with the relationships given in Eq. (15.1) or (15.6). If you've calculated the wave function, check one or more special cases for which you can guess what the results ought to be

Example 15.2 Wave on a clothesline

Your cousin Throckmorton is playing with the clothesline. He unties one end, holds it taut, and wiggles the end up and down sinusoidally with frequency 2.00 Hz and amplitude 0.075 m. The wave speed is $v = 12.0$ m/s. At time $t = 0$ the end has maximum positive displacement and is instantaneously at rest. Assume no wave bounces back from the far end to muddle up the pattern. (a) Find the amplitude, angular frequency, period, wavelength, and wave number of the wave. (b) Write a wave function describing the wave. (c) Write equations for the displacement as a function of time of Throckmorton's end of the clothesline and of a point 3.00 m from his end.

SOLUTION

IDENTIFY: This is a kinematics problem about the motion of the clothesline. Since Throcky moves his hand in a sinusoidal way, he produces a sinusoidal wave that propagates down the clothesline. Hence we can use all of the expressions we've developed in this section. Our target variables in part (a) are amplitude A, angular frequency ω, period T, wavelength λ, and wave number k, so we need to use the equations that relate these quantities. In parts (b) and (c) our target "variables" are actually expressions for displacement; to find these, we use the general equations for the wave function of a sinusoidal wave.

SET UP: A photograph of the clothesline at time $t = 0$ would look just like Fig. 15.9a, with the maximum displacement at $x = 0$ (the end that Throcky has in his hand). We take the positive x-direction to be the direction in which the wave propagates, so we can use Eqs. (15.4) and (15.7) to describe the displacement of the clothesline as a function of position x and time t. We also use the relationships $f = 1/T$, $\omega = 2\pi f$, $k = 2\pi/\lambda$, $v = \lambda f$, and $\omega = vk$.

EXECUTE: (a) The amplitude A of the wave is just the amplitude of the motion of the end of the clothesline, $A = 0.075$ m. Similarly, the wave frequency is $f = 2.00$ Hz, the same as the frequency of the end of the clothesline. The angular frequency is

$$\omega = 2\pi f = (2\pi \text{ rad/cycle})(2.00 \text{ cycles/s}) = 4.00\pi \text{ rad/s}$$
$$= 12.6 \text{ rad/s}$$

The period is $T = 1/f = 0.500$ s. We get the wavelength from Eq. (15.1):

$$\lambda = \frac{v}{f} = \frac{12.0 \text{ m/s}}{2.00 \text{ s}^{-1}} = 6.00 \text{ m}$$

We find the wave number from Eq. (15.5) or (15.6):

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{6.00 \text{ m}} = 1.05 \text{ rad/m} \quad \text{or}$$

$$k = \frac{\omega}{v} = \frac{4.00\pi \text{ rad/s}}{12.0 \text{ m/s}} = 1.05 \text{ rad/m}$$

(b) Since we found the values of A, T, and λ in part (a), we can write the wave function using Eq. (15.4):

$$\begin{aligned} y(x, t) &= A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \\ &= (0.075 \text{ m})\cos 2\pi\left(\frac{x}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}}\right) \\ &= (0.075 \text{ m})\cos[(1.05 \text{ rad/m})x - (12.6 \text{ rad/s})t] \end{aligned}$$

We can also get this same equation from Eq. (15.7) by using the values of ω and k we obtained in part (a).

(c) With our choice of the positive x-direction, the two points in question are at $x = 0$ and $x = +3.00$ m. For each point, we can find the displacement as a function of time by substituting these values of x into the wave function we found in part (b):

$$\begin{aligned} y(x = 0, t) &= (0.075 \text{ m})\cos 2\pi\left(\frac{0}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}}\right) \\ &= (0.075 \text{ m})\cos(12.6 \text{ rad/s})t \\ y(x = +3.00 \text{ m}, t) &= (0.075 \text{ m})\cos 2\pi\left(\frac{3.00 \text{ m}}{6.00 \text{ m}} - \frac{t}{0.500 \text{ s}}\right) \\ &= (0.075 \text{ m})\cos[\pi - (12.6 \text{ rad/s})t] \\ &= -(0.075 \text{ m})\cos(12.6 \text{ rad/s})t \end{aligned}$$

Continued

EVALUATE: In part (b), the quantity $(1.05 \text{ rad/m})x - (12.6 \text{ rad/s})t$ is the *phase* of a point x on the string at time t. The phases of the two points in part (c) differ by π because these points are separated by one half-wavelength $(\lambda/2 = (6.00 \text{ m})/2 = 3.00 \text{ m})$. Both points oscillate in SHM with the same frequency and amplitude, but their oscillations are one half-cycle out of phase. Thus, while a graph of y versus t for the point at $x = 0$ is a cosine curve (like Fig. 15.9b), a graph of y versus t for the point $x = 3.00$ m is a *negative* cosine (the same as a cosine curve shifted by one half-cycle).

Using the expression for $y(x = 0, t)$ in part (c), can you show that the end of the string at $x = 0$ is instantaneously at rest at $t = 0$, just as we stated at the beginning of this example? (*Hint:* Calculate the y-velocity at this point by taking the derivative of y with respect to t.)

Particle Velocity and Acceleration in a Sinusoidal Wave

From the wave function we can get an expression for the transverse velocity of any *particle* in a transverse wave. We call this v_y to distinguish it from the wave propagation speed v. To find the transverse velocity v_y at a particular point x, we take the derivative of the wave function $y(x, t)$ with respect to t, keeping x constant. If the wave function is

$$y(x, t) = A\cos(kx - \omega t)$$

then

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = \omega A \sin(kx - \omega t) \qquad (15.9)$$

The ∂ in this expression is a modified d, used to remind us that $y(x, t)$ is a function of *two* variables and that we are allowing only one (t) to vary. The other (x) is constant because we are looking at a particular point on the string. This derivative is called a *partial derivative*. If you haven't reached this point yet in your study of calculus, don't fret; it's a simple idea.

Equation (15.9) shows that the transverse velocity of a particle varies with time, as we expect for simple harmonic motion. The maximum particle speed is ωA; this can be greater than, less than, or equal to the wave speed v, depending on the amplitude and frequency of the wave.

The *acceleration* of any particle is the *second* partial derivative of $y(x, t)$ with respect to t:

$$a_y(x, t) = \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A\cos(kx - \omega t) = -\omega^2 y(x, t) \qquad (15.10)$$

The acceleration of a particle equals $-\omega^2$ times its displacement, which is the result we obtained in Section 13.2 for simple harmonic motion.

We can also compute partial derivatives of $y(x, t)$ with respect to x, holding t constant. This corresponds to studying the shape of the string at one instant of time, like a flash photo. The first derivative $\partial y(x, t)/\partial x$ is the *slope* of the string at any point. The second partial derivative with respect to x is the *curvature* of the string:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = -k^2 A\cos(kx - \omega t) = -k^2 y(x, t) \qquad (15.11)$$

From Eqs. (15.10) and (15.11) and the relationship $\omega = vk$ we see that

$$\frac{\partial^2 y(x, t)/\partial t^2}{\partial^2 y(x, t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2 \quad \text{and}$$

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2} \quad \text{(wave equation)} \qquad (15.12)$$

We've derived Eq. (15.12) for a wave traveling in the positive x-direction. You can use the same steps to show that the wave function for a sinusoidal wave propagating in the *negative* x-direction, $y(x, t) = A\cos(kx + \omega t)$, also satisfies this equation.

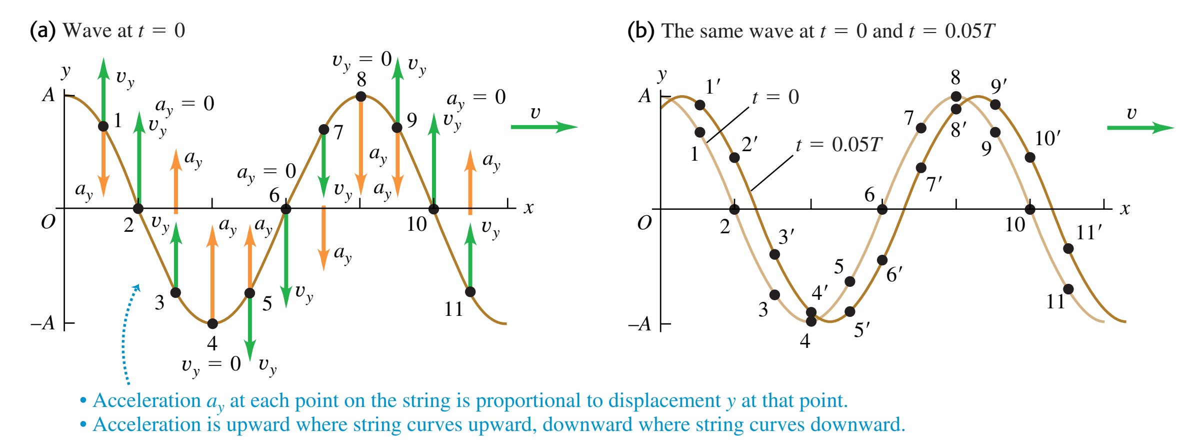
Equation (15.12), called the **wave equation,** is one of the most important equations in all of physics. Whenever it occurs, we know that a disturbance can propagate as a wave along the x-axis with wave speed v. The disturbance need not be a sinusoidal wave; we'll see in the next section that *any* wave on a string obeys Eq. (15.12), whether the wave is periodic or not (see also Problem 15.61). In Chapter 32 we will find that electric and magnetic fields satisfy the wave equation; the wave speed will turn out to be the speed of light, which will lead us to the conclusion that light is an electromagnetic wave.

Figure 15.10a shows the transverse velocity v_y and transverse acceleration a_y, given by Eqs. (15.9) and (15.10), for several points on a string as a sinusoidal wave passes along it. Note that at points where the string has an upward curvature $(\partial^2 y/\partial x^2 > 0)$, the acceleration of that point is positive $(a_y = \partial^2 y/\partial t^2 > 0)$; this follows from the wave equation, Eq. (15.12). For the same reason the acceleration is negative $(a_y = \partial^2 y/\partial t^2 < 0)$ at points where the string has a downward curvature $(\partial^2 y/\partial x^2 < 0)$, and the acceleration is zero $(a_y = \partial^2 y/\partial t^2 = 0)$ at points of inflection where the curvature is zero $(\partial^2 y/\partial x^2 = 0)$. We emphasize again that v_y and a_y are the *transverse* velocity and acceleration of points on the string; these points move along the y-direction, not along the propagation direction of the wave. Figure 15.10b shows the transverse motions of several points on the string.

The concept of wave function is equally useful with *longitudinal* waves, and everything we have said about wave functions can be adapted to this case. The quantity y still measures the displacement of a particle of the medium from its equilibrium position; the difference is that for a longitudinal wave, this displacement is *parallel* to the x-axis instead of perpendicular to it. We'll discuss longitudinal waves in detail in Chapter 16.

Test Your Understanding of Section 15.3 Figure 15.8 shows a sinusoidal wave of period T on a string at times $0, \frac{1}{8}T, \frac{2}{8}T, \frac{3}{8}T, \frac{4}{8}T, \frac{5}{8}T, \frac{6}{8}T, \frac{7}{8}T$, and T. (a) At which time is point A on the string moving upward with maximum speed? (b) At which time does point B on the string have the greatest upward acceleration? (c) At which time does point C on the string have a downward acceleration but an upward velocity?

15.10 (a) Another view of the wave at $t = 0$ in Fig. 15.9a. The vectors show the transverse velocity v_y and transverse acceleration a_y at several points on the string. (b) From $t = 0$ to $t = 0.05\,T$, a particle at point 1 is displaced to point 1′, a particle at point 2 is displaced to point 2′, and so on.



ActivPhysics ONLINE
10.2 Speed of Waves on a String

15.4 Speed of a Transverse Wave

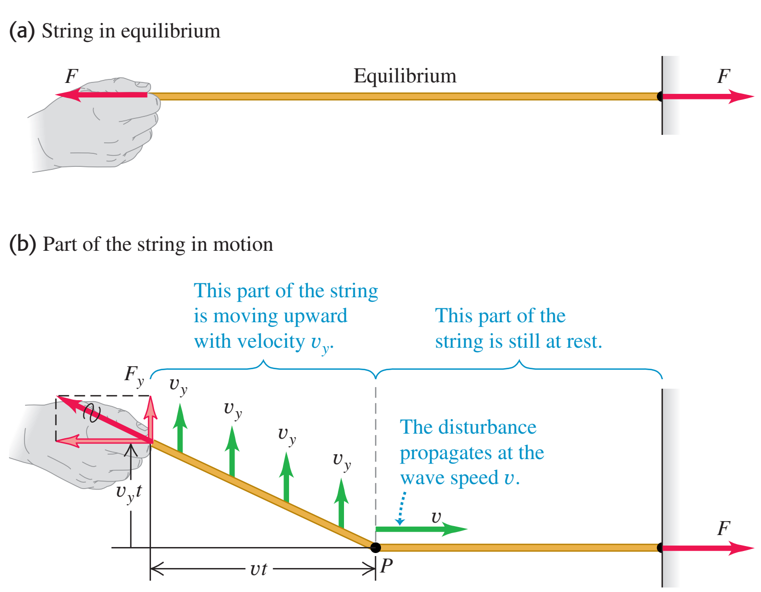
One of the key properties of any wave is the wave *speed.* Light waves in air have a much greater speed of propagation than do sound waves in air (3.00×10^8 m/s versus 344 m/s); that's why you see the flash from a bolt of lightning before you hear the clap of thunder. In this section we'll see what determines the speed of propagation of one particular kind of wave: transverse waves on a string. The speed of these waves is important to understand in its own right because it is an essential part of analyzing stringed musical instruments, as we'll discuss later in this chapter. Furthermore, the speeds of many kinds of mechanical waves turn out to have the same basic mathematical expression as does the speed of waves on a string.

The physical quantities that determine the speed of transverse waves on a string are the *tension* in the string and its *mass per unit length* (also called *linear mass density*). We might guess that increasing the tension should increase the restoring forces that tend to straighten the string when it is disturbed, thus increasing the wave speed. We might also guess that increasing the mass should make the motion more sluggish and decrease the speed. Both these guesses turn out to be right. We'll develop the exact relationship among wave speed, tension, and mass per unit length by two different methods. The first is simple in concept and considers a specific wave shape; the second is more general but also more formal. Choose whichever you like better.

Wave Speed on a String: First Method

We consider a perfectly flexible string (Fig. 15.11). In the equilibrium position the tension is F, and the linear mass density (mass per unit length) is μ. (When portions of the string are displaced from equilibrium, the mass per unit length decreases a little, and the tension increases a little.) We ignore the weight of the string so that when the string is at rest in the equilibrium position, the string forms a perfectly straight line as in Fig. 15.11a.

Starting at time $t = 0$, we apply a constant upward force F_y at the left end of the string. We might expect that the end would move with constant acceleration; that would happen if the force were applied to a *point* mass. But here the effect of the force F_y is to set successively more and more mass in motion. The wave travels with constant speed v, so the division point P between moving and nonmoving portions moves with the same constant speed v (Fig. 15.11b).



15.11 Propagation of a transverse wave on a string.

Figure 15.11b shows that all particles in the moving portion of the string move upward with constant *velocity* v_y, not constant acceleration. To see why this is so, we note that the *impulse* of the force F_y up to time t is $F_y t$. According to the impulse–momentum theorem (see Section 8.1), the impulse is equal to the change in the total transverse component of momentum $(mv_y - 0)$ of the moving part of the string. Because the system started with *no* transverse momentum, this is equal to the total momentum at time t:

$$F_y t = mv_y$$

The total momentum thus must increase proportionately with time. But since the division point P moves with constant speed, the length of string that is in motion and hence the total mass m in motion are also proportional to the time t that the force has been acting. So the *change* of momentum must be associated entirely with the increasing amount of mass in motion, not with an increasing velocity of an individual mass element. That is, mv_y changes because m, not v_y, changes.

At time t, the left end of the string has moved up a distance $v_y t$, and the boundary point P has advanced a distance vt. The total force at the left end of the string has components F and F_y. Why F? There is no motion in the direction along the length of the string, so there is no unbalanced horizontal force. Therefore F, the magnitude of the horizontal component, does not change when the string is displaced. In the displaced position the tension is $(F^2 + F_y^2)^{1/2}$ (greater than F), and the string stretches somewhat.

To derive an expression for the wave speed v, we again apply the impulse–momentum theorem to the portion of the string in motion at time t—that is, the portion to the left of P in Fig. 15.11b. The transverse *impulse* (transverse force times time) is equal to the change of transverse *momentum* of the moving portion (mass times transverse component of velocity). The impulse of the transverse force F_y in time t is $F_y t$. In Fig. 15.11b the right triangle whose vertex is at P, with sides $v_y t$ and vt, is similar to the right triangle whose vertex is at the position of the hand, with sides F_y and F. Hence

$$\frac{F_y}{F} = \frac{v_y t}{vt} \qquad F_y = F\frac{v_y}{v}$$

and

$$\text{Transverse impulse} = F_y t = F\frac{v_y}{v}t$$

The mass of the moving portion of the string is the product of the mass per unit length μ and the length vt, or μvt. The transverse momentum is the product of this mass and the transverse velocity v_y:

$$\text{Transverse momentum} = (\mu vt)v_y$$

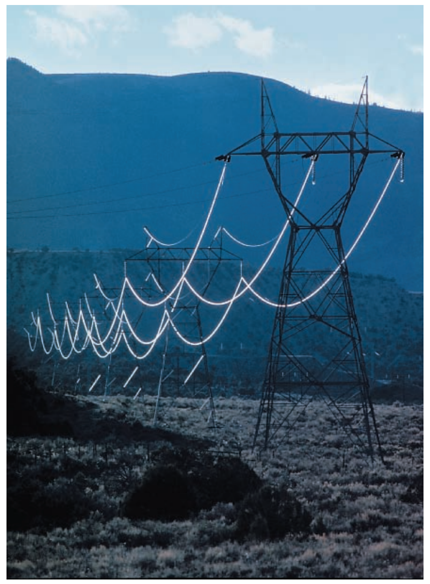
We note again that the momentum increases with time *not* because mass is moving faster, as was usually the case in Chapter 8, but because *more mass* is brought into motion. But the impulse of the force F_y is still equal to the total change in momentum of the system. Applying this relationship, we obtain

$$F\frac{v_y}{v}t = \mu vtv_y$$

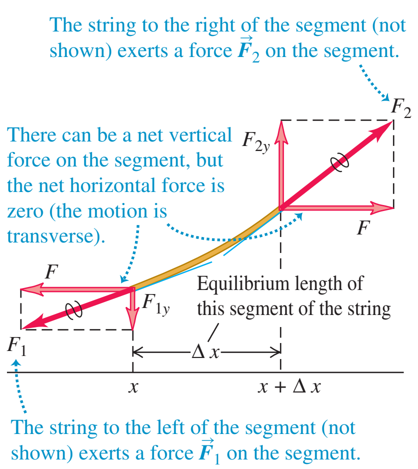
Solving this for v, we find

$$v = \sqrt{\frac{F}{\mu}} \quad \text{(speed of a transverse wave on a string)} \qquad (15.13)$$

15.12 These cables have a relatively large amount of mass per unit length (μ) and a low tension (F). If the cables are disturbed—say, by a bird landing on them—transverse waves will travel along them at a slow speed $v = \sqrt{F/\mu}$.



15.13 Free-body diagram for a segment of string. The force at each end of the string is tangent to the string at the point of application.



Equation (15.13) confirms our prediction that the wave speed v should increase when the tension F increases but decrease when the mass per unit length μ increases (Fig. 15.12).

Note that v_y does not appear in Eq. (15.13); thus the wave speed doesn't depend on v_y. Our calculation considered only a very special kind of pulse, but we can consider *any* shape of wave disturbance as a series of pulses with different values of v_y. So even though we derived Eq. (15.13) for a special case, it is valid for *any* transverse wave motion on a string, including the sinusoidal and other periodic waves we discussed in Section 15.3. Note also that the wave speed doesn't depend on the amplitude or frequency of the wave, in accordance with our assumptions in Section 15.3.

Wave Speed on a String: Second Method

Here is an alternative derivation of Eq. (15.13). If you aren't comfortable with partial derivatives, it can be omitted. We apply Newton's second law, $\sum\vec{F} = m\vec{a}$, to a small segment of string whose length in the equilibrium position is Δx (Fig. 15.13). The mass of the segment is $m = \mu\,\Delta x$; the forces at the ends are represented in terms of their x- and y-components. The x-components have equal magnitude F and add to zero because the motion is transverse and there is no component of acceleration in the x-direction. To obtain F_{1y} and F_{2y}, we note that the ratio F_{1y}/F is equal in magnitude to the *slope* of the string at point x and that F_{2y}/F is equal to the slope at point $x + \Delta x$. Taking proper account of signs, we find

$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \qquad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \tag{15.14}$$

The notation reminds us that the derivatives are evaluated at points x and $x + \Delta x$, respectively. From Eq. (15.14) we find that the net y-component of force is

$$F_y = F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] \tag{15.15}$$

We now equate F_y from Eq. (15.15) to the mass $\mu\,\Delta x$ times the y-component of acceleration $\partial^2 y/\partial t^2$. We obtain

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = \mu\,\Delta x\frac{\partial^2 y}{\partial t^2} \tag{15.16}$$

or, dividing by $F\,\Delta x$,

$$\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\Delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{15.17}$$

We now take the limit as $\Delta x \rightarrow 0$. In this limit, the left side of Eq. (15.17) becomes the derivative of $\partial y/\partial x$ with respect to x (at constant t)—that is, the *second* (partial) derivative of y with respect to x:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{15.18}$$

Now, finally, comes the punch line of our story. Equation (15.18) has exactly the same form as the *wave equation,* Eq. (15.12), that we derived at the end of Section 15.3. That equation and Eq. (15.18) describe the very same wave motion, so they must be identical. Comparing the two equations, we see that for this to be so, we must have

$$v = \sqrt{\frac{F}{\mu}} \tag{15.19}$$

which is the same expression as Eq. (15.13).

In going through this derivation, we didn't make any special assumptions about the shape of the wave. Since our derivation led us to rediscover Eq. (15.12), the wave equation, we conclude that the wave equation is valid for waves on a string that have *any* shape.

The Speed of Mechanical Waves

Equation (15.13) or (15.19) gives the wave speed for only the special case of mechanical waves on a stretched string or rope. Remarkably, it turns out that for many types of mechanical waves, including waves on a string, the expression for wave speed has the same general form:

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

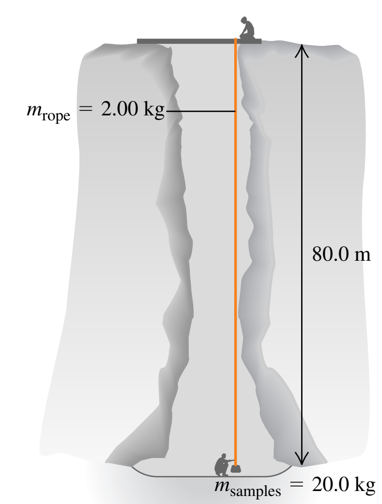
To interpret this expression, let's look at the now-familiar case of waves on a string. The tension F in the string plays the role of the restoring force; it tends to bring the string back to its undisturbed, equilibrium configuration. The mass of the string—or, more properly, the linear mass density μ—provides the inertia that prevents the string from returning instantaneously to equilibrium. Hence we have $v = \sqrt{F/\mu}$ for the speed of waves on a string.

In Chapter 16 we'll see a similar expression for the speed of sound waves in a gas. Roughly speaking, the gas pressure provides the force that tends to return the gas to its undisturbed state when a sound wave passes through. The inertia is provided by the density, or mass per unit volume, of the gas.

Example 15.3 Calculating wave speed

One end of a nylon rope is tied to a stationary support at the top of a vertical mine shaft 80.0 m deep (Fig. 15.14). The rope is stretched taut by a box of mineral samples with mass 20.0 kg attached at the lower end. The mass of the rope is 2.00 kg. The geologist at the bottom of the mine signals to his colleague at the top by jerking the rope sideways. (a) What is the speed of a transverse wave on the rope? (b) If a point on the rope is given a transverse simple harmonic motion with a frequency of 2.00 Hz, how many cycles of the wave are there in the rope's length?

15.14 Sending signals along a vertical rope using transverse waves.



SOLUTION

IDENTIFY: In part (a) the target variable is the wave speed. This part involves *dynamics*—that is, the relationship between the wave speed and the properties of the rope (tension and liner mass density). Part (b) involves *kinematics,* since we need to know how wave speed, frequency, and wavelength are related. (The target variable is actually the number of wavelengths that fit into the length of the rope.)

We'll assume that the tension in the rope is provided by the weight of the box of samples. In fact, the weight of the rope itself contributes to the tension, which means that the tension is different at the top and bottom of the rope. We'll ignore this effect here, since the weight of the rope is small compared to the weight of the samples.

SET UP: We use the relationship $v = \sqrt{F/\mu}$ in part (a). If we neglect the weight of the rope itself, the tension F is just equal to the weight of the box. In part (b) we use the equation $v = f\lambda$ to find the wavelength, which we then compare to the 80.0-m length of the rope.

EXECUTE: (a) The tension in the rope (due to the sample box) is

$$F = m_{\text{samples}}g = (20.0\text{ kg})(9.80\text{ m/s}^2) = 196\text{ N}$$

Continued

and the mass per unit length of the rope is

$$\mu = \frac{m_{rope}}{L} = \frac{2.00 \text{ kg}}{80.0 \text{ m}} = 0.0250 \text{ kg/m}$$

Hence, from Eq. (15.13), the wave speed is

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{196 \text{ N}}{0.0250 \text{ kg/m}}} = 88.5 \text{ m/s}$$

(b) From Eq. (15.1),

$$\lambda = \frac{v}{f} = \frac{88.5 \text{ m/s}}{2.00 \text{ s}^{-1}} = 44.3 \text{ m}$$

The length of the rope is 80.0 m, so the number of wave cycles in the rope is

$$\frac{80.0 \text{ m/s}}{44.3 \text{ m/cycle}} = 1.81 \text{ cycles}$$

EVALUATE: If we do account for the weight of the rope, the tension is greater at the top of the rope than at the bottom. Hence the wave speed increases and the wavelength increases as the wave travels up the rope. Can you verify that the wave speed at the top of the rope is 92.9 m/s?

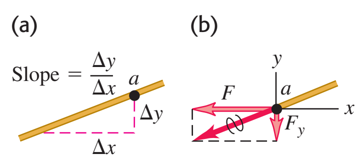
Test Your Understanding of Section 15.4 The six strings of a guitar are the same length and under nearly the same tension, but they have different thicknesses. On which string do waves travel the fastest? (i) the thickest string; (ii) the thinnest string; (iii) the wave speed is the same on all strings.

15.5 Energy in Wave Motion

Every wave motion has *energy* associated with it. The energy we receive from sunlight and the destructive effects of ocean surf and earthquakes bear this out. To produce any of the wave motions we have discussed in this chapter, we have to apply a force to a portion of the wave medium; the point where the force is applied moves, so we do *work* on the system. As the wave propagates, each portion of the medium exerts a force and does work on the adjoining portion. In this way a wave can transport energy from one region of space to another.

As an example of energy considerations in wave motion, let's look again at transverse waves on a string. How is energy transferred from one portion of string to another? Picture a wave traveling from left to right (the positive x-direction) on the string, and consider a particular point a on the string (Fig. 15.15a). The string to the left of point a exerts a force on the string to the right of it, and vice versa. In Fig. 15.15b the string to the left of a has been removed, and the force it exerts at a is represented by the components F and F_y, as we did in Figs. 15.11 and 15.13. We note again that F_y/F is equal to the negative of the *slope* of the string at a, which is also given by $\partial y/\partial x$. Putting these together, we have

15.15 (a) Point a on a string carrying a wave from left to right. (b) The components of the force exerted on the part of the string to the right of point a by the part of the string to the left of point a.



$$F_y(x, t) = -F\frac{\partial y(x, t)}{\partial x} \tag{15.20}$$

We need the negative sign because F_y is negative when the slope is positive. We write the vertical force as $F_y(x, t)$ as a reminder that its value may be different at different points along the string and at different times.

When point a moves in the y-direction, the force F_y does *work* on this point and therefore transfers energy into the part of the string to the right of a. The corresponding power P (rate of doing work) at the point a is the transverse force $F_y(x, t)$ at a times the transverse velocity $v_y(x, t) = \partial y(x, t)/\partial t$ of that point:

$$P(x, t) = F_y(x, t)v_y(x, t) = -F\frac{\partial y(x, t)}{\partial x}\frac{\partial y(x, t)}{\partial t} \tag{15.21}$$

This power is the *instantaneous* rate at which energy is transferred along the string. Its value depends on the position x on the string and on the time t. Note that energy is being transferred only at points where the string has a nonzero slope ($\partial y/\partial x$ is nonzero), so that there is a transverse component of the tension force, and where the string has a nonzero transverse velocity ($\partial y/\partial t$ is nonzero) so that the transverse force can do work.

Equation (15.21) is valid for *any* wave on a string, sinusoidal or not. For a sinusoidal wave with wave function given by Eq. (15.7), we have

$$\begin{aligned} y(x, t) &= A\cos(kx - \omega t) \\ \frac{\partial y(x, t)}{\partial x} &= -kA\sin(kx - \omega t) \\ \frac{\partial y(x, t)}{\partial t} &= \omega A\sin(kx - \omega t) \\ P(x, t) &= Fk\omega A^2\sin^2(kx - \omega t) \end{aligned} \tag{15.22}$$

By using the relationships $\omega = vk$ and $v^2 = F/\mu$, we can also express Eq. (15.22) in the alternative form

$$P(x, t) = \sqrt{\mu F}\omega^2A^2\sin^2(kx - \omega t) \tag{15.23}$$

The $\sin^2$ function is never negative, so the instantaneous power in a sinusoidal wave is either positive (so that energy flows in the positive x-direction) or zero (at points where there is no energy transfer). Energy is never transferred in the direction opposite to the direction of wave propagation (Fig. 15.16).

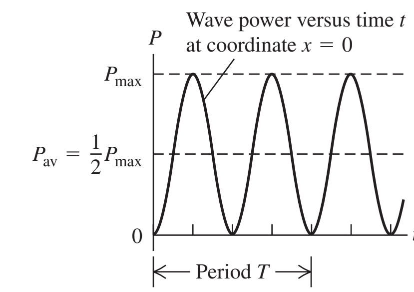
The maximum value of the instantaneous power $P(x, t)$ occurs when the $\sin^2$ function has the value unity:

$$P_{max} = \sqrt{\mu F}\omega^2A^2 \tag{15.24}$$

To obtain the *average* power from Eq. (15.23), we note that the *average* value of the $\sin^2$ function, averaged over any whole number of cycles, is $\frac{1}{2}$. Hence the average power is

$$P_{av} = \frac{1}{2}\sqrt{\mu F}\omega^2A^2 \quad \text{(average power, sinusoidal wave on a string)} \tag{15.25}$$

15.16 The instantaneous power $P(x, t)$ in a sinusoidal wave as given by Eq. (15.23), shown as a function of time at coordinate $x = 0$. The power is never negative, which means that energy never flows opposite to the direction of wave propagation.



The average power is just one-half of the maximum instantaneous power (see Fig. 15.16).

The average rate of energy transfer is proportional to the square of the amplitude and to the square of the frequency. This proportionality is a general result for mechanical waves of all types, including seismic waves (see the photo that opens this chapter). For a mechanical wave, the rate of energy transfer quadruples if the frequency is doubled (for the same amplitude) or if the amplitude is doubled (for the same frequency).

Electromagnetic waves turn out to be a bit different. While the average rate of energy transfer in an electromagnetic wave is proportional to the square of the amplitude, just as for mechanical waves, it is independent of the value of ω.

Example 15.4 Power in a wave

(a) In Example 15.2, at what maximum rate does Throcky put energy into the clothesline? That is, what is his maximum instantaneous power? Assume that the linear mass density of the clothesline is $\mu = 0.250$ kg/m and that Throcky applies tension $F = 36.0$ N. (b) What is his average power? (c) As Throcky tires, the amplitude decreases. What is the average power when the amplitude has dropped to 7.50 mm?

SOLUTION

IDENTIFY: Our target variable in part (a) is the *maximum instantaneous* power, while the target variable in parts (b) and (c) is the *average* power. As we've seen, these two quantities have different values for a sinusoidal wave. We'll be able to calculate the values of both quantities because we know all the other properties of the wave from Example 15.2.

SET UP: For part (a) we use Eq. (15.24), and for parts (b) and (c) we use Eq. (15.25).

EXECUTE: (a) The maximum instantaneous power is

$$\begin{aligned} P_{max} &= \sqrt{\mu F}\omega^2A^2 \\ &= \sqrt{(0.250 \text{ kg/m})(36.0 \text{ N})}(4.00\pi \text{ rad/s})^2(0.075 \text{ m})^2 \\ &= 2.66 \text{ W} \end{aligned}$$

Continued

(b) From Eqs. (15.24) and (15.25), the average power is one-half of the maximum instantaneous power, so

$$P_{av} = \frac{1}{2}(2.66\ \mathrm{W}) = 1.33\ \mathrm{W}$$

(c) The new amplitude is $\frac{1}{10}$ of the value we used in parts (a) and (b). The average power is proportional to the *square* of the amplitude, so now the average power is

$$P_{av} = \left(\frac{1}{10}\right)^2(1.33\ \mathrm{W}) = 0.0133\ \mathrm{W} = 13.3\ \mathrm{mW}$$

EVALUATE: The *maximum* instantaneous power in part (a) occurs when the quantity $\sin^2(kx - \omega t)$ in Eq. (15.23) is equal to 1. At any given value of x, this happens twice per period of the wave—once when the sine function is equal to $+1$, and once when it's equal to -1. The *minimum* instantaneous power is zero; this occurs when $\sin(kx - \omega t) = 0$, which also happens twice per period.

Can you confirm that the given values of μ and F give the wave speed mentioned in Example 15.2?

Wave Intensity

Waves on a string carry energy in just one dimension of space (along the direction of the string). But other types of waves, including sound waves in air and seismic waves in the body of the earth, carry energy across all three dimensions of space. For waves that travel in three dimensions, we define the **intensity** (denoted by I) to be *the time average rate at which energy is transported by the wave, per unit area,* across a surface perpendicular to the direction of propagation. That is, intensity I is average power per unit area. It is usually measured in watts per square meter $(\mathrm{W/m^2})$.

If waves spread out equally in all directions from a source, the intensity at a distance r from the source is inversely proportional to r^2 (Fig. 15.17). This follows directly from energy conservation. If the power output of the source is P, then the average intensity I_1 through a sphere with radius r_1 and surface area $4\pi r_1^2$ is

$$I_1 = \frac{P}{4\pi r_1^2}$$

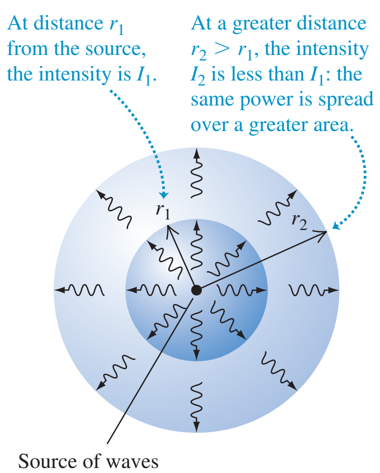
The average intensity I_2 through a sphere with a different radius r_2 is given by a similar expression. If no energy is absorbed between the two spheres, the power P must be the same for both, and

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \quad \text{(inverse-square law for intensity)} \qquad (15.26)$$

The intensity I at any distance r is therefore inversely proportional to r^2. This relationship is called the *inverse-square law* for intensity.

15.17 The greater the distance from a wave source, the greater the area over which the wave power is distributed and the smaller the wave intensity.



Example 15.5 The inverse-square law

A tornado warning siren on top of a tall pole radiates sound waves uniformly in all directions. At a distance of 15.0 m the intensity of the sound is $0.250\ \mathrm{W/m^2}$. At what distance from the siren is the intensity $0.010\ \mathrm{W/m^2}$?

SOLUTION

IDENTIFY: Because waves spread out equally in all directions, we can use the inverse-square law. Our target variable is a distance from the wave source.

SET UP: The relationship to use is Eq. (15.26). We are given the distance $r_1 = 15.0$ m at which the intensity is $I_1 = 0.250\ \mathrm{W/m^2}$, and we want to find the distance r_2 at which the intensity is $I_2 = 0.010\ \mathrm{W/m^2}$.

EXECUTE: We solve Eq. (15.26) for r_2:

$$r_2 = r_1\sqrt{\frac{I_1}{I_2}} = (15.0\ \mathrm{m})\sqrt{\frac{0.250\ \mathrm{W/m^2}}{0.010\ \mathrm{W/m^2}}} = 75.0\ \mathrm{m}$$

EVALUATE: As a check on our answer, note that r_2 is five times greater than r_1. By the inverse-square law, the intensity I_2 should be $1/5^2 = 1/25$ as great as I_1, and indeed it is.

By using the inverse-square law we've assumed that the sound waves travel in straight lines away from the siren. A more realistic solution of this problem would account for the reflection of sound waves from the ground. Such a solution is beyond our scope, however.

Test Your Understanding of Section 15.5 Four identical strings each carry a sinusoidal wave of frequency 10 Hz. The string tension and wave amplitude are different for different strings. Rank the following strings in order from highest to lowest value of the average wave power: (i) tension 10 N, amplitude 1.0 mm; (ii) tension 40 N, amplitude 1.0 mm; (iii) tension 10 N, amplitude 4.0 mm; (iv) tension 20 N, amplitude 2.0 mm.

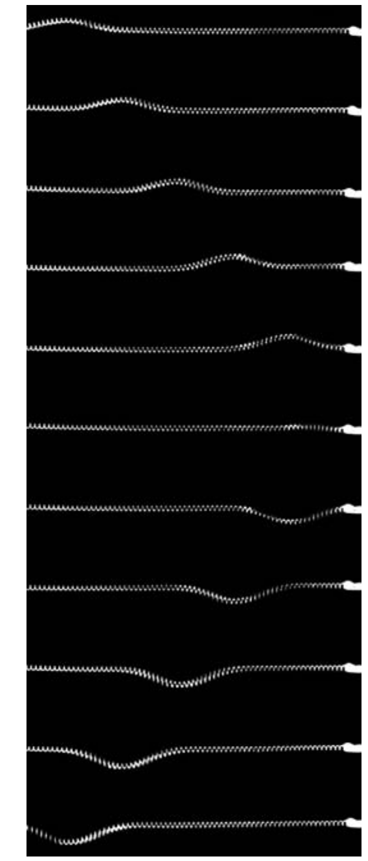
15.6 Wave Interference, Boundary Conditions, and Superposition

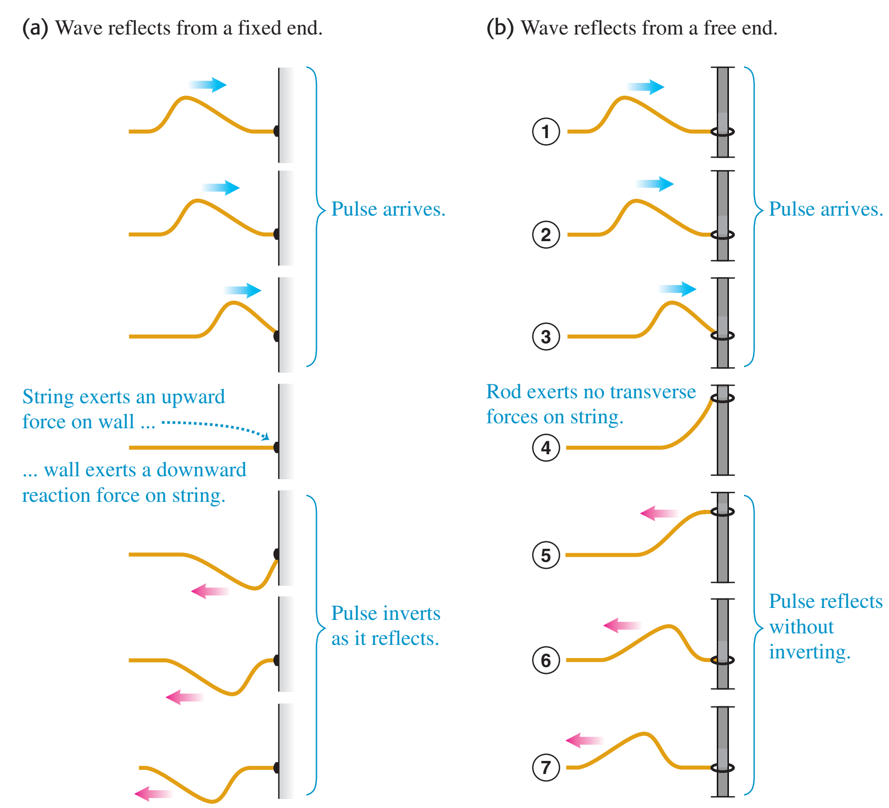
Up to this point we've been discussing waves that propagate continuously in the same direction. But when a wave strikes the boundaries of its medium, all or part of the wave is *reflected.* When you yell at a building wall or a cliff face some distance away, the sound wave is reflected from the rigid surface and you hear an echo. When you flip the end of a rope whose far end is tied to a rigid support, a pulse travels the length of the rope and is reflected back to you. In both cases, the initial and reflected waves overlap in the same region of the medium. This overlapping of waves is called **interference.** (In general, the term "interference" refers to what happens when two or more waves pass through the same region at the same time.)

As a simple example of wave reflections and the role of the boundary of a wave medium, let's look again at transverse waves on a stretched string. What happens when a wave pulse or a sinusoidal wave arrives at the *end* of the string?

If the end is fastened to a rigid support, it is a *fixed* end that cannot move. The arriving wave exerts a force on the support; the reaction to this force, exerted *by* the support *on* the string, "kicks back" on the string and sets up a *reflected* pulse or wave traveling in the reverse direction. Figure 15.18 is a series of photographs showing the reflection of a pulse at the fixed end of a long coiled spring. The reflected pulse moves in the opposite direction from the initial, or *incident,* pulse, and its displacement is also opposite. Figure 15.19a illustrates this situation for a wave pulse on a string.

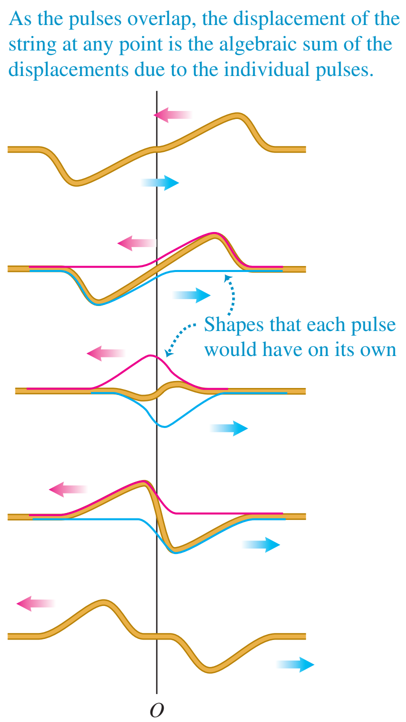
15.18 A series of images of a wave pulse, equally spaced in time from top to bottom. The pulse starts at the left in the top image, travels to the right, and is reflected from the fixed end at the right.



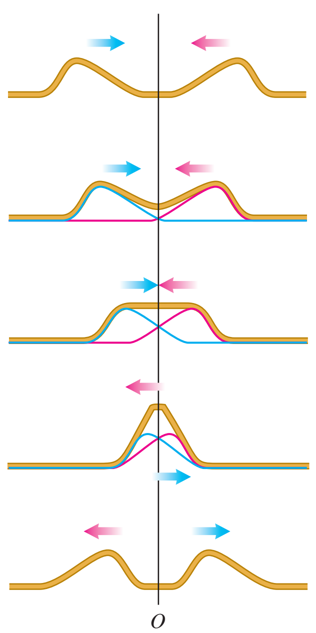


15.19 Reflection of a wave pulse (a) at a fixed end of a string and (b) at a free end. Time increases from top to bottom in each figure.

15.20 Overlap of two wave pulses—one right side up, one inverted—traveling in opposite directions. Time increases from top to bottom.



15.21 Overlap of two wave pulses—both right side up—traveling in opposite directions. Time increases from top to bottom. Compare to Fig. 15.20.



The opposite situation from an end that is held stationary is a *free* end, one that is perfectly free to move in the direction perpendicular to the length of the string. For example, the string might be tied to a light ring that slides on a frictionless rod perpendicular to the string, as in Fig. 15.19b. The ring and rod maintain the tension but exert no transverse force. When a wave arrives at this free end, the ring slides along the rod. The ring reaches a maximum displacement, and both it and the string come momentarily to rest, as in drawing 4 in Fig. 15.19b. But the string is now stretched, giving increased tension, so the free end of the string is pulled back down, and again a reflected pulse is produced (drawing 7). As for a fixed end, the reflected pulse moves in the opposite direction from the initial pulse, but now the direction of the displacement is the same as for the initial pulse. The conditions at the end of the string, such as a rigid support or the complete absence of transverse force, are called **boundary conditions.**

The formation of the reflected pulse is similar to the overlap of two pulses traveling in opposite directions. Figure 15.20 shows two pulses with the same shape, one inverted with respect to the other, traveling in opposite directions. As the pulses overlap and pass each other, the total displacement of the string is the *algebraic sum* of the displacements at that point in the individual pulses. Because these two pulses have the same shape, the total displacement at point O in the middle of the figure is zero at all times. Thus the motion of the left half of the string would be the same if we cut the string at point O, threw away the right side, and held the end at O fixed. The two pulses on the left side then correspond to the incident and reflected pulses, combining so that the total displacement at O is *always* zero. For this to occur, the reflected pulse must be inverted relative to the incident pulse.

Figure 15.21 shows two pulses with the same shape, traveling in opposite directions but *not* inverted relative to each other. The displacement at point O in the middle of the figure is not zero, but the slope of the string at this point is always zero. According to Eq. (15.20), this corresponds to the absence of any transverse force at this point. In this case the motion of the left half of the string would be the same as if we cut the string at point O and attached the end to a frictionless sliding ring (Fig. 15.19b) that maintains tension without exerting any transverse force. In other words, this situation corresponds to reflection of a pulse at a free end of a string at point O. In this case the reflected pulse is *not* inverted.

The Principle of Superposition

Combining the displacements of the separate pulses at each point to obtain the actual displacement is an example of the **principle of superposition:** When two waves overlap, the actual displacement of any point on the string at any time is obtained by adding the displacement the point would have if only the first wave were present and the displacement it would have if only the second wave were present. In other words, the wave function $y(x, t)$ that describes the resulting motion in this situation is obtained by *adding* the two wave functions for the two separate waves:

$$y(x, t) = y_1(x, t) + y_2(x, t) \qquad \text{(principle of superposition)} \qquad (15.27)$$

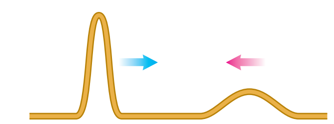
Mathematically, this additive property of wave functions follows from the form of the wave equation, Eq. (15.12) or (15.18), which every physically possible wave function must satisfy. Specifically, the wave equation is *linear;* that is, it contains the function $y(x, t)$ only to the first power (there are no terms involving $y(x, t)^2$, $y(x, t)^{1/2}$, etc.). As a result, if any two functions $y_1(x, t)$ and $y_2(x, t)$ satisfy the wave equation separately, their sum $y_1(x, t) + y_2(x, t)$ also satisfies it and is therefore a physically possible motion. Because this principle depends on the linearity of the wave equation and the corresponding linear-combination property of its solutions, it is also called the *principle of linear superposition.* For

some physical systems, such as a medium that does not obey Hooke's law, the wave equation is *not* linear; this principle does not hold for such systems.

The principle of superposition is of central importance in all types of waves. When a friend talks to you while you are listening to music, you can distinguish the sound of speech and the sound of music from each other. This is precisely because the total sound wave reaching your ears is the algebraic sum of the wave produced by your friend's voice and the wave produced by the speakers of your stereo. If two sound waves did *not* combine in this simple linear way, the sound you would hear in this situation would be a hopeless jumble. Superposition also applies to electromagnetic waves (such as light) and many other types of waves.

Test Your Understanding of Section 15.6 Figure 15.22 shows two wave pulses with different shapes traveling in different directions along a string. Make a series of sketches like Fig. 15.21 showing the shape of the string as the two pulses approach, overlap, and then pass each other.

15.22 Two wave pulses with different shapes.

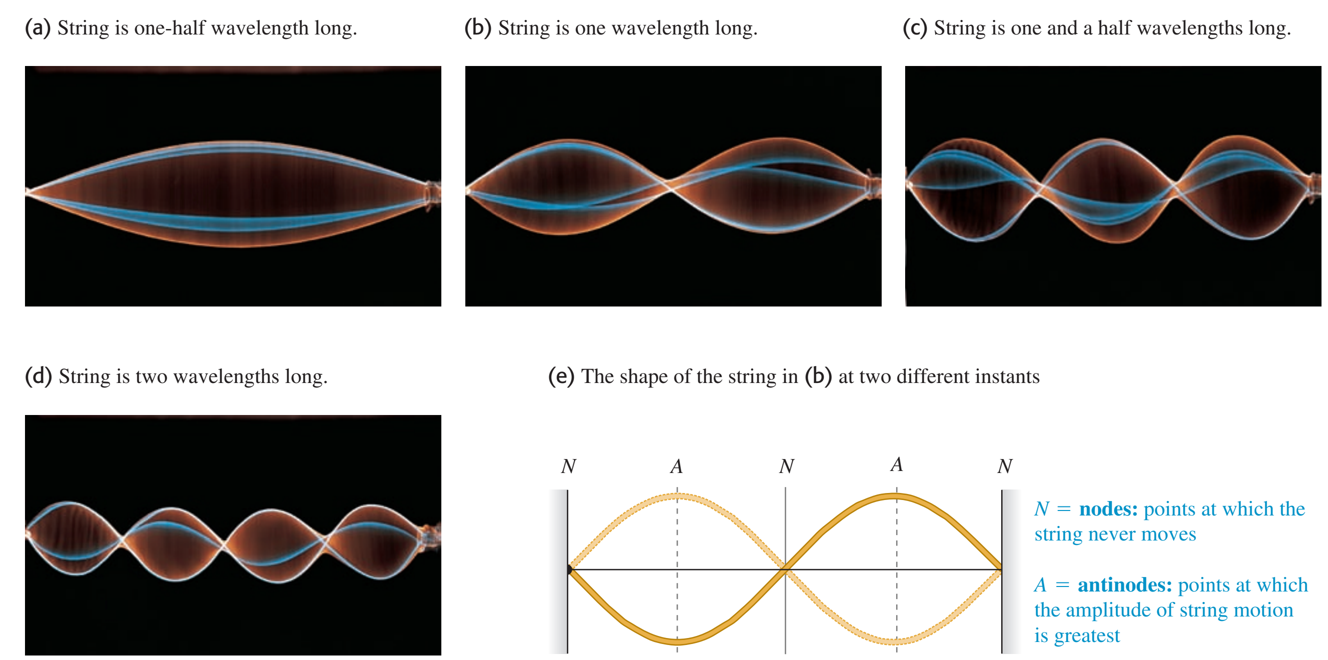


15.7 Standing Waves on a String

We have talked about the reflection of a wave *pulse* on a string when it arrives at a boundary point (either a fixed end or a free end). Now let's look at what happens when a *sinusoidal* wave is reflected by a fixed end of a string. We'll again approach the problem by considering the superposition of two waves propagating through the string, one representing the original or incident wave and the other representing the wave reflected at the fixed end.

Figure 15.23 shows a string that is fixed at its left end. Its right end is moved up and down in simple harmonic motion to produce a wave that travels to the left; the wave reflected from the fixed end travels to the right. The resulting motion when the two waves combine no longer looks like two waves traveling in opposite directions. The string appears to be subdivided into a number of segments, as in

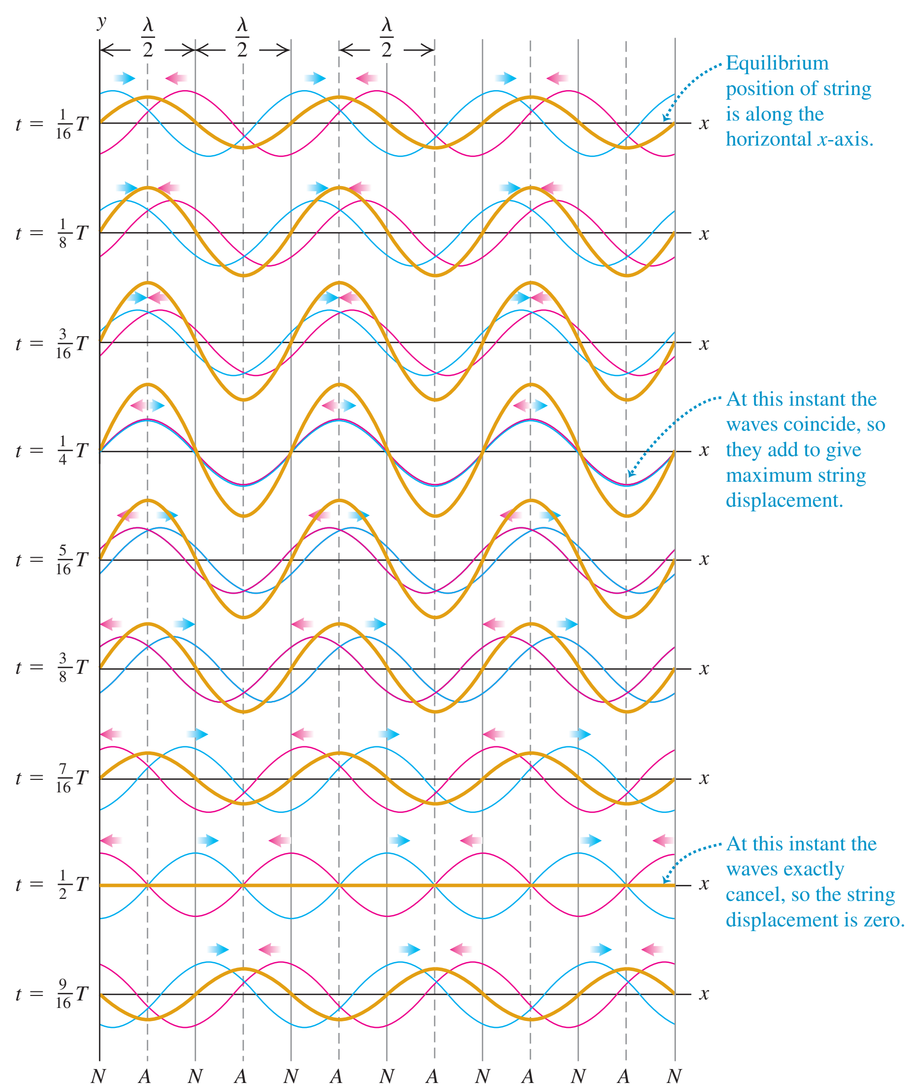
15.23 (a)–(d) Time exposures of standing waves in a stretched string. From (a) to (d), the frequency of oscillation of the right-hand end increases and the wavelength of the standing wave decreases. (e) The extremes of the motion of the standing wave in part (b), with nodes at the center and at the ends. The right-hand end of the string moves very little compared to the antinodes and so is essentially a node.



the time-exposure photographs of Figs. 15.23a, 15.23b, 15.23c, and 15.23d. Figure 15.23e shows two instantaneous shapes of the string in Fig. 15.23b. Let's compare this behavior with the waves we studied in Sections 15.1 through 15.5. In a wave that travels along the string, the amplitude is constant and the wave pattern moves with a speed equal to the wave speed. Here, instead, the wave pattern remains in the same position along the string and its amplitude fluctuates. There are particular points called **nodes** (labeled N in Fig. 15.23e) that never move at all. Midway between the nodes are points called **antinodes** (labeled A in Fig. 15.23e) where the amplitude of motion is greatest. Because the wave pattern doesn't appear to be moving in either direction along the string, it is called a **standing wave.** (To emphasize the difference, a wave that *does* move along the string is called a **traveling wave.**)

The principle of superposition explains how the incident and reflected waves combine to form a standing wave. In Fig. 15.24 the red curves show a wave traveling to the left. The blue curves show a wave traveling to the right with the same propagation speed, wavelength, and amplitude. The waves are shown at nine instants, $\frac{1}{16}$ of a period apart. At each point along the string, we add the displacements (the values of y) for the two separate waves; the result is the total wave on the string, shown in brown.

15.24 Formation of a standing wave. A wave traveling to the left (red curves) combines with a wave traveling to the right (blue curves) to form a standing wave (brown curves).



At certain instants, such as $t = \frac{1}{4}T$, the two wave patterns are exactly in phase with each other, and the shape of the string is a sine curve with twice the amplitude of either individual wave. At other instants, such as $t = \frac{1}{2}T$, the two waves are exactly out of phase with each other, and the total wave at that instant is zero. The resultant displacement is *always* zero at those places marked N at the bottom of Fig. 15.24. These are the *nodes.* At a node the displacements of the two waves in red and blue are always equal and opposite and cancel each other out. This cancellation is called **destructive interference.** Midway between the nodes are the points of *greatest* amplitude, or the *antinodes,* marked A. At the antinodes the displacements of the two waves in red and blue are always identical, giving a large resultant displacement; this phenomenon is called **constructive interference.** We can see from the figure that the distance between successive nodes or between successive antinodes is one half-wavelength, or $\lambda/2$.

We can derive a wave function for the standing wave of Fig. 15.24 by adding the wave functions $y_1(x, t)$ and $y_2(x, t)$ for two waves with equal amplitude, period, and wavelength traveling in opposite directions. Here $y_1(x, t)$ (the red curves in Fig. 15.24) represents an incoming, or *incident,* wave traveling to the left along the $+x$-axis, arriving at the point $x = 0$ and being reflected; $y_2(x, t)$ (the blue curves in Fig. 15.24) represents the *reflected* wave traveling to the right from $x = 0$. We noted in Section 15.6 that the wave reflected from a fixed end of a string is inverted, so we give a negative sign to one of the waves:

$$y_1(x, t) = -A\cos(kx + \omega t) \qquad \text{(incident wave traveling to the left)}$$

$$y_2(x, t) = A\cos(kx - \omega t) \qquad \text{(reflected wave traveling to the right)}$$

Note also that the change in sign corresponds to a shift in *phase* of 180° or π radians. At $x = 0$ the motion from the reflected wave is $A\cos\omega t$ and the motion from the incident wave is $-A\cos\omega t$, which we can also write as $A\cos(\omega t + \pi)$. From Eq. (15.27), the wave function for the standing wave is the sum of the individual wave functions:

$$y(x, t) = y_1(x, t) + y_2(x, t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$$

We can rewrite each of the cosine terms by using the identities for the cosine of the sum and difference of two angles: $\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$. Applying these and combining terms, we obtain the wave function for the standing wave:

$$y(x, t) = y_1(x, t) + y_2(x, t) = (2A\sin kx)\sin\omega t \qquad \text{or}$$

$$y(x, t) = (A_{SW}\sin kx)\sin\omega t \qquad \text{(standing wave on a string, fixed end at } x = 0\text{)} \qquad (15.28)$$

The standing wave amplitude A_{SW} is twice the amplitude A of either of the original traveling waves:

$$A_{SW} = 2A$$

Equation (15.28) has two factors: a function of x and a function of t. The factor $A_{SW}\sin kx$ shows that at each instant the shape of the string is a sine curve. But unlike a wave traveling along a string, the wave shape stays in the same position, oscillating up and down as described by the $\sin\omega t$ factor. This behavior is shown graphically by the brown curves in Fig. 15.24. Each point in the string still undergoes simple harmonic motion, but all the points between any successive pair of nodes oscillate *in phase.* This is in contrast to the phase differences between oscillations of adjacent points that we see with a wave traveling in one direction.

We can use Eq. (15.28) to find the positions of the nodes; these are the points for which $\sin kx = 0$, so the displacement is *always* zero. This occurs when $kx = 0, \pi, 2\pi, 3\pi, \ldots$, or, using $k = 2\pi/\lambda$,

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}, \ldots = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots \quad \text{(nodes of a standing wave on a string, fixed end at } x = 0\text{)} \tag{15.29}$$

In particular, there is a node at $x = 0$, as there should be, since this point is a fixed end of the string.

A standing wave, unlike a traveling wave, *does not* transfer energy from one end to the other. The two waves that form it would individually carry equal amounts of power in opposite directions. There is a local flow of energy from each node to the adjacent antinodes and back, but the *average* rate of energy transfer is zero at every point. If you evaluate the wave power given by Eq. (15.21) using the wave function of Eq. (15.28), you will find that the average power is zero (see Challenge Problem 15.84).

Problem-Solving Strategy 15.2 Standing Waves

IDENTIFY *the relevant concepts:* As with traveling waves, it's useful to distinguish between the purely kinematic quantities, such as wave speed v, wavelength λ, and frequency f, and the dynamic quantities involving the properties of the medium, such as F and μ for transverse waves on a string. Once you decide what the target variable is, try to determine whether the problem is only kinematic in nature or whether the properties of the medium are also involved.

SET UP *the problem* using the following steps:

1. To visualize nodes and antinodes in standing waves, it is always helpful to draw diagrams. For a string you can draw the shape at one instant and label the nodes N and antinodes A. The distance between two adjacent nodes or two adjacent antinodes is always $\lambda/2$, and the distance between a node and the adjacent antinode is always $\lambda/4$.
2. Decide which equations you'll need to use. The wave function for the standing wave, like Eq. (15.28), is almost always useful.
3. You can compute the wave speed if you know either λ and f (or, equivalently, $k = 2\pi/\lambda$ and $\omega = 2\pi f$) or the properties of the medium (for a string, F and μ).

EXECUTE *the solution* as follows: Solve for the unknown quantities using the equations you've selected. Once you have the wave function, you can find the value of the displacement y at any point in the wave medium (value of x) and at any time. You can find the velocity of a particle in the wave medium by taking the partial derivative of y with respect to time. To find the acceleration of such a particle, take the second partial derivative of y with respect to time.

EVALUATE *your answer:* Compare your numerical answers with your diagram. Check that the wave function is compatible with the boundary conditions (for example, the displacement should be zero at a fixed end).

Example 15.6 Standing waves on a guitar string

One of the strings of a guitar lies along the x-axis when in equilibrium. The end of the string at $x = 0$ (the bridge of the guitar) is tied down. An incident sinusoidal wave, corresponding to the red curves in Fig. 15.24, travels along the string in the $-x$-direction at 143 m/s with an amplitude of 0.750 mm and a frequency of 440 Hz. This wave is reflected from the fixed end at $x = 0$, and the superposition of the incident traveling wave and the reflected traveling wave forms a standing wave. (a) Find the equation giving the displacement of a point on the string as a function of position and time. (b) Locate the points on the string that don't move at all. (c) Find the amplitude, maximum transverse velocity, and maximum transverse acceleration at the points of maximum oscillation.

SOLUTION

IDENTIFY: This is a *kinematics* problem in which we are asked to describe the motion of the string (see Problem-Solving Strategy 15.1 in Section 15.3). The target variables are the wave function of the standing wave in part (a), the locations of the points that don't move, or the *nodes* in part (b), and the maximum values of displacement y, transverse velocity v_y, and transverse acceleration a_y in part (c). (Waves on a string are transverse waves, so *transverse* means "in the direction of the displacement"—that is, in the y-direction.) To find these quantities we use the expression that we derived in this section for a standing wave on a string with a fixed end, as well as other relationships from Sections 15.2 and 15.3.

SET UP: Since there is a fixed end at $x = 0$, we may use Eqs. (15.28) and (15.29) to describe this standing wave. We also use the relationships among ω, k, f, λ, and the wave speed v.

EXECUTE: (a) To use Eq. (15.28) we need the values of A_{SW}, ω, and k. The amplitude of the incident wave is $A = 0.750\text{ mm} = 7.50 \times 10^{-4}\text{ m}$; the reflected wave has the same amplitude, and the standing wave amplitude is $A_{SW} = 2A = 1.50 \times 10^{-3}\text{ m}$. The angular frequency ω and wave number k are

$$\omega = 2\pi f = (2\pi\text{ rad})(440\text{ s}^{-1}) = 2760\text{ rad/s}$$

$$k = \frac{\omega}{v} = \frac{2760\text{ rad/s}}{143\text{ m/s}} = 19.3\text{ rad/m}$$

Then Eq. (15.28) gives

$$y(x, t) = (A_{SW}\sin kx)\sin\omega t = [(1.50 \times 10^{-3}\text{ m})\sin(19.3\text{ rad/m})x]\sin(2760\text{ rad/s})t$$

(b) The positions of the nodes are given by Eq. (15.29): $x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$. The wavelength is

$$\lambda = \frac{v}{f} = \frac{143\text{ m/s}}{440\text{ Hz}} = 0.325\text{ m}$$

so the nodes are at the following distances from the fixed end:

$$x = 0, 0.163\text{ m}, 0.325\text{ m}, 0.488\text{ m}, \ldots$$

(c) From the expression in part (a) for $y(x, t)$, we see that the maximum displacement from equilibrium is $1.50 \times 10^{-3}\text{ m} = 1.50\text{ mm}$, which is just twice the amplitude of the incident wave. This maximum occurs at the *antinodes,* which are midway between adjacent nodes (that is, at $x = 0.081\text{ m}, 0.244\text{ m}, 0.406\text{ m}, \ldots$).

For a particle on the string at any point x, the transverse (y-) velocity is

$$v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = [(1.50 \times 10^{-3}\text{ m})\sin(19.3\text{ rad/m})x] \times [(2760\text{ rad/s})\cos(2760\text{ rad/s})t] = [(4.15\text{ m/s})\sin(19.3\text{ rad/m})x]\cos(2760\text{ rad/s})t$$

At an antinode, $\sin(19.3\text{ rad/m})x = \pm 1$ and the transverse velocity varies in value between 4.15 m/s and -4.15 m/s. As is always the case in simple harmonic motion, the maximum velocity occurs when the particle is passing through the equilibrium position ($y = 0$).

The transverse acceleration $a_y(x, t)$ is the first partial derivative of $v_y(x, t)$ with respect to time (that is, the *second* partial derivative of $y(x, t)$ with respect to time). We leave the calculation to you; the result is

$$a_y(x, t) = \frac{\partial v_y(x, t)}{\partial t} = \frac{\partial^2 y(x, t)}{\partial t^2} = [(-1.15 \times 10^4\text{ m/s}^2)\sin(19.3\text{ rad/m})x] \times \sin(2760\text{ rad/s})t$$

At the antinodes, the transverse acceleration varies in value between $+1.15 \times 10^4\text{ m/s}^2$ and $-1.15 \times 10^4\text{ m/s}^2$.

EVALUATE: The maximum transverse velocity at an antinode is quite respectable (about 15 km/h, or 9.3 mi/h). But the maximum transverse acceleration is tremendous, 1170 times the acceleration due to gravity! Guitar strings are made of sturdy stuff to be able to withstand such acceleration.

Guitar strings are actually tied down at *both* ends. We'll see the consequences of this in the next section.

Test Your Understanding of Section 15.7 Suppose the frequency of the standing wave in Example 15.6 were doubled from 440 Hz to 880 Hz. Would all of the nodes for $f = 440$ Hz also be nodes for $f = 880$ Hz? If so, would there be additional nodes for $f = 880$ Hz? If not, which nodes are absent for $f = 880$ Hz?

15.8 Normal Modes of a String

ActivPhysics Online
10.4 Standing Waves on Strings
10.5 Tuning a Stringed Instrument: Standing Waves
10.6 String Mass and Standing Waves

When we described standing waves on a string rigidly held at one end, as in Fig. 15.23, we made no assumptions about the length of the string or about what was happening at the other end. Let's now consider a string of a definite length L, rigidly held at *both* ends. Such strings are found in many musical instruments, including pianos, violins, and guitars. When a guitar string is plucked, a wave is produced in the string; this wave is reflected and re-reflected from the ends of the string, making a standing wave. This standing wave on the string in turn produces a sound wave in the air, with a frequency determined by the properties of the string. This is what makes stringed instruments so useful in making music.

To understand these properties of standing waves on a string fixed at both ends, let's first examine what happens when we set up a sinusoidal wave on such a string. The standing wave that results must have a node at *both* ends of the string. We saw in the preceding section that adjacent nodes are one half-wavelength ($\lambda/2$) apart, so the length of the string must be $\lambda/2$, or $2(\lambda/2)$, or $3(\lambda/2)$, or in general some integer number of half-wavelengths:

$$L = n\frac{\lambda}{2} \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \tag{15.30}$$

15.25 Each string of a violin naturally oscillates at one or more of its harmonic frequencies, producing sound waves in the air with the same frequencies.

That is, if a string with length L is fixed at both ends, a standing wave can exist only if its wavelength satisfies Eq. (15.30).

Solving this equation for λ and labeling the possible values of λ as λ_n, we find

$$\lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \tag{15.31}$$

Waves can exist on the string if the wavelength is *not* equal to one of these values, but there cannot be a steady wave pattern with nodes and antinodes, and the total wave cannot be a standing wave. Equation (15.31) is illustrated by the standing waves shown in Figs. 15.23a, 15.23b, 15.23c, and 15.23d; these represent $n = 1, 2, 3$, and 4, respectively.

Corresponding to the series of possible standing-wave wavelengths λ_n is a series of possible standing-wave frequencies f_n, each related to its corresponding wavelength by $f_n = v/\lambda_n$. The smallest frequency f_1 corresponds to the largest wavelength (the $n = 1$ case), $\lambda_1 = 2L$:

$$f_1 = \frac{v}{2L} \quad \text{(string fixed at both ends)} \tag{15.32}$$

This is called the **fundamental frequency.** The other standing-wave frequencies are $f_2 = 2v/2L$, $f_3 = 3v/2L$, and so on. These are all integer multiples of the fundamental frequency f_1, such as $2f_1$, $3f_1$, $4f_1$, and so on, and we can express *all* the frequencies as

$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots) \quad \text{(string fixed at both ends)} \tag{15.33}$$

These frequencies are called **harmonics,** and the series is called a **harmonic series.** Musicians sometimes call f_2, f_3, and so on **overtones;** f_2 is the second harmonic or the first overtone, f_3 is the third harmonic or the second overtone, and so on. The first harmonic is the same as the fundamental frequency (Fig. 15.25).

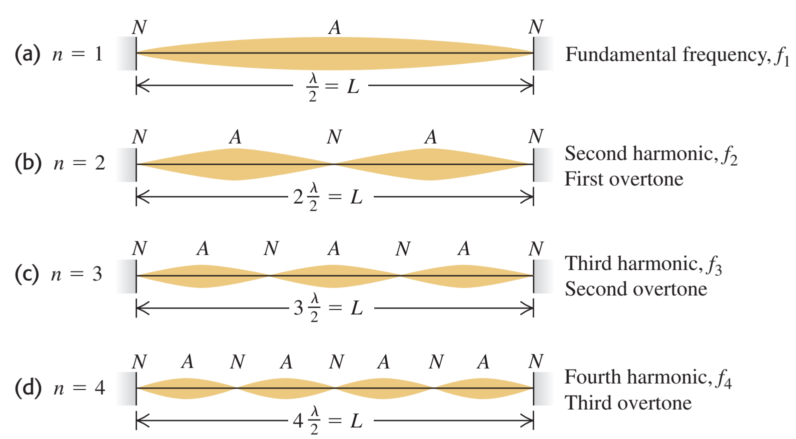
For a string with fixed ends at $x = 0$ and $x = L$, the wave function $y(x, t)$ of the nth standing wave is given by Eq. (15.28) (which satisfies the condition that there is a node at $x = 0$), with $\omega = \omega_n = 2\pi f_n$ and $k = k_n = 2\pi/\lambda_n$:

$$y_n(x, t) = A_{SW} \sin k_n x \sin \omega_n t \tag{15.34}$$

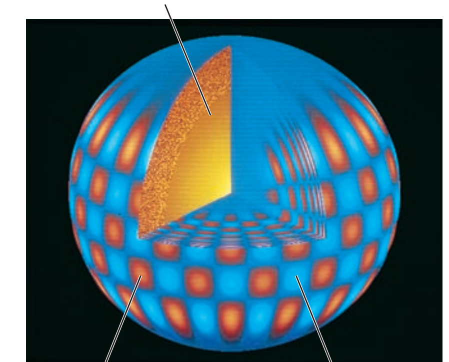
You can easily show that this wave function has nodes at both $x = 0$ and $x = L$, as it must.

A **normal mode** of an oscillating system is a motion in which all particles of the system move sinusoidally with the same frequency. For a system made up of a string of length L fixed at both ends, each of the wavelengths given by Eq. (15.31) corresponds to a possible normal-mode pattern and frequency. There are infinitely many normal modes, each with its characteristic frequency and vibration pattern. Figure 15.26 shows the first four normal-mode patterns and their associated frequencies and wavelengths; these correspond to Eq. (15.34) with $n = 1, 2, 3$, and 4. By contrast, a harmonic oscillator, which has only one oscillating particle, has only one normal mode and one characteristic frequency. The string fixed at both ends has infinitely many normal modes because it is made up of a very large (effectively infinite) number of particles. More complicated oscillating systems also have infinite numbers of normal modes, though with more complex normal-mode patterns than a string (Fig. 15.27).

15.26 The first four normal modes of a string fixed at both ends. (Compare these to the photographs in Fig. 15.23.)



15.27 Astronomers have discovered that the sun oscillates in several different normal modes. This computer simulation shows one such mode.



Complex Standing Waves

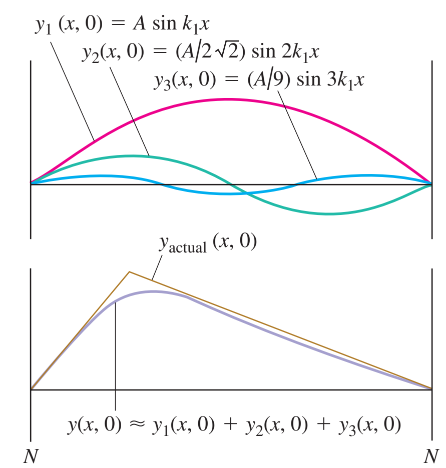
If we could displace a string so that its shape is the same as one of the normal-mode patterns and then release it, it would vibrate with the frequency of that mode. Such a vibrating string would displace the surrounding air with the same frequency, producing a traveling sinusoidal sound wave that your ears would perceive as a pure tone. But when a string is struck (as in a piano) or plucked (as is done to guitar strings), the shape of the displaced string is *not* as simple as one of the patterns in Fig. 15.26. The fundamental as well as many overtones are present in the resulting vibration. This motion is therefore a combination or *superposition* of many normal modes. Several simple-harmonic motions of different frequencies are present simultaneously, and the displacement of any point on the string is the sum (or superposition) of displacements associated with the individual modes. The sound produced by the vibrating string is likewise a superposition of traveling sinusoidal sound waves, which you perceive as a rich, complex tone with the fundamental frequency f_1. The standing wave on the string and the traveling sound wave in the air have similar **harmonic content** (the extent to which frequencies higher than the fundamental are present). The harmonic content depends on how the string is initially set into motion. If you pluck the strings of an acoustic guitar in the normal location over the sound hole, the sound that you hear has a different harmonic content than if you pluck the strings next to the fixed end on the guitar body.

ActivPhysics Online

10.10 Complex Waves: Fourier Analysis

It is possible to represent every possible motion of the string as some superposition of normal-mode motions. Finding this representation for a given vibration pattern is called *harmonic analysis.* The sum of sinusoidal functions that represents a complex wave is called a *Fourier series.* Figure 15.28 shows how a standing wave that is produced by plucking a guitar string of length L at a point $L/4$ from one end can be represented as a combination of sinusoidal functions.

15.28 When a guitar string is plucked (pulled into a triangular shape) and released, a standing wave results. The standing wave is well represented (except at the sharp maximum point) by the sum of just three sinusoidal functions. Including additional sinusoidal functions further improves the representation.



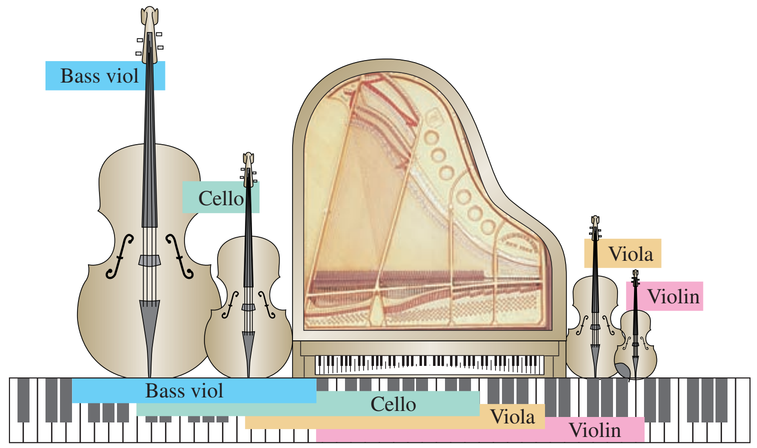
Standing Waves and String Instruments

As we have seen, the fundamental frequency of a vibrating string is $f_1 = v/2L$. The speed v of waves on the string is determined by Eq. (15.13), $v = \sqrt{F/\mu}$. Combining these equations, we find

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \quad \text{(string fixed at both ends)} \tag{15.35}$$

This is also the fundamental frequency of the sound wave created in the surrounding air by the vibrating string. Familiar musical instruments show how f_1 depends on the properties of the string. The inverse dependence of frequency on length L is illustrated by the long strings of the bass (low-frequency) section of the piano or the bass viol compared with the shorter strings on the treble section of the piano or the violin (Fig. 15.29). The pitch of a violin or guitar is usually varied by pressing a string against the fingerboard with the fingers to change the length L of the vibrating portion of the string. Increasing the tension F increases the wave speed v and thus increases the frequency (and the pitch). All string instruments are "tuned" to the correct frequencies by varying the tension; you tighten the string to raise the pitch. Finally, increasing the mass per unit length μ decreases the wave

15.29 Comparing the range of a concert grand piano to the ranges of a bass viol, a cello, a viola, and a violin. In all cases, longer strings produce bass notes and shorter strings produce treble notes.



speed and thus the frequency. The lower notes on a steel guitar are produced by thicker strings, and one reason for winding the bass strings of a piano with wire is to obtain the desired low frequency from a relatively short string.

Wind instruments such as saxophones and trombones also have normal modes. As for stringed instruments, the frequencies of these normal modes determine the pitch of the musical tones that these instruments produce. We'll discuss these instruments and many other aspects of sound in Chapter 16.

Example 15.7 A giant bass viol

In an effort to get your name in the *Guinness Book of World Records,* you set out to build a bass viol with strings that have a length of 5.00 m between fixed points. One string has a linear mass density of 40.0 g/m and a fundamental frequency of 20.0 Hz (the lowest frequency that the human ear can hear). Calculate (a) the tension of this string, (b) the frequency and wavelength on the string of the second harmonic, and (c) the frequency and wavelength on the string of the second overtone.

SOLUTION

IDENTIFY: The target variable in part (a) is the string tension; we find this from the expression for the fundamental frequency of the string, which involves the tension. In parts (b) and (c) the target variables are the frequency and wavelength of different harmonics. We determine these from the given length of the string and the fundamental frequency.

SET UP: For part (a), the equation to use is Eq. (15.35); it involves the known values of f_1, L, and μ as well as the target variable F. We solve parts (b) and (c) using Eqs. (15.31) and (15.33).

EXECUTE: (a) We solve Eq. (15.35) for the string tension F:

$$F = 4\mu L^2 f_1^2 = 4(40.0 \times 10^{-3}\ \text{kg/m})(5.00\ \text{m})^2(20.0\ \text{s}^{-1})^2$$
$$= 1600\ \text{N} = 360\ \text{lb}$$

(b) The second harmonic is denoted by $n = 2$. From Eq. (15.33), the second harmonic frequency is

$$f_2 = 2f_1 = 2(20.0\ \text{Hz}) = 40.0\ \text{Hz}$$

From Eq. (15.31), the wavelength on the string of the second harmonic is

$$\lambda_2 = \frac{2L}{2} = 5.00\ \text{m}$$

(c) The second overtone is the "second tone over" (above) the fundamental—that is, $n = 3$. Its frequency and wavelength are

$$f_3 = 3f_1 = 3(20.0\ \text{Hz}) = 60.0\ \text{Hz}$$
$$\lambda_3 = \frac{2L}{3} = 3.33\ \text{m}$$

EVALUATE: The tension in part (a) is a bit larger than in a real bass viol, for which the string tension is typically a few hundred newtons. The wavelengths in parts (b) and (c) are equal to the length of the string and two-thirds the length of the string, respectively; these results agree with the drawings of standing waves in Fig. 15.26.

Example 15.8 From waves on a string to sound waves in air

What are the frequency and wavelength of the sound waves produced in the air when the string in Example 15.7 is vibrating at its fundamental frequency? The speed of sound in air at 20°C is 344 m/s.

SOLUTION

IDENTIFY: Our target variables are f and λ for the sound wave produced by the bass viol, *not* for the standing wave on the string.

However, when the string vibrates at a particular frequency, the surrounding air is forced to vibrate at the same frequency. So the frequency of the sound wave is the same as that of the standing wave on the string. The relationship $\lambda = v/f$ shows that the *wavelength* of the sound wave is typically different from the wavelength of the standing wave on the string, because the two waves have different speeds.

SET UP: The only equation we need is $v = \lambda f$. We apply this to both the standing wave on the string (speed v_{string}) and the traveling sound wave (speed v_{sound}).

EXECUTE: The sound wave frequency is the same as the standing-wave fundamental frequency: $f = f_1 = 20.0$ Hz. The wavelength of the sound wave is

$$\lambda_{1(\text{sound})} = \frac{v_{\text{sound}}}{f_1} = \frac{344\ \text{m/s}}{20.0\ \text{Hz}} = 17.2\ \text{m}$$

EVALUATE: Note that $\lambda_{1(\text{sound})}$ is greater than the wavelength of the standing wave on the string, $\lambda_{1(\text{string})} = 2L = 2(5.00\ \text{m}) = 10.0\ \text{m}$. This is because the speed of sound is greater than the speed of waves on the string, $v_{\text{string}} = \lambda_{1(\text{string})} f_1 = (10.0\ \text{m})(20.0\ \text{Hz}) = 200\ \text{m/s}$. Hence, for *any* normal mode on this string, the sound wave that is produced has the same frequency as the wave on the string but a wavelength that is greater by a factor of $v_{\text{sound}}/v_{\text{string}} = (344\ \text{m/s})/(200\ \text{m/s}) = 1.72$.

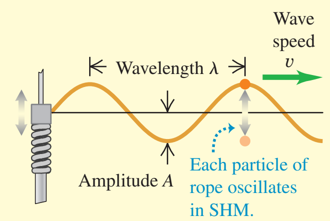
Test Your Understanding of Section 15.8 While a guitar string is vibrating, you gently touch the midpoint of the string to ensure that the string does not vibrate at that point. Which normal modes *cannot* be present on the string while you are touching it in this way?

CHAPTER 15 SUMMARY

Waves and their properties: A wave is any disturbance from an equilibrium condition that propagates from one region to another. A mechanical wave always travels within some material called the medium. The wave disturbance propagates at the wave speed v, which depends on the type of wave and the properties of the medium.

In a periodic wave, the motion of each point of the medium is periodic. A sinusoidal wave is a special periodic wave in which each point moves in simple harmonic motion. For any periodic wave, the frequency f is the number of cycles per unit time, the period T is the time for one cycle, the wavelength λ is the distance over which the wave pattern repeats, and the amplitude A is the maximum displacement of a particle in the medium. The product of λ and f equals the wave speed. (See Example 15.1.)

$$v = \lambda f \quad (15.1)$$



Wave functions and wave dynamics: The wave function $y(x, t)$ describes the displacements of individual particles in the medium. Equations (15.3), (15.4), and (15.7) give the wave equation for a sinusoidal wave traveling in the $+x$-direction. If the wave is moving in the $-x$-direction, the minus signs in the cosine functions are replaced by plus signs. (See Example 15.2.)

The wave function obeys a partial differential equation called the wave equation, Eq. (15.12).

The speed of transverse waves on a string depends on the tension F and mass per unit length μ. (See Example 15.3.)

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right] = A\cos 2\pi f\left(\frac{x}{v} - t\right) \quad (15.3)$$

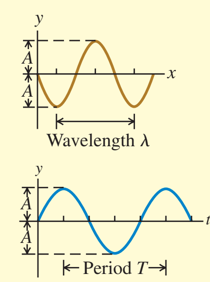
$$y(x, t) = A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \quad (15.4)$$

$$y(x, t) = A\cos(kx - \omega t) \quad (15.7)$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi f = vk$

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2} \quad (15.12)$$

$$v = \sqrt{\frac{F}{\mu}} \quad \text{(waves on a string)} \quad (15.13)$$



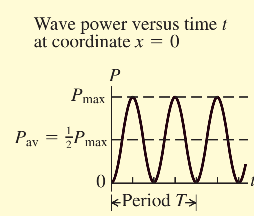
Wave power: Wave motion conveys energy from one region to another. For a sinusoidal mechanical wave, the average power P_{av} is proportional to the square of the wave amplitude and the square of the frequency. For waves that spread out in three dimensions, the wave intensity I is inversely proportional to the distance from the source. (See Examples 15.4 and 15.5.)

$$P_{av} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2 \quad (15.25)$$

(average power, sinusoidal wave)

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \quad (15.26)$$

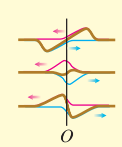
(inverse-square law for intensity)



Wave superposition: A wave that reaches a boundary of the medium in which it propagates is reflected. The principle of superposition states that the total wave displacement at any point where two or more waves overlap is the sum of the displacements of the individual waves.

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (15.27)$$

(principle of superposition)



Standing waves on a string: When a sinusoidal wave is reflected from a fixed or free end of a stretched string, the incident and reflected waves combine to form a standing sinusoidal wave with nodes and antinodes. Adjacent nodes are spaced a distance $\lambda/2$ apart, as are adjacent antinodes. (See Example 15.6.)

When both ends of a string with length L are held fixed, standing waves can occur only when L is an integer multiple of $\lambda/2$. Each frequency with its associated vibration pattern is called a normal mode. The lowest frequency f_1 is called the fundamental frequency. (See Examples 15.7 and 15.8.)

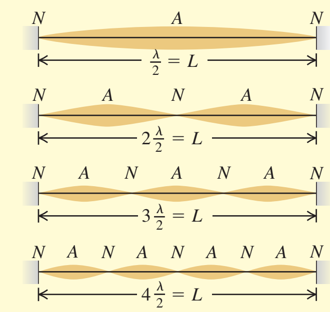
$$y(x, t) = (A_{SW}\sin kx)\sin\omega t \quad (15.28)$$

(standing on a wave on a string, fixed end at $x = 0$)

$$f_n = n\frac{v}{2L} = nf_1 \quad (n = 1, 2, 3, \ldots) \quad (15.33)$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \quad (15.35)$$

(string fixed at both ends)



Key Terms

Answer to Chapter Opening Question ?

The power of a mechanical wave depends on its frequency and amplitude [see Eq. (15.25)].

Answers to Test Your Understanding Questions

15.1 Answer: (i) The "wave" travels horizontally from one spectator to the next along each row of the stadium, but the displacement of each spectator is vertically upward. Since the displacement is perpendicular to the direction in which the wave travels, the wave is transverse.

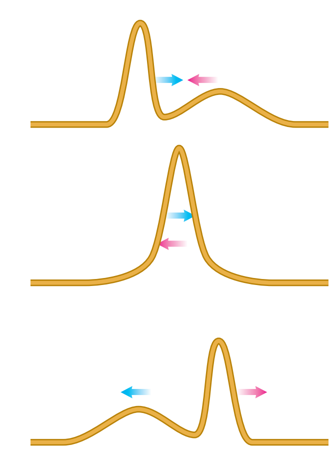
15.2 Answer: (iv) The speed of waves on a string, v, does not depend on the wavelength. We can rewrite the relationship $v = \lambda f$ as $f = v/\lambda$, which tells us that if the wavelength λ doubles, the frequency f becomes one-half as great.

15.3 Answers: (a) $\frac{2}{8}T$, **(b)** $\frac{4}{8}T$, **(c)** $\frac{5}{8}T$ Since the wave is sinusoidal, each point on the string oscillates in simple harmonic motion (SHM). Hence we can apply all of the ideas from Chapter 13 about SHM to the wave depicted in Fig. 15.8. (a) A particle in SHM has its maximum speed when it is passing through the equilibrium position ($y = 0$ in Fig. 15.8). The particle at point A is moving upward through this position at $t = \frac{2}{8}T$. (b) In vertical SHM the greatest *upward* acceleration occurs when a particle is at its maximum *downward* displacement. This occurs for the particle at point B at $t = \frac{4}{8}T$. (c) A particle in vertical SHM has a *downward* acceleration when its displacement is *upward*. The particle at C has an upward displacement and is moving downward at $t = \frac{5}{8}T$.

15.4 Answer: (ii) The relationship $v = \sqrt{F/\mu}$ [Eq. (15.13)] says that the wave speed is greatest on the string with the smallest linear mass density. This is the thinnest string, which has the smallest amount of mass m and hence the smallest linear mass density $\mu = m/L$ (all strings are the same length).

15.5 Answer: (iii), (iv), (ii), (i) Equation (15.25) says that the average power in a sinusoidal wave on a string is $P_{av} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2$. All four strings are identical, so all have the same mass, the same length, and the same linear mass density μ. The frequency f is the same for each wave, as is the angular frequency $\omega = 2\pi f$. Hence the average wave power for each string is proportional to the square root of the string tension F and the square of the amplitude A. Compared to string (i), the average power in each string is (ii) $\sqrt{4} = 2$ times greater; (ii) $4^2 = 16$ times greater; and (iv) $\sqrt{2}(2)^2 = 4\sqrt{2}$ times greater.

15.6 Answer:



15.7 Answers: yes, yes Doubling the frequency makes the wavelength half as large. Hence the spacing between nodes (equal to $\lambda/2$) is also half as large. There are nodes at all of the previous positions, but there is also a new node between every pair of old nodes.

15.8 Answers: $n = 1, 3, 5, \ldots$ When you touch the string at its center, you are demanding that there be a node at the center. Hence only standing waves with a node at $x = L/2$ are allowed. From Figure 15.26 you can see that the normal modes $n = 1, 3, 5, \ldots$ cannot be present.

PROBLEMS

Discussion Questions

Q15.1. Two waves travel on the same string. Is it possible for them to have (a) different frequencies; (b) different wavelengths; (c) different speeds; (d) different amplitudes; (e) the same frequency but different wavelengths? Explain your reasoning.

Q15.2. Under a tension F, it takes 2.00 s for a pulse to travel the length of a taut wire. What tension is required (in terms of F) for the pulse to take 6.00 s instead?

Q15.3. What kinds of energy are associated with waves on a stretched string? How could you detect such energy experimentally?

Q15.4. The amplitude of a wave decreases gradually as the wave travels down a long, stretched string. What happens to the energy of the wave when this happens?

Q15.5. For the wave motions discussed in this chapter, does the speed of propagation depend on the amplitude? What makes you say this?

Q15.6. The speed of ocean waves depends on the depth of the water; the deeper the water, the faster the wave travels. Use this to explain why ocean waves crest and "break" as they near the shore.

Q15.7. Is it possible to have a longitudinal wave on a stretched string? Why or why not? Is it possible to have a transverse wave on a steel rod? Again, why or why not? If your answer is yes in either case, explain how you would create such a wave.

Q15.8. An echo is sound reflected from a distant object, such as a wall or a cliff. Explain how you can determine how far away the object is by timing the echo.

Q15.9. Why do you see lightning before you hear the thunder? A familiar rule of thumb is to start counting slowly, once per second, when you see the lightning; when you hear the thunder, divide the number you have reached by 3 to obtain your distance from the lightning in kilometers (or divide by 5 to obtain your distance in miles). Why does this work, or does it?

Q15.10. For transverse waves on a string, is the wave speed the same as the speed of any part of the string? Explain the difference between these two speeds. Which one is constant?

Q15.11. Children make toy telephones by sticking each end of a long string through a hole in the bottom of a paper cup and knotting it so it will not pull out. When the spring is pulled taut, sound can be transmitted from one cup to the other. How does this work? Why is the transmitted sound louder than the sound traveling through air for the same distance?

Q15.12. The four strings on a violin have different thicknesses, but are all under approximately the same tension. Do waves travel faster on the thick strings or the thin strings? Why? How does the fundamental vibration frequency compare for the thick versus the thin strings?

Q15.13. A sinusoidal wave can be described by a cosine function, which is negative just as often as positive. So why isn't the average power delivered by this wave zero?

Q15.14. Two strings of different mass per unit length μ_1 and μ_2 are tied together and stretched with a tension F. A wave travels along the string and passes the discontinuity in μ. Which of the following wave properties will be the same on both sides of the discontinuity, and which ones will change? speed of the wave; frequency; wavelength. Explain the physical reasoning behind each of your answers.

Q15.15. A long rope with mass m is suspended from the ceiling and hangs vertically. A wave pulse is produced at the lower end of the rope, and the pulse travels up the rope. Does the speed of the wave pulse change as it moves up the rope, and if so, does it increase or decrease?

Q15.16. In a transverse wave on a string, the motion of the string is perpendicular to the length of the string. How, then, is it possible for energy to move along the length of the string?

Q15.17. Both wave intensity and gravitation obey inverse-square laws. Do they do so for the same reason? Discuss the reason for each of these inverse-square laws as well as you can.

Q15.18. Energy can be transferred along a string by wave motion. However, in a standing wave on a string, no energy can ever be transferred past a node. Why not?

Q15.19. Can a standing wave be produced on a string by superposing two waves traveling in opposite directions with the same frequency but different amplitudes? Why or why not? Can a standing wave be produced by superposing two waves traveling in opposite directions with different frequencies but the same amplitude? Why or why not?

Q15.20. If you stretch a rubber band and pluck it, you hear a (somewhat) musical tone. How does the frequency of this tone change as you stretch the rubber band further? (Try it!) Does this agree with Eq. (15.35) for a string fixed at both ends? Explain.

Q15.21. A musical interval of an *octave* corresponds to a factor of 2 in frequency. By what factor must the tension in a guitar or violin string be increased to raise its pitch one octave? To raise it two octaves? Explain your reasoning. Is there any danger in attempting these changes in pitch?

Q15.22. By touching a string lightly at its center while bowing, a violinist can produce a note exactly one octave above the note to which the string is tuned—that is, a note with exactly twice the frequency. Why is this possible?

Q15.23. As we discussed in Section 15.1, water waves are a combination of longitudinal and transverse waves. Defend the following statement: "When water waves hit a vertical wall, the wall is a node of the longitudinal displacement but an antinode of the transverse displacement."

Q15.24. Violins are short instruments, while cellos and basses are long. In terms of the frequency of the waves they produce, explain why this is so.

Q15.25. What is the purpose of the frets on a guitar? In terms of the frequency of the vibration of the strings, explain their use.

Exercises

Section 15.2 Periodic Waves

15.1. The speed of sound in air at 20°C is 344 m/s. (a) What is the wavelength of a sound wave with a frequency of 784 Hz, corresponding to the note G_5 on a piano, and how many milliseconds does each vibration take? (b) What is the wavelength of a sound wave one octave higher than the note in part (a)?

15.2. Audible Sound. Provided the amplitude is sufficiently great, the human ear can respond to longitudinal waves over a range of frequencies from about 20.0 Hz to about 20.0 kHz. (a) If you were to mark the beginning of each complete wave pattern with a red dot for the long-wavelength sound and a blue dot for the short-wavelength sound, how far apart would the red dots be, and how far apart would the blue dots be? (b) In reality would adjacent dots in each set be far enough apart for you to easily measure their separation with a meterstick? (c) Suppose you repeated part (a) in water, where sound travels at 1480 m/s. How far apart would the dots be in each set? Could you readily measure their separation with a meterstick?

15.3. Tsunami! On December 26, 2004, a great earthquake occurred off the coast of Sumatra and triggered immense waves (tsunami) that killed some 200,000 people. Satellites observing these waves from space measured 800 km from one wave crest to the next and a period between waves of 1.0 hour. What was the speed of these waves in m/s and km/h? Does your answer help you understand why the waves caused such devastation?

15.4. Ultrasound Imaging. Sound having frequencies above the range of human hearing (about 20,000 Hz) is called *ultrasound.* Waves above this frequency can be used to penetrate the body and to produce images by reflecting from surfaces. In a typical ultrasound scan, the waves travel through body tissue with a speed of 1500 m/s. For a good, detailed image, the wavelength should be no more than 1.0 mm. What frequency sound is required for a good scan?

15.5. Visible Light. Light is a wave, but not a mechanical wave. The quantities that oscillate are electric and magnetic fields. Light visible to humans has wavelengths between 400 nm (violet) and 700 nm (red), and all light travels through vacuum at speed $c = 3.00 \times 10^8$ m/s. (a) What are the limits of the frequency and period of visible light? (b) Could you time a single light vibration with a stopwatch?

Section 15.3 Mathematical Description of a Wave

15.6. A certain transverse wave is described by

$$y(x, t) = (6.50\ \text{mm})\cos 2\pi\left(\frac{x}{28.0\ \text{cm}} - \frac{t}{0.0360\ \text{s}}\right)$$

Determine the wave's (a) amplitude; (b) wavelength; (c) frequency; (d) speed of propagation; (e) direction of propagation.

15.7. Transverse waves on a string have wave speed 8.00 m/s, amplitude 0.0700 m, and wavelength 0.320 m. The waves travel in the $-x$-direction, and at $t = 0$ the $x = 0$ end of the string has its maximum upward displacement. (a) Find the frequency, period, and wave number of these waves. (b) Write a wave function describing the wave. (c) Find the transverse displacement of a particle at $x = 0.360$ m at time $t = 0.150$ s. (d) How much time must elapse from the instant in part (c) until the particle at $x = 0.360$ m next has maximum upward displacement?

15.8. A water wave traveling in a straight line on a lake is described by the equation

$$y(x, t) = (3.75\ \text{cm})\cos(0.450\ \text{cm}^{-1}\,x + 5.40\ \text{s}^{-1}\,t)$$

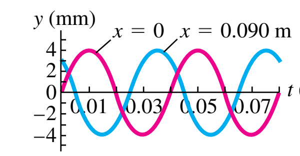
where y is the displacement perpendicular to the undisturbed surface of the lake. (a) How much time does it take for one complete wave pattern to go past a fisherman in a boat at anchor, and what horizontal distance does the wave crest travel in that time? (b) What are the wave number and the number of waves per second that pass the fisherman? (c) How fast does a wave crest travel past the fisherman, and what is the maximum speed of his cork floater as the wave causes it to bob up and down?

15.9. Which of the following wave functions satisfies the wave equation, Eq. (15.12)? (a) $y(x, t) = A\cos(kx + \omega t)$; (b) $y(x, t) = A\sin(kx + \omega t)$; (c) $y(x, t) = A(\cos kx + \cos \omega t)$. (d) For the wave of part (b), write the equations for the transverse velocity and transverse acceleration of a particle at point x.

15.10. A wave on a string is described by $y(x, t) = A\cos(kx - \omega t)$. (a) Graph y, v_y, and a_y as functions of x for time $t = 0$. (b) Consider the following points on the string: (i) $x = 0$; (ii) $x = \pi/4k$; (iii) $x = \pi/2k$; (iv) $x = 3\pi/4k$; (v) $x = \pi/k$; (vi) $x = 5\pi/4k$; (vii) $x = 3\pi/2k$; (viii) $x = 7\pi/4k$. For a particle at each of these points at $t = 0$, describe in words whether the particle is moving and in what direction, and whether the particle is speeding up, slowing down, or instantaneously not accelerating.

15.11. A sinusoidal wave is propagating along a stretched string that lies along the x-axis. The displacement of the string as a function of time is graphed in Fig.15.30 for particles at $x = 0$ and at $x = 0.0900$ m. (a) What is the amplitude of the wave? (b) What is the period of the wave? (c) You are told that the two points $x = 0$ and $x = 0.0900$ m are within one wavelength of each other. If the wave is moving in the $+x$-direction, determine the wavelength and the wave speed. (d) If instead the wave is moving in the $-x$-direction, determine the wavelength and the wave speed. (e) Would it be possible to determine definitively the wavelength in parts (c) and (d) if you were not told that the two points were within one wavelength of each other? Why or why not?

Figure 15.30 Exercise 15.11.



15.12. Speed of Propagation vs. Particle Speed. (a) Show that Eq. (15.3) may be written as

$$y(x, t) = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

(b) Use $y(x, t)$ to find an expression for the transverse velocity v_y of a particle in the string on which the wave travels. (c) Find the maximum speed of a particle of the string. Under what circumstances is this equal to the propagation speed v? Less than v? Greater than v?

15.13. A transverse wave on a string has amplitude 0.300 cm, wavelength 12.0 cm, and speed 6.00 cm/s. It is represented by $y(x, t)$ as given in Exercise 15.12. (a) At time $t = 0$, compute y at

1.5-cm intervals of x (that is, at $x = 0, x = 1.5$ cm, $x = 3.0$ cm, and so on) from $x = 0$ to $x = 12.0$ cm. Graph the results. This is the shape of the string at time $t = 0$. (b) Repeat the calculations for the same values of x at times $t = 0.400$ s and $t = 0.800$ s. Graph the shape of the string at these instants. In what direction is the wave traveling?

Section 15.4 Speed of a Transverse Wave

15.14. With what tension must a rope with length 2.50 m and mass 0.120 kg be stretched for transverse waves of frequency 40.0 Hz to have a wavelength of 0.750 m?

15.15. One end of a horizontal rope is attached to a prong of an electrically driven tuning fork that vibrates the rope transversely at 120 Hz. The other end passes over a pulley and supports a 1.50-kg mass. The linear mass density of the rope is 0.0550 kg/m. (a) What is the speed of a transverse wave on the rope? (b) What is the wavelength? (c) How would your answers to parts (a) and (b) change if the mass were increased to 3.00 kg?

15.16. A 1.50-m string of weight 1.25 N is tied to the ceiling at its upper end, and the lower end supports a weight W. When you pluck the string slightly, the waves traveling up the string obey the equation

$$y(x, t) = (8.50 \text{ mm})\cos(172 \text{ m}^{-1} x - 2730 \text{ s}^{-1} t)$$

(a) How much time does it take a pulse to travel the full length of the string? (b) What is the weight W? (c) How many wavelengths are on the string at any instant of time? (d) What is the equation for waves traveling *down* the string?

15.17. A thin, 75.0-cm wire has a mass of 16.5 g. One end is tied to a nail, and the other end is attached to a screw that can be adjusted to vary the tension in the wire. (a) To what tension (in newtons) must you adjust the screw so that a transverse wave of wavelength 3.33 cm makes 875 vibrations per second? (b) How fast would this wave travel?

15.18. Weighty Rope. If in Example 15.3 (Section 15.4) we do *not* neglect the weight of the rope, what is the wave speed (a) at the bottom of the rope; (b) at the middle of the rope; (c) at the top of the rope?

15.19. A simple harmonic oscillator at the point $x = 0$ generates a wave on a rope. The oscillator operates at a frequency of 40.0 Hz and with an amplitude of 3.00 cm. The rope has a linear mass density of 50.0 g/m and is stretched with a tension of 5.00 N. (a) Determine the speed of the wave. (b) Find the wavelength. (c) Write the wave function $y(x, t)$ for the wave. Assume that the oscillator has its maximum upward displacement at time $t = 0$. (d) Find the maximum transverse acceleration of points on the rope. (e) In the discussion of transverse waves in this chapter, the force of gravity was ignored. Is that a reasonable approximation for this wave? Explain.

Section 15.5 Energy in Wave Motion

15.20. A piano wire with mass 3.00 g and length 80.0 cm is stretched with a tension of 25.0 N. A wave with frequency 120.0 Hz and amplitude 1.6 mm travels along the wire. (a) Calculate the average power carried by the wave. (b) What happens to the average power if the wave amplitude is halved?

15.21. A jet plane at take-off can produce sound of intensity 10.0 W/m^2 at 30.0 m away. But you prefer the tranquil sound of normal conversation, which is 1.0 μW/m^2. Assume that the plane behaves like a point source of sound. (a) What is the closest distance you should live from the airport runway to preserve your peace of mind? (b) What intensity from the jet does your friend experience if she lives twice as far from the runway as you do? (c) What power of sound does the jet produce at take-off?

15.22. Threshold of Pain. You are investigating the report of a UFO landing in an isolated portion of New Mexico, and you encounter a strange object that is radiating sound waves uniformly in all directions. Assume that the sound comes from a point source and that you can ignore reflections. You are slowly walking toward the source. When you are 7.5 m from it, you measure its intensity to be 0.11 W/m^2. An intensity of 1.0 W/m^2 is often used as the "threshold of pain." How much closer to the source can you move before the sound intensity reaches this threshold?

15.23. Energy Output. By measurement you determine that sound waves are spreading out equally in all directions from a point source and that the intensity is 0.026 W/m^2 at a distance of 4.3 m from the source. (a) What is the intensity at a distance of 3.1 m from the source? (b) How much sound energy does the source emit in one hour if its power output remains constant?

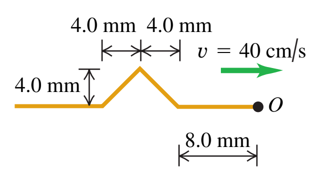
15.24. A fellow student with a mathematical bent tells you that the wave function of a traveling wave on a thin rope is $y(x, t) = 2.30 \text{ mm} \cos[(6.98 \text{ rad/m})x + (742 \text{ rad/s})t]$. Being more practical, you measure the rope to have a length of 1.35 m and a mass of 0.00338 kg. You are then asked to determine the following: (a) amplitude; (b) frequency; (c) wavelength; (d) wave speed; (e) direction the wave is traveling; (f) tension in the rope; (g) average power transmitted by the wave.

15.25. What is the total power output of the siren in Example 15.5?

Section 15.6 Wave Interference, Boundary Conditions, and Superposition

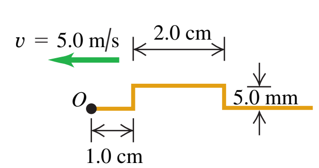
15.26. Reflection. A wave pulse on a string has the dimensions shown in Fig. 15.31 at $t = 0$. The wave speed is 40 cm/s. (a) If point O is a fixed end, draw the total wave on the string at $t = 15$ ms, 20 ms, 25 ms, 30 ms, 35 ms, 40 ms, and 45 ms. (b) Repeat part (a) for the case in which point O is a free end.

Figure 15.31 Exercise 15.26.



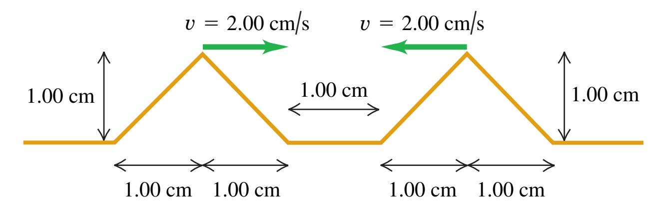
15.27. Reflection. A wave pulse on a string has the dimensions shown in Fig. 15.32 at $t = 0$. The wave speed is 5.0 m/s. (a) If point O is a fixed end, draw the total wave on the string at $t = 1.0$ ms, 2.0 ms, 3.0 ms, 4.0 ms, 5.0 ms, 6.0 ms, and 7.0 ms. (b) Repeat part (a) for the case in which point O is a free end.

Figure 15.32 Exercise 15.27.



15.28. Interference of Triangular Pulses. Two triangular wave pulses are traveling toward each other on a stretched string as shown in Fig. 15.33. Each pulse is identical to the other and travels at 2.00 cm/s. The leading edges of the pulses are 1.00 cm apart at $t = 0$. Sketch the shape of the string at $t = 0.250$ s, $t = 0.500$ s, $t = 0.750$ s, $t = 1.000$ s, and $t = 1.250$ s.

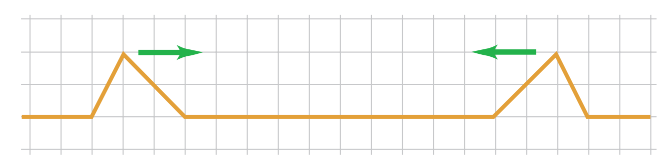
Figure 15.33 Exercise 15.28.



15.29. Suppose that the left-traveling pulse in Exercise 15.28 is *below* the level of the unstretched string instead of above it. Make the same sketches that you did in that exercise.

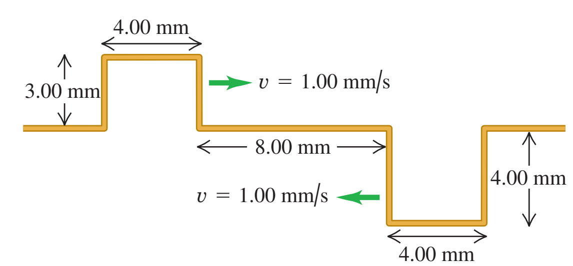
15.30. Two pulses are moving in opposite directions at 1.0 cm/s on a taut string, as shown in Fig. 15.34. Each square is 1.0 cm. Sketch the shape of the string at the end of (a) 6.0 s; (b) 7.0 s; (c) 8.0 s.

Figure 15.34 Exercise 15.30.



15.31. Interference of Rectangular Pulses. Figure 15.35 shows two rectangular wave pulses on a stretched string traveling toward each other. Each pulse is traveling with a speed of 1.00 mm/s and has the height and width shown in the figure. If the leading edges of the pulses are 8.00 mm apart at $t = 0$, sketch the shape of the string at $t = 4.00$ s, $t = 6.00$ s, and $t = 10.0$ s.

Figure 15.35 Exercise 15.31.



15.32. Two traveling waves moving on a string are identical except for opposite velocities. They obey the equation $y(x, t) = A\sin(kx \pm \omega t)$, where the plus-or-minus sign in the argument depends on the direction the wave is traveling. (a) Show that the vibrating string is described by the equation $y_{net}(x, t) = 2A\sin kx \cos\omega t$. (*Hint:* Use the trigonometric formulas for $\sin(a \pm b)$.) (b) Show that the string never moves at the places along it for which $x = n\lambda/2$, where n is a nonnegative integer.

Section 15.7 Standing Waves on a String

15.33. Standing waves on a wire are described by Eq. (15.28), with $A_{SW} = 2.50$ mm, $\omega = 942$ rad/s, and $k = 0.750\pi$ rad/m. The left end of the wire is at $x = 0$. At what distances from the left end are (a) the nodes of the standing wave and (b) the antinodes of the standing wave?

15.34. Adjacent antinodes of a standing wave on a string are 15.0 cm apart. A particle at an antinode oscillates in simple harmonic motion with amplitude 0.850 cm and period 0.0750 s. The string lies along the $+x$-axis and is fixed at $x = 0$. (a) How far apart are the adjacent nodes? (b) What are the wavelength, amplitude, and speed of the two traveling waves that form this pattern? (c) Find the maximum and minimum transverse speeds of a point at an antinode. (d) What is the shortest distance along the string between a node and an antinode?

15.35. Wave Equation and Standing Waves. (a) Prove by direct substitution that $y(x, t) = (A_{SW}\sin kx)\sin\omega t$ is a solution of the wave equation, Eq. (15.12), for $v = \omega/k$. (b) Explain why the relationship $v = \omega/k$ for *traveling* waves also applies to *standing* waves.

15.36. Give the details of the derivation of Eq. (15.28) from $y_1(x, t) + y_2(x, t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$.

15.37. Let $y_1(x, t) = A\cos(k_1 x - \omega_1 t)$ and $y_2(x, t) = A\cos(k_2 x - \omega_2 t)$ be two solutions to the wave equation, Eq. (15.12), for the same v. Show that $y(x, t) = y_1(x, t) + y_2(x, t)$ is also a solution to the wave equation.

Section 15.8 Normal Modes of a String

15.38. A 1.50-m-long rope is stretched between two supports with a tension that makes the speed of transverse waves 48.0 m/s. What are the wavelength and frequency of (a) the fundamental; (b) the second overtone; (c) the fourth harmonic?

15.39. A wire with mass 40.0 g is stretched so that its ends are tied down at points 80.0 cm apart. The wire vibrates in its fundamental mode with frequency 60.0 Hz and with an amplitude at the antinodes of 0.300 cm. (a) What is the speed of propagation of transverse waves in the wire? (b) Compute the tension in the wire. (c) Find the maximum transverse velocity and acceleration of particles in the wire.

15.40. A piano tuner stretches a steel piano wire with a tension of 800 N. The steel wire is 0.400 m long and has a mass of 3.00 g. (a) What is the frequency of its fundamental mode of vibration? (b) What is the number of the highest harmonic that could be heard by a person who is capable of hearing frequencies up to 10,000 Hz?

15.41. A thin, taut string tied at both ends and oscillating in its third harmonic has its shape described by the equation $y(x, t) = (5.60 \text{ cm})\sin[(0.0340 \text{ rad/cm})x]\sin[(50.0 \text{ rad/s})t]$, where the origin is at the left end of the string, the x-axis is along the string and the y-axis is perpendicular to the string. (a) Draw a sketch that shows the standing wave pattern. (b) Find the amplitude of the two traveling waves that make up this standing wave. (c) What is the length of the string? (d) Find the wavelength, frequency, period, and speed of the traveling waves. (e) Find the maximum transverse speed of a point on the string. (f) What would be the equation $y(x, t)$ for this string if it were vibrating in its eighth harmonic?

15.42. The wave function of a standing wave is $y(x, t) = 4.44 \text{ mm} \sin[(32.5 \text{ rad/m})x]\sin[(754 \text{ rad/s})t]$. For the two traveling waves that make up this standing wave, find the (a) amplitude; (b) wavelength; (c) frequency; (d) wave speed; (e) wave functions.

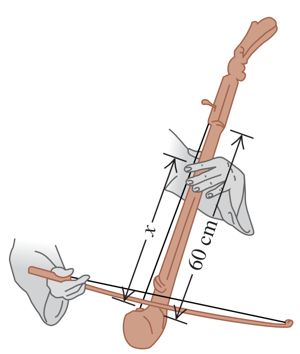
(f) From the information given, can you determine which harmonic this is? Explain.

15.43. Consider again the rope and traveling wave of Exercise 15.24. Assume that the ends of the rope are held fixed and that this traveling wave and the reflected wave are traveling in the opposite direction. (a) What is the wave function $y(x, t)$ for the standing wave that is produced? (b) In which harmonic is the standing wave oscillating? (c) What is the frequency of the fundamental oscillation?

15.44. One string of a certain musical instrument is 75.0 cm long and has a mass of 8.75 g. It is being played in a room where the speed of sound is 344 m/s. (a) To what tension must you adjust the string so that, when vibrating in its second overtone, it produces sound of wavelength 3.35 cm? (b) What frequency sound does this string produce in its fundamental mode of vibration?

15.45. The portion of the string of a certain musical instrument between the bridge and upper end of the finger board (that part of the string that is free to vibrate) is 60.0 cm long, and this length of the string has mass 2.00 g. The string sounds an A_4 note (440 Hz) when played. (a) Where must the player put a finger (what distance x from bridge) to play a D_5 note (587 Hz)? (See Fig. 15.36.) For both the A_4 and D_5 notes, the string vibrates in its fundamental mode. (b) Without retuning, is it possible to play a G_4 note (392 Hz) on this string? Why or why not?

Figure 15.36 Exercise 15.45.



15.46. (a) A horizontal string tied at both ends is vibrating in its fundamental mode. The traveling waves have speed v, frequency f, amplitude A, and wavelength λ. Calculate the maximum transverse velocity and maximum transverse acceleration of points located at (i) $x = \lambda/2$, (ii) $x = \lambda/4$, and (iii) $x = \lambda/8$ from the left-hand end of the string. (b) At each of the points in part (a), what is the amplitude of the motion? (c) At each of the points in part (a), how much time does it take the string to go from its largest upward displacement to its largest downward displacement?

15.47. Guitar String. One of the 63.5-cm-long strings of an ordinary guitar is tuned to produce the note B_3 (frequency 245 Hz) when vibrating in its fundamental mode. (a) Find the speed of transverse waves on this string. (b) If the tension in this string is increased by 1.0%, what will be the new fundamental frequency of the string? (c) If the speed of sound in the surrounding air is 344 m/s, find the frequency and wavelength of the sound wave produced in the air by the vibration of the B_3 string. How do these compare to the frequency and wavelength of the standing wave on the string?

15.48. Waves on a Stick. A flexible stick 2.0 m long is not fixed in any way and is free to vibrate. Make clear drawings of this stick vibrating in its first three harmonics, and then use your drawings to find the wavelengths of each of these harmonics. (*Hint:* Should the ends be nodes or antinodes?)

Problems

15.49. A transverse sine wave with an amplitude of 2.50 mm and a wavelength of 1.80 m travels from left to right along a long, horizontal, stretched string with a speed of 36.0 m/s. Take the origin at the left end of the undisturbed string. At time $t = 0$ the left end of the string has its maximum upward displacement. (a) What are the frequency, angular frequency, and wave number of the wave? (b) What is the function $y(x, t)$ that describes the wave? (c) What is $y(t)$ for a particle at the left end of the string? (d) What is $y(t)$ for a particle 1.35 m to the right of the origin? (e) What is the maximum magnitude of transverse velocity of any particle of the string? (f) Find the transverse displacement and the transverse velocity of a particle 1.35 m to the right of the origin at time $t = 0.0625$ s.

15.50. A transverse wave on a rope is given by

$$y(x, t) = (0.750\text{ cm})\cos\pi[(0.400\text{ cm}^{-1})x + (250\text{ s}^{-1})t]$$

(a) Find the amplitude, period, frequency, wavelength, and speed of propagation. (b) Sketch the shape of the rope at these values of t: 0, 0.0005 s, 0.0010 s. (c) Is the wave traveling in the $+x$- or $-x$-direction? (d) The mass per unit length of the rope is 0.0500 kg/m. Find the tension. (e) Find the average power of this wave.

15.51. Three pieces of string, each of length L, are joined together end to end, to make a combined string of length $3L$. The first piece of string has mass per unit length μ_1, the second piece has mass per unit length $\mu_2 = 4\mu_1$, and the third piece has mass per unit length $\mu_3 = \mu_1/4$. (a) If the combined string is under tension F, how much time does it take a transverse wave to travel the entire length $3L$? Give your answer in terms of L, F, and μ_1. (b) Does your answer to part (a) depend on the order in which the three pieces are joined together? Explain.

15.52. A 1750-N irregular beam is hanging horizontally by its ends from the ceiling by two vertical wires (A and B), each 1.25 m long and weighing 2.50 N. The center of gravity of this beam is one-third of the way along the beam from the end where wire A is attached. If you pluck both strings at the same time at the beam, what is the time delay between the arrival of the two pulses at the ceiling? Which pulse arrives first?

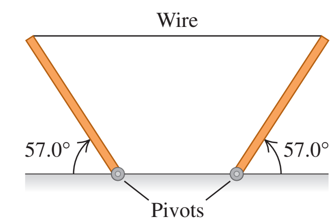
15.53. Ant Joy Ride. You place your pet ant Klyde (mass m) on top of a horizontal, stretched rope, where he holds on tightly. The rope has mass M and length L and is under tension F. You start a sinusoidal transverse wave of wavelength λ and amplitude A propagating along the rope. The motion of the rope is in a vertical plane. Klyde's mass is so small that his presence has no effect on the propagation of the wave. (a) What is Klyde's top speed as he oscillates up and down? (b) Klyde enjoys the ride and begs for more. You decide to double his top speed by changing the tension while keeping the wavelength and amplitude the same. Should the tension be increased or decreased, and by what factor?

15.54. Weightless Ant. An ant with mass m is standing peacefully on top of a horizontal, stretched rope. The rope has mass per unit length μ and is under tension F. Without warning, Cousin Throckmorton starts a sinusoidal transverse wave of wavelength λ propagating along the rope. The motion of the rope is in a vertical plane. What minimum wave amplitude will make the ant become momentarily weightless? Assume that m is so small that the presence of the ant has no effect on the propagation of the wave.

15.55. When a transverse sinusoidal wave is present on a string, the particles of the string undergo SHM. This is the same motion as that of a mass m attached to an ideal spring of force constant k', for which the angular frequency of oscillation was found in Chapter 13 to be $\omega = \sqrt{k'/m}$. Consider a string with tension F and mass per unit length μ, along which is propagating a sinusoidal wave with amplitude A and wavelength λ. (a) Find the "force constant" k' of the restoring force that acts on a short segment of the string of length Δx (where $\Delta x \ll \lambda$). (b) How does the "force constant" calculated in part (b) depend on F, μ, A, and λ? Explain the physical reasons this should be so.

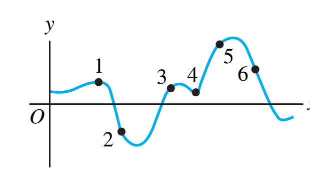
15.56. A 5.00-m, 0.732-kg wire is used to support two uniform 235-N posts of equal length (Fig. 15.37). Assume that the wire is essentially horizontal and that the speed of sound is 344 m/s. A strong wind is blowing, causing the wire to vibrate in its 7th overtone. What are the frequency and wavelength of the sound this wire produces?

Figure 15.37 Problem 15.56.



15.57. A Nonsinusoidal Wave. The shape of a wave on a string at a specific instant is shown in Fig. 15.38. The wave is propagating to the right, in the $+x$-direction. (a) Determine the direction of the transverse *velocity* of each of the six labeled points on the string. If the velocity is zero, state it as such. Explain your reasoning. (b) Determine the direction of the transverse *acceleration* of each of the six labeled points on the string. Explain your reasoning. (c) How would your answers be affected if the wave were propagating to the left, in the $-x$-direction?

Figure 15.38 Problem 15.57.



15.58. A continuous succession of sinusoidal wave pulses are produced at one end of a very long string and travel along the length of the string. The wave has frequency 40.0 Hz, amplitude 5.00 mm, and wavelength 0.600 m. (a) How long does it take the wave to travel a distance of 8.00 m along the length of the string? (b) How long does it take a point on the string to travel a distance of 8.00 m, once the wave train has reached the point and set it into motion? (c) In parts (a) and (b), how does the time change if the amplitude is doubled?

15.59. Two-Dimensional Waves. A stretched string lies along the x-axis. The string is displaced along both the y- and z-directions, so that the transverse displacement of the string is given by

$$y(x, t) = A\cos(kx - \omega t) \qquad z(x, t) = A\sin(kx - \omega t)$$

(a) Draw a graph of z versus y for a particle on the string at $x = 0$. This shows the trajectory of the particle as seen by an observer on the $+x$-axis looking back toward $x = 0$. Indicate the position of the particle at $t = 0$, $t = \pi/2\omega$, $t = \pi/\omega$, and $t = 3\pi/2\omega$. (b) Find the velocity vector of a particle at an *arbitrary* position x on the string. Show that this represents the tangential velocity of a particle moving in a circle of radius A with angular velocity ω, and show that the speed of the particle is constant (i.e., the particle is in uniform circular motion). (See Problem 3.75.) (c) Find the acceleration vector of the particle in part (b). Show that the acceleration is always directed toward the center of the circle and that its magnitude is $a = \omega^2 A$. Explain these results in terms of uniform circular motion. Suppose that the displacement of the string was instead given by

$$y(x, t) = A\cos(kx - \omega t) \qquad z(x, t) = -A\sin(kx - \omega t)$$

Describe how the motion of a particle at x would be different from the motion described in part (a).

15.60. A vertical, 1.20-m length of 18-gauge (diameter of 1.024 mm) copper wire has a 100.0-N ball hanging from it. (a) What is the wavelength of the third harmonic for this wire? (b) A 500.0-N ball now *replaces* the original ball. What is the change in the wavelength of the third harmonic caused by replacing the light ball with the heavy one? (*Hint:* See Table 11.1 for Young's modulus.)

15.61. Waves of Arbitrary Shape. (a) Explain why *any* wave described by a function of the form $y(x, t) = f(x - vt)$ moves in the $+x$-direction with speed v. (b) Show that $y(x, t) = f(x - vt)$ satisfies the wave equation, no matter what the functional form of f. To do this, write $y(x, t) = f(u)$, where $u = x - vt$. Then, to take partial derivatives of $y(x, t)$, use the chain rule:

$$\frac{\partial y(x, t)}{\partial t} = \frac{df(u)}{du}\frac{\partial u}{\partial t} = \frac{df(u)}{du}(-v)$$

$$\frac{\partial y(x, t)}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} = \frac{df(u)}{du}$$

(c) A wave pulse is described by the function $y(x, t) = De^{-(Bx - Ct)^2}$, where B, C, and D are all positive constants. What is the speed of this wave?

15.62. Equation (15.7) for a sinusoidal wave can be made more general by including a phase angle ϕ, where $0 \le \phi \le 2\pi$ (in radians). Then the wave function $y(x, t)$ becomes

$$y(x, t) = A\cos(kx - \omega t + \phi)$$

(a) Sketch the wave as a function of x at $t = 0$ for $\phi = 0$, $\phi = \pi/4$, $\phi = \pi/2$, $\phi = 3\pi/4$, and $\phi = 3\pi/2$. (b) Calculate the transverse velocity $v_y = \partial y/\partial t$. (c) At $t = 0$, a particle on the string at $x = 0$ has displacement $y = A/\sqrt{2}$. Is this enough information to determine the value of ϕ? In addition, if you are told that a particle at $x = 0$ is moving toward $y = 0$ at $t = 0$, what is the value of ϕ? (d) Explain in general what you must know about the wave's behavior at a given instant to determine the value of ϕ.

15.63. (a) Show that Eq. (15.25) can also be written as $P_{\text{av}} = \frac{1}{2}Fk\omega A^2$, where k is the wave number of the wave. (b) If the tension F in the string is quadrupled while the amplitude A is kept the same, how must k and ω each change to keep the average power constant? [*Hint:* Recall Eq. (15.6).]

15.64. Energy in a Triangular Pulse. A triangular wave pulse on a taut string travels in the positive x-direction with speed v. The tension in the string is F, and the linear mass density of the string is μ. At $t = 0$, the shape of the pulse is given by

$$y(x, 0) = \begin{cases} 0 & \text{if } x < -L \\ h(L + x)/L & \text{for } -L < x < 0 \\ h(L - x)/L & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases}$$

(a) Draw the pulse at $t = 0$. (b) Determine the wave function $y(x, t)$ at all times t. (c) Find the instantaneous power in the wave. Show that the power is zero except for $-L < (x - vt) < L$ and that in this interval the power is constant. Find the value of this constant power.

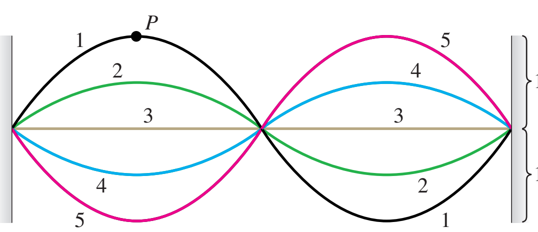
15.65. A sinusoidal transverse wave travels on a string. The string has length 8.00 m and mass 6.00 g. The wave speed is 30.0 m/s, and the wavelength is 0.200 m. (a) If the wave is to have an average power of 50.0 W, what must be the amplitude of the wave? (b) For this same string, if the amplitude and wavelength are the same as in part (a), what is the average power for the wave if the tension is increased such that the wave speed is doubled?

15.66. Instantaneous Power in a Wave. (a) Graph $y(x, t)$ as given by Eq. (15.7) as a function of x for a given time t (say, $t = 0$). On the same axes, make a graph of the instantaneous power $P(x, t)$ as given by Eq. (15.23). (b) Explain the connection between the slope of the graph of $y(x, t)$ versus x and the value of $P(x, t)$. In particular, explain what is happening at points where $P = 0$, where there is no instantaneous energy transfer. (c) The quantity $P(x, t)$ always has the same sign. What does this imply about the direction of energy flow? (d) Consider a wave moving in the $-x$-direction, for which $y(x, t) = A\cos(kx + \omega t)$. Calculate $P(x, t)$ for this wave, and make a graph of $y(x, t)$ and $P(x, t)$ as functions of x for a given time t (say, $t = 0$). What differences arise from reversing the direction of the wave?

15.67. A metal wire, with density ρ and Young's modulus Y, is stretched between rigid supports. At temperature T, the speed of a transverse wave is found to be v_1. When the temperature is increased to $T + \Delta T$, the speed decreases to $v_2 < v_1$. Determine the coefficient of linear expansion of the wire.

15.68. A vibrating string 50.0 cm long is under a tension of 1.00 N. The results from five successive stroboscopic pictures are shown in Fig. 15.39. The strobe rate is set at 5000 flashes per minute, and observations reveal that the maximum displacement occurred at flashes 1 and 5 with no other maxima in between. (a) Find the period, frequency, and wavelength for the traveling waves on this string. (b) In what normal mode (harmonic) is the string vibrating? (c) What is the speed of the traveling waves on the string? (d) How fast is point P moving when the string is in (i) position 1 and (ii) position 3? (e) What is the mass of this string? (Section 15.3).

Figure 15.39 Problem 15.68.



15.69. Clothesline Nodes. Cousin Throckmorton is once again playing with the clothesline in Example 15.2 (Section 15.3). One end of the clothesline is attached to a vertical post. Throcky holds the other end loosely in his hand, so that the speed of waves on the clothesline is a relatively slow 0.720 m/s. He finds several frequencies at which he can oscillate his end of the clothesline so that a light clothespin 45.0 cm from the post doesn't move. What are these frequencies?

15.70. A guitar string is vibrating in its fundamental mode, with nodes at each end. The length of the segment of the string that is free to vibrate is 0.386 m. The maximum transverse acceleration of a point at the middle of the segment is $8.40 \times 10^3\ \text{m/s}^2$ and the maximum transverse velocity is 3.80 m/s. (a) What is the amplitude of this standing wave? (b) What is the wave speed for the transverse traveling waves on this string?

15.71. As shown in Exercise 15.35, a standing wave given by Eq. (15.28) satisfies the wave equation Eq. (15.12). (a) Show that a standing wave given by Eq. (15.28) also satisfies the equation

$$\frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 y(x, t)$$

Interpret this equation in terms of what you know about simple harmonic motion. (b) Does a traveling wave given by $y(x, t) = A\cos(kx - \omega t)$ also satisfy the equation in part (a)? Interpret this result.

15.72. (a) The red and blue waves in Fig. 15.20 combine so that the displacement of the string at O is always zero. To show this mathematically for a wave of arbitrary shape, consider a wave moving to the right along the string in Fig. 15.20 (shown in blue) that, at time T, is given by $y_1(x, T) = f(x)$, where f is some function of x. (The form of $f(x)$ determines the shape of the wave.) If the point O corresponds to $x = 0$, explain why, at time T, the wave moving to the left in Fig. 15.20 (shown in red) is given by the function $y_2(x, T) = -f(-x)$. (b) Show that the total wave function $y(x, T) = y_1(x, T) + y_2(x, T)$ is zero at O, independent of the form of the function $f(x)$. (c) The red and blue waves in Fig. 15.21 combine so that the slope of the string at O is always zero. To show this mathematically for a wave of arbitrary shape, again let the wave moving to the right in Fig. 15.21 (shown in blue) be given by $y_1(x, T) = f(x)$ at time T. Explain why the wave moving to the left (shown in red) at this same time T is given by $y_2(x, T) = f(-x)$. (d) Show that the total wave function $y(x, T) = y_1(x, T) + y_2(x, T)$ has zero slope at O, independent of the form of the function $f(x)$, as long as $f(x)$ has a finite first derivative.

15.73. A string that lies along the $+x$-axis has a free end at $x = 0$. (a) By using steps similar to those used to derive Eq. (15.28), show that an incident traveling wave $y_1(x, t) = A\cos(kx + \omega t)$ gives rise to a standing wave $y(x, t) = 2A\cos\omega t\cos kx$. (b) Show that the standing wave has an antinode at its free end $(x = 0)$. (c) Find the maximum displacement, maximum speed, and maximum acceleration of the free end of the string.

15.74. A string with both ends held fixed is vibrating in its third harmonic. The waves have a speed of 192 m/s and a frequency of 240 Hz. The amplitude of the standing wave at an antinode is 0.400 cm. (a) Calculate the amplitude at points on the string a distance of (i) 40.0 cm; (ii) 20.0 cm; and (iii) 10.0 cm from the left end of the string. (b) At each point in part (a), how much time does it take the string to go from its largest upward displacement to its largest downward displacement? (c) Calculate the maximum transverse velocity and the maximum transverse acceleration of the string at each of the points in part (a).

15.75. A uniform cylindrical steel wire, 55.0 cm long and 1.14 mm in diameter, is fixed at both ends. To what tension must it be adjusted so that, when vibrating in its first overtone, it produces the note D# of frequency 311 Hz? Assume that it stretches an insignificant amount. (*Hint:* See Table 14.1.)

15.76. Holding Up Under Stress. A string or rope will break apart if it is placed under too much tensile stress [Eq. (11.8)]. Thicker ropes can withstand more tension without breaking because the thicker the rope, the greater the cross-sectional area and the smaller the stress. One type of steel has density 7800 kg/m^3 and will break if the tensile stress exceeds $7.0 \times 10^8\ \text{N/m}^2$. You want to make a guitar string from 4.0 g of this type of steel. In use, the guitar string must be able to withstand a tension of 900 N without breaking. Your job is the following: (a) Determine the maximum length and minimum radius the string can have. (b) Determine the highest possible fundamental frequency of standing waves on this string, if the entire length of the string is free to vibrate.

15.77. Combining Standing Waves. A guitar string of length L is plucked in such a way that the total wave produced is the sum of the fundamental and the second harmonic. That is, the standing wave is given by

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

where

$$y_1(x, t) = C\sin\omega_1 t\sin k_1 x$$
$$y_2(x, t) = C\sin\omega_2 t\sin k_2 x$$

with $\omega_1 = vk_1$ and $\omega_2 = vk_2$. (a) At what values of x are the nodes of y_1? (b) At what values of x are the nodes of y_2? (c) Graph the total wave at $t = 0$, $t = \frac{1}{8}f_1$, $t = \frac{1}{4}f_1$, $t = \frac{3}{8}f_1$, and $t = \frac{1}{2}f_1$. (d) Does the sum of the two standing waves y_1 and y_2 produce a standing wave? Explain.

15.78. When a massive aluminum sculpture is hung from a steel wire, the fundamental frequency for transverse standing waves on the wire is 250.0 Hz. The sculpture (but not the wire) is then completely submerged in water. (a) What is the new fundamental frequency? (*Hint:* See Table 14.1.) (b) Why is it a good approximation to treat the wire as being fixed at both ends?

15.79. Tuning an Instrument. A musician tunes the C-string of her instrument to a fundamental frequency of 65.4 Hz. The vibrating portion of the string is 0.600 m long and has a mass of 14.4 g. (a) With what tension must the musician stretch it? (b) What percent increase in tension is needed to increase the frequency from 65.4 Hz to 73.4 Hz, corresponding to a rise in pitch from C to D?

Challenge Problems

15.80. Longitudinal Waves on a Spring. A long spring such as a Slinky™ is often used to demonstrate longitudinal waves. (a) Show that if a spring that obeys Hooke's law has mass m, length L, and force constant k', the speed of longitudinal waves on the spring is $v = L\sqrt{k'/m}$. (b) Evaluate v for a spring with $m = 0.250$ kg, $L = 2.00$ m, and $k' = 1.50$ N/m.

15.81. (a) Show that for a wave on a string, the kinetic energy *per unit length of string* is

$$u_k(x, t) = \frac{1}{2}\mu v_y^2(x, t) = \frac{1}{2}\mu\left(\frac{\partial y(x, t)}{\partial t}\right)^2$$

where μ is the mass per unit length. (b) Calculate $u_k(x, t)$ for a sinusoidal wave given by Eq. (15.7). (c) There is also elastic potential energy in the string, associated with the work required to deform and stretch the string. Consider a short segment of string at position x that has unstretched length Δx, as in Fig. 15.13. Ignoring the (small) curvature of the segment, its slope is $\partial y(x, t)/\partial x$. Assume that the displacement of the string from equilibrium is small, so that $\partial y/\partial x$ has a magnitude much less than unity. Show that the stretched length of the segment is approximately

$$\Delta x\left[1 + \frac{1}{2}\left(\frac{\partial y(x, t)}{\partial x}\right)^2\right]$$

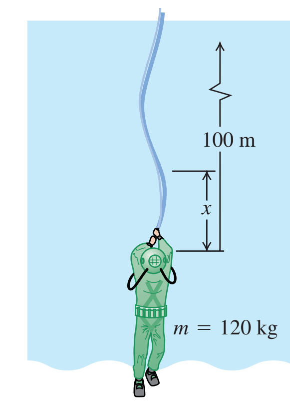
(*Hint:* Use the relationship $\sqrt{1 + u} \approx 1 + \frac{1}{2}u$, valid for $|u| \ll 1$.) (d) The potential energy stored in the segment equals the work done by the string tension F (which acts along the string) to stretch the segment from its unstretched length Δx to the length calculated in part (c). Calculate this work and show that the potential energy *per unit length of string* is

$$u_p(x, t) = \frac{1}{2}F\left(\frac{\partial y(x, t)}{\partial x}\right)^2$$

(e) Calculate $u_p(x, t)$ for a sinusoidal wave given by Eq. (15.7). (f) Show that $u_k(x, t) = u_p(x, t)$ for all x and t. (g) Show $y(x, t)$, $u_k(x, t)$, and $u_p(x, t)$ as functions of x for $t = 0$ in one graph with all three functions on the same axes. Explain why u_k and u_p are maximum where y is zero, and vice versa. (h) Show that the instantaneous power in the wave, given by Eq. (15.22), is equal to the total energy per unit length multiplied by the wave speed v. Explain why this result is reasonable.

15.82. A deep-sea diver is suspended beneath the surface of Loch Ness by a 100-m-long cable that is attached to a boat on the surface (Fig. 15.40). The diver and his suit have a total mass of 120 kg and a volume of 0.0800 m^3. The cable has a diameter of 2.00 cm and a linear mass density of $\mu = 1.10$ kg/m. The diver thinks he sees something moving in the murky depths and jerks the end of the cable back and forth to send transverse waves up the cable as a signal to his companions in the boat. (a) What is the tension in the cable at its lower end, where it is attached to the diver? Do not forget to include the buoyant force that the water (density 1000 kg/m^3) exerts on him. (b) Calculate the tension in the cable a distance x above the diver. The buoyant force on the cable must be included in your calculation. (c) The speed of transverse waves on the cable is given by $v = \sqrt{F/\mu}$ (Eq. 15.13). The speed therefore varies along the cable, since the tension is not constant. (This expression neglects the damping force that the water exerts on the moving cable.) Integrate to find the time required for the first signal to reach the surface.

Figure 15.40 Challenge Problem 15.82.



15.83. A uniform rope with length L and mass m is held at one end and whirled in a horizontal circle with angular velocity ω. You can ignore the force of gravity on the rope. Find the time required for a transverse wave to travel from one end of the rope to the other.

15.84. Instantaneous Power in a Standing Wave. From Eq. (15.21), the instantaneous rate at which a wave transmits energy along a string (instantaneous power) is

$$P(x, t) = -F\frac{\partial y(x, t)}{\partial x}\frac{\partial y(x, t)}{\partial t}$$

where F is the tension. (a) Evaluate $P(x, t)$ for a standing wave of the form given by Eq. (15.28). (b) Show that for all values of x, the *average* power P_{av} carried by the standing wave is zero. (Equation (15.25) does not apply here. Can you see why?) (c) For a standing wave given by Eq. (15.28), graph $P(x, t)$ and the displacement $y(x, t)$ as functions of x for $t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$, and $t = 3\pi/4\omega$. (Positive $P(x, t)$ means energy is flowing in the $+x$-direction; negative $P(x, t)$ means the flow is in the

$-x$-direction.) (d) The *kinetic* energy per unit length of string is greatest where the string has the greatest transverse speed, and the *potential* energy per unit length of string is greatest where the string has the steepest slope (because there the string is stretched the most). (See Challenge Problem 15.81.) Using these ideas, discuss the flow of energy along the string.

15.85. Out of Tune. The B-string of a guitar is made of steel (density 7800 kg/m^3), is 63.5 cm long, and has diameter 0.406 mm. The fundamental frequency is $f = 247.0$ Hz. (a) Find the string tension. (b) If the tension F is changed by a small amount ΔF, the frequency f changes by a small amount Δf. Show that

$$\frac{\Delta f}{f} = \frac{1}{2}\frac{\Delta f}{f}$$

(c) The string is tuned as in part (a) when its temperature is 18.5°C. Strenuous playing can make the temperature of the string rise, changing its vibration frequency. Find Δf if the temperature of the string rises to 29.5°C. The steel string has a Young's modulus of 2.00×10^{11} Pa and a coefficient of linear expansion of $1.20 \times 10^{-5}\ (C°)^{-1}$. Assume that the temperature of the body of the guitar remains constant. Will the vibration frequency rise or fall?