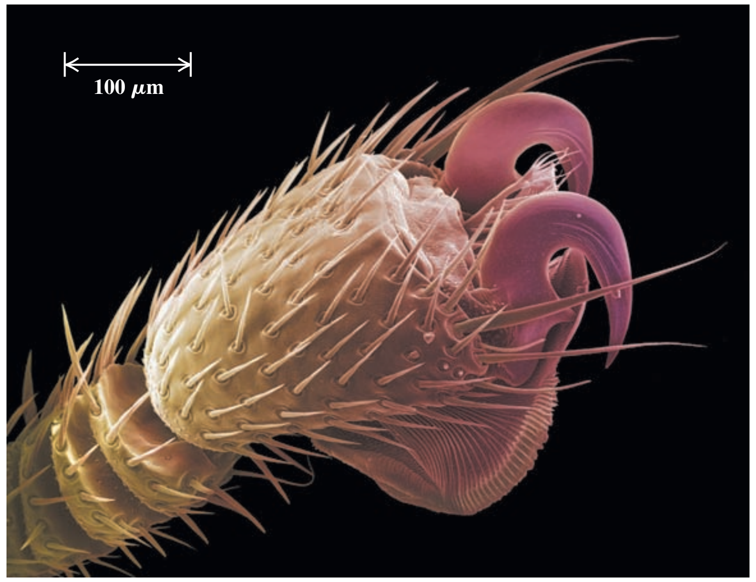
THE WAVE NATURE OF PARTICLES

39



?This image of the foot of a bat fly *(Streblidae)* was made using a beam of electrons rather than a light beam. What properties of electrons make them useful for imaging such fine details?

LEARNING GOALS

By studying this chapter, you will learn:

- De Broglie's proposal that electrons, protons, and other particles can behave like waves.
- How electron diffraction experiments provided evidence for de Broglie's ideas.
- How the Heisenberg uncertainty principle imposes fundamental limits on what can be measured.
- How electron microscopes can provide much higher magnification than visible-light microscopes.
- About the wave functions that describe the behavior of particles and the Schrödinger equation that these functions must satisfy.

In preceding chapters we've seen one aspect of nature's wave–particle duality: Light and other electromagnetic radiation sometimes act like waves and sometimes like particles. Interference and diffraction demonstrate wave behavior, while emission and absorption of photons demonstrate the particle behavior.

A complete theory should also be able to *predict,* on theoretical grounds, the energy levels of any particular atom. The 1913 Bohr model of the hydrogen atom was a step in this direction. But it combined classical principles with new ideas that were inconsistent with classical theory, and it raised as many questions as it answered. More drastic departures from classical concepts were needed.

A successful drastic departure is *quantum mechanics,* a theory that began to emerge in the 1920s. Besides waves that sometimes act like particles, quantum mechanics extends the concept of wave–particle duality to include particles that sometimes show *wavelike* behavior. In these situations a particle is modeled as an inherently spread-out entity that can't be described as a point with a perfectly definite position and velocity.

In this chapter we will explore the wavelike nature of matter and some of its applications. We'll also introduce the *Schrödinger equation,* which is as fundamental to quantum mechanics as Newton's laws are to mechanics or as Maxwell's equations are to electromagnetism. In Chapter 40 we'll see how to use this equation to find the possible energy levels of a quantum-mechanical system.

Quantum mechanics is the key to understanding atoms and molecules, including their structure, spectra, chemical behavior, and many other properties. It has the happy effect of restoring unity and symmetry to our description of both particles and radiation. We will learn about quantum mechanics and use its results throughout the remainder of this book.

39.1 De Broglie Waves

A major advance in the understanding of atomic structure began in 1924 with a proposition made by a French physicist and nobleman, Prince Louis de Broglie (pronounced “de broy”). His reasoning, freely paraphrased, went like this: Nature loves symmetry. Light is dualistic in nature, behaving in some situations like waves and in others like particles. If nature is symmetric, this duality should also hold for matter. Electrons and protons, which we usually think of as *particles,* may in some situations behave like *waves.*

39.1 Louis-Victor de Broglie, the seventh Duke de Broglie (1892–1987), broke with family tradition by choosing a career in physics rather than as a diplomat. His revolutionary proposal that particles have wave characteristics—for which de Broglie won the 1929 Nobel Prize in physics—was published in his doctoral thesis.

If a particle acts like a wave, it should have a wavelength and a frequency. De Broglie (Fig. 39.1) postulated that a free particle with rest mass m, moving with nonrelativistic speed v, should have a wavelength λ related to its momentum $p = mv$ in exactly the same way as for a photon, as expressed by Eq. (38.5): $\lambda = h/p$. The **de Broglie wavelength** of a particle is then

$$\lambda = \frac{h}{p} = \frac{h}{mv} \qquad \text{(de Broglie wavelength of a particle)} \qquad (39.1)$$

where h is Planck’s constant. If the particle’s speed is an appreciable fraction of the speed of light c, we use Eq. (37.27) to replace mv in Eq. (39.1) with $\gamma mv = mv/\sqrt{1 - v^2/c^2}$. The frequency f, according to de Broglie, is also related to the particle’s energy E in the same way as for a photon, namely,

$$E = hf \qquad (39.2)$$

Thus the relationships of wavelength to momentum and of frequency to energy, in de Broglie’s hypothesis, are exactly the same for particles as for photons.

CAUTION **Not all photon equations apply to particles with mass** The relationship $E = hf$ must be carefully applied to particles with nonzero rest mass, such as electrons and protons. Unlike a photon, they do *not* travel at speed c, so neither the equation $f = c/\lambda$ nor $E = pc$ applies to them!

To appreciate the enormous significance of de Broglie’s proposal, we have to realize that at the time there was no direct experimental evidence that particles have wave characteristics. It is one thing to suggest a new hypothesis to explain experimental observations; it is quite another to propose such a radical departure from established concepts on theoretical grounds alone. But it was clear that a radical idea was needed. The dual nature of electromagnetic radiation had led to adoption of the photon concept, also a radical idea. The limited success in understanding atomic structure indicated that a similar revolution was needed in the mechanics of particles.

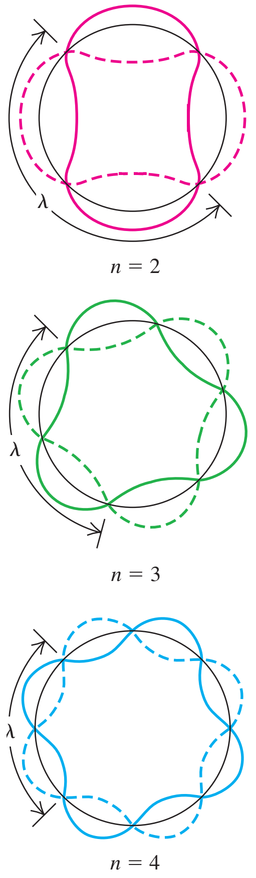
De Broglie’s hypothesis was the beginning of that revolution. Within a few years after 1924 it was developed by Heisenberg, Schrödinger, Dirac, Born, and many others into a detailed theory called **quantum mechanics.** This development was well under way even before direct experimental evidence for the wave properties of particles was found.

Quantum mechanics involves sweeping revisions of our fundamental concepts of the description of matter. A particle is not a geometric point but an entity that is spread out in space. The spatial distribution of a particle is defined by a function called a **wave function,** which is closely analogous to the wave functions we used for mechanical waves in Chapter 15 and for electromagnetic waves in Chapter 32. The wave function for a *free* particle with definite energy has a recurring wave pattern with definite wavelength and frequency. The wave and particle aspects are not inconsistent; the *principle of complementarity,* which we discussed in Section 38.9, tells us that we need both the particle model and the wave model for a complete description of nature.

The Bohr Model and de Broglie Waves

In the Bohr model, we pictured the energy levels of the hydrogen atom in terms of definite electron orbits, as shown in Fig. 38.9. This is an oversimplification and should not be taken literally. But the most important idea in Bohr’s theory was the existence of discrete energy levels and their relationship to the frequencies of emitted photons. The new quantum mechanics still assigns only certain allowed energy states to an atom, but with a more general description of the electron motion in terms of wave functions. In the hydrogen atom the energy levels predicted by quantum mechanics turn out to be the same as those given by Bohr’s theory. In more complicated atoms, for which the Bohr theory does not work, the quantum-mechanical picture is in excellent agreement with observation.

39.2 Diagrams showing the idea of fitting a standing wave around a circular orbit. For the wave to join onto itself smoothly, the circumference of the orbit must be an integral number n of wavelengths. Examples are shown for $n = 2$, $n = 3$, and $n = 4$.



The de Broglie wave hypothesis has an interesting relationship to the Bohr model. We can use Eq. (39.1) to obtain the Bohr quantum condition that the angular momentum $L = mvr$ must be an integer multiple of Planck’s constant h. The method is analogous to determining the normal-mode frequencies of standing waves. We discussed this problem in Sections 15.8 and 32.5; the central idea was to satisfy the **boundary conditions** for the waves. For example, for standing waves on a string that is fixed at both ends, the ends are always nodes, and there will be additional nodes along the string for all but the fundamental mode. For the boundary conditions to be satisfied, the total length of the string must equal some *integral* number of half-wavelengths.

A standing wave on a string transmits no energy, and electrons in Bohr’s orbits radiate no energy. So think of an electron as a standing wave fitted around a circle in one of the Bohr orbits. For the wave to “come out even” and join onto itself smoothly, the circumference of this circle must include some *whole number* of wavelengths, as suggested by Fig. 39.2. For an orbit with radius r and circumference $2\pi r$, we must have $2\pi r = n\lambda$, where $n = 1, 2, 3, \ldots$. According to the de Broglie relationship, Eq. (39.1), the wavelength λ of a particle with rest mass m, moving with nonrelativistic speed v, is $\lambda = h/mv$. Combining $2\pi r = n\lambda$ and $\lambda = h/mv$, we find $2\pi r = nh/mv$, or

$$mvr = n\frac{h}{2\pi} \qquad (39.3)$$

We recognize Eq. (39.3) as being identical to Eq. (38.10), Bohr’s result that the magnitude of the angular momentum $L = mvr$ must equal an integer n times $h/2\pi$. Thus a wave-mechanical picture leads naturally to the quantization of the electron’s angular momentum.

To be sure, the idea of fitting a standing wave around a circular orbit is a rather vague notion. But the agreement of Eq. (39.3) with Bohr’s result is much too remarkable to be a coincidence. It strongly suggests that the wave properties of electrons do indeed have something to do with atomic structure.

Later we will learn how wave functions for specific systems are determined by solution of a wave equation called the Schrödinger equation. Boundary conditions play a central role in finding solutions of this equation and thus in determining possible energy levels, values of angular momentum, and other properties.

Problem-Solving Strategy 39.1 Particles and Waves

IDENTIFY *the relevant concepts:* Jarring as it may seem, both particles and light have wavelike properties as well as particlelike properties. The wavelength of a particle is inversely proportional to the momentum, and the frequency is proportional to the energy. These concepts will help you solve many problems involving the wave character of matter.

SET UP *the problem:* As always, determine the target variable and decide which equations you will use to calculate it.

EXECUTE *the solution* as follows:

1. Use Eq. (39.1) to relate the momentum p to the wavelength λ, and use Eq. (39.2) to relate the energy E to the frequency f.

Continued

2. Nonrelativistic kinetic energy may be expressed as either $K = \frac{1}{2}mv^2$ or (because $p = mv$) $K = p^2/2m$. The latter form is often useful in calculations involving the de Broglie wavelength.
3. Be careful to use consistent units. Lengths, such as wavelengths, are always in meters if you use the other quantities consistently in SI units. If you want a length in nanometers or some other units, don't forget to convert. As in Chapter 38, energies may be expressed in either joules or electron volts. Depending on your choice, you can use either $h = 6.626 \times 10^{-34}\ \text{J}\cdot\text{s}$ or $h = 4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}$.

EVALUATE *your answer:* To check your numerical results, it helps to remember some typical orders of magnitude for quantities on the atomic scale:

Size of an atom: 10^{-10} m

Mass of an atom: 10^{-26} kg

Mass of an electron: 10^{-30} kg

Energy magnitude of an atomic state: 1 to 10 eV $(10^{-19}$ to 10^{-18} J$)$ (but some interaction energies are much smaller)

Speed of an electron in the Bohr model of a hydrogen atom: 10^6 m/s

Electron charge magnitude: 10^{-19} C

kT at room temperature: $\frac{1}{40}$ eV

You may want to add items to this list. These approximate values will also help you in Chapter 43, in which we will deal with magnitudes that are characteristic of *nuclear* structure rather than atomic structure; these are often different by factors of 10^4 to 10^6.

Example 39.1 Energy of a thermal neutron

Find the speed and kinetic energy of a neutron $(m = 1.675 \times 10^{-27}\ \text{kg})$ that has a de Broglie wavelength $\lambda = 0.200$ nm, approximately the atomic spacing in many crystals. Compare the energy with the average translational kinetic energy of a gas molecule at room temperature $(T = 20°\text{C} = 293\ \text{K})$.

SOLUTION

IDENTIFY: This problem uses several relationships: between the wavelength and speed of a particle, between particle speed and kinetic energy, and between the temperature of a gas and the average kinetic energy of a gas molecule.

SET UP: Our target variables are the speed v and kinetic energy K. We determine v using Eq. (39.1), and then calculate the kinetic energy using $K = \frac{1}{2}mv^2$. To find the average kinetic energy of a molecule at temperature T, we use the results of our discussion of kinetic theory in Section 18.3.

EXECUTE: From Eq. (39.1),

$$v = \frac{h}{\lambda m} = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{(0.200 \times 10^{-9}\ \text{m})(1.675 \times 10^{-27}\ \text{kg})}$$
$$= 1.98 \times 10^3\ \text{m/s}$$

The kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(1.675 \times 10^{-27}\ \text{kg})(1.98 \times 10^3\ \text{m/s})^2$$
$$= 3.28 \times 10^{-21}\ \text{J} = 0.0204\ \text{eV}$$

The average translational kinetic energy of a molecule of an ideal gas is given by Eq. (18.16):

$$\frac{1}{2}m(v^2)_{\text{av}} = \frac{3}{2}kT = \frac{3}{2}(1.38 \times 10^{-23}\ \text{J/K})(293\ \text{K})$$
$$= 6.07 \times 10^{-21}\ \text{J} = 0.0397\ \text{eV}$$

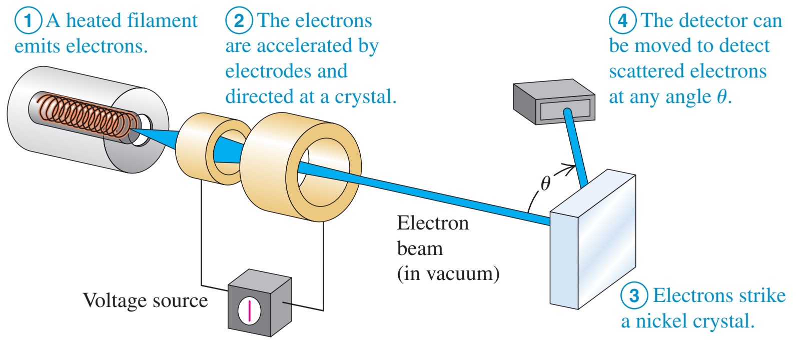
The two energies are comparable in magnitude. In fact, a neutron with kinetic energy in this range is called a *thermal neutron*. Diffraction of thermal neutrons, which we'll discuss in the next section, is used to study crystal and molecular structure in the same way as x-ray diffraction. Neutron diffraction has proved to be especially useful in the study of large organic molecules.

EVALUATE: Note that the calculated neutron speed is much less than the speed of light. This justifies our use of the nonrelativistic form of Eq. (39.1).

Test Your Understanding of Section 39.1 A proton has a slightly smaller mass than a neutron. Compared to the neutron described in Example 39.1, would a proton of the same wavelength have (i) more kinetic energy; (ii) less kinetic energy; or (iii) the same kinetic energy?

39.2 Electron Diffraction

De Broglie's wave hypothesis, radical though it seemed, almost immediately received experimental confirmation. The first direct evidence involved a diffraction experiment with electrons that was analogous to the x-ray diffraction experiments that we described in Section 36.6. In those experiments, atoms in a crystal act as a three-dimensional diffraction grating for x rays. An x-ray beam is strongly reflected when it strikes a crystal at an angle that gives constructive interference among the waves scattered from the various atoms in the crystal. These interference effects demonstrate the *wave* nature of x rays.



39.3 An apparatus similar to that used by Davisson and Germer to discover electron diffraction.

Observing the Wave Nature of Electrons

In 1927, Clinton Davisson and Lester Germer, working at the Bell Telephone Laboratories, were studying the surface of a piece of nickel by directing a beam of *electrons* at the surface and observing how many electrons bounced off at various angles. Figure 39.3 shows an experimental setup like theirs. The specimen was *polycrystalline:* Like many ordinary metals, it consisted of many microscopic crystals bonded together with random orientations. The experimenters expected that even the smoothest surface attainable would still look rough to an electron and that the electron beam would be diffusely reflected, with a smooth distribution of intensity as a function of the angle θ.

During the experiment an accident occurred that permitted air to enter the vacuum chamber, and an oxide film formed on the metal surface. To remove this film, Davisson and Germer baked the specimen in a high-temperature oven, almost hot enough to melt it. Unknown to them, this had the effect of creating large single-crystal regions with crystal planes that were continuous over the width of the electron beam.

When the observations were repeated, the results were quite different. Strong maxima in the intensity of the reflected electron beam occurred at specific angles (Fig. 39.4a), in contrast to the smooth variation of intensity with angle that Davisson and Germer had observed before the accident. The angular positions of the maxima depended on the accelerating voltage V_{ba} used to produce the electron beam. Davisson and Germer were familiar with de Broglie's hypothesis, and they noticed the similarity of this behavior to x-ray diffraction. This was not the effect they had been looking for, but they immediately recognized that the electron beam was being *diffracted.* They had discovered a very direct experimental confirmation of the wave hypothesis.

Davisson and Germer could determine the speeds of the electrons from the accelerating voltage, so they could compute the de Broglie wavelength from Eq. (39.1). The electrons were scattered primarily by the planes of atoms at the surface of the crystal. Atoms in a surface plane are arranged in rows, with a distance d that can be measured by x-ray diffraction techniques. These rows act like a reflecting diffraction grating; the angles at which strong reflection occurs are the same as for a grating with center-to-center distance d between its slits (Fig. 39.4b). From Eq. (36.13) the angles of maximum reflection are given by

$$d\sin\theta = m\lambda \qquad (m = 1, 2, 3, \ldots) \tag{39.4}$$

where θ is the angle shown in Fig. 39.3. The angles predicted by this equation, using the de Broglie wavelength, were found to agree with the observed values (Fig. 39.4a). Thus the accidental discovery of **electron diffraction** was the first direct evidence confirming de Broglie's hypothesis.

The de Broglie wavelength of a nonrelativistic particle is $\lambda = h/p = h/mv$. We can also express λ in terms of the particle's kinetic energy. For example,

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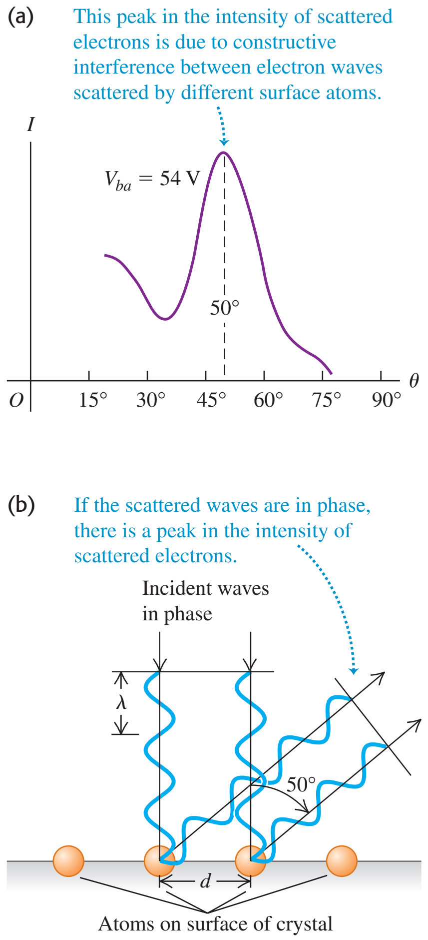
17.5 Electron Interference

39.4 (a)A graph of intensity of the scattered electron beam shown in Fig. 39.3 as a function of the scattering angle θ.
(b) Constructive interference of electron waves scattered from two adjacent atoms occurs when $d\sin\theta = m\lambda$. In the case shown here, $\theta = 50°$ and $m = 1$.



consider an electron freely accelerated from rest at point a to point b through a potential increase $V_b - V_a = V_{ba}$. The work done on the electron eV_{ba} equals its kinetic energy K. Using $K = p^2/2m$, we have

$$eV_{ba} = \frac{p^2}{2m} \qquad p = \sqrt{2meV_{ba}}$$

and the de Broglie wavelength of the electron is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV_{ba}}} \qquad \text{(de Broglie wavelength of an electron)} \qquad (39.5)$$

Example 39.2 An electron-diffraction experiment

In a particular electron-diffraction experiment using an accelerating voltage of 54 V, an intensity maximum occurs when the angle θ in Fig. 39.3 is 50° (see Fig. 39.4a). The initial kinetic energy of the electrons is negligible. The rows of atoms have been found by x-ray diffraction to have a separation $d = 2.15 \times 10^{-10}\ \text{m} = 0.215\ \text{nm}$. Find the electron wavelength.

SOLUTION

IDENTIFY: This problem uses the relationships among accelerating voltage, electron wavelength, spacing between atoms, and angle of maximum reflection for an electron-diffraction experiment.

SET UP: We are given the accelerating voltage $V_{ba} = 54\ \text{V}$, so we can determine the wavelength λ using Eq. (39.5). We are also given the atomic spacing d and the angle θ for an intensity maximum, so we can also determine λ using Eq. (39.4).

EXECUTE: From Eq. (39.5), the electron wavelength is

$$\lambda = \frac{6.626 \times 10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(9.109 \times 10^{-31}\ \text{kg})(1.602 \times 10^{-19}\ \text{C})(54\ \text{V})}}$$
$$= 1.7 \times 10^{-10}\ \text{m}$$

Alternatively, using Eq. (39.4) and assuming $m = 1$,

$$\lambda = d\sin\theta = (2.15 \times 10^{-10}\ \text{m})\sin 50° = 1.7 \times 10^{-10}\ \text{m}$$

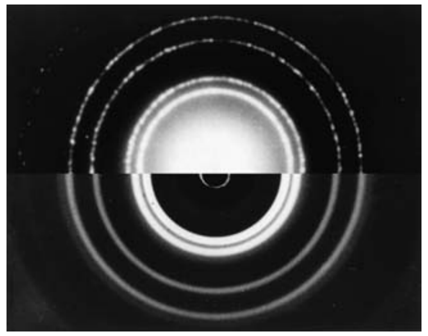
EVALUATE: The two numbers agree within the accuracy of the experimental results, which gives us an excellent check on our calculations. Note that this electron wavelength is less than the spacing between the atoms.

Diffraction of Electrons and Other Particles

In 1928, just a year after the Davisson-Germer discovery, the English physicist G. P. Thomson carried out electron-diffraction experiments using a thin polycrystalline metallic foil as a target. Debye and Sherrer had used a similar technique several years earlier to study x-ray diffraction from polycrystalline specimens. Because of the random orientations of the individual microscopic crystals in his foil, the diffraction pattern consisted of intensity maxima forming rings around the direction of the incident beam. Thomson's results again confirmed the de Broglie relationship. Figure 39.5 shows both x-ray and electron diffraction patterns for a polycrystalline aluminum foil. (G. P. Thomson was the son of J. J. Thomson, who 31 years earlier had performed the definitive experiment to establish the *particle* nature of electrons.)

Additional experiments were soon carried out in many laboratories. In Germany, Estermann and Stern demonstrated diffraction of alpha particles. Diffraction experiments have also been performed with various ions and low-energy neutrons (see Example 39.1). Thus the wave nature of particles, so strange in 1924, became firmly established in the years that followed.

39.5 X-ray and electron diffraction. The upper half of the photo shows the diffraction pattern for 71-pm x rays passing through aluminum foil. The lower half, with a different scale, shows the diffraction pattern for 600-eV electrons from aluminum.



Test Your Understanding of Section 39.2 Example 39.2 shows that to give electrons a wavelength of $1.7 \times 10^{-10}\ \text{m}$, they must be accelerated from rest through a voltage of 54 V and so acquire a kinetic energy of 54 eV. Does a photon of this same energy also have a wavelength of $1.7 \times 10^{-10}\ \text{m}$?

39.3 Probability and Uncertainty

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17.6 Uncertainty Principle

The discovery of the dual wave–particle nature of matter has forced us to reevaluate the kinematic language we use to describe the position and motion of a particle. In classical Newtonian mechanics we think of a particle as a point. We can describe its location and state of motion at any instant with three spatial coordinates and three components of velocity. But in general such a specific description is not possible. When we look on a small enough scale, there are fundamental limitations on the precision with which we can determine the position and velocity of a particle. Many aspects of a particle's behavior can be stated only in terms of *probabilities.*

Single-Slit Diffraction

To try to get some insight into the nature of the problem, let's review the single-slit diffraction of light described in Section 36.2. Suppose the wavelength λ is much less than the slit width a. Then most (85%) of the light in the diffraction pattern is concentrated in the central maximum, bounded on either side by the first intensity minimum. We use θ_1 to denote the angle between the central maximum and the first minimum. Using Eq. (36.2) with $m = 1$, we find that θ_1 is given by $\sin\theta_1 = \lambda/a$. Since we assume $\lambda \ll a$, it follows that θ_1 is very small, $\sin\theta_1$ is very nearly equal to θ_1 (in radians), and

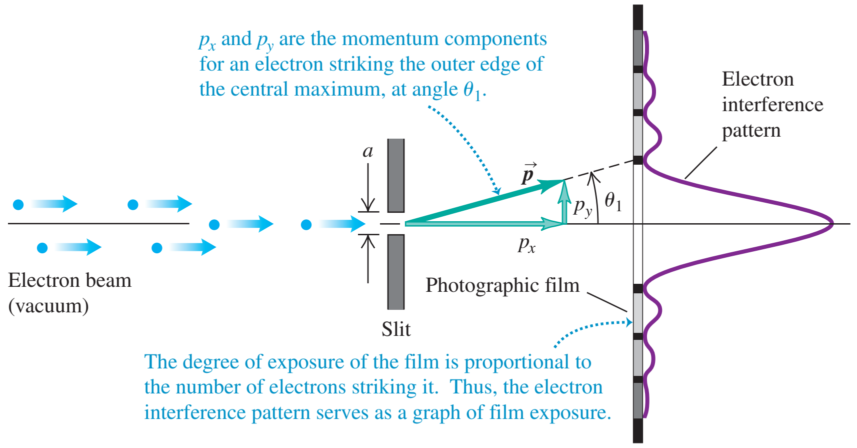
$$\theta_1 = \frac{\lambda}{a} \qquad (39.6)$$

Now we perform the same experiment again, but using a beam of *electrons* instead of a beam of monochromatic light (Fig. 39.6). We have to do the experiment in vacuum (10^{-7} atm or less) so that the electrons don't bounce off air molecules. We can produce the electron beam with a setup that is similar in principle to the electron gun in a cathode-ray tube. This produces a narrow beam of electrons that all have very nearly the same direction and speed and therefore also the same de Broglie wavelength.

The result of this experiment, recorded on photographic film or by use of more sophisticated detectors, is a diffraction pattern identical to the one shown in Fig. 39.6. This pattern gives us additional direct evidence of the *wave* nature of electrons. About 85% of the electrons strike the film within the central maximum; the remainder strike the film within the subsidiary maxima on both sides.

If we believe that electrons are waves, the wave behavior in this experiment isn't surprising. But if we try to interpret it in terms of *particles,* we run into very serious problems. First, the electrons don't all follow the same path, even though they all have the same initial state of motion. In fact, we can't predict the exact

39.6 An electron-diffraction experiment.



trajectory of any individual electron from knowledge of its initial state. The best we can do is to say that *most* of the electrons go to a certain region, *fewer* go to other regions, and so on. That is, we can describe only the *probability* that an individual electron will strike each of various areas on the film. This fundamental indeterminacy has no counterpart in Newtonian mechanics, in which the motion of a particle can always be well predicted if we know the initial position and motion with sufficient accuracy.

Second, there are fundamental *uncertainties* in both the position and the momentum of an individual particle, and these two uncertainties are related inseparably. To clarify this point, let's go back to Fig. 39.6. An electron that strikes the film at the outer edge of the central maximum, at angle θ_1, must have a component of momentum p_y in the y-direction, as well as a component p_x in the x-direction, despite the fact that initially the beam was directed along the x-axis. From the geometry of the situation the two components are related by $p_y/p_x = \tan\theta_1$. Since θ_1 is small, we may use the approximation $\tan\theta_1 = \theta_1$, and

$$p_y = p_x\theta_1 \qquad (39.7)$$

Substituting Eq. (39.6), $\theta_1 = \lambda/a$, we have

$$p_y = p_x\frac{\lambda}{a} \qquad (39.8)$$

For the 85% of the electrons that strike the film within the central maximum (that is, at angles between $-\lambda/a$ and $+\lambda/a$), we see that the y-component of momentum is spread out over a range from $-p_x\lambda/a$ to $+p_x\lambda/a$. Now let's consider *all* the electrons that pass through the slit and strike the film. Again, they may hit above or below the center of the pattern, so their component p_y may be positive or negative. However the symmetry of the diffraction pattern shows us the average value $(p_y)_{\text{av}} = 0$. There will be an *uncertainty* Δp_y in the y-component of momentum at least as great as $p_x\lambda/a$. That is,

$$\Delta p_y \geq p_x\frac{\lambda}{a} \qquad (39.9)$$

The narrower the slit width a, the broader is the diffraction pattern and the greater is the uncertainty in the y-component of momentum p_y.

The electron wavelength λ is related to the momentum $p_x = mv_x$ by the de Broglie relationship Eq. (39.1), which we can rewrite as $\lambda = h/p_x$. Using this relationship in Eq. (39.9) and simplifying, we find

$$\begin{aligned}\Delta p_y &\geq p_x\frac{h}{p_x a} = \frac{h}{a}\\ \Delta p_y a &\geq h\end{aligned} \qquad (39.10)$$

What does this result mean? The slit width a represents an uncertainty in the y-component of the *position* of an electron as it passes through the slit. We don't know exactly *where* in the slit each particle passes through. So both the y-position and the y-component of momentum have uncertainties, and the two uncertainties are related by Eq. (39.10). We can reduce the *momentum* uncertainty Δp_y only by reducing the width of the diffraction pattern. To do this, we have to increase the slit width a, which increases the *position* uncertainty. Conversely, when we *decrease* the position uncertainty by narrowing the slit, the diffraction pattern broadens and the corresponding momentum uncertainty *increases.*

You may protest that it doesn't seem to be consistent with common sense for a particle not to have a definite position and momentum. We reply that what we call *common sense* is based on familiarity gained through experience. Our usual experience includes very little contact with the microscopic behavior of particles. Sometimes we have to accept conclusions that violate our intuition when we are dealing with areas that are far removed from everyday experience.

The Uncertainty Principle

In more general discussions of uncertainty relationships, the uncertainty of a quantity is usually described in terms of the statistical concept of *standard deviation,* which is a measure of the spread or dispersion of a set of numbers around their average value. Suppose we now begin to describe uncertainties in this way [neither Δp_y nor a in Eq. (39.10) is a standard deviation]. If a coordinate x has an uncertainty Δx and if the corresponding momentum component p_x has an uncertainty Δp_x, then those standard-deviation uncertainties are found to be related in general by the inequality

$$\Delta x \Delta p_x \geq \hbar \quad \text{(Heisenberg uncertainty principle for position and momentum)} \qquad (39.11)$$

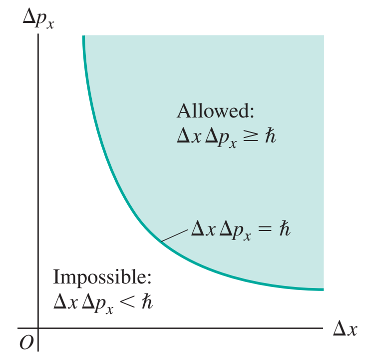
In this expression the quantity $\hbar$ (pronounced "h-bar") is Planck's constant divided by 2π:

$$\hbar = \frac{h}{2\pi} = 1.05457168(18) \times 10^{-34}\ \text{J}\cdot\text{s} \qquad (39.12)$$

We will use this quantity frequently to avoid writing a lot of factors of 2π in later equations.

Equation (39.11) is one form of the **Heisenberg uncertainty principle,** first discovered by the German physicist Werner Heisenberg (1901–1976). It states that, in general, neither the position nor the momentum of a particle can be determined with arbitrarily great precision, as classical physics would predict. Instead, the uncertainties in the two quantities play complementary roles, as we have described. Figure 39.7 shows the relationship between the two uncertainties.

39.7 The Heisenberg uncertainty principle for position and momentum components. It is impossible for the product $\Delta x \Delta p_x$ to be less than $\hbar = h/2\pi$.

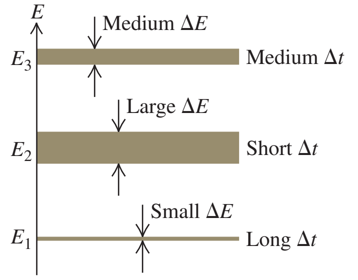


It is tempting to suppose that we could get greater precision by using more sophisticated detectors of position and momentum. This turns out not to be possible. To detect a particle, the detector must *interact* with it, and this interaction unavoidably changes the state of motion of the particle, introducing uncertainty about its original state. For example, if we were to bounce shorter-wavelength photons off a particle to better locate its position, the larger photon momentum h/λ would make the particle recoil more, giving us greater uncertainty in its momentum. A more detailed analysis of such hypothetical experiments shows that the uncertainties we have described are fundamental and intrinsic. They *cannot* be circumvented *even in principle* by any experimental technique, no matter how sophisticated.

There is nothing special about the x-axis. In a three-dimensional situation with coordinates (x, y, z) there is an uncertainty relationship for each coordinate and its corresponding momentum component: $\Delta x \Delta p_x \geq \hbar$, $\Delta y \Delta p_y \geq \hbar$, and $\Delta z \Delta p_z \geq \hbar$. However, the uncertainty in one coordinate is *not* related to the uncertainty in a different component of momentum. For example, Δx is not related directly to Δp_y.

For a particle moving along a radius, we can replace x in Eq. (39.11) with r, giving $\Delta r \Delta p_r \geq \hbar$. In the Bohr model, an electron moves in a circle of exact radius r, giving $\Delta r = 0$ and $\Delta p_r = 0$. Thus the Bohr model violates the Heisenberg uncertainty principle. We'll give a more correct description of atomic structure in Chapter 41; happily, it turns out that the energy-level predictions of the Bohr model *are* correct.

39.8 The longer the lifetime Δt of a state, the smaller is its spread in energy (shown by the width of the energy levels).



Uncertainty in Energy

There is also an uncertainty principle for *energy.* It turns out that the energy of a system also has inherent uncertainty. The uncertainty ΔE depends on the *time interval* Δt during which the system remains in the given state. The relationship is

$$\Delta E \Delta t \geq \hbar \qquad \text{(Heisenberg uncertainty principle for energy and time interval)} \qquad (39.13)$$

A system that remains in a metastable state for a very long time (large Δt) can have a very well-defined energy (small ΔE), but if it remains in a state for only a short time (small Δt), the uncertainty in energy must be correspondingly greater (large ΔE). Figure 39.8 illustrates this idea.

Example 39.3 The uncertainty principle: momentum and position

An electron is confined within a region of width 1.0×10^{-10} m. (a) Estimate the minimum uncertainty in the x-component of the electron's momentum. (b) If the electron has momentum with magnitude equal to the uncertainty found in part (a), what is its kinetic energy? Express the result in joules and in electron volts.

SOLUTION

IDENTIFY: This problem uses the Heisenberg uncertainty principle for position and momentum, as well as the relationship between a particle's momentum and its kinetic energy.

SET UP: We know that the electron is *somewhere* within the region to which it is confined. Since we don't know exactly where within this region the electron is, the width of the region is the position uncertainty: $\Delta x = 1.0 \times 10^{-10}$ m. (You could argue that the uncertainty should actually be half the width of the region. All we want is an estimate, however, so we won't worry about factors of 2.) We then find the momentum uncertainty Δp_x using Eq. (39.11), and we find the kinetic energy using $K = p^2/2m$.

EXECUTE: (a) For a given value of Δx, there is minimum uncertainty in momentum when the product $\Delta x \Delta p_x$ in Eq. (39.11) is equal to $\hbar$. Using the value of $\hbar$ from Eq. (39.12),

$$(\Delta p_x)_{\text{min}} = \frac{\hbar}{\Delta x} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{1.0 \times 10^{-10}\ \text{m}} = 1.1 \times 10^{-24}\ \text{kg} \cdot \text{m/s}$$

(b) An electron with this magnitude of momentum has kinetic energy

$$K = \frac{p^2}{2m} = \frac{(1.1 \times 10^{-24}\ \text{kg} \cdot \text{m/s})^2}{2(9.11 \times 10^{-31}\ \text{kg})}$$
$$= 6.1 \times 10^{-19}\ \text{J} = 3.8\ \text{eV}$$

EVALUATE: The region is roughly the same width as an atom, and the energy is of the same order of magnitude as typical electron energies in atoms. This is a very rough calculation, but it is reassuring that the energy has a reasonable order of magnitude.

In a type of radioactive decay called *beta-minus decay,* an electron is emitted from a nucleus. Although it seems reasonable to assume that the electron was confined within the nucleus before the decay, that would require the position of the electron to be known to within an uncertainty of $\Delta x = 10^{-14}$ m or so. But this would give the confined electron a value of Δp_x that is 10^4 times greater than for the electron in this example and an expected kinetic energy so large that we would need to use relativistic equations to calculate it. This high energy is one reason to believe that there are no electrons confined in nuclei. In beta-minus decay the electron is actually *produced* within the nucleus when a neutron changes into a proton.

Example 39.4 The uncertainty principle: energy and time

A sodium atom is in one of the states labeled "Lowest excited levels" in Fig. 38.10 (Section 38.3). It remains in that state for an average time of 1.6×10^{-8} s before it makes a transition back to a ground state, emitting a photon with wavelength 589.0 nm and energy 2.105 eV. What is the uncertainty in energy of that excited state? What is the wavelength spread of the corresponding spectrum line?

SOLUTION

IDENTIFY: This problem uses the Heisenberg uncertainty principle for energy and time interval, as well as the relationship between the energy and wavelength of a photon.

SET UP: The average time that the atom spends in this excited state is equal to Δt in Eq. (39.13). We find the uncertainty in energy by replacing the $\geq$ sign in Eq. (39.13) with an equals sign and solving for ΔE.

EXECUTE: From Eq. (39.13),

$$\Delta E = \frac{\hbar}{\Delta t} = \frac{1.055 \times 10^{-34}\ \text{J} \cdot \text{s}}{1.6 \times 10^{-8}\ \text{s}}$$
$$= 6.6 \times 10^{-27}\ \text{J} = 4.1 \times 10^{-8}\ \text{eV}$$

The atom remains an indefinitely long time in the ground state, so there is *no* fundamental uncertainty there. The fractional uncertainty of the photon energy is

$$\frac{4.1 \times 10^{-8}\ \text{eV}}{2.105\ \text{eV}} = 1.95 \times 10^{-8}$$

The corresponding spread in wavelength, or "width," of the spectrum line is approximately

$$\Delta\lambda = (1.95 \times 10^{-8})(589.0\ \text{nm}) = 0.000011\ \text{nm}$$

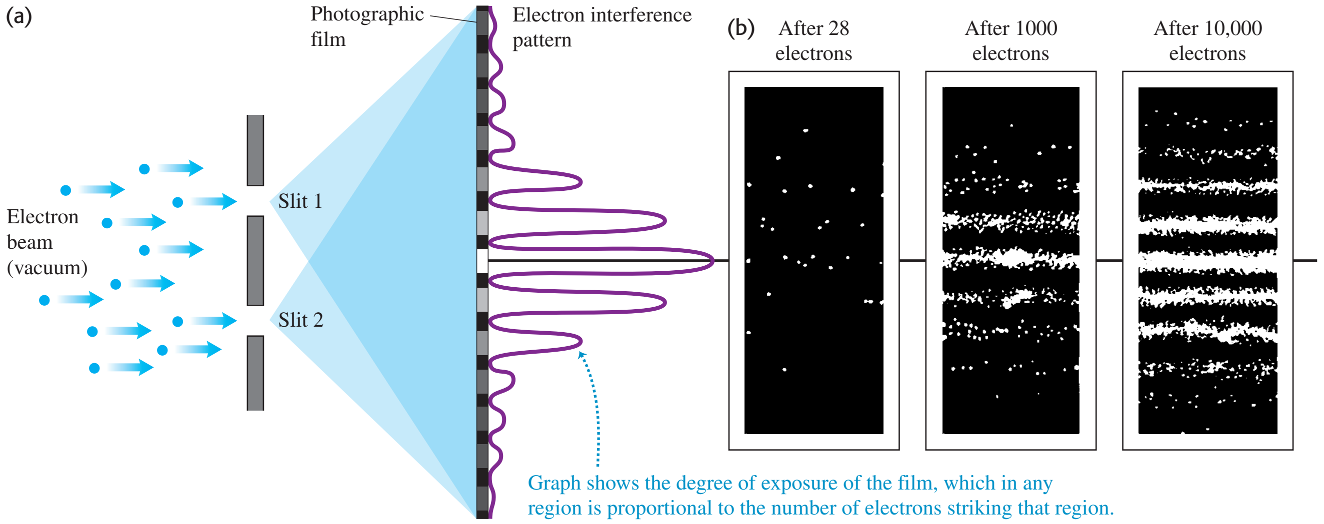
EVALUATE: The irreducible uncertainty $\Delta\lambda$ that we have calculated is called the *natural line width* of this particular spectral line. Though very small, it is within the limits of resolution of present-day spectrometers. Ordinarily, the natural line width is much smaller than the line width from other causes such as the Doppler effect and collisions among the rapidly moving atoms.

Two-Slit Interference

Now let's take a brief look at a quantum interpretation of a *two-slit* optical interference pattern. We studied these patterns in detail for light in Sections 35.2 and 35.3, and in Section 38.9 we discussed their interpretation in terms of the probability that photons strike various regions of the screen where the pattern is formed.

It is natural to ask what happens when we do a two-slit interference experiment with electrons. The answer is: exactly the same thing we saw in Section 38.9 with photons! We can again use photographic film (Fig. 39.9) or electronic detectors to trace out the interference pattern, as we did with photons. The principle of complementarity, introduced in Section 38.9, again tells us that we cannot simultaneously attempt to apply the wave model and the particle model to describe a single part of this experiment. Thus we *cannot* predict exactly where in the pattern (a wave phenomenon) any individual electron (a particle) will land. We can't even ask which slit an individual electron passed through in building up the two-slit interference pattern. If we do determine which slits that the electrons pass through by scattering photons off them, the electrons recoil, and the two-slit interference pattern is not built up.

39.9 (a) Formation of an interference pattern for electrons incident on two slits, (b) after 28, 1000, and 10,000 electrons.



Graph shows the degree of exposure of the film, which in any region is proportional to the number of electrons striking that region.

Test Your Understanding of Section 39.3 Rank the following situations according to the uncertainty in x-momentum, from largest to smallest. The mass of the proton is 1836 times the mass of the electron. (i) an electron whose x-coordinate is known to within 2×10^{-15} m; (ii) an electron whose x-coordinate is known to within 4×10^{-15} m; (iii) a proton whose x-coordinate is known to within 2×10^{-15} m; (iv) a proton whose x-coordinate is known to within 4×10^{-15} m.

39.4 The Electron Microscope

The **electron microscope** offers an important and interesting example of the interplay of wave and particle properties of electrons. An electron beam can be used to form an image of an object in much the same way as a light beam. A ray of light can be bent by reflection or refraction, and an electron trajectory can be bent by an electric or magnetic field. Rays of light diverging from a point on an object can be brought to convergence by a converging lens or concave mirror, and electrons diverging from a small region can be brought to convergence by electric and/or magnetic fields.

The analogy between light rays and electrons goes deeper. The *ray* model of geometric optics is an approximate representation of the more general *wave* model. Geometric optics (ray optics) is valid whenever interference and diffraction effects can be neglected. Similarly, the model of an electron as a point particle following a line trajectory is an approximate description of the actual behavior of the electron; this model is useful when we can neglect effects associated with the wave nature of electrons.

How is an electron microscope superior to an optical microscope? The *resolution* of an optical microscope is limited by diffraction effects, as we discussed in Section 36.7. Using wavelengths around 500 nm, an optical microscope can't resolve objects smaller than a few hundred nanometers, no matter how carefully its lenses are made. The resolution of an electron microscope is similarly limited by the wavelengths of the electrons, but these wavelengths may be many thousands of times *smaller* than wavelengths of visible light. As a result, the useful magnification of an electron microscope can be thousands of times as great as that of an optical microscope.

Note that the ability of the electron microscope to form a magnified image *does not* depend on the wave properties of electrons. Within the limitations of the Heisenberg uncertainty principle, we can compute the electron trajectories by treating them as classical charged particles under the action of electric- and magnetic-field forces (in analogy to ray optics). Only when we talk about *resolution* do the wave properties become important.

Example 39.5 An electron microscope

The nonrelativistic electron beam in an electron microscope is formed by a setup similar to the electron gun used in the Davisson-Germer experiment (Fig. 39.3). What accelerating voltage is needed to produce electrons with wavelength 10 pm $= 0.010$ nm (roughly 50,000 times smaller than typical visible-light wavelengths)? The initial kinetic energy of the electrons is negligible.

SOLUTION

IDENTIFY: Since this situation is similar to the Davisson-Germer experiment, we can use all of the concepts that we developed to understand that experiment.

SET UP: The accelerating voltage (our target variable) is the quantity V_{ba} in Eq. (39.5). We are given the values of all the other quantities in that equation, so we can solve for V_{ba}.

EXECUTE: Solving Eq. (39.5) for V_{ba} and inserting the appropriate numbers, we find

$$\begin{aligned} V_{ba} &= \frac{h^2}{2me\lambda^2} \\ &= \frac{(6.626 \times 10^{-34}\ \mathrm{J \cdot s})^2}{2(9.109 \times 10^{-31}\ \mathrm{kg})(1.602 \times 10^{-19}\ \mathrm{C})(10 \times 10^{-12}\ \mathrm{m})^2} \\ &= 1.5 \times 10^4\ \mathrm{V} = 15{,}000\ \mathrm{V} \end{aligned}$$

EVALUATE: This is approximately equal to the accelerating voltage for the electron beam in a TV picture tube. This example shows, incidentally, that the sharpness of a TV picture is *not* limited by electron diffraction effects. Also note that this 15-kV voltage will increase the kinetic energy of the electrons from a relatively small value to 15 keV. Since electrons have a rest energy of 0.511 MeV = 511 keV, we can accurately describe these 15-keV electrons as nonrelativistic.

The Transmission Electron Microscope

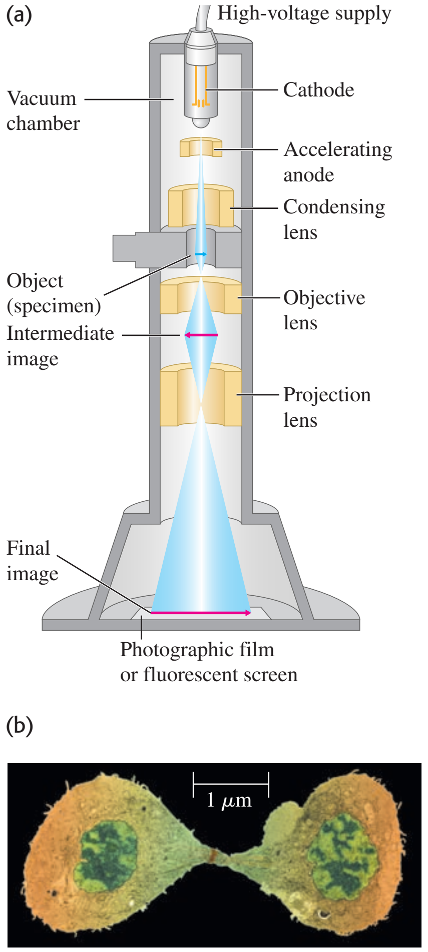
Except within their electron guns, most practical electron microscopes use magnetic fields rather than electric fields as "lenses" for focusing the beam. A common setup for *transmission electron microscopes* includes three such lenses in a compound-microscope arrangement, as shown in Fig. 39.10a. Electrons are emitted from a hot cathode and accelerated by a potential difference, typically 10 to 100 kV. The electrons pass through a condensing lens and are formed into a parallel beam before passing through the specimen or object to be viewed. The specimen to be viewed is very thin, typically 10 to 100 nm, so the electrons are not slowed appreciably as they pass through. The objective lens then forms an intermediate image of this object, and the projection lens produces a final real image of that image. The objective and projection lenses play the roles of the objective and eyepiece lenses, respectively, of a compound optical microscope (see Section 34.8). The final image is recorded on photographic film or projected onto a fluorescent screen for viewing or photographing. The entire apparatus, including the specimen, must be enclosed in a vacuum container, just as with the cathode-ray tube; otherwise, electrons would scatter off air molecules and muddle the image. Figure 39.10b shows an image made with a transmission electron microscope.

We might think that when the electron wavelength is 0.01 nm (as in Example 39.5), the resolution would also be about 0.01 nm. In fact, it is seldom better than 0.5 nm, in part because the focal length of a magnetic lens depends on the electron speed, which is never exactly the same for all electrons in the beam.

The Scanning Electron Microscope

An important variation is the *scanning electron microscope*. The electron beam is focused to a very fine line and is swept across the specimen, just as the electron beam in a TV picture tube traces out the picture. As the beam scans the specimen, electrons are knocked off and are collected by a collecting anode that is kept at a potential a few hundred volts positive with respect to the specimen. The current in the collecting anode is amplified and used to modulate the electron beam in a cathode-ray tube, which is swept in synchronization with the microscope beam. Thus the cathode-ray tube traces out a greatly magnified image of the specimen. This scheme has several advantages. The specimen can be thick because the beam does not need to pass through it. Also, the knock-off electron production depends on the *angle* at which the beam strikes the surface. Thus scanning electron micrographs have an appearance that is much more three-dimensional than conventional visible-light micrographs. The resolution is typically of the order of 10 nm, still much finer than the best optical microscopes. The image that opens this chapter was made with a scanning electron microscope.

39.10 (a) Schematic diagram of a transmission electron microscope (TEM). (b) This TEM image shows a skin cell in the process of dividing into two daughter cells. False color has been used to show the genetic material (dark green).



39.5 Wave Functions and the Schrödinger Equation

We have now seen persuasive evidence that on an atomic or subatomic scale, a particle such as an electron cannot be described simply as a point. Instead we use a *wave function* to describe the state of a particle. Let's describe more specifically the kinematic language we must use to replace the classical scheme of describing a particle by its coordinates and velocity components.

Our new scheme for describing the state of a particle has a lot in common with the language of classical wave motion. In Chapter 15, we described transverse waves on a string by specifying the position of each point in the string at each instant of time by means of a *wave function* (see Section 15.3). If y represents the displacement from equilibrium, at time t, of a point on the string at a distance x from the origin, then the function $y(x, t)$ represents the displacement of any point x at any time t. Once we know the wave function for a particular wave motion, we know everything there is to know about the motion. We can find the position and velocity of any point on the string at any time, and so on (Fig. 39.11). We worked out specific forms for these functions for *sinusoidal* waves, in which each particle undergoes simple harmonic motion.

39.11 These children are talking over a cup-and-string "telephone." The displacement of the string is completely described by a wave function $y(x, t)$. In an analogous way, a particle is completely described by a quantum-mechanical wave function $\Psi(x, y, z, t)$.

We followed a similar pattern for sound waves in Chapter 16. The wave function $p(x, t)$ for a wave traveling along the x-direction represented the pressure variation at any point x at any time t. We used this language once more in Section 32.3, where we used *two* wave functions to describe the electric and magnetic fields of *electromagnetic* waves at any point in space at any time.

Thus it is natural to use a wave function as the central element of our new language. The symbol customarily used for this wave function is Ψ or ψ. In general, Ψ is a function of all the space coordinates and time, whereas ψ is a function of the space coordinates only—*not* of time. Just as the wave function $y(x, t)$ for mechanical waves on a string provides a complete description of the motion, so the wave function $\Psi(x, y, z, t)$ for a particle contains all the information that can be known about the particle. The mathematical theory of quantum mechanics describes how to use $\Psi(x, y, z, t)$ to determine the average values of the particle's position, velocity, momentum, energy, and angular momentum. The required techniques are beyond the scope of this discussion, but they are well established and well supported by experimental results.

CAUTION **Particle waves vs. mechanical waves** Unlike for mechanical waves on a string or sound waves in air, the wave function for a particle is *not* a mechanical wave that needs some material medium in order to propagate. The wave function describes the particle, but we cannot define the function itself in terms of anything material. We can only describe how it is related to physically observable effects.

Interpretation of the Wave Function

The wave function describes the distribution of a particle in space, just as the wave functions for an electromagnetic wave describe the distribution of the electric and magnetic fields. When we worked out interference and diffraction patterns in Chapters 35 and 36, we found that the intensity I of the radiation at any point in a pattern is proportional to the square of the electric-field magnitude—that is, to E^2. In the photon interpretation of interference and diffraction (see Section 38.9), the intensity at each point is proportional to the number of photons striking around that point or, alternatively, to the *probability* that any individual photon will strike around the point. Thus the square of the electric-field magnitude at each point is proportional to the probability of finding a photon around that point.

In exactly the same way, the square of the wave function of a particle at each point tells us about the probability of finding the particle around that point. More precisely, we should say the square of the *absolute value* of the wave function, $|\Psi|^2$. This is necessary because, as we'll see later, Ψ may be a *complex* quantity with real and imaginary parts. (The imaginary part of the function is a real function multiplied by the imaginary number $i = \sqrt{-1}$.)

For a particle moving in three dimensions, the quantity $|\Psi(x, y, z, t)|^2\,dV$ is the probability that the particle will be found at time t within a volume dV around the point (x, y, z). The particle is most likely to be found in regions where $|\Psi|^2$ is large, and so on. This interpretation, first made by the German physicist Max Born (Fig. 39.12), requires that the wave function Ψ be *normalized.* That is, the integral of $|\Psi|^2\,dV$ over all space must equal exactly 1. In other words, the probability is exactly 1, or 100%, that the particle is *somewhere* in the universe.

39.12 In 1926, the German physicist Max Born (1882–1970) devised the interpretation that $|\Psi|^2\,dV$ is the probability of finding a particle within a given volume dV. He also coined the term "quantum mechanics" (in the original German, *Quantenmechanik*). For his contributions, Born shared (with Walther Bothe) the 1954 Nobel Prize in physics.

CAUTION **Interpreting $|\Psi|^2$** Note that $|\Psi(x, y, z, t)|^2$ itself is *not* a probability. Rather, $|\Psi(x, y, z, t)|^2\,dV$ is the probability of finding the particle within a volume dV around the point (x, y, z) at time t. If the volume is made smaller, it becomes less likely that the particle will be found within that volume, so the probability decreases. A better name for $|\Psi(x, y, z, t)|^2$ is the **probability distribution function,** since it describes how the probability of finding the particle at different locations is distributed over space.

Stationary States

In general, the value of $|\Psi(x, y, z, t)|^2$ at a particular point varies with time. This makes sense: As an electron in a television tube flies from the cathode toward the screen, the place it is most likely to be found changes with time. But if the particle is in a state of a definite energy, such as an atomic electron in an atom in a definite energy level, the value of $|\Psi|^2$ at each point is *independent* of time. Because the particle's probability distribution in such a state doesn't change with time, a state with a definite energy is called a **stationary state.** Such states are of tremendous importance in quantum mechanics. For example, for each definite-energy, stationary state in a hydrogen atom, there is a specific wave function. It is possible for an atom to be in a state that is not identical to any one of these stationary-state wave functions and that does not have a definite energy. Nevertheless, the wave function for any state can always be written as a combination of stationary-state wave functions. We saw something similar in Section 15.8: Any standing wave on a string, no matter how complex, can be expressed as a combination of the normal-mode vibrations of the string. We will devote most of Chapter 40 to finding the stationary-state wave functions for different physical situations.

Why *is* $|\Psi|^2$ independent of time if the particle is in a state of definite energy? That is, why are such states stationary? To answer this question, we first note the following result from quantum mechanics: For a particle in a state of definite energy E, the time-dependent wave function $\Psi(x, y, z, t)$ can be written as a product of a time-*independent* function $\psi(x, y, z)$ and a simple function of time:

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-iEt/\hbar} \qquad \text{(time-dependent wave function for a stationary state)} \qquad (39.14)$$

The exponential function in Eq. (39.14) is defined by *Euler's formula,* which states that for any angle θ,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta \qquad (39.15)$$

Thus Eq. (39.14) shows that the wave function for any stationary state is a *complex* function.

Let's now look at the probability distribution function $|\Psi|^2$ obtained from Eq. (39.14). Note that $|\Psi|^2$ is the product of Ψ and its *complex conjugate* Ψ^*. To find the complex conjugate of a complex number, simply replace all i with $-i$. For example, the complex conjugate of $c = a + ib$, where a and b are real, is $c^* = a - ib$, so $|c|^2 = c^*c = (a + ib)(a - ib) = a^2 + b^2$ (recall that $i^2 = -1$). The complex conjugate of Eq. (39.14) is

$$\Psi^*(x, y, z, t) = \psi^*(x, y, z)e^{+iEt/\hbar} \qquad (39.16)$$

Hence

$$\begin{aligned} |\Psi(x, y, z, t)|^2 &= \Psi^*(x, y, z, t)\Psi(x, y, z, t) \\ &= \psi^*(x, y, z)\psi(x, y, z)e^{+iEt/\hbar}e^{-iEt/\hbar} \\ &= \psi^*(x, y, z)\psi(x, y, z)e^0 = |\psi(x, y, z)|^2 \end{aligned} \qquad (39.17)$$

Since $|\psi(x, y, z)|^2$ does not depend on time, Eq. (39.17) shows that the same must be true for the probability distribution function $|\Psi(x, y, z, t)|^2$. This justifies the term "stationary state" for a state of definite energy.

The Schrödinger Equation

We have stressed the importance of stationary states in describing quantum-mechanical systems. As Eq. (39.14) shows, to describe a stationary state we must know its spatial wave function $\psi(x, y, z)$ and its energy E. To determine these

39.13 Erwin Schrödinger (1887–1961) developed the equation that bears his name in 1926, an accomplishment for which he shared (with the British physicist P. A. M. Dirac) the 1933 Nobel Prize in physics. In later life he did research into the philosophy and history of science.

values we use a tool developed in 1926 by the Austrian physicist Erwin Schrödinger (Fig. 39.13) and known as the *Schrödinger equation.* The Schrödinger equation plays the same central role in quantum mechanics as do Newton's laws in mechanics and Maxwell's equations in electromagnetism. Our understanding of every quantum-mechanical system, including atoms, molecules, atomic nuclei, and electrons in solids, is based on the solutions to this equation for that system.

We can't *derive* the Schrödinger equation from other principles; it is a new principle of its own. But we can show how it is related to the de Broglie relationships, and we can make it seem plausible.

The simplest form of the Schrödinger equation is for a particle of mass m that moves in one dimension only, parallel to the x-axis, so that the spatial wave function ψ is a function only of x. We assume that the particle moves in the presence of a conservative force that has only an x-component, so there is a corresponding potential energy $U(x)$. The **Schrödinger equation** for such a particle with a definite energy E is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \qquad \text{(one-dimensional Schrödinger equation)} \tag{39.18}$$

In this equation E is a constant. How do we know that this equation is correct? Because it works. Predictions made using this equation agree with experimental results. In Chapter 40 we'll apply Eq. (39.18) to several physical situations, each with a different form of the function $U(x)$.

Wave Equation for a Free Particle

As an example, let's consider a *free* particle that experiences no force at all. If there is no force, $U(x)$ is independent of x; for simplicity we choose $U(x) = 0$. If such a free particle is moving in the $+x$-direction with momentum of magnitude p, its kinetic energy (and hence its total energy) is $E = p^2/2m$. Such a particle is in a state of definite energy (a stationary state).

From the de Broglie equations, the particle has a definite wavelength $\lambda = h/p$ and a definite frequency $f = E/h$. By analogy to the traveling mechanical waves we studied in Section 15.3, we write the wave function for the particle as

$$\Psi(x,t) = A\cos(kx - \omega t) + B\sin(kx - \omega t) \tag{39.19}$$

where A and B are constants. As we did for mechanical waves in Chapter 15 and electromagnetic waves in Chapter 32, we use the wave number $k = 2\pi/\lambda$ and the angular frequency $\omega = 2\pi f$ in Eq. (39.19). These are related to the momentum and energy by

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{h}\frac{h}{\lambda} = \frac{p}{\hbar} \qquad \omega = 2\pi f = \frac{2\pi}{h}hf = \frac{E}{\hbar} \tag{39.20}$$

The wave function in Eq. (39.19) doesn't seem to be of the stationary-state form given in Eq. (39.14). We can cast it into this form, however, if we let $B = iA$. Then

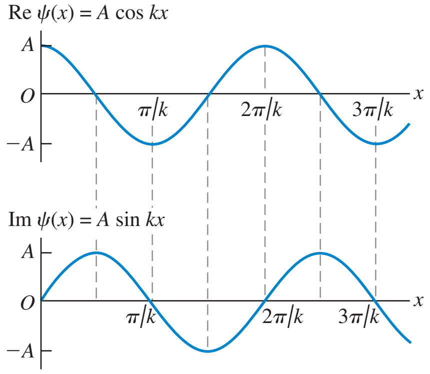
$$\begin{aligned}\Psi(x,t) &= A\cos(kx - \omega t) + iA\sin(kx - \omega t)\\ &= A[\cos(kx - \omega t) + i\sin(kx - \omega t)]\\ &= Ae^{i(kx - \omega t)} = Ae^{ikx}e^{-i\omega t}\end{aligned} \tag{39.21}$$

In Eq. (39.21) we have again used Euler's formula, which states that for any angle θ,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta \tag{39.22}$$

Comparing Eq. (39.21) and Eq. (39.14), we see that our wave function does indeed represent a stationary state with energy $E = \hbar\omega$ and spatial wave function $\psi(x) = Ae^{ikx}$. Figure 39.14 graphs the real and imaginary parts of this function.

39.14 The spatial wave function $\psi(x) = Ae^{ikx}$ for a free particle of definite momentum $p = \hbar k$ is a *complex* function: It has both a real part and an imaginary part.



If we substitute $\psi(x)$ into the left-hand side of the Schrödinger equation, Eq. (39.18), we obtain

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2(Ae^{ikx})}{dx^2} + (0)(Ae^{ikx}) &= -\frac{\hbar^2}{2m}(ik)^2Ae^{ikx} = \frac{\hbar^2k^2}{2m}Ae^{ikx}\\ &= \frac{p^2}{2m}\psi(x)\end{aligned} \tag{39.23}$$

(We used the result that the second derivative of e^{ax}, where a is a constant, is a^2e^{ax}.) The right-hand side of Eq. (39.18) is $E\psi(x)$. Since $E = p^2/2m$ from Eq. (39.18), we see from Eq. (39.23) that the free-particle wave function in Eq. (39.21) satisfies the Schrödinger equation.

If k is positive in Eq. (39.21), the wave function represents a free particle moving in the positive x-direction. If k is negative, the momentum and hence the motion are in the negative x-direction. (With a negative value of k, the wavelength is $\lambda = 2\pi/|k|$.)

For a free particle, there is no restriction on the value of p and hence no restriction on the value of the energy $E = p^2/2m$. We will find in Chapter 40 that if $U(x)$ is not a constant, then solutions of the Schrödinger equation are possible only for certain values of E. These values represent the allowed *energy levels* of the system described by $U(x)$. This discovery is of the utmost importance. Before the development of the Schrödinger equation, there was no way to predict energy levels from any fundamental theory except the Bohr model, which had very limited success.

There is also a version of the Schrödinger equation that includes time dependence. It is needed to study states that are *not* stationary and for which the probability distribution function $|\Psi(x, y, z, t)|^2$ depends on time. However, we don't need that version to calculate the energy levels and wave functions of stationary states. If we do need a time-dependent wave function for a stationary state of energy E, we simply use Eq. (39.14). The time-dependent Schrödinger equation is essential for studying the details of *transitions* between states. In more advanced courses you may study the quantum mechanics of time-dependent phenomena such as photon emission and absorption and the lifetimes of states.

Example 39.6 The Schrödinger equation

Consider the wave function $\psi(x) = A_1e^{ikx} + A_2e^{-ikx}$, where k is positive. Is this a valid stationary-state wave function for a free particle? What is the energy?

SOLUTION

IDENTIFY: A valid stationary-state wave function for a free particle must satisfy the Schrödinger equation, Eq. (39.18), with $U(x) = 0$.

SET UP: To test the given function $\psi(x)$, we simply substitute it into the left-hand side of the equation. If the result is a constant times $\psi(x)$, then the wave function is indeed a solution and the constant is equal to the energy E.

EXECUTE: Substituting $\psi(x) = A_1e^{ikx} + A_2e^{-ikx}$ and $U(x) = 0$ into Eq. (39.18), we obtain

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} &= -\frac{\hbar^2}{2m}\frac{d^2(A_1e^{ikx} + A_2e^{-ikx})}{dx^2}\\ &= -\frac{\hbar^2}{2m}[(ik)^2A_1e^{ikx} + (-ik)^2A_2e^{ikx}]\\ &= \frac{\hbar^2k^2}{2m}(A_1e^{ikx} + A_2e^{-ikx}) = \frac{\hbar^2k^2}{2m}\psi(x)\end{aligned}$$

We see that the left-hand side of the Schrödinger equation is equal to a constant times $\psi(x)$, so this $\psi(x)$ is indeed a valid stationary-state wave function for a free particle. The constant is just equal to the energy: $E = \hbar^2k^2/2m$.

EVALUATE: Note that $\psi(x)$ is a *superposition* of two different wave functions: one for a particle with a given magnitude of momentum $p = \hbar k$ moving in the positive x-direction and one for a particle with the same magnitude of momentum moving in the negative x-direction. While the combined wave function represents a stationary state with a definite energy, this state does *not* have a definite momentum. We will see in Chapter 40 that such a wave function can represent a *standing wave,* and we will explore the situations in which such standing matter waves can arise.

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17.7 Wave Packets

Wave Packets

The wave function for a free particle given by Eq. (39.21) has a definite momentum p in the x-direction. For such a state, there is *no* uncertainty in momentum: $\Delta p_x = 0$. The Heisenberg uncertainty principle, Eq. (39.11), says that $\Delta x \Delta p_x \geq \hbar$. If Δp_x is zero, then Δx must be infinite. The price we pay for knowing the particle's momentum precisely is that we have no idea *where* the particle is! We can show this by calculating the probability distribution function $|\Psi(x, t)|^2$:

$$\begin{aligned}|\Psi(x,t)|^2 &= \Psi^*(x,t)\Psi(x,t) = (A^* e^{-ikx} e^{+i\omega t})(A e^{ikx} e^{-i\omega t})\\ &= A^* A e^0 = |A|^2\end{aligned}$$

The probability distribution function doesn't depend on time, as we would expect because this is a stationary state (a state of definite energy). It also doesn't depend on position, which says that we are equally likely to find the particle *anywhere* in space! Mathematically, this situation occurs because the spatial wave function $\psi(x) = Ae^{ikx} = A\cos kx + iA\sin kx$ is a sinusoidal function that extends all the way from $x = -\infty$ to $x = +\infty$ with the same amplitude A. (This also means that the wave function can't be normalized: The integral of $|\Psi(x, t)|^2$ over all space would be infinite for any value of A.)

In practical situations, we always have *some* idea where a particle is. To describe this situation, we need a wave function that is more localized in space. We can create one by superimposing two or more sinusoidal functions, as we did in Example 39.6. In that example we added two wave functions with opposite wave numbers k and $-k$; to make a localized wave function, we must make a different choice.

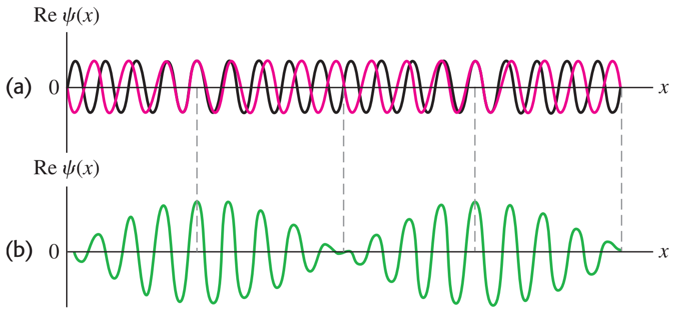
To keep things simple, we'll imagine wave functions that depend on only one spatial coordinate (x), and we'll consider them at one instant of time (say, $t = 0$). Our wave functions are then functions only of x, so we'll denote them as ψ. For example, let's add together two different stationary-state wave functions for a free particle as given by Eq. (39.21), but with slightly different values k_1 and k_2 of the wave number. At $t = 0$ the time factors $e^{-i\omega_1 t}$ and $e^{-i\omega_2 t}$ are both equal to $e^0 = 1$, so the wave function at $t = 0$ is

$$\begin{aligned}\psi(x) &= A_1 e^{ik_1 x} + A_2 e^{ik_2 x}\\ &= [A_1\cos(k_1x) + iA_1\sin(k_1x)] + [A_2\cos(k_2x) + iA_2\sin(k_2x)] \qquad (39.24)\\ &= [A_1\cos(k_1x) + A_2\cos(k_2x)] + i[A_1\sin(k_1x) + A_2\sin(k_2x)]\end{aligned}$$

We have used Euler's formula, Eq. (39.22), to separate the real and imaginary parts of $\psi(x)$.

Figure 39.15a is a graph of the real parts of the individual wave functions for the case $A_2 = -A_1$; Fig. 39.15b graphs the real part of the combined wave function $\psi(x)$ given by Eq. (39.24). We saw something very similar to Fig. 39.15b in our discussion of beats in Section 16.7: When we superimposed two sinusoidal waves with slightly different frequencies (Fig. 16.24), the result was a wave with a lumpy character that the individual waves did not possess. In the same way, a particle represented by the lumpy wave function given in Eq. (39.24) is more likely to be found in some regions than in others; that is, it is *localized.* However, the particle's momentum no longer has a definite value because we began with two different wave numbers. This agrees with the Heisenberg uncertainty principle: By decreasing the uncertainty in the particle's position, we have increased the uncertainty in its momentum.

39.15 (a) The real parts of two sinusoidal waves with slightly different wave numbers, shown at one instant of time. (b) The superposition of these waves has a wave number equal to the average of the two individual wave numbers. The amplitude varies, giving the total wave a lumpy character not possessed by either individual wave.



It's not hard to imagine superposing two additional sinusoidal waves with different wave numbers and amplitudes so as to reinforce alternate lumps in Fig. 39.15b and cancel out the in-between ones. Finally, if we superpose waves with a very large number of different wave numbers, we can construct a wave with only one lump (Fig. 39.16). Then, finally, we have something that begins to look like both a particle and a wave. It is a particle in the sense that it is localized in space; if we look from a distance, it may look like a point. But it also has a periodic structure that is characteristic of a wave.

Such a wave pulse is called a **wave packet.** We can represent such a superposition by an expression such as

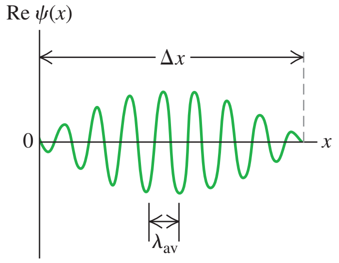
$$\psi(x) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk \qquad (39.25)$$

This integral represents a superposition of a very large number of waves, each with a different wave number and each with an amplitude $A(k)$ that depends on k.

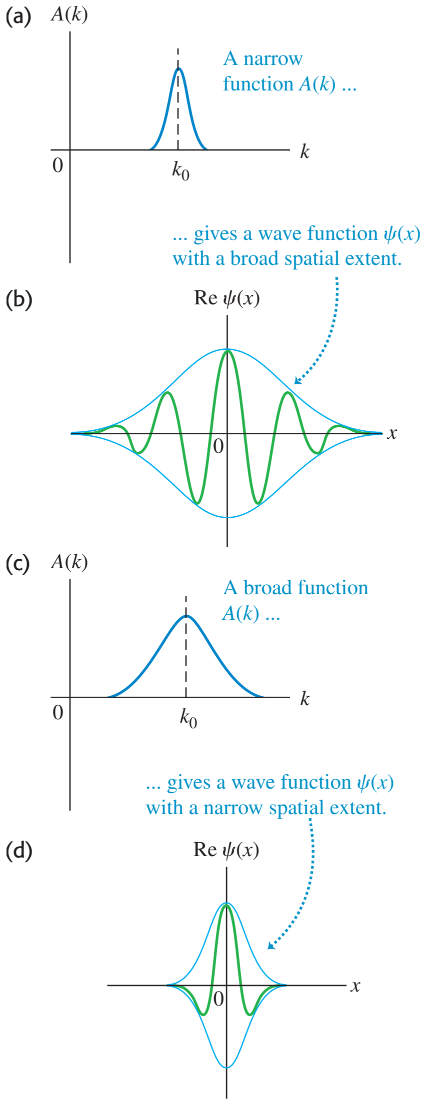
There is an important relationship between the two functions $\psi(x)$ and $A(k)$. This relationship is shown qualitatively in Fig. 39.17. If the function $A(k)$ is sharply peaked, as in Fig. 39.17a, we are superposing only a narrow range of wave numbers. The resulting wave pulse is then relatively broad (Fig. 39.17b). But if we use a wider range of wave numbers, so that the function $A(k)$ is broader (Fig. 39.17c), then the wave pulse is more narrowly localized (Fig. 39.17d).

What we are seeing is the uncertainty principle in action. A narrow range of k means a narrow range of $p_x = \hbar k$ and thus a small Δp_x; the result is a relatively large Δx. A broad range of k corresponds to a large Δp_x, and the resulting Δx is smaller. Thus we see that the uncertainty principle $\Delta x \Delta p_x \geq \hbar$ is an inevitable consequence of the de Broglie relationship and the properties of integrals such as Eq. (39.25). These integrals are called *Fourier integrals;* they are a generalization of the concept of Fourier series, which we mentioned in Section 15.8. In both cases we are representing a complex wave form as a superposition of sinusoidal functions. With Fourier series we use a sequence of frequencies (or values of k) that are integer multiples of some basic value, while with Fourier integrals we superpose functions with a *continuous* distribution of values of k.

39.16 Superimposing a large number of sinusoidal waves with different wave numbers and appropriate amplitudes can produce a wave pulse with a wavelength $\lambda_{av} = 2\pi/k_{av}$ that is localized in a region of space of length Δx. This localized pulse has aspects of both particle and wave.



39.17 How varying the function $A(k)$ in the wave-packet expression, Eq. (39.25), changes the character of the wave function $\psi(x)$.



Test Your Understanding of Section 39.5 Does a wave packet given by Eq. (39.25) represent a stationary state?

CHAPTER 39 SUMMARY

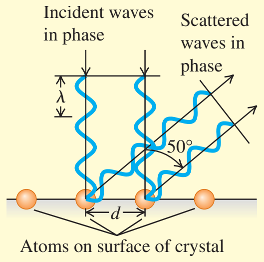
De Broglie waves: Electrons and other particles have wave properties. The wavelength of the wave depends on the particle's momentum in the same way as for photons. The state of a particle is described not by its coordinates and velocity components but rather by a wave function, which in general is a function of the three space coordinates and time. (See Example 39.1.)

$$\lambda = \frac{h}{p} = \frac{h}{mv} \qquad (39.1)$$

$$E = hf \qquad (39.2)$$

Electron diffraction and electron microscopes: Diffraction of an electron beam from the surface of a metallic crystal provided the first direct confirmation of the wave nature of particles. If a nonrelativistic electron has been accelerated from rest through a potential difference V_{ba}, its wavelength is given by Eq. (39.5). Electron microscopes use the very small wavelengths of fast-moving electrons to make images with resolution thousands of times finer than is possible with visible-light microscopes. (See Examples 39.2 and 39.5.)

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV_{ba}}} \qquad (39.5)$$



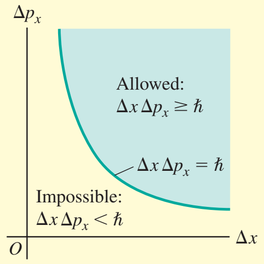
The Heisenberg uncertainty principle: It is impossible to make precise determinations of a coordinate of a particle and of the corresponding momentum component at the same time. The precision of such measurements for the x-components is limited by the Heisenberg uncertainty principle, Eq. (39.11); there are corresponding relationships for the y- and z-components. The uncertainty ΔE in the energy of a state that is occupied for a time Δt is given by Eq. (39.13). In these expressions, $\hbar = h/2\pi$. (See Examples 39.3 and 39.4.)

$$\Delta x \Delta p_x \geq \hbar \qquad (39.11)$$

(Heisenberg uncertainty principle for position and momentum)

$$\Delta E \Delta t \geq \hbar \qquad (39.13)$$

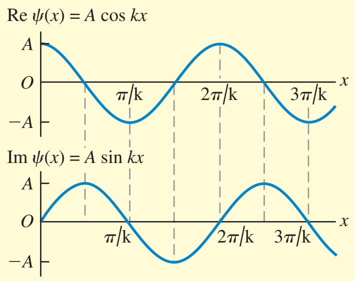
(Heisenberg uncertainty principle for energy and time)



Wave functions: The wave function $\Psi(x, y, z, t)$ for a particle contains all of the information about that particle. The quantity $|\Psi(x, y, z, t)|^2$, called the probability distribution function, determines the relative probability of finding a particle near a given position at a given time. If the particle is in a state of definite energy, called a stationary state, $\Psi(x, y, z, t)$ is a product of a function ψ that depends only on spatial coordinates and a function $e^{-iEt/\hbar}$ that depends only on time. For a stationary state, the probability distribution function is independent of time.

$$\Psi(x, y, z, t) = \psi(x, y, z)\, e^{-iEt/\hbar} \qquad (39.14)$$

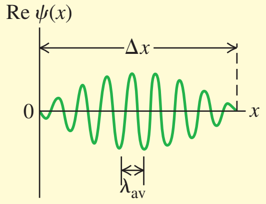
(stationary state)



The Schrödinger equation: For a particle that moves in one dimension in the presence of a potential-energy function $U(x)$, the wave function for a stationary state of energy E satisfies the Schrödinger equation. More complex wave functions can be made by superposing stationary-state wave functions. These can represent particles that are localized in a certain region and still have wave properties, giving them both particle and wave aspects. (See Example 39.6.)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \qquad (39.18)$$

(one-dimensional Schrödinger equation)



Key Terms

de Broglie wavelength, 1350
quantum mechanics, 1350
wave function, 1350
boundary condition, 1351
electron diffraction, 1353
Heisenberg uncertainty principle, 1357
electron microscope, 1360
probability distribution function, 1362
stationary state, 1363
Schrödinger equation, 1364
wave packet, 1367

Answer to Chapter Opening Question

The smallest detail visible in an image is comparable to the wavelength used to make the image. Electrons can easily be given a large momentum p and hence a short wavelength $\lambda = h/p$, and so can be used to resolve extremely fine details. (See Section 39.4.)

Answers to Test Your Understanding Questions

39.1 Answer: (i) From Example 39.1, the speed of a particle is $v = h/\lambda m$ and the kinetic energy is $K = \frac{1}{2}mv^2 = (m/2)(h/\lambda m)^2 = h^2/2\lambda^2 m$. This shows that for a given wavelength, the kinetic energy is inversely proportional to the mass. Hence the proton, with a smaller mass, has more kinetic energy than the neutron.

39.2 Answer: no The energy of a photon is $E = hf$, and the frequency of a photon is $f = c/\lambda$. Hence $E = hc/\lambda$ and $\lambda = hc/E = (4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}) \times (2.998 \times 10^8\ \text{m/s})/(54\ \text{eV}) = 2.3 \times 10^{-8}$ m. This is more than 100 times greater than the wavelength of an electron of the same energy. While both photons and electrons have wavelike properties, they have different relationships between their energy and momentum and hence between their frequency and wavelength.

39.3 Answer: (i) and (iii) (tie), (ii) and (iv) (tie) According to the Heisenberg uncertainty principle, the smaller the uncertainty Δx in the x-coordinate, the greater the uncertainty Δp_x in the x-momentum. The relationship between Δx and Δp_x does not depend on the mass of the particle, and so is the same for a proton as for an electron.

39.5 Answer: no Equation (39.25) represents a superposition of wave functions with different values of wave number k and hence different values of energy $E = \hbar^2 k^2/2m$. The state that this combined wave function represents is not a state of definite energy, and therefore not a stationary state. Another way to see this is to note that the time-dependent wave function $\Psi(x, t)$ would include a factor $e^{-iEt/\hbar}$ inside the integral in Eq. (39.25), with a different value of E for each value of k. This wave function therefore has a very complicated time dependence, and the probability distribution function $|\Psi(x, t)|^2$ does depend on time.

PROBLEMS

Discussion Questions

Q39.1. In attempting to reconcile the wave and particle models of light, some people have suggested that the photon rides up and down on the crests and troughs of the electromagnetic wave. What things are *wrong* with this description?

Q39.2. If a proton and an electron have the same speed, which has the longer de Broglie wavelength? Explain.

Q39.3. If a proton and an electron have the same kinetic energy, which has the longer de Broglie wavelength? Explain.

Q39.4. Does a photon have a de Broglie wavelength? If so, how is it related to the wavelength of the associated electromagnetic wave? Explain.

Q39.5. When an electron beam goes through a very small hole, it produces a diffraction pattern on a screen, just like that of light. Does this mean that an electron spreads out as it goes through the hole? What does this pattern mean?

Q39.6. You have been asked to design a magnet system to steer a beam of 54-eV electrons like those described in Example 39.2 (Section 39.2). The goal is to be able to direct the electron beam to a specific target location with an accuracy of ±1.0 mm. In your design, do you need to take the wave nature of electrons into account? Explain.

Q39.7. The upper half of the electron diffraction pattern shown in Fig. 39.6 is the mirror image of the lower half of the pattern. Would it be correct to say that the upper half of the pattern is caused by electrons that pass through the upper half of the slit? Explain.

Q39.8. A particular electron in the experimental setup shown in Fig. 39.6 lands on the photographic film a distance x above the center of the pattern. Given the value of x, is it possible to calculate the precise trajectory that the electron followed? Explain.

Q39.9. Does the uncertainty principle have anything to do with marksmanship? That is, is the accuracy with which a bullet can be aimed at a target limited by the uncertainty principle? Explain.

Q39.10. Suppose a two-slit interference experiment is carried out using an electron beam. Would the same interference pattern result if one slit at a time is uncovered instead of both at once? If not, why not? Doesn't each electron go through one slit or the other? Or does every electron go through both slits? Discuss the latter possibility in light of the principle of complementarity.

Q39.11. Equation (39.13) states that the energy of a system can have uncertainty. Does this mean that the principle of conservation of energy is no longer valid? Explain.

Q39.12. Laser light results from transitions from long-lived metastable states. Why is it more monochromatic than ordinary light?

Q39.13. Could an electron-diffraction experiment be carried out using three or four slits? Using a grating with many slits? What sort of results would you expect with a grating? Would the uncertainty principle be violated? Explain.

Q39.14. As the lower half of Fig. 39.5 shows, the diffraction pattern made by electrons that pass through aluminum foil is a series of concentric rings. But if the aluminum foil is replaced by a single crystal of aluminum, only certain points on these rings appear in the pattern. Explain.

Q39.15. Why can an electron microscope have greater magnification than an ordinary microscope?

Q39.16. If quantum mechanics replaces the language of Newtonian mechanics, why don't we have to use wave functions to describe the motion of macroscopic bodies such as baseballs and cars?

Q39.17. A student remarks that the relationship of ray optics to the more general wave picture is analogous to the relationship of Newtonian mechanics, with well-defined particle trajectories, to quantum mechanics. Comment on this remark.

Q39.18. As Eq. (39.14) indicates, the time-dependent wave function for a stationary state is a complex number having a real part and an imaginary part. How can this function have any physical meaning, since part of it is *imaginary?*

Q39.19. When you check the air pressure in a tire, a little air always escapes; the process of making the measurement changes the quantity being measured. Think of other examples of measurements that change or disturb the quantity being measured.

Q39.20. Why must the wave function of a particle be normalized?

Q39.21. If a particle is in a stationary state, does that mean that the particle is not moving? If a particle moves in empty space with constant momentum $\vec{p}$ and hence constant energy $E = p^2/2m$, is it in a stationary state? Explain your answers.

Q39.22. Some lasers emit light in pulses that are only 10^{-12} s in duration. The length of such a pulse is $(3 \times 10^8 \text{ m/s})(10^{-12} \text{ s}) = 3 \times 10^{-4} \text{ m} = 0.3 \text{ mm}$. Can pulsed laser light be as monochromatic as light from a laser that emits a steady, continuous beam? Explain.

Exercises

Section 39.1 De Broglie Waves

39.1. (a) An electron moves with a speed of 4.70×10^6 m/s. What is its de Broglie wavelength? (b) A proton moves with the same speed. Determine its de Broglie wavelength.

39.2. For crystal diffraction experiments (discussed in Section 39.2), wavelengths on the order of 0.20 nm are often appropriate. Find the energy in electron volts for a particle with this wavelength if the particle is (a) a photon; (b) an electron; (c) an alpha particle $(m = 6.64 \times 10^{-27} \text{ kg})$.

39.3. An electron has a de Broglie wavelength of 2.80×10^{-10} m. Determine (a) the magnitude of its momentum and (b) its kinetic energy (in joules and in electron volts).

39.4. Wavelength of an Alpha Particle. An alpha particle $(m = 6.64 \times 10^{-27} \text{ kg})$ emitted in the radioactive decay of uranium-238 has an energy of 4.20 MeV. What is its de Broglie wavelength?

39.5. In the Bohr model of the hydrogen atom, what is the de Broglie wavelength for the electron when it is in (a) the $n = 1$ level and (b) the $n = 4$ level? In each case, compare the de Broglie wavelength to the circumference $2\pi r_n$ of the orbit.

39.6. (a) A nonrelativistic free particle with mass m has kinetic energy K. Derive an expression for the de Broglie wavelength of the particle in terms of m and K. (b) What is the de Broglie wavelength of an 800-eV electron?

39.7. Why Don't *We* Diffract? (a) Calculate the de Broglie wavelength of a typical person walking through a doorway. Make reasonable approximations for the necessary quantities. (b) Will the person in part (a) exhibit wave-like behavior when walking through the "single slit" of a doorway? Why?

39.8. What is the de Broglie wavelength for an electron with speed (a) $v = 0.480c$ and (b) $v = 0.960c$? (*Hint:* Use the correct relativistic expression for linear momentum if necessary.)

39.9. (a) If a photon and an electron each have the same energy of 20.0 eV, find the wavelength of each. (b) If a photon and an electron each have the same wavelength of 250 nm, find the energy of each. (c) You want to study an organic molecule that is about 250 nm long using either a photon or an electron microscope. Approximately what wavelength should you use, and which probe, the electron or the photon, is likely to damage the molecule the least?

39.10. Hydrogen gas (H_2) is at 0°C. The mass of a hydrogen *atom* is 1.67×10^{-27} kg. (a) What is the average de Broglie wavelength of the hydrogen molecules? (b) How fast would an electron have to move to have the same de Broglie wavelength as the hydrogen? Do we need to consider relativity for this electron? (c) What would be the energy of a photon having the same wavelength as the H_2 molecules and the electrons? Compare it to the kinetic energy of the hydrogen molecule in part (a) and the electron in part (b).

39.11. Wavelength of a Bullet. Calculate the de Broglie wavelength of a 5.00-g bullet that is moving at 340 m/s. Will the bullet exhibit wavelike properties?

Section 39.2 Electron Diffraction

39.12. Through what potential difference must electrons be accelerated so they will have (a) the same wavelength as an x ray of wavelength 0.150 nm and (b) the same energy as the x ray in part (a)?

39.13. (a) Approximately how fast should an electron move so it has a wavelength that makes it useful to measure the distance between adjacent atoms in typical crystals (about 0.10 nm)? (b) What is the kinetic energy of the electron in part (a)? (c) What would be the energy of a photon of the same wavelength as the electron in part (b)? (d) Which would make a more effective probe of small-scale structures, electrons or photons? Why?

39.14. A beam of electrons is accelerated from rest through a potential difference of 0.100 kV and then passes through a thin slit. The diffracted beam shows its first diffraction minima at ±11.5° from the original direction of the beam when viewed far from the slit. (a) Do we need to use relativity formulas? How do you know? (b) How wide is the slit?

39.15. A beam of neutrons that all have the same energy scatters from the atoms that have a spacing of 0.0910 nm in the surface plane of a crystal. The $m = 1$ intensity maximum occurs when the angle θ in Fig. 39.3 is 28.6°. What is the kinetic energy (in electron volts) of each neutron in the beam?

39.16. A beam of 188-eV electrons is directed at normal incidence onto a crystal surface as shown in Fig. 39.4b. The $m = 2$ intensity maximum occurs at an angle $\theta = 60.6°$. (a) What is the spacing between adjacent atoms on the surface? (b) At what other angle or angles is there an intensity maximum? (c) For what electron energy (in electron volts) would the $m = 1$ intensity maximum occur at $\theta = 60.6°$? For this energy, is there an $m = 2$ intensity maximum? Explain.

39.17. A CD-ROM is used instead of a crystal in an electron-diffraction experiment like that shown in Fig. 39.3. The surface of the CD-ROM has tracks of tiny pits with a uniform spacing of 1.60 μm. (a) If the speed of the electrons is 1.26×10^4/s, at which values of θ will the $m = 1$ and $m = 2$ intensity maxima appear? (b) The scattered electrons in these maxima strike at normal incidence a piece of photographic film that is 50.0 cm from the CD-ROM. What is the spacing on the film between these maxima?

Section 39.3 Probability and Uncertainty

39.18. A pesky 1.5-mg mosquito is annoying you as you attempt to study physics in your room, which is 5.0 m wide and 2.5 m high. You decide to swat the bothersome insect as it flies toward you, but you need to estimate its speed to make a successful hit. (a) What is the maximum uncertainty in the horizontal position of the mosquito? (b) What limit does the Heisenberg uncertainty principle place on your ability to know the horizontal velocity of this mosquito? Is this limitation a serious impediment to your attempt to swat it?

39.19. By extremely careful measurement, you determine the x-coordinate of a car's center of mass with an uncertainty of only 1.00 μm. The car has a mass of 1200 kg. (a) What is the minimum uncertainty in the x-component of the velocity of the car's center of mass as prescribed by the Heisenberg uncertainty principle? (b) Does the uncertainty principle impose a practical limit on our ability to make simultaneous measurements of the positions and velocities of ordinary objects like cars, books, and people? Explain.

39.20. A 10.0-g marble is gently placed on a horizontal tabletop that is 1.75 m wide. (a) What is the maximum uncertainty in the horizontal position of the marble? (b) According to the Heisenberg uncertainty principle, what is the minimum uncertainty in the horizontal velocity of the marble? (c) In light of your answer to part (b), what is the longest time the marble could remain on the table? Compare this time to the age of the universe, which is approximately 14 billion years. (*Hint:* Can you know that the horizontal velocity of the marble is *exactly* zero?)

39.21. A scientist has devised a new method of isolating individual particles. He claims that this method enables him to detect simultaneously the position of a particle along an axis with a standard deviation of 0.12 nm and its momentum component along this axis with a standard deviation of $3.0 \times 10^{-25} \text{ kg} \cdot \text{m/s}$. Use the Heisenberg uncertainty principle to evaluate the validity of this claim.

39.22. (a) The x-coordinate of an electron is measured with an uncertainty of 0.20 mm. What is the x-component of the electron's velocity, v_x, if the minimum percentage uncertainty in a simultaneous measurement of v_x is 1.0%? (b) Repeat part (a) for a proton.

39.23. An atom in a metastable state has a lifetime of 5.2 ms. What is the uncertainty in energy of the metastable state?

39.24. (a) The uncertainty in the y-component of a proton's position is 2.0×10^{-12} m. What is the minimum uncertainty in a simultaneous measurement of the y-component of the proton's velocity? (b) The uncertainty in the z-component of an electron's velocity is 0.250 m/s. What is the minimum uncertainty in a simultaneous measurement of the z-coordinate of the electron?

39.25. The ψ ("psi") particle has a rest energy of 3097 MeV $(1 \text{ MeV} = 10^6 \text{ eV})$. The ψ particle is unstable with a lifetime of 7.6×10^{-21} s. Estimate the uncertainty in rest energy of the ψ particle. Express your answer in MeV and as a fraction of the rest energy of the particle.

39.26. Particle Lifetime. The unstable W^+ particle has a rest energy of 80.41 GeV $(1 \text{ GeV} = 10^9 \text{ eV})$ and an uncertainty in rest energy of 2.06 GeV. Estimate the lifetime of the W^+ particle.

Section 39.4 The Electron Microscope

39.27. You want to study a biological specimen by means of a wavelength of 10.0 nm, and you have a choice of using electromagnetic waves or an electron microscope. (a) Calculate the ratio of the energy of a 10.0-nm-wavelength photon to the kinetic energy of a 10.0-nm-wavelength electron. (b) In view of your answer to part (a), which would be less damaging to the specimen you are studying: photons or electrons?

39.28. (a) In an electron microscope, what accelerating voltage is needed to produce electrons with wavelength 0.0600 nm? (b) If protons are used instead of electrons, what accelerating voltage is needed to produce protons with wavelength 0.0600 nm? (*Hint:* In each case the initial kinetic energy is negligible.)

Section 39.5 Wave Functions and the Schrödinger Equation

39.29. Consider a wave function given by $\psi(x) = A \sin kx$, where $k = 2\pi/\lambda$ and A is a real constant. (a) For what values of x is there the highest probability of finding the particle described by this wave function? Explain. (b) For which values of x is the probability *zero?* Explain.

39.30. Compute $|\Psi|^2$ for $\Psi = \psi \sin \omega t$, where ψ is time independent and ω is a real constant. Is this a wave function for a stationary state? Why or why not?

39.31. Normalization of the Wave Function. Consider a particle moving in one dimension, which we shall call the x-axis. (a) What does it mean for the wave function of this particle to be *normalized?* (b) Is the wave function $\psi(x) = e^{ax}$, where a is a positive real number, normalized? Could this be a valid wave function? (c) If the particle described by the wave function $\psi(x) = Ae^{bx}$, where A and b are positive real numbers, is confined to the range $x \geq 0$, determine A (including its units) so that the wave function is normalized.

39.32. A particle is described by a wave function $\psi(x) = Ae^{-\alpha x^2}$, where A and α are real, positive constants. If the value of α is increased, what effect does this have on (a) the particle's uncertainty in position and (b) the particle's uncertainty in momentum? Explain your answers.

39.33. Consider the complex-valued function $f(x, y) = (x - iy)/(x + iy)$. Calculate $|f|^2$.

39.34. Particle A is described by the wave function $\psi(x, y, z)$. Particle B is described by the wave function $\psi(x, y, z)e^{i\phi}$, where ϕ is a real constant. How does the probability of finding particle A within a volume dV around a certain point in space compare with the probability of finding particle B within this same volume?

39.35. A particle moving in one dimension (the x-axis) is described by the wave function

$$\psi(x) = \begin{cases} Ae^{-bx}, & \text{for } x \geq 0 \\ Ae^{bx}, & \text{for } x < 0 \end{cases}$$

where $b = 2.00 \text{ m}^{-1}$, $A > 0$, and the $+x$-axis points toward the right. (a) Determine A so that the wave function is normalized. (b) Sketch the graph of the wave function. (c) Find the probability of finding this particle in each of the following regions: (i) within 50.0 cm of the origin, (ii) on the left side of the origin (can you first guess the answer by looking at the graph of the wave function?), (iii) between $x = 0.500$ m and $x = 1.00$ m.

39.36. Linear Combinations of Wave Functions. Let ψ_1 and ψ_2 be two solutions of Eq. (39.18) with the same energy E. Show that $\psi = B\psi_1 + C\psi_2$ is also a solution with energy E, for any values of the constants B and C.

39.37. Let ψ_1 and ψ_2 be two solutions of Eq. (39.18) with energies E_1 and E_2, respectively, where $E_1 \neq E_2$. Is $\psi = A\psi_1 + B\psi_2$, where A and B are nonzero constants, a solution to Eq. (39.18)? Explain your answer.

Problems

39.38. A beam of 40-eV electrons traveling in the $+x$-direction passes through a slit that is parallel to the y-axis and 5.0 μm wide. The diffraction pattern is recorded on a screen 2.5 m from the slit. (a) What is the de Broglie wavelength of the electrons? (b) How much time does it take the electrons to travel from the slit to the screen? (c) Use the width of the central diffraction pattern to calculate the uncertainty in the y-component of momentum of an electron just after it has passed through the slit. (d) Use the result of part (c) and the Heisenberg uncertainty principle (Eq. 39.11 for y) to estimate the minimum uncertainty in the y-coordinate of an electron just after it has passed through the slit. Compare your result to the width of the slit.

39.39. (a) What is the energy of a photon that has wavelength $0.10\ \mu\text{m}$? (b) Through approximately what potential difference must electrons be accelerated so that they will exhibit wave nature in passing through a pinhole $0.10\ \mu\text{m}$ in diameter? What is the speed of these electrons? (c) If protons rather than electrons were used, through what potential difference would protons have to be accelerated so they would exhibit wave nature in passing through this pinhole? What would be the speed of these protons?

39.40. Electrons go through a single slit 150 nm wide and strike a screen 24.0 cm away. You find that at angles of $\pm 20.0°$ from the center of the diffraction pattern, no electrons hit the screen but electrons hit at all points closer to the center. (a) How fast were these electrons moving when they went through the slit? (b) What will be the next larger angles at which no electrons hit the screen?

39.41. A beam of electrons is accelerated from rest and then passes through a pair of identical thin slits that are 1.25 nm apart. You observe that the first double-slit interference dark fringe occurs at $\pm 18.0°$ from the original direction of the beam when viewed on a distant screen. (a) Are these electrons relativistic? How do you know? (b) Through what potential difference were the electrons accelerated?

39.42. A beam of protons and a beam of alpha particles (of mass 6.64×10^{-27} kg and charge $+2e$) are accelerated from rest through the same potential difference and pass through identical circular holes in a very thin, opaque film. When viewed far from the hole, the diffracted proton beam forms its first dark ring at 15° with respect to its original direction. When viewed similarly, at what angle will the alpha particle form its first dark ring?

39.43. An electron beam and a photon beam pass through identical slits. On a distant screen, the first dark fringe occurs at the same angle for both of the beams. The electron speeds are much slower than that of light. (a) Express the energy of a photon in terms of the kinetic energy K of one of the electrons. (b) Which is greater, the energy of a photon or the kinetic energy of an electron?

39.44. Coherent light is passed through two narrow slits whose separation is $40.0\ \mu\text{m}$. The second-order bright fringe in the interference pattern is located at an angle of 0.0300 rad. If electrons are used instead of light, what must the kinetic energy (in electron volts) of the electrons be if they are to produce an interference pattern for which the second-order maximum is also at 0.0300 rad?

39.45. What is the de Broglie wavelength of a red blood cell, with mass 1.00×10^{-11} g, that is moving with a speed of 0.400 cm/s? Do we need to be concerned with the wave nature of the blood cells when we describe the flow of blood in the body?

39.46. High-speed electrons are used to probe the interior structure of the atomic nucleus. For such electrons the expression $\lambda = h/p$ still holds, but we must use the relativistic expression for momentum, $p = mv/\sqrt{1 - v^2/c^2}$. (a) Show that the speed of an electron that has de Broglie wavelength λ is

$$v = \frac{c}{\sqrt{1 + (mc\lambda/h)^2}}$$

(b) The quantity h/mc equals 2.426×10^{-12} m. (As we saw in Section 38.7, this same quantity appears in Eq. (38.23), the expression for Compton scattering of photons by electrons.) If λ is small compared to h/mc, the denominator in the expression found in part (a) is close to unity and the speed v is very close to c. In this case it is convenient to write $v = (1 - \Delta)c$ and express the speed of the electron in terms of Δ rather than v. Find an expression for Δ valid when $\lambda \ll h/mc$. [*Hint:* Use the binomial expansion $(1 + z)^n = 1 + nz + n(n - 1)z^2/2 + \cdots$, valid for the case $|z| < 1$. (c) How fast must an electron move for its de Broglie wavelength to be 1.00×10^{-15} m, comparable to the size of a proton? Express your answer in the form $v = (1 - \Delta)c$, and state the value of Δ.

39.47. (a) What is the de Broglie wavelength of an electron accelerated from rest through a potential increase of 125 V? (b) What is the de Broglie wavelength of an alpha particle $(q = +2e,\ m = 6.64 \times 10^{-27}\ \text{kg})$ accelerated from rest through a potential drop of 125 V?

39.48. Suppose that the uncertainty of position of an electron is equal to the radius of the $n = 1$ Bohr orbit for hydrogen. Calculate the simultaneous minimum uncertainty of the corresponding momentum component, and compare this with the magnitude of the momentum of the electron in the $n = 1$ Bohr orbit. Discuss your results.

39.49. (a) A particle with mass m has kinetic energy equal to three times its rest energy. What is the de Broglie wavelength of this particle? (*Hint:* You must use the relativistic expressions for momentum and kinetic energy: $E^2 = (pc)^2 + (mc^2)^2$ and $K = E - mc^2$.) (b) Determine the numerical value of the kinetic energy (in MeV) and the wavelength (in meters) if the particle in part (a) is (i) an electron and (ii) a proton.

39.50. Proton Energy in a Nucleus. The radii of atomic nuclei are of the order of 5.0×10^{-15} m. (a) Estimate the minimum uncertainty in the momentum of a proton if it is confined within a nucleus. (b) Take this uncertainty in momentum to be an estimate of the magnitude of the momentum. Use the relativistic relationship between energy and momentum, Eq. (37.39), to obtain an estimate of the kinetic energy of a proton confined within a nucleus. (c) For a proton to remain bound within a nucleus, what must the magnitude of the (negative) potential energy for a proton be within the nucleus? Give your answer in eV and in MeV. Compare to the potential energy for an electron in a hydrogen atom, which has a magnitude of a few tens of eV. (This shows why the interaction that binds the nucleus together is called the "strong nuclear force.")

39.51. Electron Energy in a Nucleus. The radii of atomic nuclei are of the order of 5.0×10^{-15} m. (a) Estimate the minimum uncertainty in the momentum of an electron if it is confined within a nucleus. (b) Take this uncertainty in momentum to be an estimate of the magnitude of the momentum. Use the relativistic relationship between energy and momentum, Eq. (37.39), to obtain an estimate of the kinetic energy of an electron confined within a nucleus. (c) Compare the energy calculated in part (b) to the magnitude of the Coulomb potential energy of a proton and an electron separated by 5.0×10^{-15} m. On the basis of your result, could there be electrons within the nucleus? (*Note:* It is interesting to compare this result to that of Problem 39.50.)

39.52. In a TV picture tube the accelerating voltage is 15.0 kV, and the electron beam passes through an aperture 0.50 mm in diameter to a screen 0.300 m away. (a) Calculate the uncertainty in the component of the electron's velocity perpendicular to the line between aperture and screen. (b) What is the uncertainty in position of the point where the electrons strike the screen? (c) Does this uncertainty affect the clarity of the picture significantly? (Use nonrelativistic expressions for the motion of the electrons. This is fairly accurate and is certainly adequate for obtaining an estimate of uncertainty effects.)

39.53. The neutral pion (π^0) is an unstable particle produced in high-energy particle collisions. Its mass is about 264 times that of the electron, and it exists for an average lifetime of 8.4×10^{-17} s before decaying into two gamma-ray photons. Using the relationship $E = mc^2$ between rest mass and energy, find the uncertainty in the mass of the particle and express it as a fraction of the mass.

39.54. Quantum Effects in Daily Life? A 1.25-mg insect flies through a 4.00-mm-diameter hole in an ordinary window screen. The thickness of the screen is 0.500 mm. (a) What should be the approximate wavelength and speed of the insect for her to show wave behavior as she goes through the hole? (b) At the speed found in part (a), how long would it take the insect to pass through the 0.500-mm thickness of the hole in the screen? Compare this time to the age of the universe (about 14 billion years). Would you expect to see "insect diffraction" in daily life?

39.55. Doorway Diffraction. If your wavelength were 1.0 m, you would undergo considerable diffraction in moving through a doorway. (a) What must your speed be for you to have this wavelength? (Assume that your mass is 60.0 kg.) (b) At the speed calculated in part (a), how many years would it take you to move 0.80 m (one step)? Will you notice diffraction effects as you walk through doorways?

39.56. Atomic Spectra Uncertainties. A certain atom has an energy level 2.58 eV above the ground level. Once excited to this level, the atom remains in this level for 1.64×10^{-7} s (on average) before emitting a photon and returning to the ground level. (a) What is the energy of the photon (in electron volts)? What is its wavelength (in nanometers)? (b) What is the smallest possible uncertainty in energy of the photon? Give your answer in electron volts. (c) Show that $|\Delta E/E| = |\Delta\lambda/\lambda|$ if $|\Delta\lambda/\lambda| \ll 1$. Use this to calculate the magnitude of the smallest possible uncertainty in the wavelength of the photon. Give your answer in nanometers.

39.57. You intend to use an electron microscope to study the structure of some crystals. For accurate resolution, you want the electron wavelength to be 1.00 nm. (a) Are these electrons relativistic? How do you know? (b) What accelerating potential is needed? (c) What is the kinetic energy of the electrons you are using? To see if it is great enough to damage the crystals you are studying, compare it to the potential energy of a typical NaCl molecule, which is about 6.0 eV. (d) If you decided to use electromagnetic waves as you probe, what energy should their photons have to provide the same resolution as the electrons? Would this energy damage the crystal?

39.58. For x rays with wavelength 0.0300 nm, the $m = 1$ intensity maximum for a crystal occurs when the angle θ in Fig. 36.23c is 35.8°. At what angle θ does the $m = 1$ maximum occur when a beam of 4.50-keV electrons is used instead? Assume that the electrons also scatter from the atoms in the surface plane of this same crystal.

39.59. Electron diffraction can also take place when there is interference between electron waves that scatter from atoms on the surface of a crystal and waves that scatter from atoms in the next plane below the surface, a distance d from the surface (see Fig. 36.23c). (a) Find an equation for the angles θ at which there is an intensity maximum for electron waves of wavelength λ. (b) The spacing between crystal planes in a certain metal is 0.091 nm. If 71.0-eV electrons are used, find the angle at which there is an intensity maximum due to interference between scattered waves from adjacent crystal planes. The angle is measured as shown in Fig. 36.23c. (c) The actual angle of the intensity maximum is slightly different from your result in part (b). The reason is the work function ϕ of the metal (see Section 38.2), which changes the electron potential energy by $-e\phi$ when it moves from vacuum into the metal. If the effect of the work function is taken into account, is the angle of the intensity maximum larger or smaller than the value found in part (b)? Explain.

39.60. Zero-Point Energy. Consider a particle with mass m moving in a potential $U = \frac{1}{2}kx^2$, as in a mass-spring system. The total energy of the particle is $E = p^2/2m + \frac{1}{2}kx^2$. Assume that p and x are approximately related by the Heisenberg uncertainty principle, $px \approx h$. (a) Calculate the minimum possible value of the energy E, and the value of x that gives this minimum E. This lowest possible energy, which is not zero, is called the *zero-point energy.* (b) For the x calculated in part (a), what is the ratio of the kinetic to the potential energy of the particle?

39.61. A particle with mass m moves in a potential $U(x) = A|x|$, where A is a positive constant. In a simplified picture, quarks (the constituents of protons, neutrons, and other particles, as will be described in Chapter 44) have a potential energy of interaction of approximately this form, where x represents the separation between a pair of quarks. Because $U(x) \to \infty$ as $x \to \infty$, it's not possible to separate quarks from each other (a phenomenon called *quark confinement*). (a) Classically, what is the force acting on this particle as a function of x? (b) Using the uncertainty principle as in Problem 39.60, determine approximately the zero-point energy of the particle.

39.62. The discussion in Section 39.5 shows that the wave function $\Psi = \psi e^{-i\omega t}$ is a stationary state, where ψ is time independent and ω is a real (not complex) constant. Consider the wave function $\Psi = \psi_1 e^{-i\omega_1 t} + \psi_2 e^{-i\omega_2 t}$, where ψ_1 and ψ_2 are different time-independent functions and ω_1 and ω_2 are different real constants. Assume that ψ_1 and ψ_2 are real-valued functions, so that $\psi_1{}^* = \psi_1$ and $\psi_2{}^* = \psi_2$. Is this Ψ a wave function for a stationary state? Why or why not?

39.63. The Time-Dependent Schrödinger Equation. Equation (39.18) is the time-independent Schrödinger equation in one dimension. The time-dependent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

If $\psi(x)$ is a solution to Eq. (39.18) with energy E, show that the time-dependent function $\Psi(x,t) = \psi(x)e^{-i\omega t}$ is a solution to the time-dependent Schrödinger equation if ω is chosen appropriately. What is the value of ω that makes Ψ a solution?

39.64. Time-Dependent Wave Function for a Free Particle. One example of a time-dependent wave function is that of a free particle [one for which $U(x) = 0$ for all x] of energy E and x-component of momentum p. From the de Broglie relationships (see Section 39.1), such a particle has associated with it a frequency $f = E/h$ and a wavelength $\lambda = h/p$. A reasonable first guess for the time-dependent wave function for such a particle is $\Psi(x,t) = A\cos(kx - \omega t)$, where A is a constant, $\omega = 2\pi f$ is the angular frequency, and $k = 2\pi/\lambda$ is the wave number. This is the same function we used to describe a mechanical wave [see Eq. (15.7)] or an electromagnetic wave propagating in the x-direction [see Eq. (32.16)]. (a) Show that $\omega = E\hbar, k = p/\hbar$, and $\omega = \hbar k^2/2m$. (*Hint:* The energy is purely kinetic, so $E = p^2/2m$.) (b) To check this guess for the time-dependent wave function, substitute $\Psi(x,t) = A\cos(kx - \omega t)$ into the time-dependent Schrödinger equation (see Problem 39.63) with $U(x) = 0$ (so the particle is free). Show that this guess for $\Psi(x,t)$ does *not* satisfy this equation and so is not a suitable wave function for a free particle. (c) Use the procedure described in part (b) to show that a second guess, $\Psi(x,t) = A\sin(kx - \omega t)$, is also not a suitable wave function for a free particle. (d) Consider a combination of the functions proposed in parts (b) and (c):

$$\Psi(x, t) = A\cos(kx - \omega t) + B\sin(kx - \omega t)$$

By using the procedure described in part (b), show that this wave function is a solution to the time-dependent Schrödinger equation with $U(x) = 0$, but only if $B = iA$. (*Hint:* To satisfy the time-dependent Schrödinger equation for all x and t, the coefficients of $\cos(kx - \omega t)$ on both sides of the equation must be equal. The same is true for the coefficients of $\sin(kx - \omega t)$ on both sides of the equation.) This is an example of the general result that time-dependent wave functions always have both a real part and an imaginary part.

39.65. Imagine another universe in which the value of Planck's constant is 0.0663 J · s, but in which the physical laws and all other physical constants are the same as in our universe. In this universe, two physics students are playing catch. They are 12 m apart, and one throws a 0.25-kg ball directly toward the other with a speed of 6.0 m/s. (a) What is the uncertainty in the ball's horizontal momentum, in a direction perpendicular to that in which it is being thrown, if the student throwing the ball knows that it is located within a cube with volume 125 cm^3 at the time she throws it? (b) By what horizontal distance could the ball miss the second student?

39.66. A particle is described by the normalized wave function $\psi(x, y, z) = Axe^{-\alpha x^2}e^{-\beta y^2}e^{-\gamma z^2}$, where A, α, β, and γ are all real, positive constants. The probability that the particle will be found in the infinitesimal volume $dx\,dy\,dz$ centered at the point (x_0, y_0, z_0) is $|\psi(x_0, y_0, z_0)|^2\,dx\,dy\,dz$. (a) At what value of x_0 is the particle most likely to be found? (b) Are there values of x_0 for which the probability of the particle being found is zero? If so, at what x_0?

39.67. A particle is described by the normalized wave function $\psi(x, y, z) = Ae^{-\alpha(x^2+y^2+z^2)}$, where A and α are real, positive constants. (a) Determine the probability of finding the particle at a distance between r and $r + dr$ from the origin. (*Hint:* See Problem 39.66. Consider a spherical shell centered on the origin with inner radius r and thickness dr.) (b) For what value of r does the probability in part (a) have its maximum value? Is this the same value of r for which $|\psi(x, y, z)|^2$ is a maximum? Explain any differences.

39.68. Consider the wave packet defined by

$$\psi(x) = \int_0^\infty B(k)\cos kx\,dk$$

Let $B(k) = e^{-\alpha^2k^2}$. (a) The function $B(k)$ has its maximum value at $k = 0$. Let k_h be the value of k at which $B(k)$ has fallen to half its maximum value, and define the width of $B(k)$ as $w_k = k_\mathrm{h}$. In terms of α, what is w_k? (b) Use integral tables to evaluate the integral that gives $\psi(x)$. For what value of x is $\psi(x)$ maximum? (c) Define the width of $\psi(x)$ as $w_x = x_\mathrm{h}$, where x_h is the positive value of x where $\psi(x)$ has fallen to half its maximum value. Calculate w_x in terms of α. (d) The momentum p is equal to $hk/2\pi$, so the width of B in momentum is $w_p = hw_k/2\pi$. Calculate the product w_pw_x and compare to the Heisenberg uncertainty principle.

39.69. (a) Using the integral in Problem 39.68, determine the wave function $\psi(x)$ for a function $B(k)$ given by

$$B(k) = \begin{cases} 0 & k < 0 \\ 1/k_0, & 0 \le k \le k_0 \\ 0, & k > k_0 \end{cases}$$

This represents an equal combination of all wave numbers between 0 and k_0. Thus $\psi(x)$ represents a particle with average wave number $k_0/2$, with a total spread or uncertainty in wave number of k_0. We will call this spread the *width* w_k of $B(k)$, so $w_k = k_0$. (b) Graph $B(k)$ versus k and $\psi(x)$ versus x for the case $k_0 = 2\pi/L$, where L is a length. Locate the point where $\psi(x)$ has its maximum value and label this point on your graph. Locate the two points closest to this maximum (one on each side of it) where $\psi(x) = 0$, and define the distance along the x-axis between these two points as w_x, the width of $\psi(x)$. Indicate the distance w_x on your graph. What is the value of w_x if $k_0 = 2\pi/L$? (c) Repeat part (b) for the case $k_0 = \pi/L$. (d) The momentum p is equal to $hk/2\pi$, so the width of B in momentum is $w_p = hw_k/2\pi$. Calculate the product w_pw_x for each of the cases $k_0 = 2\pi/L$ and $k_0 = \pi/L$. Discuss your results in light of the Heisenberg uncertainty principle.

Challenge Problems

39.70. The wave nature of particles results in the quantum-mechanical situation that a particle confined in a box can assume only wavelengths that result in standing waves in the box, with nodes at the box walls. (a) Show that an electron confined in a one-dimensional box of length L will have energy levels given by

$$E_n = \frac{n^2h^2}{8mL^2}$$

(*Hint:* Recall that the relationship between the de Broglie wavelength and the speed of a nonrelativistic particle is $mv = h/\lambda$. The energy of the particle is $\frac{1}{2}mv^2$.) (b) If a hydrogen atom is modeled as a one-dimensional box with length equal to the Bohr radius, what is the energy (in electron volts) of the lowest energy level of the electron?

39.71. You have entered a contest in which the contestants drop a marble with mass 20.0 g from the roof of a building onto a small target 25.0 m below. From uncertainty considerations, what is the typical distance by which you will miss the target, given that you aim with the highest possible precision? (*Hint:* The uncertainty Δx_f in the x-coordinate of the marble when it reaches the ground comes in part from the uncertainty Δx_i in the x-coordinate initially and in part from the initial uncertainty in v_x. The latter gives rise to an uncertainty Δv_x in the horizontal motion of the marble as it falls. The values of Δx_i and Δv_x are related by the uncertainty principle. A small Δx_i gives rise to a large Δv_x, and vice versa. Find the value of Δx_i that gives the smallest total uncertainty in x at the ground. Ignore any effects of air resistance.)