THE INSTITUTION OF ELECTRICAL ENGINEERS

Electronics Division
Tutorial Meeting organised by
Professional Groups E15 (Radar, sonar and navigation systems)/
E5 (Signal processing) and the Institute of Acoustics on
"DIGITAL SIGNAL PROCESSING FOR RADAR AND SONAR APPLICATIONS"
to be held at the University of Birmingham from
Tuesday, 11 September to Wednesday, 12 September 1990

PROGRAMME

Tuesday, 11 September

Contribution No	Time	
	11.00	Registration
	1.00	LUNCH
1	2.00	"Similarities and differences in signal processing in sonar and radar": Professor D J Creasey (University of Birmingham)
	3.10	TEA
2	3.30	"Introduction to matrix techniques": P J Hargrave (STC Technology)
3	4.30	"Sonar signal processing - I": T E Curtis (ARE)
	7.00 for 7.30	Formal Tutorial Dinner

Wednesday, 12 September

Contribution No	Time	
4	9.00	"Radar signal processing": P Matthewson (GEC-Marconi Research Centre)
	10.30	COFFEE
5	10.45	"Sonar signal processing - II": T E Curtis (ARE)
6	11.45	"Adaptive signal processing": J G McWhirter (RSRE)
	1.00	LUNCH
7	2.00	"Synthetic aperture techniques": S Watts (Thorn EMI)
	3.30	TEA
	3.45	CLOSE

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SIMILARITIES AND DIFFERENCES IN SIGNAL PROCESSING FOR RADAR AND SONAR

David Creasey[*]

1 INTRODUCTION

1.1 Information

Both sonar and radar systems are types of communication systems. Shannon states that in a single ideal communication channel, the maximum information output is given by [1]

$$I < 2BT\log_2(1 + s/n) \text{ bit} \qquad (1)$$

where B = the bandwidth of the channel,
T = the observation time and
s/n = the signal-to-noise ratio.

Neither radar nor sonar are ideal information channels, but Equation 1 does indicate what we must do in order to get more information from the system.

Firstly, note that if the signal-to-noise ratio approaches zero then the information that can be derived from the system is also virtually zero, ($\log_2 1 = 0$). Put another way, if you put rubbish in you get rubbish out. The system designer and the system user must always ensure that there is signal to process.

Secondly, a long observation time will produce more information than a short observation time. Thus if we can integrate the output of a system over along time then we shall obtain some processing gain.

Thirdly, always use a signal with a wide bandwidth. In some ways, this contradicts the requirement for a high signal-to-noise ratio since noise increases as the bandwidth increases. However, the signal-to-noise term is included in a logarithm, so there is a gain in an information sense by using a high bandwidth.

Often radar and sonar systems use an array of elements. If the array elements are sufficiently spaced, more than half a wavelength apart, then each element can be considered to form a single channel. This means in a system with an array of M elements, each suitably spaced, there are M separate channels. Shannon's simple information limit can then be increased by a factor of M to give

$$I < 2BTM\log_2(1 + s/n) \text{ bit} \qquad (2)$$

1.2 Active and Passive Systems
1.2.1 Active systems.

If energy is propagated into a medium, an object within that medium will intercept and re-radiate some of the transmitted energy. A receiver will then be able to detect the presence of the object by observing the reflected energy. Such systems are known as echo-ranging systems since they measure the two-way propagation time and, from a knowledge of the speed of propagation, the range of the target can be calculated.

The most obvious difference between a radar and a sonar system is the difference between the speed of electromagetic propagation for a radar (300×10^6 m/s) compared to the speed of sound in water (1.5×10^3 m/s). For a target at a range of 15 km the two-way propagation time for a radar is only 100 μs.

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For a sonar target at the same range the two-way propagation time is 20 s. Things happen very much more slowly in sonar than they do in a radar.

There are three sources of interference in both radar and sonar systems. Firstly, there is the background noise. A signal that is always present due to many causes. These causes can be naturally occuring noise, or, in the other extreme, it may be noise deliberately generated to confuse the echo-ranging system. The system must ensure that the output of the receiver can have a sufficiently large signal component so that it may be detected in the presence of noise. This can often be accomplished by increasing the quantity of energy radiated by the transmitter.

There are often many scatterers both within the medium and at its boundaries. These scatterers are often individually small but there are many of them so that they can combine to produce large but very variable signals at the receiver. Such signals are called clutter in radar and reverberation in sonar. Increasing the transmitted signal level does not improve the signal-to-reverberation level since they are both proportional to the level of the signal transmitted. A large amount of directionality in the receiver or the differences in Doppler shift between the signals backscattered from the target and the clutter/reverberation can be used to differentiate between signal and clutter/reverberation.

The third form of interference signal is very similar to clutter/reverberation. Energy can often propagate over different paths,for example by a direct path and by reflection from a boundary. Differences in propagation time can be used to differentiate between the direct path and the reflected path.

The signal-to-noise ratio at the receiver can be evaluated from a number of factors. These result in an energy balance equation where the level of the echo signal is compared to the interference (noise). The resulting radar equation and sonar equation are both derived by taking the transmitted signal level multiplying by the losses expected in the two-way transmission path and further multiplying by the ratio of the energy reflected from the target to that intercepted by the target. This gives the level of the echo signal at the receiver. This is then divided by the level of the noise to give the signal-to-noise ratio. Often the series of multiplications and divisions provides a very cumbersome representation. Alternatively, the ratio can be expressd in a logarithmic form and usually this is done by taking the terms individually and expressing them in logarithmic form. This results in an equation of the type

$$S/N = SL - 2PL + TS - NL \text{ dB} \qquad (3)$$

Equation 3 appears very simple but it hides many components. For example the source level, SL, will depend upon the directivity of the transmitter; the propagation loss, PL, will contain components that comprise spreading losses and absorption losses; the target strength, TS, is aspect dependent for many practical targets; and the noise level, NL, will depend upon the noise spectral density, the bandwidth and possibly the directional properties of the receiver. In radar and sonar the basic formulation of Equation 3 is identical but the detailed components within each of the terms differ.

Active sonars operate over frequencies from a few hundred hertz to about 1 MHz. Radar on the other hand operates from r.f. frequencies of say 10 MHz up to frequencies just below the infra-red region at about 30 GHz. Thus a second major difference

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between radar and sonar is the operating frequency. Strangely the wavelengths used overlap because of the major difference of propagation speed. The relationship between velocity, c, frequency, f, and wavelength, λ, is

$$\lambda = c/f \qquad (4)$$

Thus, the wavelengths for an active sonar range from about 3 m down to 1.5 mm. The range of radar wavelengths are from 30 m down to 10 mm or so.

1.2.2 Passive operations

The need for covert operations underwater has lead to the development of sophisticated systems that rely upon the noise created at the target by its propellers, on-board rotating and reciprocating machinery and flow noise.

Radio telescopes are effectively passive radar systems although they are not often viewed as such. The nearest direct passive equivalent usually associated with radar area electronic support measures (ESM) where listening equipment detects radiation from active radars and communications systems. Sonar intercept equipments are of course used in the underwater environment.

The operating frequencies of ESM and sonar intercept equipments are dictated by the frequencies used in active radars and active sonars. Ideally, passive sonars would operate at frequencies from 10 Hz or so and up to about 3 kHz so as to encompass all the useful spectrum of the radiated signatures of passive targets. The difficulty of operating at the frequencies of around 10 Hz is the high ambient sea noise and the difficulty of operating with large apertures so as to obtain directional information.

The major interference component in a passive sonar is the ambient noise of the sea. Multi-path propagation can cause problems when narrow-band operation results from such operations as spectrum analysis. Here the signals from different paths can combine either constructively of destructively and the signal levels perceived by the sonar can fluctuate wildly. However, the basic premise of a passive sonar is often wideband operation so that over the wideband the signal level is more constant.

1.3 Objectives

The objectives of radar and sonar are threefold. They must detect, locate and classify targets so that effective actions,such as the derivation of a fire-control solution, can be obtained. All of these objectives require high signal-to-noise ratios to be effective.

Detection requires that the target signal should be well above any interference. If that interference is clutter or reverberation, it may be necessary to look for differences in Doppler between the target and the interference. When an active system is noise limited and it may be impossible to increase the peak transmitted signal level. One solution is to code the transmitted signal, for example, changing the frequency during the pulse. On reception the echo signal plus noise is cross correlated agianst a replica of the transmitted signal. The echo should produce a high degree of correlation while the noise should be poorly correlated.

The range, bearing and heading are the three basic parameters that an echo ranging system can produce. The range is obtained simply from the time between pulse transmission and echo reception. Bearing is obtained from the directional response of

the system arrays. Heading information is obtained from the changes in range and bearing observed over a period of time. Doppler shift will give radial speed directly.
In a passive system, range is not directly available and must be computed from a number of sets of data samples either separated in time or space.

Classification is a very sensitive topic. Suffice to say, if a very short pulse is used the amplitude highlights in the echo signal may give clues about the target shape. Alternatively the behaviour of the target observed over a period of time may be the only clue available in an active system. In passive sonar, the received signal spectrum will contain a number of discrete lines that may be used to classify the target.

Thus radars and sonars both need to generate signals in the receiver above interference signals. They need to look at the spectrum of the received signals. Time integration is used in replica correlators and both need to process the data received from arrays of elements.

1.4 Bandwidth

The speed of propagation causes the major difference between the digital processing of radar and sonar signals. This speed difference is directly responsible for the differences in system bandwidth. The bandwidth in an active system is fixed by the need to obtain a good range resolution. The range-resolution cell is equal 0.5cT where T is the effective pulse length and c is the velocity of propagation. The system is matched when it has a bandwidth, B, just sufficiently wide to accommodate the pulse length. This is usually expressed in the simple equation

$$B = 1/T \qquad \qquad \dots\dots(5)$$

Table 1.1 gives comparative parameters for typical radars and active sonars.

TABLE 1.1

Activity		Operating frequency	Pulse length	Bandwidth	Range res'n cell
Long Range:	Radar	1.25 GHz	1 µs	1 MHz	150 m
	Sonar	4 kHz	1 ms	1 kHz	0.75 m
Weapons System	Radar	15 GHz	30 ns	33 MHz	4.5 m
	Sonar	40 kHz	100 µs	10 kHz	75 mm

This bring us to the question about how signals should be sampled prior to quantisation and being processed digitally. It is possible for signals from an active soanr and a passive sonar to be sampled directly as baseband signals at a rate something slightly above the Nyquist rate. For the long range sonar operating over the band 3.5 kHz to 4.5 kHz then a sampling frequency of about 10 kHz will suffice. Even the signals from a weapons sonar could be sampled as a baseband signal at about sampling rate of 100 kHz.

However, analog-to-digital converters operating in the gigahertz region are very rare. To accommodate radar signals the signals have to be modulated down to baseband frequencies. When this is done, it is usual to modulate down to baseband using a local oscillator working at the centre frequency of the radar signal. The signal then needs to be modulated with both the sine and the cosine of the carrier in separate channels to produce

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in-phase (I) and quadrature (Q) signals. The sampling frequency
in each channel can then be halved but as there are two channels
the number of samples is still equal to those rquired by the
Nyquist sampling theorem. Table 1.2 compares sampling rates for
radar and sonar signals.

<div align="center">TABLE 1.2</div>

Activity		Operating frequency	Bandwidth	Min. Sampling Frequency
Long Range	Radar	1.25 GHz	1 MHz	1 MHz in both I & Q channels
	Sonar	4 kHz	1 kHz	2 kHz as base-band signal or 1 kHz in both I & Q channels
Weapons System	Radar	15 GHz	33 MHz	33 MHz in both I & Q channels
	Sonar	40 kHz	10 kHz	20 kHz as base-band signal or 10 kHz in both I & Q channels

Note that for many situations sonar signals need to be sampled in
quadrature and apart from the sampling frequencies involved radar
and sonar signals are often treated similarly.

2 SPECTRUM ANALYSIS

2.1 System Model
Spectrum analysis is one of the basic processes used in
signal analysis. The technique is based upon a linear-system
model that assumes all signals consist of a summation of cosine
waveforms of different frequencies. So that time delays can be
incorporated each cosine term has an associated phase shift.
Mathematically this can be written simply as

$$x(t) = \Sigma X_n \cos(\omega_n t + \phi_n) \qquad \ldots \ldots (6)$$

Alternatively, the cosine term can be expanded to give

$$x(t) = \Sigma(X_n \cos\phi_n . \cos\omega_n t - X_n \sin\phi_n . \sin\omega_n t) \ldots (7)$$

$$= \Sigma a_n \cos\omega_n t - \Sigma b_n \sin\omega_n t \qquad \ldots \ldots (8)$$

Complex waveforms, such as noise, can consist of an infinite
number of sines and cosines. The problem in spectrum analysis is
to determine the terms X_n and ϕ_n or the alternative pair a_n and
b_n. We will now discuss how a practical spectrum analyser could
be constructed.

2.2 Spectrum Analysers
The simplest method of realisation would be to feed the
signals through a set of band pass filters each with a narrow
pass band and centred at different frequencies, see Figure 2.1.
This has three disadvantages:

    i) cost
    ii) the required phase term is not available
    iii) such filters may be impossible to design.

The system could be made simpler by mixing the signals with
a local oscillator and selecting one of the sidebands in a single
filter, see Figure 2.2. This has the advantage that only one
filter is required. To check all frequencies the local oscillator
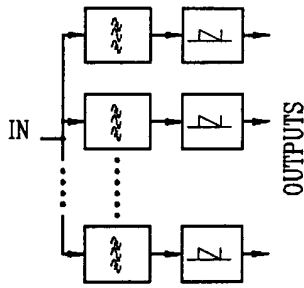
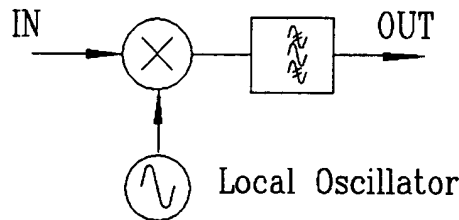<div align="center">1/5</div>

Fig. 2.1

Fig. 2.2

frequency would need to be altered. This is how a radio or television receiver works.

The design of the band pass filter could still be a problem if the centre frequency is too high. A low pass filter could be substituted for the band pass filter if the local oscillator frequency is made equal to the frequency of the signal being measured, see Figure 2.3.
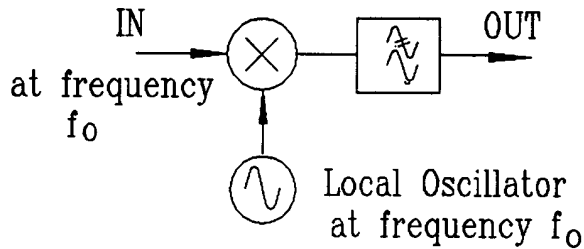

Fig. 2.3

If the local oscillator is say $2.\cos\omega_n t$, and the input signal is $a_n\cos\omega_n t - b_n\sin\omega_n t$, the modulation process produces a waveform

$$a_n + a_n\cos2\omega_n nt - b_n\sin2\omega_n t$$

The low pass filter removes the last two terms (the upper sidebands) and only passes a_n (the lower sideband). The problem with this realisation is that there is no measure of b_n. This may

be overcome by incorporating a second channel where the local oscillator is $2.\sin\omega_n t$, see Figure 2.4. There are now two outputs one from each channel, a_n and b_n.
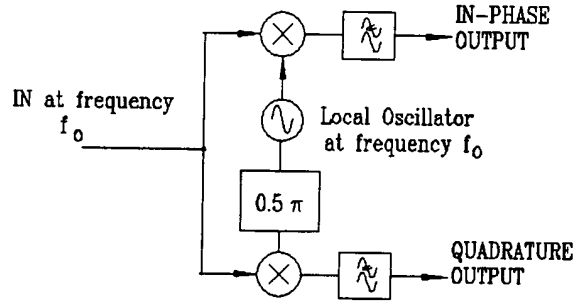

Fig. 2.4

2.3 Mathematical Representatiom

To represent this process mathematically, we need to express the modulation processes as simple multipliers and the low pass filters as integrators. The two channels also need separating by using a multiplier, say α, in front of the sine channel. Hence, for the time waveform $x(t)$, the component, $X(\omega)$, with angular frequency ω can be written

$$X(\omega) = \int_{-\infty}^{\infty} x(t).(\cos\omega t + \alpha.\sin\omega t).dt \qquad \ldots\ldots(9)$$

Note that the factor 2 in the multiplier has not been used. This is because in Eqn 7, there are two outputs, one at an angular frequency $-\omega$ as well as the expected component at $+\omega$.

2.4 The Fourier Integral Equation

The α indicates that two channels are being used and that the resulting two terms are not added directly. The two channels act in quadrature to one and other. Of course, engineers and mathematicians commonly use "j" or "i", the square root of -1 to represent quadrature components. In fact we can write $\alpha = -j$ so that Eqn.9 becomes

$$X(\omega) = \int_{-\infty}^{\infty} x(t).(\cos\omega t - j.\sin\omega t).dt \qquad \ldots\ldots(10)$$

Remembering that by de Moivre's theorem

$$\cos\omega t - j.\sin\omega t = e^{-j\omega t}$$

a more compact form of Eqn.10 becomes

$$X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}.dt \qquad \ldots\ldots(11)$$

This is the Fourier integral equation, one of a pair of equations that form the Fourier transform pair [2]. The second equation forming the pair is

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$$x(t) = (1/2\pi).\int_{-\infty}^{\infty} X(\omega).e^{j\omega t}.d\omega \qquad \ldots\ldots(12)$$

Eqn.11 enables a time waveform to be represented by a collection of sine and cosine waveforms and produces the so-called spectrum. Eqn.12, on the other hand provides the reverse operation where the spectrum can be converted back into the time waveform.

2.5 The Discrete Fourier Transform (DFT) Pair

To calculate the Fourier transform in a digital form requires that the input signal is a sampled set of data. Thus $x(t)$ becomes $x(k.\Delta t)$, where samples are taken at instants of time Δt apart and for brevity we write

$$x(t) = x(k\Delta t) = x_k$$

2.5.1 Band limiting

If we assume that we are dealing with a baseband signal with bandwidth B and sampled at the Nyquist rate, the time samples are taken at instants $\Delta t = 1/2B$. Thus in Eqn. 11 and Eqn. 12 we can replace the time term t in the exponential by k/2B.

2.5.2 Time limiting

Any set of data being processed by a computer must be limited in length. Time signals found in radar and sonar have to be time limited. Suppose that we place a time window around our time waveform so that

$$x(t) = 0 \text{ for } t < -0.5T,$$
$$x(t) = x_k \text{ for } -0.5T < t < +0.5T, \text{ and}$$
$$x(t) = 0 \text{ for } t > 0.5T.$$

The Fourier transform is a linear process so that superposition applies. This means that we can apply one component of signal at a time to Eqn. 11. The result using this signal can be evaluated and then added to similar results obtained by applying other signal components to Eqn. 11. Hence, suppose that we consider that within the time window -0.5T to +0.5T

$$x(t) = A_1\cos\omega_1 t$$

Applying this to Eqn.11 and evaluating with the integral limits +/-0.5T gives the standard result

$$X(\omega) = 0.5A_1T.[\sin\{0.5(\omega-\omega_1)T\}/\{0.5(\omega-\omega_1)T\} +$$
$$\sin\{0.5(\omega+\omega_1)T\}/\{0.5(\omega+\omega_1)T\}] \ldots\ldots(13)$$

Eqn. 13 is sketched in Fig. 2.5. It can be seen that the function $X(\omega)$ has two peaks, one at $\omega = -\omega_1$ and the second at $\omega = +\omega_1$. This gives rise to the concept of negative frequency which we shall deal with in a moment. However, if we consider Eqn. 13 more closely we note that the positve frequency component peaks at $\omega = +\omega_1$ and that this component first passes through zero when the sine function in the sinX/X term equals +/-π. Thus zeros occur at at $\omega = +\omega_1 - (2\pi/T)$ and at $\omega = +\omega_1 + (2\pi/T)$. The size of the frequency resolution cell can be defined as 1/T Hz.

Remembering the 2π multiplier involved between frequency and angular frequency, the resolution cell size is equivalent to placing sinX/X functions along the frequency axis so that the peak of one function coincides with zeros for the other functions, see Fig.2.6.

Returning now to Eqn. 11 and Eqn. 12, the continuous spectrum $X(\omega)$ must be replaced by samples of that spectrum at frequency spacing $2\pi/T$ radian. These samples we shall call Fourier coefficients, A_r. In Eqn. 11 and Eqn. 12 the ω in the exponential terms must also be represented by samples. Hence, we replace ω by $2\pi r/T$.
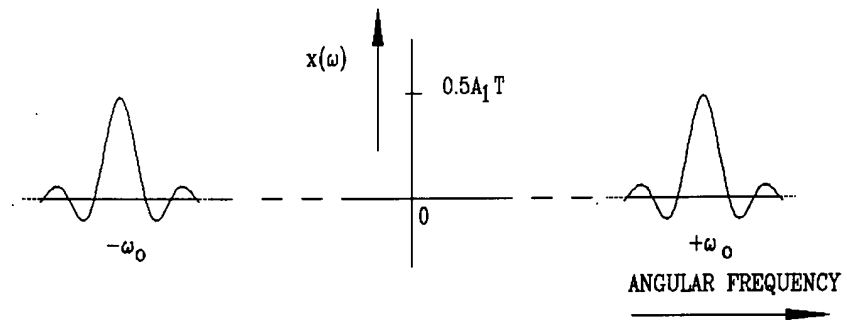
$x(\omega)$

$0.5A_1T$

0

$-\omega_0$

$+\omega_0$

ANGULAR FREQUENCY

Fig. 2.5
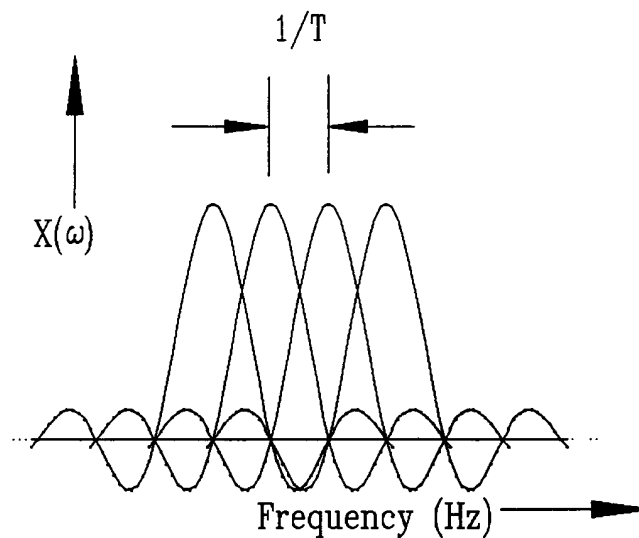
$1/T$

$X(\omega)$

Frequency (Hz)

Fig. 2.6

2.5.3 The number of samples.

It is impossible [2] to limit the frequency bandwidth of a signal while simultaneously time limiting that same signal. A time-limited signal requires an infinite bandwidth and vice versa. However, if the number of samples taken is sufficiently large the resulting errors are sufficiently small for the equations to provide a good approximation. Combining the ideas of Section 2.5.1 and Section 2.5.2, we see that the time waveform of duration T has been sampled at instants separated by 1/2B. Thus the number of time samples is 2BT. If the one-sided bandwidth is B we also note that there is a band of positive signals and a band of negative signals. This means that the total two-sided bandwidth is 2B. Samples of the spectrum are taken 1/T apart so the total number of Fourier coefficients is also 2BT. This indicates that the total information in the system remains unaltered by the transformation.

The integral equations that define the Fourier transform

pairs must be replaced by summations in the discrete format. If we write 2BT = N, the total number of data samples in the time and frequency domains, we obtain equations of the form

$$A_r = 1/N \sum_{k=0}^{N-1} x_k.e^{-2\pi jrk/N} \qquad \ldots\ldots\ldots(14)$$

and

$$x_k = \sum_{r=0}^{N-1} A_r.e^{+2\pi jrk/N} \qquad \ldots\ldots\ldots(15)$$

These two equations represent the discrete Fourier pair [2].

The 1/N multiplier can be found in the equation for x_k rather than in front of the equation for A_r. Alternatively, some authors use $(1/N)^{0.5}$ [3]. The reason for using Eqn. 14 and Eqn. 15 is that Eqn. 14 does at least have a physical significance. For example, with r = 0, A_0 is simply the average value. Thus the zero-frequency term is equivalent to the d.c. value. For other values of r, the time samples, x_k, are multiplied by samples of cosine and sine in orthogonal channels. If a frequency component equivalent to r is present, the summation and divide by N, then averages the lower sideband or base-banded component. Hopefully, the upper sideband components resulting from the multiplication will average to zero.

2.5.4 Relationship to the Fourier series.

If a continuous signal, x(t), is periodic with a period $2\pi/\omega_0$, the signal may be expressed as a complex Fourier series [2]

$$x(t) = \sum_{r=-\infty}^{+\infty} c_r.e^{jr\omega_0 t} \qquad \ldots\ldots\ldots(16)$$

Comparing Eqn. 15 and Eqn. 16, it can be seen that the two equations are almost identical. In Eqn. 15, the time waveform is continuous whereas in Eqn. 14 the time wavefrom is sampled and time is represented by the variable k/2B. The fundamental frequency in Eqn. 16 is $\omega_0/2\pi$. The equivalent fundamental frequency in Eqn.15 is r/T.

It is important to note that the discrete Fourier transform is simply a band-limited discrete version of the complex Fourier series. Such a representation requires that the signal being analysed is a PERIODIC signal with a period T.

In radar and sonar it is very unlikely that the signals being analysed will be periodic and as such the DFT is only an approximation. The non-periodic nature of the signals analysed gives rise to errors in the estimation of the Fourier coefficients. In particular, the Fourier coeficients are spaced along the frequency axis at spacings of 1/T Hz. If a component does not fall at one of these discrete frequencies, r/T, it will cause all the frequency bins (all values of r) to have an output. These outputs will be modulated in amplitude by the sinX/X envelope. Thus if the frequency component is at $(r+\delta)/T$ Hz, where δ is a positve fraction less than unity, bins r and r+1 will have major outputs from the forward DFT operation. Bins r-1 and r+2 will also have significant outputs and these might be only about -13 dB below the expected output.

2.6 Negative Frequencies

The concept of negative frequency is sometimes difficult to

comprehend. It may be explained pictorially by reference to Fig. 2.7. In Fig. 2.7(a), the arrow represents a vector of unit amplitude with a phase ϕ. If a perpendicular is dropped to the x axis the distance along the x axis is cosϕ. A second line drawn to the y axis will be offset along the y axis by an amount sinϕ. If $\phi = \omega t$ the vector becomes a phasor that rotates at an angular frequency ω. In representing cosωt as a phasor, the problem is to remove the sinωt component on the quadrature axis. This is done by having two phasors each of amplitude 0.5 rotating in opposite directions, see Fig. 2.7(b). Along the x axis the phasors add vectorially to give cosωt. Along the y axis the resulting components add to give zero.
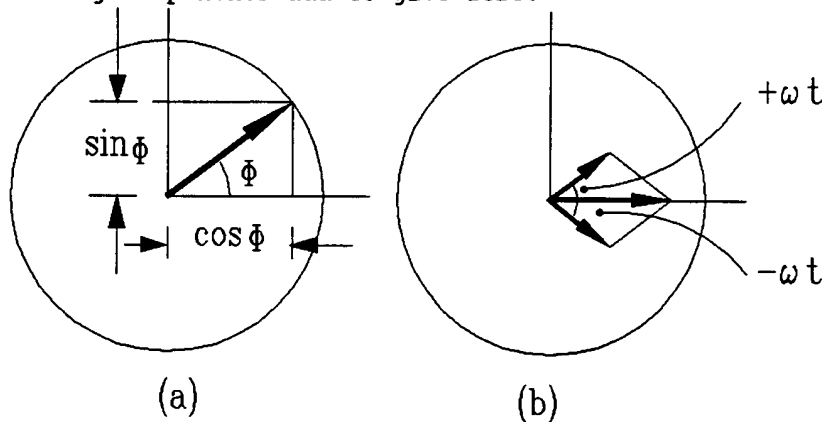

Fig. 2.7

Effectively, Fig. 2.7(b) represents the function
$$0.5[e^{j\omega t} + e^{-j\omega t}].$$
This is of course the expression for cosϕ derived from de Moivre's theorem.

A similar phasor diagram will produce a representation for sinϕ. From this second phasor diagram sinϕ can be seen to be equal to

$$[e^{j\omega t} - e^{-j\omega t}]/2j \qquad\qquad \dots\dots\dots(17).$$

2.7 Doppler Effects

In active systems, relative movement between the the source and receiver on one hand and the target on the other will produce a Doppler shift. The received pulse will be lengthened or shortened relative to the transmitted pulse. This in turn produces a change in the received signal spectrum compared to the spectrum of the transmitted pulse. For a relative radial velocity u (considered to be positive for a closing target), the fractional change in angular frequency is given by
$$\Delta\omega/\omega_0 = 2u/c \qquad\qquad \dots\dots(18)$$
where c is the velocity of energy propagation.

Table 2.1 shows the Doppler shifts expected for a relative radial velocity of 1m/s in typical sonar and radar systems.

Radial velocities in sonar can be from zero to 40 m/s. In radars the radial velocities can be from zero to values in excess of 600 m/s. Thus, the Doppler shifts in radar and sonar are similar in magnitude.

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TABLE 2.1

System	Carrier frequency (Hz)	Doppler shift per 1m/s radial velocity (Hz)
Sonar	3×10^3	4
	30×10^3	40
Radar	1.5×10^9	0.1
	15×10^9	1

It is important to note that differences arise when the
spectrum of the signals are considered. Eqn. 13 can be adapted
for this purpose. Consider a pulse of a carrier signal of angular
frequency ω_0 and of pulse width T. Eqn. 13 is then the spectrum
of this pulse, see Fig. 2.5. However, in sonar and radar systems,
the pulse itself is repeated at a regular pulse repetition
frequency. This pulse repetition frequency has to be small
enoegh to allow the energy to propagate to maximum range and be
reflected from targets at that range. Thus the pulse repetion
period, T_p, for a system operating to a maximum range R is given
by

$$T_p = 2R/c \qquad \qquad \ldots \ldots (19)$$

The periodic pulsing of the transmit signal produces a line
spectrum with a spacing between the lines equal to $1/T_p$. The
spectrum of Fig. 2.8 shows the positive frequency components of
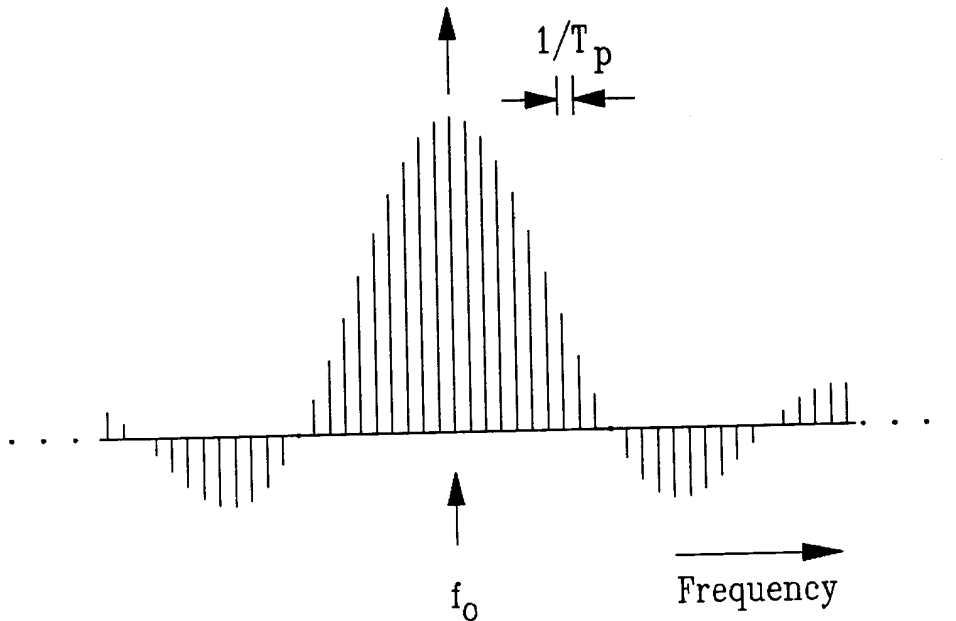the spectrum of a pulse of carrier signal of frequency f_0.

Fig. 2.8

Consider
    i) a sonar operating at a frequency of 3 kHz and R = 15 km
and ii) a radar operating at 1.5 GHz and R = 150 km.
The values of pulse repetition frequency are then 1/20 Hz and 500
kHz respectively. By reference to Fig.2.8 it can be seen that in
sonar the Doppler shift for a modest radial velocity of 1 m/s (4
Hz) exceeds the line spacing in the spectrum (1/20 Hz). In the
case of the radar, the Doppler shift for a mach 2 (600 m/s)
target is only 60 Hz with a line spacing of 500 kHz. To detect
the Doppler in the radar requires detection of the shift in the
spectral line associated wth the carrier. In sonar, it is
necessary to detect changes in the whole sinX/X spectrum.

3 ARRAY SIGNAL PROCESSING

3.1 System Model

A receiving array with a large aperture is directional and
is able to discriminate against noise sources that fall outside
the array's directional pattern. An array with an area A operat-
ing at a wavelength λ has a field of view λ^2/A steradian. Such an
array is also beneficial as far as clutter (or reverberation) is
concerned. Because the array has a restricted field of view, the
number of clutter (reverberation) scatterers is reduced by com-
parison to an omnidirectional array whose field of view is 4π
steradian. Thus, an array is able to improve the signal-to-
interference level simply by virtue of its size.

As was stated in Section 1.1, an array containing M elements
suitably spaced is capable of increasing the information rate of
a system by the factor M. This increase is due to the increase in
data fed into the system in an M-element array by comparison to a
single element. If this increased data is processed correctly a
number of individual beams can be formed each of which is steered
in a different direction. The M elements sample the spatial field
and if the spacing is more than half wavelength, the resulting
beams are said to be independent of each other. This spacing
requirement is equivalent to the Nyquist sampling theorem.

Imagine an array in the form of a straight line with the
elements equally spaced at a distance d apart, see Fig. 3.1. Such
an array is common in modern sonars where to obtain a sufficient-
ly large aperture operating at low frequencies a towed array is
used. In radar also the synthetic aperture that is formed in many
space-bourne systems forms such a line array. A major difference
in the processing needs of radar and sonar arrays results again
from the difference in propagation speeds.

In sonar it is vital to steer beams because of the otherwise
slow data rate. For example, consider a sonar with a maximum
range of 15 km, a sector to interrogate of 360° and resolution of
10°.   It would require 20 s for the sound to travel to maximum
range and back to the receiver. If the array were to be mechanic-
ally scanned, the array could not be moved mechanically during
this time. Thus it would take 36 sequential transmissions of
sound in each of the 36 bearing resolution cells to interrogate
the complete 360° sector. The minimum time taken for a single
look in each bearing cell over all ranges is 720 s or 12 minutes.

A radar, on the other hand, operating to a range of 150 km
only requires 10 μs for the electromagnetic energy to travel from
the transmitter, to maximum range and back to the receiver. Thus,
for a 1° resolution cell in bearing it is theoretically possible
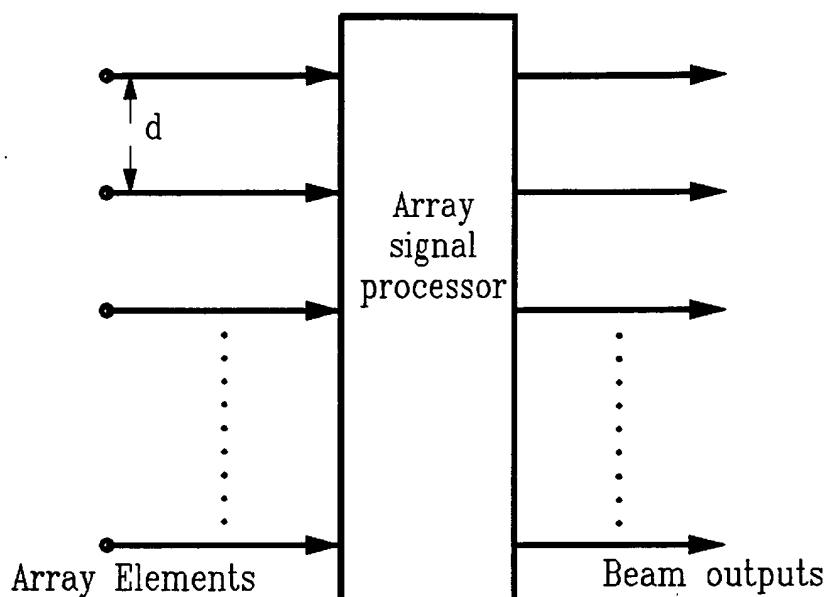to interrogate the complete sector in only 3.6 ms. In practice of

Fig. 3.1

course, the array could not be rotated sufficiently fast and
simple radars dwell in the same bearing resolution cell for many
pings of the transmitter.

Beam steering by electronic means in radar is used to scan
in elevation while allowing mechanical scanning to provide azim-
uthal information. The Plessey 3-D AR radar is such a radar. Some
radars operating at in the h.f. frequency band (where the wave-
lengths are tens or even hundreds of meters) use electronic
methods to steer the beams. When such h.f. radars require a high
angular resolution, it is impossible to rotate the massive arrays
employed. Sonar must rely upon electronic beam steering in all
but very short-range situations.

Suppose that a plain wave approaches the array shown in Fig.
3.1 from a bearing θ. There will be difference between adjacent
elements in path lenth travelled by the plain wave equal to
d.sinθ. This results in

      i) a difference in time delay equal to (d.sinθ)/c
  or ii) a difference in phase of (2π.sinθ)/λ.

The processing necessary for a simple beam steering system is to
place either the time delays or phase shifts in the appropriate
signal paths to make all the signals in phase before they are
added to form a beam output.

Time-delay beam steering methods are wide-band in their
operation. This is essential in most sonar operations. Even
though sonar signals only have bandwidths that are typically say
5 kHz they are base-band signals and the ratio of centre
frequency to bandwidth is very low. Time delays in sonar systems
is readily achieved by using random-access memories that are
addressed so that they behave like serial shift registers. Some
radars still use coaxial lines to produce the differential time
delays. In digital processing schemes, the penalty for using time
delays to steer beams is that the signals need to be sampled at a
minimum of 0.5M times the the Nyquist rate. Interpolation tech-
niques are used to avoid this high sampling rate, but these all

use much addition processing power.

Many phased arrays only produce coherence at a single frequency. Hence phasing techniques are generally only narrow band. Radars operating with a pulsed-carrier signal of a single frequency use phasing techniques to steer beams. One advantage of a phased array is that in theory the signals from each channel need only be sampled at the Nyquist rate. It should be stressed here that the specific case where the phase shift $\phi = -\omega T$ produces a pure time delay T at all frequencies. However, most phasing techniques do not have this property. When such general phasing techniques need to be used with broad-band signals, the signals need to be separated by filtering into a number of narrow bands. Each band is applied to a beam-steering unit where the inter-element phase shifts are made proportional to frequency. The outputs of these individual beam-steering units are then combined to produce the required wide-band output.

3.2 Narrow-band Operations
3.2.1 Physical interpretation.

Suppose that a plain wave approaches an array of elements from the broadside direction. If the elements form a staight line and are equally spaced as in Fig. 3.1, the waveforms produced at the output of each element are as shown in Fig. 3.2. If these waveforms are sampled at the instant indicated, it can be seen that the outputs of the sampling operations produce a signal of equal amplitude in each channel.

Fig. 3.3 shows the effect of sampling the signals received on the same multi-element array when the single-frequency input waveform produces sinusoidal outputs on the array elements that are delayed with respect to each other. It can be seen that if these samples are placed so as to form a data series, the series is a sampled sine wave. The frequency of the sine wave increases as the bearing of the signal increases. As already shown in Fig. 3.2 a broadside target (bearing = 0) produces samples of a zero-frequency signal. Thus, one way of estimating the bearing of the target is via spectrum analysis of the sampled data series.

A problem does arise because the simple process illustrated in Fig. 3.2 and Fig. 3.3 produces the same frequency component in the sampled data series for targets arising in the port and starboard sectors. This left-right ambiguity is overcome by sampling the input signals in quadrature. The negative-frequency components represent beams formed in one half sector. Those from the other half sector are represented by the positive-frequencies.

3.2.2 Mathematical representation

The range of inter-element phase shifts that can be inserted in the signal paths before addition to form the individual beams is limited to 2π rad. If this value of 2π rad is exceeded, the phases repeat themselves because of the periodicity of the sine and cosine waveforms involved. With M elements sampling the spatial field, no more than M independent beams can be expected. If these beams are formed by inserting equal increments of inter-element phase shifts, the rth beam will be formed by using an inter-element phase shift of $2\pi r/M$ rad. The phase shifts used to form the rth beam produce a linear phase taper across the array. It is sensible to label the array elements starting at one end 0, 1, 2,........,k , ...,(M-1) and the phase shift applied to the kth element is $2\pi rk/M$.
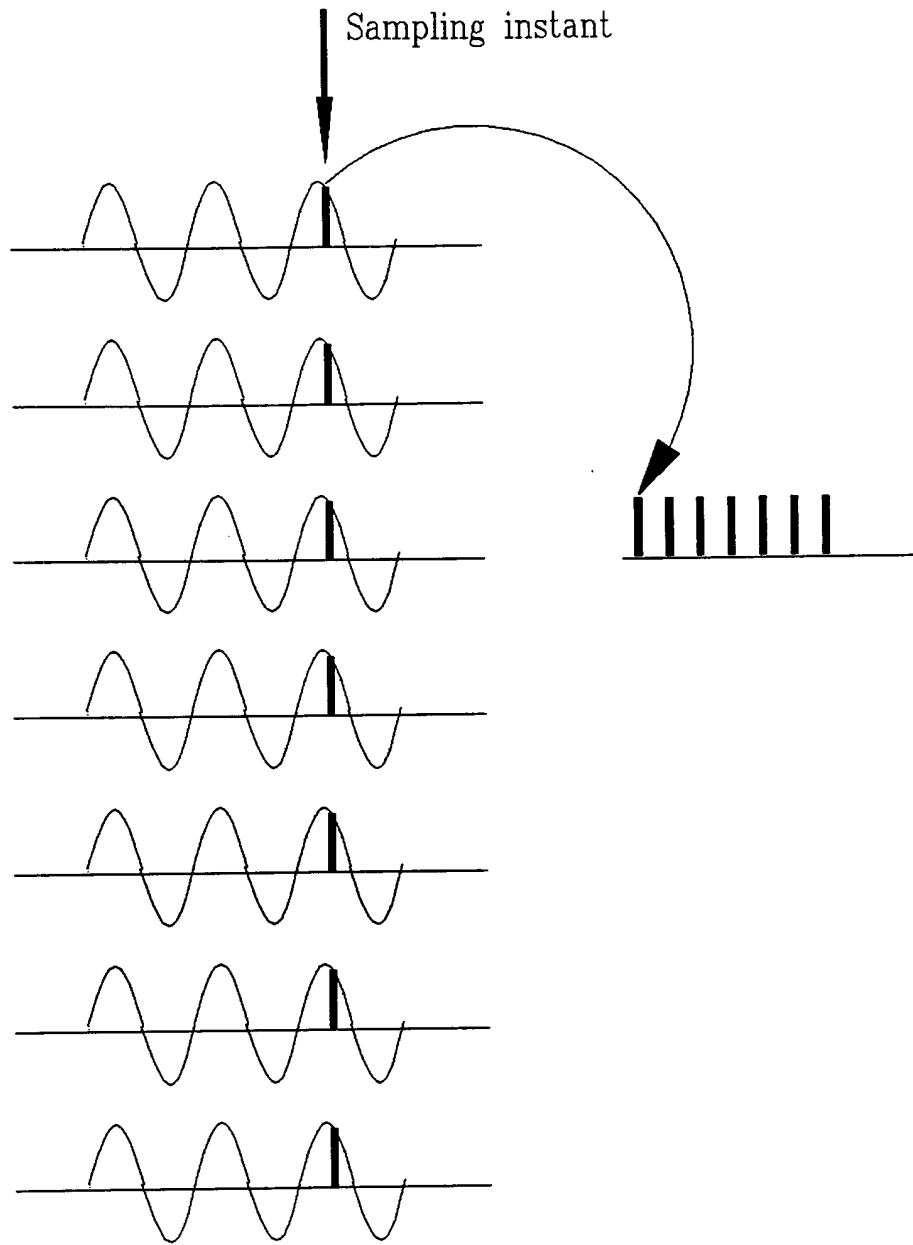
1/15
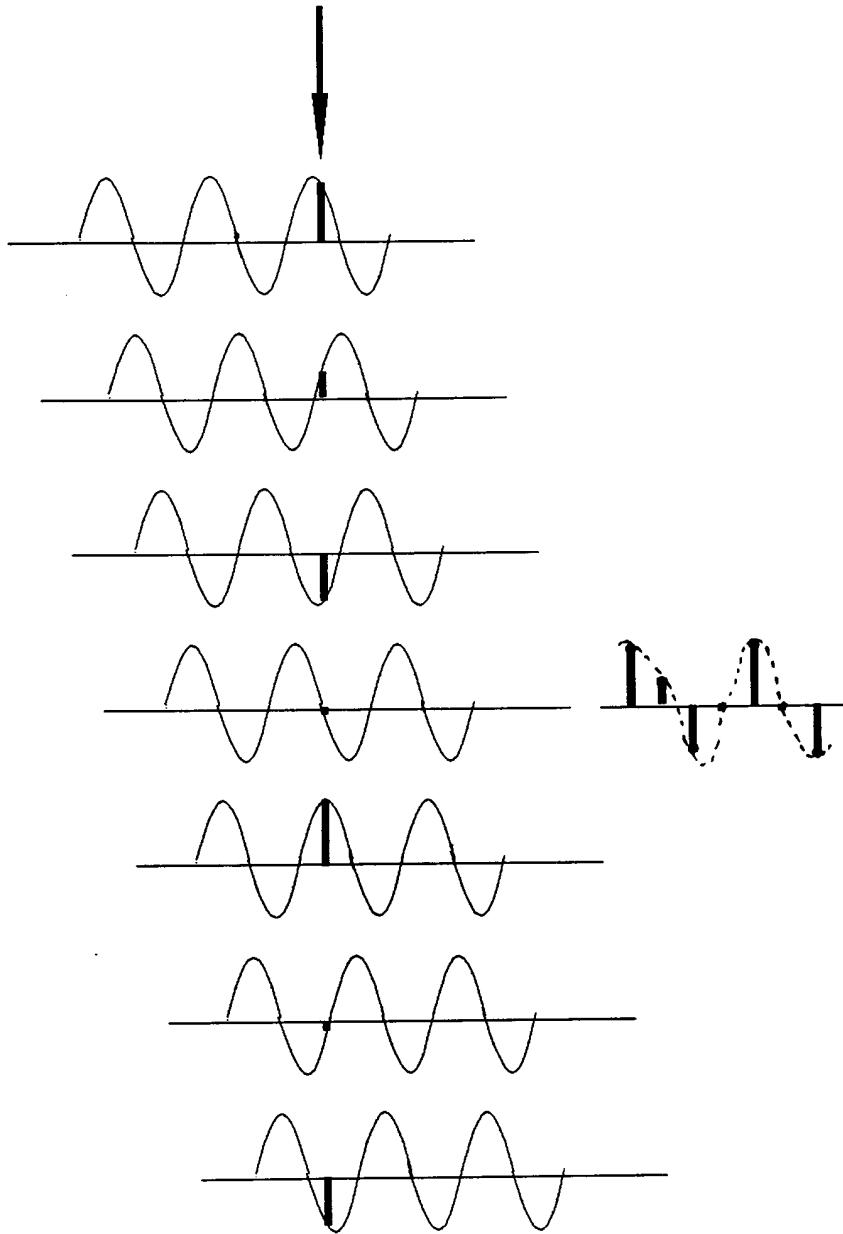
Sampling instant

Fig.3.2

Fig. 3.3

The rth beam is then formed by summing the phase-shifted signals. If a target exists in the direction θ given by the equation $(2\pi d.\sin\theta)/\lambda = 2\pi rk/M$, the phase-shifted signals will add coherently to produce a maximum response. This phase shifting and subsequent addition may be represented by

$$B_r = \sum_{k=0}^{M-1} x_k . \angle 2\pi rk/M \qquad \ldots\ldots(20)$$

Note that x_k is a complex number. This was discussed in Section 3.2.1. Hence, the spatial samples must be formed by quadrature sampling. A more elegant way of representing the phase shifts is to use multiplications by cosine and sine in quadrature channels. This can be condensed into a single complex exponential form. Hence:

$$B_r = \sum_{k=0}^{M-1} x_k . e^{-2\pi jrk/M} \qquad \ldots\ldots(21)$$

Of course, this is the same expression as Eqn. 14, the discrete Fourier transform. Thus, the process of beam steering in narrow-band systems is to form discrete Fourier coefficients via the discrete Fourier transform. For this reason, the beams are often called spatial frequencies. Algorithms, such as the fast Fourier transform, devised to perform the discrete Fourier transform efficiently and quickly, are equally applicable to both sampled time series and to spatial samples.

4 Correlation Processing

4.1 Averaging
One method of improving the signal-to-interference ratio is to average signals. This assumes that the interference is incoherent and that the averaging will only increase the power associated with the interference component in a mean-square manner. The signal on the other hand is assumed to be coherent from sample to sample. The averaging process will then combine the signal components together coherently so that the power is proportional to the square of the sum of the magnitudes. Ideally, this results in an increase in signal-to-interference ratio proportional to the number of samples averaged.
Correlation processing comes in this general area of time averaging. Sonar and radar signals vary with time. For example in narrow-band systems the signals can be expected to be sinusoidal. Time averaging is only advantageous if the signal being averaged is constant from sample to sample. To achieve this the signals received in sonar and radar systems must first be modified. The method employed is to create a model or replica of the expected signal. This is then multiplied by the received signal and the averaging process then removes all high frequency components leaving an average value of the product with frequencies at or very near to zero frequency. This averaged product is the cross-correlation between the two waveforms at a particular value of time delay.
Consider an active signal where the transmitted signal is $y(t)$. Let the received signal be $x(t)$. This will be delayed due to the time of flight, T, to and from the target. The recieved signal is assumed to have the same form as that transmitted. If a Doppler shift is expected, $y(t)$ must be modified to account for

the Doppler shift. Hence, the model for the process can be the transmitted signal or a Doppler-shifted version of this signal. To account for the time delay in the received signal, $y(t)$ must be delayed by T before being multiplied by $x(t)$ and the product averaged. Hence, we can write the cross correlation coefficient

$$c(T) = \int_{-\infty}^{\infty} x(t).y(t+T).dt \qquad \ldots\ldots(22)$$

This represents the correlation between $x(t)$ and $y(t)$. Strictly Eqn. 22 should be divided by a normalising factor to represent a true average but this factor is often ignored. In discrete form with N samples present and a time delay of d units the discrete correlation coefficient becomes

$$c_d = \sum_{k=0}^{N-1} x_k.y_{k+d} \qquad \ldots\ldots(23)$$

Again Eqn. 23 should be divided by N for it to be a true average. This factor is usually neglected.

Radar signals are often processed using a surface acoustic waveform correlator. This correlator uses a replica in sampled-data form and a continuous version of the received signal. Similarly, sonar systems have used correlators made from charge-coupled devices. Here both the signal and the replica are sampled data signals but they are not coded in a binary fashion. Neither is then of the form normally recognised as digital signal processors. However, both sonar and radar regularly use equations such as Eqn. 23 to carry out cross correlation using true digtal signal-processing techniques. Signals with time-bandwith products of the order of 5×10^3 are quite easily processed.

4.2 Relationship with the Fourier Transform and Beam Steering

Comparison of Eqn. 11 with Eqn. 22 and Eqn. 14 with Eqn. 23 reveals that the Fourier and discrete Fourier transforms are in fact correlation processes. The delay terms T and d respectively in Eqn. 11 and Eqn. 14 are zero. Thus, these Fourier transforms are the cross correlations of the signal with

(i) a cosine waveform of zero delay in the in-phase channel and (ii) a sine waveform of zero delay in the quadrature channel. The ratio of the signals in the two channels provides for the delay. It is of interest to note that the chirp-z transform [4] is an algorithm based upon a sampled data correlator to evaluate the discrete Fourier transform.

Using the same arguements, beam steering is also a correlation process. Eqn. 21 can be compared with Eqn. 23. It is seen that the model assumed for the beam-steering system is that the spatial samples will consist of a number of equally-spaced, discrete spatial frequencies. The system model is then cross correlated with the spatial samples to produce the beams, B_r.

4.3 Correlation Processing in the Frequency Domain

It is often advantageous to evaluate correlation coefficients by working with the spectra of the signals. In particular, by using efficient algorithms such as the fast Fourier transform, the correlation process may be evaluated very quickly and with a much reduced computational load. Consider the Fourier transform of Eqn. 22

$$C(\omega) = \int_{-\infty}^{\infty} c(T).\exp(-j\omega T).dT = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t).y(t+T).dt.\exp(-j\omega T).dT$$

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Write (t+T) = s so that ds = dT and separate the variables so that

$$C(\omega) = \int_{-\infty}^{\infty} x(t).\exp(-j\omega t).dt \; . \int_{-\infty}^{\infty} y(s).\exp(+j\omega s).ds$$

The first integral is the Fourier transform $X(\omega)$ of $x(t)$. The second integral is very similar except that the complex exponential multiplier has a positive sign in the exponent. This simply changes the sign of the imaginary part of the Fourier transform. Hence the second integral is the complex conjugate of the Fourier transform. This is written as $Y^*(\omega)$. Thus, we have the simple relationship

$$C(\omega) = X(\omega). \; Y^*(\omega) \qquad\qquad \ldots\ldots(24)$$

This equation is known as the Weiner-Khinchine algorithm and it is used regularly in radar and sonar signal processing to reduce the computational load. The advantage of using the algorithm is often missed. In the discrete form, with both data sequences of length N, N separate correlation coefficients are evaluated. Care must be exercised when using the discrete form of Eqn.(24),

$$C_r = X_r.Y_r^*, \qquad\qquad \ldots\ldots(25)$$

since the discrete Fourier transform assumes the data is periodic. Most signals in radar and sonar are aperiodic. To overcome this lack of periodicty the data in one of the data sequences can have zeros added (zero padding). If this is not done only the coefficient with zero time lag is meaningful. When increasing numbers of zeros are added to one of the sequences an increasing number of coefficients are meaningful. It is usual to take two data sequences with N data points each. One consists of 0.5N non-zero data samples and 0.5N zeros. The second sequence contains N non-zero data points. The application of Eqn. 25 then produces 0.5N valid coefficients. The remaining 0.5 invalid coefficients are discarded. The time windows are made to overlap and in this way all the required coefficients are evaluated.

5 Concluding Remarks

This paper has looked at the differences and similarities between the reqirements and execution of digital signal processing in radar and sonar systems. The major differences are caused by the 2×10^5 ratio in the velocities of propagation of electromagnetics and sound. This results in differences in the bandwidths of the signals employed, the durations of the time windows that can be processed, differences in the ways that Doppler shifts are processed and the urgent need in sonar for the use of beam steering using electronic methods.

In spite of the vast difference in the velocities of propagation, surprisingly the wavelengths involved in radar and sonar overlap. At h.f. radars also need to employ electronic methods to steer beams. At higher frequencies it is becoming fashionable to process the signals received by the radar array to avoid the necessity of mechanical rotation. The time-bandwidth products involved in radar and sonar are also very similar.

The paper has also reviewed the basic methods of processing in the time, frequency and spatial domains. Spectrum analysis is often the key to the algoritms used. Where it is relevant a physical explaination of the methods described have been given together with a brief interpretation of the mathematics involved.

The paper has been purposely restricted to a description of techniques based on Fourier methods. More modern methods of analysis [5] using alternative models follow in subsequent papers. Similarly, details of the methods employed will also be covered in other papers.

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THE INSTITUTION
OF ELECTRICAL
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LECTURE 1

SIMILARITIES AND DIFFERENCES IN SIGNAL PROCESSING
FOR RADAR AND SONAR

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INTRODUCTION TO MATRIX TECHNIQUES

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1.   INTRODUCTION

The Digital Signal Processing algorithms employed in radar and sonar applications invariably involve the manipulation of time series samples taken from a multiplicity of sensor elements. Matrix algebra provides a powerful symbology to represent such samples and to analyze their manipulation in a concise manner. Once candidate algorithms have been expressed in this algebra there are a wealth of techniques and theorems available to assist with the prediction of performance, and thereby optimise the design of the finally implemented signal processing structure.

In what follows we review the use of matrix algebra to represent sampled signals and to analyze Digital Signal processing algorithms. We also present some of the key theorems of matrix algebra that find applications in such analysis.

2.   THE VECTOR REPRESENTATION OF SAMPLED SIGNALS

Consider an arbitrary band-limited waveform, $x(t)$, originating from a single sensor element. If v_{max} denotes the maximum frequency component in $x(t)$, it follows from the Nyquist sampling theorem that the waveform can be re-constructed for all values of t from samples taken at times $t_k = k\Delta t$, with integral k and $\Delta t \leq 1/2v_{max}$. If the bandwidth of the waveform, B, is less than v_{max} one may seek to reduce this sampling interval by a suitable down-conversion process. In the limit one can in principle down convert such that the waveform spans the baseband range $0 \leq v \leq B$. In this case the Nyquist sampling interval is reduced to $\Delta t = 1/2B$.

A series of p such samples with $m-p+1 \leq k \leq m$ may be represented by a p component column vector, $\underline{x}$ given by

$$\underline{x} = (x_{m-p+1}, x_{m-p+2} \ldots x_m)^T$$

where x_k is the sample obtained at time $k\Delta t$. The notation ^{T} denotes the transpose of a matrix. For an arbitrary matrix this operation involves the interchange of rows and columns. It is used here because it is more compact to represent vectors in row form.

An alternative to using real samples is to represent the waveform in complex notation. The waveform $x(t)$ can be viewed as being composed of sinusoidal components and expressed in the form

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$$x(t) - \int_0^\infty A(w)\cos[wt+\phi(w)]dv.$$

We may then define the complex representation of the waveform by

$$\tilde{x}(t) - \int_0^\infty A(w).\exp\{ j \ [wt+\phi(w)] \}dv.$$

We then have that

$$x(t) - R_e \{ \tilde{x}(t) \}.$$

We also have that

$$I_m \{ \tilde{x}(t) \} - \int_0^\infty A(w) \sin [wt+\phi(w)]dv.$$
$$- \int_0^\infty A(w) \cos [wt+\phi(w)-\pi/_2]dv.$$

Samples of the imaginary component of $\tilde{x}(t)$ can therefore in principle be obtained from $x(t)$ by passing the latter waveform through a filter with a flat unit amplitude response and a frequency-independent phase shift of $-\pi/_2$. Alternatively samples of the imaginary component can be directly computed from samples of $x(t)$ by means of the Discrete Hilbert Transform which performs this filtering operation digitally.

We may therefore alternatively represent the series of p samples with $m-p+1 \le k \le m$ by a complex column vector given by

$$\tilde{x} - (\tilde{x}_{m-p+1}, \tilde{x}_{m-p+2} \cdots \tilde{x}_m)^T.$$

A third representation that is commonly employed involves the use of complex baseband notation. The complex baseband representation of the waveform $x(t)$ may be defined by

$$\hat{x}(t) = \tilde{x}(t).\exp\{-j\ w_o t\}.$$

Here w_o is the angular frequency at the centre of the band of interest.

$\tilde{x}(t)$ may be expressed in the form

$$\tilde{x}(t) = \int_{-\infty}^{+\infty} \hat{A}(w)\ \exp\{jwt\}dv,$$

where $\hat{A}(w)$ is a complex function of w spanning both positive and negative frequencies.

The real and imaginary components of $\tilde{x}(t)$ have spectra which span the band $0 \le \nu \le B/2$. The Nyquist sampling interval appropriate to each is therefore $\Delta t = 1/B$. This is twice the interval required for real baseband samples, but each complex baseband sample has both a real and imaginary component.

The real and imaginary components of $\tilde{x}(t)$ can be obtained by mixing x(t) with orthogonal local oscillator signals of the form $\cos[w_o t]$ and $-\sin[w_o t]$ and subjecting the resulting waveforms to low pass filters which span the frequency range $0 \le \nu \le B/2$. This mixing and filtering may be undertaken in the analogue domain, in which case appropriate samples need to be taken at the outputs from the low pass filters, or in the digital domain in which case the input to the process will be the sequence x_k of real sample values of x(t).

We have thus far developed three possible vector representations of the last p samples obtained from a single element sensor. If we have n such sensor elements we may similarly represent the set of samples obtained at time $t = m\Delta t$ across the array by an n component column vector of the form

$$\underline{x} = (x_{1,m},\ x_{2,m}\ ....\ x_{n,m})^T.$$

Here $x_{i,m}$ is the sampled output from the ith sensor at time $t = m\Delta t$. Such x_{im} can either be real samples, or samples in one of the two forms of complex representation we have considered. It is now a simple extrapolation to represent the last p samples from all the n sensors by an np component column vector.

3.   DIGITAL FILTERING IN VECTOR NOTATION

Now that we have a column vector representation of signals sampled both in time and, via an array of sensor elements, in space, we may consider the development of sampled frequency and spatial filters in vector notation.

Consider the sampled output from the ith sensor. We may apply a sampled finite impulse response filter to the output from this sensor by forming a weighted combination of the last p samples. We then have for the output, y, appropriate to time $t = m\Delta t$

$$y = \sum_{j=o}^{j-p-1} w_{m-j} \, x_{i,m-j}$$

Here the w_j are the components of the impulse response of the filter. We may represent this impulse response by a p component column vector, $\underline{w}$, given by

$$\underline{w} = (w_{m-p+1}, \, w_{m-p+2}, \dots \, w_m)^T.$$

We then have in vector notation

$$y = \underline{w}^T \underline{x}.$$

where

$$\underline{x} = (x_{i,m-p+1}, \, x_{i,m-p+2}, \dots \, x_{i,m})^T.$$

If we are working in one of the two complex notations, the components of $\underline{w}$ will be complex, representing amplitude scaling factors and phase shifts applied to the complex representations of the sensor samples. Conversely, if we are working in real notation the components of $\underline{w}$ will be real. The vector notation for the filter output is equally valid in all three cases. From now on we will develop the matrix algebra for the more general complex case. The processing of real samples can then be treated as an appropriate special case.

The finite impulse response filter we have applied to the output from the ith sensor implements a filtering operation in the frequency domain. If we replace $\underline{x}$ by the vector of samples obtained across the array at time $t = m\Delta t$ we may similarly implement a spatial filter. Such a filter corresponds to a far field reception pattern for the sensor array which varies with bearing. In general, with a column vector, $\underline{x}$, containing both space and time samples we may implement a combination of frequency and spatial filtering. In all cases we may employ the simple inner vector product

$$y = \underline{w}^T \underline{x}$$

to represent the filtering operation.

4. FREQUENCY AND SPATIAL FILTERING

When the components of the column vector $\underline{x}$ represent the last p samples taken from a sensor element, the transformation $y = \underline{w}^T\underline{x}$ corresponds to that of implementing a discrete time finite impulse response filter in the frequency domain. If the samples are in complex baseband notation, a particularly important form for $\underline{w}$ is one of the set $\underline{w}_q$, with $o \leq q \leq p-1$, given by

$$\underline{w}_q - \frac{1}{\sqrt{p}} \; (1, \; \exp\{-j\frac{2\pi}{p} \; q\}, \; \exp\{-j\frac{2\pi}{p}.2q\},....\exp\{-j\frac{2\pi}{p} \; (p-1)q\})^T.$$

Such an impulse response seeks to combine coherently the time series samples of any component of the sensor waveform that has a frequency given by

$$\upsilon - \frac{q}{p\Delta t} + \frac{k}{\Delta t}$$

where Δt is the sampling interval, and k is an appropriate integer selected such that

$$-\frac{1}{2\Delta t} \leq \upsilon \leq +\frac{1}{2\Delta t}.$$

For a more general component of the form $S_o \exp\{jwt\}$, the output from the filter at time $t = k\Delta t$ follows as

$$y_q - \frac{S_o}{\sqrt{p}} \cdot \frac{\sin \left[p\pi(q/p - \upsilon\Delta t)\right]}{\sin \left[\pi(q/p - \upsilon\Delta t)\right]}.\exp\{jwk\Delta t\}.$$

The complete set of $\underline{w}_q$ thus corresponds to a set of filters with centre frequencies separated by $\Delta\upsilon = 1/p\Delta t$ which span the band

$$-\frac{1}{2\Delta t} \leq \upsilon \leq +\frac{1}{2\Delta t}.$$

If we define

$$F - (\underline{w}_o, \; \underline{w}_1,... \; \underline{w}_{p-1})^T$$

and

$$\underline{y} - (y_0, \; y_2,.... \; y_{p-1})^T,$$

we may write

$$y - F x.$$

The column vector y then has components equal to the outputs from each of the filters in the set. This equation expresses, in matrix notation, the operation of taking the Discrete Fourier Transform of the samples that are the components of x. It is straightforward to show that F is a full rank transformation. It therefore has an inverse, F^{-1}, defined by $F^{-1} F = I$ where I denotes the identity matrix. We may therefore write

$$x - F^{-1} y.$$

It follows from Fourier Transform theory that if we set

$$F^{-1} - (\tilde{w}_0, \tilde{w}_1, \dots \tilde{w}_{p-1})^T,$$

then

$$\tilde{w}_l - \frac{1}{\sqrt{p}} \left(1, \exp\left\{+j\frac{2\pi}{p} l\right\}, \exp\left\{+j\frac{2\pi}{p} 2l\right\}, \dots \exp\left\{+j\frac{2\pi}{p}(p-1)l\right\}\right)^T.$$

That is

$$F^{-1} - F^*.$$

If the components of x correspond to a single time snapshot of samples from an array of sensors, the transformation $y = w^T x$ corresponds to that of implementing a spatial beamforming operation. The spatial domain equivalent to a single frequency component is a plane wave incident on the array from a specific bearing. Let such a wave have an angular frequency w. The output from the ith sensor may then be expressed in the form $\alpha_i \exp\{jw[t-\tau_i]\}$. If the receiver chain associated with each sensor produces noise of equal power, but uncorrelated from sensor to sensor, it follows from matched filter theory that the optimum weight vector to combine these outputs is given by

$$w - \eta \left(\alpha_1 \exp\{j w \tau_1\}, \alpha_2 \exp\{j w \tau_2\}, \dots \alpha_n \exp\{j w \tau_n\}\right)^T,$$

where η is an arbitrary scaling factor. For the case of a linear array of equally spaced isotropic elements the Discrete Fourier Transform forms a useful set of receiving beams spanning all space.

5.   THE REPRESENTATION OF POWER THROUGH COVARIANCE MATRICES

The instantaneous value of the magnitude squared of the voltage at the output of an arbitrary filter follows as

$$y^* \, y = (\underline{w}^T \, \underline{x})^* \, (\underline{w}^T \, \underline{x}).$$

Here the notation * denotes the operation of complex conjugate.

Since $\underline{w}^T \underline{x}$ is, in general, a complex scalar, it follows that

$$\underline{w}^T \, \underline{x} = (\underline{w}^T \, \underline{x})^T.$$

Now, for arbitrary matrices $\underline{A}$ and $\underline{B}$ it may readily be shown that

$$(\underline{A} \, \underline{B})^T = \underline{B}^T \, \underline{A}^T.$$

Thus

$$\underline{w}^T \, \underline{x} = (\underline{w}^T \, \underline{x})^T = \underline{x}^T \, \underline{w}.$$

.

We thus have that

$$y^* y = (\underline{w}^T \, \underline{x})^* \, \underline{x}^T \, \underline{w} = \underline{w}^H \, \underline{x}^* \, \underline{x}^T \, \underline{w}.$$

Here the notation ^{H} denotes the operation of taking the complex conjugate of the transpose of a matrix.

The expectation of the magnitude squared of the output voltage therefore follows as

$$P = E \, \{ \, y^* \, y \, \} = \underline{w}^H \, E \, \{ \, \underline{x}^* \, \underline{x}^T \, \} \, \underline{w}.$$

Here E{...} denotes the expectation operator, and it has been assumed that $\underline{w}$ is time invariant. We may define the covariance matrix, $\underline{R}$, characterising the components of $\underline{x}$ by

$$\underline{R} = E \, \{ \, \underline{x}^* \, \underline{x}^T \, \}.$$

We then have that

$$\underline{R}^H = E \, \{ \, (\underline{x}^* \, \underline{x}^T)^H \, \} = E \, \{ \, (\underline{x} \, \underline{x}^H)^T \, \} = E \, \{ \, \underline{x}^* \, \underline{x}^T \, \} = \underline{R}.$$

Such a matrix is said to be Hermitian, or in the case of real variables orthogonal.

In terms of the co-variance matrix we thus have

$$P = \underline{w}^H R \underline{w}.$$

P is termed the filter output power. In the case of real variables this is an exact definition if the sampled voltages are referenced to a transmission line of unit characteristic impedance. In the case of complex notation, P is strictly twice the power, but this factor of two is conventionally ignored.

The diagonal components of the co-variance matrix $\underline{R}$ represent the powers in the individual components of $\underline{x}$. The off-diagonal elements represent correlations between distinct components of $\underline{x}$.

6. CONSTRAINED MINIMISATION IN MATRIX ALGEBRA

A powerful tool much used in the solution of constrained minimisation problems with real functions of several variables is that of the method of Lagrange Multipliers. If, for example, we wish to compute the values of x and y that minimise $\phi = f(x,y)$ subject to the real constraint y=y(x), rather than eliminating one of the variables in f(x,y) with the aid of the constraint equation, we can instead consider the function

$$\hat{\phi} = f(x,y) + \lambda [y - y(x)],$$

where λ is a Lagrange multiplier. A necessary condition for the constrained minimisation of ϕ is that

$$\frac{\partial \hat{\phi}}{\partial x} = \frac{\partial \hat{\phi}}{\partial y} = 0.$$

These two equations, together with that defining the constraint, can be used to solve for the three unknowns x,y and λ. The procedure can be extended to deal with multiple constraints by adding additional Lagrange Multipliers. Each additional constraint introduces an additional unknown Lagrange Multiplier and an additional constraint equation such that the number of equations remains equal to the number of unknowns.

Quantities to be minimised are necessarily real, but constraints are in general complex. Suppose for example we wish to find the weight vector, $\underline{w}$, that minimises the output power $\underline{w}^H R\underline{w}$, from a filter, subject to the constraint that a certain weighted sum of elements of the weight vector is equal to a given, in general complex, number. Such a constraint may be written in the form

$$\sum_i c_i \, w_i - \beta.$$

If we define

$$\underline{c} - (c_1, c_2, ...)^T$$

the constraint equation can be expressed in matrix form as

$$\underline{c}^T \, \underline{w} - \beta.$$

This complex constraint can be viewed as two real constraints, given by

$$R_e \, \{\underline{c}^T \, \underline{w}\} - R_e \{\beta\}$$

and

$$I_m \, \{\underline{c}^T \, \underline{w}\} - I_m \, \{\beta\}.$$

We are therefore led to consider the function

$$\hat{\sigma} - \underline{w}^H \, R \, \underline{w} + \tilde{\lambda}_1 \, R_e \{\underline{c}^T \, \underline{w} - \beta\} + \tilde{\lambda}_2 \, I_m \, \{\underline{c}^T \, \underline{w} - \beta\},$$

where $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are two real undetermined Lagrange Multipliers. This augmented equation can be re-expressed in the form

$$\hat{\sigma} - \underline{w}^H \, R \, \underline{w} + \frac{\tilde{\lambda}_1}{2} \, [\underline{c}^T \, \underline{w} - \beta + \underline{c}^H \, \underline{w}^* - \beta^*]$$

$$-j \frac{\tilde{\lambda}_2}{2} [\underline{c}^T \, \underline{w} - \beta - \underline{c}^H \, \underline{w}^* + \beta^*]$$

$$- \underline{w}^H \, R \, \underline{w} + [\frac{\tilde{\lambda}_1}{2} - j \, \frac{\tilde{\lambda}_2}{2}] \, [\, \underline{c}^T \, \underline{w} - \beta]$$

$$+ [\frac{\tilde{\lambda}_1}{2} + j \, \frac{\tilde{\lambda}_2}{2}] \, [\underline{c}^H \, \underline{w}^* - \beta^*]$$

Hence

$$\hat{\partial} = \underline{w}^H R \underline{w} + \lambda[\underline{c}^T \underline{w} - \beta] + \lambda^*[\underline{c}^H \underline{w}^* - \beta^*].$$

Here

$$\lambda = \frac{\tilde{\lambda}_1}{2} - j \frac{\tilde{\lambda}_2}{2}$$

is a single complex Lagrange Multiplier.

In order to proceed to obtain the solution to the constrained power minimisation problem we must now evaluate the complex vector $\underline{w}$ that minimises ϕ.

We may write

$$\hat{\partial} = \hat{\partial}(\underline{w}, \underline{w}^*)$$

where

$$\underline{w} = \underline{w}_I + j \underline{w}_Q$$

and

$$\underline{w}^* = \underline{w}_I - j \underline{w}_Q$$

Thus

$$\hat{\partial} = \hat{\partial} \ (\underline{w}_I + j \underline{w}_Q, \ \underline{w}_I - j \underline{w}_Q)$$
$$= \eta \ (\underline{w}_I, \underline{w}_Q)$$

Now

$$\delta \hat{\partial} = \sum_i \left[\frac{\partial \eta}{\partial w_{Ii}} \delta w_{Ii} + \frac{\partial \eta}{\partial w_{Qi}} \delta w_{Qi} \right]$$

where w_{Ii} is the ith component of $\underline{w}_I$, and $\underline{w}_{Qi}$ the ith component of $\underline{w}_Q$.

We may now define a complex vector gradient, $\tilde{\nabla} \phi$, by

$$\tilde{\nabla} \hat{\partial} - \left(\frac{\partial \eta}{\partial w_{II}} +j \frac{\partial \eta}{\partial w_{QI}}, \frac{\partial \eta}{\partial w_{I2}} +j \frac{\partial \eta}{\partial w_{Q2}}, \cdots \right)^T ,$$

which is a generalisation of the three dimensional real gradient operator denoted by grad or $\underline{\nabla}$. For a small change $\delta\underline{w} = \delta\underline{w}_I +j\, \delta\underline{w}_Q$ in the weight vector $\underline{w}$ the corresponding changes in ϕ follows as

$$\delta\hat{\partial} - R_e \{(\tilde{\nabla} \hat{\partial})^T\}\, \delta\, \underline{w}_I + I_m \{(\tilde{\nabla} \hat{\partial})^T\}\, \delta\, \underline{w}_Q$$
$$- \frac{1}{2} [(\tilde{\nabla}\hat{\partial})^T + (\tilde{\nabla}\hat{\partial})^H]\, \delta\, \underline{w}_I - \frac{1}{2}\, j\, [(\tilde{\nabla} \hat{\partial})^T - (\tilde{\nabla} \hat{\partial})^H]\, \delta\, \underline{w}_Q$$
$$- \frac{1}{2} \{(\tilde{\nabla} \hat{\partial})^T [\delta\, \underline{w}_I - j\, \delta\, \underline{w}_Q] + (\tilde{\nabla}\hat{\partial})^H [\delta\underline{w}_I + j\, \delta\, \underline{w}_Q]\}$$
$$- R_e \{(\tilde{\nabla}\hat{\partial})^T\, \delta\, \underline{w}^*\} - R_e \{\delta\, \underline{w}^H.\tilde{\nabla}\, \hat{\partial}\}$$

It therefore follows that, if $\tilde{\nabla} \phi = \underline{o},\ \delta\, \phi = o$ for all $\delta\underline{w}$, the condition required to find the desired minimum of ϕ.

Since ϕ is expressed as a function of $\underline{w}$ and $\underline{w}^*$, rather than $\underline{w}_I$ and $\underline{w}_Q$, it is convenient to re-express $\tilde{\nabla} \phi$ in terms of derivatives with respect to the components of $\underline{w}$ and $\underline{w}^*$.

Now

$$\frac{\partial \eta}{\partial w_{Ii}} - \frac{\partial \hat{\partial}}{\partial w_i} \frac{\partial w_i}{\partial w_{Ii}} + \frac{\partial \hat{\partial}}{\partial w_i^*} \frac{\partial w_i^*}{\partial w_{Ii}}$$
$$- \frac{\partial \hat{\partial}}{\partial w_i} + \frac{\partial \hat{\partial}}{\partial w_i^*}.$$

We also have that

$$\frac{\partial \eta}{\partial w_{Qi}} - \frac{\partial \hat{\partial}}{\partial w_i} \cdot \frac{\partial w_i}{\partial w_{Qi}} + \frac{\partial \oslash}{\partial w_i^*} \cdot \frac{\partial w_i^*}{\partial w_{Qi}}$$
$$- j \left[\frac{\partial \hat{\partial}}{\partial w_i} - \frac{\partial \hat{\partial}}{\partial w_i^*} \right].$$

Hence

$$(\tilde{\nabla}\hat{\partial})_i - \frac{\partial \eta}{\partial w_{Ii}} + j \frac{\partial \eta}{\partial w_{Qi}} - 2 \frac{\partial \hat{\partial}}{\partial w_i^*}.$$

Thus

$$\tilde{\nabla}\hat{\partial} - 2 \left(\frac{\partial \hat{\partial}}{\partial w_1^*}, \frac{\partial \hat{\partial}}{\partial w_2^*} \ldots \ldots \right)^T.$$

Returning to the expression for ϕ, we have that

$$\hat{\partial} - \underline{w}^H \underline{R} \, \underline{w} + \lambda \, (\underline{c}^T \underline{w} - \beta) + \lambda^* (\underline{c}^H \underline{w}^* - \beta^*).$$

Hence

$$\tilde{\nabla}\hat{\partial} - 2 \left[\underline{R} \, \underline{w} + \lambda^* \, \underline{c}^* \right].$$

The required condition for the minimisation is therefore

$$\tilde{\nabla}\hat{\partial} - 2 \left[\underline{R} \, \underline{w} + \lambda^* \, \underline{c}^* \right] - \underline{O}.$$

Hence we require

$$\underline{R} \, \underline{w} - -\lambda^* \, \underline{c}^*.$$

Pre-multiplying both sides by the inverse of $\underline{R}$, we thus have that

$$\underline{w} - -\lambda^* \, \underline{R}^{-1} \, \underline{c}^*.$$

But

$$\underline{c}^T \underline{w} - -\lambda^* \, \underline{c}^T \underline{R}^{-1} \, \underline{c}^* - \beta.$$

Hence

$$\lambda^* - \frac{-\beta}{\underline{c}^T \underline{R}^{-1} \underline{c}^*}.$$

Thus

$$\underline{w} = \beta \frac{R^{-1} \underline{c}^*}{\underline{c}^T R^{-1} \underline{c}^*}.$$

7. EIGENVALUES AND EIGENVECTORS

Let us consider a further constrained minimisation problem, namely that of finding the weight vector $\underline{w}$ that minimises the output power subject to the sum of the squares of the moduli of the components of $\underline{w}$ being constrained to unity. This constraint can be expressed as

$$\underline{w}^H \underline{w} = 1$$

and is necessarily real. We therefore consider the real function

$$\hat{o} = \underline{w}^H R \underline{w} - \lambda (\underline{w}^H \underline{w} - 1),$$

where $-\lambda$ is a real Lagrange multiplier. We thus require

$$\underline{\nabla}\hat{o} = 2 [R \underline{w} - \lambda \underline{w}] = \underline{0},$$

or

$$R \underline{w} = \lambda \underline{w}.$$

We are thus seeking a column vector $\underline{w}$ which when pre-multiplied by the matrix $\underline{R}$ produces a scaled version of $\underline{w}$. Such a vector is called an eigenvector of the matrix $\underline{R}$, and the scaling factor is termed the corresponding eigenvalue. In general there are a number of eigenvectors of a matrix. Let the ith be denoted by $\underline{w}_i$ and its corresponding eigenvalue by λ_i. We then have that

$$R \underline{w}_i = \lambda_i \underline{w}_i.$$

Clearly, if $\underline{w}_i$ is as eigenvector of $\underline{R}$ so is any scaled version of $\underline{w}_i$. We may therefore, without loss of generality, choose to work with normalised eigenvectors satisfying $\underline{w}_i^H \underline{w}_i = 1$.

Pre-multiplying both sides of the defining equation for $\underline{w}_i$ by $\underline{w}_i^H$ we find that

$$\underline{w}_i^H R \underline{w}_i = \lambda_i \underline{w}_i^H \underline{w}_i.$$

Taking the complex conjugate of the transpose of both sides we then have that

$$\underline{w}_i^H R^H \underline{w}_i = \lambda_i^* \underline{w}_i^H \underline{w}_i.$$

When, as in the case of a covariance matrix, the matrix $\underline{R}$ is Hermitian we have that

$$R^H = R.$$

Hence

$$\underline{w}_i^H R \underline{w}_i = \lambda_i^* \underline{w}_i^H \underline{w}_i.$$

We therefore conclude that $\lambda_i^* = \lambda_i$. The eigenvalues of an Hermitian matrix are therefore always real.

Let $\underline{w}_j$ be a second eigenvector of $\underline{R}$, with eigenvalue λ_j. Then

$$R \underline{w}_j = \lambda_j \underline{w}_j.$$

Thus

$$\underline{w}_i^H R \underline{w}_j = \lambda_j \underline{w}_i^H \underline{w}_j.$$

But

$$\underline{w}_i^H R \underline{w}_j = (R \underline{w}_i)^H \underline{w}_j = (\lambda_i \underline{w}_i)^H \underline{w}_j = \lambda_i^* \underline{w}_i^H \underline{w}_j = \lambda_i \underline{w}_i^H \underline{w}_j.$$

Hence

$$\lambda_j \underline{w}_i^H \underline{w}_j = \lambda_i \underline{w}_i^H \underline{w}_j.$$

Thus, if $\lambda_i \neq \lambda_j$, we must have that

$$\underline{w}_i^H \underline{w}_j = 0.$$

Thus eigenvectors of an Hermitian matrix corresponding to distinct eigenvalues satisfy $\underline{w}_i^H \underline{w}_j = O$. By analogy with the expression for the dot product of two real vectors, eigenvectors satisfying such a condition are said to be orthogonal.

Even if the eigenvectors have the same eigenvalue, in which case the eigenvalue is said to be degenerate, we may still incorporate them into a set of orthogonal eigenvectors. As an example let $\underline{w}_i$ and $\underline{w}_j$ both have the same eigenvalue λ. Then $\alpha \underline{w}_i + \beta \underline{w}_j$ is also an eigenvector with eigenvalue λ. We may take as our two eigenvectors

$$\underline{w}_i \ \text{and} \ \alpha \underline{w}_i + \beta \underline{w}_j$$

we may require that

$$\underline{w}_i^H (\alpha \underline{w}_i + \beta \underline{w}_j) - O \quad (orthogonality)$$

and

$$(\alpha \underline{w}_i + \beta \underline{w}_j)^H (\alpha \underline{w}_i + \beta \underline{w}_j) - 1 \quad (normalization).$$

If we solve these two expressions for α and β we will have constructed two orthogonal and normalized eigenvectors. We may generalize this approach to greater than two-fold degeneracy to prove that, even in the case of degeneracy, the eigenvectors of an Hermitian matrix can be constructed to be normalized and orthogonal to each other. They are then said to constitute an orthonormal set.

We may solve for the eigenvalues of a matrix $\underline{R}$ by means of the following procedure. Since

$$R \underline{w}_i - \lambda_i \underline{w}_i$$

we have that

$$[R - \lambda I] \underline{w}_i - Q$$

where $\underline{I}$ is the identity matrix with elements δ_{ij}.

If $[\underline{R} - \lambda\underline{I}]^{-1}$ exists, we have that

$$\underline{w}_i - [R - \lambda I]^{-1} Q - Q.$$

In this case the only solution for $\underline{w}_i$ is the trivial solution $\underline{w}_i = \underline{O}$. We may therefore demand that $[\underline{R} - \lambda\underline{I}]^{-1}$ does not exist to gain information about the non-trivial solutions.

But

$$[R - \lambda I]^{-1} = \frac{adj\ [R - \lambda I]}{det\ [R - \lambda I]}$$

Here adj[...] denotes the adjoint of a matrix, and det [...] the determinant of a matrix. Since the adjoint of a matrix may always be evaluated, we conclude that the condition for the existence of non-trivial solutions reduces to

$$det\ [R - \lambda I] = O.$$

The expansion of this determinant when $\underline{R}$ is an (n x n) matrix will yield an nth order polynomial. There are therefore in general n eigenvalues, and hence n eigenvectors, of such a matrix. As we have seen above, we may construct these eigenvectors to form an orthonormal set.

The n orthonormal eigenvectors of the matrix $\underline{R}$ may then be written as the rows of a matrix $\underline{B}$. We will then have that

$$R\ B^T = B^T\ D,$$

where $\underline{D}$ is a diagonal matrix containing the n eigenvalues λ_i.

Since the rows of $\underline{B}$ form an orthonormal set, it follows that

$$B^*\ B^T = I,$$

and hence that

$$B\ B^H = I.$$

A matrix $\underline{B}$ satisfying such a condition is said to be a unitary matrix. The inverse of a unitary matrix is therefore equal to the complex conjugate of its tranpose.

It therefore follows that

$$B^*\ R\ B^T = B^*\ B^T\ D = D.$$

A transformation of a matrix $\underline{R}$ by means of a unitary matrix $\underline{B}$ as in this last equation is called a unitary transformation. We have thus shown that any Hermitian matrix can be transformed into a purely diagonal form by means of a unitary transformation. Such diagonalization of an Hermitian matrix is a powerful tool which can be used to uncouple matrix equations as we shall now see.

8. THE SOLUTION OF TIME DEPENDENT COUPLED EQUATIONS

In Section 6 we considered the determination of the weight vector corresponding to the minimum output power subject to the constraint $\underline{c}^T \underline{w} = \beta$. One method of evaluating such a vector is to implement a recursive procedure in which the gradient of the output power with respect to the currently applied weight vector is computed, and a small step applied in the direction of steepest descent until the minimum condition is reached. Clearly such steps must not be allowed to violate the constraint condition.The gradient can, for example, be evaluated by monitoring the effects of small perturbations in the applied weight vector components.

Such a recursive update procedure may be expressed in the form

$$\underline{w}_{k+1} = \underline{w}_k - \mu\, \underline{P}\, \underline{\tilde{v}}\, (\underline{w}_k^H\, R\, \underline{w}_k).$$

Here

$$\underline{P} = \left[\underline{I} - \frac{\underline{c}\,\underline{c}^T}{\underline{c}^T\,\underline{c}} \right].$$

Now

$$\underline{c}^T\, \underline{P}\, \Delta\underline{w} = \underline{c}^T \left[\underline{I} - \frac{\underline{c}\,\underline{c}^T}{\underline{c}^T\underline{c}} \right] \Delta\underline{w}$$

Hence

$$\underline{c}^T\, \underline{P}\, \Delta\underline{w} = [\underline{c}^T - \underline{c}^T]\, \Delta\underline{w} = \underline{0}.$$

The matrix $\underline{P}$ ensures that the desired constraint is maintained by such a procedure provided that $\underline{w}_0$ satisfies $\underline{c}^T \underline{w}_0 = \beta$. $\underline{P}$ is known as a projection matrix.

We therefore have that

$$\underline{w}_{k+1} = \underline{w}_k - 2\mu\, \underline{P}\, R\, \underline{w}_k.$$

The recursive procedure will reach a steady-state condition in which

$$\underline{w}_{k+1} = \underline{w}_k = \underline{w}_\infty.$$

We therefore have that

$$P R \underline{w}_o - \underline{O}.$$

We may therefore write that

$$\underline{w}_{k+1} - \underline{w}_o - \underline{w}_k - \underline{w}_o - 2 \mu P R [\underline{w}_k - \underline{w}_o].$$

If we define

$$\Delta \underline{w}_k - \underline{w}_k - \underline{w}_o,$$

the equation can be expressed in the form.

$$\Delta \underline{w}_{k+1} - \Delta \underline{w}_k - 2 \mu P R \Delta \underline{w}_k.$$

Since

$$\underline{c}^T \underline{w}_k - \underline{c}^T \underline{w}_o - \beta$$

It follows that

$$\underline{c}^T [\underline{w}_k - \underline{w}_o] - \underline{c}^T \Delta \underline{w}_k - 0$$

Hence

$$P \Delta \underline{w}_k - \Delta \underline{w}_k.$$

We therefore have that

$$\Delta \underline{w}_{k+1} - \Delta \underline{w}_k - 2 \mu P R P \Delta \underline{w}_k.$$

This represents a set of coupled equations. We may seek to uncouple these equations by noting that $\underline{P} \ \underline{R} \ \underline{P}$ is an Hermitian matrix. If $\underline{\tilde{B}}$ is a matrix with rows equal to n orthonormal eigenvectors of $\underline{P} \ \underline{R} \ \underline{P}$ we then have that

$$\tilde{B}^* (\underline{P} \ \underline{R} \ \underline{P}) \tilde{B}^T - \tilde{D},$$

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where $\underline{\hat{D}}$ is a diagonal matrix with elements equal to the eigenvalues of $\underline{P} \ \underline{R} \ \underline{P}$.

Premultiplying both sides of the coupled equation set by $\underline{\hat{B}}^*$, we find that

$$\hat{B}^* \ \Delta \ \underline{w}_{k+1} - \hat{B}^* \ \Delta \ \underline{w}_k - 2 \ \mu \ \hat{B}^* \ (\underline{P} \ \underline{R} \ \underline{P}) \ \Delta \ \underline{w}_k$$

$$- \hat{B}^* \ \Delta \ \underline{w}_k - 2 \ \mu \ \hat{B}^* \ (\underline{P} \ \underline{R} \ \underline{P}) \ \hat{B}^T \ (\hat{B}^* \ \Delta \ \underline{w}_k)$$

$$- \hat{B}^* \ \Delta \ \underline{w}_k - 2 \ \mu \ \hat{D} \ \hat{B}^* \ \Delta \ \underline{w}_k$$

since

$$\hat{B}^T - (\hat{B}^*)^{-1}.$$

We now have a set of uncoupled equations which we can solve for $\underline{\hat{B}}^* \ \Delta \ \underline{w}_k$, and hence for $\Delta \ \underline{w}_k$. The time constants of convergence of this equation set are directly related to the eigenvalues of the matrix $\underline{P} \ \underline{R} \ \underline{P}$.

9.   LEAST SQUARES PROCESSING

Suppose we wish to find the weight vector which minimizes the output power from an arbitrary spatial/frequency filter, subject to the constraint of one of the components of the weight vector being held fixed at unity. Although we may address this problem using the techniques described in Section 6, it is instructive to re-couch it as follows.

Let successive snapshots across the filter inputs to which the variable weight components are to be applied be denoted by column vectors $\underline{x}_i$. Let the corresponding scalar snapshots associated with the input to which the fixed weight is to be applied be denoted by y_i. Let N such snapshots be available, with $1 \leq i \leq N$. We may then define

$$X - (\ \underline{x}_1, \ \underline{x}_2 \ \cdots \ \underline{x}_N \ )^T$$

and

$$\underline{y} - (\ y_1, \ y_2, \ \cdots \ y_N \ )^T.$$

Let $\underline{w}$ denote the vector of variable weight components we are seeking. If we consider

$$\underline{\varepsilon} - X \ \underline{w} + \underline{y},$$

the best estimate of $\underline{w}$ after the N snapshots is that which minimizes $\underline{e}^H \underline{e}$. The problem has therefore been reduced to that of classical least squares processing.

Now

$$\underline{e}^H \underline{e} = (\underline{X} \underline{w} + \underline{y})^H (\underline{X} \underline{w} + \underline{y})$$
$$= \underline{w}^H \underline{X}^H \underline{X} \underline{w} + \underline{w}^H \underline{X}^H \underline{y} + \underline{y}^H \underline{X} \underline{w} + \underline{y}^H \underline{y}.$$

Hence

$$\bar{\nabla} (\underline{e}^H \underline{e}) = 2 [\underline{X}^H \underline{X} \underline{w} + \underline{X}^H \underline{y}].$$

The required value of $\underline{w}$ therefore follows as

$$\underline{w} = -(\underline{X}^H \underline{X})^{-1} (\underline{X}^H \underline{y}).$$

A problem with using this expression to evaluate $\underline{w}$ is that the equation set may be ill conditioned. That is, small perturbations applied to the input data may produce large fluctuations in the solution. Such perturbations arise from the finite step size associated with the analogue to digital conversion process. The problem is made worse by the particular equation set we are considering because it involves the computation of $\underline{X}^H \underline{X}$, which means that the input data samples are effectively subjected to a squaring process.

Instead of employing such a squaring process, we can instead consider multiplying both sides of the equation defining $\underline{e}$ by a series of unitary matrix transformations. Let the first of this series be denoted by $\hat{Q}$. Since $\hat{Q}^H \hat{Q} = \underline{I}$ for a unitary matrix, we have that

$$(\hat{Q} \underline{e})^H (\hat{Q} \underline{e}) = \underline{e}^H \hat{Q}^H \hat{Q} \underline{e} = \underline{e}^H \underline{e}.$$

The length of the vector $\hat{Q} \underline{e}$ is therefore the same as that of $\underline{e}$. By means of an appropriate choice of the series of rotations, the compound effect of which we denote by the matrix $\underline{Q}$, we may reduce the equation set to the form

$$\underline{Q} \underline{e} = \underline{Q} \underline{X} \underline{w} + \underline{Q} \underline{y}$$
$$= \left[\frac{\underline{R}}{\underline{Q}} \right] \underline{w} + \left[\frac{\underline{h}_1}{\underline{h}_2} \right].$$

Here $\underline{R}$ now denotes a matrix with zeros below the main diagonal. It is clear that the condition corresponding to the minimum length of $Q\,\underline{e}$, and hence of $\underline{e}$, is

$$R\,\underline{w} = -\underline{b}_1.$$

The vector $\underline{w}$ can then be found by the process of back substitution. This method of solving the least squares problem is known as "QR processing".

A particularly important form for the constituent rotations Q is when each is constrained to operate on only a pair of rows of the equation set at a time, leaving all the other rows unaltered. We may represent each such a transformation by a [2x2] matrix, $\underline{q}$, which operates on its associated pair of rows, and is given by

$$\hat{q} = \begin{pmatrix} c & s^* \\ -s & c \end{pmatrix}.$$

We require that $\underline{q}^H\,\underline{q} = \underline{I}$, and hence that

$$s^*s + c^*c = 1$$

and

$$c^* = c.$$

These two conditions will be satisfied if we select

$$c = \frac{|x|}{\sqrt{|x|^2 + |y|^2}},$$

and

$$s = \frac{y}{x}\,c.$$

With these forms for c and s we also have that

$$\hat{q}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx + s^*y \\ O \end{pmatrix}.$$

A series of such Givens' rotations may therefore be used to annihilate successive components of the data matrix $\underline{X}$, until it is reduced to the desired upper triangular form.

By avoiding the explicit computation of $\underline{X}^H \underline{X}$ the data domain QR processing technique offers a numerically superior algorithm for determining the required weight vector.

10.   CONCLUDING COMMENTS

In the preceding sections we have given an introduction to the use of matrix techniques in representing and analyzing time series samples taken from the elements of a sensor array. Matrix representation provides a concise notation with which to represent the samples taken with such systems, and the wealth of available theorems relating to manipulations in this algebra means that it also represents an extremely powerful analytical tool.

11.   ACKNOWLEDGEMENT

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LECTURE 2

INTRODUCTION TO MATRIX TECHNIQUES

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