

# **SLIDING MODE CONTROL IN ENGINEERING**

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Many textbooks have been written on control engineering, describing new techniques for controlling systems, or new and better ways of mathematically formulating existing methods to solve the ever-increasing complex problems faced by practicing engineers. However, few of these books fully address the applications aspects of control engineering. It is the intention of this new series to redress this situation.

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There are already many exciting examples of the application of control techniques in the established fields of electrical, mechanical (including aerospace), and chemical engineering. We have only to look around in today's highly automated society to see the use of advanced robotics techniques in the manufacturing industries; the use of automated control and navigation systems in air and surface transport systems; the increasing use of intelligent control systems in the many artifacts available to the domestic consumer market; and the reliable supply of water, gas, and electrical power to the domestic consumer and to industry. However, there are currently many challenging problems that could benefit from wider exposure to the applicability of control methodologies, and the systematic systems-oriented basis inherent in the application of control techniques.

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*SERIES INTRODUCTION*

*in Engineering* is another outstanding entry to Dekker's Control Engineering series.

*Neil Munro*

# Preface

Many physical systems naturally require the use of discontinuous terms in their dynamics. This is, for instance, the case of mechanical systems with friction. This fact was recognized and advantageously exploited since the very beginning of the 20<sup>th</sup> century for the regulation of a large variety of dynamical systems. The keystone of this new approach was the theory of differential equations with discontinuous right-hand sides pioneered by academic groups of the former Soviet Union.

On this basis, discontinuous feedback control strategies appeared in the middle of the 20<sup>th</sup> century under the name of theory of variable-structure systems. Within this viewpoint, the control inputs typically take values from a discrete set, such as the extreme limits of a relay, or from a limited collection of prespecified feedback control functions. The switching logic is designed in such a way that a contracting property dominates the closed-loop dynamics of the system thus leading to a stabilization on a switching manifold, which induces desirable trajectories. Based on these principles, one of the most popular techniques was created, developed since the 1950s and popularized by the seminal paper by Utkin (see [30] in chapter 7): the sliding mode control. The essential feature of this technique is the choice of a switching surface of the state space according to the desired dynamical specifications of the closed-loop system. The switching logic, and thus the control law, are designed so that the state trajectories reach the surface and remain on it.

The main advantages of this method are:

- its robustness against a large class of perturbations or model uncertainties
- the need for a reduced amount of information in comparison to classical control techniques
- the possibility of stabilizing some nonlinear systems which are not stabilizable by continuous state feedback laws

The first implementations had an important drawback: the actuators had to cope with the high frequency bang-bang type of control actions that could produce premature wear, or even breaking. This phenomenon was the main obstacle to the success of these techniques in the industrial community. However, this main disadvantage, called chattering, could be reduced, or even suppressed, using techniques such as nonlinear gains, dynamic extensions, or by using more recent strategies, such as higher-order sliding mode control (see Chapter 3).

Once the constraint sliding function (CSF) was chosen according to some design specifications (stabilizing dynamics or tracking), then two difficulties may appear:

*D1*) the CSF should be of relative degree one (differentiating once for this function with respect to time: the control should appear) in order to provide the existence of a sliding motion; and

*D2*) the CSF may depend on the whole state (and not only on the measured outputs).

To circumvent *D1*) one may use a new CSF of relative degree one (see the introduction of Chapter 3 and the choice of the CSF in subsection 13.3.1). Another promising alternative to this difficulty is based on higher-order sliding mode controller design (see Chapter 3). Concerning *D2*) when the CSF depends on other variables than the measured outputs, a natural solution is provided by observer design. This approach has one advantage which concerns the natural filtering of the measurements (see Chapter 4 p. 121). But the drawback is that the class of admissible perturbations is reduced, since the perturbation should match two conditions: one for the control (see Chapter 1, p. 20) and the other for the observer (see Section 4.5).

We are currently living in an important time for these types of techniques. Now they may become more popular in the industrial community: they are relatively simple to implement, they show a great robustness, and they are also applicable to complex problems. Finally, many applications have been developed (see the Table of Contents):

- Control of electrical motors, DTC
- Observers and signal reconstruction
- Mechanical systems
- Control of robots and manipulators
- Magnetic bearings

Based on these facts, several active researchers in this field combined their efforts, thanks to the support of many French institutions<sup>1</sup>, to present new trends in sliding mode control.

In order to clearly present new trends, it is necessary to first give an historical overview of classical sliding mode (Chapter 1).

In the same manner of thinking, it is important to recall and introduce, from a very clear educational standpoint, a mathematical background for discontinuous differential equations, which is done in Chapter 2.

Next, a new concept in variable structure systems is introduced in Chapter 3 : the higher-order sliding mode. Such control design is naturally motivated by the limits of classical sliding mode (see Chapter 1) and completely validated by the mathematical background (see Chapter 2).

On the basis of these chapters, some control domains and methods are discussed with a sliding mode point of view:

- Chapter 4 deals with observer design for a large class of nonlinear systems.
- Chapter 5 presents a complementary point of view concerning the design of dynamical output controllers, instead of observer and state controllers.
- Chapter 6 presents the link between three of the most popular nonlinear control methods (i.e., sliding mode, passivity, and flatness) illustrated through power converter examples.
- Chapter 7 is dedicated to stability and stabilization. The domain of sliding mode motion is particularly investigated and the usefulness of the regular form is pointed out.
- Chapter 8 recalls some problems due to the discretization of the sliding mode controller. Some solutions are recalled and the usefulness under sampling of the higher-order sliding mode is highlighted.
- Chapter 9 deals with adaptive control design. Here, some basic features of control algorithms derived from a suitable combination of sliding mode and adaptive control theory are presented.
- Chapters 10 and 11 are dedicated to time delay effects. They deal, respectively, with relay control systems and with changes of behavior due to the delay presence.

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<sup>1</sup>CNRS, GdR Automatique, GRAISyHM, LAIL-UPRESA CNRS 8021, ECE-ENSEA and Ecole Centrale de Lille.

- Chapter 12 is dedicated to the control of infinite-dimensional systems. A disturbance rejection for such systems is particularly presented.

In order to increase interest in the proposed methods, the book ends with two applicative fields. Chapter 13 is dedicated to robotic applications and Chapter 14 deals with sliding mode control for induction motors.

Wilfrid PERRUQUETTI  
Jean-Pierre BARBOT  
FRANCE

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# Chapter 1

## Introduction: An Overview of Classical Sliding Mode Control

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### 1.1 Introduction and historical account

Sliding mode control has long proved its interests. Among them, relative simplicity of design, control of independent motion (as long as sliding conditions are maintained), invariance to process dynamics characteristics and external perturbations, wide variety of operational modes such as regulation, trajectory control [14], model following [30] and observation [24]. Although the subject has already been treated in many papers [5, 6, 13, 20], surveys [3], or books [7, 17, 28], it remains the object of many studies (theoretical or related to various applications). The main purpose of this chapter is to introduce the most basic and elementary concepts such as attractivity, equivalent control and dynamics in sliding mode, which will be illustrated by examples and applications.

Sliding mode control is fundamentally a consequence of discontinuous control. In the early sixties, discontinuous control (at least in its simplest form of bang-bang control) was a subject of study for mechanics and control engineers. Just recall, as an example, Hamel's work [15] in France, or Cypkin's [27] and Emelyanov's [9] in the USSR, solving in a rigorous way the

problem of oscillations appearing in bang-bang control systems. These first studies, more concerned with analysis and where the phenomena appeared rather as nuisances to be avoided, turned rapidly to synthesis problems in various ways. One of them was related to (time) optimal control, another to linearization and robustness. In the first case, discontinuities in the control, occurring at given times, resulted from the solution of a variational problem. In the second, which is of interest here, the use of a discontinuous control was an a priori choice. The more or less high frequency of the commutations used depended on the goal pursued (linearization), as produced by the beating spoilers used in the early sixties to control the lift of a wing, conception of corrective nonlinear networks enabling them to bypass the Bode's law limitations and, of course, generation of sliding modes. Although both approaches and objectives were at the beginning quite different, it is interesting to note that they turned out to have much in common.

In fact, it was when looking for ways to design what we now call robust control laws that sliding mode was discovered at the beginning of the sixties. For the needs of military aeronautics, and even before the term of robustness was used, control engineers were looking for control laws insensitive to the variations of the system to be controlled. The linear networks used at these time did not bring enough compensation to use high gains required to get parametric insensitivity: they match the Bode's law according to which phase and amplitude effects are coupled and antagonistic.

At the beginning of 1962, on B. Hamel's idea, studies of nonlinear compensators were initiated, whose aim was to overcome previous limitations. Typically, these networks, acting on the error signal  $x$  of the feedback system, were defined by the relation

$$u = |F_1(x, \dot{x}, \dots)| \operatorname{sgn}(F_2(x, \dot{x}, \dots))$$

where  $||$  denotes the absolute value and  $F_1$  and  $F_2$  are appropriate linear filters. Hence the output was discontinuous but modulated by a function of  $x$  and its derivatives. Under the simplest form, one had, for instance,

$$u = -|x| \operatorname{sgn}(x + k\dot{x}) \tag{1.1}$$

instead of the classical PD corrector.

It is easy to see that, under the approximation of the first harmonic:

- the equivalent gain of such a network (for a sinusoidal input  $x = x_0 \sin \omega t$ ) is independent of the amplitude  $x_0$  and only depends on the pulsation  $\omega$  (as a linear network), hence the denomination of pseudo linear network;

- it produces a lead phase without any increase (and even decrease) of the dynamics amplitude.

For instance, in the previous case, if  $\varphi(\omega)$  is the phase of  $1 + kp$  ( $p$  denoting the Laplace operator) at  $\omega$ , the real  $Re$  and imaginary parts  $Im$  of the equivalent gain are given by

$$Re = 1 - \frac{2}{\pi}(\varphi - \sin \varphi \cos \varphi)$$

$$Im = \frac{2}{\pi} \sin^2 \varphi$$

leading to the generalized transfer locus of Figure 1.1 where, for comparison, the loci for a simple PD (dotted line) and a classical lead phase network (thin line) are given. This shows that a lead phase can be obtained (theoretically till  $\frac{\pi}{2}$ ), not only without increase of the dynamics rate but also with a small reduction (from 1 to  $\frac{2}{\pi}$ ).

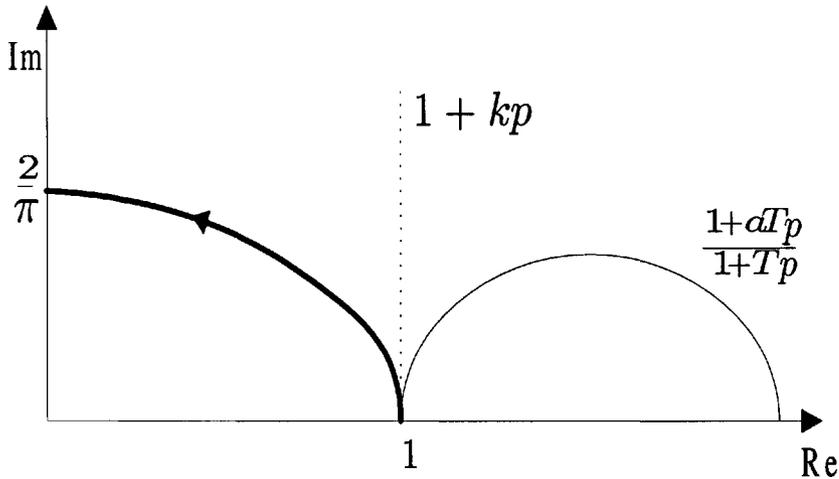


Figure 1.1: Nyquist plots

In fact, it appeared simultaneously in France and in the former USSR, that these laws presented two different aspects:

- pseudo linear compensation: astute combinations of linear and non-linear signals, including commutations, can lead to appreciable advantages while being freed from the disadvantages specific to purely linear systems;

- they generate a sliding motion by controlling the evolution of the system through commutations. This mode is certainly nonoptimal but exhibits a rather interesting sensitivity.

## 1.2 An introductory example

By way of illustration, let us take the simple example of a variable inertia  $\frac{a^2}{p^2}$  [1], as shown in Figure 1.2.

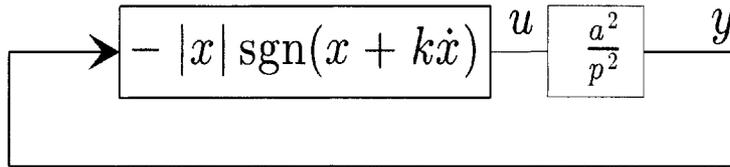


Figure 1.2: Variable inertia

Taking as state variables  $x_1 = x$ ,  $x_2 = \dot{x}$ , the system can be put in the following state space representation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= a^2 u\end{aligned}\tag{1.2}$$

where the control law  $u$  is designed as in (1.1) and is given by

$$u = -|x_1| \operatorname{sgn}(x_1 + kx_2)\tag{1.3}$$

In the following,  $\sigma = x_1 + kx_2 = 0$  will be called the *switching surface*. The term *switching* illustrates the fact that the control law  $u$  commutes while crossing the line  $\sigma = 0$ .

Then, one can easily see that (Figure 1.3):

- the phase plane is divided into four regions;
- in regions I and III (where  $x_1 \operatorname{sgn}(x_1 + kx_2) > 0$ ), trajectories are ellipses given by  $a^2 x_1^2 + x_2^2 = cst$ ;
- in regions II and IV (where  $x_1 \operatorname{sgn}(x_1 + kx_2) < 0$ ), trajectories are hyperbolas with asymptotes  $x_2 = \pm ax_1$ ;
- the control only commutes on the boundary surface  $x_1 + kx_2 = 0$ ;

- by a suitable choice of  $k$ , all trajectories are directed toward this surface (regardless of which side of the surface they are). Consequently, once it is reached, a new phenomenon appears: the trajectories are “sliding” along this surface.

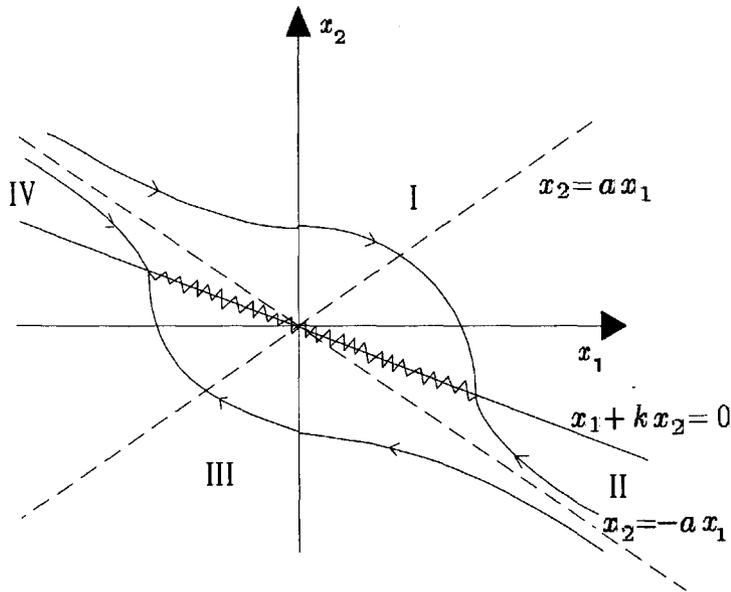


Figure 1.3: Trajectories in the portrait phase

The classical theory of ordinary differential equations however is unable to explain what occurs here (the solution of the system (1.2) is known to exist and be unique if  $u$  is a Lipschitz function, and so continuous). Consequently, the design of appropriate mathematical tools appears necessary and alternative approaches and construction of solutions can be found in Filippov’s work [11] and in other’s using the theory of differential inclusions [2]. Those results are not developed here since they are the subject of the chapter *Differential Inclusions and Sliding Mode Control*.

To understand more “physically” what is happening, a very simple interpretation can be given just by introducing some kind of imperfections in the switching devices, for instance a time delay  $\tau$ . Under such an assumption, the motion proceeds along a succession of small arcs (sequentially ellipsoidal and hyperbolic) between the lines  $x_1 + k'x_2 = 0$  and  $x_1 + k''x_2 = 0$ ,

crossing the origin, with

$$k' = \frac{k - \tau}{1 + a^2 k \tau}$$

$$k'' = \frac{k - \tau}{1 - a^2 k \tau}$$

When  $\tau$  tends to zero, the amplitude of these oscillations tends to zero, whereas the frequency increases indefinitely and the representative point “slides” along the line  $x + k\dot{x} = 0$  (Figure 1.4).

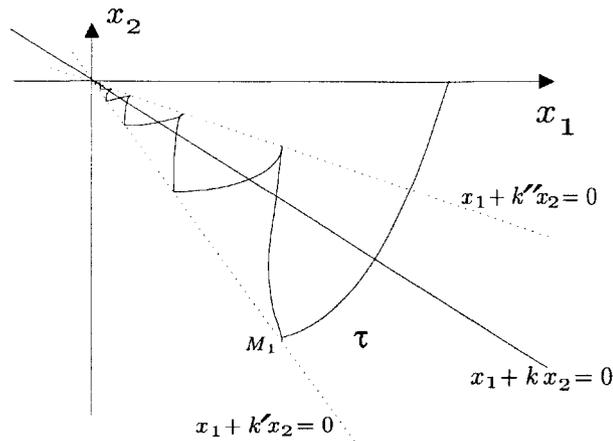


Figure 1.4: Trajectories with time delay

Further important remarks must be made:

In the *sliding motion*,  $\sigma \equiv 0$ , which implies that the dynamics is now defined by

$$\dot{x} = -\frac{1}{k}x$$

Therefore, the second-order system behaves then like a first-order system, with time constant  $k$  and independent of the inertia  $a$ , and the trajectory will slide along  $\sigma = 0$  to the origin (thus  $\sigma = 0$  is also called the *sliding surface*). Note also that, with the discontinuous control, the system is equivalent to a proportional-derivative feedback associated with an infinite gain.

As  $\dot{\sigma} = 0$ ,  $x_2 + ka^2u = 0$ . On the sliding surface, the motion is consequently the same as if, instead of the discontinuous control, an “equivalent”

continuous control defined by

$$u_e = -\frac{x_2}{ka^2} \quad (1.4)$$

had been used. This equivalent control can be considered as the mean value of the discontinuous control  $u$  on the sliding surface, modulated in width and amplitude. Yet, in sliding motion, the control switches with a high frequency between the values  $-|x_1|$  and  $|x_1|$ . This phenomenon is known as chattering and is a drawback of sliding modes (see section 1.3.3).

The latter dynamical behavior is called the *ideal sliding mode*, that is to say that there exists a finite time  $t_e$  such that for all  $t \geq t_e$ ,

$$s(x(t)) = 0$$

Of course, the ideal sliding mode along  $x + k\dot{x} = 0$  only exists for a time-continuous system and without delay, which is not the case in real system. Attention is drawn to the fact that, under sampling, the situation is much more complicated. The problem is beyond the scope of this introductory chapter and the interested reader will find developments in subsequent chapters, for instance *Discretization Issues* or *Sliding Mode Control for Systems with Time Delay*.

This simple example allowed us to enhance some characteristics of the sliding phenomenon and it has been shown that the sliding mode was initiated at the first switching. Of course, this is not always the case unless some precautions are taken. For instance, if the discontinuous control

$$u = -\operatorname{sgn}(x_1 + kx_2)$$

is used instead of (1.3), the sliding mode only occurs in the layer

$$|x_2| < ka^2$$

as can be seen in Figure 1.5.

This comes from the fact that the switching surface is known to be attractive if the condition  $s\dot{s} < 0$  is fulfilled. This will be detailed in the following sections, as well as the dynamics in sliding motion, the notion of equivalent control, the chattering phenomenon and the robustness properties of the sliding mode.

## 1.3 Dynamics in the sliding mode

### 1.3.1 Linear systems

Let us consider a linear process, eventually a multi-input system, defined by

$$\dot{x} = Ax + Bu \quad (1.5)$$

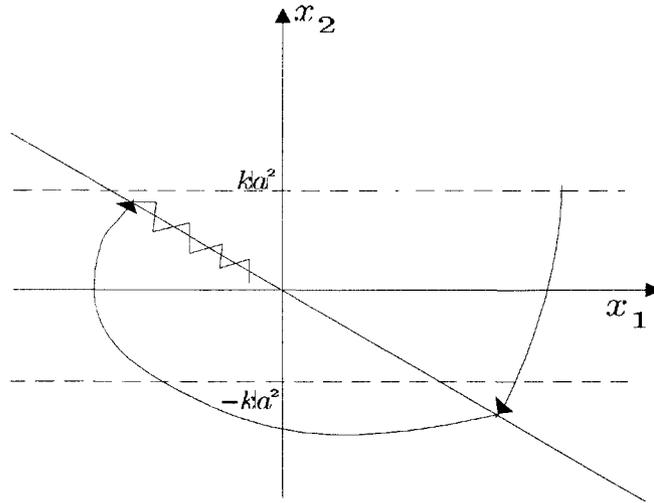


Figure 1.5: Portrait phase and sliding mode domain

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $\text{rank } B = m$ .

Let us also define the sliding surface as the intersection of  $m$  linear hyperplanes

$$\mathcal{S} = \{x \in \mathbb{R}^n : s(x) = Cx = 0\}$$

where  $C$  is a full rank ( $m \times n$ ) matrix and let us assume that a sliding motion occurs on  $\mathcal{S}$ .

In sliding mode,  $s \equiv 0$  and  $\dot{s} = CAx + CBu = 0$ . Assuming that  $CB$  is invertible (which is reasonable since  $B$  is assumed to be full rank and  $s$  is a chosen function), the sliding motion is affected by the so-called equivalent control

$$u_e = -(CB)^{-1} CAx$$

Consequently, the equivalent dynamics, in the sliding phase, is defined by

$$\dot{x}_e = [I - B(CB)^{-1}C] Ax_e = A_e x_e \quad (1.6)$$

The physical meaning of the equivalent control can be interpreted as follows. The discontinuous control  $u$  consists of a high frequency component ( $u_{hf}$ ) and a low frequency one ( $u_s$ ):  $u = u_{hf} + u_s$ .

$u_{hf}$  is filtered out by the bandwidth of the system and the sliding motion is only affected by  $u_s$ , which can be viewed as the output of the low pass filter

$$\tau \dot{u}_s + u_s = u, \quad \tau \ll 1$$

This means that  $u_e \simeq u_s$  and represents the mean value of the discontinuous control  $u$ .

$C$  being full rank,  $Cx = 0$  implies that  $m$  states of the system can be expressed as a linear combination of the remaining  $(n - m)$  states. Thus, in sliding motion, the dynamics of the system evolves on a reduced order state space (whose dimension is  $(n - m)$ ).

It is easy to verify that  $A_e$  is independent of the control and has at most  $(n - m)$  nonzero eigenvalues, depending on the chosen switching surface, while the associated eigenvectors belong to  $\ker(C)$ . As  $B$  is full rank, there exists a basis where it is equivalent to the matrix

$$\tilde{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

where  $B_2$  is an invertible  $(m \times m)$  matrix. Let us decompose the state as  $x = [x_1^T, x_2^T]^T$ , where  $x_1 \in \mathbb{R}^{n-m}$ ,  $x_2 \in \mathbb{R}^m$ . Thus, the system (1.5) becomes

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 &= A_{12}x_1 + A_{22}x_2 + B_2u \end{aligned}$$

and

$$C = [ C_1 \quad C_2 ]$$

the  $(m \times m)$  matrix  $C_2$  being assumed invertible (which is the necessary and sufficient condition for  $CB$  to be invertible since  $\det(CB) = \det(C_2B_2)$ ). Then one can compute  $A_e$  as following

$$\begin{aligned} A_e &= \begin{bmatrix} A_{11} & A_{12} \\ -C_2^{-1}C_1A_{21} & -C_2^{-1}C_1A_{22} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -C_2^{-1}C_1 & I \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}C_2^{-1}C_1 & A_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ C_2^{-1}C_1 & I \end{bmatrix} \end{aligned}$$

Under this form, the characteristic polynomial of  $A_e$  clearly appears to be

$$P(A_e) = \lambda^m P(A_{11} - A_{12}C_2^{-1}C_1)$$

Thus  $A_e$  has at least  $m$  null eigenvalues and the sliding dynamics is defined by

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}C_2^{-1}C_1)x_1 \\ \dot{x}_2 &= -C_2^{-1}C_1x_1 \end{aligned}$$

These last equations are interesting since they show that:

- designing  $C$  is analogous to design a state feedback matrix ensuring the desired behavior for the reduced order system  $(A_{11}, A_{12})$ , provided that the pair  $(A_{11}, A_{12})$  is controllable (which is the case if and only if the original pair  $(A, B)$  is controllable). Then the problem is a classical one which can be solved by the usual control techniques of direct eigenvalue and eigenvector placement or quadratic minimization [4], [28];
- the dynamics only depends on the matrix  $A_{11}$ ,  $A_{12}$ , and not on  $A_{21}$ ,  $A_{22}$ . For a single-input system, this means in particular, that if the system is written under the canonical controllability form,

$$\dot{x} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & \cdots & \cdots & \cdots & -a_{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

then the sliding dynamics is independent from the parameters  $a_i$  of the system.

Note that this remark can be generalized to multi-input systems. However, observe that, for this kind of system, the design of the control law is more complex than in the single-input case as the required sliding motion must take place at the intersection of the  $m$  switching surfaces. Broadly speaking, at least three strategies can be considered:

- the first one uses a hierarchical procedure where the system is gradually brought to the intersection of all the surfaces. Denoting  $\mathcal{S}_1, \dots, \mathcal{S}_m$  the  $m$  linear hyperplanes such that  $\mathcal{S} = \bigcap_{i=1}^m \mathcal{S}_i$ , and starting from an arbitrary initial condition, the control  $u_1$  is designed to induce a sliding mode on the surface  $\mathcal{S}_1$ , for any control  $u_2, \dots, u_m$ . This done, the second control  $u_2$  (while the system is still sliding on  $\mathcal{S}_1 = 0$ ) leads to  $\mathcal{S}_1 \cap \mathcal{S}_2$  and generates a sliding mode on this surface, and so on till a sliding motion takes place at the intersection of the  $m$  switching surfaces (Figure 1.6);
- another solution lies in reducing the system in  $m$  single-input subsystems such that every surface  $\mathcal{S}_i$  only depends on the  $i^{th}$  component of the discontinuous part of the control.

These first two policies lead to a rather simple procedure. However this implies a high prompting and wear of the actuators of the system since

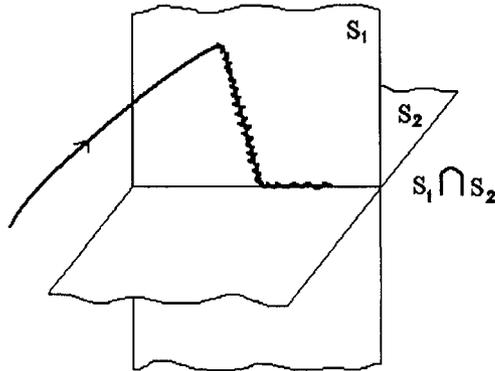


Figure 1.6: A sliding mode motion with two control functions

the control commutes at many more points of the state space than those constituting the sliding surface  $\mathcal{S}$ . Situations where one control drives the state away from the required intersection by imposing a sliding motion on a subset of surfaces can also occur. A way to face these problems is to make the sliding motion appear only at the intersection of all the manifolds. The control is continuous at the crossing of any separate surface and discontinuous only at the intersection of all of them. For this, the following control laws were proposed (see [7]) [called the unit vector approach],

$$u = u_e - \frac{\rho Cx}{\|Cx\|}$$

or

$$u = u_e - \frac{\rho Mx}{\|Nx\|}$$

where the matrix  $M$  and  $N$  are such that

$$\ker M = \ker N = \ker C$$

### 1.3.2 Nonlinear systems

Let us now consider the following nonlinear system affine in the control:

$$\dot{x} = f(x) + g(x)u(t) \quad (1.7)$$

and a set of  $m$  switching surfaces

$$\mathcal{S} = \{x \in \mathbb{R}^n : s(x) = [s_1(x), \dots, s_m(x)]^T = 0\} \quad (1.8)$$

An extension of the previous results leads to:

- the associated equivalent control

$$u_e = - \left[ \frac{\partial s}{\partial x} g(x) \right]^{-1} \frac{\partial s}{\partial x} f(x)$$

obtained by writing that  $\dot{s}(x) = \frac{\partial s}{\partial x} [f(x) + g(x)u(t)] = 0$ ;

- the resulting dynamics, in sliding mode

$$\dot{x}_e = \left[ I - g(x_e) \left[ \frac{\partial s}{\partial x_e} g(x_e) \right]^{-1} \frac{\partial s}{\partial x_e} \right] f(x_e)$$

Note that  $\sigma$  must be designed such that  $\frac{\partial s}{\partial x} g(x)$  is regular.

However, it is clear that, outside specific cases, the determination of the switching surfaces, in order to get a prescribed dynamics, is not as easy as in the linear case. One of these specific cases is when the system (1.7) can be transform into the so-called regular form [18], [19]:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned} \quad (1.9)$$

with  $x_1 \in \mathbb{R}^{n-m}$ ,  $x_2 \in \mathbb{R}^m$  and  $g_2$  regular. Suppose that the control problem is to stabilize the system at a prescribed point with the following dynamics

$$\dot{x}_1 = f(x_1, h(x_1))$$

Defining  $s(x) = x_2 - h(x_1)$  and a control  $u$  such that a sliding mode occurs on  $s = 0$  solves the problem, and the resulting sliding motion then evolves on a reduced order manifold of dimension  $(n - m)$  ( $x_2$  can be viewed as the input of the subsystem whose state is  $x_1$ ). This can be illustrated by the example of the two-arm manipulators which can be found in [25]. Yet, the transformation of the system into the regular form can induce complex diffeomorphisms. An alternative is to proceed by pseudo linearization as in [21].

### 1.3.3 The chattering phenomenon

An ideal sliding mode does not exist in practice since it would imply that the control commutes at an infinite frequency. In the presence of switching imperfections, such as switching time delays and small time constants in the actuators, the discontinuity in the feedback control produces a particular

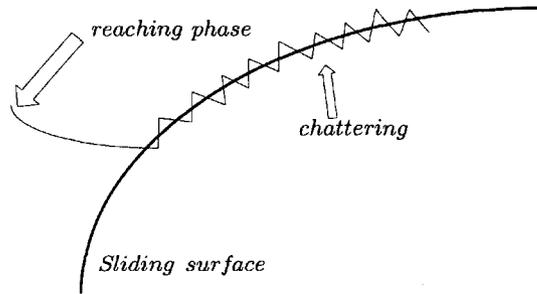


Figure 1.7: The chattering phenomenon

dynamic behavior in the vicinity of the surface, which is commonly referred to as chattering (Figure 1.7).

This phenomenon is a drawback as, even if it is filtered at the output of the process, it may excite unmodeled high frequency modes, which degrades the performance of the system and may even lead to instability [16]. Chattering also leads to high wear of moving mechanical parts and high heat losses in electrical power circuits. That is why many procedures have been designed to reduce or eliminate this chattering. One of them consists in a regulation scheme in some neighborhood of the switching surface which, in the simplest case, merely consists of replacing the signum function by a continuous approximation with a high gain in the boundary layer: for instance, sigmoid functions (see [23]) or saturation functions as shown in Figure 1.8. However, although the chattering can be removed, the robustness of sliding mode is also compromised. Another solution to cope with chattering is based on the recent theory of higher-order sliding modes (see Chapter 3).

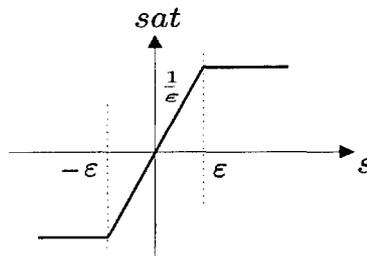


Figure 1.8: Saturation function  $sat(s)$

The real motion near the surface can be seen as the superposition of a “slow” movement, along the surface, and a “fast” one, perpendicular to this surface (the chattering phenomenon). To put in a prominent position these two movements, let us consider again our introductory example and let us approximate, in an  $\varepsilon$ -neighborhood of the surface, the signum function by a saturation function whose slope is  $\frac{1}{\varepsilon}$ . Taking  $\varepsilon$  as a (small) perturbation parameter, the behavior in the boundary layer can be described, under the standard singularly perturbed form, by

$$\begin{cases} \dot{x}_1 = x_2 \\ \varepsilon \dot{x}_2 = -a^2 x_1 (x_1 + k x_2) \end{cases}$$

The slow motion is defined by setting  $\varepsilon = 0$ , hence

$$\dot{x}_{1s} = x_{2s} = -\frac{1}{k} x_{1s}$$

and

$$\begin{aligned} x_{1s} &= x_{10} e^{-\frac{t}{k}} \\ x_{2s} &= -\frac{1}{k} x_{10} e^{-\frac{t}{k}} \end{aligned}$$

with  $x_{10}$  being the value of  $x_1$  at point  $M_1$  (see Figure 1.9). As it has been seen in section 1.2, this corresponds to the dynamics in the sliding motion.

In the time scale  $\frac{t}{\varepsilon}$ , the fast motion is defined by

$$\dot{x}_{2f} = -a^2 x_{10}^2 - a^2 k x_{10} x_{2f}$$

that is

$$x_{2f} = -\frac{1}{k} x_{10} (1 - e^{-\frac{a^2 k}{\varepsilon} x_{10} t}) + x_{20} e^{-\frac{a^2 k}{\varepsilon} x_{10} t}$$

and the global motion is approximated by

$$\begin{aligned} x_1 &= x_{1s} = x_{10} e^{-\frac{t}{k}} \\ x_2 &= x_{2s} + x_{2f} - x_{20s} = -\frac{1}{k} x_{10} e^{-\frac{t}{k}} + \left( x_{20} + \frac{1}{k} x_{10} \right) e^{-\frac{a^2 k}{\varepsilon} x_{10} t} \end{aligned}$$

which gives the trajectories in Figure 1.9.

## 1.4 Sliding mode control design

### 1.4.1 Reachability condition

It has been said that, in the sliding, the motion was independent from the control. Nonetheless, it is obvious that the control must be designed such

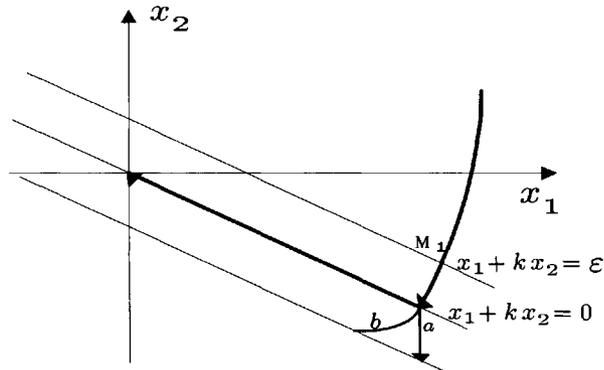


Figure 1.9: a) Singular perturbed motion  $\varepsilon = 0$  ; b) Real motion

that it drives the trajectories to the switching surface and maintains it on this surface once it has been reached. The local attractivity of the sliding surface can be expressed by the condition

$$\lim_{\sigma \rightarrow 0^+} \frac{\partial s}{\partial x} (f + gu) < 0 \quad \text{and} \quad \lim_{\sigma \rightarrow 0^-} \frac{\partial s}{\partial x} (f + gu) > 0$$

or, in a more concise way,

$$s\dot{s} < 0 \tag{1.10}$$

which is called the reachability condition [17].

**Example 1** *In a way of illustration, let us consider a dc-motor modeled by the following transfer function*

$$Y(p) = \frac{1}{p(p+1)}U(p)$$

*that is, in a state-space representation:*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u \\ y = x_1 \end{cases} \tag{1.11}$$

*Let us assume that the sliding surface is designed as*

$$s = x_2 + \alpha x_1 = 0, \quad \alpha > 0$$

Thus

$$\dot{s} = (\alpha - 1)x_2 + u \quad (1.12)$$

Using the control law  $u = -k \operatorname{sgn}(s)$ ,  $k > 0$ , the reachability condition is satisfied in the domain

$$\Omega = \{x : |(\alpha - 1)x_2| < k\}$$

since

$$s\dot{s} < |s| (|(\alpha - 1)x_2| - k) < 0$$

One should note that condition (1.10) is not sufficient to ensure a finite time convergence to the surface. Indeed, in the latter example, the control

$$u = (1 - \alpha)x_2 - ks$$

provides  $\dot{s} = -ks$ , but the convergence to  $s = 0$  is only asymptotic since

$$s(t) = s(0)e^{-kt}$$

where  $s(0)$  is the initial value of  $s$ . Condition (1.10) is often replaced by the so-called  $\eta$ -reachability condition

$$s\dot{s} \leq -\eta|s| \quad (1.13)$$

which ensures a finite time convergence to  $s = 0$ , since by integration

$$|s(t)| - |s(0)| \leq -\eta t$$

showing that the time required to reach the surface, starting from initial condition  $s(0)$ , is bounded by

$$t_e = \frac{|s(0)|}{\eta}$$

In a practical way, the control law is generally displayed as  $u = u_e + u_d$  where  $u_e$  is the equivalent control (allowing us to cancel the known terms on the right hand side of (1.12)) and where  $u_d$  is the discontinuous part, ensuring a finite time convergence to the chosen surface.

The example (1.11) was simulated using the following control law

$$u = (1 - \alpha)x_2 - k \operatorname{sgn} s$$

where the term  $(1 - \alpha)x_2$  represents the equivalent control (since  $\dot{s} = 0$  implies  $u + (\alpha - 1)x_2 = 0$ ). One can also note that the  $\eta$ -reachability condition is satisfied. Figures 1.10 and 1.12 show obviously that the sliding

motion takes place after about 1.3 sec. Indeed, after this time, the dynamics of the system is represented by the reduced order system given by the chosen surface, i.e.:

$$\dot{x}_1 = -\alpha x_1 = x_2$$

and the control switches at high frequency. In Figure 1.12 one can see that the equivalent control, in sliding motion, represents the mean value of the control  $u$ . The portrait phase, in Figure 1.11, illustrates the two steps of the dynamics behavior: first, a parabolic trajectory before the surface is reached (which is called the reaching phase) and then the sliding along the designed line  $s = 0$  ( $x_2 = -\alpha x_1$ ) to the origin.

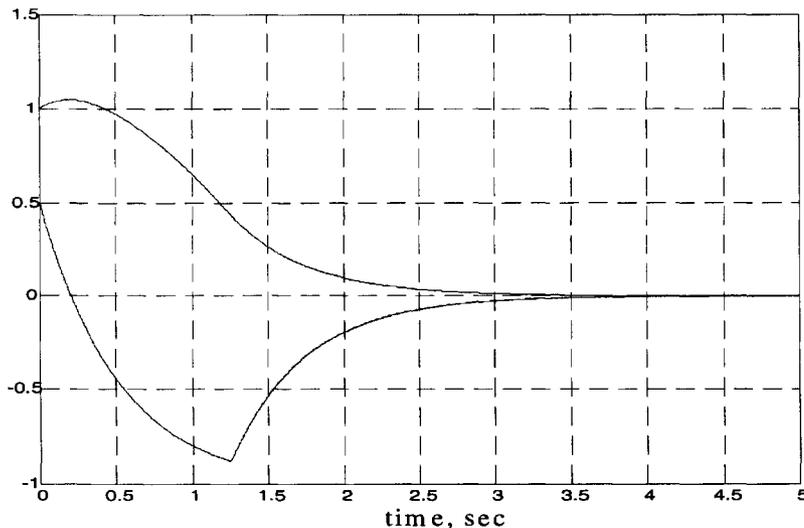


Figure 1.10: Evolution of the states versus time  $x_1$  (dotted) and  $x_2$  (solid)

## 1.4.2 Robustness properties

An important feature of sliding mode control is its robustness properties with respect to uncertainties. In the case of invariant and nonperturbed systems, recall first that the use of a continuous component, equal to  $u_e$ , allows the use of a discontinuous component as small as desired. Indeed, for the sake of simplicity, consider the linear system (1.5) and choose the following controller

$$u = u_e - k (CB)^{-1} \text{sgn}(s)$$

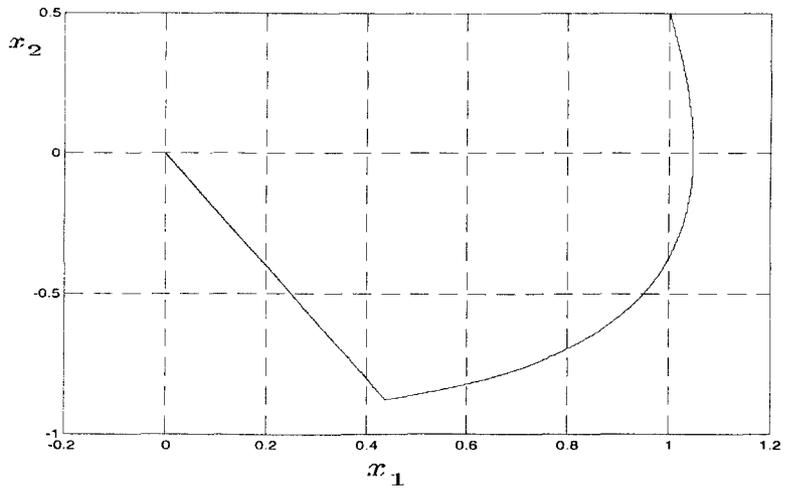


Figure 1.11: Portrait phase of the sliding motion

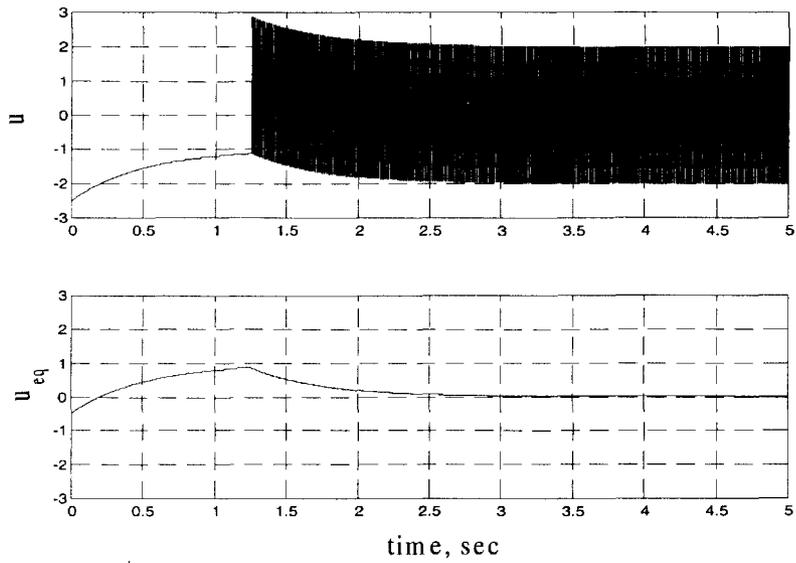


Figure 1.12: Discontinuous and equivalent control

with  $u_e = -(CB)^{-1}CAx$ . This implies

$$s\dot{s} = sC\dot{x} = s[CAx + CBu_e - k \operatorname{sgn}(s)] = -k|s| < 0$$

and consequently  $k$  might be taken high enough when the trajectory is far from the switching surface (so that the reaching time is short) and then as small as desired in order to limit the chattering.

Actually the use of a large enough discontinuous signal is necessary to complete the reachability condition despite parametric uncertainties and exogenous perturbations. Still, to be as simple as possible, consider the system under the canonical controllable form but with parametric uncertainties  $\Delta a_i$

$$\dot{x} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -a_0 - \Delta a_0 & \dots & \dots & \dots & -a_{n-1} - \Delta a_{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

where the  $\Delta a_i$  are all supposed to be bounded such that

$$\alpha_i^- < |\Delta a_i| < \alpha_i^+$$

Let the switching surface be

$$s = [c_0 \ c_1 \ c_{n-2} \ 1]x = 0$$

(corresponding to the sliding dynamics  $p^{n-1} + c_{n-2}p^{n-2} + \dots + c_0 = 0$ ).

The control law is chosen as follows

$$u = \sum_{i=1}^n k_{i-1}x_i - k_n \operatorname{sgn}(s)$$

The  $\eta$ -reachability condition, (1.13), can be satisfied by two ways, and thus despite the uncertainties:

- if constant gains are set as  $k_0 = a_0$ ,  $k_i = a_i - c_{i-1}$ ,  $i = 1, \dots, n-1$ , one gets  $s\dot{s} = -\sum_{i=1}^n \Delta a_{i-1}x_i s - k_n|s|$

and thus setting

$$k_n > \eta + \sum_{i=1}^n |\Delta a_{i-1}x_i|$$

is sufficient to satisfy (1.13). The magnitude of the discontinuity in the control is a function of the state and of the uncertainties on the process. The control law is easy to design but the discontinuity can be important (and consequently the chattering).

- another solution relies on using commuting gains. Taking  $k_0 = \tilde{k}_0 + a_0$ ,  $k_i = \tilde{k}_i + a_i - c_{i-1}$ ,  $i = 1, \dots, n - 1$  leads to

$$s\dot{s} = \sum_{i=1}^n (\tilde{k}_{i-1} - \Delta a_{i-1}) x_i s - k_n |s|$$

and the condition  $s\dot{s} < -\eta |s|$  can be satisfied by choosing  $k_n = \eta$  as a small positive scalar and

$$\tilde{k}_{i-1} = \begin{cases} \alpha_{i-1}^- & \text{if } x_i s > 0 \\ \alpha_{i-1}^+ & \text{if } x_i s \leq 0 \end{cases} \quad i = 1, \dots, n$$

The structure of the control law is a little more complex but the amplitude of the discontinuity in the control is reduced.

Sliding modes are also known to be insensitive to exogenous perturbations satisfying the so-called *matching condition* (originally stated and proved by Drazenovic in [6]), that is to say that these perturbations act exactly in the input channels. Considering the perturbed linear system

$$\dot{x} = Ax + Bu + \Delta(x, t)$$

where  $\Delta$  is an unknown but bounded function, the matching condition means that the sliding mode is insensitive to the uncertain function  $\Delta$  if it is in the range space of the input matrix  $B$ : that is, there exists a known matrix  $D$  and an unknown function  $\delta$  such that  $\Delta = D\delta$  and  $\text{rank}[B D] = \text{rank} B$ . Indeed, it is easy to show that, in that case,

$$\left( I - B(CB)^{-1}C \right) \Delta = 0$$

since

$$\left( I - B(CB)^{-1}C \right) B = 0$$

and thus the dynamics in sliding motion remains independent of the exogenous input  $\Delta$  ( $\dot{x}_e = \left[ I - B(CB)^{-1}C \right] Ax = A_e x$ ).

It is important to note that the system only becomes insensitive to those perturbations during sliding mode but remains affected by the perturbations during the reaching phase (that is to say before the sliding surface has been reached).

## 1.5 Trajectory and model following

In the previous sections, variable structure control and sliding modes have been designed for regulation purposes but they can also be used for trajectory and model following.

### 1.5.1 Trajectory following

Without going into the details, and with the aim of outlining the interest of sliding mode controls in trajectory following, let us consider a simple linear single-input system

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1\dot{y} + a_0y = u$$

written in the canonical controllable representation

$$\dot{x} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -a_0 & \dots & \dots & \dots & -a_{n-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

where  $x = [y, \dot{y}, \dots, y^{(n-1)}]^T$ .

Assume that the control problem is to constrain the output  $y$  to follow a prescribed trajectory  $y_d(t)$  and set

$$x_d = [y_d, \dot{y}_d, \dots, y_d^{(n-1)}]^T$$

Defining the sliding surface to be  $s(t) = C(x - x_d)$  and designing a control law leading to a sliding motion on this surface gives  $x = x_d$ . It should be noted, in comparison with the regulation case, that here, the surface is time-varying and that the dynamics of the response is imposed by the desired trajectory (and not by the coefficients of the surface).

It should also be noted that this idea can be enlarged to nonlinear multi-input systems. Consider for instance the system

$$\begin{aligned} \ddot{x}_1 &= 3x_1 + x_2^2 + x_1\dot{x}_2 \cos x_2 + u_1 \\ \ddot{x}_2 &= x_1^3 - x_2 \cos x_1 + u_2 \end{aligned}$$

whose outputs are

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \end{aligned}$$

The control problem is to constrain these outputs to follow trajectories corresponding to second order responses with respect to step inputs. It is sufficient to take the sliding surfaces

$$s_i = c_i e_i + \dot{e}_i, \quad i = 1, 2$$

with  $e_i = x_i - x_{id}$ , and to generate controls  $u_i$  such that  $s_i \dot{s}_i < 0$ . Taking

$$u_1 = k_{11} \dot{e}_1 + \alpha_{11} x_1 + \alpha_{12} x_2^2 + \alpha_{13} x_1 \dot{x}_2 + \ddot{x}_{1d} - k_1 \operatorname{sgn} s_1$$

gives

$$\dot{s}_1 = (c_1 + k_{11}) \dot{e}_1 + (3 + \alpha_{11}) x_1 + (1 + \alpha_{12}) x_2^2 + (\cos x_2 + \alpha_{13}) x_1 \dot{x}_2 - k_1 \operatorname{sgn} s_1$$

so that with  $k_{11} = -c_1$ ,  $\alpha_{11} = -3$ ,  $\alpha_{12} = -1$

$$s_1 \dot{s}_1 = (\cos x_2 + \alpha_{13}) x_1 \dot{x}_2 s_1 - k_1 |s_1|$$

Thus, taking

$$\alpha_{13} = -\operatorname{sgn}(x_1 \dot{x}_2 s_1)$$

implies

$$s_1 \dot{s}_1 < 0, \quad \forall k_1 > 0$$

The control  $u_2$  can be designed similarly such that  $s_2 \dot{s}_2 < 0$ . Then each output follows the predefined trajectories.

### 1.5.2 Model following

Variable structure control and sliding mode can also be used for model following, that is to control the process in such a way that it behaves like a given model (of the same order). The idea is to force a sliding motion on the surfaces

$$S = K_e(x_m - x) = K_e x_e = 0$$

where  $x$  and  $x_m$  are respectively, the process and model state vectors. It is easy to see that, in sliding motion, the error dynamics is given by

$$\dot{x}_e = (1 - \Theta) A K_e$$

with  $\Theta = B(K_e B)^{-1} K_e$ .

Except for the case of perfect matching, which supposes that

$$\operatorname{rank}[B, B_m] = \operatorname{rank}[B, A_m - A] = \operatorname{rank} B$$

there exists a steady-state error which can be computed by the equation

$$\begin{pmatrix} [(1 - \Theta) A]_T \\ K_e \end{pmatrix} \bar{x}_e = \begin{pmatrix} -[(1 - \Theta) A]_T A_m^{-1} B_m \bar{u}_m \\ 0 \end{pmatrix}$$

where  $[(1 - \Theta) A]_T$  denotes the matrix constituted by the  $(n - m)$  independent lines of  $(1 - \Theta) A$ .

In the general case, when the conditions can not be met, one will only focus on the outputs and integrators to be added on the error  $y_m - y$  so that the steady state error is null (Figure 1.13).

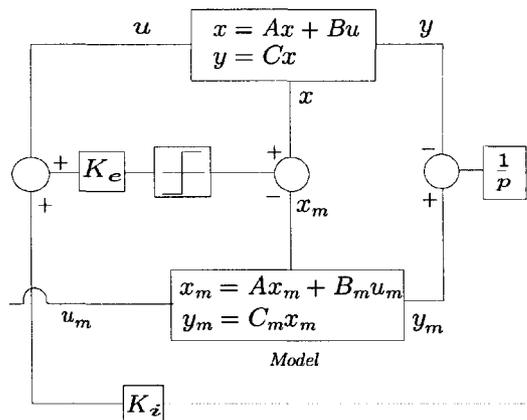


Figure 1.13: Model following

By way of illustration, let us consider the following case of a process given by the transfer function

$$G(s) = \frac{4\rho(s+1)}{s^2 + 4\delta s + 4}$$

where  $\rho$  and  $\delta$  are parameters which may vary. The control problem is to follow a model corresponding to

$$G_m(s) = \frac{1}{s^2 + 1.4s + 1}$$

The following figure shows the results of simulations enhancing the fact that the model following scheme is able to cope with important parametric variations. In Figure 1.14, continuous variations of  $\delta$  and  $\rho$  have been assumed such that  $\delta = \frac{t}{9}$  and  $\rho = 2 + \frac{4}{3}t$  (that is to say, for the span time of 9 seconds,  $\delta$  is varying from 0 to 1 and  $\rho$  from 2 to 14).

As far as the problem of model following is concerned, variable structure control laws using sliding modes can also be found in [12], [29] or [30].

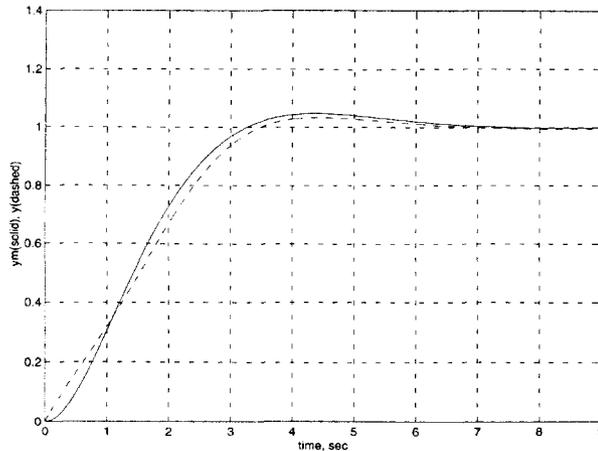


Figure 1.14: Example of model following

## 1.6 Conclusion

In this introductory chapter, the basic properties and interests of sliding modes have been enhanced. Since this technique involves differential equations with discontinuous right-hand sides, the concept of solution needs to be redefined and alternative approaches to the classical ordinary differential equation theory must be developed. One concerns differential inclusions and is presented in Chapter 2. The main benefits of sliding mode control are the invariance properties and the ability to decouple high dimensional problems into sub-tasks of lower dimensionality. However, it has been shown that imperfections in switching devices and delays were inducing a high-frequency motion called chattering (the states are repeatedly crossing the surface rather than remaining on it), so that no ideal sliding mode can occur in practice. Yet, solutions have been developed to reduce the chattering and so that the trajectories remain in a small neighborhood of the surface, like the higher-order sliding modes developed in Chapter 3. The continuous case has been considered in this introduction, but the problems induced by sliding modes under sampling and in the presence of delays are treated in Chapters 8, 10, 11.

The control problem given here was a regulation one and the illustrative examples were quite simple. However, sliding modes find their application in many other area such as observers (Chapter 4), output feedback (Chapter 5) or trajectory following (Chapter 6), and in practical applications such

as robotics (Chapter 13) and control of induction motors (Chapter 14).

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# Chapter 2

# Differential Inclusions and Sliding Mode Control

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## 2.1 Introduction

A basic problem in the field of variable structure control is the following. We are given a controlled system of ordinary differential equations with prescribed initial value

$$\dot{x} = f(t, x, u), x(0) = a \quad (2.1)$$

where the dynamics are defined by a given function

$$f : [0, +\infty) \times \Omega \times U \rightarrow \mathbb{R}^N$$

and the fixed initial condition  $a \in \Omega$  which is an open set in  $\mathbb{R}^N$  and  $U$  is a closed set in  $\mathbb{R}^M$ . The  $M$ -dimensional control variable

$$u \in U \quad (2.2)$$

is constrained to belong to the given control region  $U$ ; the  $N$ -dimensional state variable  $x$  is required to fulfill a given sliding condition

$$s[x(t)] = 0 \text{ for all } t \quad (2.3)$$

where  $s : \mathbb{R}^N \rightarrow \mathbb{R}^P$  is a fixed mapping which defines the sliding manifold

$$s(x) = 0 \quad (2.4)$$

The overall problem is to select an admissible control law  $u = u(t, x)$ , usually in the feedback form, such that through the corresponding state  $x$  issued from  $a$  at time 0 sends in finite time the initial position  $a$  to some point  $x(t^*)$  fulfilling (2.3) and keeps the state vector  $x(t)$  on the sliding manifold (2.4) for all  $t \geq t^*$  (in a prescribed time interval).

For simplicity we treat the case that  $s$  does not depend on time, even if what follows can be extended to the more general sliding condition

$$s(t, x) = 0$$

In these notes we deal only with the mathematical description of the sliding motion, assuming that a suitable control law has been found which solves the attainability problem, to reach in finite time the sliding manifold (hence we assume  $t^* = 0$ ).

At least three methods are known to control the given system in order to fulfill the state constraint (2.3).

*Componentwise sliding control.* Let  $P = M$ , then a suitably defined pair of feedback control laws

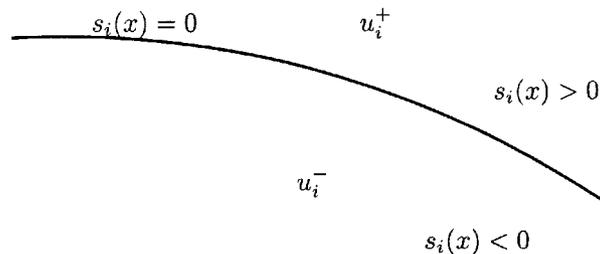
$$u_i^+ = u_i^+(t, x), u_i^- = u_i^-(t, x), i = 1, \dots, M$$

are used for each component of  $s$  to obtain the control law

$$u_i^*(t, x) = u_i^+(t, x) \text{ if } s_i(x) > 0$$

$$u_i^*(t, x) = u_i^-(t, x) \text{ if } s_i(x) < 0$$

Here  $s_i(x)$  denotes the  $i$ -th component of the vector  $s(x)$ . Proper choice of  $u_i^+$  and of  $u_i^-$  allows us to keep  $x(t)$  on the sliding manifold (2.3).



*Unit control.*

Let

$$f(t, x, u) = A(t, x) + B(t, x)u$$

and denote by  $Ds$  the Jacobian matrix of  $s$ . Let  $E(t, x) = Ds(x)B(t, x)$ . Then, under suitable nonsingularity assumptions, the control law

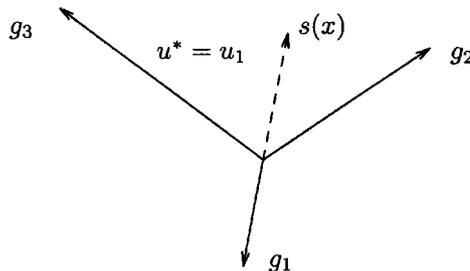
$$u(t, x) = -\alpha(t, x)E(t, x)'s(x)/|E(t, x)'s(x)|$$

with a proper choice of the gain  $\alpha$  allows us to reach the sliding manifold (2.3) and to keep the state vector on it.

*Sliding mode simplex method.* For every  $t, x$ , points  $u_1(t, x), \dots, u_{P+1}$  in  $U$  are found such that the vectors

$$g_i(t, x) = Ds(x)f[t, x, u_i(t, x)]$$

form a simplex in  $\mathbb{R}^P$ .



For every  $x$ ,  $s(x)$  belongs to some cone generated by the edges  $g_i(t, x)$ ,  $i \neq h$ , for the smallest index  $h$ . Then the choice of the control law

$$u^*(t, x) = u_h(t, x)$$

guarantees the sliding mode condition under suitable assumptions about the shape of the simplex.

## 2.2 Discontinuous differential equations and differential inclusions

All the above control methods share the following basic feature: the corresponding control law  $u^*$  undergoes discontinuities as a function of  $x$ . More precisely,  $u^*$  is (quite often) a piecewise continuous function of  $x$ . By inserting  $u^*$  into (2.1) we are forced to consider states  $x$  of the control system such that

$$\dot{x}(t) = f(t, x(t), u^*[t, x(t)])$$

and the corresponding dynamics

$$g(t, x) = f[t, x, u^*(t, x)] \quad (2.5)$$

is a discontinuous function of  $x$ . A basic issue of the mathematical description of the sliding mode control method is then the following. Which is the meaning of the solution concept of the differential equation

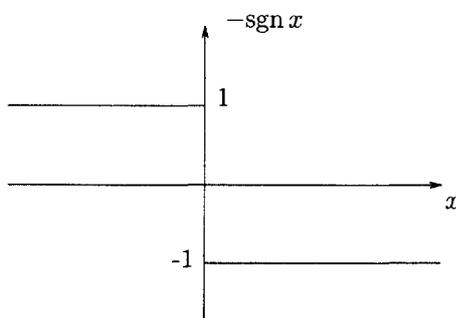
$$\dot{x} = g(t, x), x(0) = a \quad (2.6)$$

with a discontinuous  $g(t, \cdot)$ ?

**Example 2** *There exists (almost everywhere) no solution of the scalar equation*

$$\dot{x} = -\operatorname{sgn} x, x(0) = 0 \quad (2.7)$$

Here  $\operatorname{sgn} x = x/|x|, x \neq 0, \operatorname{sgn}(0) = 1$ .



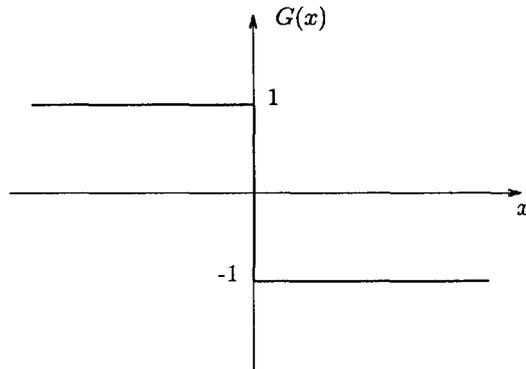
The previous example shows that, in general, discontinuous initial value problems (2.6) fail to possess classical (i.e., almost everywhere) solutions. A generalization of the concept of solution is required. A natural way of modifying the solution concept to (2.7) is to enlarge the right-hand side at 0, taking into account the behavior of  $g(x) = -\operatorname{sgn} x$  when  $x \neq 0$ . This leads us to consider the multifunction  $G : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$G(x) = \{g(x)\} = \{-\operatorname{sgn} x\}, x \neq 0; G(0) = [-1, 1]$$

and the initial-value problem for the differential inclusion

$$\dot{y}(t) \in G[y(t)], y(0) = 0 \quad (2.8)$$

which has the constant solution  $y(t) = 0$  for every  $t$ . The set-valued function  $G$  agrees with the singleton  $\{g(x)\}$  whenever  $g$  is continuous; at 0,  $G(0)$  is obtained by taking the set of all values of  $g(x)$  as  $|x|$  is sufficiently small and  $> 0$ , that is  $\{-1, 1\}$ , then its convex hull  $[-1, 1]$  and finally the intersection when  $|x| \rightarrow 0$  (which here has no effect). In this way we restore existence without losing contact with the original equation (2.7).



Let us remark that the existence behavior of (2.7) is very sensitive to changes of the initial value.

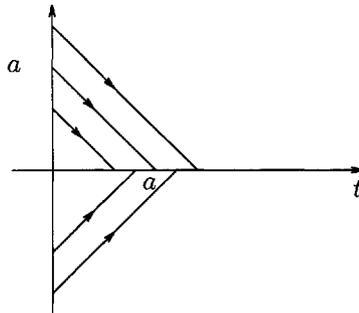
**Example 3** *The scalar equation*

$$\dot{x} = - \operatorname{sgn} x, x(0) = a \quad (2.9)$$

has (everywhere) local solutions for each  $a \neq 0$  given by  $x(t) = a - t, 0 \leq t < a$  or  $x(t) = t + a, 0 \leq t < -a$ . If we consider

$$g(x) = 0, x = 0; g(x) = - \operatorname{sgn} x, x \neq 0$$

then (2.9) has (almost everywhere) global solutions (i.e., on the whole time interval  $[0, +\infty)$  for every initial value  $a$ , namely  $x(t) = (a - t)^+$  if  $a > 0, x(t) = (a + t)^-$  if  $a < 0, x(t) = 0$  if  $a = 0$ .



## 2.3 Differential inclusions and Filippov solutions

We consider first initial value problems for differential inclusions and briefly review some existence theorems.

We are given a multifunction (set-valued mapping)

$$G : \Omega \rightarrow \mathbb{R}^N$$

where  $\Omega$  is an open set of  $\mathbb{R}^N$ , which takes on nonempty values  $G(x) \subset \mathbb{R}^N$ . Existence of classical (i.e., almost everywhere) solutions to the initial value problem

$$\dot{x} \in G(x), x(0) = a \quad (2.10)$$

is related to continuity properties of  $G$ , as follows.  $G$  is called

- *upper semicontinuous* at  $x_o \in \Omega$  if for every open set  $A$  such that  $G(x_o) \subset A$ , we have  $G(x) \subset A$  for all  $x$  sufficiently close to  $x_o$ ;
- *lower semicontinuous* at  $x_o \in \Omega$  if for every open set  $A$  such that  $G(x_o) \cap A \neq \emptyset$  we have  $G(x) \cap A \neq \emptyset$  for all  $x$  sufficiently close to  $x_o$ .

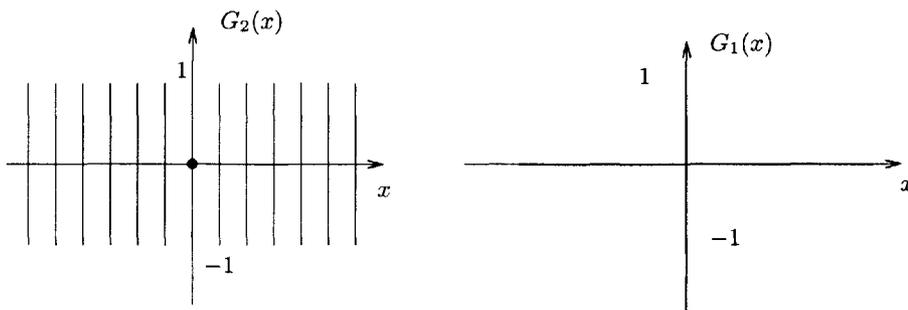
**Example 4** Consider

$$G_1(0) = [-1, 1]; G_1(x) = \{0\} \text{ if } x \neq 0$$

Then  $G_1$  is upper semicontinuous, not lower semicontinuous at 0. Consider

$$G_2(0) = \{0\}; G_2(x) = [-1, 1] \text{ if } x \neq 0$$

Then  $G_2$  is lower semicontinuous, not upper semicontinuous at 0.



A solution to the initial-value problem (2.10) is a function

$$y : [0, T) \rightarrow \Omega$$

for some positive  $T \leq +\infty$  such that its derivative exists for almost all  $t \in (0, T)$  and it is locally integrable,  $\int_a^b \dot{y} dt = y(b) - y(a)$  for every pair  $a, b$  in  $(0, T)$ , and

$$\dot{y}(t) \in G[y(t)] \text{ for almost all } t \in (0, T) \quad (2.11)$$

The conditions imposed on  $y$  in the previous definition [except (2.11)] amount to local absolute continuity of  $y$ .

Control problems quite often require examining the behavior over a prescribed time interval, for example  $[0, +\infty)$  if asymptotic stability is the main issue. For this reason, global existence theorems are most significant.

**Theorem 5 (Existence)** *Let  $G$  be nonempty compact convex valued and upper semicontinuous. Suppose there exist constants  $A, B$  such that*

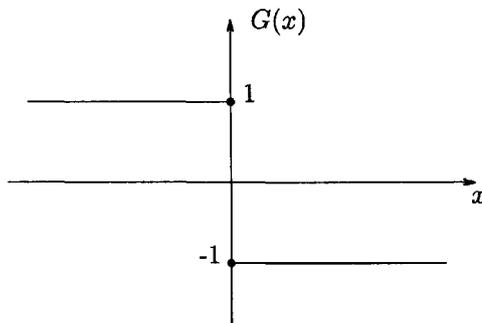
$$\sup \{|u| : u \in G(x)\} \leq A|x| + B \text{ for every } x$$

*Then problem (2.10) has solutions on  $[0, +\infty)$  for every  $a \in \Omega$ .*

**Example 6** *Let  $G(x) = \{-\operatorname{sgn} x\}$  if  $x \neq 0, G(0) = \{-1, 1\}$ . Then  $G$  is upper semicontinuous, compact valued, convex valued except at 0 (and  $G$  fails to be lower semicontinuous at 0). The initial value problem*

$$\dot{x} \in G(x), x(0) = 0$$

*lacks existence.*



The previous example shows that convexity of  $G(x)$  for every  $x$  cannot be omitted in the existence theorem 5. The previous theorem can be extended to time-varying right-hand sides

$$\dot{x} \in G(t, x), x(0) = a \tag{2.12}$$

under suitable measurability and growth properties of  $G$ .

**Theorem 7 (Existence)** *Let the multifunction*

$$G : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

*be nonempty closed convex valued and such that  $G(t, \cdot)$  is upper semicontinuous for all  $t$ , for every  $x$  there exists a measurable function  $h$  such*

that  $h(t) \in G(t, x)$  for almost all  $t \geq 0$ , and there exist locally integrable functions  $b, c$  such that

$$\sup \{|u| : u \in G(t, x)\} \leq b(t)|x| + c(t)$$

for almost all  $t \geq 0$  and every  $x$ . Then (2.12) has solutions on  $[0, +\infty)$ .

Both existence theorems 5 and 7 require upper semicontinuity of the right-hand side. If the right-hand side is lower semicontinuous (a case of less interest in the next developments) with respect to the state variable, then an existence theorem similar to theorem 7 holds without requiring convexity of the values.

We come back to initial-value problems for discontinuous differential equations

$$\dot{x} = g(t, x), x(0) = a \quad (2.13)$$

We have seen that the concept of solution to (2.6) needs to be properly redefined in order to guarantee existence. The basic definition we are going to review is due to Filippov, as follows. Let

$$g : [0, +\infty) \times \Omega \rightarrow \mathbb{R}^N$$

be measurable and such that for every  $A$  there exists  $B = B(t)$  locally integrable such that almost everywhere

$$\sup \{|g(t, x)| : t + |x| \leq A\} \leq B(t)$$

We associate to  $g$  the multifunction  $G$ , as follows. Denote by  $B(x, \epsilon)$  the ball in  $\mathbb{R}^N$  of center  $x$  and radius  $\epsilon$ . Consider the set

$$\{g(t, y) : y \in B(x, \epsilon)\} = g[t, B(x, \epsilon)], t \geq 0, x \in \Omega$$

Then let

$$G(t, x) = \cap \text{cl co } \{g[t, B(x, \epsilon) \setminus L] : \epsilon > 0, \text{ meas } L = 0\} \quad (2.14)$$

where  $\text{cl co } A$  denotes the closed convex hull of  $A$ , i.e., the intersection of all closed convex sets containing the set  $A$ .

**Definition 8** A Filippov solution  $y$  to (2.6) is a locally absolutely continuous function  $y : [0, T) \rightarrow \mathbb{R}^N$  such that

$$\dot{y}(t) \in G[t, y(t)] \quad (2.15)$$

for almost every  $t \in (0, T)$ .

Thus the Filippov definition replaces the discontinuous differential equation (2.6) by the differential inclusion (2.15). The construction of  $G$  from  $g$  generalizes what we have seen after example 2. Removing sets of measure 0 from the values taken by  $g$  corresponds, roughly speaking, to purposely ignoring possible misbehavior of the right-hand side in (2.6) on small sets. For every  $t$  and  $x$ ,  $G(t, x)$  defined by (2.14) turns out to be a nonempty closed convex set and the multifunction  $G(t, \cdot)$  is upper semicontinuous, moreover if  $g(t, \cdot)$  is continuous at  $z$  then  $G(t, z) = \{g(t, z)\}$ . It follows that, if  $g(\cdot, x)$  is a measurable function, and  $g(t, \cdot)$  is everywhere continuous, then  $y$  is a classical solution to (2.6) if and only if  $y$  is a Filippov solution.

For control systems (2.1) with discontinuous feedback control  $u^* = u^*(t, x)$  we obtain as a particular case the notion of Filippov solutions to (2.5). Hence

$$\dot{x} = f(t, x(t), u^*[t, x(t)])$$

if  $x$  is a Filippov solution to (2.1) with  $u = u^*$ ,  $f$  is smooth and  $u^*(t, \cdot)$  is continuous at  $x(t)$ . Quite often in applications,  $f$  is a smooth function and the discontinuous behavior is due to the insertion of the discontinuous control feedback  $u^*$  inside  $f$ . The properties of the multivalued function  $G$  given by (2.14) allows us to apply the existence theorems for initial value problems of differential inclusions.

**Theorem 9 (Existence)** *There exist Filippov solutions to (2.1) with  $u = u^*(t, x)$  on  $[0, +\infty)$  provided:*

- $\Omega = \mathbb{R}^N$ ,  $f(\cdot, x, u)$  is measurable for every  $x$  and  $u$ ,  $f(t, \cdot, \cdot)$  is continuous for almost every  $t \geq 0$ ;
- there exist locally integrable functions  $b, c$  such that

$$|f(t, x, u)| \leq b(t)|x| + c(t)$$

for almost all  $t \geq 0$ , every  $x$  and every  $u \in U$ ;

- $u^*$  is measurable.

We refer the reader to [2] as far as the physical meaning of Filippov solutions is involved, showing that this notion has not only a proper mathematical meaning but, as documented in [2], also a physical significance, which is relevant for control applications. (However there are stabilization problems in nonlinear control via discontinuous feedback in which Filippov solutions are not adequate, and something different must be used.)

Using directly the Filippov definition based on (2.14) is often rather complicated. In the following we describe an explicit formula which allows

us to obtain, in a simple yet useful case in practice, an explicit expression for the Filippov dynamics. We shall consider scalar control, smooth sliding manifold of codimension one, and piecewise smooth dynamics.

More precisely, suppose that

$$M = P = 1$$

$s$  is continuously differentiable, its gradient  $Ds(x) \neq 0$  whenever  $s(x) = 0$ , and the smooth surface  $S$  defined by (2.4) partitions  $\Omega$  in two disjoint open sets  $G^-$ ,  $G^+$  (with common boundary  $S$ ). Assume that  $g$  given by (2.5) is bounded and its restriction to both  $G^+$ ,  $G^-$  converges as  $x \rightarrow x_o \in S$  to limiting values  $g^+(t, x_o)$ ,  $g^-(t, x_o)$ , respectively, for all  $x_o \in S$ . Denote by

$$g_N^+, g_N^-$$

the projections of  $g^+$ ,  $g^-$  on the unit normal vector  $N$  to  $S$  at each point, oriented from  $G^-$  to  $G^+$ . Let  $y$  be absolutely continuous in a given time interval such that, for every  $t$ ,

$$s[y(t)] = 0, g_N^-[t, y(t)] \geq 0, g_N^+[t, y(t)] \leq 0, g_N^-[t, y(t)] > g_N^+[t, y(t)]$$

Then  $y$  is a Filippov solution to (2.6) if and only if for almost every  $t$

$$\dot{y}(t) = \alpha g^+[t, y(t)] + (1 - \alpha) g^-[t, y(t)]$$

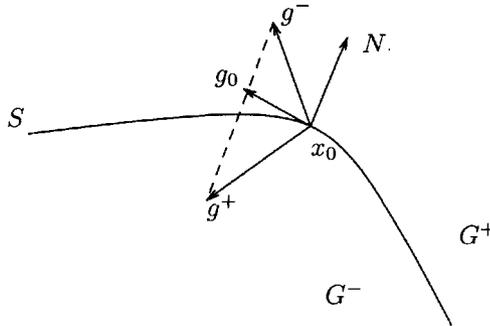
where

$$\alpha = \frac{g_N^-}{(g_N^- - g_N^+)}[t, y(t)]$$

or explicitly

$$\dot{y}(t) = \frac{[(Ds \cdot g^-)g^+ - (Ds \cdot g^+)g^-]}{[Ds \cdot (g^- - g^+)]} \quad (2.16)$$

with everything on the right being evaluated at  $t, y(t)$ . Thus, in this case, the Filippov dynamics is obtained explicitly as a convex combination of the vectors  $g^+$ ,  $g^-$ .



## 2.4 Viability and equivalent control

The basic problem we started with, was to find a feedback control law  $u^*$  such that the solution to (2.1) corresponding to  $u^*$  fulfilled the sliding condition (2.3). The appropriate meaning of state variable corresponding to the possibly discontinuous feedback  $u^*$  through (2.1) was obtained via the Filippov definition discussed in the previous section.

In this section we take into account the state constraint (2.3) and consider a more general version of the resulting problem, to find solutions to the following problem

$$\dot{x} \in F(x), x(t) \in K \text{ for all } t \quad (2.17)$$

where the multifunction

$$F : \Omega \rightarrow \mathbb{R}^N$$

and the closed set  $K \subset \mathbb{R}^N$  are fixed ( $\Omega$  being an open set of  $\mathbb{R}^N$ ).

In order to avoid technical points we consider only autonomous differential equations (2.17), i.e., we assume that  $F$  does not depend upon  $t$ . Our control problem (2.1), (2.2), (2.3) obtains as a particular case provided  $f = f(x, u)$ , i.e., the given dynamics is time-invariant, by taking

$$F(x) = f(x, U) = \{f(x, u) : u \in U\}, x \in \Omega \quad (2.18)$$

which is the set of all admissible velocities (so to speak) of the given control system, and

$$K = \{z \in \Omega : s(z) = 0\} \quad (2.19)$$

Let us pause to remark that (by a known result), if  $f$  is continuous and  $U$  is compact, then the set of all trajectories of the control system

$$\dot{x}(t) = f[t, x(t), u(t)], u(t) \in U \text{ almost everywhere}$$

corresponding to open loop (measurable) control laws  $u(\cdot)$ , coincides with the set of all solutions to the differential inclusion

$$\dot{x}(t) \in f(t, x(t), U)$$

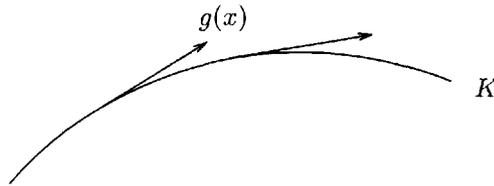
Even if this result has no relevance for us here, it shows that differential inclusions can provide a convenient mathematical framework for the study of certain control problems.

A solution  $\gamma$  to the differential inclusion

$$\dot{x} \in F(x)$$

is called viable for  $K$  if  $y$  fulfils (2.17), i.e.,  $y(t) \in K$  for all  $t$ . So we are interested in those special solutions (if any) of the differential inclusion in (2.17) which fulfill the sliding condition defined by the constraint set  $K$ . More precisely we are given the initial value  $a = x(0) \in K$  and we look for conditions guaranteeing that there exists at least one viable solution, i.e., a solution to (2.17) issued from  $a$  at 0.

If  $F$  is single-valued, i.e., we are considering a system of ordinary differential equations  $\dot{x} = g(x)$ , it is natural to impose, as a sufficient condition to viability, that the dynamics be tangent to the set  $K$ . This will force a solution starting on  $K$  to remain there forever.



Then we are led to consider the tangent cone to  $K$  at a given point  $x \in K$ ,

$$T(K, x)$$

which is the set of all points  $w = \lim_{n \rightarrow \infty} (x_n - x)/t_n$  where the sequence  $x_n \in K, x_n \rightarrow x$  and the sequence of positive numbers  $t_n \rightarrow 0$ . If  $K$  is a smooth surface  $S$  obtained by (2.19), then the tangent cone  $T(K, x)$  turns out to be the tangent space of  $S$  at  $x$ . We denote by

$$Ds(x)$$

the  $P \times N$  Jacobian matrix of  $s$  at  $x$ , whose  $(j, h)$  element is given by

$$\frac{\partial s_j}{\partial x_h}(x)$$

where  $s_j$  is the  $j$ -th component of  $s$ . We have

**Proposition 10** *Let  $K$  be given by (2.19). Let  $s \in C^1(\mathbb{R}^N, \mathbb{R}^P)$  be with  $Ds(x)$  of maximum rank if  $s(x) = 0$ . Then*

$$T(K, x) = \{w \in \mathbb{R}^N : Ds(x)w = 0\}$$

Now we consider the autonomous control problem

$$\dot{x} \in F(x), x(0) = a, x(t) \in K \tag{2.20}$$

where  $F$  is given by (2.18) and  $K$  is a given closed set. The differential inclusion (2.20) models (as a particular case) the control problem. Indeed, all states from (2.1) (if time-invariant) corresponding to arbitrary control laws fulfilling (2.2) obey (2.20) for almost all  $t$ . Moreover, all Filippov dynamics  $x$  corresponding to discontinuous feedback control laws from  $U$  fulfill (2.20) provided

$$U \text{ is compact, } f \text{ is continuous, and } f(x, U) \text{ is convex for all } x \in \Omega. \quad (2.21)$$

The main point is the following. The tangency condition we discussed before turns out to be a necessary and sufficient viability condition, which shows (in principle) how to control the system in order to fulfill the sliding condition (2.3).

**Theorem 11 (Viability)** *Suppose that (2.21) is verified and assume that  $|f(x, u)| \leq a|x| + b$  for suitable constants  $a, b$  and for every  $x \in \Omega, u \in U$ . Then the following are equivalent*

*for every  $a \in K$  there exists a solution  $x$  to (2.20) on  $[0, +\infty)$ ;*

$$F(x) \cap T(K, x) \neq \emptyset \text{ for every } x \in K \quad (2.22)$$

Let us write down the viability condition (2.22) in the case of interest, i.e.,  $K$  is defined by (2.19) and  $s$  is as in Proposition 10. Then (2.22) is true if and only if for every  $x \in K$  there exists some point  $\bar{u} = \bar{u}(x) \in U$  such that

$$Ds(x)f(\bar{x}, \bar{u}) = 0 \quad (2.23)$$

Condition (2.23) can be obtained formally by differentiating the sliding condition

$$s[x(t)] = 0$$

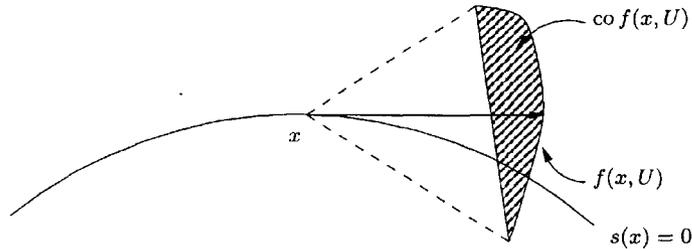
and working as  $x$  were a classical solution of (2.10), which could be false as we know, since Filippov solutions are not pointwise solutions. Then an *equivalent control*  $\bar{u}$  for (2.1), (2.2) and (2.3) (in the time invariant case we are discussing), is any feedback control law  $\bar{u}$  such that (2.23) holds and the classical solution to (2.1) corresponding to  $\bar{u}$  verifies the sliding condition (2.3). This last requirement is automatically true provided  $s[x(0)] = 0$  since for almost every  $t$

$$d/dt s[x(t)] = Ds[x(t)]f[x(t), \bar{u}(x(t))] = 0$$

hence  $s[x(t)]$  is constant. If  $U$  is compact,  $f$  is continuous and in addition we assume that the mapping

$$Ds f(x, \cdot)$$

for every  $x \in \Omega$  is one-to-one on a neighborhood of  $U$  to  $\mathbb{R}^P$  and its range contains 0, then the equivalent control  $\bar{u} = \bar{u}(x)$  defined by (2.23) is uniquely defined and is a continuous function of  $x$  in  $\Omega$ . Unfortunately it is not true (even if it is tempting to admit) that the sliding dynamics corresponding to the equivalent control agree with those obtained via Filippov's concept of solution.



**Example 12** The control system is ( $N = 2$ )

$$\dot{x}_1 = 0.3x_2 + ux_1, \dot{x}_2 = -0.7x_1 + 4u^3x_1$$

the sliding manifold ( $P = 1$ ) is defined by

$$s(x) = x_1 + x_2$$

and the scalar control  $u \in [-1, 1]$ . The discontinuous feedback control we consider is given by

$$u^*(x) = -\operatorname{sgn}(s(x)x_1)$$

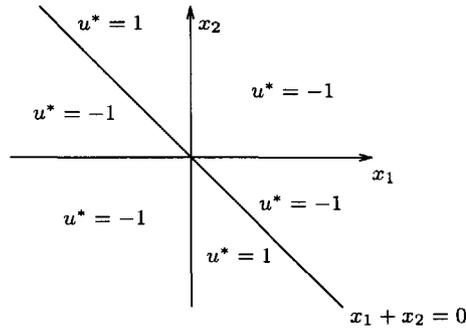
which can be shown to guarantee the sliding condition. Thus the control where  $s(x) > 0$  is given by  $u^+ = -\operatorname{sgn} x_1$ , while  $u^- = \operatorname{sgn} x_1$  is the control law where  $s(x) < 0$ . Here the equivalent control is the constant  $\bar{u} = 0.5$  obtained as the unique real root of  $u + 4u^3 = 1$ , giving rise to the dynamics

$$\dot{x}_1 = 0.2x_1 \text{ on } x_1 + x_2 = 0$$

By applying (2.16) we get the Filippov dynamics

$$\dot{x}_1 = -0.1x_1 \text{ on } x_1 + x_2 = 0$$

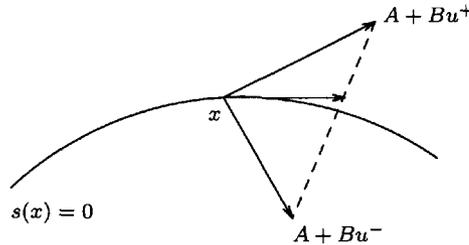
which is different from that corresponding to the equivalent control.



However, the sliding dynamics obtained by using the equivalent control agree with Filippov's dynamics in the particular case of control systems which are affine in the control signal, i.e.,

$$f(t, x, u) = A(t, x) + B(t, x)u \quad (2.24)$$

where  $A, B$  are matrices of the appropriate dimensions.



**Example 13** If  $f$  is given by (2.23) with scalar control  $u$ ,  $M = P = 1$ ,  $A$  and  $B$  being measurable with respect to  $t$  and continuous with respect to  $x$ , and  $s$  is continuously differentiable, then a sufficient condition to existence and uniqueness of the equivalent control is that

$$Ds(x) \cdot B(t, x) \neq 0$$

for almost all  $t$  and every  $x$ . Then the equivalent control is given by

$$\bar{u}(t, x) = -Ds(x) \cdot A(t, x) / Ds(x) \cdot B(t, x)$$

and is again measurable in  $t$  and continuous in  $x$ . More generally, for multivariable control systems (2.24) with  $M = P$ , a sufficient condition for the existence and uniqueness of the equivalent control is that the  $M \times M$  matrix  $Ds(x)B(t, x)$  is everywhere nonsingular. In this case the equivalent control is

$$\bar{u}(t, x) = -[Ds(x)B(t, x)]^{-1} Ds(x)A(t, x)$$

The equivalence between Filippov and equivalent control states deals with the following situation (componentwise sliding mode control described in section 2.1). Let (2.24) hold,  $M = P$ , and each  $s_i$  be continuously differentiable,  $i = 1, \dots, M$ . Then for each  $x$  with  $s(x) = 0$ , every sufficiently small neighborhood of  $x$  turns out to be a disjoint union of open regions  $G_1, \dots, G_q$  and points of the sliding surface. We are given  $q = 2^M$  feedback control laws  $u_i(t, x)$ , which are measurable in  $t$  and continuous in  $x$ . Let  $y$  be absolutely continuous in the given time interval  $[0, T]$  such that  $s[y(t)] = 0$  for every  $t$ .

**Theorem 14 (Equivalence)**  *$y$  is a Filippov solution to (2.1) corresponding to the feedback  $u^*$  defined by  $u_1$  on  $G_1, \dots, u_q$  on  $G_q$  if and only if  $y$  is a classical solution to (2.1), corresponding to the equivalent control, provided  $U$  is closed convex and  $Ds(x)B(t, x)$  is nonsingular for every  $t$  and  $x$  close to  $S$ .*

**Proof** of a particular case of Theorem 14. Let  $M = P = 1$  and suppose that the conditions leading to (2.16) are met. Then the dynamics corresponding to the equivalent control are as in Example 13, namely

$$\dot{y} = A - B(Ds \cdot A)/Ds \cdot B$$

In order to compare this with (2.16) we write  $u^* = u^+$  if  $s(x) > 0$ ,  $u^* = u^-$  if  $s(x) < 0$ , and compute

$$(Ds \cdot g^-)g^+ - (Ds \cdot g^+)g^- = [(Ds \cdot A)B - (Ds \cdot B)A](u^+ - u^-)$$

thus, by (2.16), the conclusion.

The practical value of Theorem 14 is obvious. For control systems (2.24) (under the above conditions), all calculations involving Filippov sliding mode controls can be correctly performed by formally differentiating the sliding condition (2.3) and working with states corresponding in the pointwise (classical) sense to the equivalent control; no discontinuous differential equation is involved at this stage.

An interesting property of the equivalent control, assuming (2.24) and suitable smoothness properties, involves the convergence of states, fulfilling only approximately the sliding condition, to the sliding state corresponding to the equivalent control, when the boundary layer width tends to disappear (regularization procedure). This fact will be discussed from a more general point of view in the next section.

## 2.5 Robustness and discontinuous control

Feedback control is important, among other reasons, mainly because of its robustness properties. In this section we briefly summarize a mathematical

interpretation of a form of robustness which deals with the dynamic behavior of sliding mode control systems under discontinuous feedback, and lies at the roots of practical control methods.

Given the variable structure control system (2.1), (2.2) and (2.3) we distinguish between

- real states which are solutions to (2.1) fulfilling only approximately the sliding condition and
- ideal states which solve (2.1) and fulfill exactly condition (2.3).

The following problem is relevant in this connection. Find conditions on the variable structure control system (2.1), (2.2) and (2.3) such that the following two properties hold:

- for every sequence of real states, whenever their initial values converge to the sliding manifold, then they converge towards a well-defined ideal state;
- one can approximate any ideal sliding state by real states fulfilling only approximately the sliding condition as the sliding error tends to zero.

We would like to obtain such robustness properties, no matter what the reasons are of violating the sliding condition (like disturbances, control errors, uncertainties, delays, etc.). Taking into account the discussion of Section 2.4 we assume that  $s$  is continuously differentiable and the mapping

$$Ds f(t, x, \cdot)$$

takes on the value 0, and is one-to-one on  $U$  for all  $x$  in some neighborhood  $V$  of the sliding manifold (2.4) and almost every  $t$ . We denote by

$$\bar{u}(t, x, w)$$

the unique solution  $u \in U$  of  $Ds(x)f(t, x, u) = w$  for a given  $w$ , hence the equivalent control is now  $\bar{u}(t, x, 0)$ . Given  $p > 1, T > 0$  and  $m(t) \geq 0$  such that  $\int_0^T [m(t)]^p dt$  is finite, let  $H$  denote the set of all parametrized functions  $a_\epsilon(t), \epsilon > 0$ , such that

$$|a_\epsilon(t)| \leq m(t) \text{ and } \sup \left\{ \left| \int_0^t a_\epsilon(s) ds \right|; 0 \leq t \leq T \right\} \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

Given  $a_\epsilon \in H$  suppose that  $x_\epsilon$  solves almost everywhere (2.1) with  $u = \bar{u}[t, x, a_\epsilon(t)]$ . Then

$$d/dt s[x_\epsilon(t)] = Ds[x_\epsilon(t)]f[t, x_\epsilon(t), u_\epsilon(t)] = a_\epsilon(t) \quad (2.25)$$

where  $u_\epsilon = \bar{u}[t, x_\epsilon(t), a_\epsilon(t)]$ . Integrating (2.25) between 0 and  $t$  we get

$$s[x_\epsilon(t)] \rightarrow 0 \text{ uniformly on } [0, T] \text{ as } \epsilon \rightarrow 0$$

The parameter  $\epsilon$  describes the amount of violation of the sliding condition (2.3) due to some imperfection (whatever they be). The sliding error is measured by  $a_\epsilon$ . Let  $y$  be a classical solution on  $[0, T]$  of (2.1) corresponding to the equivalent control  $\bar{u}(\cdot, \cdot, 0)$  such that  $s[y(0)] = 0$ , hence  $s[y(t)] = 0$  for all  $t \in [0, T]$  (because of the definition of  $\bar{u}$ ). The required robustness conditions are then satisfied provided the control system fulfills the following *approximability property* in  $(0, T)$ :

for every  $a_\epsilon$  in  $H$  such that  $\bar{u}[t, x, a_\epsilon(t)]$  exists for almost every  $t$  and  $x \in V$ , if we have

$$s[x_\epsilon(0)] \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

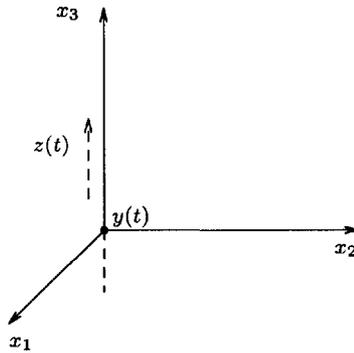
then  $x_\epsilon(0) \rightarrow y(0)$  implies  $x_\epsilon \rightarrow y$  uniformly on  $[0, T]$ .

Thus we have the following behavior provided (2.1), (2.2) and (2.3) satisfies the approximability property. If the control law we are employing yields small sliding errors, then reduction of the sliding error at the initial time implies uniformly small deviations from the desired (sliding) dynamical behavior (described by the equivalent control). Thus all real states converge to a well defined (uniquely determined) sliding state of the control system as the disturbances disappear, provided the initial values tend to the sliding manifold. Therefore approximability holds if and only if we can uniformly approximate any ideal sliding state by real states, disregarding the particular nature of the disturbances which are responsible for the sliding errors.

**Example 15** *The control system is*

$$\dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = u_1 u_2,$$

$N = 3, M = P = 2$ , with control constraint  $|u_1| \leq 1, |u_2| \leq 1$ ; the sliding manifold is given by  $s_1(x) = x_1, s_2(x) = x_2$ ; the initial condition is  $x(0) = 0$ . Here the equivalent control  $\bar{u} = 0$  gives rise to the motion  $y(t) = 0$  for all  $t$ . Partition the time interval in  $2^n$  equal subintervals and consider the control laws  $u_{1n}(t) = u_{2n}(t) = -1$  or  $+1$  alternatively. Then for the corresponding states, as  $n \rightarrow +\infty, x_{1n}(t) \rightarrow 0, x_{2n}(t) \rightarrow 0$ , however  $x_{3n}(t) = t$  for every  $n$ . Approximability fails (and the sliding state  $z(t) = (0, 0, t)'$  does not correspond to the equivalent control). The dynamic behavior of the system on the sliding manifold is in some sense ambiguous, and lacks robustness: by reducing the sliding error the corresponding real states do not converge to  $y$ .



It can be proved that, under suitable smoothness and nonsingularity conditions, approximability is verified in each of the following cases:

$$f(t, x, u) = A(t, x) + B(t, x)u \quad (2.26)$$

$$f(t, x, u) = (x_2, x_3, \dots, x_N, g)$$

where  $g = g(t, x_1, x_2, \dots, x_N, u)$  is strictly monotone with respect to the scalar control variable  $u$ .

Approximability is a theoretical basis to justify on rigorous grounds several sliding mode control procedures as far as their robustness properties are involved.

## 2.6 Numerical treatment

The simplest way to solve numerically the initial value problem

$$\dot{x} \in G(t, x), x(0) = a, 0 \leq t \leq T$$

is to look for a suitable extension of the classical Euler method, as follows. Choose a uniform grid

$$0 < t_1 < t_2 < \dots < t_n = T$$

with step size  $h = T/n$ ,  $n$  a given positive integer, hence

$$t_j = jT/n, j = 0, 1, \dots, n$$

Let  $x_0 = a$  and for  $j = 0, 1, \dots, n - 1$  compute any point  $x_{j+1}$  such that

$$x_{j+1} \in x_j + (T/n)G(t_j, x_j)$$

Consider the corresponding piecewise affine continuous function

$$y_n(t) = x_j + (n/T)(t - t_j)(x_{j+1} - x_j), t_j \leq t \leq t_{j+1}, j = 0, 1, \dots, n - 1$$

Then  $y_n$  can be considered as an approximate solution to (2.12) on  $[0, T]$ .

**Theorem 16 (Convergence)** *As  $n \rightarrow +\infty$ ,  $y_n$  converges uniformly on  $[0, T]$ , up to subsequences, to some solution of (2.12) provided  $G$  is upper semicontinuous with nonempty compact convex values and*

$$\sup \{|z| : z \in G(t, x)\} \leq k|x| + h$$

for every  $t, x$  and some constants  $k, h$ .

Thus convergence of the Euler method is guaranteed for discontinuous feedback control systems (under the previous assumptions). More refined methods, known to have better convergence properties when applied to smooth differential equations, cannot be guaranteed to converge when extended more or less directly to apply to, say, (2.15). Indeed, smoothness properties under which convergence is guaranteed for differential inclusions, are usually not satisfied for piecewise continuous differential equations. If applicable, such methods require special care to handle discontinuous differential equations. See also Chapter 8 of this book (Discretization Issues, by J-P. Barbot et al.).

## 2.7 Mathematical appendix

We collect here a few mathematical definitions which have been used in these notes.

A bounded subset  $A$  of the real numbers has Lebesgue measure zero if for every  $\varepsilon > 0$  there exists a countable collection  $B$  of disjoint intervals  $B_n, n = 1, 2, \dots$ , such that  $A \subset \cup\{B_n : n = 1, 2, \dots\}$  and the total length of  $B$ , i.e.  $\sum_{n=1}^{+\infty}(\text{length } B_n)$  is  $\leq \varepsilon$ . Any finite set, the set of points of any sequence, the set of all decimal numbers in a given bounded interval are all examples of sets of measure 0 in  $\mathbb{R}$ . Almost everywhere means except of a set of measure 0. Hence (Section 2.2) if  $x$  is an almost everywhere solution of the differential equation  $\dot{x} = g(t, x)$  on some bounded interval, then  $\dot{x}(t) = g[t, x(t)]$  for all  $t$  except those in a set of measure 0 (which could be empty of course).

The family of all Lebesgue measurable subsets of  $\mathbb{R}^N$  contains all compact and all open sets, all subsets of sets of measure 0 (which are de-

ned similarly as the case  $N = 1$ ), and it is invariant under complementation, countable unions and intersections. A given real-valued function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is Lebesgue measurable if and only if all sublevel sets  $\{x \in \mathbb{R}^N : f(x) \leq c\}$  are measurable (for all real  $c$ ). A vector-valued function is measurable if and only if its components are. Very roughly speaking, most of the functions we encounter in the control sciences are indeed measurable.

A function  $y : [p, q] \rightarrow \mathbb{R}^N$  is absolutely continuous if and only if  $y$  has a derivative  $\dot{y}(t)$  at almost every point  $t$  of the interval  $[p, q]$ ,  $\dot{y}$  is integrable there and for all pairs of points  $a, b$  in  $[p, q]$  we have  $\int_a^b \dot{y} dt = y(b) - y(a)$ . Hence  $y(t) = y(p) + \int_p^t \dot{y}(s) ds, p \leq t \leq q$ , which allows us to represent the absolutely continuous function  $y$  via its derivative. Of course every continuously differentiable function is absolutely continuous (e.g., any classical solution of (2.6) with a continuous  $g$ ).

A set  $C \subset \mathbb{R}^N$  is convex if and only if for every pair of points  $u, v \in C$  we have that all points  $\alpha u + (1 - \alpha)v, 0 \leq \alpha \leq 1$  belong to  $C$  as well: i.e., if  $u, v$  are in  $C$  then the whole segment with ends  $u, v$  belongs to  $C$ .

## 2.8 Bibliographical comments

Section 2.1. A comprehensive treatment of the whole subject of sliding mode control with several applications can be found in [2]. Basic points of design of variable structure control are described in [8], see also [13]. The simplex method was discovered by Bajda-Isozimov (*Automation Remote Control* 46, 1985) and further developed by Bartolini-Parodi-Utkin-Zolezzi (to appear in *Mathematical Problems in Engineering*).

Sections 2.2, 2.3. The basic definition and the mathematical properties of Filippov solutions are in [3], see also the treatise [6]. Further definitions are compared in [7]. In [1] we find an exposition of the basic mathematical results about differential inclusions, see also [11]. An interesting discussion about the very beginning of relating the theory of discontinuous differential equations with control problems is in [9]. The physical meaning of Filippov solutions is discussed in [2].

Stabilization of control systems via discontinuous control require notions of solution of control systems which are different from Filippov's, see Clarke-Ledyaev-Sontag in *IEEE Trans. Autom. Control* 42 (1997), and Bressan, preprint SISSA (Trieste) n. 144 (1998).

Section 2.4. The theory of viability is discussed in [1] (and at a greater length in [10]). The concept of equivalent control and its physical meaning can be found in [2]. A survey of several concepts related to viability is in [5].

Section 2.5. Approximability was introduced in [4], see Bartolini-Zolezzi in [13] for further developments.

Section 2.6 See the survey [12], which among other things presents some computer plots of numerical solutions to a discontinuous differential equation.

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## Chapter 3

# Higher-Order Sliding Modes

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### 3.1 Introduction

One of the most important control problems is control under heavy uncertainty conditions. While there are a number of sophisticated methods like adaptation based on identification and observation, or absolute stability methods, the most obvious way to withstand the uncertainty is to keep some constraints by "brutal force". Indeed any strictly kept equality removes one "uncertainty dimension". The most simple way to keep a constraint is to react immediately to any deviation of the system stirring it back to the constraint by a sufficiently energetic effort. Implemented directly, the approach leads to so-called sliding modes, which become main operation modes in the variable structure systems (VSS) [55]. Having proved their high accuracy and robustness with respect to various internal and external disturbances, they also reveal their main drawback: the so-called chattering effect, i.e., dangerous high-frequency vibrations of the controlled system. Such an effect was considered as an obvious intrinsic feature of the very idea of immediate powerful reaction to the minutest deviation from the chosen constraint. Another important feature is proportionality of the maximal deviation from the constraint to the time interval between the measurements (or to the switching delay).

To avoid chattering some approaches were proposed [15, 51]. The main idea was to change the dynamics in a small vicinity of the discontinuity surface in order to avoid real discontinuity and at the same time to preserve the main properties of the whole system. However, the ultimate accuracy and robustness of the sliding mode were partially lost. Recently invented higher order sliding modes (HOSM) generalize the basic sliding mode idea, acting on the higher order time derivatives of the system deviation from the constraint instead of influencing the first deviation derivative as it happens in standard sliding modes. Along with keeping the main advantages of the original approach, at the same time they totally remove the chattering effect and provide for even higher accuracy in realization. A number of such controllers were described in the literature [16, 34, 35, 38, 3, 5].

HOSM is actually a movement on a discontinuity set of a dynamic system understood in Filippov's sense [22]. The sliding order characterizes the dynamics smoothness degree in the vicinity of the mode. If the task is to provide for keeping a constraint given by equality of a smooth function  $s$  to zero, the sliding order is a number of continuous total derivatives of  $s$  (including the zero one) in the vicinity of the sliding mode. Hence, the  $r$ th order sliding mode is determined by the equalities

$$s = \dot{s} = \ddot{s} = \dots = s^{(r-1)} = 0 \quad (3.1)$$

forming an  $r$ -dimensional condition on the state of the dynamic system. The words " $r$ th order sliding" are often abridged to " $r$ -sliding".

The standard sliding mode on which most variable structure systems (VSS) are based is of the first order ( $\dot{s}$  is discontinuous). While the standard modes feature finite time convergence, convergence to HOSM may be asymptotic as well.  $r$ -sliding mode realization can provide for up to the  $r$ th order of sliding precision with respect to the measurement interval [35, 38, 41]. In that sense  $r$ -sliding modes play the same role in sliding mode control theory as Runge-Kutta methods in numerical integration. Note that such utmost accuracy is observed only for HOSM with finite-time convergence.

Trivial cases of asymptotically stable HOSM are easily found in many classic VSSs. For example there is an asymptotically stable 2-sliding mode with respect to the constraint  $x = 0$  at the origin  $x = \dot{x} = 0$  (at the one point only) of a 2-dimensional VSS keeping the constraint  $x + \dot{x} = 0$  in a standard 1-sliding mode. Asymptotically stable or unstable HOSMs inevitably appear in VSSs with fast actuators [23, 25, 26, 27, 30]. Stable HOSM reveals itself in that case by spontaneous disappearance of the chattering effect. Thus, examples of asymptotically stable or unstable sliding modes of any order are well known [16, 14, 50, 35, 30]. On the contrary, examples of  $r$ -sliding modes attracting in finite time are known for  $r = 1$

(which is trivial), for  $r = 2$  [34, 16, 17, 35, 4, 5] and for  $r = 3$  [30]. Arbitrary order sliding controllers with finite-time convergence were only recently presented [38, 41]. Any new type of higher-order sliding controller with finite-time convergence is unique and requires thorough investigation.

The main problem in implementation of HOSMs is increasing information demand. Generally speaking, any  $r$ -sliding controller keeping  $s = 0$  needs  $s, \dot{s}, \dots, s^{(r-1)}$  to be available. The only known exclusion is a so-called "super-twisting" 2-sliding controller [35, 37], which needs only measurements of  $s$ . First differences of  $s^{(r-2)}$  having been used, measurements of  $s, \dot{s}, \dots, s^{(r-2)}$  turned out to be sufficient, which solves the problem only partially. A recently published robust exact differentiator with finite-time convergence [37] allows that problem to be solved in a theoretical way. In practice, however, the differentiation error proves to be proportional to  $\varepsilon^{(2-k)}$ , where  $k < r$  is the differentiation order and  $\varepsilon$  is the maximal measurement error of  $s$ . Yet the optimal one is proportional to  $\varepsilon^{(r-k)/r}$  ( $s^{(r)}$  is supposed to be discontinuous, but bounded [37]). Nevertheless, there is another way to approach HOSMs.

It was mentioned above that  $r$ -sliding mode realization provides for up to the  $r$ th order of sliding precision with respect to the switching delay  $\tau$ , but the opposite is also true [35]: keeping  $|s| = O(\tau^r)$  implies  $|s^{(i)}| = O(\tau^{r-i}), i = 0, 1, \dots, r - 1$ , to be kept, if  $s^{(r)}$  is bounded. Thus, keeping  $|s| = O(\tau^r)$  corresponds to approximate  $r$ -sliding. An algorithm providing for fulfillment of such relation in finite time, independent on  $\tau$ , is called  $r$ th order real-sliding algorithm [35]. Few second order real sliding algorithms [35, 52] differ from 2-sliding controllers with discrete measurements. Almost all  $r$ th order real sliding algorithms known to date require measurements of  $s, \dot{s}, \dots, s^{(r-2)}$  with  $r > 2$ . The only known exceptions are two real-sliding algorithms of the third order [7, 39], which require only measurements of  $s$ .

Definitions of higher order sliding modes (HOSM) and order of sliding are introduced in Section 3.2 and compared with other known control theory notions in Section 3.3. Stability of relay control systems with higher sliding orders is discussed in Section 3.4. The behavior of sliding mode systems with dynamic actuators is analyzed from the sliding-order viewpoint in Section 3.5. A number of main 2-sliding controllers with finite time convergence are listed in Section 3.6. A family of arbitrary-order sliding controllers with finite time convergence is presented in Section 3.7. The main notions are illustrated by simulation results.

## 3.2 Definitions of higher order sliding modes

Regular sliding mode features few special properties. It is reached in finite time, which means that a number of trajectories meet at any sliding point. In other words, the shift operator along the phase trajectory exists, but is not invertible in time at any sliding point. Other important features are that the manifold of sliding motions has a nonzero codimension and that any sliding motion is performed on a system discontinuity surface and may be understood only as a limit of motions when switching imperfections vanish and switching frequency tends to infinity. Any generalization of the sliding mode notion must inherit some of these properties.

First let us recall what Filippov's solutions [21, 22] are of a discontinuous differential equation

$$\dot{x} = v(x)$$

where  $x \in \mathbb{R}^n$ ,  $v$  is a locally bounded measurable (Lebesgue) vector function. In that case, the equation is replaced by an equivalent differential inclusion

$$\dot{x} \in \mathcal{V}(x)$$

In the particular case when the vector-field  $v$  is continuous almost everywhere, the set-valued function  $\mathcal{V}(x)$  is the convex closure of the set of all possible limits of  $v(y)$  as  $y \rightarrow x$ , while  $\{y\}$  are continuity points of  $v$ . Any solution of the equation is defined as an absolutely continuous function  $x(t)$ , satisfying the differential inclusion almost everywhere.

The following Definitions are based on [34, 16, 17, 19, 35, 30]. Note that the word combinations "rth order sliding" and "r-sliding" are equivalent.

### 3.2.1 Sliding modes on manifolds

Let  $\mathcal{S}$  be a smooth manifold. Set  $\mathcal{S}$  itself is called the 1-sliding set with respect to  $\mathcal{S}$ . The 2-sliding set is defined as the set of points  $x \in \mathcal{L}$ , where  $\mathcal{V}(x)$  lies entirely in tangential space  $\mathbf{T}_x$  to manifold  $\mathcal{S}$  at point  $x$  (Figure 3.1).

**Definition 17** *It is said that there exists a first (or second) order sliding mode on manifold  $\mathcal{S}$  in a vicinity of a first (or second) order sliding point  $x$ , if in this vicinity of point  $x$  the first (or second) order sliding set is an integral set, i.e., it consists of Filippov's sense trajectories.*

Let  $\mathcal{S}_1 = \mathcal{S}$ . Denote by  $\mathcal{S}_2$  the set of 2-sliding points with respect to manifold  $\mathcal{S}$ . Assume that  $\mathcal{S}_2$  may itself be considered as a sufficiently smooth manifold. Then the same construction may be considered with respect to  $\mathcal{S}_2$ . Denote by  $\mathcal{S}_3$  the corresponding 2-sliding set with respect

to  $S_2$ .  $S_3$  is called *the 3-sliding set with respect to manifold  $S$* . Continuing the process, we can achieve sliding sets of any order.

**Definition 18** *It is said that there exists an  $r$ -sliding mode on manifold  $S$  in a vicinity of an  $r$ -sliding point  $x \in S_r$ , if in this vicinity of point  $x$  the  $r$ -sliding set  $S_r$  is an integral set, i.e., it consists of Filippov's sense trajectories.*

### 3.2.2 Sliding modes with respect to constraint functions

Let a constraint be given by an equation  $s(x) = 0$ , where  $s : \mathbb{R}^n \rightarrow \mathbb{R}$  is a sufficiently smooth constraint function. It is also supposed that total time derivatives along the trajectories  $s, \dot{s}, \ddot{s}, \dots, s^{(r-1)}$  exist and are single-valued functions of  $x$ , which is not trivial for discontinuous dynamic systems. In other words, this means that discontinuity does not appear in the first  $r - 1$  total time derivatives of the constraint function  $s$ . Then the  $r$ th order sliding set is determined by the equalities

$$s = \dot{s} = \ddot{s} = \dots = s^{(r-1)} = 0 \quad (3.2)$$

Here (3.2) is an  $r$ -dimensional condition on the state of the dynamic system.

**Definition 19** *Let the  $r$ -sliding set (3.2) be non-empty and assume that it is locally an integral set in Filippov's sense (i.e., it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion satisfying (3.2) is called an  $r$ -sliding mode with respect to the constraint function  $s$  (Figure 3.1).*

To exhibit the relation with the previous Definitions, consider a manifold  $S$  given by the equation  $s(x) = 0$ . Suppose that  $s, \dot{s}, \ddot{s}, \dots, s^{(r-2)}$  are differentiable functions of  $x$  and that

$$\text{rank}\{\nabla s, \nabla \dot{s}, \dots, \nabla s^{(r-2)}\} = r - 1 \quad (3.3)$$

holds locally (here  $\text{rank } \mathcal{V}$  is a notation for the rank of vector set  $\mathcal{V}$ ). Then  $S_r$  is determined by (3.2) and all  $S_i, i = 1, \dots, r - 1$  are smooth manifolds. If in its turn  $S_r$  is required to be a differentiable manifold, then the latter condition is extended to

$$\text{rank}\{\nabla s, \nabla \dot{s}, \dots, \nabla s^{(r-1)}\} = r \quad (3.4)$$

Equality (3.4) together with the requirement for the corresponding derivatives of  $s$  to be differentiable functions of  $x$  will be referred to as *the sliding*

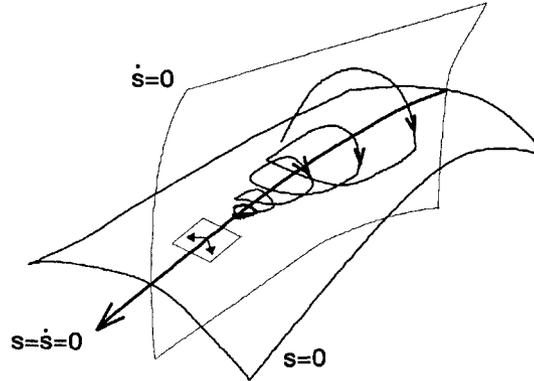


Figure 3.1: Second order sliding mode trajectory

regularity condition, whereas condition (3.3) will be called *the weak sliding regularity condition*.

With the weak regularity condition satisfied and  $S$  given by equation  $s = 0$ , Definition 19 is equivalent to Definition 18. If regularity condition (3.4) holds, then new local coordinates may be taken. In these coordinates the system will take the form

$$\begin{aligned}
 y_1 = s, \quad \dot{y}_1 = y_2; \quad \dots; \quad \dot{y}_{r-1} = y_r \\
 s^{(r)} = \dot{y}_r = \Phi(y, \xi) \\
 \dot{\xi} = \Psi(y, \xi), \quad \xi \in \mathbf{R}^{n-r}
 \end{aligned}$$

**Proposition 20** *Let regularity condition (3.4) be fulfilled and  $r$ -sliding manifold (3.2) be non-empty. Then an  $r$ -sliding mode with respect to the constraint function  $s$  exists if and only if the intersection of the Filippov vector-set field with the tangential space to manifold (3.2) is not empty for any  $r$ -sliding point.*

**Proof.** The intersection of the Filippov set of admissible velocities with the tangential space to the sliding manifold (3.2), mentioned in the Proposition, induces a differential inclusion on this manifold. This inclusion satisfies all the conditions by Filippov [21, 22] for solution existence. Therefore manifold (3.2) is an integral one.

Let  $s$  now be a smooth vector function,  $s : \mathbf{R}^n \rightarrow \mathbf{R}^m$ ,  $s = (s_1, \dots, s_m)$ , and also  $r = (r_1, \dots, r_m)$ , where  $r_i$  are natural numbers.

**Definition 21** Assume that the first  $r_i$  successive full time derivatives of  $s_i$  are smooth functions, and a set given by the equalities

$$s_i = \dot{s}_i = \ddot{s}_i = \dots = s_i^{(r_i-1)} = 0, \quad i = 1, \dots, m$$

is locally an integral set in Filippov's sense. Then the motion mode existing on this set is called a sliding mode with vector sliding order  $r$  with respect to the vector constraint function  $s$ .

The corresponding sliding regularity condition has the form

$$\text{rank}\{\nabla s_1, \dots, \nabla s_i^{(r_i-1)} | i = 1, \dots, m\} = r_1 + \dots + r_m$$

Definition 21 corresponds to Definition 18 in the case when  $r_1 = \dots = r_m$  and the appropriate weak regularity condition holds.

A sliding mode is called *stable* if the corresponding integral sliding set is stable.

### Remarks

1. These definitions also include trivial cases of an integral manifold in a smooth system. To exclude them we may, for example, call a sliding mode "not trivial" if the corresponding Filippov set of admissible velocities  $V(x)$  consists of more than one vector.
2. The above definitions are easily extended to include non-autonomous differential equations by introduction of the fictitious equation  $\dot{t} = 1$ . Note that this differs slightly from the Filippov definition considering time and space coordinates separately.

## 3.3 Higher order sliding modes in control systems

Single out two cases: *ideal sliding* occurring when the constraint is ideally kept and *real sliding* taking place when switching imperfections are taken into account and the constraint is kept only approximately.

### 3.3.1 Ideal sliding

All the previous considerations are translated literally to the case of a process controlled

$$\dot{x} = f(t, x, u), \quad s = s(t, x) \in \mathbb{R}, \quad u = U(t, x) \in \mathbb{R}$$

where  $x \in \mathbb{R}^n$ ,  $t$  is time,  $u$  is control, and  $f$  and  $s$  are smooth functions. Control  $u$  is determined here by a feedback  $u = U(t, x)$ , where  $U$  is a discontinuous function. For simplicity we restrict ourselves to the case when  $s$  and  $u$  are scalars. Nevertheless, all statements below may also be formulated for the case of vector sliding order.

Standard sliding modes satisfy the condition that the set of possible velocities  $V$  does not lie in tangential vector space  $T$  to the manifold  $s = 0$ , but intersects with it, and therefore a trajectory exists on the manifold with the velocity vector lying in  $T$ . Such modes are the main operation modes in variable structure systems [54, 55, 12, 57] and according to the above definitions they are of the first order. When a switching error is present the trajectory leaves the manifold at a certain angle. On the other hand, in the case of second order sliding all possible velocities lie in the tangential space to the manifold, and even when a switching error is present, the state trajectory is tangential to the manifold at the time of leaving.

To see connections with some well-known results of control theory, consider at first the case when

$$\dot{x} = a(x) + b(x)u, \quad s = s(x) \in \mathbb{R}, \quad u \in \mathbb{R}$$

where  $a, b, s$  are smooth vector functions. Let the system have a relative degree  $r$  with respect to the output variable  $s$  [31] which means that Lie derivatives  $L_b s, L_b L_a s, \dots, L_b L_a^{r-2} s$  equal zero identically in a vicinity of a given point and  $L_b L_a^{r-1} s$  is not zero at the point. The equality of the relative degree to  $r$  means, in a simplified way, that  $u$  first appears explicitly only in the  $r$ th total time derivative of  $s$ . It is known that in that case  $s^{(i)} = L_a^i s$  for  $i = 1, \dots, r-1$ , regularity condition (3.4) is satisfied automatically and also  $\frac{\partial}{\partial u} s^{(r)} = L_b L_a^{r-1} s \neq 0$ . There is a direct analogy between the relative degree notion and the sliding regularity condition. Loosely speaking, it may be said that the sliding regularity condition (3.4) means that the "relative degree with respect to discontinuity" is not less than  $r$ . Similarly, the  $r$ th order sliding mode notion is analogous to the zero-dynamics notion [31].

The relative degree notion was originally introduced for the autonomous case only. Nevertheless, we will apply this notion to the non-autonomous case as well. As was done above, we will introduce for the purpose a fictitious variable  $x_{n+1} = t$ ,  $\dot{x}_{n+1} = 1$ . It should be mentioned that some results by Isidori will not be correct in this case, but the facts listed in the previous paragraph will still be true.

Consider a dynamic system of the form

$$\dot{x} = a(t, x) + b(t, x)u, \quad s = s(t, x), \quad u = U(t, x) \in \mathbb{R}$$

**Theorem 22** *Let the system have relative degree  $r$  with respect to the output function  $s$  at some  $r$ -sliding point  $(t_0, x_0)$ . Let, also, the discontinuous function  $U$  take on values from sets  $[K, \infty)$  and  $(-\infty, -K]$  on some sets of non zero measure in any vicinity of any  $r$ -sliding point near point  $(t_0, x_0)$ . Then it provides, with sufficiently large  $K$ , for the existence of  $r$ -sliding mode in some vicinity of point  $(t_0, x_0)$ .  $r$ -sliding motion satisfies the zero-dynamics equations.*

**Proof.** This Theorem is an immediate consequence of Proposition 20, nevertheless, we will detail the proof. Consider some new local coordinates  $y = (y_1, \dots, y_n)$ , where  $y_1 = s, y_2 = \dot{s}, \dots, y_r = s^{(r-1)}$ . In these coordinates manifold  $L_r$  is given by the equalities  $y_1 = y_2 = \dots = y_r = 0$  and the dynamics of the system is as follows:

$$\begin{aligned} \dot{y}_1 &= y_2, \quad \dots, \quad \dot{y}_{r-1} = y_r \\ \dot{y}_r &= h(t, y) + g(t, y)u, \quad g(t, y) \neq 0 \\ \dot{\xi} &= \Psi_1(t, y) + \Psi_2(t, y)u, \quad \xi = (y_{r+1}, \dots, y_n) \end{aligned} \quad (3.5)$$

Denote  $U_{eq} = -h(t, y)/g(t, y)$ . It is obvious that with initial conditions being on the  $r$ -th order sliding manifold  $\mathcal{S}_r$  equivalent control  $u = U_{eq}(t, y)$  provides for keeping the system within manifold  $\mathcal{S}_r$ . It is also easy to see that the substitution of all possible values from  $[-K, K]$  for  $u$  gives us a subset of values from Filippov's set of the possible velocities. Let  $|U_{eq}|$  be less than  $K_0$ , then with  $K > K_0$  the substitution  $u = U_{eq}$  determines Filippov's solution of the discontinuous system which proves the Theorem.

The trivial control algorithm  $u = -K \text{ sign } s$  satisfies Theorem 22. Usually, however, such a mode will not be stable. It follows from the proof above that the equivalent control method [54] is applicable to  $r$ -sliding mode and produces equations coinciding with the zero-dynamics equations for the corresponding system.

The sliding mode order notion [11, 14] seems to be understood in a very close sense (the authors had no opportunity to acquaint themselves with the work by Chang). A number of papers approach the higher order sliding mode technique in a very general way from the differential-algebraic point of view [48, 49, 50, 43]. In these papers so-called "dynamic sliding modes" are not distinguished from the algorithms generating them.

Consider that approach. Let the following equality be fulfilled identically as a consequence of the dynamic system equations [50]:

$$P(s^{(r)}, \dots, \dot{s}, s, x, u^{(k)}, \dots, \dot{u}, u) = 0 \quad (3.6)$$

Equation (3.6) is supposed to be solvable with respect to  $s^{(r)}$  and  $u^{(k)}$ . Function  $s$  may itself depend on  $u$ . The  $r$ th order sliding mode is considered

as a steady state  $s \equiv 0$  to be achieved by a controller satisfying (3.6). In order to achieve for  $s$  some stable dynamics

$$\sigma = s^{(r-1)} + a_1 s^{(r-2)} + \dots + a_{r-1} s = 0$$

the discontinuous dynamic

$$\dot{\sigma} = -\text{sign } \sigma \quad (3.7)$$

is provided. For this purpose the corresponding value of  $s^{(r)}$  is evaluated from (3.7) and substituted into (3.6). The obtained equation is solved for  $u^{(k)}$ .

Thus, a dynamic controller is constituted by the obtained differential equation for  $u$  which has a discontinuous right hand side. With this controller successive derivatives  $s, \dots, s^{(r-1)}$  will be smooth functions of closed system state space variables. The steady state of the resulting system will satisfy at least (3.2) and under some relevant conditions also the regularity requirement (3.4), and therefore Definition 19 will hold.

Hence, it may be said that the relation between our approach and the approach by Sira-Ramírez is a classical relation between geometric and algebraic approaches in mathematics. Note that there are two different sliding modes in system (3.6) and (3.7): a standard sliding mode of the first order which is kept on the manifold  $\sigma = 0$ , and an asymptotically stable  $r$ -sliding mode with respect to the constraint  $s = 0$  which is kept in the points of the  $r$ -sliding manifold  $s = \dot{s} = \ddot{s} = \dots = s^{(r-1)} = 0$ .

### 3.3.2 Real sliding and finite time convergence

Recall that the objective is synthesis of such a control  $u$  that the constraint  $s(t, x) = 0$  holds. The quality of the control design is closely related to the sliding accuracy. In reality, no approach to this design problem provides for ideal keeping of the prescribed constraint. Therefore, there is a need to introduce some means in order to provide a capability for comparison of different controllers.

Any ideal sliding mode should be understood as a limit of motions when switching imperfections vanish and the switching frequency tends to infinity (Filippov [21, 22]). Let  $\varepsilon$  be some measure of these switching imperfections. Then sliding precision of any sliding mode technique may be featured by a sliding precision asymptotics with  $\varepsilon \rightarrow 0$  [35]:

**Definition 23** *Let  $(t, x(t, \varepsilon))$  be a family of trajectories, indexed by  $\varepsilon \in \mathbb{R}^{\mu}$ , with common initial condition  $(t_0, x(t_0))$ , and let  $t \geq t_0$  (or  $t \in [t_0, T]$ ). Assume that there exists  $t_1 \geq t_0$  (or  $t_1 \in [t_0, T]$ ) such that on every segment*

$[t', t'']$ , where  $t' \geq t_1$  (or on  $[t_1, T]$ ), the function  $s(t, x(t, \varepsilon))$  tends uniformly to zero with  $\varepsilon$  tending to zero. In that case we call such a family a *real-sliding family on the constraint  $s = 0$* . We call the motion on the interval  $[t_0, t_1]$  a *transient process*, and the motion on the interval  $[t_1, \infty)$  (or  $[t_1, T]$ ) a *steady state process*.

**Definition 24** A control algorithm, dependent on a parameter  $\varepsilon \in \mathbb{R}^\mu$ , is called a *real-sliding algorithm on the constraint  $s = 0$* , with  $\varepsilon \rightarrow 0$ , if it forms a *real-sliding family for any initial condition*.

**Definition 25** Let  $\gamma(\varepsilon)$  be a real-valued function such that  $\gamma(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . A *real-sliding algorithm on the constraint  $s = 0$*  is said to be of order  $r$  ( $r > 0$ ) with respect to  $\gamma(\varepsilon)$  if for any compact set of initial conditions and for any time interval  $[T_1, T_2]$  there exists a constant  $C$ , such that the steady state process for  $t \in [T_1, T_2]$  satisfies

$$|s(t, x(t, \varepsilon))| \leq C|\gamma(\varepsilon)|^r$$

In the particular case when  $\gamma(\varepsilon)$  is the smallest time interval of control smoothness, the words "with respect to  $\gamma$ " may be omitted. This is the case when real sliding appears as a result of switching discretization.

As follows from [35], with the  $r$ -sliding regularity condition satisfied, in order to get the  $r$ th order of real sliding with discrete switching it is necessary to get at least the  $r$ th order in ideal sliding (provided by infinite switching frequency). Thus, the real sliding order does not exceed the corresponding sliding mode order. The standard sliding modes provide, therefore, for the first-order real sliding only. The second order of real sliding was really achieved by discrete switching modifications of the second-order sliding algorithms [34, 16, 17, 18, 19, 35]. Any arbitrary order of real sliding can be achieved by discretization of the same order sliding algorithms from [38, 39, 41] (see Section 3.7).

Real sliding may also be achieved in a way different from the discrete switching realization of sliding mode. For example, high gain feedback systems [47] constitute real sliding algorithms of the first order with respect to  $k^{-1}$ , where  $k$  is a large gain. A special discrete-switching algorithm providing for the second order real sliding were constructed in [52]. Another example of a second order real sliding controller is the drift algorithm [18, 35]. A third order real-sliding controller exploiting only measurements of  $s$  was recently presented [7].

It is true that in practice the final sliding accuracy is always achieved in finite time. Nevertheless, besides the pure theoretical interest there are also some practical reasons to search for sliding modes attracting in finite time. Consider a system with an  $r$ -sliding mode. Assume that with minimal

switching interval  $\tau$  the maximal  $r$ -th order of real sliding is provided. That means that the corresponding sliding precision  $|s| \sim \tau^r$  is kept, if the  $r$ -th order sliding condition holds at the initial moment. Suppose that the  $r$ -sliding mode in the continuous switching system is asymptotically stable and does not attract the trajectories in finite time. It is reasonable to conclude in that case that with  $\tau \rightarrow 0$  the transient process time for fixed general case initial conditions will tend to infinity. If, for example, the sliding mode were exponentially stable, the transient process time would be proportional to  $r \ln(\tau^{-1})$ . Therefore, it is impossible to observe such an accuracy in practice, if the sliding mode is only asymptotically stable. At the same time, the time of the transient process will not change drastically if it was finite from the very beginning. It should be mentioned, also, that the authors are not aware of a case when a higher real-sliding order is achieved with infinite-time convergence.

### 3.4 Higher order sliding stability in relay systems

In this section we present classical results by Tsytkin [53] (published in Russian in 1956) and Anosov (1959) [1]. They investigated the stability of relay control systems of the form

$$\begin{aligned} \dot{y}_1 &= y_2, \dots, \dot{y}_{l-1} = y_l \\ \dot{y}_l &= \sum_{j=1}^n a_{l,j} y_j + k \operatorname{sign} y_1 \\ \dot{y}_i &= \sum_{j=1}^n a_{i,j} y_j, \quad i = l+1, \dots, n \end{aligned} \quad (3.8)$$

where  $a_{i,j} = \text{const}$ ,  $k \neq 0$ , and  $y_1 = y_2 = \dots = y_l = 0$  is the  $l$ -th order sliding set. The main result is as follows:

- for stability of equilibrium point of relay control system (3.8) with second order sliding ( $l = 2$ ), three main cases are singled out: exponentially stable, stable, and unstable;
- it is shown that the equilibrium point of the system (3.8) is always unstable with  $l \geq 3$ . Consequently, all higher order sliding modes arriving in the relay control systems are unstable with an order of sliding more than 2.

Consider the ideas of the proof.

### 3.4.1 2-sliding stability in relay systems

Consider a simple example of a second-order dynamic system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = ay_1 + by_2 + k \operatorname{sign} y_1 \quad (3.9)$$

The 2-sliding set is given here by  $y_1 = y_2 = 0$ . At first, let  $k < 0$ . Consider the Lyapunov function

$$E = \frac{y_2^2}{2} - a\frac{y_1^2}{2} + |k||y_1| - \frac{b}{2}y_1y_2 \quad (3.10)$$

Function  $E$  is an energy integral of system (3.9) Computing the derivative of function  $E$ , we achieve

$$\dot{E} = \frac{b}{2}y_2^2 + \frac{b}{2}|y_1|(|k| - a|y_1| - by_2 \operatorname{sign} y_1)$$

It is obvious that for some positive  $\alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2$

$$\alpha_1|y_1| + \beta_1y_2^2 \leq E \leq \alpha_2|y_1| + \beta_2y_2^2$$

Thus, the inequalities  $-\gamma_2E \leq \dot{E} \leq -\gamma_1E$  or  $\gamma_1E \leq \dot{E} \leq \gamma_2E$  hold for  $b < 0$  or  $b > 0$ , respectively, in a small vicinity of the origin with some  $\gamma_2 \geq \gamma_1 > 0$ .

Now let  $k > 0$ . It is easy to see in that case that the trajectories cannot leave the set  $y_1 > 0, y_2 = \dot{y}_1 > 0$  if  $a \geq 0$ . The same is true with  $y_1 < k/|a|$  if  $a < 0$ . Starting with an infinitesimally small  $y_1 > 0, y_2 > 0$ , any trajectory inevitably leaves some fixed origin vicinity.

It allows three main cases to be singled out for investigation of the stability of the system (3.9).

- *Exponentially stable case.* Under the conditions

$$b < 0, \quad k < 0 \quad (3.11)$$

the equilibrium point  $y_1 = y_2 = 0$  is exponentially stable.

- *Unstable case.* Under the condition

$$k > 0 \quad \text{or} \quad b > 0$$

the equilibrium point  $y_1 = y_2 = 0$  is unstable.

- *Critical case.*

$$k \leq 0, \quad b \leq 0, \quad bk = 0$$

With  $b = 0, k < 0$  the equilibrium point  $y_1 = y_2 = 0$  is stable.

It is easy to show that if the matrix  $A$  consisting of  $a_{i,j}, i, j > 2$  is Hurvitz and conditions  $a_{2,2} < 0, k < 0$  are true, then the equilibrium point of system (3.8) is exponentially stable.

### 3.4.2 Relay system instability with sliding order more than 2

Let us illustrate the idea of the proof on an example of a simple third-order system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = a_{31}y_1 + a_{32}y_2 + a_{33}y_3 - k \operatorname{sign} y_1, \quad k > 0 \quad (3.12)$$

Consider the Lyapunov function<sup>1</sup>

$$V = y_1y_3 - \frac{1}{2}y_2^2$$

Thus,

$$\dot{V} = -k|y_1| + y_1(a_{31}y_1 + a_{32}y_2 + a_{33}y_3)$$

and  $\dot{V}$  is negative at least in a small neighborhood of origin  $(0, 0, 0)$ . That means that the zero solution of system (3.12) is unstable.

On the other hand, in relay control systems with an order of sliding more than 2, a stable periodic solution can occur [46, 32].

## 3.5 Sliding order and dynamic actuators

Let the constraint be given by the equality of some constraint function  $s$  to zero and let the sliding mode  $s \equiv 0$  be provided by a relay control. Taking into account an actuator conducting a control signal to the process controlled, we achieve more complicated dynamics. In that case the relay control  $u$  enters the actuator and continuous output variables of the actuator  $z$  are transmitted to the plant input (Figure 3.2). As a result, discontinuous switching is hidden now in the higher derivatives of the constraint function [55, 23, 24, 25, 26, 27, 9].

### 3.5.1 Stability of 2-sliding modes in systems with fast actuators

Condition (3.11) is used in [25, 26, 27, 9] for analysis of sliding mode systems with fast dynamic actuators. Here is a simple outline of these reasonings. One of the actuator output variables is formally replaced by  $\dot{s}$  after application of some coordinate transformation. Let the system under consideration be rewritten in the following form:

$$\begin{aligned} \mu \dot{z} &= Az + B\eta + D_1x \\ \mu \dot{\eta} &= Cz + b\eta + D_2x + k \operatorname{sign} s \\ \dot{s} &= \eta \\ \dot{x} &= F(z, \eta, s, x) \end{aligned} \quad (3.13)$$

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<sup>1</sup>This function was suggested by V.I. Utkin in private communications.

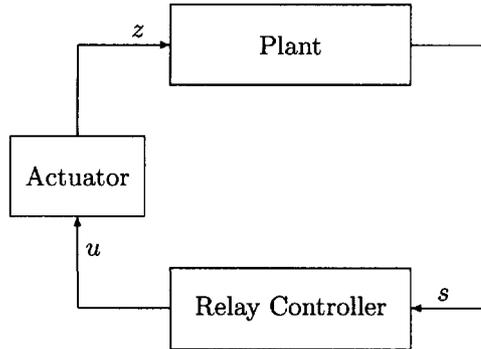


Figure 3.2: Control system with actuator

where  $z \in \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$ ,  $\eta$  and  $s \in \mathbb{R}$ .

With (3.11) fulfilled and  $\text{ReSpec } A < 0$ , system (3.13) has an exponentially stable integral manifold of slow motions which is a subset of the second-order sliding manifold and given by the equations

$$z = H(\mu, x) = -A^{-1}D_1x + O(\mu), \quad s = \eta = 0.$$

Function  $H$  may be evaluated with any desired precision with respect to the small parameter  $\mu$ .

Therefore, according to [25, 26, 27, 9], under the conditions

$$\text{ReSpec } A < 0, \quad b < 0, \quad k < 0. \quad (3.14)$$

the motions in such a system with a fast actuator of relative degree 1 consists of fast oscillations, vanishing exponentially, and slow motions on a submanifold of the second-order sliding manifold.

Thus, if conditions (3.14) of chattering absence hold, the presence of a fast actuator of relative degree 1 does not lead to chattering in sliding mode control systems.

### Remark

The stability of the fast actuator and of the second-order sliding mode in (3.13) still does not guarantee absence of chattering if  $\dim z > 0$  and  $\frac{\partial F}{\partial z} \neq 0$ , for in that case fast oscillations may still remain in the 2-sliding mode itself. Indeed, the stability of a fast actuator corresponds to the stability of the fast actuator matrix

$$\text{Re Spec} \begin{pmatrix} A & B \\ C & b \end{pmatrix} < 0$$

Consider the system

$$\begin{aligned} \mu \dot{z}_1 &= z_1 + z_2 + \eta + D_1 x \\ \mu \dot{z}_2 &= 2z_2 + z_3 + D_2 x \\ \mu \dot{\eta} &= 24z_1 - 60z_2 - 9\eta + D_3 x + k \text{ sign } s \\ \dot{s} &= \eta \\ \dot{x} &= F(z_1, z_2, \eta, s, x) \end{aligned}$$

where  $z_1, z_2, \eta, s$  are scalars. It is easy to check that the spectrum of the matrix is  $\{-1, -2, -3\}$  and condition (3.11) holds for this system. On the other hand the motions in the second-order sliding mode are described by the system

$$\begin{aligned} \mu \dot{z}_1 &= z_1 + z_2 + D_1 x \\ \mu \dot{z}_2 &= 2z_2 + D_2 x \\ \dot{x} &= F(z_1, z_2, 0, 0, x) \end{aligned}$$

The fast motions in this system are unstable and the absence of chattering in the original system cannot be guaranteed.

### Example

Without loss of generality we illustrate the approach by some simple examples. Consider, for instance, sliding mode usage for the tracking purpose. Let the process be described by the equation  $\dot{x} = u$ ,  $x, u \in \mathbb{R}$ , and the sliding variable be

$$s = x - f(t), \quad f : \mathbb{R} \rightarrow \mathbb{R}$$

so that the problem is to track a signal  $f(t)$  given in real time, where  $|f|, |\dot{f}|, |\ddot{f}| < 0.5$ . Only values of  $x, f, u$  are available.

The problem is successfully solved by the controller  $u = -\text{sign } s$ ,

keeping  $s = 0$  in a 1-sliding mode. In practice, however, there is always some actuator between the plant and the controller, which inserts some additional dynamics and removes the discontinuity from the real system. With respect to Figure 3.2 let the system be described by the equation

$$\dot{x} = v$$

where  $v \in \mathbb{R}$  is an output of some dynamic actuator. Assume that the actuator has some fast first order dynamics. For example

$$\mu \dot{v} = u - v$$

The input  $u$  of the actuator is the relay control

$$u = -\text{sign } s$$

where  $\mu$  is a small positive number. The second order sliding manifold  $\mathcal{S}_2$  is given here by the equations

$$s = x - f(t) = 0, \quad \dot{s} = v - \dot{f}(t) = 0$$

The equality

$$\ddot{s} = \frac{1}{\mu}(u - v) - \ddot{f}(t)$$

shows that the relative degree here equals 2 and, according to Theorem 22, a 2-sliding mode exists, provided  $\mu < 1$ . The motion in this mode is described by the equivalent control method or by zero-dynamics, which is the same: from  $s = \dot{s} = \ddot{s} = 0$  we achieve  $u = \mu\ddot{f}(t) + v$ ,  $v = \dot{f}(t)$  and therefore

$$x = f(t), \quad v = \dot{f}(t), \quad u = \mu\ddot{f}(t) + v$$

It is easy to prove that the 2-sliding mode is stable here with  $\mu$  small enough. Note that the latter equality describes the equivalent control [54, 55] and is kept actually only in the average, while the former two are kept accurately in the 2-sliding mode.

Let

$$f(t) = 0.08 \sin t + 0.12 \cos 0.3t, \quad x(0) = 0, \quad v(0) = 0$$

The plots of  $x(t)$  and  $f(t)$  with  $\mu = 0.2$  are shown in Figure 3.3, whereas the plot of  $v(t)$  is demonstrated in Figure 3.4.

### 3.5.2 Systems with fast actuators of relative degree 3 and higher

The equilibrium point of any relay system with relative degree  $\geq 3$  is always unstable [1, 53] (Section 3.4.2). That leads to an important conclusion: even being stable, higher order actuators do not suppress chattering in the closed-loop relay systems. For investigation of chattering phenomena in such systems, the averaging technique was used [25, 29]. Higher-order actuators may give rise to high-frequency periodic solutions. The general model of sliding mode control systems with fast actuators has the form [10]

$$\begin{aligned} \dot{x} &= h(x, s, \eta, z, u(s)), \quad \dot{s} = \eta \\ \mu\dot{\eta} &= g_2(x, s, \eta, z), \quad \mu\dot{z} = g_1[x, s, \eta, z, u(s)] \end{aligned} \quad (3.15)$$

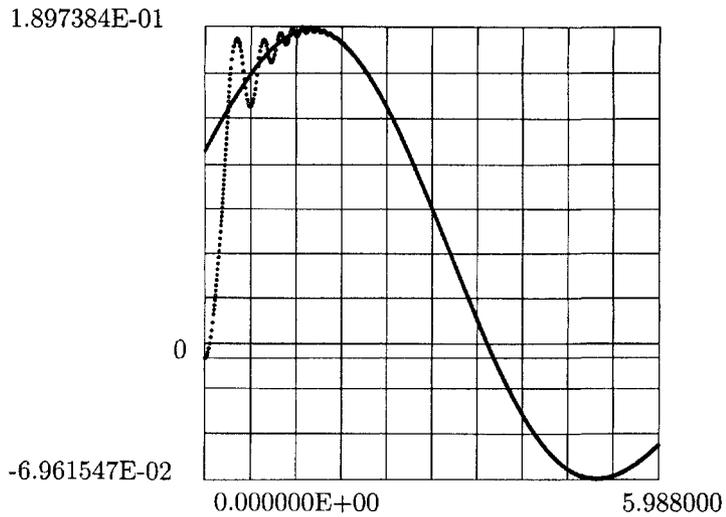


Figure 3.3: Asymptotically stable second-order sliding mode in a system with a fast actuator. Tracking:  $x(t)$  and  $f(t)$

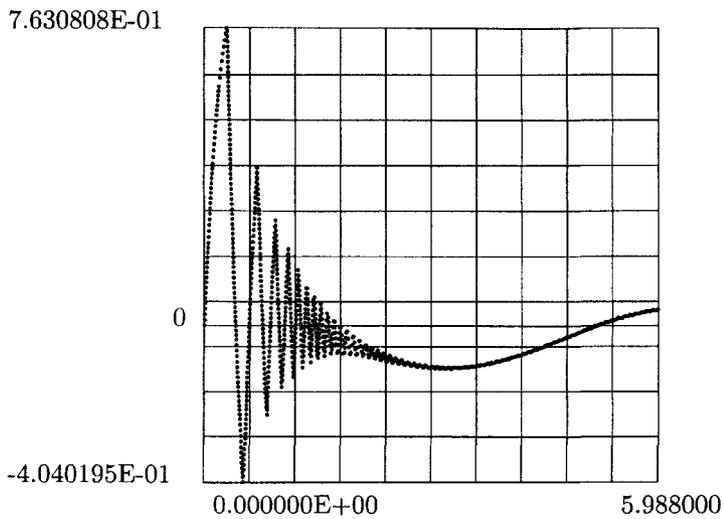


Figure 3.4: Asymptotically stable second-order sliding mode in a system with a fast actuator: actuator output  $v(t)$

where  $z \in \mathbf{R}^m$ ,  $\eta, s \in \mathbf{R}$ ,  $x \in X \subset \mathbf{R}^n$ ,  $u(s) = \text{sign}s$ , and  $g_1, g_2, h$  are smooth functions of their arguments. Variables  $s$  and  $x$  may be considered as the state coordinates of the plant.  $\eta, z$  are the fast-actuator coordinates, and  $\mu$  being the actuator time constant.

Suppose that following conditions are true:

1. The fast-motion system

$$\frac{ds}{d\tau} = \eta, \quad \frac{d\eta}{d\tau} = g_2(x, 0, \eta, z), \quad \frac{dz}{d\tau} = g_1[x, 0, \eta, z, u(s)] \quad (3.16)$$

has a  $T(x)$ -periodic solution  $(s_0(\tau, x), \eta_0(\tau, x), z_0(\tau, x))$  for any  $x \in X$ . System (3.16) generates a point mapping  $\Psi(x, \eta, z)$  of the switching surface  $s = 0$  into itself which has a fixed point  $(\eta^*(x), z^*(x))$ ,  $\Psi(x, \eta^*(x), z^*(x)) = (\eta^*(x), z^*(x))$ . Moreover, the Frechet derivative of  $\Psi(x, \eta, z)$  with respect to variables  $(\eta, z)$  calculated at  $(\eta^*(x), z^*(x))$  is a contractive matrix for any  $x \in \bar{X}$ .

2. The averaged system

$$\dot{x} = \bar{h}(x) = \frac{1}{T(x)} \int_0^{T(x)} h\{x, 0, \eta_0(\tau, x), z_0(\tau, x), u[s_0(\tau, x)]\} d\tau \quad (3.17)$$

has an unique equilibrium point  $x = x_0$ . This equilibrium point is exponentially stable.

**Theorem 26** [29]. *Under conditions 1 and 2 system (3.15) has an isolated orbitally asymptotically stable periodic solution with the period  $\mu(T(x_0) + O(\mu))$  near the closed curve*

$$(x_0, \mu s_0(t/\mu, x_0), \eta_0(t/\mu, x_0), z_0(t/\mu, x_0))$$

### Example

Consider a mathematical model of a control system with actuator and the overall relative degree 3

$$\dot{x} = -x - u, \quad \dot{s} = z_1 \quad (3.18)$$

$$\mu \dot{z}_1 = z_2, \quad \mu \dot{z}_2 = -2z_1 - 3z_2 - u \quad (3.19)$$

Here  $z_1, z_2, s, x \in \mathbf{R}$ ,  $u(s) = \text{sign}s$ ,  $\mu$  is the actuator time constant. The fast motions taking place in (3.18), (3.19) are described by the system

$$\frac{d\xi}{d\tau} = z_1, \quad \frac{dz_1}{d\tau} = z_2,$$

$$\frac{dz_2}{d\tau} = -2z_1 - 3z_2 - u, \quad u = \text{sign } \xi \quad (3.20)$$

Then the solution of system (3.20) for  $\xi > 0$  with initial condition  $\xi(0) = 0$ ,  $z_1(0) = z_{10}$ ,  $z_2(0) = z_{20}$  is as follows

$$\begin{aligned}\xi(\tau) &= \frac{3}{2}z_{10} - 2z_{10}e^{-\tau} + \frac{1}{2}z_{10}e^{-2\tau} + \frac{1}{2}z_{20} - z_{20}e^{-\tau} + \frac{1}{2}z_{20}e^{-2\tau} \\ &\quad - \frac{1}{2}\tau + \frac{3}{4} - e^{-\tau} + \frac{1}{4}e^{-2\tau} \\ z_1(\tau) &= 2z_{10}e^{-\tau} - z_{10}e^{-2\tau} + z_{20}e^{-\tau} - z_{20}e^{-2\tau} - \frac{1}{2} + e^{-\tau} - \frac{1}{2}e^{-2\tau} \\ z_2(\tau) &= 2z_{10}e^{-2\tau} - 2z_{10}e^{-\tau} - z_{20}e^{-\tau} + 2z_{20}e^{-2\tau} - e^{-\tau} + e^{-2\tau}\end{aligned}$$

Consider the point mapping  $\Xi(z_1, z_2)$  of the domain  $z_1 > 0$ ,  $z_2 > 0$  on the switching surface  $\xi = 0$  into the domain  $z_1 < 0$ ,  $z_2 < 0$  with  $\text{sign } \xi > 0$  made by system (3.20). Then

$$\begin{aligned}\Xi(z_1, z_2) &= (\Xi_1(z_1, z_2), \Xi_2(z_1, z_2)) \\ \Xi_1(z_1, z_2) &= 2z_1e^{-T} - z_1e^{-2T} + z_2e^{-T} - z_2e^{-2T} - \frac{1}{2} + e^{-T} - \frac{1}{2}e^{-2T} \\ \Xi_2(z_1, z_2) &= 2z_1e^{-2T} - 2z_1e^{-T} - z_2e^{-T} + 2z_2e^{-2T} - e^{-T} + e^{-2T}\end{aligned}$$

where  $T(z_1, z_2)$  is the smallest root of equation

$$\begin{aligned}\xi(T(z_1, z_2)) &= \frac{3}{2}z_1 - 2z_1e^{-T} + \frac{1}{2}z_1e^{-2T} + \frac{1}{2}z_2 - z_2e^{-T} \\ &\quad + \frac{1}{2}z_2e^{-2T} - \frac{1}{2}T + \frac{3}{4} - e^{-T} + \frac{1}{4}e^{-2T} = 0\end{aligned}$$

System (3.20) is symmetric with respect to the point  $\xi = z_1 = z_2 = 0$ . Thus, the initial condition  $(0, z_1^*, z_2^*)$  and the semi-period  $T^* = T(z_1^*, z_2^*)$  for the periodic solution of (3.20) are determined by the equations  $\Xi(z_1^*, z_2^*) = -(z_1^*, z_2^*)$  and  $\xi(T(z_1^*, z_2^*)) = 0$ , and consequently

$$\begin{aligned}\frac{3}{2}z_1^* - 2z_1^*e^{-T^*} + \frac{1}{2}z_1^*e^{-2T^*} + \frac{1}{2}z_2^* - z_2^*e^{-T^*} + \frac{1}{2}z_2^*e^{-2T^*} \\ - \frac{1}{2}T^* + \frac{3}{4} - e^{-T^*} + \frac{1}{4}e^{-2T^*} = 0 \\ 2z_1^*e^{-T^*} - z_1^*e^{-2T^*} + z_2^*e^{-T^*} - z_2^*e^{-2T^*} - \frac{1}{2} + e^{-T^*} - \frac{1}{2}e^{-2T^*} = -z_1^* \\ 2z_1^*e^{-2T^*} - 2z_1^*e^{-T^*} - z_2^*e^{-T^*} + 2z_2^*e^{-2T^*} - e^{-T^*} + e^{-2T^*} = -z_2^*\end{aligned}\quad (3.21)$$

Expressing  $z_1^*$ ,  $z_2^*$  from the latter two equations of (3.21), we achieve

$$T^*(e^{T^*} + e^{3T^*} + 1 + e^{2T^*}) - 5e^{T^*} - 3e^{3T^*} + 3 + 5e^{2T^*} = 0. \quad (3.22)$$

Equations (3.22) and (3.21) have positive solution

$$T^* \approx 2.2755, z_1^* \approx 0.3241, z_2^* \approx 0.1654$$

corresponding to the existence of a  $2T^*$ -periodic solution in system (3.20).

Thus

$$\left( \frac{\partial T}{\partial z_1}, \frac{\partial T}{\partial z_2} \right) = \left( -\frac{\frac{3}{2} - 2e^{-T} + \frac{1}{2}e^{-2T}}{(2z_1 + z_2 + 1)e^{-T} - (z_1 + z_2 + \frac{1}{2})e^{-2T} - \frac{1}{2}}, \frac{\frac{1}{2} - e^{-T} + \frac{1}{2}e^{-2T}}{(2z_1 + z_2 + 1)e^{-T} - (z_1 + z_2 + \frac{1}{2})e^{-2T} - \frac{1}{2}} \right)$$

and

$$\frac{\partial \Xi_1}{\partial z_1} = 2e^{-T} - e^{-2T} + [e^{-2T}(2z_1 + 2z_2 + 1) - e^{-T}(2z_1 + z_2 + 1)] \frac{\partial T}{\partial z_1}$$

$$\frac{\partial \Xi_1}{\partial z_2} = e^{-T} - e^{-2T} + [e^{-2T}(2z_1 + 2z_2 + 1) - e^{-T}(2z_1 + z_2 + 1)] \frac{\partial T}{\partial z_2}$$

$$\frac{\partial \Xi_2}{\partial z_1} = 2e^{-2T} - 2e^{-T} - [(e^{-T}(2z_1 + z_2 + 1) - 2e^{-2T}(2z_1 + 2z_2 + 1))] \frac{\partial T}{\partial z_1}$$

$$\frac{\partial \Xi_2}{\partial z_2} = 2e^{-2T} - e^{-T} - [(e^{-T}(2z_1 + z_2 + 1) - 2e^{-2T}(2z_1 + 2z_2 + 1))] \frac{\partial T}{\partial z_2}$$

Calculating the value of Frechet derivative  $\frac{\partial \Xi}{\partial z}$  at  $(z_1^*, z_2^*)$ , using the found value of  $T^*$ , achieve

$$\frac{\partial \Xi}{\partial z}(z_1^*, z_2^*) = J = \begin{bmatrix} -0.4686 & -0.1133 \\ 0.3954 & 0.0979 \end{bmatrix}$$

The eigenvalues of matrix  $J$  are  $-0.3736$  and  $0.0029$ . That implies existence and asymptotic stability of the periodic solution of (3.20). The averaged equation for system (3.18) and (3.19) is

$$\dot{x} = -x$$

and it has the asymptotically stable equilibrium point  $x = 0$ . Hence, system (3.18) and (3.19) has an orbitally asymptotically stable periodic solution which lies in the  $O(\mu)$ -neighborhood of the switching surface.

## 3.6 2-sliding controllers

We follow here [36, 35, 6].

### 3.6.1 2-sliding dynamics

Return to the system

$$\dot{x} = f(t, x, u), \quad s = s(t, x) \in \mathbb{R}, \quad u = U(t, x) \in \mathbb{R} \quad (3.23)$$

where  $x \in \mathbf{R}^n$ ,  $t$  is time,  $u$  is control, and  $f, s$  are smooth functions. The control task is to keep output  $s \equiv 0$ . Differentiating successively the output variable  $s$ , we achieve functions  $\dot{s}, \ddot{s}, \dots$ . Depending on the relative degree [31] of the system, different cases should be considered

- a) relative degree  $r = 1$ , i.e.,  $\frac{\partial}{\partial u} \dot{s} \neq 0$
- b) relative degree  $r \geq 2$ , i.e.,  $\frac{\partial}{\partial u} s^{(i)} = 0$  ( $i = 1, 2, \dots, r - 1$ ),  $\frac{\partial}{\partial u} s^{(r)} \neq 0$

In case a) the classical VSS approach solves the control problem by means of 1-sliding mode control, nevertheless 2-sliding mode control can also be used in order to avoid chattering. For that purpose  $u$  will become an output of some first-order dynamic system [35]. For example, the time derivative of the plant control  $\dot{u}(t)$  may be considered as the actual control variable. A discontinuous control  $\dot{u}$  steers the sliding variable  $s$  to zero, keeping  $s = 0$  in a 2-sliding mode, so that the plant control  $u$  is continuous and the chattering is avoided [35, 5]. In case b) the  $p$ -sliding mode approach (with  $p \geq r$ ) is the control technique of choice.

#### Chattering avoidance: the generalized constraint fulfillment problem

When considering classical VSS the control variable  $u(t)$  is a feedback-designed relay output. The most direct application of 2-sliding mode control is that of attaining sliding motion on the sliding manifold by means of a continuous bounded input  $u(t)$  being a continuous output of a suitable first-order dynamic system driven by a proper discontinuous signal. Such first-order dynamics can be either inherent to the control device or specially introduced for chattering elimination purposes.

Assume that  $f$  and  $s$  are respectively  $\mathcal{C}^1$  and  $\mathcal{C}^2$  functions, and that the only available current information consists of the current values of  $t, u(t), s(t, x)$  and, possibly, of the sign of the time derivative of  $s$ . Differentiating the sliding variable  $s$  twice, the following relations are derived:

$$\dot{s} = \frac{\partial}{\partial t} s(t, x) + \frac{\partial}{\partial x} s(t, x) f(t, x, u) \quad (3.24)$$

$$\ddot{s}(t) = \frac{\partial}{\partial t} \dot{s}(t, x, u) + \frac{\partial}{\partial x} \dot{s}(t, x, u) f(t, x, u) + \frac{\partial}{\partial u} \dot{s}(t, x, u) \dot{u}(t) \quad (3.25)$$

The control goal for a 2-sliding mode controller is that of steering  $s$  to zero in finite time by means of control  $u(t)$  continuously dependent on time. In order to state a rigorous control problem, the following conditions are assumed:

- 1) Control values belong to the set  $\mathcal{U} = \{u : |u| \leq U_M\}$ , where  $U_M > 1$  is a real constant; furthermore the solution of the system is well defined for all  $t$ , provided  $u(t)$  is continuous and  $\forall t \ u(t) \in \mathcal{U}$ .
- 2) There exists  $u_1 \in (0, 1)$  such that for any continuous function  $u(t)$  with  $|u(t)| > u_1$ , there is  $t_1$ , such that  $s(t)u(t) > 0$  for each  $t > t_1$ . Hence, the control  $u(t) = -\text{sign}(s(t_0))$ , where  $t_0$  is the initial value of time, provides hitting the manifold  $s = 0$  in finite time.
- 3) Let  $\dot{s}(t, x, u)$  be the total time derivative of the sliding variable  $s(t, x)$ . There are positive constants  $s_0, u_0 < 1, \Gamma_m, \Gamma_M$  such that if  $|s(t, x)| < s_0$  then

$$0 < \Gamma_m \leq \frac{\partial}{\partial u} \dot{s}(t, x, u) \leq \Gamma_M, \quad \forall u \in \mathcal{U}, x \in \mathcal{X} \quad (3.26)$$

and the inequality  $|u| > u_0$  entails  $\dot{s}u > 0$ .

- 4) There is a positive constant  $\Phi$  such that within the region  $|s| < s_0$  the following inequality holds  $\forall t, x \in \mathcal{X}, u \in \mathcal{U}$

$$\left| \frac{\partial}{\partial t} \dot{s}(t, x, u) + \frac{\partial}{\partial x} \dot{s}(t, x, u) f(t, x, u) \right| \leq \Phi \quad (3.27)$$

The above condition 2 means that starting from any point of the state space it is possible to define a proper control  $u(t)$  steering the sliding variable within a set such that the boundedness conditions on the sliding dynamics defined by conditions 3 and 4 are satisfied. In particular they state that the second time derivative of the sliding variable  $s$ , evaluated with fixed values of the control  $u$ , is uniformly bounded in a bounded domain.

It follows from the theorem on implicit function that there is a function  $u_{eq}(t, x)$  which can be considered as equivalent control [55], satisfying the equation  $\dot{s} = 0$ . Once  $s = 0$  is attained, the control  $u = u_{eq}(t, x)$  would provide for the exact constraint fulfillment. Conditions 3 and 4 mean that  $|s| < s_0$  implies  $|u_{eq}| < u_0 < 1$ , and that the velocity of the  $u_{eq}$  changing is bounded. This provides for a possibility to approximate  $u_{eq}$  by a Lipschitzian control.

The unit upper bound for  $u_0$  and  $u_1$  is actually a scaling factor. Note also that linear dependence on control  $u$  is not required here. The usual form of the uncertain systems dealt with by the VSS theory, i.e., systems affine in  $u$  and possibly in  $x$ , are a special case of the considered system and the corresponding constraint fulfillment problem may be reduced to the considered one [35, 20].

*Relative degree two.* In case of relative degree two the control problem statement could be derived from the above by considering the variable  $u$  as a state variable and  $\dot{u}$  as the actual control. Indeed, let the controlled system be

$$f(t, x, u) = a(t, x) + b(t, x)u(t) \quad (3.28)$$

where  $a : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  and  $b : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  are sufficiently smooth uncertain vector functions,  $[\frac{\partial}{\partial x}s(t, x)]b(t, x) \equiv 0$ . Calculating, we find that

$$\ddot{s} = \varphi(t, x) + \gamma(t, x)u \quad (3.29)$$

It is assumed that  $|\varphi| \leq \Phi, 0 < \Gamma_m \leq \gamma \leq \Gamma_M, \Phi > 0$

Thus in a small vicinity of the manifold  $s = 0$  the system is described by (3.28), (3.29) if the relative degree is 2, or by (3.23) and

$$\ddot{s} = \varphi(t, x) + \gamma(t, x)\dot{u} \quad (3.30)$$

if the relative degree is 1.

### 3.6.2 Twisting algorithm

Let relative degree be 1. Consider local coordinates  $y_1 = s$  and  $y_2 = \dot{s}$ , then after a proper initialization phase, the second order sliding mode control problem is equivalent to the finite time stabilization problem for the uncertain second-order system with  $|\varphi| \leq \Phi, 0 < \Gamma_m \leq \gamma \leq \Gamma_M, \Phi > 0$ .

$$\begin{cases} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \varphi(t, x) + \gamma(t, x)\dot{u} \end{cases} \quad (3.31)$$

with  $y_2(t)$  immeasurable but with a possibly known sign, and  $\varphi(t, x)$  and  $\gamma(t, x)$  uncertain functions with

$$\Phi > 0, |\varphi| \leq \Phi, 0 < \Gamma_m \leq \gamma \leq \Gamma_M \quad (3.32)$$

Being historically the first known 2-sliding controller [34], that algorithm features twisting around the origin of the 2-sliding plane  $y_1 O y_2$  (Figure 3.5). The trajectories perform an infinite number of rotations while converging in finite time to the origin. The vibration magnitudes along the axes as well as the rotation times decrease in geometric progression. The control

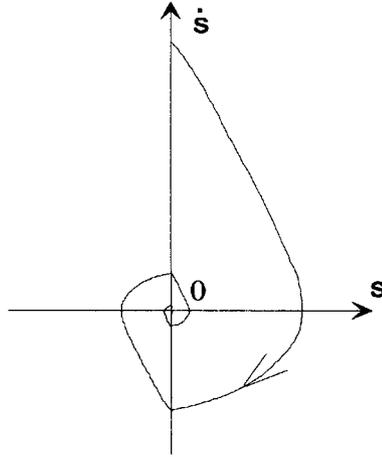


Figure 3.5: Twisting algorithm phase trajectory

derivative value commutes at each axis crossing, which requires availability of the sign of the sliding-variable time-derivative  $y_2$ .

The control algorithm is defined by the following control law [34, 35, 17, 20], in which the condition on  $|u|$  provides for  $|u| \leq 1$  :

$$\dot{u}(t) = \begin{cases} -u & \text{if } |u| > 1 \\ -V_m \text{sign}(y_1) & \text{if } y_1 y_2 \leq 0; |u| \leq 1 \\ -V_M \text{sign}(y_1) & \text{if } y_1 y_2 > 0; |u| \leq 1 \end{cases} \quad (3.33)$$

The corresponding sufficient conditions for the finite time convergence to the sliding manifold are [35]

$$\begin{aligned} V_M &> V_m \\ V_m &> \frac{4\Gamma_M}{\Gamma_m} \\ V_m &> \frac{\Phi^0}{\Gamma_m} \\ \Gamma_m V_M - \Phi &> \Gamma_M V_m + \Phi \end{aligned} \quad (3.34)$$

The similar controller

$$u(t) = \begin{cases} -V_m \text{sign}(y_1) & \text{if } y_1 y_2 \leq 0 \\ -V_M \text{sign}(y_1) & \text{if } y_1 y_2 > 0 \end{cases}$$

is to be used in order to control system (3.28) when the relative degree is 2.

By taking into account the different limit trajectories arising from the uncertain dynamics of (3.29) and evaluating time intervals between successive crossings of the abscissa axis, it is possible to define the following

upper bound for the convergence time [6]

$$t_{tw\infty} \leq t_{M_1} + \Theta_{tw} \frac{1}{1 - \theta_{tw}} \sqrt{|y_{1M_1}|} \quad (3.35)$$

Here  $y_{1M_1}$  is the value of the  $y_1$  variable at the first abscissa crossing in the  $y_1Oy_2$  plane,  $t_{M_1}$  is the corresponding time instant and

$$\begin{aligned} \Theta_{tw} &= \sqrt{2} \frac{\Gamma_m V_M + \Gamma_M V_m}{(\Gamma_m V_M - \Phi) \sqrt{\Gamma_M V_m + \Phi}} \\ \theta_{tw} &= \sqrt{\frac{\Gamma_M V_m + \Phi}{\Gamma_m V_M - \Phi}} \end{aligned}$$

In practice when  $y_2$  is immeasurable, its sign can be estimated by the sign of the first difference of the available sliding variable  $y_1$  in a time interval  $\tau$ , i.e.,  $\text{sign}(y_2(t))$  is estimated by  $\text{sign}(y_1(t) - y_1(t - \tau))$ . In that case the 2-sliding precision with respect to the measurement time interval is provided, and the size of the boundary layer of the sliding manifold is  $\Delta \sim \mathcal{O}(\tau^2)$  [35]. Recall that it is the best possible accuracy asymptotics with discontinuous  $\dot{y}_2 = \dot{s}$ .

### 3.6.3 Sub-optimal algorithm

That 2-sliding controller was developed as a sub-optimal feedback implementation of a classical time-optimal control for a double integrator. Let the relative degree be 2. The auxiliary system is

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \varphi(t, x) + \gamma(t, x)u \end{cases} \quad (3.36)$$

The trajectories on the  $y_1Oy_2$  plane are confined within limit parabolic arcs which include the origin, so that both twisting and leaping (when  $y_1$  and  $y_2$  do not change sign) behaviors are possible (Figure3.6). Also here the coordinates of the trajectory intersections with axis  $y_1$  decrease in geometric progression. After an initialization phase the algorithm is defined by the following control law [4, 5, 6]:

$$\begin{aligned} v(t) &= -\alpha(t)V_M \text{sign}(y_1(t) - \frac{1}{2}y_{1M}) \\ \alpha(t) &= \begin{cases} \alpha^* & \text{if } [y_1(t) - \frac{1}{2}y_{1M}][y_{1M} - y_1(t)] > 0 \\ 1 & \text{if } [y_1(t) - \frac{1}{2}y_{1M}][y_{1M} - y_1(t)] \leq 0 \end{cases} \end{aligned} \quad (3.37)$$

where  $y_{1M}$  is the latter singular value of the function  $y_1(t)$ , i.e. the latter value corresponding to the zero value of  $y_2 = \dot{y}_1$ . The corresponding sufficient conditions for the finite-time convergence to the sliding manifold are as follows [4]:

$$\begin{aligned} \alpha^* &\in (0, 1] \cap (0, \frac{3\Gamma_m}{\Gamma_M}) \\ V_M &> \max\left(\frac{\Phi}{\alpha^* \Gamma_m}, \frac{4\Phi}{3\Gamma_m - \alpha^* \Gamma_M}\right) \end{aligned} \quad (3.38)$$

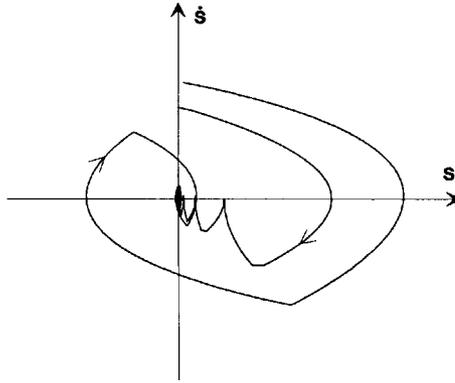


Figure 3.6: Sub-optimal algorithm phase trajectories

Also in that case an upper bound for the convergence time can be determined [4]

$$t_{opt\infty} \leq t_{M_1} + \Theta_{opt} \frac{1}{1 - \theta_{opt}} \sqrt{|y_{1M_1}|} \quad (3.39)$$

Here  $y_{1M_1}$  and  $t_{M_1}$  are defined as for the twisting algorithm, and

$$\Theta_{opt} = \frac{(\Gamma_m + \alpha^* \Gamma_M) V_M}{(\Gamma_m V_M - \Phi) \sqrt{\alpha^* \Gamma_M V_M + \Phi}}$$

$$\theta_{opt} = \sqrt{\frac{(\alpha^* \Gamma_M - \Gamma_m) V_M + 2\Phi}{2(\Gamma_m V_M - \Phi)}}$$

The effectiveness of the above algorithm was extended to larger classes of uncertain systems [6]. It was proved [5] that in case of unit gain function the control law (3.37) can be simplified by setting  $\alpha = 1$  and choosing  $V_M > 2\Phi$ .

The sub-optimal algorithm requires some device in order to detect the singular values of the available sliding variable  $y_1 = s$ . In the most practical case  $y_{1M}$  can be estimated by checking the sign of the quantity  $D(t) = [y_1(t - \tau) - y_1(t)] y_1(t)$  in which  $\frac{\tau}{2}$  is the estimation delay. In that case the control amplitude  $V_M$  must belong to an interval instead of a half-line:

$$V_M \in \left( \max \left( \frac{\Phi}{\alpha^* \Gamma_m}, V_{M_1}(\tau, y_{1M}) \right), V_{M_2}(\tau, y_{1M}) \right) \quad (3.40)$$

Here  $V_{M_1} < V_{M_2}$  are the solutions of the second-order algebraic equation

$$\left[ \frac{(3\Gamma_m - \alpha^* \Gamma_M) V_{M_i}}{\Phi} - 4 \right] \frac{y_{1M}}{\Phi \delta^2} - \frac{V_{M_i}}{8\Phi} [\Gamma_m + \Gamma_M (2 - \alpha^*)] \left( \frac{\Gamma_M V_{M_i}}{\Phi} + 1 \right) = 0$$

In the case of approximated evaluation of  $y_{1M}$  the second order real sliding mode is achieved, and the size of the boundary layer of the sliding manifold is  $\Delta \sim \mathcal{O}(\tau^2)$ . It can be minimized by choosing  $V_M$  as follows [6]:

$$V_M = \frac{4\Phi}{3\Gamma_m - \alpha^*\Gamma_M} \left[ 1 + \sqrt{1 + \frac{3\Gamma_m - \alpha^*\Gamma_M}{4\Gamma_M}} \right]$$

An extension of the sub-optimal 2-sliding controller to a class of sampled data systems such that the gain function in (3.29) is constant, i.e.,  $\gamma(\cdot) = 1$ , was recently presented [6].

### 3.6.4 Super-twisting algorithm

This algorithm has been developed to control systems with relative degree one in order to avoid chattering in VSC. Also in this case the trajectories on the 2-sliding plane are characterized by twisting around the origin (Figure 3.7), but the continuous control law  $u(t)$  is constituted by two terms. The first is defined by means of its discontinuous time derivative, while the other is a continuous function of the available sliding variable.

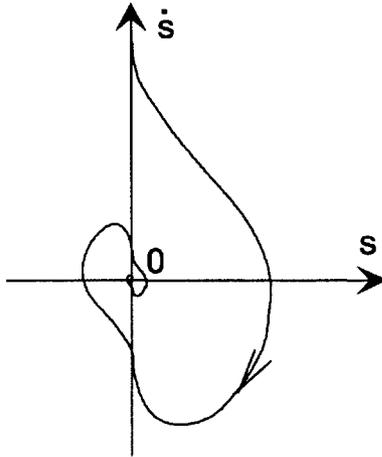


Figure 3.7: Super-twisting algorithm phase trajectory

The control algorithm is defined by the following control law [35]:

$$\begin{aligned} u(t) &= u_1(t) + u_2(t) \\ \dot{u}_1(t) &= \begin{cases} -u & \text{if } |u| > 1 \\ -W\text{sign}(y_1) & \text{if } |u| \leq 1 \end{cases} \\ u_2(t) &= \begin{cases} -\lambda|s_0|^\rho\text{sign}(y_1) & \text{if } |y_1| > s_0 \\ -\lambda|y_1|^\rho\text{sign}(y_1) & \text{if } |y_1| \leq s_0 \end{cases} \end{aligned} \quad (3.41)$$

and the corresponding sufficient conditions for the finite time convergence to the sliding manifold are [35]

$$\begin{aligned} W &> \frac{\Phi}{\Gamma_m} \\ \lambda^2 &\geq \frac{4\Phi}{\Gamma_m^2} \frac{\Gamma_M(W+\Phi)}{\Gamma_m(W-\Phi)} \\ 0 &< \rho \leq 0.5 \end{aligned} \quad (3.42)$$

That controller may be simplified when controlled systems (3.28) are linearly dependent on control,  $u$  does not need to be bounded and  $s_0 = \infty$ :

$$\begin{aligned} u &= -\lambda|s|^\rho \text{sign}(y_1) + u_1 \\ \dot{u}_1 &= -W \text{sign}(y_1) \end{aligned}$$

The super-twisting algorithm does not need any information on the time derivative of the sliding variable. An exponentially stable 2-sliding mode arrives if the control law (3.41) with  $\rho = 1$  is used. The choice  $\rho = 0.5$  ensures that the maximal possible for 2-sliding realization real-sliding order 2 is achieved. Being extremely robust, that controller is successfully used for real-time robust exact differentiation [37] ( see further).

### 3.6.5 Drift algorithm

The idea of the controller is to steer the trajectory to the 2-sliding mode  $s = 0$  while keeping  $\dot{s}$  relatively small, i.e., to cause "drift" towards the origin along axis  $y_1$ . When using the drift algorithm, the phase trajectories on the 2-sliding plane are characterized by loops with constant sign of the sliding variable  $y_1$  (Figure 3.8). That controller intentionally yields real 2-sliding and uses sample values of the available signal  $y_1$  with sampling period  $\tau$ . The control algorithm is defined by the following control law [35, 16, 18] (relative degree is 1):

$$\dot{u} = \begin{cases} -u & \text{if } |u| > 1 \\ -V_m \text{sign}(\Delta y_{1,i}) & \text{if } y_1 \Delta y_{1,i} \leq 0; |u| \leq 1 \\ -V_M \text{sign}(\Delta y_{1,i}) & \text{if } y_1 \Delta y_{1,i} > 0; |u| \leq 1 \end{cases} \quad (3.43)$$

where  $V_m$  and  $V_M$  are proper positive constants such that  $V_m < V_M$  and  $\frac{V_M}{V_m}$  is sufficiently large, and  $\Delta y_{1,i} = y_1(t_i) - y_1(t_i - \tau)$ ,  $t \in [t_i, t_{i+1})$ . The corresponding sufficient conditions for the convergence to the sliding manifold are rather cumbersome [18] and are omitted here for the sake of simplicity. Also here a similar controller corresponds to relative degree 2:

$$\dot{u} = \begin{cases} -V_m \text{sign}(\Delta y_{1,i}) & \text{if } y_1 \Delta y_{1,i} \leq 0 \\ -V_M \text{sign}(\Delta y_{1,i}) & \text{if } y_1 \Delta y_{1,i} > 0 \end{cases}$$

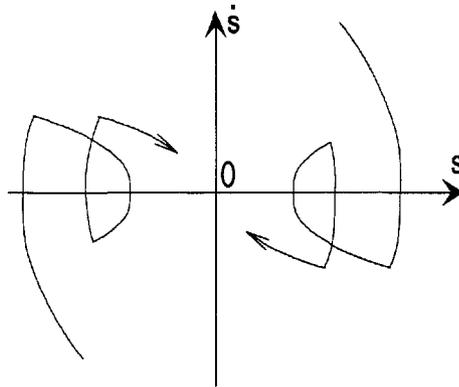


Figure 3.8: Drift algorithm phase trajectories

After substituting  $y_2$  for  $\Delta y_1$ , a first order sliding mode on  $y_2 = 0$  would be achieved. That implies  $y_1 = \text{const}$ , but since an artificial switching time delay appears, we ensure a real sliding on  $y_2$  with most of the time spent in the region  $y_1 y_2 < 0$ . Therefore,  $y_1 \rightarrow 0$ . The accuracy of the real sliding on  $y_2 = 0$  is proportional to the sampling time interval  $\tau$ ; hence, the duration of the transient process is proportional to  $\tau^{-1}$ . Such an algorithm does not satisfy the definition of a real sliding algorithm (Section 3.3) requiring the convergence time to be uniformly bounded with respect to  $\tau$ . Consider a variable sampling time  $\tau_{i+1}[y_1(t_i)] = t_{i+1} - t_i$ ,  $i = 0, 1, 2, \dots$  with  $\tau = \max(\tau_M, \min(\tau_m, \eta|y_1(t_i)|^\rho))$ , where  $0.5 \leq \rho \leq 1$ ,  $\tau_M > \tau_m > 0$ ,  $\eta > 0$ . Then with  $\eta, \frac{V_m}{V_M}$  sufficiently small and  $V_m$  sufficiently large, the drift algorithm constitutes a second-order real sliding algorithm with respect to  $\tau \rightarrow 0$ . That algorithm has no overshoot if the parameters are chosen properly [18].

### 3.6.6 Algorithm with a prescribed convergence law

This class of sliding controllers features the possibility of choosing a transient process trajectory: the switching of  $\dot{u}$  depends on a suitable function of  $s$ . The general formulation of such a class of 2-sliding control algorithms is as follows:

$$\dot{u} = \begin{cases} -u & \text{if } |u| > 1 \\ -V_M \text{sign}(y_2 - g(y_1)) & \text{if } |u| \leq 1 \end{cases} \quad (3.44)$$

Here  $V_M$  is a positive constant and the continuous function  $g(y_1)$  is smooth everywhere but in  $y_1 = 0$ . A controller for the relative degree 2 is formed

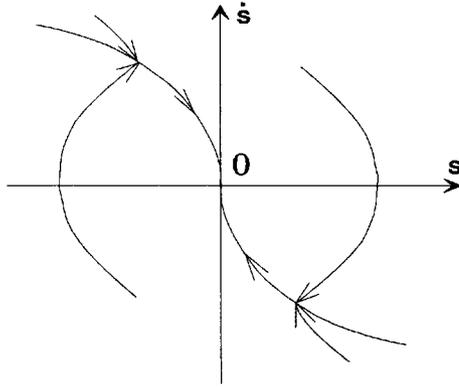


Figure 3.9: Phase trajectories for the algorithm with prescribed law of variation of  $s$

in an obvious way:

$$\dot{u} = -V_M \text{sign}[y_2 - g(y_1)]$$

Function  $g$  must be chosen in such a way that all solutions of the equation  $\dot{y}_1 = g(y_1)$  vanish in finite time and the function  $g' \cdot g$  be bounded. For example, the following function can be used

$$g(y_1) = -\lambda_1 |y_1|^\rho \text{sign}(y_1), \quad \lambda > 0, \quad 0.5 \leq \rho < 1$$

The sufficient condition for the finite time convergence to the sliding manifold is defined by the following inequality

$$V_M > \frac{\Phi + \sup[g'(y_1)g(y_1)]}{\Gamma_m} \quad (3.45)$$

and the convergence time depends on the function  $g$  [16, 35, 56].

That algorithm needs  $y_2$  to be available, which is not always the case. The substitution of the first difference of  $y_1$  for  $y_2$  i.e.,  $\text{sign}[\Delta y_{1_i} - \tau_i g(y_1)]$  instead of  $\text{sign}[y_2 - g(y_1)]$  ( $t \in [t_i, t_{i+1})$ ,  $\tau_i = t_i - t_{i-1}$ ), turns the algorithm into a real sliding algorithm. The real sliding order equals two if  $g(\cdot)$  is chosen as in the above example with  $\rho = 0.5$  [35].

**Important remark.** All the above-listed discretized 2-sliding controllers, except for the super-twisting one, are sensitive to the choice of the measurement interval  $\tau$ . Indeed, given any measurement error magnitude, any information of significance of the first difference  $\Delta y_{1_i}$  is eliminated with sufficiently small  $\tau$ , and the algorithm convergence is disturbed. That problem was shown to be solved [40] by a special feedback determining

$\tau$  as a function of the real-time measured value of  $y_1$ . In particular, it was shown that the feedback  $\tau = \max(\tau_M, \min(\tau_m, \eta|y_1(t_i)|^\rho))$ ,  $0.5 \leq \rho \leq 1$ ,  $\tau_M > \tau_m > 0$ ,  $\eta > 0$ , makes the twisting controller robust with respect to measurement errors. Moreover, the choice  $\rho = 1/2$  is proved to be the best one. It provides for keeping the second-order real-sliding accuracy  $s = O(\tau^2)$  in the absence of measurement errors and for sliding accuracy proportional to the maximal error magnitude otherwise. Note that the super-twisting controller is robust due to its own nature and does not need such auxiliary constructions.

### 3.6.7 Examples

Practical implementation of 2-sliding controllers is described in [42]. Continue the example series 3.5.1 and 3.5.2. The process is given by

$$\dot{x} = u, \quad x, u \in \mathbb{R}, \quad s = x - f(t), \quad f : \mathbb{R} \rightarrow \mathbb{R}$$

so that the problem is to track a signal  $f(t)$  given in real time, where  $|f|, |\dot{f}|, |\ddot{f}| < 0.5$ . Only values of  $x, f, u$  are available. Following is the appropriate discretized twisting controller:

$$\dot{u} = \begin{cases} -u(t_i), & |u(t_i)| > 1 \\ -5 \operatorname{sign} s(t_i), & s(t_i)\Delta s_i > 0, \quad |u(t_i)| \leq 1 \\ -\operatorname{sign} s(t_i), & s(t_i)\Delta s_i \leq 0, \quad |u(t_i)| \leq 1 \end{cases}$$

Here  $t_i \leq t < t_{i+1}$ . Let function  $f$  be chosen as in examples 3.5.1 and 3.5.2:

$$f(t) = 0.08 \sin t + 0.12 \cos 0.3t, \quad x(0) = 0, \quad v(0) = 0.$$

The corresponding simulation results are shown in Figure 3.10 and 3.11.

The discretized super-twisting controller [19, 35, 37] serving the same goal is the algorithm

$$u = -2\sqrt{|s(t_i)|} \operatorname{sign} s(t_i) + u_1, \quad \dot{u}_1 = \begin{cases} -u(t_i), & |u(t_i)| > 1 \\ -\operatorname{sign} s(t_i), & |u(t_i)| \leq 1 \end{cases}$$

Its simulation results are shown in Figures 3.12 and 3.13.

## 3.7 Arbitrary-order sliding controllers

We follow here [38, 39, 41].

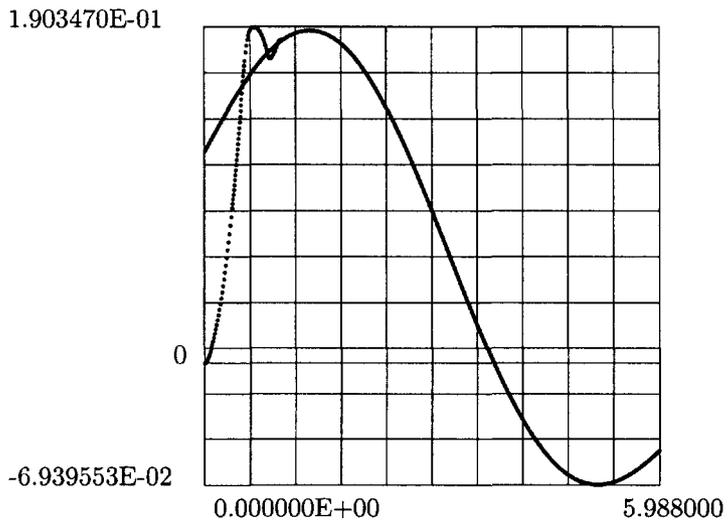


Figure 3.10: Twisting 2-sliding algorithm. Tracking:  $x(t)$  and  $f(t)$

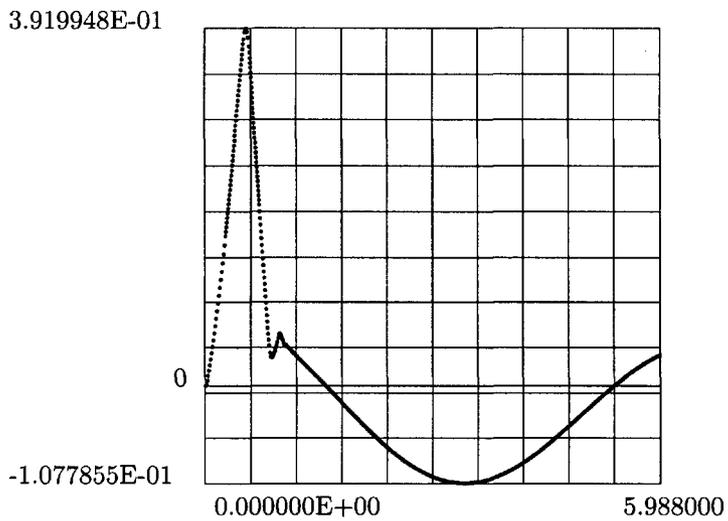


Figure 3.11: Twisting 2-sliding algorithm. Control  $u(t)$

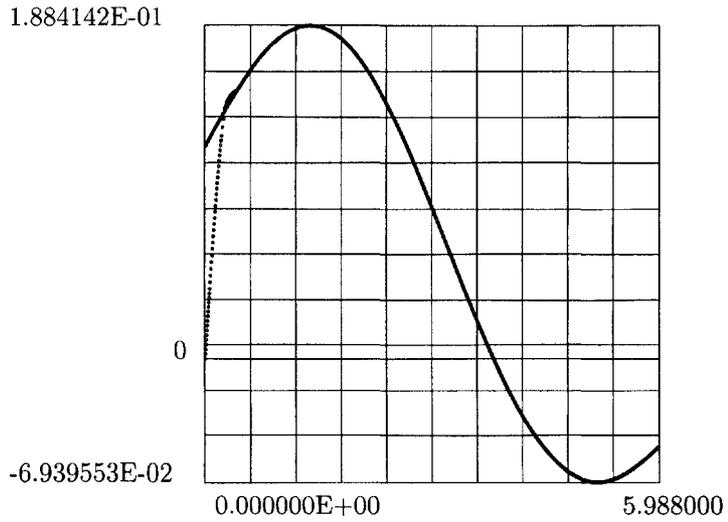


Figure 3.12: Super-twisting 2-sliding controller. Tracking:  $x(t)$  and  $f(t)$ .

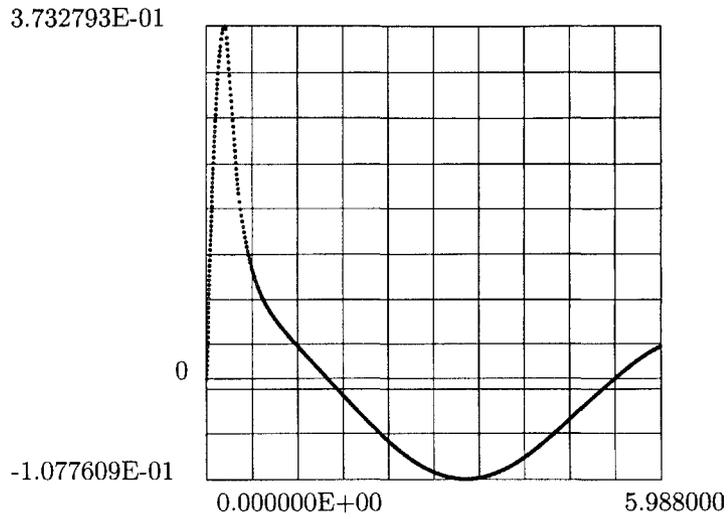


Figure 3.13: Super-twisting 2-sliding controller. Control  $u(t)$ .

### 3.7.1 The problem statement

Consider a dynamic system of the form

$$\dot{x} = a(t, x) + b(t, x)u, s = s(t, x) \quad (3.46)$$

where  $x \in \mathbb{R}^n$ ,  $a, b, s$  are smooth functions,  $u \in \mathbb{R}$ . The relative degree  $r$  of the system is assumed to be constant and known. That means, in a simplified way, that  $u$  first appears explicitly only in the  $r$ -th total derivative of  $s$  and  $\frac{d}{du}s^{(r)} \neq 0$  at the given point. The task is to fulfill the constraint  $s(t, x) = 0$  in finite time and to keep it exactly by discontinuous feedback control. Since  $s, \dot{s}, \dots, s^{(r-1)}$  are continuous functions of  $t$  and  $x$ , the corresponding motion will correspond to an  $r$ -sliding mode. Introduce new local coordinates  $y = (y_1, \dots, y_n)$ , where  $y_1 = s, y_2 = \dot{s}, \dots, y_r = s^{(r-1)}$ . Then

$$s^{(r)} = h(t, y) + g(t, y)u, \quad g(t, y) \neq 0$$

$$\xi = \eta(t, s, \dot{s}, \dots, s^{(r-1)}, \xi) + \gamma(t, s, \dot{s}, \dots, s^{(r-1)}, \xi)u, \quad \xi = (y_{r+1}, \dots, y_n) \quad (3.47)$$

Let a trivial controller  $u = -K$  signs be chosen with  $K > \sup|u_{eq}|$ ,  $u_{eq} = -h(t, y)/g(t, y)$  [55]. Then the substitution  $u = u_{eq}$  defines a differential equation on the  $r$ -sliding manifold of (3.46). Its solution provides for the  $r$ -sliding motion. Usually, however, such a mode is not stable. It is easy to check that  $g = L_b L_a^{r-1} s = \frac{d}{du} s^{(r)}$ . Obviously,  $h = L_a^r s$  is the  $r$ th total time derivative of  $s$  calculated with  $u = 0$ . In other words, functions  $h$  and  $g$  may be defined in terms of input-output relations. Therefore, dynamic system (3.46) may be considered as a "black box".

The problem is to find a discontinuous feedback  $u = U(t, x)$  causing finite-time convergence to an  $r$ -sliding mode. That controller must generalize the 1-sliding relay controller  $u = -K$  signs. Hence,  $g(t, y)$  and  $h(t, y)$  in (3.47) are to be bounded,  $h > 0$ . Thus, we require that for some  $K_m, K_M, C > 0$

$$0 < K_m \leq \frac{\partial}{\partial u} s^{(r)} \leq K_M, \quad |L_a^r s| \leq C \quad (3.48)$$

### 3.7.2 Controller construction

Let  $p$  be a positive number. Denote

$$N_{1,r} = |s|^{(r-1)/r}$$

$$N_{i,r} = (|s|^{p/r} + |\dot{s}|^{p/(r-1)} + \dots + |s^{(i-1)}|^{p/(r-i+1)})^{(r-i)/p}, \quad i = 1, \dots, r-1$$

$$N_{r-1,r} = (|s|^{p/r} + |\dot{s}|^{p/(r-1)} + \dots + |s^{(r-2)}|^{p/2})^{1/p}$$

$$\psi_{0,r} = s$$

$$\psi_{1,r} = \dot{s} + \beta_1 N_{1,r} \text{sign}(s)$$

$$\psi_{i,r} = s^{(i)} + \beta_i N_{i,r} \text{sign}(\psi_{i-1,r}), \quad i = 1, \dots, r-1$$

where  $\beta_1, \dots, \beta_{r-1}$  are positive numbers.

**Theorem 27** *Let system (3.46) have relative degree  $r$  with respect to the output function  $s$  and (3.48) be fulfilled. Then with properly chosen positive parameters  $\beta_1, \dots, \beta_{r-1}$  controller*

$$u = -\alpha \text{sign} \left[ \psi_{r-1,r}(s, \dot{s}, \dots, s^{(r-1)}) \right]. \quad (3.49)$$

*provides for the appearance of  $r$ -sliding mode  $s \equiv 0$  attracting trajectories in finite time.*

The positive parameters  $\beta_1, \dots, \beta_{r-1}$  are to be chosen sufficiently large in the index order. Each choice determines a controller family applicable to all systems (3.46) of relative degree  $r$ . Parameter  $\alpha > 0$  is to be chosen specifically for any fixed  $C, K_m, K_M$ . The proposed controller is easily generalized: coefficients of  $N_{i,r}$  may be any positive numbers, etc. Obviously,  $\alpha$  is to be negative with  $\frac{\partial}{\partial u} s^{(r)} < 0$ .

Certainly, the number of choices of  $\beta_i$  is infinite. Here are a few examples with  $\beta_i$  tested for  $r \leq 4$ ,  $p$  being the least common multiple of  $1, 2, \dots, r$ . The first is the relay controller and the second is listed in Section 3.6.

1.  $u = -\alpha \text{sign } s$
2.  $u = -\alpha \text{sign}(s + |s|^{1/2} \text{sign } s)$ ,
3.  $u = -\alpha \text{sign}(\ddot{s} + 2(|\dot{s}|^3 + |s|^2)^{1/6} \text{sign}(\dot{s} + |s|^{2/3} \text{sign } s))$ ,
4.  $u = -\alpha \text{sign}\{s^{(3)} + 3(\ddot{s}^6 + \dot{s}^4 + |s|^3)^{1/12} \text{sign}[\ddot{s} + (s^4 + |s|^3)^{1/6} \text{sign}(\dot{s} + 0.5|s|^{3/4} \text{sign } s)]\}$ ,
5.  $u = -\alpha \text{sign}(s^{(4)} + \beta_4(|s|^{12} + |\dot{s}|^{15} + |s|^{20} + |s^{(3)}|^{30})^{1/60} \text{sign}(s^{(3)}) + \beta_3(|s|^{12} + |\dot{s}|^{15} + |\ddot{s}|^{20})^{1/30} \text{sign}(\ddot{s}) + \beta_2(|s|^{12} + |\dot{s}|^{15})^{1/20} \text{sign}(\dot{s} + \beta_1|s|^{4/5} \text{sign } s))$

The idea of the controller is that a 1-sliding mode is established on the smooth parts of the discontinuity set  $\Gamma$  of (3.49) (Figure 3.14). That sliding mode is described by the differential equation  $\psi_{r-1,r} = 0$  providing in its turn for the existence of a 1-sliding mode  $\psi_{r-1,r} = 0$ . But the primary sliding mode disappears at the moment when the secondary one is to appear. The resulting movement takes place in some vicinity of the subset of  $\Gamma$  satisfying  $\psi_{r-2,r} = 0$ , transfers in finite time into some vicinity of the

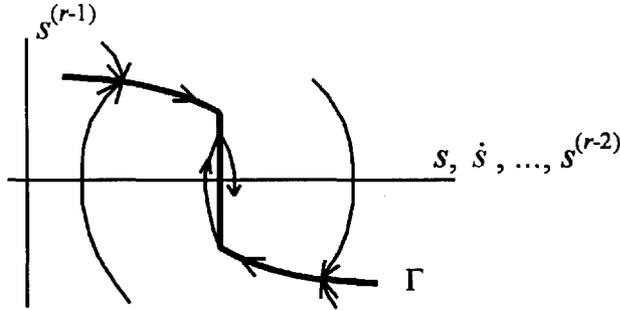


Figure 3.14: The idea of  $r$ -sliding controller

subset satisfying  $\psi_{r-3,r} = 0$  and so on. While the trajectory approaches the  $r$ -sliding set, set  $\Gamma$  retracts to the origin in the coordinates  $s, \dot{s}, \dots, s^{(r-1)}$ . Set  $\Gamma$  with  $r = 3$  is shown in Figure 3.15.

An interesting controller, so-called “terminal sliding mode controller”, was proposed by [56]. In the 2-dimensional case it coincides with a particular case of the 2-sliding controller with given convergence law (Section 3.6). In the  $r$ -dimensional case a mode is produced at the origin similar to the  $r$ -sliding mode. The problem is that a closed-loop system with terminal sliding mode does not satisfy the Filippov conditions [22] for the solution existence with  $r > 2$ . Indeed, the control influence is unbounded in vicinities of a number of hyper-surfaces intersecting at the origin. The corresponding Filippov velocity sets are unbounded as well. Thus, some special solution definition is to be elaborated, the stability of the corresponding quasi-sliding mode at the origin and the very existence of solutions are to be shown.

Controller (3.49) requires the availability of  $s, \dot{s}, \dots, s^{(r-1)}$ . The needed information may be reduced if the measurements are carried out at times  $t_i$  with constant step  $\tau > 0$ . Consider the controller

$$u(t) = -\alpha \text{sign}(\Delta s_i^{(r-2)} + \beta_{r-1} \tau N_{r-1,r}(s_i, \dot{s}_i, \dots, s_i^{(r-2)}) \text{sign} [\psi_{r-2,r}(s_i, \dot{s}_i, \dots, s_i^{(r-2)})]) \quad (3.50)$$

**Theorem 28** *Under conditions of Theorem 27 with discrete measurements both algorithms (3.49) and (3.50) provide in finite time for some positive constants  $a_0, a_1, \dots, a_{r-1}$  for fulfillment of inequalities*

$$|s| < a_0 \tau^r, |\dot{s}| < a_1 \tau^{r-1}, \dots, |s^{(r-1)}| < a_{r-1} \tau$$

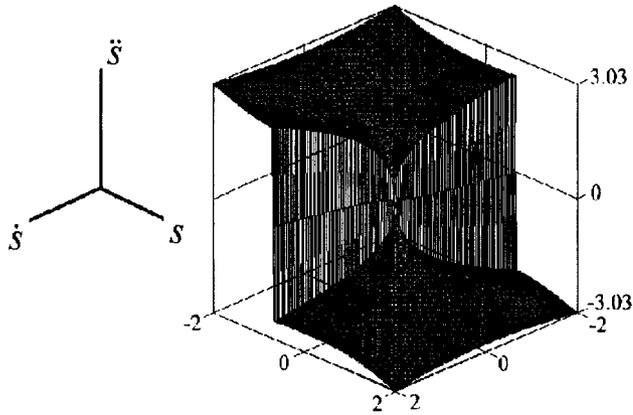


Figure 3.15: The discontinuity set of the 3-sliding controller

That is the best possible accuracy attainable with discontinuous  $s^{(r)}$ . Convergence time may be reduced by changing coefficients  $\beta_j$ . Another way is to substitute  $\lambda^{-j}s^{(j)}$  for  $s^{(j)}$ ,  $\lambda^r\alpha$  for  $\alpha$  and  $\alpha\tau$  for  $\tau$  in (3.49) and (3.50),  $\lambda > 0$ , causing convergence time to be diminished approximately by  $\lambda$  times.

*Implementation of  $r$ -sliding controller when the relative degree is less than  $r$ .* Introducing successive time derivatives  $u, \dot{u}, \dots, u^{(r-k-1)}$  as new auxiliary variables and  $u^{(r-k)}$  as a new control, achieve different modifications of each  $r$ -sliding controller

intended to control systems with relative degrees  $k = 1, 2, \dots, r$ . The resulting control is  $(r - k - 1)$ -smooth function of time with  $k < r$ , a Lipschitz function with  $k = r - 1$  and a bounded "infinite-frequency switching" function with  $k = r$ .

*Chattering removal.* The same trick removes the chattering effect. For example, substituting  $u^{(r-1)}$  for  $u$  in (3.50), receive a local  $r$ -sliding controller to be used instead of the relay controller  $u = -\text{signs}$  and attain  $r$ th order sliding precision with respect to  $\tau$  by means of  $(r - 2)$ -smooth control with Lipschitz  $(r - 2)$ th time derivative. It must be modified for global usage.

*Controlling systems nonlinear on control.* Consider a system

$$\dot{x} = f(t, x, u)$$

nonlinear in the control variable  $u$ . Let  $\frac{\partial}{\partial u} s^{(i)}(t, x, u) = 0$  for  $i = 1, \dots, r-1$ ,  $\frac{\partial}{\partial u} s^{(r)}(t, x, u) > 0$ . It is easy to check that

$$s^{(r+1)} = \Lambda_u^{r+1} s + \frac{\partial}{\partial u} s^{(r)} \dot{u}, \quad \Lambda_u(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot) f(t, x, u)$$

The problem is now reduced to that considered above with relative degree  $r+1$  by introducing a new auxiliary variable  $u$  and a new control  $v = \dot{u}$ .

*Discontinuity regularization.* The complicated discontinuity structure of the above-listed controllers may be smoothed by replacing the discontinuities under the sign-function with their finite-slope approximations. As a result, the transient process becomes smoother. Consider, for example, the above-listed 3-sliding controller. The function  $\text{sign}(\dot{s} + |s|^{2/3} \text{sign}s)$  may be replaced by the function  $\max[-1, \min(1, |s|^{-2/3}(\dot{s} + |s|^{2/3} \text{sign}s)/\varepsilon)]$  for some sufficiently small  $\varepsilon > 0$ . For  $\varepsilon = 0.1$  the resulting tested controller is

$$u = -\alpha \text{sign}(\ddot{s} + 2(|\dot{s}|^3 + |s|^2)^{\frac{1}{6}} \max[-1, \min(1, 10|s|^{-\frac{2}{3}}(\dot{s} + |s|^{\frac{2}{3}} \text{sign}s)])) \quad (3.51)$$

Controller (3.51) provides for the existence of a standard 1-sliding mode on the corresponding continuous piece-wise smooth surface.

**Theorem 29** *Theorems 27 and 28 remain valid for controller (3.51).*

### Real-time control of output variables

The implementation of the above-listed  $r$ -sliding controllers requires real-time observation of the successive derivatives  $s, \dot{s}, \dots, s^{(r-1)}$ . Thus, theoretically no model of the controlled process needs to be known. Only the relative degree and 3 constants are needed in order to adjust the controller. Unfortunately, the problem of successive real-time exact differentiation is usually considered to be practically unsolvable. Nevertheless, under some assumptions the real-time exact robust differentiation is possible. Indeed, let input signal  $\eta(t)$  be a Lebesgue-measurable locally bounded function defined on  $[0, \infty)$  and let it consist of a base signal  $\eta_0(t)$  having a derivative with Lipschitz's constant  $C > 0$  and a bounded measurable noise  $N(t)$ . Then the following system realizes a *real-time differentiator* [37]:

$$\dot{v} = v, \quad v = \nu_1 - \lambda|\nu - \eta(t)|^{1/2} \text{sign}[\nu - \eta(t)], \quad \dot{\nu}_1 = -\mu \text{sign}[\nu - \eta(t)]$$

where  $\mu, \lambda > 0$ . Here  $v(t)$  is the output of the differentiator. Solutions of the system are understood in the Filippov sense. Parameters may be chosen in the form  $\mu = 1.1C$ ,  $\lambda = 1.5C^{1/2}$ , for example (it is only one of possible choices). That differentiator provides for finite time convergence to the exact derivative of  $\eta_0(t)$  if  $N(t) = 0$ . Otherwise, if  $\text{sup } N(t) = \varepsilon$

it provides for accuracy proportional to  $C^{1/2}\varepsilon^{1/2}$ . Therefore, having been implemented  $k$  times successively, that differentiator will provide for  $k$ th order differentiation accuracy of the order of  $\varepsilon^{(2^{-k})}$ . Thus, full local real-time robust control of output variables is possible, using only output variable measurements and knowledge of the relative degree [41].

When the base signal  $\eta_0(t)$  has  $(r-1)$ th derivative with Lipschitz's constant  $C > 0$ , the best possible  $k$ th order differentiation accuracy is  $d_k C^{k/r} \varepsilon^{(r-k)/r}$ , where  $d_k > 1$  may be estimated (the asymptotics may be improved with additional restrictions on  $\eta_0(t)$ ). Moreover, it is proved that such a robust exact differentiator really exists [37]. The corresponding differentiator has been submitted by A. Levant for possible presentation at the European Control Conference in Portugal (2001).

**Theorem 30** *An optimal  $k$ -th order differentiator having been applied,  $r$ -sliding controller (3.49) provides locally for the sliding accuracy  $|s^{(i)}| < c_i \varepsilon^{(r-i)/r}$ ,  $i = 0, 1, \dots, r-1$ , where  $\varepsilon$  is the maximal possible error of real-time measurements of  $s$  and  $c_i$  are some positive constants.*

Theorem 30 probably determines the best sliding asymptotics attainable when only  $s$  is available.

### 3.7.3 Examples

#### Car control

Consider a simple kinematic model of car control [45]

$$\begin{aligned}\dot{x} &= v \cos \varphi, \quad \dot{y} = v \sin \varphi \\ \dot{\varphi} &= \frac{v}{l} \tan \delta \\ \dot{\delta} &= u\end{aligned}$$

where  $x$  and  $y$  are Cartesian coordinates of the rear-axle middle point,  $\varphi$  is the orientation angle,  $v$  is the longitudinal velocity,  $l$  is the length between the two axles, and  $\delta$  is the steering angle. The task is to steer the car from a given initial position to the trajectory  $y = g(x)$ , while  $x, y$ , and  $\varphi$  are assumed to be measured in real time. Define

$$s = y - g(x)$$

Let  $v = \text{const} = 10\text{m/s}$ ,  $l = 5\text{m}$ ,  $g(x) = 10 \sin 0.05x + 5$ ,  $x = y = \varphi = \delta = 0$  at  $t = 0$ . The relative degree of the system is 3 and both 3-sliding controller No. 3 and its regularized form (3.51) may be applied here. It was taken  $\alpha = 20$ . The corresponding trajectories are the same, but the performance

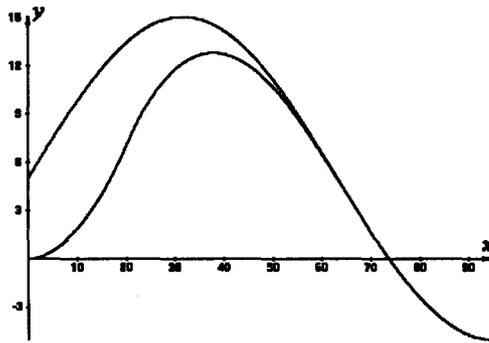


Figure 3.16: Car trajectory tracking

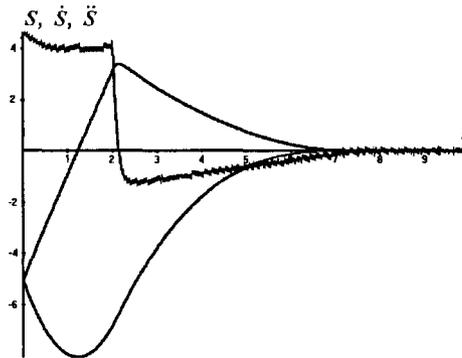


Figure 3.17: Regularized 3-sliding controller

is different. The trajectory and function  $y = g(x)$  with measurement step  $\tau = 2 \cdot 10^{-4}$  are shown in Figure 3.16. Graphs of  $s$ ,  $\dot{s}$ ,  $\ddot{s}$  are shown in Figure 3.16 and 3.17 for regularized and not regularized controllers, respectively.

#### 4-sliding control

Consider a model example of a tracking system. Let input  $z(t)$  and the control system satisfy equations

$$z^{(4)} + 3\ddot{z} + 2z = 0$$

$$x^{(4)} = u$$

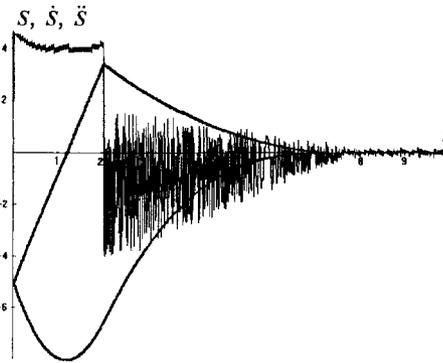


Figure 3.18: Standard 3-sliding controller

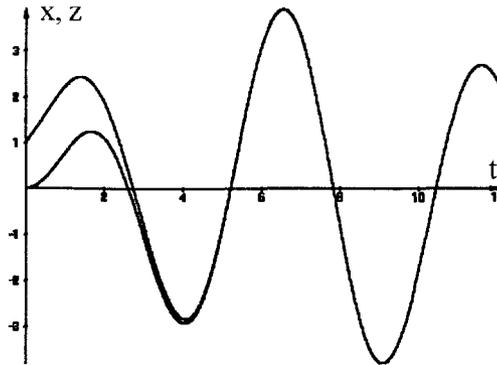


Figure 3.19: 4-sliding tracking

The task is to track  $z$  by  $x$ ,  $s = x - z$ , thus the 4th controller with  $\alpha = 40$  is used. Initial conditions for  $z$  and  $x$  at time  $t = 0$  are

$$z(0) = 0, \dot{z}(0) = 0, \ddot{z}(0) = 2, z^{(3)}(0) = 0$$

$$x(0) = 1, \dot{x}(0) = 1, \ddot{x}(0) = 1, x^{(3)}(0) = 1$$

A mutual graph of  $x$  and  $z$  with  $\tau = 0.01$  is shown in Figure 3.19.

A mutual graph of  $x^{(3)}$  and  $z^{(3)}$  with  $\tau = 0.001$  is shown in Figure 3.20. Mutual graphs of  $s, \dot{s}, \ddot{s}, s^{(3)}$  with  $\tau = 0.001$  are demonstrated in Figure 3.21. The attained accuracies are  $|s| \leq 1.33 \cdot 10^{-4}$  with  $\tau = 0.01$  and  $|s| \leq 1.49 \cdot 10^{-12}$  with  $\tau = 0.0001$ .

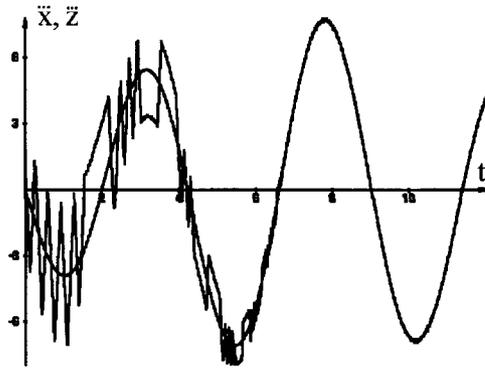


Figure 3.20: Third derivative tracking

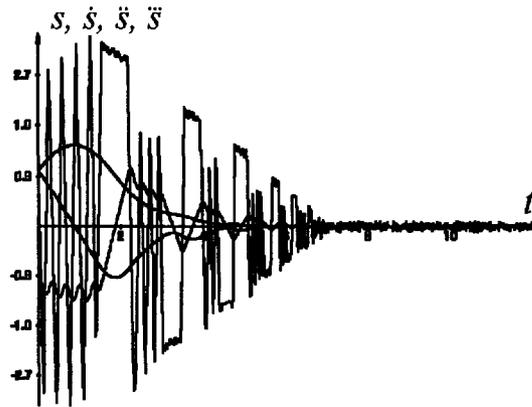


Figure 3.21: Tracking deviation and its three derivatives

The authors are grateful to A. Stotsky for helpful discussions on VSS car control.

### 3.8 Conclusions

- A general review of the current state of the higher order sliding theory, its main notions and results were presented.
- It was demonstrated that higher order sliding modes are natural phenomena for relay control systems if the relative degree of the system is more than 1 or a dynamic actuator is present.
- Stability was studied of second order sliding modes in relay systems with fast stable dynamic actuators of relative degree 1.
- Instability of higher order sliding modes was shown in relay systems with dynamic actuators of relative degree 2 and more.
- A number of the most popular 2-sliding controllers were listed and compared.
- A family of arbitrary order sliding controllers with finite time convergence was presented.
- The discrete switching modification of presented sliding controllers provided for the sliding precision of their order with respect to the measurement time interval.
- A robust exact differentiator was presented allowing for full control of output variables using only measurements of their current values.
- A number of simulation examples were presented.

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# Chapter 4

## Sliding Mode Observers

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### 4.1 Introduction

Sliding mode techniques have been widely studied and developed for the control problem and observation in the occidental countries<sup>1</sup> since the works of Utkin [43]. As discussed by many authors [22, 40, 21, 37, 49, 50, 20, 4, 31, 24, 33], this methodology has several drawbacks in control design, adaptive control and observation. More particularly, several authors have used sliding observer for linear and nonlinear systems, and in many applications such as robotics [41, 12, 13, 28], mobile robots [5], AC motors [16, 17, 18] and converters [36].

This kind of observer is very useful and was developed for many reasons:

- to work with reduced observation error dynamics
- for the possibility of obtaining a step-by-step design
- for a finite time convergence for all the observables states
- to design, under some conditions, an observer for nonsmooth systems, and
- robustness under parameter variations is possible, if the condition (dual of the well-known matching condition) is verified.

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<sup>1</sup>It is important to highlight the paternity and the major contribution of the Russian school in the sliding mode domain.

Here, we highlight a few advantages of the sliding observer. One advantage is the possibility to design an observer for a system with an undetermined but bounded specific variable structure, however, throughout this chapter we choose to focus our attention on widening the class of considered systems in the design of the observer.

Historically, in nonlinear control theories, the problem of a nonlinear observer design with linearization of the observation error dynamics for a class of nonlinear systems, called *the input injection form*, has been investigated ([29, 45, 46]...). Some necessary and sufficient conditions to obtain such a form are given in [46]. From this form, it is “easy” to design an observer. Unfortunately, the geometric conditions to obtain this form are very often too restrictive with respect to the system considered. Thus, in [11] we have given an extension of the results obtained in [29, 30, 35, 45, 46], for systems that can be written in an output injection form to systems which can be written in the form of the output and the output’s derivative injection. We first recall this result and then we deal with a more general case, which is the *triangular observer form* [1]. Here, aiming for simplicity, we only present the case of single output system. The multi-output case may be found in [6], where the implicit triangular observer form is introduced in order to take into account the fact that the information quantity given by one output and its derivatives may change along the state space. Roughly speaking, in the nonlinear case, in the neighborhood of  $x_0$ , information about the state can be given by the output  $y_1$  (one component of the output) and its derivative, and in another neighborhood of  $x_1$ , information can be given by  $y_2$  (another component of the output) and its derivative. In both forms considered in this presentation, input derivatives are prohibited. Indeed, if they are allowed it is possible to use the observer form proposed in [25] and in that case a sliding observer is also widely used (see for example [34]).

As in other chapters, some recall on high order sliding mode are given [31], then for the sake of clarity we do not present the high order sliding observer [7, 3, 7]. Moreover, we deferred some technical proofs to the appendix.

We find that it is important to end this introduction with the following warning: in this chapter we omit many interesting aspects, for example, the observer design without coordinate change [14], high gain [10], and noise sensibility [47]. The subject is too large and open, to be able to squeeze it in an introductory presentation. The main purpose of this chapter is to highlight the utilities and difficulties of sliding mode technique for the observer design.

## 4.2 Preliminary example

In this section, the sliding observer is introduced based on a simple academic example. Let  $\Sigma$  be the system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2) \\ y &= x_1\end{aligned}\tag{4.1}$$

where  $x \in \mathbb{R}^2$  and  $y \in \mathbb{R}$  is the output and the function  $f(x_1, x_2)$  is bounded ( $|f(x_1, x_2)| < B$ ) but not necessary smooth, thus (4.1) is a particular case of variable structure dynamics.

One wants to observe the state  $x$  with the additional constraint to obtain the real value of  $x_2$  in finite time. To do this, one uses a classical sliding mode observer, but completed with a new component  $\tilde{x}_2$ .

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \lambda_1 \operatorname{sgn}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= f(x_1, \tilde{x}_2) + E_1 \lambda_2 \operatorname{sgn}(\tilde{x}_2 - \hat{x}_2) \\ \hat{y} &= \hat{x}_1 \\ \dot{\tilde{x}}_2 &= \hat{x}_2 + E_1 \lambda_1 \operatorname{sgn}(x_1 - \hat{x}_1)\end{aligned}\tag{4.2}$$

where  $\hat{x}$  represents the estimated value of  $x$  and  $E_1 = 1$  if  $x_1 = \hat{x}_1$  else  $E_1 = 0$  and sign denote the usual sign function.

From (4.1) and (4.2), the error observation ( $e = x - \hat{x}$ ) dynamics are:

$$\begin{aligned}\dot{e}_1 &= e_2 - \lambda_1 \operatorname{sgn}(e_1) \\ \dot{e}_2 &= f(x_1, x_2) - f(x_1, \tilde{x}_2) - E_1 \lambda_2 \operatorname{sgn}(\tilde{x}_2 - \hat{x}_2)\end{aligned}\tag{4.3}$$

Considering the nonempty manifold  $S = \{e/e_1 = 0\}$  and the Lyapunov function  $V = \frac{1}{2}e_1^2$ , one proves the attractivity of  $S$  as follows. One gets:  $\dot{V} = e_1 e_2 - \lambda_1 e_1 \operatorname{sgn}(e_1)$ , which verifies the inequality  $\dot{V} < 0$  when  $\lambda_1$  is chosen such that  $\lambda_1 > |e_2|_{max}$  (where  $|e|_{max}$  denotes the maximal value of  $e$ ,  $\forall t \in [0, \infty]$ ). As one uses a *sgn* function and as the Lyapunov function  $V$  is decreasing, one obtains the convergence to the sliding surface  $S = 0$  in finite time  $t_0$  (and moreover, we have  $|e|_{max} = |e|_{max}^{t_0}$  and  $|e|_{max}^{t_0}$  is the maximal value of  $e$ ,  $\forall t \in [0, t_0]$ ). Thus, for  $\lambda_1 > |e_2|_{max}$ ,  $\hat{x}_1$  converges to  $x_1$  in finite time and remains equal to  $x_1$  for  $t > t_0$ .

Moreover, one also has that  $\dot{e}_1 = 0 \forall t > t_0$ , so that from (4.3),

$$e_2 = \lambda_1 \operatorname{sgn}(e_1)\tag{4.4}$$

Therefore, the observer output,  $\tilde{x}_2 = \hat{x}_2 + \lambda_1 \operatorname{sgn}(e_1)$  is equal to  $x_2 \forall t > t_0$ .

**Remark 31** *This is obviously only true without any noise measurement, but this difficulty may be partially overcome by a sgn function modification (see [47] for analysis and design of observer with respect to noise) or by high order sliding mode [31].*

Up to now, we proved for the system (4.1) that the observer (4.2) is suitable to give all the values of the state in finite time.

The condition  $\lambda_1 > |e_2|_{max}$  can only be verified if  $e_2$  has stable dynamics, which is fulfilled after  $t_0$  for  $\lambda_2 > 0$ , where we have

$$\dot{e}_2 = f(x_1, x_2) - f(x_1, \tilde{x}_2) - E_1 \lambda_2 \operatorname{sgn}(\tilde{x}_2 - \hat{x}_2)$$

with  $\tilde{x}_2 = x_2$  and  $E_1 = 1$  then

$$\dot{e}_2 = -\lambda_2 \operatorname{sgn}(e_2)$$

Therefore, one gets  $|e_2|_{max}^{t_0}$ , which is bounded by the way that  $t_0$  and  $f(x_1, x_2)$  are bounded. The observer (4.2) with assumptions  $\lambda_1 > |e_2|_{max}^{t_0}$  and  $\lambda_2 > 0$  ensures a finite time convergence of  $(e_1, e_2)$  to  $(0, 0)$ .

**Remark 32** *The time  $t_0$  can be very short because it is natural to initialize  $\hat{x}_1 = x_1$ .*

### 4.3 Output and output derivative injection form

Following, we recall some classical results on nonlinear observer theory.

#### 4.3.1 Nonlinear observer

First of all, we recall the definition of *observability indices*.

**Definition 33** [29] *Let the system*

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{4.5}$$

*which is observable at  $x_0$  if there exists a neighborhood  $\mathcal{U}$  of  $x_0$  and  $p$ -tuple of integers  $(\mu_1, \dots, \mu_p)$  such that*

$$1) \mu_1 \geq \mu_2 \geq \dots \geq \mu_p \geq 0 \text{ and } \sum_{i=1}^p \mu_i = n.$$

- 2) After suitable reordering of the  $h_i$  at each  $x \in \mathcal{U}$ , the  $n$  row vectors  $\{L_f^{j-1}(dh_i) : i = 1, \dots, p; j = 1, \dots, \mu_i\}$  are linearly independent.
- 3) If  $l_1, \dots, l_p$  satisfies (i) and after suitable reordering the  $n$  row vectors  $\{L_f^{j-1}(dh_i) : i = 1, \dots, p; j = 1, \dots, l_i\}$  are linearly independent at some  $x \in \mathcal{U}$

then  $(l_1, \dots, l_p) \geq (\mu_1, \dots, \mu_p)$  in the lexicographic ordering  $[(l_1 > \mu_1)$  or  $(l_1 = \mu_1$  and  $l_2 > \mu_2)$  or... or  $(l_1 = \mu_1, \dots, l_p = \mu_p)]$ . The integers  $(\mu_1, \dots, \mu_p)$  are called the observability indices at  $x_0$ .

**Remark 34** In the nonlinear case, the previous notion of observability index is local. In the linear case, this notion is global.

As it is shown in [29, 30, 45], an interesting nonlinear systems is the output injection form without forced terms:

$$\begin{aligned} \dot{x} &= Ax + \phi(y) \\ y &= Cx \end{aligned} \quad (4.6)$$

where:

$$A = \left( \begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_p \end{array} \right)$$

$$A_i \text{ is a } \mathbb{R}^{\mu_i \times \mu_i} \text{ matrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.7)$$

$$\text{and} \quad \left( \begin{array}{c|c|c} C_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & C_p \end{array} \right)$$

$C_i$  is a line vector  $\in \mathbb{R}^{\mu_i}$ , such that :  $C_i = (1, 0, \dots, 0)$ .

This is interesting because for such a class, one can design an observer that allows us to obtain an observation error with stable linear dynamics.

In fact, for the nonlinear observable system:

$$\begin{aligned} \dot{\xi} &= f(\xi) \\ y &= h(\xi) \end{aligned} \quad (4.8)$$

where  $f$  and  $h$  are smooth functions, necessary and sufficient conditions for the existence of a diffeomorphism  $x = \Phi(\xi)$  to transform the system (4.8) into (4.6) are given in [46].

**Theorem 35** [46] *There exists a change of coordinates transforming (4.8) into (4.6) only if there exists a  $p$ -tuple of integers  $(\mu_1, \dots, \mu_p)$ ,  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$  such that we have the following:*

1) *If one denotes (with a possible reordering of the  $h_i$ )*

$$Q = \left\{ L_f^{j-1}(dh_i) : i = 1, \dots, p; j = 1, \dots, \mu_i \right\}$$

*then  $\dim \text{span } Q = n$  in a neighborhood of  $\xi^0$ .*

2) *If one denotes for  $j = 1, \dots, p$ ,*

$$Q_j = \left\{ L_f^{k-1}(dh_i) : \begin{array}{l} i = 1, \dots, p; \\ k = 1, \dots, \mu_j \end{array} \right\} - \left\{ L_f^{\mu_j-1}(dh_j) \right\}$$

*then  $\text{span } Q_j = \text{span } Q \cap Q_j$  for  $j = 1, \dots, p$ .*

**Theorem 36** [46] *There exists a change of coordinates transforming (4.8) to (4.6) if and only if 1. and 2. in the previous Theorem hold and, moreover, if there exists vector fields  $g^1, \dots, g^p$  satisfying:*

$$L_{g^i} L_f^{l-1}(h_j) = \delta_{i,j} \delta_{l,\mu_i}, \quad i, j = 1, \dots, p, \quad l = 1, \dots, \mu_i$$

*such that:  $[\text{ad}_{(-f)}^k g^i, \text{ad}_{(-f)}^l g^j] = \mathbf{0}$  for  $i, j = 1, \dots, p; k = 0, \dots, \mu_i - 1; l = 0, \dots, \mu_j - 1$ .*

Thus, it immediately follows:

**Corollary 37** *The conditions of Theorem 36 are sufficient to construct an observer that is asymptotically locally stable.*

### 4.3.2 Sliding observer for output and output derivative nonlinear injection form

In this section, one first constructs an asymptotically stable observer for the following class of systems called *output and output derivative nonlinear injection form*:

$$\begin{aligned} \dot{x}_i &= A_i x + \phi_i(y, \dot{y}) \\ y_i &= x_{i,1} \\ \dot{y}_i &= x_{i,2} \end{aligned} \quad \text{for } i = 1, \dots, p \quad (4.9)$$

with

$$\phi_i(y, \dot{y}) = \begin{pmatrix} \phi_{i,1}(y) \\ \phi_{i,2}(y, \dot{y}) \\ \vdots \\ \phi_{i,\mu_i}(y, \dot{y}) \end{pmatrix}$$

and all  $A_i$  matrix are of appropriated dimensions. Secondly, one exhibits the necessary and sufficient conditions under which the system (4.8) may be rewritten as (4.9). For the sake of simplicity, one introduces the following notations:

$$\begin{aligned} x_i &= (x_{i,1}, x_{i,2}, \dots, x_{i,\mu_i})^T \\ \tilde{x} &= (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_p)^T \\ \hat{x}_i &= (\hat{x}_{i,1}, \hat{x}_{i,2}, \dots, \hat{x}_{i,\mu_i})^T \quad \text{for } i = 1, \dots, p \end{aligned}$$

where  $\tilde{x}_i = \hat{x}_{i,2} + E_1 \lambda_{i,1} \text{sgn}(y_i - \hat{x}_{i,1})$  and  $E_1 = 1$  if  $(x_{1,1} - \hat{x}_{1,1}) = \dots = (x_{1,p} - \hat{x}_{1,p}) = 0$ , else  $E_1 = 0$ .

Let us construct for the system (4.9) the sliding observer:

$$\begin{aligned} \dot{\hat{x}}_{i,1} &= \hat{x}_{i,2} + \phi_1(y) + \lambda_{i,1} \text{sgn}(y_i - \hat{y}_i) \\ \dot{\hat{x}}_{i,2} &= \hat{x}_{i,3} + \phi_2(y, \dot{y}) + E_1 \lambda_{i,2} \text{sgn}(y_i - \hat{y}_i) \\ &\vdots \\ \dot{\hat{x}}_{i,\mu_i-1} &= \hat{x}_{i,\mu_i} + \phi_{\mu_i-1}(y, \dot{y}) + E_1 \lambda_{i,\mu_i-1} \text{sgn}(y_i - \hat{y}_i) \\ \dot{\hat{x}}_{i,\mu_i} &= \phi_{\mu_i}(y, \dot{y}) + E_1 \lambda_{i,\mu_i} \text{sgn}(y_i - \hat{y}_i) \\ \dot{\hat{y}}_i &= \hat{x}_{i,1} \end{aligned} \tag{4.10}$$

for  $i = 1, \dots, p$  where:  $\dot{\hat{y}}_i \triangleq \hat{x}_{i,2} + E_1 \lambda_{i,1} \text{sgn}(y_i - \hat{y}_i)$

From this, one deduces a part of the error's observation dynamic ( $e_{i,1} = (y_i - \hat{x}_{i,1})$  and  $e_{i,2} = \hat{y}_i - \hat{x}_{i,2}$ ):

$$\dot{e}_{i,1} = e_{i,2} + \lambda_{i,1} \text{sgn}(e_{i,1})$$

Therefore, using the same method as in the previous section one obtains:

**Theorem 38** *Under the conditions:*

- 1)  $\lambda_{i,1} > |e_{2,i}|_{\max}$  for  $i = 1, \dots, p$ .
- 2) All the  $\lambda_{i,j}$   $i = 1, \dots, p$ ,  $j = 2, \dots, \mu_i$  are such that  $\left[ sI - \left( A_i + \frac{\Lambda_i}{\lambda_{i,1}} u_1 \right) \right]$  is a Hurwitz polynomial. Where  $u_1 = (1, 0, \dots, 0)^T$  and  $A_i$  is the

$(\mu_i - 1) \times (\mu_i - 1)$  matrix defined by

$$A_i \text{ is a } \mathbb{R}^{(\mu_i-1) \times (\mu_i-1)} \text{ matrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The observer (4.10) gives, in finite time  $t_0$ , the convergence of  $\hat{y}$  (respectively  $\dot{y}$ ) to  $y$  (respectively to  $\dot{y}$ ), and an asymptotic linear stable observation error dynamics on the sliding surface ( $e_{i,1} = 0$ ).

**Proof** The dynamics of the observation error are

$$\dot{e}_i = A_i e_i + \phi(y, \dot{y}) - \phi(y, \dot{\hat{y}}) - \Lambda_i \text{sgn}(y_i - \hat{y}_i)$$

for  $i = 1, \dots, p$ . It is clear that, after a finite time  $t_0$ , one has  $\dot{y} = \dot{\hat{y}}$ , so  $\phi(y, \dot{y}) - \phi(y, \dot{\hat{y}}) = 0$ . So that, for  $\forall t > t_0$  the error dynamics will be on the reduced manifold ( $e_{i,1} = 0$ ),  $\forall i \in \{1, \dots, p\}$ , and given by

$$\dot{\bar{e}}_i = A_i \bar{e}_i - \Lambda_i \frac{e_{i,2}}{\lambda_{i,1}} \quad \text{for } i = 1, \dots, p \quad (4.11)$$

with  $\bar{e}_i = (e_{i,2}, e_{i,3}, \dots, e_{i,\mu_i})$  which is linear. If  $\left[ sI - (A_i + \frac{\Lambda_i}{\lambda_{i,1}} u_1) \right]$  is Hurwitz, this dynamic is asymptotically stable.

One has shown that using a sliding mode observer (4.10), the system (4.9) may be, under an appropriate choice of  $\lambda_{i,j}$ , observed with a linear asymptotic stable observation error dynamics (4.11).

In the next proposition, one characterizes the observability indices of the output  $\bar{y} \triangleq (y, \dot{y}) = (h, L_f h)$ .

**Proposition 39** *Considering the system (4.8) with the extended output:  $\bar{y} = (y, \dot{y}) = (h, L_f h)$ :*

$$\begin{aligned} \dot{\xi} &= f(\xi) \\ \bar{y} &= (h(\xi), L_f h(\xi)) \end{aligned} \quad (4.12)$$

*the indices of observability become:*

$$\bar{\mu}_i = \begin{cases} 1 & \text{if } i \in \{1, \dots, p\} \\ & \text{one has } \bar{y}_i = y_j \text{ with } j = i \\ \mu_j - 1 & \text{if } i \in \{p+1, \dots, 2p\} \\ & \text{one has } \bar{y}_i = \dot{y}_j \text{ with } j = i - p \end{cases}$$

*where  $\mu_i$  is the observability indices of the output  $y_i$  in the system (4.8).*

For the proof see the appendix, page 123.

**Remark 40** *The necessary and sufficient conditions to obtain output and output derivative form are the same as those in Theorem 35 for the extended output  $\bar{y} = (y, \dot{y})$ .*

From the last remark, necessary and sufficient conditions for the existence of a diffeomorphism transforming (4.8) into (4.12) are given by applying Theorem 36 to system (4.12) rewritten only in terms of the real output  $y$ .

**Theorem 41** *There exists a change of coordinates transforming (4.12) into (4.9) if and only if*

1) *If one denotes (with a possible reordering of the  $h_i$ )*

$$Q = \left\{ L_f^{j-1}(dh_i) \text{ with } i = 1, \dots, p \text{ and } j = 1, \dots, \mu_i \right\}$$

*then  $\dim \text{span } Q = n$  in a neighborhood of  $\xi^0$ .*

2) *If one denotes for  $j = 1, \dots, p$*

$$Q_j = \left\{ L_f^{k-1}(dh_i) \begin{array}{l} i = 1, \dots, p \\ k = 1, \dots, \mu_j \end{array} \right\} - \left\{ L_f^{\mu_j-1}(dh_j) \right\}$$

*then for  $j = 1, \dots, p$   $\text{span } Q_j = \text{span } Q \cap Q_j$*

3) *There exists vector fields  $\bar{g}^1, \bar{g}^2, \dots, \bar{g}^{2p}$  satisfying:*

$$L_{\bar{g}^l} L_f^{l-1}(h_j) = \delta_{i,j} \delta_{l,\mu_i}, \quad \text{with } \begin{cases} i, j = 1, \dots, p, \\ l = 1, \dots, \mu_i \end{cases}$$

$$L_{\bar{g}^i}(h_j) = \delta_{i,j+p}, \quad \text{with } \begin{cases} i = p+1, \dots, 2p, \\ j = 1, \dots, p \end{cases}$$

$$\text{and } L_{\bar{g}^i}(L_f(h_j)) = 0, \quad \text{with } \begin{cases} i = p+1, \dots, 2p, \\ j = 1, \dots, p \end{cases}$$

4) *Setting :*

$$\bar{\Delta} = \left\{ \text{ad}_{(-f)}^k \bar{g}^i, \begin{array}{l} i = 1, \dots, p \\ k = 0, \dots, \mu_i - 2 \end{array} \right\} \cup \{ \bar{g}^i, i = p+1, \dots, 2p \}$$

$$\forall u, v \in \bar{\Delta}, \quad u \neq v \Rightarrow [u, v] = \mathbf{0}$$

For the proof see the appendix, page 124.

**Remark 42** *From the proof of Theorem 41, one can see that the Definition of  $\bar{g}^i$  for  $i = 1, \dots, p$  is the same as the definition of  $g^i$ . However, condition 3. is less restrictive than the one given in Theorem 35.*

**Example 43** *Let us consider the following system:*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + x_2^2 \\ \dot{x}_3 &= -x_3 - x_2 - x_1 \\ y &= x_1\end{aligned}\tag{4.13}$$

*which is in output and output derivative nonlinear injection form and can not be transformed into output injection form. In fact, as defined in Theorem 35, the vector  $g^1$  is such that:*

$$\begin{aligned}L_{g^1}(x_1) &= L_{g^1}(x_2) = 0 \\ L_{g^1}(x_3 + x_2^2) &= 1\end{aligned}\tag{4.14}$$

*So,  $g^1 = (0, 0, 1)^T$ ,  $ad_{(-f)}^1 g^1 = (0, 1, -1)^T$  and  $ad_{(-f)}^2 g^1 = (1, 2x_2 - 1, 0)^T$ .*

*The Lie brackets of these vectors are equal to*

$$\begin{aligned}\left[ g^1, ad_{(-f)}^1 g^1 \right] &= \left[ g^1, ad_{(-f)}^2 g^1 \right] = 0 \\ \left[ ad_{(-f)}^1 g^1, ad_{(-f)}^2 g^1 \right] &= (0, 2, 0)^T \neq 0\end{aligned}$$

*Consequently, this system does not verify the conditions of Theorem 35. Looking now at the conditions of Theorem 38, one has for the vectors  $\bar{g}^1$  and  $\bar{g}^2$ :*

$$\begin{aligned}L_{\bar{g}^1}(x_1) &= L_{\bar{g}^1}(x_2) = 0, \quad L_{\bar{g}^1}(x_3 + x_2^2) = 1 \\ L_{\bar{g}^2}(x_1) &= 1, \quad L_{\bar{g}^2}(x_2) = 0\end{aligned}$$

*So,  $\bar{g}^1 = (0, 0, 1)^T$ ,  $ad_{(-f)}^1 \bar{g}^1 = (0, 1, -1)^T$ , and  $\bar{g}^2 = (1, 0, *)^T$ . Then if one chooses  $* = 0$  for example, one obtains:*

$$\left[ \bar{g}^1, ad_{(-f)}^1 \bar{g}^1 \right] = \left[ \bar{g}^2, \bar{g}^1 \right] = \left[ \bar{g}^2, ad_{(-f)}^1 \bar{g}^1 \right] = 0$$

*Thus, this system verifies all the conditions of Theorem 38. Choosing  $z_1 = x_1$ ;  $z_2 = x_2$ ;  $z_3 = x_2 + x_3$ , one obtains in the new coordinates the following system:*

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + \phi_2(z_1, z_2) \\ \dot{z}_3 &= \phi_3(z_1, z_2) \\ y &= z_1\end{aligned}$$

**Remark 44** Every system in the form of (4.6) is obviously on the form (4.9). One important consequence of the previous remark and the example is that the conditions of Theorem 35 imply conditions of Theorem 38, but the converse is false.

In the next section we consider an actuated system but for the sake of simplicity only in a single input single output (SISO) form.

## 4.4 Triangular input observer form

Let us consider the following SISO analytic system  $\Sigma$

$$\begin{aligned}\dot{x} &= f(x) + g(x, u) \\ y &= h(x)\end{aligned}\tag{4.15}$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}$  is the input,  $y \in \mathbb{R}$  is the output and  $f, g, h$  are analytical function vectors of appropriate dimensions. Moreover for any  $x \in \mathbb{R}^n$  the function  $g(x, 0)$  is equal to zero and the system (4.15) is assumed bounded input bounded state in finite time. In order to transform (4.15) in a triangular input observer form, we modified the classical *observation rank condition*:

**Condition 45**

$$\text{rank} \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \\ dL_f^\infty h \end{pmatrix} = n$$

where  $L_f$  denotes the classical Lie derivative in  $f$  and  $dh$  is the classical one form.

**Remark 46** Condition 45 is the classical one for an autonomous system. In the nonlinear context, we can't refer to the Cayley–Hamilton theorem.

But in the next we assume

**Condition 47**

$$\text{rank} \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \end{pmatrix} = n$$

From condition 47 it is known that the codistribution

$$\Omega^i = \text{span}\{dh, \dots, dL_f^i h\} \quad 0 \leq i \leq n-1$$

is involutive. We also need the following condition

**Condition 48** *The vector field  $g$  verifies for any  $u \in \mathbb{R}$*

$$dL_g L_f^i h \in \Omega^i \quad \forall i \in \{0, \dots, n-1\}$$

Now we can set the following Theorem :

**Theorem 49** *System (4.15) may be transformed, by diffeomorphism, in the neighborhood of  $x$  in a triangular input observer form*

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} \xi_2 + \bar{g}_1(\xi_1, u) \\ \xi_3 + \bar{g}_2(\xi_1, \xi_2, u) \\ \vdots \\ \xi_n + \bar{g}_{n-1}(\xi_1, \dots, \xi_{n-1}, u) \\ f_n(\xi) + \bar{g}_n(\xi, u) \end{pmatrix} \quad (4.16)$$

$$y = \xi_1$$

with  $\bar{g}_i(\cdot, u=0) = 0$  for any  $i \in \{1, \dots, n\}$ , if and only if conditions 47 and 48 hold in the neighborhood of  $x$ .

For the proof see the appendix, page 125.

#### 4.4.1 Sliding mode observer design for triangular input observer form

From the work of Drakunov and Utkin [14, 15] and our previous work [28, 16, 6], we propose the *sliding observer for triangular input observer form*

$$\begin{pmatrix} \dot{\hat{\xi}}_1 \\ \dot{\hat{\xi}}_2 \\ \vdots \\ \dot{\hat{\xi}}_{n-1} \\ \dot{\hat{\xi}}_n \end{pmatrix} = \begin{pmatrix} \hat{\xi}_2 + \bar{g}_1(\xi_1, u) + \lambda_1 \text{sgn}(\xi_1 - \hat{\xi}_1) \\ \hat{\xi}_3 + \bar{g}_2(\xi_1, \tilde{\xi}_2, u) + \lambda_2 \text{sgn}_1(\tilde{\xi}_2 - \hat{\xi}_2) \\ \vdots \\ \hat{\xi}_n + \bar{g}_{n-1}(\xi_1, \tilde{\xi}_2, \dots, \tilde{\xi}_{n-1}, u) + \lambda_{n-1} \text{sgn}_{n-2}(\tilde{\xi}_{n-1} - \hat{\xi}_{n-1}) \\ f_n(\xi_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n) + \bar{g}_n(\xi_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n, u) + \lambda_n \text{sgn}_{n-1}(\tilde{\xi}_{n-1} - \hat{\xi}_n) \end{pmatrix} \quad (4.17)$$

where

$$\begin{aligned}
\tilde{\xi}_2 &= \hat{\xi}_2 + \lambda_1 \text{sgn}_1(\xi_1 - \hat{\xi}_1) \\
\tilde{\xi}_3 &= \hat{\xi}_3 + \lambda_2 \text{sgn}_2(\tilde{\xi}_2 - \hat{\xi}_2) \\
&\vdots \\
\tilde{\xi}_n &= \hat{\xi}_n + \lambda_n \text{sgn}_{n-1}(\tilde{\xi}_{n-1} - \hat{\xi}_{n-1})
\end{aligned}$$

and the  $\text{sgn}_i(\xi)$  function denotes the usual  $\text{sgn}$  function but with a low pass filter of the  $\xi$  variable [15] and an anti-peaking structure [6]. This anti-peaking structure follows the idea that we do not inject the observation error information before reaching the sliding manifold linked with this information (i.e.,  $\text{sign}_i = E_i \text{sign}$ , with  $E_i = 1$  if  $E_1 = \dots = E_{i-1} = 1$  and  $\xi_1 - \hat{\xi}_1 = 0$  else  $E_i = 0$ ). Moreover we reach the manifold one by one. Doing this we obtain a “high gain” dynamic (i.e., see the equivalence between the sliding mode and the high gain [32]) of dimension one and consequently we do not have a peaking phenomena [42]. More precisely  $\text{sgn}_i(\cdot)$  is equal to zero if there exists  $0 < j < i - 1$  such that  $\tilde{\xi}_j - \hat{\xi}_j \neq 0$  (by definition  $\tilde{\xi}_1 = \xi_1$ ), else  $\text{sgn}_i(\cdot)$  is equal to the usual  $\text{sgn}(\cdot)$  function. In the observer structure, this particular  $\text{sgn}$  function allows that  $\tilde{\xi}_i - \hat{\xi}_i$  converges to zero if all the  $\tilde{\xi}_j - \hat{\xi}_j$  with  $j < i$  have converged to zero before.

**Theorem 50** *Considering a bounded input bounded state (BIBS) in finite time system (4.16) and observer (4.17), for any initial state  $\xi(0)$ ,  $\hat{\xi}(0)$  and any bounded input  $u$ , there exists a choice of  $\lambda_i$  such that the observer state  $\hat{\xi}$  converges in finite time to  $\xi$ .*

**Proof** From (4.16) and (4.17) and considering the initial state condition such that  $\xi_1(0) \neq \hat{\xi}_1(0)$  (if this is not the case, we directly move on to the next step of the proof). Thus we are in the

• **first step** of our proof and we obtain the following observation error dynamics  $e = \xi - \hat{\xi}$ .

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 - \lambda_1 \text{sgn}(\xi_1 - \hat{\xi}_1) \\ e_3 + \bar{g}_2(\xi_1, \xi_2, u) - \bar{g}_2(\xi_1, \hat{\xi}_2, u) \\ \vdots \\ e_n + \bar{g}_{n-1}(\xi_1, \xi_2, \dots, \xi_{n-1}, u) - \bar{g}_{n-1}(\xi_1, \hat{\xi}_2, \dots, \hat{\xi}_{n-1}, u) \\ (\bar{f}_n(\xi) - \bar{f}_n(\xi_1, \hat{\xi}_2, \dots, \hat{\xi}_n) + \bar{g}_n(\xi, u) - \bar{g}_n(\xi_1, \hat{\xi}_2, \dots, \hat{\xi}_n, u)) \end{pmatrix}$$

Thus as the input  $u$  is bounded the state  $\xi$  does not go to infinity in finite time. Moreover if  $\hat{\xi}_1$  is bounded all the states of the observer are also

bounded during step 1. Consequently the observation error state is also bounded. Now, setting  $V_1 = \frac{e_1^2}{2}$ , we have

$$\dot{V}_1 = e_1(e_2 - \lambda_1 \operatorname{sgn}(e_1))$$

Thus choosing  $\lambda_1 > |e_2|_{max}$  the observation error  $e_1$  goes to zero in finite time  $t_1$ . Moreover, if after  $t_1$  the observation error stays equal to zero (i.e.,  $\lambda_1 > |e_2|_{max}$ ) we have  $e_2 = \lambda_1 \operatorname{sgn}(\xi_1 - \hat{\xi}_1)$  and consequently  $\tilde{\xi}_2 = \xi_2$ . Now we pass to the:

• **second step.** Here, we ensure that the observation error  $e_2$  is bounded in order to remain on the manifold  $e_1 = 0$ . Moreover, we want to reach the submanifold  $e_1 = e_2 = 0$ . Using the same argument as in [14, 15] the equivalent vector is obtained in finite time via a low pass filtering of  $\lambda_1 \operatorname{sgn}(\xi_1 - \hat{\xi}_1)$  which is equal to  $e_2$ . Thus, as at  $t_1$ , we have  $e_1 = 0$ , and the observation error is now equal to

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 - \lambda_1 \operatorname{sgn}(\xi_1 - \hat{\xi}_1) = 0 \\ e_3 - \lambda_2 \operatorname{sgn}(\xi_2 - \hat{\xi}_2) \\ e_4 + \bar{g}_3(\xi_1, \xi_2, \xi_3, u) - \bar{g}_3(\xi_1, \xi_2, \hat{\xi}_3, u) \\ \vdots \\ e_n + \bar{g}_{n-1}(\xi_1, \xi_2, \dots, \xi_{n-1}, u) - \bar{g}_{n-1}(\xi_1, \xi_2, \hat{\xi}_3, \dots, \hat{\xi}_{n-1}, u) \\ (\bar{f}_n(\xi) - \bar{f}_n(\xi_1, \xi_2, \dots, \xi_n) + \bar{g}_n(\xi, u) - \bar{g}_n(\xi_1, \xi_2, \dots, \xi_n, u)) \end{pmatrix}$$

Setting  $V_2 = \frac{e_1^2}{2} + \frac{e_2^2}{2}$ , we obtain

$$\dot{V}_2 = e_1(e_2 - \lambda_1 \operatorname{sgn}(e_1)) + e_2(e_3 - \lambda_2 \operatorname{sgn}(e_2))$$

Moreover, if the condition  $\lambda_1 > |e_2|_{max}$  holds for  $t > t_1$ , we have  $e_1 = 0$  and  $e_2 - \lambda_1 \operatorname{sgn}(e_1) = 0$ , thus we find

$$\dot{V}_2 = e_2(e_3 - \lambda_2 \operatorname{sgn}(e_2))$$

Consequently  $e_2$  goes to zero in finite time  $t_2 > t_1$  if  $\lambda_2 > |e_3|_{max}$ . Moreover, from  $V_2$  we obtain that the observation error is strictly decreasing during the period of time  $[t_1, t_2]$ . This implies that the condition on  $\lambda_1$  is verified after  $t_1$  if it is verified before  $t_1$ . Moreover as the input is bounded, the state  $\xi$  stays bounded during the period  $[0, t_2]$  and from the structure of the observation error the dynamics  $e$  is also bounded and consequently  $\hat{\xi}$  is too.

Now let us assume that we are at the step  $j < n$ . This step starts at time  $t_{j-1}$  and at  $t_{j-1}$ , all the  $e_k = 0$  and all the conditions on  $\lambda_k$  are

verified for  $k < j$ . Thus, we proceed to

• **step  $j$** . The observation error dynamic is equal to

$$\begin{pmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{j-1} \\ \dot{e}_j \\ \dot{e}_{j+1} \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 - \lambda_1 \operatorname{sgn}(\xi_1 - \hat{\xi}_1) = 0 \\ \vdots \\ e_j - \lambda_{j-1} \operatorname{sgn}(\xi_{j-1} - \hat{\xi}_{j-1}) = 0 \\ e_{j+1} + \lambda_j \operatorname{sgn}(\xi_j - \hat{\xi}_j) \\ e_{j+2} + \bar{g}_{j+1}(\xi_1, \dots, \xi_{j+1}, u) - \bar{g}_{j+1}(\xi_1, \dots, \xi_j, \hat{\xi}_{j+1}, u) \\ \vdots \\ e_n + \bar{g}_{n-1}(\xi_1, \xi_2, \dots, \xi_{n-1}, u) \\ -\bar{g}_{n-1}(\xi_1, \dots, \xi_j, \hat{\xi}_{j+1}, \dots, \hat{\xi}_{n-1}, u) \\ (\bar{f}_n(\xi) - \bar{f}_n(\xi_1, \dots, \xi_j, \hat{\xi}_{j+1}, \dots, \hat{\xi}_n)) \\ +\bar{g}_n(\xi, u) - \bar{g}_n(\xi_1, \dots, \xi_j, \hat{\xi}_{j+1}, \dots, \hat{\xi}_n, u) \end{pmatrix}$$

Setting  $V_j = \sum_{i=1}^j \frac{e_i^2}{2}$  we deduce from  $e_k = 0 \forall i < j$  that

$$\dot{V}_j = e_j(e_{j+1} - \lambda_j \operatorname{sgn}(e_j))$$

Consequently  $e_j$  goes to zero in finite time  $t_j > t_{j-1}$  if  $\lambda_j > |e_{j+1}|_{\max}$  and all  $\lambda_k$  conditions are verified for  $k < j$ . As the input is bounded  $\xi$  is bounded and from the observer structure  $\hat{\xi}_j$  is also bounded during the period  $[0, t_j]$ . It follows that  $e_j$  is bounded and we can find  $\lambda_j$  such that  $\lambda_j > |e_{j+1}|_{\max}$  is verified. Moreover, as  $e_j$  is decreasing during the period  $[t_{j-1}, t_j]$ ,  $\lambda_{j-1} > |e_j|_{\max}$  is verified during this period and therefore all the  $e_k$  remain equal to zero for any  $k < j$ .

Now we go to :

• **step  $n$** . This step starts at the time  $t_{n-1}$  and at this time  $e_k = 0$  for any  $k < n$ . Thus we obtain the following observation error dynamics

$$\begin{pmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} e_2 + \lambda_1 \operatorname{sgn}(\xi_1 - \hat{\xi}_1) = 0 \\ \vdots \\ e_n + \lambda_{n-1} \operatorname{sgn}(\xi_{n-1} - \hat{\xi}_{n-1}) = 0 \\ \lambda_n \operatorname{sgn}(\xi_n - \hat{\xi}_n) \end{pmatrix}$$

Setting  $V_n = \sum_{i=1}^n \frac{e_i^2}{2}$  we deduce from  $e_k = 0 \forall i < n$  that

$$\dot{V}_n = e_n [-\lambda_n \operatorname{sgn}(e_n)]$$

So,  $e_n$  go to zero in finite time  $t_n > t_{n-1}$  for any  $\lambda_n > 0$  and if all the conditions on the  $\lambda_k$  for  $k < n$  are verified after  $t_{n-1}$ . Condition on  $\lambda_{n-1}$  is always verified because  $e_n$  is decreasing after  $t_{n-1}$  and by induction all conditions follow.

#### 4.4.2 Observer matching condition

It is well known from the work [19] that roughly speaking, a condition in order to reject a perturbation, is that the perturbation act in the same direction of the control.

In the same manner of thinking, for observer design we obtain the condition in order to observe the state under unknown perturbation. Consider the linear observable bounded perturbed system:

$$\dot{x} = Ax + Bu + Pw \quad (4.18)$$

and the output equation is  $y = Cx$  with  $y \in \mathbb{R}$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $w \in [-B_w, B_w]$

$$\mathcal{O}(A, C) = \begin{pmatrix} C \\ \vdots \\ CA^{n-2} \end{pmatrix} = n$$

A condition in order to cancel the perturbation effect on the state observation is that

$$\begin{pmatrix} C \\ \vdots \\ CA^{n-2} \end{pmatrix} P = 0$$

which is called the *observer matching condition*.

**Remark 51** *Necessity is obvious such that the perturbation derivative time does not act on the state observations.*

Sufficiency is clear: considering for example, an observer for triangular input observer.

Generalizing the previous observer matching condition to the bounded input bounded state single input single output (BIBS-SISO) local weakly observable nonlinear perturbed system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w := F(x, u) + p(x)w \\ y &= h(x) \end{aligned} \quad (4.19)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and the bounded perturbation  $w \in [-B_w, B_w]$ , and  $f$ ,  $g$ ,  $p$  are functions vector fields of appropriate dimensions.

We immediately obtain the following sufficient conditions in order to reject the perturbation effect on the observer.

**Proposition 52** *If the system (4.19) without perturbation verifies conditions (47) and Condition 48 of Theorem 49 and the observer matching condition*

$$\begin{pmatrix} dh \\ dL_F h \\ \vdots \\ dL_F^{n-2} h \end{pmatrix} p(x) = 0 \quad (4.20)$$

*in the neighborhood of  $x$ , and where the Lie derivative is done with respect to  $x$  and  $u$ . Then it is possible to locally design an observer which estimates all state components and does this in both cases: with and without perturbation.*

**Proof** The proof is a direct consequence of Theorem 49 and sliding mode triangular observer design.

## 4.5 Simulations and comments

Let us consider the following system  $\Sigma$  which is in the triangular input observer form

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3 u \\ \dot{x}_2 &= x_3 + x_2 x_1 u \\ \dot{x}_3 &= -3x_3 - 3x_2 - x_1 - x_3^3 - u \\ y &= x_1 \end{aligned} \quad (4.21)$$

For this system, the observer 4.17 takes the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - \hat{x}_1^3 u + \lambda_1 \operatorname{sgn}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + \tilde{x}_2 x_1 u + \lambda_2 \operatorname{sgn}_1(\tilde{x}_2 - \hat{x}_2) \end{aligned} \quad (4.22)$$

$$\begin{aligned} \dot{\hat{x}}_3 &= -3\tilde{x}_3 - 3\tilde{x}_2 - x_1 - \tilde{x}_3^3 - u + \lambda_3 \operatorname{sgn}_2(\tilde{x}_3 - \hat{x}_3) \\ y &= x_1 \end{aligned} \quad (4.23)$$

with  $\tilde{x}_2 = \hat{x}_2 + \lambda_1 \operatorname{sgn}_1(x_1 - \hat{x}_1)$  and  $\tilde{x}_3 = \hat{x}_3 + \lambda_2 \operatorname{sgn}_2(\tilde{x}_2 - \hat{x}_2)$ , and where  $\operatorname{sgn}_i$  functions are designed as noted in Section 4.3.

This approach has been tested by simulation with the following initial conditions  $x = (1, 0.5, 0.5)^T$  and  $\hat{x} = (0, 0, 0)^T$ . Moreover, we have chosen a first-order low pass filter with a cut frequency equal to 100Hz and observation gain  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  respectively equal to 4, 2, and 2. Moreover the function “ $\operatorname{sgn}$ ” is approximated by a saturation function with a slow rate equal to  $10^4$ .

In Figure 4.1, we see that  $\hat{x}_1$  reaches  $x_1$  in finite time  $\simeq 0.25s$ . In Figure 4.2, we see that  $\hat{x}_2$  also reaches  $x_2$  in finite time  $\simeq 0.75s$ . But  $\hat{x}_2$  will only reach  $x_2$  when  $\hat{x}_1$  will be equal to  $x_1$ . In Figure 4.3, we see that  $\hat{x}_3$  reaches  $x_3$  in finite time  $\simeq 1s$ .

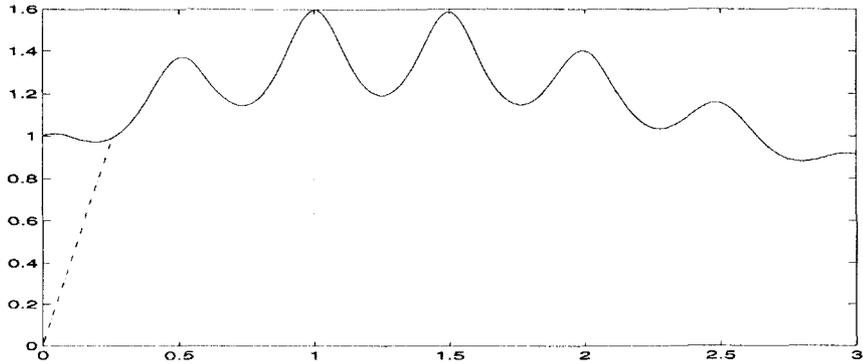


Figure 4.1:  $x_1(-)$  and  $\hat{x}_1(-)$

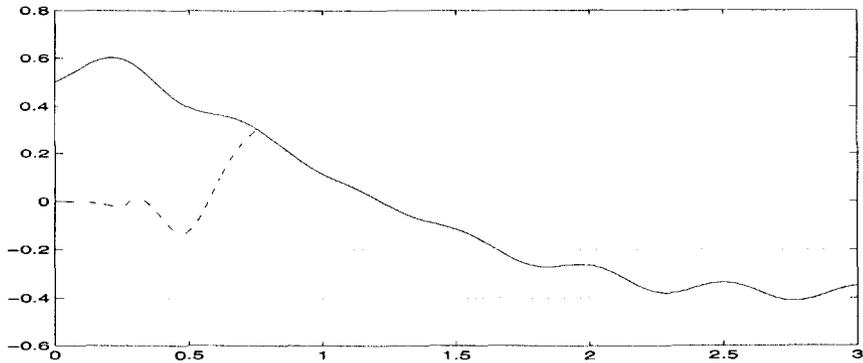


Figure 4.2:  $x_2(-)$  and  $\hat{x}_2(-)$

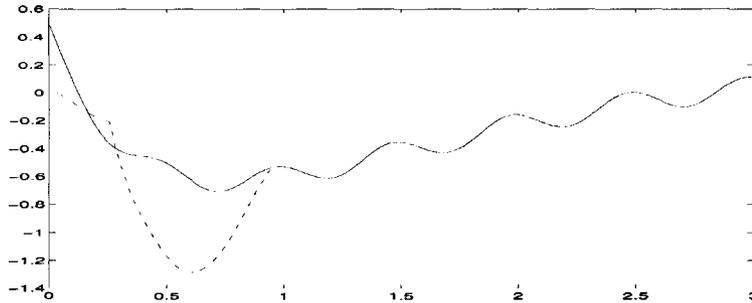


Figure 4.3:  $x_3(-)$  and  $\hat{x}_3 (-)$

Now, starting from the same initial conditions, we add an output noise in order to show the behavior of the observer in this case. In [6], following the work of Yaz and Azemi [47], the author proposed to use a saturation function with dead zone for observer in the case of the extended injection form. This reduces the observer sensitivity to the noise, but we were obliged to change the observer gain as follows  $\lambda_1 = \lambda_2 = \lambda_3 = 4$  in order to recover a time response quite similar to the previous simulation.

In Figures 4.4, 4.5, 4.6 and 4.7, we see that the observer state  $\hat{x}$  reaches the neighborhood of the system state  $x$  in finite time. But we also see that the noise is not totally suppressed in the observer. We can reduce this noise with some minor modifications by introducing an asymptotic gain or a *sgn* function modified with respect to the noise output knowledge [47], for example.

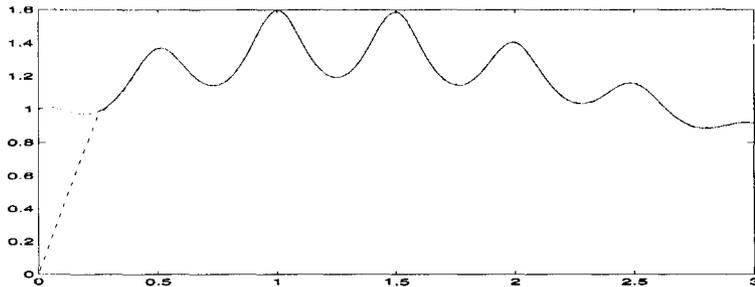


Figure 4.4:  $x_1(-)$  and  $\hat{x}_1 (-)$

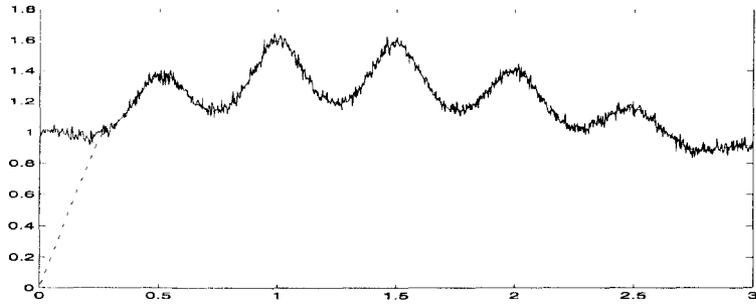


Figure 4.5: Measured  $x_1(-)$  and  $\hat{x}_1(-)$

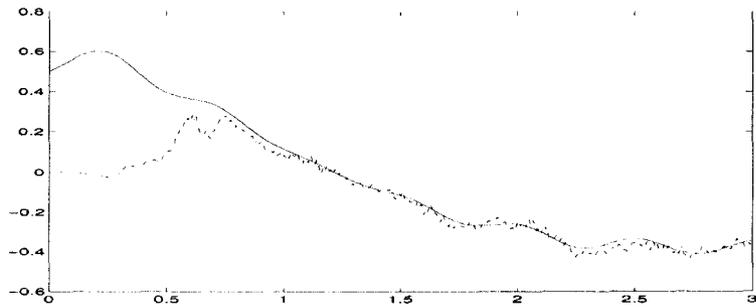


Figure 4.6:  $x_2(-)$  and  $\hat{x}_2(-)$

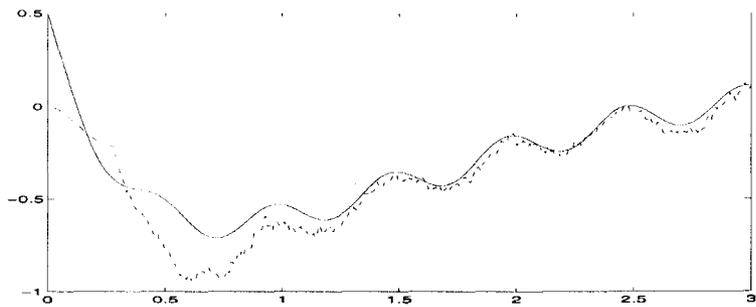


Figure 4.7:  $x_3(-)$  and  $\hat{x}_3(-)$

## 4.6 Conclusion

In this chapter, we introduced a sliding observer that does not depend on the derivative of  $u$ . This is due to the fact that our main application domain is the AC motor where the derivative of the input does not exist or is not easy to obtain. This appears, for example, when we consider the converter in the observer and control scheme. But, if it exists, and if it is technologically possible to obtain  $\dot{u}$ ,  $\ddot{u}$ , ..., and so on. A very clever observer form was given in [23]. For this form, many observer designs work well, and in this case, advantages of the sliding mode observer were principally the design simplicity and the finite time convergence. In practical observer design, we always take into account the output noise, thus generally we replace the *sgn* function by a modified *sgn* function or higher order sliding mode. In the latter, we think that it is important, when it is possible, as it is proposed in [15], to design an observer without the use of diffeomorphism, because the observer validity domain is restricted to the diffeomorphism validity domain.

## 4.7 Appendix

### 4.7.1 Proof of Proposition 39

From Definition 33, the indices  $\mu_i$  verify:

- $\sum_{i=1}^p \mu_i = n$ , so from the Definition of  $\bar{\mu}_i$ , one has:  $\sum_{i=1}^{2p} \bar{\mu}_i = n$ .

- $\Delta = \left\{ L_f^{j-1}(dh_i) : i = 1, \dots, p; j = 1, \dots, \mu_i \right\}$  are linearly independent.

As  $L_f^j(dh_i) = L_f^{j-1}(L_f(dh_i)) = L_f^{j-1}(\dot{y}_i)$ .  $\Delta$  will be rewritten as

$$\left\{ L_f^{j-1}(d\bar{y}_i) : i = 1, \dots, 2p; j = 1, \dots, \bar{\mu}_i \right\}$$

- Thus, if  $\mu_i$  verify 3. of Definition 33, it is easy to see that: If  $l_1, \dots, l_{2p}$  satisfies  $\sum_{i=1}^{2p} l_i = n$  and  $\left\{ L_f^{j-1}(d\bar{y}_i) : i = 1, \dots, 2p; j = 1, \dots, l_i \right\}$  are linearly independent at some  $\xi \in \mathcal{U}$ , then  $(l_1, \dots, l_{2p}) \geq (\bar{\mu}_1, \dots, \bar{\mu}_{2p})$  in the lexicographic reordering  $[(l_1 > \bar{\mu}_1) \text{ or } (l_1 = \bar{\mu}_1 \text{ and } l_2 > \bar{\mu}_2) \text{ or } \dots \text{ or } (l_1 = \bar{\mu}_1, \dots, l_p = \bar{\mu}_{2p})]$ .

Then, the  $2p$ -tuple  $\bar{\mu}_1, \dots, \bar{\mu}_{2p}$  satisfies the three conditions of Definition 33.

## 4.7.2 Proof of Theorem 41

First, starting from Theorem 36, where  $y$  is substituted by  $\bar{y}$ , one proves hereafter that conditions 1. and 2. of Theorem 35 (which are required in Theorem 36) are equivalent to conditions 1. and 2. of Theorem 41. For the Theorem 36, let us define:  $\bar{Q} = \left\{ L_f^{j-1}(d\bar{y}_i) : i = 1, \dots, 2p; j = 1, \dots, \bar{\mu}_i \right\}$

and for  $j = 1, \dots, 2p$   $\bar{Q}_j = \left\{ L_f^{k-1}(d\bar{y}_i) : \begin{array}{l} i = 1, \dots, 2p; \\ k = 1, \dots, \bar{\mu}_j \end{array} \right\} - \left\{ L_f^{\bar{\mu}_j-1}(d\bar{y}_j) \right\}$

but

$\bar{Q} = \left\{ L_f^{j-1}(L_f(dh_i)) : \begin{array}{l} i = 1, \dots, p; \\ j = 1, \dots, \mu_i - 1 \end{array} \right\} \cup \{h_1, \dots, h_p\}$  and  $L_f^{j-1}(L_f(dh_i)) = L_f^j(dh_i)$ , then:

$$\bar{Q} = Q$$

So the equivalence of condition 1. is proved. Now, for condition 2., for Theorem 36 one computes  $\bar{Q}_j$ .

• For  $j = 1, \dots, p$  one has:

$$\bar{Q}_j = \left\{ L_f^{k-1}[L_f(d\bar{y}_i)] : \begin{array}{l} i = 1, \dots, 2p; \\ k = 1, \dots, \bar{\mu}_j - 1 \end{array} \right\}$$

and  $\bar{y}_j = L_f(h_j)$ , so

$$\bar{Q}_j = \left\{ \left[ L_f^{k-1}(L_f(dh_i)) : \begin{array}{l} i = 1, \dots, p; \\ k = 1, \dots, \mu_j - 1 \end{array} \right] \cup \{h_1, \dots, h_p\} \right\} - \left\{ L_f^{\bar{\mu}_j-1-1}(L_f(dh_j)) \right\}$$

as  $L_f^k(L_f(d\bar{y}_{i+p})) = L_f^{k-1}[L_f(d\bar{y}_i)]$ , one immediately has  $\bar{Q}_j = Q_j$ .

• For  $j = p+1, \dots, 2p$  one has  $\bar{y}_j = h_j$ , so

$\bar{Q}_j = \{dh_1, dh_2, \dots, dh_p, L_f(dh_1), \dots, L_f(dh_p)\} - \{dh_j\}$  then, as  $\mu_i \geq 2$ , one obtains  $\bar{Q}_j \cap Q = Q_j$  for  $j = 1, \dots, p$ .

Thus, the condition 2. of Theorem 41 is equivalent to condition 2. of Theorem 35.

Secondly, in the same way, one proves the equivalence between conditions 3. and 4. of Theorem 41, and the last conditions of Theorem 36, where  $y$  is substituted by  $\bar{y}$ . Theorem 36 applied to  $\bar{y}$  gives:

There exists a change of coordinates transforming (4.12) into (4.9) if and only if the previous conditions hold and there exist vector fields  $\bar{g}^1, \bar{g}^2, \dots, \bar{g}^{2p}$  satisfying

$$L_{\bar{g}^i} L_f^{l-1}(\bar{y}_j) = \delta_{i,j} \delta_{l, \bar{\mu}_i}, \quad \begin{array}{l} i, j = 1, \dots, 2p, \\ l = 1, \dots, \bar{\mu}_i, \end{array}$$

such that

$$[ad_{(-f)}^k \bar{g}^i, ad_{(-f)}^l \bar{g}^j] = 0 \quad (4.24)$$

for  $i, j = 1, \dots, 2p$ ;  $k = 0, \dots, \bar{\mu}_i - 1$ ;  $l = 0, \dots, \bar{\mu}_j - 1$ .

Now, one wants to rewrite this condition only as a function of  $y$ . Therefore, the  $p$  first vector fields  $\bar{g}^i$  are defined such that

$$L_{\bar{g}^i} L_f^{l-1}(\bar{y}_j) = \delta_{i,j} \delta_{l, \mu_i - 1}, \quad \begin{array}{l} i = 1, \dots, p, \quad j = 1, \dots, 2p \\ l = 1, \dots, \mu_i - 1, \end{array}$$

with the Definition of  $\bar{y}_j$ , this is equivalent to the real output  $y$  to

$$L_{\bar{g}^i} L_f^{l-1}(y_j) = \delta_{i,j} \delta_{l, \mu_i}, \quad \begin{array}{l} i, p = 1, \dots, p, \\ l = 1, \dots, \mu_i, \end{array} \quad (4.25)$$

Now, the  $p$ -last vector fields  $\bar{g}^i$  are defined such that

$$L_{\bar{g}^i}(\bar{y}_j) = \delta_{i,j}, \quad \begin{array}{l} i = p + 1, \dots, 2p, \\ j = 1, \dots, 2p \end{array}$$

which can be rewritten as:

$$\begin{array}{l} L_{\bar{g}^i}(h_j) = \delta_{i,j}, \quad i = p + 1, \dots, 2p, \\ L_{\bar{g}^i}(L_f(h_j)) = 0, \quad j = 1, \dots, p \end{array} \quad (4.26)$$

Thus, from (4.25) and (4.26), one obtains condition 3. of Theorem 41. Moreover, from this and (4.24) one immediately finds condition 4. of Theorem 41 and reciprocally.

### 4.7.3 Proof of Theorem 49

#### Sufficiency

If condition 47 holds, then

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} h \\ L_f(h) \\ \vdots \\ l_f^{n-1} h \end{pmatrix}$$

is a diffeomorphism and transforms system (4.15) in

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{n-1} \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} \xi_2 + \bar{g}_1(\xi, u) \\ \xi_3 + \bar{g}_2(\xi, u) \\ \vdots \\ \xi_n + \bar{g}_{n-1}(\xi, u) \\ f_n(\xi) + \bar{g}_n(\xi, u) \end{pmatrix}$$

with  $\bar{g}_i(\xi, u = 0) = 0$  for any  $i \in \{1, \dots, n\}$ . Moreover, in the  $x$  coordinate, condition 48 is equal to

$$d\bar{g}_i \in \text{span}\{dx_1, \dots, dx_i\} \quad \forall i \in \{1, n\} \quad (4.27)$$

this implies that the system is in form 4.16.

### **Necessity**

If there exists a diffeomorphism  $\xi = \phi(x)$  which transforms (4.15) into (4.16) condition 47 is directly verified by the existence of  $\phi$ . Moreover as (4.27) is a necessary condition, this implies that condition 48 is a necessary condition too.

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# Chapter 5

# Dynamic Sliding Mode Control and Output Feedback

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## 5.1 Introduction

The sliding mode design approach involves two distinct stages. The first considers the design of a switching function which provides desirable system performance in the sliding mode. The second consists of designing a control law which will ensure the sliding mode, and thus the desired performance, is attained and maintained. The first stage is often termed the *existence problem* and the second the *reachability problem*. Traditionally much of the work in the area of sliding mode control considered uncertain, often linear, state-space systems and the solution of both the existence and reachability problems assumed full state information was available to the control law. Thus, a switching function would be determined that was a function of the system states and an associated state-dependent control law would result.

Clearly the assumption of full state availability is restrictive; it may be impossible or impractical to measure all the states for many processes. One possible solution is to use an observer to estimate the system states and sliding mode techniques for such observation have been illustrated in the previous chapter. The alternative is to consider solutions to the existence and reachability problems which are dependent on system outputs alone.

Uncertain linear systems, represented by a nominal  $(A, B, C)$  triple, will be the initial focus of this chapter. A straightforward solution to the problem of sliding mode control via output feedback will be seen to be possible if the nominal triple is relative degree one, i.e., the product  $CB$  is full rank, and the transmission zeros of the nominal system are in the left hand plane, i.e., the triple is minimum phase. As may be expected from the full state scenario, these transmission zeros will appear as poles of the dynamics in the sliding mode. In fact, when the number of outputs and inputs is equal, these transmission zeros will wholly determine the sliding mode performance in general and the existence problem is trivial. If there are more measured process outputs than control inputs, then it will be seen that the solution to the existence problem may be formulated as the design of a static output feedback controller for a particular sub-system triple. It is well known that any triple is stabilizable via static output feedback if it is both controllable and observable and satisfies a certain inequality which is a function of the system dimensions. This latter result is often termed the *Kimura–Davison Condition*. It will be shown that a sufficient condition to solve the existence problem can be formulated. This depends on the satisfaction of a similar inequality relating to the system dimensions and the number of transmission zeros of the original triple. If this inequality does not hold for the process of interest, then the existence problem can always be solved by introducing a compensator. This effectively amounts to augmenting the system with some extra dynamics that are driven by the outputs of the plant. In this case it will be seen that the existence problem and the design of the compensator may be effectively accomplished by solving a particular output feedback problem. Here the inequality which must be satisfied will be seen to relate to the dimension of the compensator as well as the dimensionality and number of transmission zeros of the system. The first is a design variable which will ensure that a switching function can be found to make the sliding motion stable. This chapter will go on to present output dependent reachability conditions that will ensure that the sliding mode is ultimately attractive and that the designed dynamics are attained.

It has been seen in the above that the use of dynamic feedback is desirable to broaden the class of linear systems for which sliding mode controllers dependent only on system outputs may be designed. A second area where the use of dynamic feedback yields useful properties is in the sliding mode control of nonlinear systems. The results described above are only applicable where the process of interest may be modelled by a linear uncertain system. However, some processes are so nonlinear that such a modelling assumption is invalid. Many results in the literature in the area of sliding mode control for nonlinear systems are either based on particular applica-

tion areas, such as robotics, or assume that the process satisfies often quite restrictive structural properties, for example feedback linearisability. It will be seen that the use of a particular canonical form, the Fliess generalized controller canonical form, enables the existence and reachability problems to be solved for a relatively broad class of nonlinear systems. It will be shown that the resulting method has the additional advantage of providing a natural way of designing dynamic sliding mode controllers, which effectively filter the discontinuous control usually associated with sliding mode control methods. The method may also be applied to certain processes which are not stabilizable by continuous feedback alone. The use of sliding mode control methods involving dynamic feedback has proved to yield useful results.

## 5.2 Static output feedback of uncertain systems

Consider an uncertain dynamical system of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + f(t, x, u) \\ y(t) &= Cx(t)\end{aligned}\tag{5.1}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$  with  $m \leq p < n$ . Assume that the nominal linear system  $(A, B, C)$  is known, the pair  $(A, B)$  is controllable and the input and output matrices  $B$  and  $C$  are both of full rank. The unknown function  $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ , which represents the system nonlinearities plus any model uncertainties in the system, is assumed to satisfy the usual matching condition

$$f(t, x, u) = B\xi(t, x, u)\tag{5.2}$$

where the bounded function  $\xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  satisfies

$$\|\xi(t, x, u)\| < k_1\|u\| + \alpha(t, y)\tag{5.3}$$

for some known function  $\alpha : \mathbb{R}_+ \times \mathbb{R}^p \rightarrow \mathbb{R}_+$  and positive constant  $k_1 < 1$ .

The intention is to develop a control law which induces an ideal sliding motion on the surface

$$\mathcal{S} = \{x \in \mathbb{R}^n : FCx = 0\}\tag{5.4}$$

for some selected matrix  $F \in \mathbb{R}^{m \times p}$ . A control law of the form

$$u(t) = Gy(t) - \nu_y\tag{5.5}$$

will be sought where  $G$  is a fixed gain matrix and the discontinuous vector

$$\nu_y = \begin{cases} \rho(t, y) \frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.6)$$

where  $\rho(t, y)$  is some positive scalar function of the outputs.

Consider first the choice of hyperplane to ensure a stable reduced-order motion. To guarantee the existence of a unique equivalent control

$$u_{eq}(t) = -(FCB)^{-1} FCAx(t),$$

it is necessary that  $\det(FCB) \neq 0$ . It is well known that

$$\text{rank}(FCB) \leq \min\{\text{rank}(F), \text{rank}(CB)\}$$

and so in order for  $FBC$  to have full rank both  $F$  and  $CB$  must have rank  $m$ . The matrix  $F$  is a design parameter and can be chosen to be of full rank. A necessary condition therefore for the matrix  $FBC$  to be full rank is that  $\text{rank}(CB) = m$ .

The following canonical form will be the key to the developments that follow.

**Lemma 53** *Let  $(A, B, C)$  be a linear system with  $p > m$  and  $\text{rank}(CB) = m$ . Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:*

a) *The system matrix can be written as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{where } A_{11} \in \mathbb{R}^{(n-m) \times (n-m)} \quad (5.7)$$

*and the sub-block  $A_{11}$  when partitioned has the structure*

$$A_{11} = \left[ \begin{array}{cc|c} A_{11}^o & A_{12}^o & A_{12}^m \\ 0 & A_{22}^o & \\ \hline 0 & A_{21}^o & A_{22}^m \end{array} \right] \quad (5.8)$$

*where  $A_{11}^o \in \mathbb{R}^{r \times r}$ ,  $A_{22}^o \in \mathbb{R}^{(n-p-r) \times (n-p-r)}$  and  $A_{21}^o \in \mathbb{R}^{(p-m) \times (n-p-r)}$  for some  $r \geq 0$  and the pair  $(A_{22}^o, A_{21}^o)$  is completely observable.*

b) *The input distribution matrix has the form*

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad (5.9)$$

*where  $B_2 \in \mathbb{R}^{m \times m}$  and is nonsingular.*

c) The output distribution matrix has the form

$$C = [ 0 \quad T ] \quad (5.10)$$

where  $T \in \mathbb{R}^{p \times p}$  and is orthogonal.

For a proof see [1].

Let

$$\begin{bmatrix} \overset{p-m}{\leftrightarrow} & \overset{m}{\leftrightarrow} \\ F_1 & F_2 \end{bmatrix} = FT$$

where  $T$  is the matrix from equation (5.10). As a result

$$FC = [ F_1 C_1 \quad F_2 ] \quad (5.11)$$

where

$$C_1 := [ 0_{(p-m) \times (n-p)} \quad I_{(p-m)} ] \quad (5.12)$$

Therefore  $FCB = F_2 B_2$  and the square matrix  $F_2$  is nonsingular. By assumption the uncertainty is matched and therefore the sliding motion is independent of the uncertainty. In addition, because the canonical form in Lemma 53 can be viewed as a special case of the regular form normally used in sliding mode controller design, the reduced-order sliding motion is governed by a free motion with system matrix

$$A_{11}^s := A_{11} - A_{12} F_2^{-1} F_1 C_1 \quad (5.13)$$

which must therefore be stable. If  $K \in \mathbb{R}^{m \times (p-m)}$  is defined as  $K = F_2^{-1} F_1$  then

$$A_{11}^s = A_{11} - A_{12} K C_1 \quad (5.14)$$

and the problem of hyperplane design is equivalent to a *static output feedback problem* for the system  $(A_{11}, A_{12}, C_1)$ .

In the case where  $r > 0$ , the intention is to construct a new system  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  which is both controllable and observable with the property that

$$\lambda(A_{11}^s) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_1)$$

To this end, partition the matrices  $A_{12}$  and  $A_{12}^m$  as

$$A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^m = \begin{bmatrix} A_{121}^m \\ A_{122}^m \end{bmatrix} \quad (5.15)$$

where  $A_{122} \in \mathbb{R}^{(n-m-r) \times m}$  and  $A_{122}^m \in \mathbb{R}^{(n-p-r) \times (p-m)}$  and form a new sub-system represented by the triple  $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$  where

$$\tilde{A}_{11} := \begin{bmatrix} A_{22}^o & A_{122}^m \\ A_{21}^o & A_{22}^m \end{bmatrix} \tilde{C}_1 := \begin{bmatrix} 0_{(p-m) \times (n-p-r)} & I_{(p-m)} \end{bmatrix} \quad (5.16)$$

It can be shown that the spectrum of  $A_{11}^s$  decomposes as

$$\lambda(A_{11} - A_{12}KC_1) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_1)$$

and the spectrum of  $A_{11}^o$  represents the invariant zeros of  $(A, B, C)$ [1]. It follows directly that for a stable sliding motion, the invariant zeros of the system  $(A, B, C)$  must lie in the open left-half plane and the triple  $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$  must be stabilizable with respect to output feedback.

It should be noted that the matrix  $A_{122}$  is not necessarily full rank. Suppose  $\text{rank}(A_{122}) = m'$ , then it is possible to construct a matrix of elementary column operations  $T_{m'} \in \mathbb{R}^{m \times m}$  such that

$$A_{122}T_{m'} = \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} \quad (5.17)$$

where  $\tilde{B}_1 \in \mathbb{R}^{(n-m-r) \times m'}$  and is of full rank. If  $K_{m'} = T_{m'}^{-1}K$  and  $K_{m'}$  is partitioned compatibly as

$$K_{m'} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{matrix} \uparrow m' \\ \uparrow m-m' \end{matrix}$$

then

$$\tilde{A}_{11} - A_{122}K\tilde{C}_1 = \tilde{A}_{11} - \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} K_{m'}\tilde{C}_1 = \tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$$

and  $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$  is stabilizable by output feedback if and only if the triple  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  is stabilizable by output feedback. By using PBH tests it can be verified that the pair  $(\tilde{A}_{11}, \tilde{B}_1)$  is completely controllable and the pair  $(\tilde{A}_{11}, \tilde{C}_1)$  may be shown to be completely observable [1]. If the Kimura-Davison conditions

$$m' + p + r \geq n + 1 \quad (5.18)$$

are met, the triple  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  is stabilizable.

Having established conditions to guarantee existence of a stable sliding motion, a controller to guarantee reachability must now be sought. Assume there exists a  $K_1 \in \mathbb{R}^{m' \times (p-m)}$  such that  $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$  is stable. Let

$$K = T_{m'} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \quad (5.19)$$

where  $K_2 \in \mathbb{R}^{(m-m') \times (p-m)}$  and is arbitrary and the matrix  $T_{m'} \in \mathbb{R}^{m \times m}$  is defined in equation (5.17). Then providing any invariant zeros are stable, it follows that the matrix  $A_{11} - A_{12}KC_1$  is stable. Choose

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^T$$

where  $F_2 \in \mathbb{R}^{m \times m}$  is nonsingular and will be defined later. Introduce a nonsingular state transformation  $x \mapsto \bar{T}x$  where

$$\bar{T} = \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix} \quad (5.20)$$

and  $C_1$  is defined in (5.12). In this new coordinate system, the system triple  $(\bar{A}, \bar{B}, F\bar{C})$  has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad F\bar{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix} \quad (5.21)$$

where  $\bar{A}_{11} = A_{11} - A_{12}KC_1$  and is therefore stable. Let  $P$  be a symmetric positive definite matrix partitioned conformably with the matrices in (5.21) so that

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad (5.22)$$

where the symmetric positive definite sub-block  $P_2$  is a design matrix and the symmetric positive definite sub-block  $P_1$  satisfies the Lyapunov equation

$$P_1 \bar{A}_{11} + \bar{A}_{11}^T P_1 = -Q_1 \quad (5.23)$$

for some symmetric positive definite matrix  $Q_1$ . If

$$F := B_2^T P_2 \quad (5.24)$$

then the matrix  $P$  satisfies the structural constraint

$$P\bar{B} = \bar{C}^T F^T \quad (5.25)$$

For notational convenience let

$$Q_2 := P_1 \bar{A}_{12} + \bar{A}_{21}^T P_2 \quad (5.26)$$

$$Q_3 := P_2 \bar{A}_{22} + \bar{A}_{22}^T P_2 \quad (5.27)$$

and define

$$\gamma_0 := \frac{1}{2} \lambda_{max} \left( (F^{-1})^T (Q_3 + Q_2^T Q_1^{-1} Q_2) F^{-1} \right) \quad (5.28)$$

This scalar is well defined since the matrix on the right is symmetric and therefore has no complex eigenvalues. It can be shown that the symmetric matrix  $L(\gamma) := PA_0 + A_0^T P$  where  $A_0 = \bar{A} - \gamma \bar{B} F \bar{C}$  is negative definite if and only if  $\gamma > \gamma_0$  [1]. A variable structure control law, depending only on outputs, which will ensure reachability of the sliding mode for appropriate square systems is thus given by

$$u(t) = -\gamma Fy(t) - \nu_y \quad (5.29)$$

where  $\gamma > \gamma_0$  and  $\nu_y$  is the discontinuous vector given by

$$\nu_y = \begin{cases} \rho(t, y) \frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.30)$$

and  $\rho(t, y)$  is the positive scalar function

$$\rho(t, y) = (k_1 \gamma \|Fy\| + \alpha(t, y) + \gamma_2) / (1 - k_1) \quad (5.31)$$

where  $\gamma_2$  is a positive design scalar which defines the region in which sliding takes place. It can be shown [1] that the variable structure control law above will quadratically stabilize the uncertain system and a Lyapunov function is

$$V(\bar{x}) := \bar{x}^T P \bar{x} \quad (5.32)$$

Furthermore an ideal sliding motion is induced on  $\mathcal{S}$  in finite time.

### Numerical example

Consider the nominal linear system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \frac{1}{3} & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & \frac{8}{3} & 1 \\ 4 & \frac{2}{3} & -2 \end{bmatrix} \quad (5.33)$$

taken from [4]. By defining appropriate transformation matrices the system may be expressed in the appropriate canonical form as

$$A = \begin{bmatrix} -1.5816 & 0.0192 & 0.1457 \\ 1.4071 & 0.3845 & -1.7080 \\ 0.2953 & 0.3400 & 0.1971 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ -3.9016 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 0.3417 & -0.9398 \\ 0 & 0.9398 & 0.3417 \end{bmatrix}$$

It can be verified that  $B_2 = -3.9016$ , the orthogonal matrix

$$T = \begin{bmatrix} 0.3417 & -0.9398 \\ 0.9398 & 0.3417 \end{bmatrix}$$

and the triple  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  is given by

$$\tilde{A}_{11} = \begin{bmatrix} -1.5816 & 0.0192 \\ 1.4071 & 0.3845 \end{bmatrix} \quad \tilde{B}_1 = \begin{bmatrix} 0.1457 \\ -1.7080 \end{bmatrix} \quad \tilde{C}_1 = [ 0 \quad 1 ]$$

Here  $r = 0$ , hence the original system does not possess any invariant zeros. Arbitrary placement of the poles of  $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$  is not possible since only a single scalar is available as design freedom. For the single-input single-output system  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  the variation in the poles of  $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$  with respect to  $K_1$  can be examined by *root locus* techniques. In this case if the gain matrix  $K = K_1 = -1.0556$ , then  $\lambda(\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_1) = \{-1, -2\}$ , from which

$$\begin{aligned} F &= F_2 [ K \quad 1 ] T^T \\ &= F_2 [ -1.3005 \quad -0.6503 ] \end{aligned} \quad (5.34)$$

where  $F_2$  is a nonzero scalar that will be computed later. Transforming the system into the canonical form using  $\tilde{T}$  defined in (5.20) generates

$$\bar{A}_{11} = \begin{bmatrix} -1.5816 & 0.1729 \\ 1.4071 & -1.4184 \end{bmatrix}$$

where  $\lambda(\bar{A}_{11}) = \{-1, -2\}$  by construction. It can be verified that

$$P_1 = \begin{bmatrix} 0.3368 & 0.1891 \\ 0.1891 & 0.5401 \end{bmatrix}$$

is a Lyapunov matrix for  $\bar{A}_{11}$  and if  $P_2 = 1$ , the parameter  $F_2 = -3.9016$ . It can be checked that  $\gamma_0 = 0.2452$  and substituting for  $F_2$  in (5.34) gives

$$F = [ 5.0741 \quad 2.5370 ]$$

The following closed-loop simulation represents the regulation of the initial states  $[1 \ 0 \ 0]$  to the origin. Figure 5.1 represents a plot of the switching function versus time. The hyperplane is not globally attractive since at approximately 0.3 second it is pierced and a sliding motion cannot be maintained. Only after approximately 1 sec is sliding established. Figure 5.2 shows the decay of the states to the origin.

In summary, for the case of a non-square system, there exists a matrix  $F$  defining a surface  $\mathcal{S}$  which provides a stable sliding motion with a unique equivalent control if and only if

- the rank  $(CB) = m$
- the invariant zeros of  $(A, B, C)$  lie in  $\mathbb{C}_-$
- the triple  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  is stabilizable with respect to output feedback.

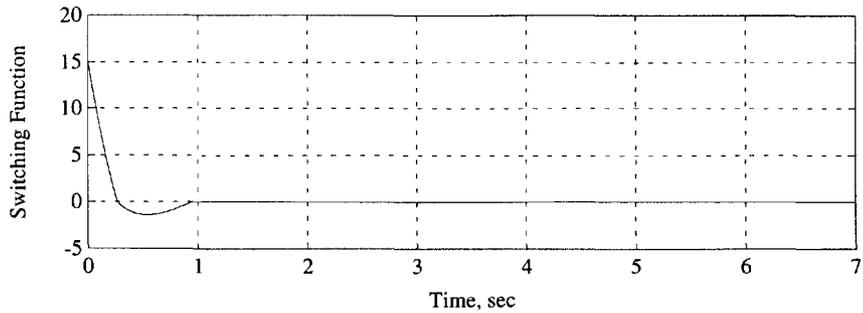


Figure 5.1: Switching function versus time

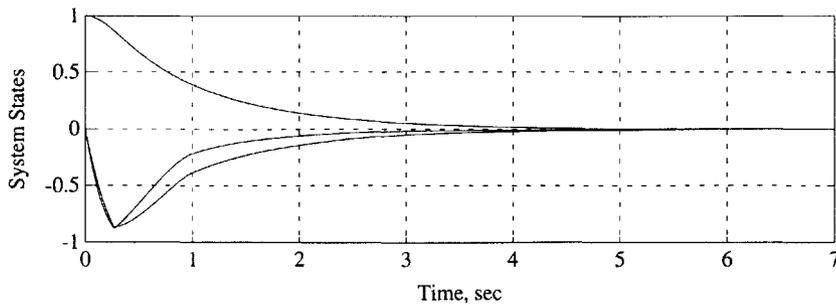


Figure 5.2: Evolution of system states with respect to time

The invariant zeros are a property of the system under consideration which must usually be regarded as fixed. The next section will explore how a dynamic approach can be used to extend the class of uncertain systems for which output feedback sliding mode controllers can be developed. This will be achieved by eliminating the stabilizability restriction.

### 5.3 Output feedback sliding mode control for uncertain systems via dynamic compensation

In the analysis above, it was assumed that the triple  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$  was stabilizable with respect to output feedback. This property can be guaranteed if the so-called Kimura–Davison conditions hold. If it is not possible

to synthesize a  $K_1$  to stabilize the triple  $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ , then it is natural to explore the effect of introducing a *compensator* – i.e., a dynamical system driven by the output of the plant – to introduce extra dynamics to provide additional degrees of freedom.

Consider the uncertain system from equation (5.1) together with a compensator given by

$$\dot{x}_c(t) = Hx_c(t) + Dy(t) \quad (5.35)$$

where the matrices  $H \in \mathbb{R}^{q \times q}$  and  $D \in \mathbb{R}^{q \times p}$  are to be determined. Define a new hyperplane in the augmented state space, formed from the plant and compensator state spaces, as

$$\mathcal{S}_c = \{(x, x_c) \in \mathbb{R}^{n+q} : F_c x_c + FCx = 0\} \quad (5.36)$$

where  $F_c \in \mathbb{R}^{m \times q}$  and  $F \in \mathbb{R}^{m \times p}$ . As in Section 5.2, assume that the nominal linear system  $(A, B, C)$  is in the canonical form of Lemma 53 and partition the matrix  $FT$ , where  $T$  is the orthogonal matrix from (5.10), as

$$\begin{array}{cc} \begin{array}{c} p-m \\ \leftarrow \\ \rightleftarrows \\ \rightarrow \\ \end{array} & \begin{array}{c} m \\ \rightleftarrows \\ \rightarrow \\ \end{array} \\ [ F_1 & F_2 ] & = FT \end{array}$$

In an analogous way define  $D_1 \in \mathbb{R}^{q \times (p-m)}$  and  $D_2 \in \mathbb{R}^{q \times m}$  as

$$[ D_1 \quad D_2 ] = DT \quad (5.37)$$

If the states of the uncertain system in the coordinates of Lemma 53 are partitioned as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{array}{l} \downarrow n-m \\ \downarrow m \end{array} \quad (5.38)$$

then the compensator can be written as

$$\dot{x}_c(t) = Hx_c(t) + D_1 C_1 x_1(t) + D_2 x_2(t) \quad (5.39)$$

where  $C_1$  is defined in Equation (5.12). Assume that a control action exists which forces and maintains motion on the hyperplane  $\mathcal{S}_c$  given in (5.36). As usual, in order for a unique equivalent control to exist, the square matrix  $F_2$  must be invertible. By writing  $K = F_2^{-1} F_1$  and defining  $K_c = F_2^{-1} F_c$  then the system matrix governing the reduced-order sliding motion, obtained by eliminating the coordinates  $x_2$ , can be written as

$$\dot{x}_1(t) = (A_{11} - A_{12} K C_1) x_1(t) - A_{12} K_c x_c(t) \quad (5.40)$$

$$\dot{x}_c(t) = (D_1 - D_2 K) C_1 x_1(t) + (H - D_2 K_c) x_c(t) \quad (5.41)$$

From the above equations it is clear that the introduction of the compensator has produced more design freedom. As would be expected, the

invariant zeros of the uncertain system are still embedded in the dynamics, since from the definition of the partition of  $A_{12}$  given in (5.15) and from an appropriately partitioned form of  $A_{11} - A_{12}KC_1$ , it follows that

$$\begin{bmatrix} A_{11} - A_{12}KC_1 & -A_{12}K_c \\ (D_1 - D_2K)C_1 & H - D_2K_c \end{bmatrix} = \begin{bmatrix} A_{11}^o & [A_{12}^o \mid A_{121}^m - A_{121}K] & -A_{121}K_c \\ 0 & \tilde{A}_{11} - A_{122}K\tilde{C}_1 & -A_{122}K_c \\ 0 & (D_1 - D_2K)\tilde{C}_1 & H - D_2K_c \end{bmatrix}$$

As in the uncompensated case, it is necessary for the eigenvalues of  $A_{11}^o$  to have negative real parts. The design problem becomes one of selecting a compensator, represented by the matrices  $D_1$ ,  $D_2$  and  $H$ , and a hyperplane represented by the matrices  $K$  and  $K_c$  so that the matrix

$$A_c := \begin{bmatrix} \tilde{A}_{11} - A_{122}K\tilde{C}_1 & -A_{122}K_c \\ (D_1 - D_2K)\tilde{C}_1 & H - D_2K_c \end{bmatrix} \quad (5.42)$$

is stable. Again if there is rank deficiency in the matrix  $A_{122}$ , then the problem is over-parameterized. As in Section 5.2, suppose  $\text{rank}(A_{122}) = m' < m$  and let  $T_{m'} \in \mathbb{R}^{m \times m}$  be a matrix of elementary column operations such that

$$A_{122}T_{m'} = [\tilde{B}_1 \quad 0]$$

where  $\tilde{B}_1 \in \mathbb{R}^{(n-m-r) \times m'}$  and is of full rank. Define partitions of the transformed hyperplane matrices as

$$T_{m'}^{-1}K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{matrix} \uparrow m' \\ \downarrow m-m' \end{matrix} \quad \text{and} \quad T_{m'}^{-1}K_c = \begin{bmatrix} K_{c1} \\ K_{c2} \end{bmatrix} \begin{matrix} \uparrow m' \\ \downarrow m-m' \end{matrix}$$

then it follows that

$$A_c = \begin{bmatrix} \tilde{A}_{11} - \tilde{B}_1K_1\tilde{C}_1 & -\tilde{B}_1K_{c1} \\ (D_1 - D_2K)\tilde{C}_1 & H - D_2K_c \end{bmatrix} \quad (5.43)$$

As before, the matrix given in (5.43) will be written as the result of an output feedback problem for a certain system triple. Unfortunately, a degree of over-parametrization is still present in (5.43), which for simplicity will be removed by defining

$$\tilde{D}_1 := D_1 - D_2K \quad \text{and} \quad \tilde{H} := H - D_2K_c \quad (5.44)$$

This is comparable to the situation which occurred in the uncompensated case where  $K_2$  was found to have no effect on  $\tilde{A}_{11} - \tilde{B}_1K\tilde{C}_1$ . The key observation is that Equation (5.43) can now be written as

$$\begin{bmatrix} \tilde{A}_{11} - \tilde{B}_1K_1\tilde{C}_1 & -\tilde{B}_1K_{c1} \\ \tilde{D}_1\tilde{C}_1 & \tilde{H} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} K_1 & K_{c1} \\ \tilde{D}_1 & \tilde{H} \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & I_q \end{bmatrix}$$

Thus by defining

$$A_q := \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0_{q \times q} \end{bmatrix} \quad B_q := \begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & -I_q \end{bmatrix} \quad C_q := \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & I_q \end{bmatrix}$$

the parameters  $K_1, K_{c1}, \tilde{D}_1$ , and  $\tilde{H}$  can be obtained from output feedback pole placement of the triple  $(A_q, B_q, C_q)$ . In order to use standard output feedback results it is necessary for the triple  $(A_q, B_q, C_q)$  to be both controllable and observable. From the definition of  $(A_q, B_q)$  it follows that

$$\text{rank} \begin{bmatrix} zI - A_q & B_q \end{bmatrix} = \text{rank} \begin{bmatrix} zI - \tilde{A}_{11} & \tilde{B}_1 \end{bmatrix} + q$$

for all  $z \in \mathbb{C}$ . As argued earlier, the pair  $(\tilde{A}_{11}, \tilde{B}_1)$  is controllable and therefore from the PBH rank test  $(A_q, B_q)$  is controllable. Using the fact that the pair  $(\tilde{A}_{11}, \tilde{C}_1)$  is observable, a PBH argument proves that  $(A_q, C_q)$  is observable.

The Kimura–Davison conditions for the triple  $(A_q, B_q, C_q)$  amount to requiring that

$$m' + q + p \geq n - r + 1 \quad (5.45)$$

Thus for a large enough  $q$ , the Kimura–Davison conditions can always be satisfied and the static output feedback method can be employed.

### 5.3.1 Dynamic compensation (observer based)

It is well known that numerical solutions to the static output feedback problem often invoke the use of optimization routines which may not be guaranteed to converge. This subsection explores an observer-based methodology for hyperplane design. Consider the compensator defined in (5.35) then, as in the previous section (eliminating any invariant zeros), the assignable dynamics of the sliding motion are given by the system matrix

$$A_c = \begin{bmatrix} \tilde{A}_{11} - A_{122}K\tilde{C}_1 & -A_{122}K_c \\ (D_1 - D_2K)\tilde{C}_1 & H - D_2K_c \end{bmatrix} \quad (5.46)$$

An alternative method for choosing appropriate compensator variables  $H, D_1$  and  $D_2$ , and the hyperplane matrix gains  $K$  and  $K_c$  will now be sought.

Consider the fictitious system

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}_{11}\tilde{x}(t) + A_{122}\tilde{u}(t) \\ \tilde{y}(t) &= \tilde{C}_1\tilde{x}(t) \end{aligned} \quad (5.47)$$

with associated triple  $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ . The structure of

$$\tilde{C}_1 = \begin{bmatrix} 0_{(p-m) \times (n-p-r)} & I_{(p-m)} \end{bmatrix}$$

means that the second  $(p - m)$ th dimensional component of the 'state' is known. A reduced order observer would thus only be required to estimate the first  $(n - p - r)$ th dimensional component. If the input distribution matrix is partitioned conformably so that

$$A_{122} = \begin{bmatrix} A_{1221} \\ A_{1222} \end{bmatrix} \begin{matrix} \downarrow_{n-p-r} \\ \downarrow_{p-m} \end{matrix} \quad (5.48)$$

then a reduced-order observer for the fictitious system (5.47) is given by

$$\begin{aligned} \dot{z} &= (A_{22}^o + L^o A_{21}^o)z + (A_{122}^m + L^o A_{22}^m - (A_{22}^o + L^o A_{21}^o)L^o)\tilde{y} \\ &+ (A_{1221} + L^o A_{1222})\tilde{u} \end{aligned} \quad (5.49)$$

where  $L^o \in \mathbb{R}^{(n-p-r) \times (p-m)}$  is any gain matrix so that  $A_{22}^o + L^o A_{21}^o$  is stable. Let  $\mathcal{K}$  be any state feedback matrix for the controllable pair  $(\tilde{A}_{11}, A_{122})$  so that  $\tilde{A}_{11} - A_{122}\mathcal{K}$  is stable, and partition the state feedback matrix so that

$$\begin{matrix} \begin{matrix} \leftarrow & \leftarrow \\ \leftrightarrow & \leftrightarrow \\ \rightarrow & \rightarrow \end{matrix} \\ \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_2 \end{bmatrix} \end{matrix} = \mathcal{K}$$

The state feedback law can be implemented using the observer states and the outputs in the form

$$\tilde{u} = -\mathcal{K}_1 z - (\mathcal{K}_2 - \mathcal{K}_1 L^o)\tilde{y} \quad (5.50)$$

and the closed-loop system comprising (5.47) and (5.49) is stable. Define

$$H = A_{22}^o + L^o A_{21}^o \quad (5.51)$$

$$D_1 = A_{122}^m + L^o A_{22}^m - (A_{22}^o + L^o A_{21}^o)L^o \quad (5.52)$$

$$D_2 = A_{1221} + L^o A_{1222} \quad (5.53)$$

$$K = \mathcal{K}_2 - \mathcal{K}_1 L^o \quad (5.54)$$

$$K_c = \mathcal{K}_1 \quad (5.55)$$

then equation (5.49) can be written

$$\dot{z}(t) = Hz(t) + D_1\tilde{y} + D_2\tilde{u} \quad (5.56)$$

where

$$\tilde{u} = -K_c z(t) - K\tilde{y} \quad (5.57)$$

It can easily be verified that the closed-loop system formed from (5.47) and (5.49) is given by

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} - A_{122}K\tilde{C}_1 & -A_{122}K_c \\ (D_1 - D_2K)\tilde{C}_1 & H - D_2K_c \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} \quad (5.58)$$

and from the separation principle the closed-loop poles are given by

$$\lambda(H) \cup \lambda(\tilde{A}_{11} - A_{122}\mathcal{K})$$

The system matrix associated with (5.58) is identical to the system matrix of the reduced-order sliding motion given in (5.42). Therefore the choice of compensator matrices in (5.51) to (5.53) and the hyperplane matrices (5.54) and (5.55) give rise to a stable sliding mode.

### 5.3.2 Control law construction

Having investigated design procedures to determine the compensator and associated sliding surface, it is necessary to construct a control which will render the defined sliding mode attractive. Assume that there are  $r$  (stable) invariant zeros and partition the state vector  $x_1$  as in (5.38) so that

$$x_1 = \begin{bmatrix} x_r \\ x_{11} \\ x_{12} \end{bmatrix} \quad \begin{array}{l} \uparrow r \\ \uparrow n-p-r \\ \uparrow p-m \end{array}$$

As a result, the (original) compensator can be written as

$$\dot{x}_c(t) = Hx_c(t) + D_1x_{12}(t) + D_2x_2(t) \quad (5.59)$$

Define a new dynamical system by

$$\dot{z}_r(t) = A_{11}^o z_r(t) + A_{12}^o x_c(t) + (A_{121}^m - A_{12}^o L^o)x_{12}(t) + A_{121}x_2(t) \quad (5.60)$$

and augment (5.59) with (5.60) to form a new compensator

$$\dot{\hat{x}}_c(t) = \hat{H}\hat{x}_c(t) + \hat{D}y(t) \quad (5.61)$$

where

$$\hat{H} := \begin{bmatrix} A_{11}^o & A_{12}^o \\ 0 & H \end{bmatrix} \quad \text{and} \quad \hat{D} := \begin{bmatrix} (A_{121}^m - A_{12}^o L^o) & A_{121} \\ D_1 & D_2 \end{bmatrix} T^T$$

Using the partitions (5.7), (5.8), (5.15) and (5.48), the original dynamics can be written as

$$\dot{x}_r(t) = A_{11}^o x_r(t) + A_{12}^o x_{11}(t) + A_{121}^m x_{12}(t) + A_{121}x_2(t) \quad (5.62)$$

$$\dot{x}_{11}(t) = A_{22}^o x_{11}(t) + A_{122}^m x_{12}(t) + A_{1221}x_2(t) \quad (5.63)$$

$$\dot{x}_{12}(t) = A_{21}^o x_{11}(t) + A_{22}^m x_{12}(t) + A_{1222}x_2(t) \quad (5.64)$$

$$\begin{aligned} \dot{x}_2(t) &= A_{211}x_r(t) + A_{212}x_{11}(t) + A_{213}x_{12}(t) + A_{22}x_2(t) \\ &\quad + B_2(u(t) + \xi(t)) \end{aligned} \quad (5.65)$$

where the lower left sub-block of  $A$  from (5.7) has been partitioned so that

$$\begin{array}{ccc} \begin{array}{c} r \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} & \begin{array}{c} n-p-r \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} & \begin{array}{c} p-m \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \\ [ A_{211} & A_{212} & A_{213} ] = A_{21} \end{array} \quad (5.66)$$

Define two error states

$$e_r = z_r - x_r \quad (5.67)$$

and

$$e_c = x_c - x_{11} - L^o x_{12} \quad (5.68)$$

then straightforward algebra reveals

$$\dot{e}_r(t) = A_{11}^o e_r(t) + A_{12}^o e_c \quad (5.69)$$

and also

$$\dot{e}_c(t) = H e_c(t) \quad (5.70)$$

These stable error systems result from the fact that, by construction, the compensator states  $x_c$  and  $z_r$  are observations of  $x_{11} + L^o x_{12}$  and  $x_r$ , respectively. Define a state matrix

$$\hat{x} = \begin{bmatrix} z_r \\ x_c \\ x_{12} \\ x_2 \end{bmatrix} \quad (5.71)$$

then standard algebra reveals

$$\dot{\hat{x}}(t) = \hat{A} \hat{x}(t) - \hat{A}_e \hat{e}(t) + B [u(t) + \xi(t, x, u)] \quad (5.72)$$

where

$$\hat{A} = \begin{bmatrix} A_{11}^o & A_{12}^o & A_{121}^m - A_{12}^o L^o & A_{121} \\ 0 & H & D_1 & D_2 \\ 0 & A_{21}^o & A_{22}^m - A_{21}^o L^o & A_{1222} \\ A_{211} & A_{212} & A_{213} - A_{212} L^o & A_{22} \end{bmatrix} \quad \text{and}$$

$$\hat{A}_e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & A_{21}^o \\ A_{211} & A_{212} \end{bmatrix}$$

and the augmented error state

$$\hat{e} = \begin{bmatrix} e_r \\ e_c \end{bmatrix} \quad (5.73)$$

Note that the triple  $(\hat{A}, B, C)$  can be obtained from the canonical form  $(A, B, C)$  via a similarity transformation. Thus the original system together with the compensator can be written as

$$\dot{\hat{e}}(t) = \hat{H}\hat{e}(t) \quad (5.74)$$

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) - \hat{A}_e\hat{e}(t) + B[u(t) + \xi(t, x, u)] \quad (5.75)$$

Note also that the sliding surface  $\mathcal{S}_c$  can be written as

$$\{\hat{x} \in \mathbb{R}^n : S\hat{x} = 0\}$$

where

$$S = F_2 \begin{bmatrix} 0_{m \times r} & K_c & K & I_m \end{bmatrix} \quad (5.76)$$

Define a switching function

$$s(t) = S\hat{x}(t) \quad (5.77)$$

and define a linear feedback component

$$u_l(t) = -\Lambda^{-1}S\hat{A}\hat{x}(t) + \Lambda^{-1}\Phi S\hat{x}(t) \quad (5.78)$$

where  $\Lambda = SB$  and  $\Phi \in \mathbb{R}^{m \times m}$  is a stable design matrix. Let  $P$  be the unique positive definite solution to the Lyapunov equation

$$P\Phi + \Phi^T P = -I \quad (5.79)$$

A control law to induce a sliding motion on the sliding surface  $\mathcal{S}_c$  is given by

$$u(t) = u_l(t) - \nu_y \quad (5.80)$$

where

$$\nu_y = \begin{cases} \rho(t, y)\Lambda^{-1} \frac{Ps(t)}{\|Ps(t)\|} & \text{if } s(t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.81)$$

and  $\rho(\cdot)$  is the positive scalar function

$$\rho(t, y) = (k_1\|\Lambda\|\|u_l(t)\| + \|\Lambda\|\alpha(t, y) + \gamma_2) / [1 - k_1\kappa(\Lambda)] \quad (5.82)$$

where  $\gamma_2$  is a small positive constant.

By considering a Lyapunov candidate of the form  $V(s) = s^T P s$  where  $s(t) = S\hat{x}(t)$ , it may be shown that the control law defined in (5.78) to (5.82) induces a sliding motion on the sliding surface  $\mathcal{S}_c$ .

This control law is effectively a state feedback controller since the components  $z_r$  and  $x_c$  are estimates of the true states  $x_r$  and  $x_{11}$  (up to a coordinate transformation).

### 5.3.3 Design example

Consider the nominal linear system

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -6 & -9 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (5.83)$$

This system is already in the appropriate canonical form and thus

$$A_{11} = \begin{bmatrix} A_{11}^o & [A_{12}^o & A_{121}^m] \\ 0 & \tilde{A}_{11} \end{bmatrix} = \left[ \begin{array}{c|cc} -2 & 1 & 0 \\ \hline 0 & 0 & 4 \\ 0 & 1 & 0 \end{array} \right] \quad A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In terms of the compensator design, the triple of interest is given by

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \quad A_{122} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_1 = [0 \quad 1] \quad (5.84)$$

It can be shown by direct computation that for  $K = k$

$$\lambda(\tilde{A}_{11} - A_{122}KC_1) = \{\pm\sqrt{4 - k^2}\}$$

and so the triple  $(\tilde{A}_{11}, A_{122}, C_1)$  is not stabilizable by static output feedback and a compensator-based approach must be employed. It follows from (5.84) that

$$\begin{bmatrix} A_{22}^o & A_{122}^m \\ A_{21}^o & A_{22}^m \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

and so from Equations (5.51) to (5.53) an appropriate parametrization for the compensator is

$$H = L^o \quad D_1 = 4 - (L^o)^2 \quad D_2 = 1$$

where  $L^o$  is any negative scalar which will appear as one of the eigenvalues of (5.42). In the simulation which follows  $L^o = -2.5$  and  $\lambda(\tilde{A}_{11} - A_{122}K) = \{-1, -1.5\}$ . Since the system has an invariant zero at  $-2$ , the sliding motion will have poles at  $\{-1, -1.5, -2\}$ . The pole represented by  $\Phi$  which governs the range space dynamics has been chosen to be  $-5$ . For simplicity the scaling factor for the sliding surface is  $F_2 = 1$ . All the available degrees of freedom have now been assigned.

Figure 5.3 is a plot of the switching function against time; it can be seen that sliding occurs after approximately 1 second. Figure 5.4 shows the

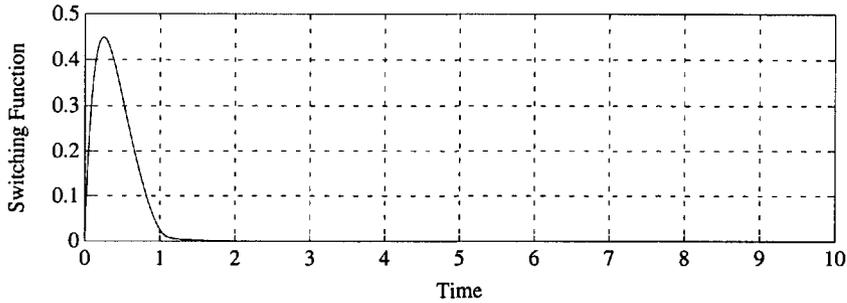


Figure 5.3: Switching function versus time

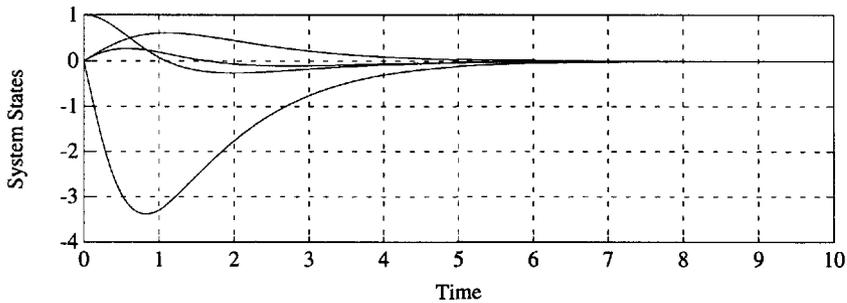


Figure 5.4: Evolution of the system states

evolution of the states against time. Initially the states of the compensator have been set to zero. The states of the system have a nonzero initial condition which needs to be regulated to zero.

Figures 5.5 and 5.6 show the evolution of the error states  $e_c$  and  $e_r$ . Initially  $e_c$  is nonzero since the state  $x_{11}$  was given a nonzero initial condition. As indicated in Equation (5.70), this error system is completely decoupled and decays away to zero (Figure 5.5).

The error states  $e_r$ , shown in Figure 5.6, although initially zero, are coupled to the state  $e_c$  as shown in Equation (5.69). However, this also decays asymptotically to zero in accordance with the theory.

Notice from Figure 5.3 that, although the states initially lie on the sliding surface, a sliding motion is not maintained. This is due to the fact that the error term  $\hat{e}$  is initially too large. A sliding motion occurs after approximately 1 second, by which time the error  $\hat{e}$  has decayed sufficiently.

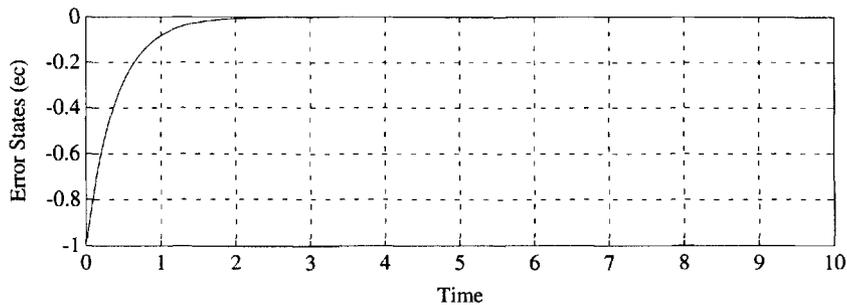


Figure 5.5: Evolution of the error states  $e_c$

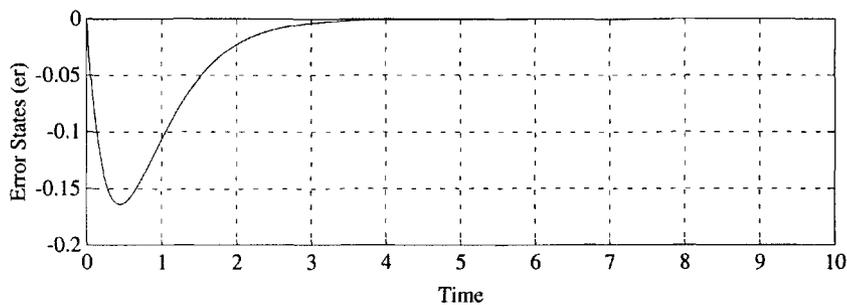


Figure 5.6: Evolution of the errors states  $e_r$

## 5.4 Dynamic sliding mode control for nonlinear systems

Sliding mode control is known to provide an appropriate solution to the robust control problem. However, the majority of design methodologies, whether reliant on state or output feedback, have been based around linear uncertain systems, as described earlier in this chapter, or specific types of nonlinear systems. The latter may involve particular application areas, such as robotics [10], or require that relatively stringent conditions are met by members of the system class: for example the system class may be required to be feedback linearizable [11]. It is obviously desirable to have a sliding mode control methodology that will be applicable to a fairly broad class of nonlinear system representations, exhibit robustness while yielding appropriate performance, and lend itself to the development of appropriate

tool boxes for controller design. It will be shown in the remainder of the chapter that the dynamic sliding mode policies which result from considering differential input-output (I-O) system representations are sufficiently general to meet this remit [5, 6, 9].

Dynamic sliding mode control methods assume that all the systems states, or equivalently, the derivatives of the outputs to some appropriate order, are available for use by the control law. Thus a state estimator is necessary for implementation if only measured outputs are available.

The following notation will be used throughout:

$$N_\delta(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$$

where  $\delta > 0$ , or simply  $N_\delta$  if  $x_0 = 0$ .

For sliding mode controller design using static feedback, it is necessary that the system assumes a regular form and that the control variables appear linearly in the system in order to recover the control parameters from the chosen sliding condition [13]. In general, this is not practically implementable for general nonlinear systems with nonlinear control. In order to develop the sliding mode control method to include dynamic policies, and hence to ensure it becomes applicable to an extended class of nonlinear systems, differential I-O system representations will be employed.

For a given system in state-space form that is locally observable,

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} &= h(\mathbf{x}, \mathbf{u}, t)\end{aligned}\tag{5.85}$$

where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^m$ , and  $f(\mathbf{x}, \mathbf{u})$  and  $h(\mathbf{x}, \mathbf{u})$  are smooth vector functions, the following locally equivalent differential I-O system exists [14]

$$\begin{aligned}y_1^{(n_1)} &= \varphi_1(\hat{\mathbf{y}}, \hat{\mathbf{u}}, t) \\ &\vdots \\ y_p^{(n_p)} &= \varphi_p(\hat{\mathbf{y}}, \hat{\mathbf{u}}, t)\end{aligned}\tag{5.86}$$

where

$$\hat{\mathbf{u}} = (u_1, \dots, u_1^{(\beta_1)}, \dots, u_m, \dots, u_m^{(\beta_m)})^T\tag{5.87}$$

and

$$\hat{\mathbf{y}} = (y_1, \dots, y_1^{(n_1-1)}, \dots, y_p, \dots, y_p^{(n_p-1)})^T\tag{5.88}$$

with  $n_1 + \dots + n_p = n$ .

**Definition 54** *A differential I-O system (5.86) is called proper if*

- 1)  $p = m$ ;

2) All  $\varphi_i(\cdot, \cdot, \cdot)$ ,  $i = 1, \dots, m$ , are  $C^3$  functions;

3) Regularity condition

$$\det \left[ \frac{\partial(\varphi_1, \dots, \varphi_m)}{\partial(u_1^{(\beta_1)}, \dots, u_m^{(\beta_m)})} \right] \neq 0 \quad (5.89)$$

is satisfied with  $\hat{\mathbf{y}} \in N_\delta(0)$  for all  $t \geq 0$ , some  $\delta > 0$  and generically for  $\hat{\mathbf{u}}$ .

Throughout this chapter it is assumed that all the differential I-O systems considered are proper.

Whether or not the resulting system is minimum phase will again be shown to be pertinent to the stability of the closed-loop system.

**Definition 55** The zero dynamics, corresponding to (5.86), is defined as

$$\begin{aligned} \varphi_1(0, \hat{\mathbf{u}}, t) &= 0 \\ &\vdots \\ \varphi_p(0, \hat{\mathbf{u}}, t) &= 0. \end{aligned} \quad (5.90)$$

The system (5.86) is called minimum phase if there exist  $\delta > 0$  and  $\tilde{\mathbf{u}}_0 \in \mathbb{R}^\beta$  where  $\beta = \beta_1 + \dots + \beta_m$ , such that (5.90) is uniformly asymptotically (exponentially) stable for an initial condition  $\tilde{\mathbf{u}}(0) \in N_\delta(\tilde{\mathbf{u}}_0)$ , where

$$\tilde{\mathbf{u}} = (u_1, \dots, u_1^{(\beta_1-1)}, \dots, u_m, \dots, u_m^{(\beta_m-1)})$$

Otherwise, it is non-minimum phase. Note that, in this case, the “minimum phase-ness” is a property of the chosen control signal.

In order to address robustness, uncertain systems of the following form may be considered.

$$\begin{aligned} y_1^{(n_1)} &= \varphi_1(\hat{\mathbf{y}}, \hat{\mathbf{u}}, t) + \Delta_1(\hat{\mathbf{y}}, t) \\ &\vdots \\ y_p^{(n_p)} &= \varphi_p(\hat{\mathbf{y}}, \hat{\mathbf{u}}, t) + \Delta_p(\hat{\mathbf{y}}, t) \end{aligned} \quad (5.91)$$

The uncertainties are Lebesgue measurable and satisfy

$$\|\Delta_i(\hat{\mathbf{y}}, t)\| \leq \rho_i \|\hat{\mathbf{y}}\| + l_i, \quad \rho_i \geq 0, \quad l_i \geq 0, \quad i = 1, \dots, p \quad (5.92)$$

The uncertainty may be due to external uncertainties, internal parameter uncertainties, measurement noise, system identification error, or indeed the elimination procedure used to generate a differential input-output model from a state space model as in [14].

It is often convenient to consider the *Generalized Controller Canonical Form* (GCCF) representation of (5.86). Without loss of generality, suppose that  $n_1, \dots, n_{m_1} > 1$  and  $n_{m_1} = \dots = n_{m_1+m_2} = 1, m_1 + m_2 = m$ . The system (5.86) may be expressed in the following GCCF [2]

$$\begin{aligned}
 \dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
 &\vdots \\
 \dot{\zeta}_{n_1-1}^{(1)} &= \zeta_{n_1}^{(1)} \\
 \zeta_{n_1}^{(1)} &= \varphi_1(\zeta, \hat{\mathbf{u}}, t) \\
 &\vdots \\
 \dot{\zeta}_{n_m-1}^{(m)} &= \zeta_{n_m}^{(m)} \\
 \zeta_{n_m}^{(m)} &= \varphi_m(\zeta, \hat{\mathbf{u}}, t)
 \end{aligned} \tag{5.93}$$

where

$$\zeta^{(i)} = (\zeta_1^{(i)}, \dots, \zeta_{n_i}^{(i)}) = (y_i, \dots, y_i^{(n_i-1)}), i = 1, \dots, m$$

and

$$\zeta = (\zeta^{(1)}, \dots, \zeta^{(m)})^T$$

represent the system outputs and their derivatives.

It has been seen that it is necessary to solve existence and reachability problems in order to determine the sliding mode controller. In the nonlinear case, the two popular choices of sliding surface are:

(1) Direct sliding surface [8, 6, 9]

$$s_i = \sum_{j=1}^{n_i} a_j^{(i)} \zeta_j^{(i)}, i = 1, \dots, m \tag{5.94}$$

where  $\sum_{j=1}^{n_i} a_j^{(i)} \lambda^{j-1}$  are Hurwitz polynomials with  $a_{n_i}^{(i)} = 1$ . This will provide a reduced order sliding motion whose dynamics are prescribed by the roots of the polynomials.

(2) Indirect sliding surface [5]

$$s_i = \sum_{j=1}^{n_i+1} a_j^{(i)} \zeta_j^{(i)} + \varphi_i(\zeta, \hat{\mathbf{u}}, t), i = 1, \dots, m \tag{5.95}$$

where  $\sum_{j=1}^{n_i+1} a_j^{(i)} \lambda^{j-1}$  are Hurwitz polynomials with  $a_{n_i+1}^{(i)} = 1$ . With this choice, the system (when sliding) becomes equivalent to an  $n$ th order linear system, with dynamics prescribed by the choice of the Hurwitz polynomial. This may be regarded as an alternative model.

An appropriate algorithm for robust dynamic sliding mode control is described below. The system (5.91) may be expressed in the following generalized controller canonical form

$$\begin{aligned}
\dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\
&\vdots \\
\dot{\zeta}_{n_1-1}^{(1)} &= \zeta_{n_1}^{(1)} \\
\dot{\zeta}_{n_1}^{(1)} &= \varphi_1(\zeta, \hat{\mathbf{u}}, t) + \Delta_1(\zeta, t) \\
&\vdots \\
\dot{\zeta}_1^{(m)} &= \zeta_2^{(m)} \\
&\vdots \\
\dot{\zeta}_{n_m-1}^{(m)} &= \zeta_{n_m}^{(m)} \\
\dot{\zeta}_{n_m}^{(m)} &= \varphi_m(\zeta, \hat{\mathbf{u}}, t) + \Delta_m(\zeta, t)
\end{aligned} \tag{5.96}$$

where

$$\zeta^{(i)} = (\zeta_1^{(i)}, \dots, \zeta_{n_i}^{(i)}) = (y_i, \dots, y_i^{(n_i-1)}), i = 1, \dots, m$$

and

$$\zeta = (\zeta^{(1)}, \dots, \zeta^{(m)})^T$$

**Step 1:** Choose design parameters to define the sliding surface (5.94). For  $i = 1, \dots, m$ , if  $n_i > 1$ , choose  $(a_1^{(i)}, \dots, a_{n_i-1}^{(i)}, 1)$  and  $(a_1^{(i)}, \dots, a_{n_i-1}^{(i)})$ , both Hurwitz. This is always possible according to the result in [3]. Without loss of generality, suppose  $n_1, \dots, n_{m_1} > 1$  and  $n_{m_1+1} = \dots = n_{m_1+m_2}$  where  $m_1 + m_2 = m$ .

**Step 2:** Estimate the uncertainty bound as in (5.92) when the system is in the GCCF. Choose  $\theta_0$  and  $\theta$  where  $0 < \theta < 1$ ,  $\theta_0 + \theta = 1$  and define

$$\rho = \left( \sum_{i=1}^m \rho_i^2 \right)^{1/2} + \left( \rho^{(1)} \right)^2 / (4\theta) \tag{5.97}$$

where

$$\rho^{(1)} = \left( \sum_{i=1}^m \rho_i^2 \right)^{1/2} \left( 1 + \max_{1 \leq i \leq m_1, 1 \leq j \leq n_i} \{ |a_j^{(i)}| \} \max_{1 \leq i \leq m_1} \{ \sqrt{n_i - 1} \} \right) \tag{5.98}$$

**Step 3:** Define

$$A_i := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & \vdots \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & & & 0 & 1 \\ -a_1^{(i)} & -a_2^{(i)} & \cdots & \cdots & \cdots & -a_{n_i-1}^{(i)} \end{bmatrix} \quad D_i := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

for  $i = 1, \dots, m_1$ . Let  $D := \text{diag}[D_1, \dots, D_{m_1}]$  and  $A := \text{diag}[A_1, \dots, A_{m_1}]$ . Since the  $A_i$  are stable, then  $A$  is stable, and define  $P$  to be the unique positive definite solution to the Lyapunov equation

$$A^T P + P A = -I$$

Next choose  $K \in \mathbb{R}^{m \times m}$  as a positive definite matrix which satisfies

$$\lambda_{\min}(K) - \left[ \frac{1}{\theta_0} (PD)^T (PD) + \rho I_{m_1} \right] > 0 \quad (5.99)$$

**Step 4:** Differentiating (5.94) with respect to time  $t$  along the trajectories of (5.96) leads to

$$\dot{s}_i \Big|_{(5.96)} = \sum_{j=1}^{n_i-1} a_j^{(i)} \zeta_{j+1}^{(i)} + \varphi_i(\zeta, \hat{u}, t) + \Delta_i(\zeta, t) \quad (5.100)$$

for  $i = 1, \dots, m$ . Now set

$$\sum_{j=1}^{n_i-1} a_j^{(i)} \zeta_{j+1}^{(i)} + \varphi_i(\zeta, \hat{u}, t) = -(Ks)_i - k_{0i} \text{sat}_\varepsilon(s_i) \quad (5.101)$$

where  $k_{0i} > l_0 := (\sum_{i=1}^m l_i^2)^{1/2}$  and

$$\text{sat}_\varepsilon(x) = \varepsilon \cdot \text{sat}\left(\frac{x}{\varepsilon}\right) = \begin{cases} 1, & x > \varepsilon \\ x, & |x| \leq \varepsilon \\ -1, & x < -\varepsilon \end{cases}$$

Equation (5.100) becomes

$$\dot{s} = -Ks - \mathbf{K}_0 \text{sat}_\varepsilon(s) + \Delta(\zeta, t) \quad (5.102)$$

where  $\mathbf{K}_0 = \text{diag}[k_{01}, \dots, k_{0m}]$ ,  $\text{sat}_\varepsilon(s) = [\text{sat}_\varepsilon(s_1), \dots, \text{sat}_\varepsilon(s_m)]^T$ .

**Step 5:** From (5.101) the highest order derivatives of the control, namely  $[u_1^{(\beta_1)}, \dots, u_m^{(\beta_m)}]$ , can be solved out by the implicit function theorem as

$$u^{(\beta_i)} = p_i(\zeta, \hat{\mathbf{u}}, t), \quad i = 1, \dots, m$$

if the regularity condition is satisfied. Note that  $p_i(\zeta, \hat{\mathbf{u}}, t)$  is a continuous function if  $s_i \neq 0$  because  $\varphi_i$  is  $\mathcal{C}^1$  and  $\gamma_i$  is  $\mathcal{C}^0$  if  $s_i \neq 0$ . This dynamic feedback can be realized in canonical form by introducing the pseudo-state variables as

$$\begin{aligned} \dot{z}_1^{(1)} &= z_2^{(1)} \\ &\vdots \\ \dot{z}_{\beta_1-1}^{(1)} &= z_{\beta_1}^{(1)} \\ \dot{z}_{\beta_1}^{(1)} &= p_1(\zeta, z, t) \\ &\vdots \\ \dot{z}_1^{(m)} &= z_2^{(m)} \\ &\vdots \\ \dot{z}_{\beta_m-1}^{(m)} &= z_{\beta_m}^{(m)} \\ \dot{z}_{\beta_m}^{(m)} &= p_m(\zeta, z, t) \end{aligned} \tag{5.103}$$

where

$$z^{(i)} = (z_1^{(i)}, \dots, z_{\beta_i}^{(i)}) = (u_i, \dot{u}_i, \dots, u_i^{(\beta_i-1)}), \quad i = 1, \dots, m \tag{5.104}$$

and

$$z = (z^{(1)}, \dots, z^{(m)})^T. \tag{5.105}$$

The system in (5.103) together with (5.96) yields a closed-loop system of dimension  $\sum_{i=1}^m n_i + \sum_{i=1}^m \beta_i$ , where  $\beta_i$  is the highest order derivative of  $u_i$ .

**Step 6:** Choose  $\hat{\mathbf{u}}_0 \in \mathbb{R}^\beta$  and a  $\delta > 0$  such that, for initial condition  $\hat{\mathbf{u}}(0) \in N_\delta(\hat{\mathbf{u}}_0)$ , and

- (1) the regularity condition is satisfied;
- (2) the zero dynamics (5.90), (or (5.103) when  $\zeta = 0$ ) are uniformly asymptotically stable; and
- (3) all the initial conditions for (5.96, 5.103) are compatible.

It was shown in [6] that the procedure outlined above will effect uniformly ultimately bounded motion of the uncertain system (5.91) if it is minimum phase.

### Remarks

The proof in [6] relies on first showing that the closed-loop subsystem associated with the states  $\zeta$  is stable. This is demonstrated by considering the

system  $(\tilde{\zeta}, s)$  obtained from the linear coordinate transformation resulting from substituting for  $\zeta_{n_i}^{(i)}$  according to the formula

$$\zeta_{n_i}^{(i)} = s_i - \sum_{j=1}^{n_i-1} a_j^{(i)} \zeta_j^{(i)}$$

[which is a rearrangement of (5.94)]. Defining  $\tilde{\zeta}^{(i)} := (\zeta_1^{(i)}, \dots, \zeta_{n_i-1}^{(i)})^T$  it follows that

$$\dot{\tilde{\zeta}}^{(i)} = A_i \tilde{\zeta} + D_i s_i \quad (5.106)$$

for  $i = 1, \dots, m_1$ . Using the candidate Lyapunov function

$$\tilde{V} = \tilde{\zeta}^T P \tilde{\zeta} + \frac{1}{2} s^T s$$

ultimate boundedness of the  $(\tilde{\zeta}, s)$  subsystem, with respect to an arbitrary neighborhood of the origin, can be shown. The overall closed-loop system is given by (5.106) and (5.102) together with equations of the form

$$\dot{z} = \eta(\tilde{\zeta}, s, z, t) \quad (5.107)$$

where the right hand side is such that  $\dot{z} = \eta(0, 0, z, t)$  represents the zero dynamics. Using the stability properties of the states  $s$  and  $\tilde{\zeta}$ , stability of the overall closed-loop system can be shown by using a modification of the results for ‘triangular systems’ in [15].

Equation (5.102) can be shown to represent a *strong reachability* condition in the sense of [12].

The dynamic sliding mode control method above assumes that all the system states, or equivalently, the derivatives of the outputs to some appropriate order, are available for use by the control law. Thus a state estimator is necessary for implementation if only measured outputs are available. A particular high gain observer was shown to be particularly appropriate for this estimation task [7].

### 5.4.1 Design example

Consider the following nonlinear model

$$y^{(2)} = u \sin(y^{(1)}) + (1 + y^2)u^{(2)} + uy + u + \mu u^{(1)}(u^2 - 1) + \Delta \quad (5.108)$$

$$\Delta = y \sin(y^{(1)}) + \text{rand}(1) \quad (5.109)$$

Here  $\Delta$  represents the uncertainty and  $\text{rand}(1)$  is the one dimensional random variable from MATLAB. The corresponding zero dynamics are obtained by setting  $y^{(2)} = y^{(1)} = y = 0$ . This yields

$$u^{(2)} + u + \mu u^{(1)}(u^2 - 1) = 0 \quad (5.110)$$

which is the Van der Pol equation. This is uniformly asymptotically stable for  $\mu < 0$  with  $[u(0)]^2 + [u^{(1)}(0)]^2 < 1$  as shown by the phase plane portrait in Figure 5.7 where  $\mu = -1$  and  $u(0) = \dot{u}(0) = 0.5$ . The system is thus minimum phase and the closed-loop dynamic sliding mode control scheme will be stable for appropriately chosen initial parameters.

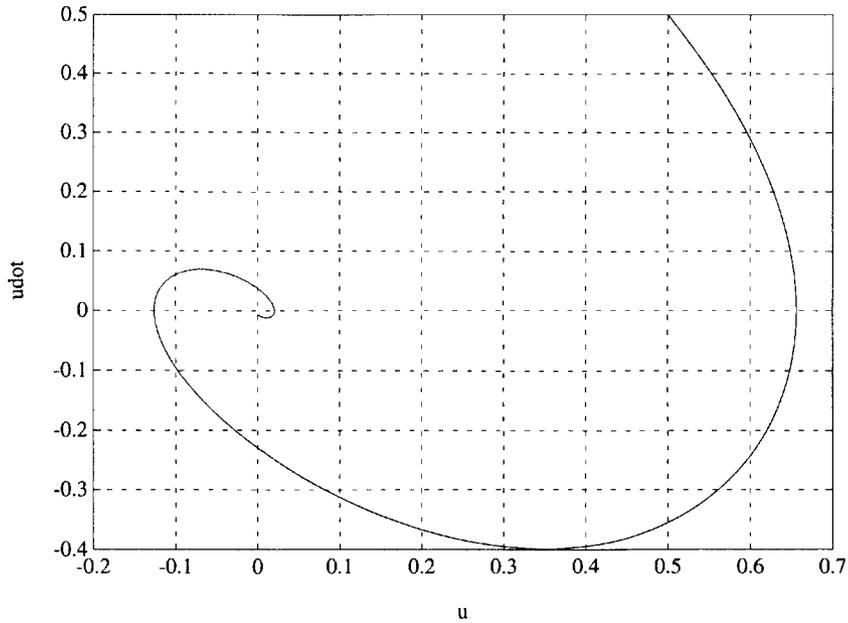


Figure 5.7: Phase plane portrait showing the typical evolution of the zero dynamics

**Step 1:** Choose the direct sliding surface  $s = ay + y^{(1)}$  where  $a = 2$ .

**Step 2:** To estimate the uncertainty bound, choose  $\theta_0 = 0.25$  so that  $\theta = 0.75$ . From  $|\Delta| \leq |y| + 1$ , it follows that  $l_0 = 1$  and  $\rho^{(1)} = 3$  implies  $\rho = 4$ .

**Step 3:** As  $A = -2$ , it follows that  $P = 0.25$ . Thus

$$\left(\frac{1}{\theta_0}\right)(PD)^T(PD) + \rho = 4.25 \quad (5.111)$$

and appropriate choice for  $k$  in the reachability condition is  $k = 5 > 4.25$ . Let  $k_0 = 1.5 > l_0 = 1$ .

**Step 4:** The controller is then solved out from  $\dot{s} = -ks - k_0 \text{sat}_\varepsilon(s)$  as

$$u^{(2)} = -(ks + k_0 \text{sat}_\varepsilon(s) + u \sin(y^{(1)}) + uy + u + \mu u^{(1)}(u^2 - 1)) / (1 + y^2) \quad (5.112)$$

Simulation results for initial conditions  $\dot{y}(0) = 0$ ,  $y(0) = 0.5$  are illustrated. The level of uncertainty present is shown in Figure 5.8. The output response is shown in Figure 5.9. It is seen that the closed-loop system rejects the uncertainty and effective output regulation is achieved. Figure 5.10 shows that a sliding mode is attained and maintained. It is seen from Figure 5.11 that this performance is achieved with out the switched control action, which is often associated with sliding mode control. The dynamic control strategy act as a natural filter for the control signal and its robustness to the prescribed uncertainty results.

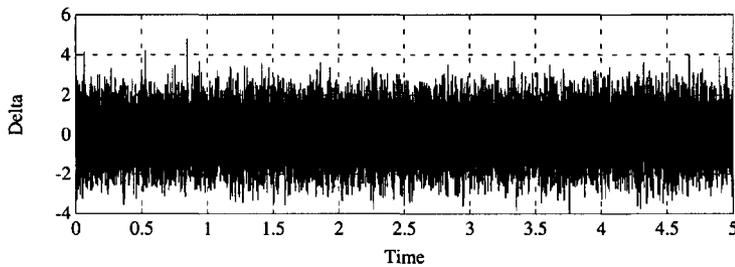


Figure 5.8: Evolution of the uncertainty contribution to the dynamics

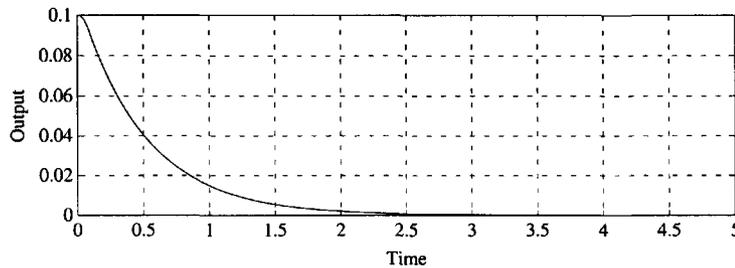


Figure 5.9: Evolution of the system output with respect to time

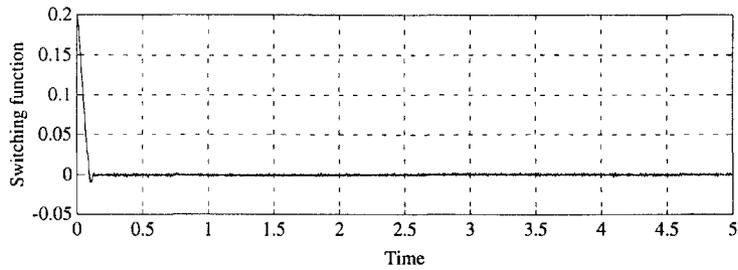


Figure 5.10: Evolution of the switching function

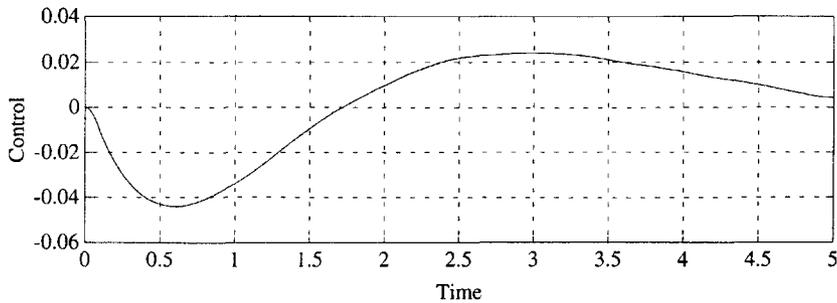


Figure 5.11: Evolution of the control signal with respect to time

## 5.5 Conclusions

In this chapter design procedures have been presented to synthesize robust output feedback controllers for linear uncertain systems. The class of systems to which the results apply has been identified, and includes the requirement that the nominal linear system is minimum phase. It has been shown that certain dimensionality requirements must be satisfied if the sliding surface is to be designed using a straightforward static output feedback pole placement, which is dependent only on a particular subsystem of the original plant dynamics. This restriction can be overcome using a dynamic feedback approach. A reduced-order Luenberger observer approach was shown to yield a convenient methodology for designing the sliding surface and compensator dynamics in this case. An output dependent controller which guarantees attainment of a sliding mode by the linear uncertain system was presented.

This chapter also addressed the problem of designing sliding mode con-

trollers for nonlinear systems. A particular canonical form was used to render the results applicable to a fairly broad class of systems. This method was shown to produce controllers that are dynamic in nature and thus avoid the chattering which often characterizes the sliding mode approach.

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# Chapter 6

## Sliding Modes, Passivity, and Flatness

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### 6.1 Introduction

In this chapter, we present two discontinuous feedback controller design examples that are solved by a combination of off-line trajectory planning (OLTP), based on flatness, passivity-based control (PBC) and either pulse width modulation (PWM), or sliding modes (SM). The examples are presented in the context of two electrical systems. A permanent magnet (PM) stepper motor which is a weakly minimum phase multivariable nonlinear system and a dc-to-dc power converter of the “boost” type, which is a single input, non-minimum phase system. Both systems are switched systems, i.e., their control inputs take values on discrete sets.

The OLTP process for nonlinear differentially flat systems is not only natural but it is also quite flexible and powerful, as already demonstrated by many application examples and solid theoretical developments (see the work of Fliess and his colleagues [3][4] for interesting details and far reaching developments). The PM stepper motor and the “boost” converter, treated in this chapter, are differentially flat (see [6] and [9]).

Stabilization tasks for single input and for multivariable flat systems can be easily achieved, even in the case of non-minimum phase output requirements, thanks to the fact that the differential parameterization provided by flatness, degenerates, under equilibrium conditions, into a static

parameterization which allows to reconcile the non-minimum phase output controlled maneuver objective with an equivalent objective for the flat output variable [5].

A more complex problem is that of having a non-minimum phase output follow a prespecified trajectory that leads to a desired, permanent, oscillation, as in the case of dc-to-ac power conversion. Part of the difficulty now arises from the fact that, in some cases, the differential parameterization is no longer helpful in directly establishing the corresponding signal to be tracked by the flat output or, alternatively, by a minimum phase output. The problem may be solved by resorting to an approximating sequence of finite differential parameterizations of the minimum phase output in terms of the non-minimum phase output. In the limit, this sequence may be interpreted to lead to an *infinite order* flatness. With the aid of digital computer simulations, we show that for the normalized model of the boost converter this sequence of parameterizations enjoys a rather fast convergence property and only one or two of its terms are required in order to obtain a tight solution to the tracking problem.

Section 6.2 presents a PM stepper motor controller design example solving a stabilization task requiring an equilibrium-to-equilibrium transfer via trajectory planning, exact linearization, and PWM control. Section 6.3 presents a boost converter design example. The general properties of flatness and passivity of the “boost” converter circuit are established and a general derivation is provided for a sliding mode solution (based on passivity and flatness), of a trajectory tracking task for the non-minimum phase output. The proposed solution is suitable for both the stabilization and trajectory tracking problems. The last section of this chapter presents some conclusions and suggestions for further research.

## 6.2 The permanent magnet stepper motor

Consider a nonlinear model of a permanent magnet (PM) stepper motor, taken from Bodson and Chiasson [1],

$$\begin{aligned}
 \frac{di_a}{dt} &= \frac{1}{L} (v_a - Ri_a + K_m \omega \sin(N_r \theta)) \\
 \frac{di_b}{dt} &= \frac{1}{L} (v_b - Ri_b - K_m \omega \cos(N_r \theta)) \\
 \frac{d\omega}{dt} &= \frac{1}{J} (-K_m i_a \sin(N_r \theta) + K_m i_b \cos(N_r \theta) - B\omega - \tau) \\
 \frac{d\theta}{dt} &= \omega
 \end{aligned} \tag{6.1}$$

where  $i_a$  represents the current in phase  $A$  of the motor,  $i_b$  is the current in the phase  $B$  of the motor,  $\theta$  is the angular displacement of the shaft of the motor, and  $v_a$  and  $v_b$ , are the voltage applied on the windings of the phase  $A$  and phase  $B$ , respectively. The parameters  $R$  and  $L$ , the resistance and self inductances in each of the phase windings, are constant and assumed to be known. Similarly the number of rotor teeth  $N_r$ , the torque constant of the motor  $K_m$ , the rotor load inertia  $J$ , and the viscous friction  $B$  are assumed known and constant. The load torque perturbation, denoted by  $\tau$ , is, for analysis purposes, assumed to be zero.

The nature of the control inputs  $v_a$  and  $v_b$  is that of *switched* inputs respectively taking values in the discrete control sets,  $\mathcal{U}_a = \{+V_a, -V_a\}$  and  $\mathcal{U}_b = \{+V_b, -V_b\}$ , as obtained from a diode-based PWM operated bridge inverter. However, our design developments treat  $v_a$  and  $v_b$  as if they were continuous valued inputs. The interpretation of this procedure is that the model (6.1) is being regarded as an infinite frequency *PWM average model* [8]. We shall consider the obtained feedback control law as a feedback *duty ratio synthesizer* and implement the derived control law through an actual switching law of the PWM type, taking values in  $\mathcal{U}_a$  and  $\mathcal{U}_b$ . Trajectory planning is shown to naturally avoid the possible saturation of the computed duty ratio functions.

### 6.2.1 The simpler D-Q nonlinear model of the PM stepper motor

The, so called *D-Q* (direct-to-quadrature) transformation gets rid of all the trigonometric terms appearing in the motor model. This transformation is given by

$$\begin{aligned} \begin{bmatrix} i_d \\ i_q \end{bmatrix} &= \begin{bmatrix} \cos(N_r\theta) & \sin(N_r\theta) \\ -\sin(N_r\theta) & \cos(N_r\theta) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix} \\ \begin{bmatrix} v_d \\ v_q \end{bmatrix} &= \begin{bmatrix} \cos(N_r\theta) & \sin(N_r\theta) \\ -\sin(N_r\theta) & \cos(N_r\theta) \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} \end{aligned} \quad (6.2)$$

The current  $i_d$  is the *direct current* and  $i_q$  is the *quadrature current*. Also,  $v_d$  and  $v_q$  are addressed as the *direct* and *quadrature voltages*, respectively and act as the new control inputs to the system.

The transformed system is given by

$$\begin{aligned} \frac{di_d}{dt} &= \frac{1}{L} (v_d - Ri_d + N_r\omega Li_q) \\ \frac{di_q}{dt} &= \frac{1}{L} (v_q - Ri_q - N_r\omega Li_d - K_m\omega) \end{aligned}$$

$$\begin{aligned}\frac{d\omega}{dt} &= \frac{1}{J} (K_m i_q - B\omega) \\ \frac{d\theta}{dt} &= \omega\end{aligned}\tag{6.3}$$

## 6.2.2 The control problem

The control objective is to drive the system from a given initial equilibrium value towards a final equilibrium value achieving, as a result, a desired final value for the position variable  $\theta$ .

The equilibrium point  $(\bar{i}_d, \bar{i}_q, \bar{\omega}, \bar{\theta})$  of the transformed system, for a given constant value of the direct voltage,  $v_d = \bar{v}_d$ , is given by

$$\bar{v}_q = 0, \quad \bar{i}_d = \frac{\bar{v}_d}{R}, \quad \bar{i}_q = 0, \quad \bar{\omega} = 0, \quad \bar{\theta} = \text{arbitrary constant}$$

We assume that the equilibrium value of  $i_d$  is not zero. In fact, we will keep  $i_d$  bounded away from zero throughout the equilibrium transfer maneuver. As will be shown, this is quite easy to guarantee.

Suppose, for a moment, that the vector relative degree  $\{1, 1\}$  outputs  $i_d$  and  $i_q$  are held *constant* at some value  $(i_d, i_q) = (\bar{i}_d, 0)$ . Then the *zero dynamics* corresponding to this set of values is given by the linear system

$$\frac{d\theta}{dt} = \omega \quad ; \quad \frac{d\omega}{dt} = -B\omega\tag{6.4}$$

which exhibits two eigenvalues; one located at the origin, and the other located in the left half portion of the complex plane, at the point  $(-B, 0)$ . The system outputs,  $(i_d, i_q)$ , are then *weakly minimum phase* and, according to the results of [2], they are passive outputs.

## 6.2.3 A passivity canonical model of the PM stepper motor

Consider the following positive definite (Lyapunov) *energy storage function*

$$H(i_d, i_q, \omega, \theta) = \frac{1}{2} [L(i_d^2 + i_q^2) + J\omega^2 + \gamma\theta^2]\tag{6.5}$$

The time derivative of the storage function, along the controlled motions of the system, satisfies

$$\begin{aligned}\dot{H} &= -[R(i_d^2 + i_q^2) + B\omega^2] + v_d i_d + v_q i_q + \gamma\theta\omega \\ &\leq i_d \left( v_d + \gamma \frac{\theta\omega}{i_d} \right) + i_q v_q\end{aligned}\tag{6.6}$$

This last expression, plus the weakly minimum phase character of the outputs  $i_d$  and  $i_q$ , reveals that the system is a *passive* operator between the modified inputs  $(\vartheta_d, \vartheta_q) = (v_d + \theta\omega/i_d, v_q)$  and the system outputs  $(i_d, i_q)$ . This justifies the following additional input coordinate transformation,

$$\vartheta_d = v_d + \gamma \frac{\theta\omega}{i_d} ; \quad \vartheta_q = v_q \quad (6.7)$$

We write the system, in matrix form, as

$$\begin{aligned} \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \frac{di_d}{dt} \\ \frac{di_q}{dt} \\ \frac{d\omega}{dt} \\ \frac{d\theta}{dt} \end{bmatrix} &= \begin{bmatrix} 0 & N_r L \omega & 0 & -\gamma \frac{\omega}{i_d} \\ -N_r L \omega & 0 & -K_m & 0 \\ 0 & K_m & 0 & 0 \\ \gamma \frac{\omega}{i_d} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ \omega \\ \theta \end{bmatrix} \\ + \begin{bmatrix} -R & 0 & 0 & 0 \\ 0 & -R & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ \omega \\ \theta \end{bmatrix} &+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_d \\ \vartheta_q \end{bmatrix} \end{aligned} \quad (6.8)$$

The obtained model, clearly exhibits the *conservative* and the *dissipative* structure of the system.

## 6.2.4 A controller based on “energy shaping plus damping injection”

The “energy shaping plus damping injection” dynamic feedback controller design method, extensively treated in [7], yields the following dynamical feedback controller specification,

$$\begin{aligned} \vartheta_d &= L \frac{d}{dt} i_d^*(t) - N_r L \omega i_q^*(t) + \gamma \frac{\omega}{i_d} \zeta_2 + R i_d^*(t) \\ \vartheta_q &= L \frac{d}{dt} i_q^*(t) + N_r L \omega i_d^*(t) + K_m \zeta_1 + R i_q^*(t) \end{aligned} \quad (6.9)$$

with  $\zeta_1$  and  $\zeta_2$  satisfying

$$\begin{aligned} J \dot{\zeta}_1 &= K_m i_q^*(t) - B \zeta_1 + R_B (\omega - \zeta_1) \\ \gamma \dot{\zeta}_2 &= \gamma \frac{\omega}{i_d} i_d^*(t) + R_\theta (\theta - \zeta_2) \end{aligned} \quad (6.10)$$

with  $R_B$  and  $R_\theta$  positive design constants.

The transformed control inputs to the system are determined from the equalities

$$v_d = \vartheta_d - \gamma \frac{\theta \omega}{i_d} ; \quad v_q = \vartheta_q \quad (6.11)$$

The feedback controller, in terms of the original inputs  $v_a, v_b$  and the phase currents  $i_a, i_b$ , is obtained from (6.2) and (6.10).

We state the tracking error stabilization properties of the feedback controller (6.9), (6.10), as follows.

**Proposition 56** *The passivity-based dynamic feedback controller yields a state vector tracking error dynamics, described by the vector,*

$$e = [i_d - i_d^*(t), i_q - i_q^*(t), \omega - \zeta_1, \theta - \zeta_2]$$

*which is globally exponentially asymptotically stable to zero.*

### Proof

Substituting the control input expressions, given in (6.9), into the D-Q system model (6.3), we obtain, using the following definitions of the state tracking error variables;  $e_1 = i_d - i_d^*(t)$  and  $e_2 = i_q - i_q^*(t)$ ,  $e_3 = \omega - \zeta_1$  and  $e_4 = \theta - \zeta_1$ ,

$$\begin{aligned} \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} &= \begin{bmatrix} 0 & N_r L \omega & 0 & 0 \\ -N_r L \omega & 0 & -K_m & 0 \\ 0 & K_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ &+ \begin{bmatrix} -R & 0 & 0 & 0 \\ 0 & -R & 0 & 0 \\ 0 & 0 & -B - R_B & 0 \\ 0 & 0 & 0 & -R_\theta \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \end{aligned} \quad (6.12)$$

Using the modified energy function  $H(e) = \frac{1}{2} (Le_1^2 + Le_2^2 + Je_3^2 + \gamma e_4^2)$ , with  $\gamma$  being an arbitrary strictly positive constant, one establishes that,  $\dot{H}(e) \leq \alpha H(e)$ , with  $\alpha < 0$  as a constant dependent on the system parameters  $L, J, R, B, \gamma$ , and the design parameters,  $R_B$  and  $R_\theta$ . The tracking error vector  $e$  is globally exponentially asymptotically stable to zero, i.e.,

$$i_d \rightarrow i_d^*(t), \quad i_q \rightarrow i_q^*(t), \quad \omega \rightarrow \zeta_1, \quad \theta \rightarrow \zeta_2 \quad (6.13)$$

In the absence of load torque perturbations, the desired current  $i_q^*(t)$  is made to converge to zero and, then,  $i_q$  also converges to zero. The planned flat current  $i_d^*(t)$  is made to converge to a nonzero constant. Then,  $i_d$

converges to the same value. The outputs  $i_d$  and  $i_q$  are passive, thus,  $\omega$  and  $\zeta_1$  converge to zero. The angle  $\theta$ , and  $\zeta_2$ , both converge to a constant to be established. The flatness property allows the final value of  $\theta$  to be completely determined at will, as will be shown in the following section.

### 6.2.5 Differential flatness of the system

The PM stepper motor is easily seen to be differentially flat, since all variables in the system can be completely parameterized in terms of *differential functions* of the independent variables constituted by the direct current  $i_d$  and the motor shaft angular position  $\theta$ . (see [6] and [11] and the references therein ). The flat outputs, denoted by  $F = (F_1, F_2) = (i_d, \theta)$ , yield,

$$\begin{aligned}
 i_d &= F_1, \quad \theta = F_2, \quad \omega = \dot{F}_2, \quad i_q = \frac{J}{K_m} \ddot{F}_2 + \frac{B}{K_m} \dot{F}_2, \\
 v_d &= L\dot{F}_1 + RF_1 - N_r L \dot{F}_2 \left( \frac{J}{K_m} \ddot{F}_2 + \frac{B}{K_m} \dot{F}_2 \right) \\
 v_q &= \frac{LJ}{K_m} F_2^{(3)} + \frac{LB}{K_m} \ddot{F}_2 + R \left( \frac{J}{K_m} \ddot{F}_2 + \frac{B}{K_m} \dot{F}_2 \right) + N_r L \dot{F}_2 F_1 + K_m \dot{F}_2
 \end{aligned} \tag{6.14}$$

All systems properties, in particular those concerning the ones needed for passivity-based controller design, are already reflected in the above complete differential parameterization, as it can be easily verified.

### 6.2.6 A dynamic passivity plus flatness based controller

The passivity-based controller (6.10) requires pre-specified passive outputs trajectories  $i_d^*(t)$  and  $i_q^*(t)$ . Instead of directly specifying those trajectories, it was proposed to specify them in terms of the flat outputs, i.e., we take advantage of the fact that the passive outputs are differentially related to the flat outputs (which, incidentally, are devoid of zero dynamics). The (off-line) specification of such flat outputs already determines the rest of the system variables. The advantage of this approach resides in fact that the flat outputs are *fundamental* system outputs devoid of internal dynamics and correspond to the *hidden* linear controllability properties of the system.

The passivity-based controller, exploiting the flatness property of the

system, is then given by

$$\begin{aligned}
\vartheta_d &= L\dot{F}_1^*(t) - N_r L\omega \left[ \frac{J}{K_m} \ddot{F}_2^*(t) + \frac{B}{K_m} \dot{F}_2^*(t) \right] - \gamma \frac{\omega}{i_d} \zeta_2 + R F_1^*(t) \\
\vartheta_q &= L \left[ \frac{J}{K_m} (F_2^*(t))^{(3)} + \frac{B}{K_m} \ddot{F}_2^*(t) \right] + N_r L\omega F_1^*(t) + K_m \zeta_1 \\
&\quad + R \left[ \frac{J}{K_m} \ddot{F}_2^*(t) + \frac{B}{K_m} \dot{F}_2^*(t) \right]
\end{aligned} \tag{6.15}$$

with  $\zeta_1$  and  $\zeta_2$  satisfying

$$\begin{aligned}
J\dot{\zeta}_1 &= K_m \left[ \frac{J}{K_m} \ddot{F}_2^*(t) + \frac{B}{K_m} \dot{F}_2^*(t) \right] - B\zeta_1 + R_B (\omega - \zeta_1) \\
\gamma\dot{\zeta}_2 &= \gamma \frac{\omega}{i_d} F_1^*(t) + R_\theta (\theta - \zeta_2)
\end{aligned} \tag{6.16}$$

## 6.2.7 Simulation results

We consider a PM stepper motor with the following parameters

$$R = 8.4 \, \Omega \quad L = 0.010 \, \text{H}, \quad K_m = 0.05 \, \text{Vs/rad}, \quad J = 3.6 \times 10^{-6} \, \text{Nms}^2/\text{rad},$$

$$B = 1 \times 10^{-4} \, \text{Nms/rad}, \quad N_r = 50 \quad R_b = 0.05\Omega$$

It is desired to transfer the angular position  $\theta$  from the initial value of  $\theta_0 = 0$  rad, towards the final value  $\theta_F = 0.03$  rad, following a trajectory specified by means of an interpolating time polynomial of the form  $\psi(t, t_0, t_f)$  satisfying

$$\psi(t_0, t_0, t_f) = 0, \quad \psi(t_f, t_0, t_f) = 1 \tag{6.17}$$

Thus,

$$\theta^*(t) = \theta_0 + \psi(t, t_0, t_f) [\theta_F - \theta_0] \tag{6.18}$$

One such possible expression, including a particular interpolating polynomial  $\psi(t, t_0, t_f)$ , is given by

$$\begin{aligned}
\theta^*(t) &= \theta_0 + \left( \frac{t-t_0}{t_f-t_0} \right)^5 \left[ r_1 - r_2 \left( \frac{t-t_0}{t_f-t_0} \right) + r_3 \left( \frac{t-t_0}{t_f-t_0} \right)^2 - \dots \right. \\
&\quad \left. \dots - r_6 \left( \frac{t-t_0}{t_f-t_0} \right)^5 \right] (\theta_F - \theta_0)
\end{aligned} \tag{6.19}$$

with

$$r_1 = 252, \quad r_2 = 1050, \quad r_3 = 1800, \quad r_4 = 1575, \quad r_5 = 700, \quad r_6 = 126$$

and  $t_0 = 0.01$  s and  $t_f = 0.02$  s.

The flat output variable,  $i_d$ , is also made to follow a similar time trajectory  $i_d^*(t)$ , taking the d-current coordinate from the value  $i_d(t_0) = i_{d0} = 0.3$  A, towards the final value  $i_d(t_f) = i_{df} = 0.5$  A, during the same previous time interval  $[t_0, t_f]$ . In other words we specified  $i_d^*(t)$  as

$$i_d^*(t) = i_{d0} + \psi(t, t_0, t_f) (i_{df} - i_{d0}) \quad (6.20)$$

The passivity-based feedback controller, proposed in the previous section, is used with the passive output reference trajectories given by

$$\begin{aligned} i_d^*(t) &= i_{d0} + \psi(t, t_0, t_f) [i_{df} - i_{d0}] \\ i_q^*(t) &= \frac{J}{K_m} \ddot{\theta}^*(t) + \frac{B}{K_m} \dot{\theta}^*(t), \end{aligned} \quad (6.21)$$

The design constants  $R_B$ ,  $R_{theta}$ , and  $\gamma$ , are

$$R_B = 0.05, \quad R_{\theta} = 2, \quad \gamma = 1$$

Figure 6.1 shows the simulations of the closed loop performance of the stepper motor in original  $a - b$  coordinates. The load torque was set to zero in these simulations.

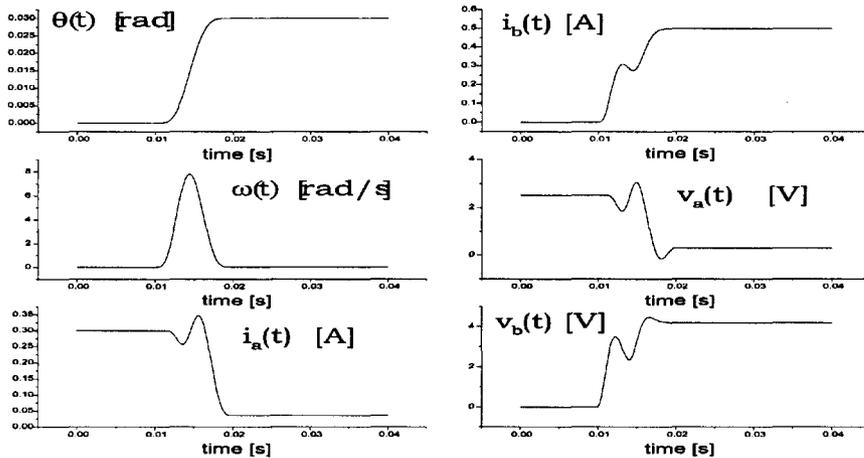


Figure 6.1: PM Stepper motor closed-loop response to passivity plus flatness based controller ( $a-b$  variables)

In order to account for unmodeled constant load torque perturbations, entering the angular velocity dynamics as  $\tau$ , we use an *outer loop* proportional-integral-derivative (PID) controller, feeding back the *dynamic controller* angular velocity tracking error  $\epsilon(t) = \zeta_1 - \dot{\theta}^*(t)$ . This controller guarantees that  $\zeta_1$  actually tracks  $\dot{\theta}^*(t)$ , in spite of the perturbation load torque. Since  $\omega$  is guaranteed to track  $\zeta_1$ , by the previous arguments, the net result is that  $\omega$  tracks  $\dot{\theta}^*(t)$  in spite of the unknown but constant perturbations. The integral action of the PID controller corrects the angular position deviations.

The modified controller is

$$\begin{aligned} \vartheta_d &= L \frac{d}{dt} i_d^*(t) - N_r L \omega i_q^*(t) + \gamma \frac{\omega}{i_d} \zeta_2 + R i_d^*(t) - k_{Pd} \epsilon + k_{Id} \eta + k_{Dd} \dot{\epsilon} \\ \vartheta_q &= L \frac{d}{dt} i_q^*(t) + N_r L \omega i_d^*(t) + K_m \zeta_1 + R i_q^*(t) + k_{Pq} \epsilon - k_{Iq} \eta - k_{Dq} \dot{\epsilon} \\ \dot{\eta} &= \epsilon \end{aligned} \quad (6.22)$$

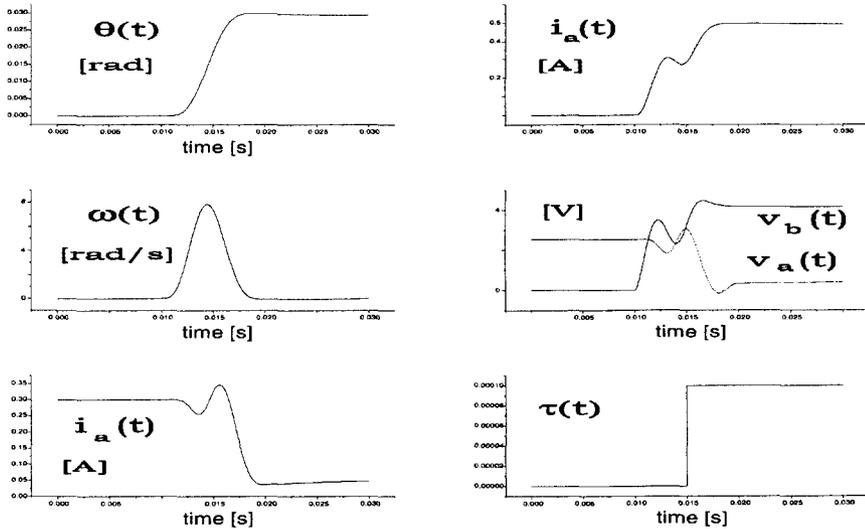


Figure 6.2: PM stepper motor closed-loop response to passivity plus flatness based controller including perturbation torque

Figure 6.2 shows the performance of the modified passivity-based controller in the presence of constant but unknown load torque perturbations. We used  $k_{Pd} = k_{Pq} = 0.01$ ,  $k_{Id} = k_{Iq} = 60$  and  $k_{Dd} = k_{Dq} = 0.001$ . The load torque amplitude was taken to be  $10^{-4}$  N-m.

## 6.2.8 A pulse width modulation implementation

The controller design, and the obtained simulation results, may be regarded as those corresponding to an *average PWM model* of a corresponding switched model of the PM stepper motor in which the input voltages,  $v_a$  and  $v_b$ , are assumed to only take values, respectively, on the discrete sets  $\{-V_a, V_a\}$  and  $\{-V_b, V_b\}$  with  $V_a$  and  $V_b$  being constant values representing the maximum available input voltages.

Consider then the average PWM model of the PM stepper motor, obtained by simply substituting the control input voltages  $v_a$  and  $v_b$ , respectively, by the expressions

$$v_a = \mu_a V_a; \quad v_b = \mu_b V_b \quad (6.23)$$

with  $\mu_a$  and  $\mu_b$  acting effectively as the *average* independent control inputs to the system, also known as *duty ratios* (see [8]). These control inputs are constrained to the open intervals  $(-1, 1)$ .  $V_a$  and  $V_b$  are positive constant values determined on the basis of specified maximum absolute values of the actual control input variables  $v_a$  and  $v_b$ , respectively.

The average PWM model of the PM stepper motor is then given by

$$\begin{aligned} \frac{di_a}{dt} &= \frac{1}{L} [\mu_a V_a - Ri_a + K_m \omega \sin(N_r \theta)] \\ \frac{di_b}{dt} &= \frac{1}{L} [\mu_b V_b - Ri_b - K_m \omega \cos(N_r \theta)] \\ \frac{d\omega}{dt} &= \frac{1}{J} [-K_m i_a \sin(N_r \theta) + K_m i_b \cos(N_r \theta) - B\omega - \tau] \\ \frac{d\theta}{dt} &= \omega. \end{aligned} \quad (6.24)$$

The actual switching control inputs  $v_a$  and  $v_b$  are specified according to a ‘‘PWM switching policy’’, which entitles the sampling of the nonlinear system states at time instants  $t_k$ , with a sampling period given by the fixed positive scalar  $T$ . The pulsed control inputs  $v_d$  and  $v_q$  are then decided, at the beginning of each sampling interval, according to

$$\begin{aligned} v_a(t) &= \begin{cases} V_a \text{sign}[\mu_a(t_k)] & \text{for } t_k < t \leq t_k + |\mu_a(t_k)| T \\ 0 & \text{for } t_k + |\mu_a(t_k)| T < t \leq t_k + T \end{cases} \\ v_b(t) &= \begin{cases} V_b \text{sign}[\mu_b(t_k)] & \text{for } t_k < t \leq t_k + |\mu_b(t_k)| T \\ 0 & \text{for } t_k + |\mu_b(t_k)| T < t \leq t_k + T \end{cases} \end{aligned} \quad (6.25)$$

with the duty ratio values  $\mu_a(t_k)$  and  $\mu_b(t_k)$  obtained, in a feedback manner, as follows,

$$\mu_a(t_k) = \left( \frac{v_a(t_k)}{V_a} \right) ; \quad \mu_b(t_k) = \left( \frac{v_b(t_k)}{V_b} \right) \quad (6.26)$$

with  $v_a(t_k)$  and  $v_b(t_k)$  obtained by sampling the (average) feedback control laws, obtained in Section 6.3, from the controller design procedure, based on passivity and flatness. In the simulations, we used a sampling frequency of 5 KHz and complemented the dynamic feedback controller. The *outer loop* PID controller also managed to compensate for the constant errors arising from the finite frequency PWM sampling process.

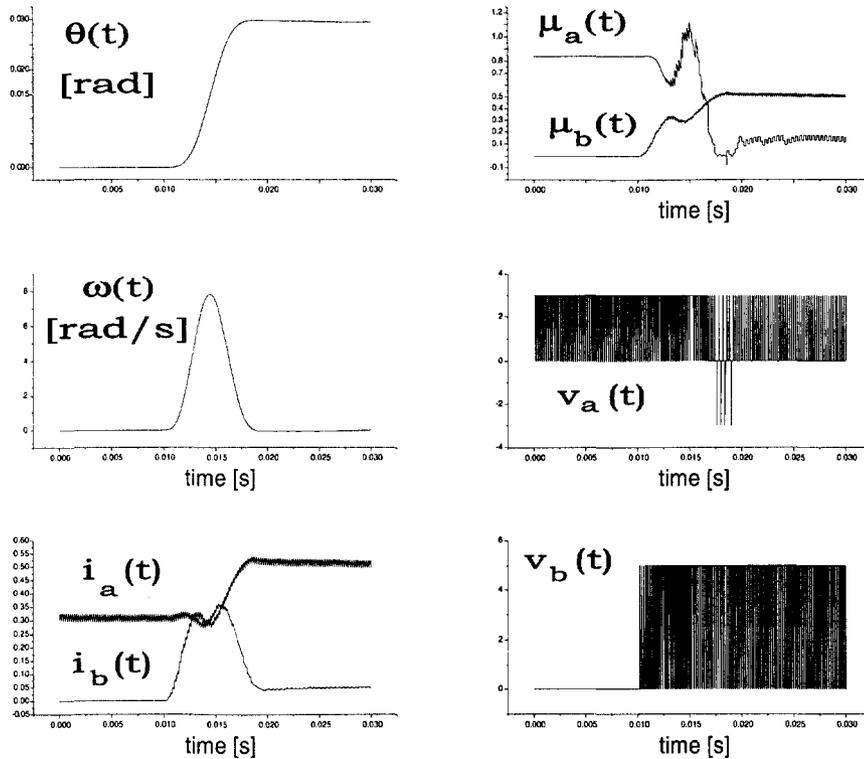


Figure 6.3: PM stepper motor closed-loop response to passivity plus flatness based controller (PWM implementation)

Figure 6.3 presents the actual switched responses of the system according to the previously described PWM control policy. In this instance we used  $V_a = 7$  V. and  $V_b = 5$  V. For simplicity, in these simulations, the load torque,  $\tau$ , was set to be zero.

### 6.3 The “boost” DC-to-DC power converter

Consider the “boost” converter circuit, shown in Figure 6.4. The system given by the following bilinear switched model

$$\begin{aligned} \dot{x}_1 &= -u\frac{1}{L}x_2 + \frac{E}{L} \\ \dot{x}_2 &= u\frac{1}{C}x_1 - \frac{x_2}{RC} \end{aligned} \quad (6.27)$$

where  $u \in \mathcal{U}$  is the control input, taking values on the discrete set  $\mathcal{U} = \{0, 1\}$ ,  $x_1$  is the inductor current,  $x_2$  is the capacitor voltage, and the parameters  $R$ ,  $L$ ,  $C$ , and  $E$ , are known constants.

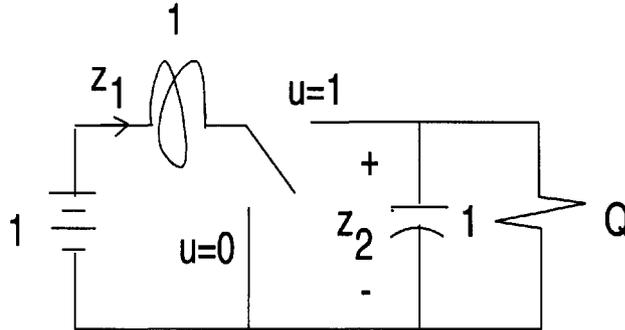


Figure 6.4: Normalized “boost” converter

In order to simplify matters, we use, as in Zinober *et al* [10], a *per-unit* normalized model of the above converter. This was achieved by setting the following state, and time variable, coordinates transformation,  $z_2 = x_2/E$ ,  $z_1 = x_1/E\sqrt{C/L}$ , and  $\tau = t/\sqrt{LC}$ . This yields,

$$\begin{aligned} \frac{d}{d\tau}z_1 &= -uz_2 + 1 \\ \frac{d}{d\tau}z_2 &= uz_1 - \frac{z_2}{Q} \end{aligned} \quad (6.28)$$

where  $Q$  is the circuit “quality”, given by  $Q = R\sqrt{C/L}$ .

The “boost” converter is a dc voltage amplifier. This is translated into the fact that the normalized output capacitor voltage is greater than 1 under its normal “amplifying mode” operating condition. To see this, we give an heuristic argument. Suppose that we manage to hold  $z_2$  ideally constant, at some value  $\bar{z}_2$  (this will entitle infinite frequency switchings, of course). The average value of the control input  $u$ , sustaining this condition, would be given by  $u = \mu = \bar{z}_2/(Qz_1)$ . Since the actual  $u$  only takes values in the set  $\{0, 1\}$ , this average value is necessarily bounded by the interval  $[0, 1]$ , given that neither of the extreme control values,  $u = 0$ , or  $u = 1$ , yields a prespecified constant value for the converter state variable  $z_2$ . The resulting differential equation for  $z_1$  is given by  $\dot{z}_1 = -\bar{z}_2^2/(Qz_1) + 1$ , whose (unstable) equilibrium value is given by  $\bar{z}_1 = \bar{z}_2^2/Q$ . The equilibrium value for  $\mu$  is then computed as  $\bar{\mu} = \bar{z}_2/Q\bar{z}_1 = 1/\bar{z}_2$ . Since  $\bar{\mu} \in [0, 1]$ ,  $\bar{z}_2$  is necessarily larger than 1. An equivalent reasoning is achieved, starting with constant values of the normalized inductor current  $z_1$ .

### 6.3.1 Flatness of the “boost” converter

The total stored energy, given in this case by,

$$F = \frac{1}{2} (z_1^2 + z_2^2) \quad (6.29)$$

qualifies as a *flat output*, since all system variables can be obtained as differential functions of such an output. Indeed, derivation with respect to  $\tau$ , denoted also by means of a “dot”, of the expression (6.29), yields

$$\dot{F} = z_1 - \frac{z_2^2}{Q} \quad (6.30)$$

From the set of Equations (6.27), (6.30) one can solve (uniquely up to physical considerations) for the state variables  $z_1$  and  $z_2$ , in terms of  $F$  and  $\dot{F}$ . One obtains

$$\begin{aligned} z_1 &= -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + Q\dot{F} + 2F} \\ z_2 &= \sqrt{-Q\dot{F} - \frac{Q^2}{2} + Q\sqrt{\frac{Q^2}{4} + Q\dot{F} + 2F}} \end{aligned} \quad (6.31)$$

These equations point to the fact that the state variables are *differential functions* of the flat output  $F$ . In the case of switched systems, the parameterization of the control input by differential functions of the flat output is to be understood only in an *average sense* as if representing the

*duty ratio* function of a PWM control scheme or an *equivalent control*. In this case we obtain,

$$u = \frac{Q}{(2z_1(F, \dot{F}) + Q)z_2(F, \dot{F})} \left[ 1 + \frac{2}{Q^2} z_2^2(F, \dot{F}) - \ddot{F} \right] \quad (6.32)$$

The differential parameterization (6.31) immediately allows for a convenient static parameterization of the state equilibria in terms of the flat output constant values. Indeed, letting  $F = \bar{F}$  be a constant, one obtains

$$\begin{aligned} \bar{z}_1 &= -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + 2\bar{F}} \\ \bar{z}_2 &= \sqrt{-\frac{Q^2}{2} + Q\sqrt{\frac{Q^2}{4} + 2\bar{F}}} \end{aligned} \quad (6.33)$$

The parameterization (6.33), immediately leads to the following alternative parameterization of the normalized capacitor voltage equilibrium in terms of the normalized inductor current equilibrium,

$$\bar{z}_2 = \sqrt{\bar{z}_1 Q} \quad \longrightarrow \quad \bar{z}_1 = \frac{\bar{z}_2^2}{Q} \quad (6.34)$$

### 6.3.2 Passivity properties through flatness

The system properties, especially those pertinent to passivity-based control, can be readily established from the differential parameterization (6.31) and (6.32).

Let  $z_1 = \bar{z}_1$  be a constant in the first equation of (6.31). We obtain the following corresponding *zero dynamics*,

$$\dot{F} = -\frac{2}{Q} \left[ F - \frac{1}{2} (\bar{z}_1^2 + \bar{z}_1 Q) \right] = -\frac{2}{Q} \left[ F - \frac{1}{2} (\bar{z}_1^2 + \bar{z}_2^2) \right] \quad (6.35)$$

whose trajectories are exponentially asymptotically stable to the equilibrium value  $1/2(\bar{z}_1^2 + \bar{z}_2^2)$ . This establishes that the output  $z_1$ , the normalized inductor current, is a *minimum phase* output. Since it is also relative degree one, it is a *passive* output [2].

Let  $z_2 = \bar{z}_2$  be a constant value. From (6.31), we obtain, after some algebraic manipulations, that the corresponding zero dynamics is given by the following *implicit* first-order differential equation,

$$\left( \dot{F} + \frac{\bar{z}_2^2}{Q} \right)^2 = 2F - \bar{z}_2^2 \quad (6.36)$$

which, in the phase plane  $(F, \dot{F})$ , can be represented by a parabola, opening to the right, with the vertex located at the point  $(0.5\bar{z}_2^2, -\bar{z}_2^2/Q)$ . Incidentally, the vertex of the parabola is an *impasse* point since the differential equation degenerates into an algebraic equation. The equilibrium point for this differential equation is given by

$$\bar{F} = \frac{1}{2} \left[ \left( \frac{\bar{z}_2^2}{Q} \right)^2 + \bar{z}_2^2 \right] = \frac{1}{2} [\bar{z}_1^2 + \bar{z}_2^2] \quad (6.37)$$

A phase diagram of Equation (6.36) readily reveals that this equilibrium point is unstable. The normalized output capacitor voltage  $z_2$  is thus a *non-minimum phase* output.

The *constant input equilibrium state detectability* (see Byrnes et al. [2] for definitions and general results) of the output  $z_1$  also readily follows from the differential parameterization (6.31) and (6.32). This last fact implies that the system is also stabilizable by means of *output feedback* (see [2]).

### 6.3.3 A passivity-based sliding mode controller

The “energy shaping plus damping injection” controller design (see the recent book by Ortega et al. [7]) is based on the creation of a linear in-the-state, time-varying, “copy” of the plant, sharing the same control input as the given system. This reference model of the plant is provided with appropriate supplementary damping enhancing the corresponding “dissipation structure”. The *virtual*, or auxiliary, model of the system shares all the important properties of the original plant (such as passivity of the corresponding outputs and flatness over a larger ground field) has the property of “pulling” the systems state trajectories towards the desired prespecified trajectories. In our “boost” converter case, such an auxiliary model is given by

$$\begin{aligned} \dot{\xi}_1 &= -u\xi_2 + 1 + Q_c(z_1 - \xi_1) \\ \dot{\xi}_2 &= u\xi_1 - \frac{\xi_2}{Q} \end{aligned} \quad (6.38)$$

where  $Q_c$  is the added damping we impose on the auxiliary inductor current dynamics.

Notice that the reference model “tracking error” state  $e_1 = z_1 - \xi_1$ ,  $e_2 = z_2 - \xi_2$ , satisfies the following controlled dynamics

$$\begin{aligned} \dot{e}_1 &= -ue_2 - Q_c e_1 \\ \dot{e}_2 &= ue_1 - \frac{1}{Q} e_2 \end{aligned} \quad (6.39)$$

Along the trajectories of the system (6.39), the time derivative of the modified energy function  $V(e) = (1/2)(e_1^2 + e_2^2)$  satisfies

$$\dot{V}(e) = -Q_c e_1^2 - \frac{1}{Q} e_2^2 \leq \alpha V(e) \quad (6.40)$$

with  $\alpha = 2 \min\{Q_c, 1/Q\}$ .

Thus, as stated, the tracking error state trajectories asymptotically exponentially converge to zero, independently of the control input. The system state trajectories asymptotically track the auxiliary system controlled trajectories. Thus, regulating the auxiliary system along a given desired trajectory results, in turn, in an effective regulation of the original plant. Notice that the auxiliary system initial states are entirely at our disposal.

Let  $z_1^*(\tau)$  be a desired trajectory for the auxiliary variable  $\xi_1$ . A sliding mode controller that forces the auxiliary state  $\xi_1$  to track the specified trajectory is given by

$$u = \frac{1}{2} (1 + \text{sign}(\xi_1 - z_1^*(\tau))) \quad (6.41)$$

The existence of a sliding mode on the time-varying sliding surface

$$S_0 = \{(\xi_1, \xi_2) \mid \sigma = \xi_1 - z_1^*(\tau) = 0\} \quad (6.42)$$

is assessed in the following manner.

Suppose that the initial the auxiliary state  $\xi_1$ , is set to coincide with the plant's state  $z_1$ . Thus, due to linearity of the model reference tracking error, the error state  $e = z - \xi$  remains constrained to zero. If such is not the case, this error, as it was already shown, exponentially decreases to zero. Suppose now that the quantity  $\xi_1 - z_1^*(\tau)$  is initially negative. According to (6.41), the control input  $u$  is initially set to zero, and it will remain clamped at this value until  $\xi_1$  reaches the desired trajectory  $z_1^*(t)$ . The time derivative of the "sliding surface" coordinate,  $\sigma = \xi_1 - z_1^*(\tau)$ , throughout this phase is given by

$$\dot{\sigma} = 1 + Q_c(z_1 - \xi_1) - \dot{z}_1^*(\tau) \quad (6.43)$$

where the second term is either identically zero, or exponentially approaching zero, as already described. In order to reach the time-varying sliding surface, from *below*, i.e.,  $\xi_1 < z_1^*(\tau)$ , the time derivative of  $\sigma$  needs to be positive. This is achieved as long as the initial value of  $\xi_1$  is chosen close to, or equal, that of  $z_1$ , and the desired trajectory  $z_1^*(\tau)$  has a normalized time derivative which is absolutely bounded above by 1, i.e.  $|\dot{z}_1^*(\tau)| < 1$ . Thus, we should specify the desired trajectory  $z_1^*(\tau)$  and initialize the auxiliary system state  $\xi_1$  with these conditions in mind.

If  $\xi_1$  is at certain time  $\tau_1$  above the value of  $z_1^*(\tau)$ ,  $\sigma$  is positive. Then, the control input  $u$ , according to (6.41), is set to adopt the value 1 and the time derivative of the sliding surface coordinate  $\sigma = \xi_1 - z_1^*(\tau)$  is given by

$$\dot{\sigma} = -\xi_2 + Q_c(z_1 - \xi_1) - \dot{z}_1^*(\tau) \quad (6.44)$$

Notice that if we are already above the sliding surface,  $\xi_2$  is necessarily larger than 1, due to the amplifying feature of the converter, and the tracking error  $(z_1 - \xi_1)$  may be considered to be negligible or already zero. This means, under the same previous assumption regarding the absolute value of the time derivative of the proposed trajectory,  $|\dot{z}_1^*(\tau)| < 1$ , that  $\dot{\sigma}$  will be locally negative and the sliding surface  $S_0$  is guaranteed to be reached from above.

The sliding mode controller (6.41) achieves the convergence of  $\xi_1$  towards the desired trajectory  $z_1^*(\tau)$ . The *equivalent control*, defined as the *virtual* control action,  $u_{eq}(\tau)$ , responsible for *ideally* maintaining the evolution of the sliding surface coordinate  $\sigma$  at the value zero, and is obtained from the *invariance* condition  $\dot{\sigma} = 0$ , which is evaluated at the ideal sliding condition  $\sigma = 0$ ,

$$u_{eq} = \frac{-\dot{z}_1^*(\tau) + 1 + Q_c(z_1 - z_1^*(\tau))}{\xi_2} \quad (6.45)$$

The *ideal sliding dynamics*, or *remaining dynamics* is obtained by substituting the equivalent control expression (6.45) into the auxiliary capacitor voltage dynamic (6.38)

$$\dot{\xi}_2 = \left( \frac{-\dot{z}_1^*(\tau) + 1 + Q_c(z_1 - z_1^*(\tau))}{\xi_2} \right) z_1^*(\tau) - \frac{\xi_2}{Q} \quad (6.46)$$

Since,  $z_1$  also converges towards  $z_1^*(\tau)$ , due to the fact that  $\xi_1$  is forced to follow  $z_1^*(\tau)$  in finite time and, as we have seen,  $z_1$  exponentially asymptotically converges towards  $\xi_1$ , the equivalent control and the ideal sliding dynamic asymptotically converge towards values given by the following expressions

$$\begin{aligned} u_{eq} &= \frac{1 - \dot{z}_1^*(\tau)}{\xi_2} \\ \dot{\xi}_2 &= \left( \frac{1 - \dot{z}_1^*(\tau)}{\xi_2} \right) z_1^*(\tau) - \frac{\xi_2}{Q} \end{aligned} \quad (6.47)$$

Equations (6.45) and (6.46) can be regarded as a *dynamic* feedback equivalent controller, which only requires the feedback of the *passive* output  $z_1$  from the plant.

### 6.3.4 Non-minimum phase output stabilization

The control objective is to perform an equilibrium-to-equilibrium transfer for the output capacitor voltage,  $z_2$ , of the converter. As will be shown, due to the non-minimum phase character of the output voltage, the corresponding regulation problem is not directly feasible. However, by embedding this problem into a corresponding equilibrium transfer for the flat output, the underlying internal stability problem is easily circumvented.

### 6.3.5 Trajectory planning

Suppose we specify a trajectory  $z_1^*(\tau)$  for the auxiliary, minimum phase output variable  $\xi_1$ , in full accordance with the desired equilibrium to equilibrium transfer for the plant capacitor voltage. In other words, we assume that the capacitor voltage initial equilibrium,  $\bar{z}_{20}$ , is to be transferred, over a time period  $\Delta T = \tau_f - \tau_0 > 0$ , towards the final equilibrium value  $z_{2F}$ . Due to the non-minimum phase character of  $z_2$ , this stabilization task needs to be reformulated in terms of a transfer defined on the corresponding equilibria for the minimum phase variable  $z_1$ . This is achieved by specifying a suitable trajectory  $z_1^*(\tau)$  for the auxiliary variable  $\xi_1$ . If the auxiliary system state  $\xi_1$  is forced to track the trajectory  $z_1^*(\tau)$ , as it has been previously demonstrated, the plant state trajectory  $z_1(\tau)$  will follow suit. Thus, we specify

$$z_1^*(\tau) = \bar{z}_{10} + (\bar{z}_{1F} - \bar{z}_{10}) \psi(\tau, \tau_0, \tau_f) \quad (6.48)$$

with  $\psi(\tau, \tau_0, \tau_f)$  being a time polynomial smoothly interpolating between 0 and 1, satisfying

$$\psi(\tau_0, \tau_0, \tau_f) = 0 \quad ; \quad \psi(\tau_f, \tau_0, \tau_f) = 1$$

and use this specification in the sliding mode controller expression (6.41).

This scheme yields desirable results and good quality of responses, for any suitably defined reference trajectory  $z_1^*(\tau)$ . However, we claim that the most natural choice for specifying this trajectory is to do so by resorting to the flatness property present in the original plant. Instead of directly specifying the trajectory for the auxiliary variable  $\xi_1$  in terms of  $z_1^*(\tau)$ , we specify  $z_1^*(t)$  through its relation with the flat output, as given by (6.33)

$$z_1^*(\tau) = -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + Q\dot{F}^*(\tau) + 2F^*(\tau)} \quad (6.49)$$

The described choice has a justification in terms of the simplicity of off-line trajectory planning tasks, when made in terms of the flat output,

and the induced response on the non-minimum phase normalized capacitor voltage variable  $z_2$ . The specification of a flat output reference trajectory enjoys a direct differentially parameterized relationship with the corresponding induced trajectory for  $z_2$ , as evidenced by (6.31). This, in a sense, is a static relationship, devoid of any dynamics, in which all features of the corresponding desired time response of  $z_2$  can be *assessed, predicted, and corrected* without solving differential equations. On the other hand, the specification of  $z_1$  undergoes an *implicit* dynamic relationship with the corresponding response trajectory of  $z_2$ , which needs to invoke not only the feedback expression for the equivalent control input but it also requires the solution of a differential equation. As a result, pre-specifying the flat output is not only more efficient from the design viewpoint but it also allows for simple off-line experimentation and evaluation that results in a better quality of response for the capacitor voltage variable. In fact, the typical “undershoot” response of the non-minimum phase variable  $z_2$  can be effectively avoided by a reasonable off-line designed specification of the flat output trajectory. It is not clear how to achieve the same goal with a direct choice for  $z_1^*(t)$ .

### 6.3.6 Simulation results

Figure 6.5 shows the closed-loop responses of the state and input variables for a desired capacitor voltage equilibrium transfer on a typical dc-to-dc power converter of the “boost” type, with parameter values given by

$$L = 0.02 \text{ H}, \quad C = 1 \text{ } \mu\text{F}, \quad R = 200 \text{ } \Omega, \quad E = 15 \text{ V}$$

A sliding mode passivity based controller was designed to increase the output capacitor voltage from the initial equilibrium value of 30 V, towards a final desired value of 60 V. The corresponding equilibrium values for the inductor current are 0.3 A, and 1.2 A, respectively. The flat output must be transferred from the initial value of 0.00135 towards the final value of 0.0162. In terms of the normalized state variables, the digital simulations are carried out more efficiently with corresponding simulation values.

$$Q = \sqrt{2}, \quad z_{10} = 2\sqrt{2}, \quad z_{1F} = 8\sqrt{2}, \quad z_{20} = 2, \quad z_{2F} = 4,$$

$$\tau = \frac{\sqrt{2}}{2} \times 10^4 t$$

The corresponding trajectory for the normalized flat output was designed using the expression (6.48) with the following *Bézier* polynomial,

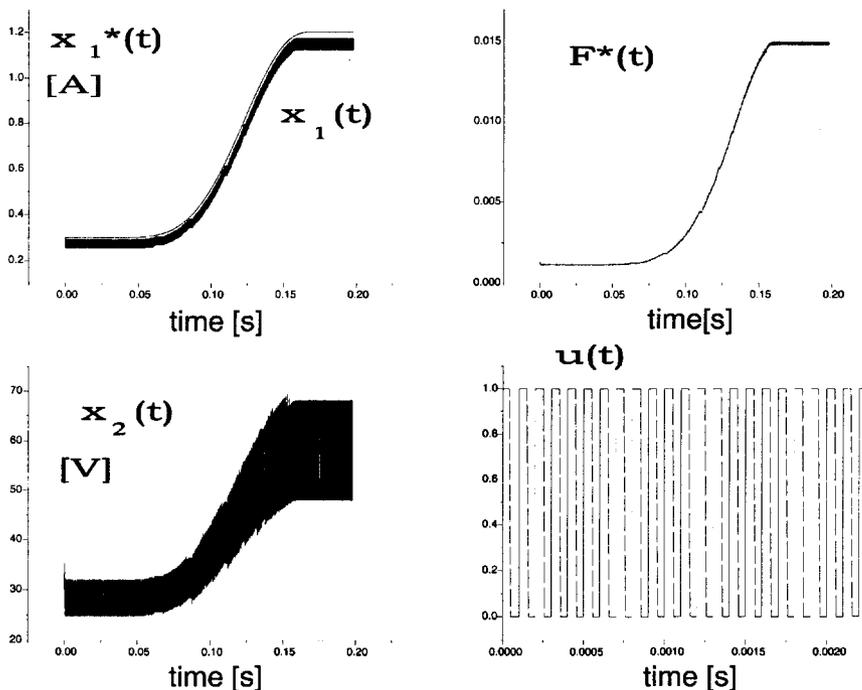


Figure 6.5: Closed-loop response of sliding mode controlled “boost” converter with planned current reference trajectory in terms of the flat output

smoothly interpolating between 0 and 1,

$$\psi(\tau, \tau_0, \tau_f) = \left(\frac{\Delta\tau}{\Delta T}\right)^5 \left[ 21 - 35 \left(\frac{\Delta\tau}{\Delta T}\right) + 15 \left(\frac{\Delta\tau}{\Delta T}\right)^2 \right]$$

where  $\Delta T = \tau_f - \tau_0$ , and  $\Delta\tau = \tau - \tau_0$  where  $\tau_0 = 200$  and  $\tau_f = 1200$ , dimensionless units, which correspond, after the change of time scale, to  $t_1 = 0.028$  s and  $t_2 = 0.144$  s.

### 6.3.7 Dc-to-ac power conversion

A discontinuous feedback control law for  $u$  was desired, such that the normalized capacitor voltage,  $z_2$ , tracks a given desired voltage signal  $z_2^*(\tau)$ , which never becomes constant. This signal was assumed to be bounded and sufficiently differentiable. In fact, we assumed that  $z_2^*(\tau)$  was smooth, i.e., infinitely differentiable. Specifically, we were interested in generating a

normalized output voltage of the form  $z_2(t) = A + (B/2) \sin \omega\tau$  with  $A > 0$  and  $\omega > 0$  and  $B$  being a constant of arbitrary sign.

The non-minimum phase properties of the output capacitor voltage, joined to the “control acquisition” structure of the converter equations and the discrete-valued nature of the control input  $u$ , made it especially difficult for the synthesis of a switching feedback control law that results in a stable ac capacitor voltage reference signal tracking scheme. It should be clear that the main task to be solved was to obtain a procedure by which an inductor current reference signal was approximately, or exactly, computed whose corresponding “remaining dynamics” trajectories are either given by the desired output capacitor voltage or by some reasonable approximation to it.

In order to obtain a suitable reference trajectory  $z_1^*(\tau)$  for  $z_1$ , given that  $z_2$  is of a particular form  $z_2^*(\tau)$ , one should proceed to *eliminate* the flat output  $F^*$  from the set of relations

$$F^* = \frac{1}{2} [(z_1^*(\tau))^2 + (z_2^*(\tau))^2]; \dot{F}^* = z_1^*(\tau) - \frac{(z_2^*(\tau))^2}{Q} \quad (6.50)$$

Such an elimination yields a *differential relation* between  $z_1^*(\tau)$  and  $z_2^*(\tau)$ , i.e., one which, necessarily, involves an infinite number of time derivatives of  $z_2^*(\tau)$ . We will exploit this elimination idea in order to generate an approximating sequence of static differential algebraic relationships yielding the normalized input inductor current reference signal  $z_1^*$ , exclusively in terms of the output capacitor voltage reference trajectory  $z_2^*$  and a *finite* number of its time derivatives. This *finite differential parameterization* of  $z_1$  in terms of  $z_2$  will, of course, allow for the indirect sliding mode generation of a large class of bounded ac output capacitor voltage profiles, which are sufficiently differentiable.

### 6.3.8 An iterative procedure for generating a suitable inductor current reference

In order to simplify the notation we will temporarily suppress the asterisks and the time argument in the developments of this subsection. Consider then the set of relations (6.50). Those relations can be alternatively viewed in the following manner:

$$\begin{aligned} z_1 &= \frac{z_2^2}{Q} + \dot{F} \\ F &= \frac{1}{2} (z_1^2 + z_2^2) \end{aligned} \quad (6.51)$$

Evidently, one may “embed” the set of relations (6.51) as the outcome of a convergent iterative procedure, aimed at eliminating  $F$ , where the value of  $z_1 = z_{1,\infty}$  was computed exclusively in terms of a given fixed function  $z_2$  and, possibly, an *infinite* number of its time derivatives. In other words  $z_1$ , viewed as the outcome of such an iterative procedure, could be represented, after convergence, by

$$z_{1,\infty} = \frac{z_2^2}{Q} + \dot{F}_\infty \quad (6.52)$$

$$F_\infty = \frac{1}{2} (z_{1,\infty}^2 + z_2^2) \quad (6.53)$$

Equations (6.52) and (6.53) immediately suggest the consideration of the following iterative procedure,

$$\begin{aligned} z_{1,k} &= \frac{z_2^2}{Q} + \dot{F}_k \\ F_{k+1} &= \frac{1}{2} (z_{1,k}^2 + z_2^2) \end{aligned} \quad (6.54)$$

This algorithm sequentially yields an approximation of a static relationship between  $z_1$  and  $z_2$ , which only involves polynomial expression of  $z_2$  and of its time derivatives. The algorithm of course should be “initialized” by an arbitrary but reasonable trajectory  $F_0(\tau)$  for the flat output  $F$ .

Starting from the equilibrium condition,  $F_0(\tau) = \text{constant}$ , one obtains the following sequence of approximating expressions for the normalized inductor current reference trajectory  $z_1$ ,

$$z_{1,0} = \frac{z_2^2}{Q} \implies F_1 = \frac{1}{2} \frac{z_2^4}{Q^2} + \frac{1}{2} z_2^2 \quad (6.55)$$

$$\begin{aligned} z_{1,1} &= \frac{z_2^2}{Q} + z_2 \dot{z}_2 \left( 1 + \frac{2}{Q^2} z_2^2 \right) \\ &\implies F_2 = \frac{1}{2} \left[ \frac{z_2^2}{Q} + z_2 \dot{z}_2 \left( 1 + \frac{2}{Q^2} z_2^2 \right) \right]^2 + \frac{1}{2} z_2^2 \end{aligned} \quad (6.56)$$

$$\begin{aligned} z_{1,2} &= \frac{z_2^2}{Q} + \left( \frac{z_2^2}{Q} + z_2 \dot{z}_2 + \frac{2z_2^3}{Q^2} \dot{z}_2 \right) \times \\ &\quad \left( \frac{2}{Q} z_2 \dot{z}_2 + (\dot{z}_2)^2 + z_2 \ddot{z}_2 + \frac{6z_2^2}{Q^2} (\dot{z}_2)^2 + \frac{2z_2^3}{Q^2} \ddot{z}_2 \right) + z_2 \dot{z}_2 \end{aligned} \quad (6.57)$$

⋮

$$z_{1,\infty} = \psi(z_2, \dot{z}_2, z_2^{(2)}, \dots, z_2^{(k)}, \dots)$$

### 6.3.9 Simulation results

The generated normalized inductor current reference signals (6.55)-(6.57) were used in a passivity-based sliding mode control scheme, carried out in the same manner for the stabilization problem described in Section 6.2.

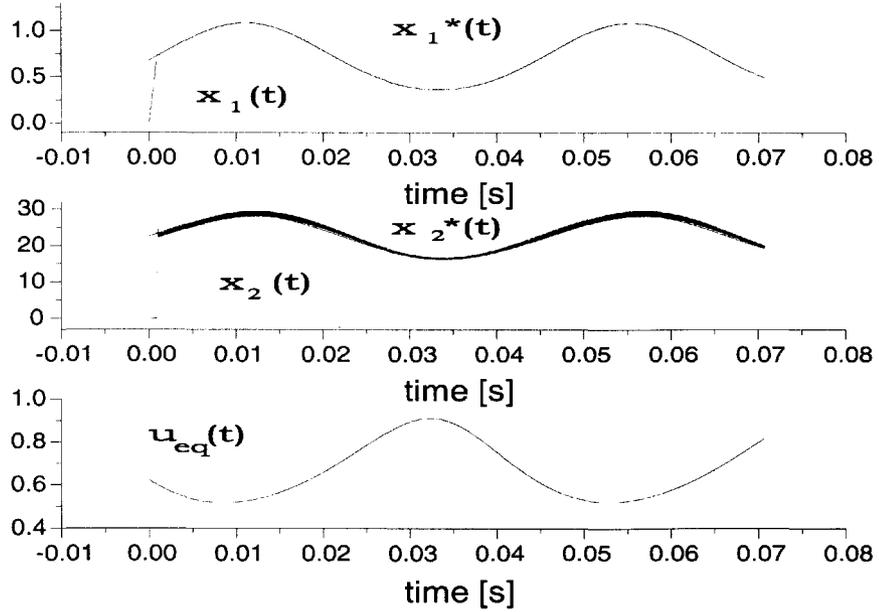


Figure 6.6: Closed-loop response for output capacitor voltage using  $z_{1,0}^*$  as the normalized inductor current reference trajectory (normalized  $\omega = 0.02$ )

A typical “boost” converter was chosen, with circuit parameters  $L = 20$  mH,  $C = 1 \mu\text{F}$ ,  $R = 50 \Omega$ , and  $E = 15$  V. For the normalized “boost” converter dynamics, the dimensionless circuit quality turns out to be  $Q = 0.3535$ . We took the following candidates as sliding surfaces  $\sigma_k = z_1 - z_{1,k}^*(\tau)$  for  $k = 0, 1$ , with  $z_{1,0}^*(\tau)$  and  $z_{1,1}^*(\tau)$  as given by (6.55)-(6.56), respectively. As a desired output capacitor voltage signal, we chose  $z_2^*(\tau) = A + B/2 \sin \omega\tau$ . The constants  $A > 0$ ,  $B$  and  $\omega$  were set so that the sliding mode existence conditions were satisfied. The parameters of the desired normalized sinusoidal voltage reference signal,  $A + (B/2) \sin(\omega\tau)$ , were set to be  $A = 1.5$ ,  $B = 0.8$ ,  $\omega = 0.02$ , i.e.,

$$z_2^*(\tau) = 1.5 + 0.4 \sin(0.02\tau)$$

which corresponds with an denormalized sinusoidal voltage of the form

$$x_2^*(t) = 22.5 + 6 \sin(\sqrt{2} \times 10^2 t) \text{ V}$$

The corresponding time basis for the adopted normalization was  $t_b = 0.1414$  ms. The underlying sampling process for the simulation used normalized sampling periods of 0.1 time units, which corresponded to an actual sampling frequency of about 70.71 KHz.

Figure 6.6 shows the closed-loop output voltage response of the proposed sliding mode tracking controller for a sliding surface candidate of the form  $\sigma_k = z_1 - z_{1,k}^*(\tau)$ , with  $k = 0$ . The simulated output voltage response is shown, for comparison purposes, along with the desired output capacitor voltage signal  $z_2^*(\tau)$ . These signals can hardly be distinguished from each other. Figure 6.6 also shows the trajectory of the off-line generated inductor current signal, along with the actual inductor current response. The equivalent control trajectory, also shown along with the denormalized voltage response, are bounded, after sliding starts, by the closed interval  $[0, 1]$ . The simulations corresponding to the sliding surface candidate obtained for  $k = 1$ , depicted in Figure 6.7, show that the agreement with the desired trajectories, obtained for  $k = 0$ , were not substantially improved.

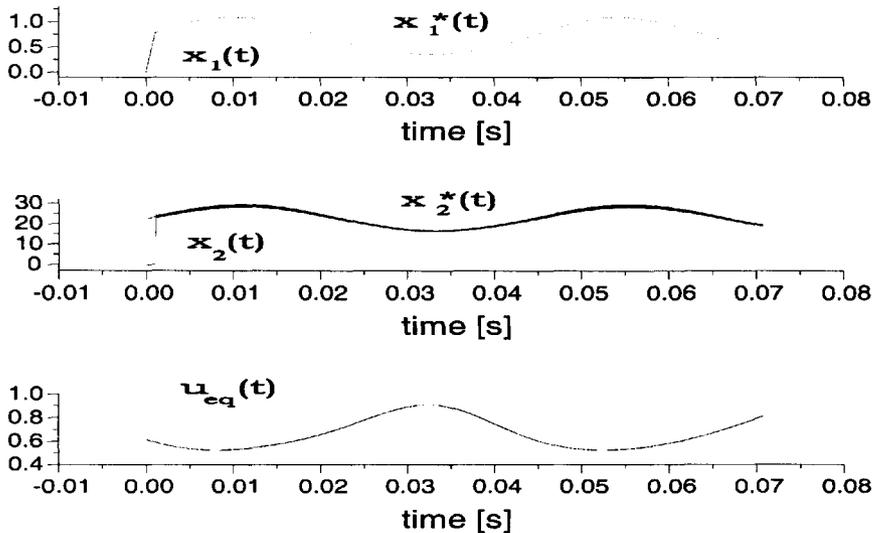


Figure 6.7: Closed-loop response for output capacitor voltage using  $z_{1,1}^*$  as the normalized inductor current reference trajectory (normalized  $\omega = 0.02$ )

## 6.4 Conclusions

In this chapter we proposed a flatness-based approach for the passivity-based stabilization and trajectory tracking of two switched electrical systems. One was weakly minimum phase while the other was a non-minimum phase system. The approach used a suitable combination of flatness-based trajectory tracking, passivity, and sliding mode control. The more complex problem of minimum phase output signal reference tracking, not leading to equilibrium, but sustaining a desired oscillatory behavior, required a new approach based on consideration, as reference trajectories candidates, those emerging from a sequence of finite-order differential parameterizations of the minimum-phase output in terms of the non-minimum phase output.

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# Chapter 7

## Stability and Stabilization

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### 7.1 Introduction

Sliding mode control design techniques are based on a two stage procedure:

- hitting phase (or reaching phase), and the
- sliding phase.

Both of them are concerned with stability/attractivity concepts because:

- in the first step, the condition ensuring the sliding motions is a contraction property (at least locally around the sliding manifold),
- in the second one, the choice of the surface (shaping procedure) is mostly related to some stabilization problem: one has to compute (or “tune”) the parameters involved in the shape of the sliding surface such that the sliding motions achieve some convergence and/or stabilization problem.

To make this explicit, let us consider the following example

$$\begin{cases} \dot{x}_1 = \frac{x_2}{(1+x_2^2)} - 2\frac{x_1x_2}{1+x_2^2}u \\ \dot{x}_2 = u \end{cases} \quad (7.1)$$

This system “seems” complex, however, if we set

$$\begin{aligned} z_1 &= x_1(1+x_2^2) \\ z_2 &= x_2 \end{aligned}$$

(note that it defines a global diffeomorphism), then one obtains

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = u \end{cases} \quad (7.2)$$

and it becomes obvious that if in sliding mode  $z_2 = -z_1$ , then  $z_1$  converges asymptotically to zero ( $\dot{z}_1 = z_2 = -z_1$ ) and thus  $z_2$  also converges. In this step of design (the “sliding phase”), the shape of the sliding manifold arises naturally.

Now, we need to force the system to evolve on the constraint  $z_2 = -z_1$ . For this, let us define the sliding surface as

$$\mathcal{S} = \{z \in \mathbb{R}^2 : s(z) = 0\} \quad (7.3)$$

$$s(z) = z_2 + z_1 \quad (7.4)$$

Then, according to the equivalent control method [32, 31], we need the control to satisfy

$$u(z) = \begin{cases} u^+(z) & \text{if } s(z) > 0 \\ u^-(z) & \text{if } s(z) < 0 \end{cases}$$

$$\min(u^+(z), u^-(z)) < u_{\text{eq}} = -z_2 < \max[u^+(z), u^-(z)]$$

in order to ensure that a sliding mode exists on  $\mathcal{S}$ . This leads to various design controls, for example,

$$u(z) = \begin{cases} -1 & \text{if } s(z) > 0 \\ 1 & \text{if } s(z) < 0 \end{cases}$$

which ensures a finite time convergence to  $\mathcal{S}$  as soon as the initial conditions are close enough to the surface and satisfy  $|z_2| < 1$ . But, can we provide a better characterization of the initial conditions leading to a sliding mode?

An alternative to this control is

$$u(z) = \begin{cases} -z_2 - 1 & \text{if } s(z) > 0 \\ -z_2 + 1 & \text{if } s(z) < 0 \end{cases} \quad (7.5)$$

which ensures a finite time convergence to  $\mathcal{S}$ , whatever the initial conditions. But since the chattering problem remains, can we stabilize the system while reducing the chattering?

In this chapter, three problems related to the above-mentioned phases are developed:

- to transform the initial system into an appropriate form which provides a guide for the choice of the sliding surface: Section 7.3,

- to determine the set of all admissible initial conditions which are leading to sliding motions on the sliding domain (the useful part of the sliding manifold on which a sliding regime can take place): Section 7.4, and
- stabilization issue reducing the chattering phenomenon: Section 7.5 (see also [1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 17, 24, 28, 29]).

## 7.2 Notation

In this chapter, MIMO nonlinear systems are considered and are of the following type:

$$(M) \equiv \begin{cases} \dot{x} = f(x) + G(x)u \\ y = h(x) \end{cases} \quad (7.6)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}^m$  is the control vector ( $m$  inputs),  $y \in \mathbb{R}^p$  is the output vector ( $p$  outputs:  $y = [h_1(x), \dots, h_p(x)]^\top$ ),  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , a smooth drift vector field,  $G(x) = ([g_1(x), \dots, g_m(x)])$ , is an  $(n \times m)$ -matrix and  $g_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$  are smooth vector fields, with  $g_{ij}(x)$  the control gain of the  $j^{\text{th}}$  input acting on the  $i^{\text{th}}$  state space variable. We assume that  $h(0) = 0$ , in such a way that the problem of driving the outputs to zero is translated into the problem of driving the state asymptotically to the zero equilibrium.

Furthermore, the sliding motions are studied for a control defined as follows:

$$u_i = \begin{cases} u_i^+(x) & \text{if } s_i(x) > 0 \\ u_i^-(x) & \text{if } s_i(x) < 0 \end{cases} \quad (7.7)$$

with  $s_i = s_i(x)$  defining the manifold of commutation  $s_i(x) = 0$  and  $s(x) \in \mathbb{R}^m$ .

Throughout the chapter the following notation will be used:

- $\bar{\mathcal{S}}$ ,  $\overset{\circ}{\mathcal{S}}$ ,  $\partial(\mathcal{S})$ , respectively denote the closure, the interior, and the boundary of the set  $\mathcal{S}$
- $\rho$  is the Euclidean distance
- $\mathcal{N}(\mathcal{S}, \varepsilon) = \{x \in \mathbb{R}^n : \rho(\mathcal{S}, x) < \varepsilon\}$  is the  $\varepsilon$ -neighborhood of the set  $\mathcal{S}$
- if  $z \in \mathbb{R}^k$  then  $|z| = [|z_1|, \dots, |z_k|]^\top$
- $\mathcal{C}^\alpha(\mathcal{S}, \mathbb{R}^k)$  is the set of  $\alpha$ -times continuously differentiable functions from  $\mathcal{S}$  into  $\mathbb{R}^k$

- for smooth  $n$ -vector fields,  $f(x)$ ,  $g(x)$ , [20]:  $[f, g](x) \triangleq \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)$ , (Lie product or Lie Bracket), generate a vector field as the  $Ad$  operator defined by  $Ad_f^0 g(x) \triangleq g(x)$ ,  $Ad_f^1 g(x) \triangleq [f, g](x)$ ,  $Ad_f^k g(x) \triangleq [f, Ad_f^{k-1} g](x)$
- for a smooth real-valued function  $\lambda(x)$ , [20]:

$$d\lambda(x) \triangleq \left( \frac{\partial \lambda}{\partial x_1}, \dots, \frac{\partial \lambda}{\partial x_i}, \dots, \frac{\partial \lambda}{\partial x_n} \right) \quad (7.8)$$

(the “gradient” of  $\lambda$ )

- $\text{sign}(\zeta)$  and  $\text{SIGN}(z)$  are, respectively, real and vector signum functions defined as follows:

$$\text{sign}(\zeta) = \begin{cases} -1 & \text{if } \zeta < 0 \\ 0 & \text{if } \zeta = 0 \\ 1 & \text{if } \zeta > 0 \end{cases} \quad (7.9)$$

$$\text{SIGN}(z) = (\text{sign}(z_1), \dots, \text{sign}(z_l)), z \in \mathbb{R}^l \quad (7.10)$$

When the signum function is not defined at zero, then the above defined functions will be denoted respectively by  $\text{sgn}$  and  $\text{SGN}$ .

## 7.3 Generalized regular form

In this section, hypotheses are given for the existence of a regular change of coordinates transforming the initial system into a so-called “regular form”. The original result was presented in [21] and recalled in [26]. Here it is enlarged to the following expected regular form:

$$(RF) \equiv \begin{cases} \dot{z}_1 = f_1^R(z_1, z_2) + G^R(z_1, z_2)u \\ \dot{z}_2 = f_2^R(z_1, z_2) \\ z_1 \in \mathbb{R}^d, z_2 \in \mathbb{R}^{(n-d)} \end{cases} \quad (7.11)$$

where the number  $d$  may be greater or smaller than the number of input  $m$ , where as in [21]  $d = m$ . On the basis of such a regular form, we will investigate our two basic problems cited in the introduction.

### 7.3.1 Obtention of the regular form

The problem is to find a diffeomorphic state space transformation  $z = \phi(x)$  changing  $(M)$  (7.6) into  $(RF)$  (7.11). First, if  $G(x)$  is not full rank then one can find a pre-static feedback in order to obtain a new system with full rank input gain matrix.

## On the rank of the input gain matrix

In [21], the classical case a fundamental hypothesis relies on the  $\text{rank}(G(x))$  which should be maximal ( $m$ ): there is as inputs as the rank of the input gain matrix. The next theorem show how to recover that classical hypothesis using a pre-static state feedback.

**Theorem 57** [23, 25] *If  $\text{rank}(G(x_0)) = r$ , then there is a static feedback*

$$u = W(x)(v^T, 0, \dots, 0)^T, v \in \mathbb{R}^r$$

with  $W$  nonsingular in a neighborhood  $\mathcal{N}(x_0)$  of  $x_0$ , such that:

$$G(x)W(x) = \begin{pmatrix} \overbrace{X \quad 0 \quad \dots \quad 0}^r & \overbrace{0 \quad 0 \quad \dots \quad 0}^{m-r} \\ \vdots & \vdots \\ \vdots & \vdots \\ X & 0 \quad \dots \quad 0 \\ X & 0 \quad \dots \quad 0 \\ \vdots & \vdots \\ X & 0 \quad \dots \quad 0 \end{pmatrix} \quad (7.12)$$

Note that  $W(x)$  is not unique, so it can be used to “balance” the control on each physical input. This argument is important when, for safety sake, the process has more physical inputs ( $m$  actuators) than necessary free controls ( $r$ ). In the following, we consider that  $\text{rank}(G(x_0)) = m$ . If it is not the case ( $\text{rank}(G(x_0)) = r < m$ ), using the previous theorem one can consider the system  $(M)$  (7.6) with static feedback (7.12), which is transformed into:

$$(M') \equiv \begin{cases} \dot{x} = f(x) + G'(x)v \\ y = h(x) \end{cases} \quad (7.13)$$

where  $v \in \mathbb{R}^r$  is the new control vector and  $G'(x)$  is an  $(n \times r)$  matrix of full rank  $r$ .

## Results on the existence of a regular form

The given results are local, but when assumption H1) (see theorem statements) holds everywhere in the state space, then the diffeomorphism is global and so are the results. The following result is an extension of classical results for nonlinear systems using a differential geometric approach (see [20]):

**Theorem 58** [23, 25] Let  $\Delta$  be a distribution such that

H1)  $\Delta$  is nonsingular at  $x_0$  (i.e., of constant dimension  $\dim \Delta = d_\Delta \leq n$ )

H2)  $\Delta$  is involutive, which is

$$\forall \tau_1 \in \Delta, \forall \tau_2 \in \Delta : [\tau_1, \tau_2] \in \Delta \quad (7.14)$$

H3)  $\text{span} \{g_1(x), \dots, g_m(x)\} \subset \Delta$

Then there exists a neighborhood  $\mathcal{N}(x_0)$  of  $x_0$  and a local diffeomorphism  $z = \phi(x)$  defined on  $\mathcal{N}(x_0)$ , such that (M) (7.6) is transformed into (RF) (7.11) with  $d = d_\Delta \leq n$ . Moreover, if  $d_\Delta < n$ , then the conclusion holds for  $n \geq d \geq d_\Delta$ .

Note that (RF) is not the classical local decomposition for “controllability/reachability” (see [20]) because the term  $\dot{z}_2 = f_2^R(z_1, z_2)$  depends on the variable  $z_1$ .

**Remark 59** For single input systems,  $G \equiv g$  a smooth vector field and  $u \in \mathbb{R}$ . If  $g(x_0) \neq 0$ , then the distribution  $\Delta = \text{span} \{g(x)\}$  ( $\dim \Delta = d_\Delta = 1$ ) is involutive and so the system (M) (7.6) can be transformed into (RF) (7.11) with  $d = d_\Delta = 1$ . This was originally treated in [21].

**Remark 60** Note, that if the candidate distribution  $\Delta$  in the previous theorem is  $\text{span} \{g_1(x), \dots, g_m(x)\}$ , then one obtains the classical result of [21] ( $d = d_\Delta = m$ ). So, this result is an extension which, as we will see in the following sections, provides guidance for the design of a sliding mode controller in the general case:  $d$  may be greater or smaller than  $m$ .

### Algorithm: explicit construction of a candidate distribution to Theorem 58

The following algorithm allows the construction of a candidate distribution satisfying the assumptions of Theorem 58. Its structure is based on classical computation of the involutive closure of a distribution (see [20]):

**1<sup>st</sup> Step:** Let  $\Delta_1 = \text{span} \{g_1(x), \dots, g_m(x)\}$ . Check the  $d_1$  ( $\dim \Delta_1 = d_1$ ) linearly independent vector fields of  $\Delta_1$  (denoted  $\tau_i$ ):

$$\Delta_1 = \text{span} \{\tau_1(x), \dots, \tau_{d_1}(x)\}$$

**2<sup>nd</sup> Step:**  $\forall \tau_i \in \Delta_1, \forall \tau_j \in \Delta_1$ , compute  $[\tau_i, \tau_j]$  and test whether it belongs to  $\Delta_1$ : if not, add these vector fields to the new distribution  $\Delta_2$  under construction.

**k<sup>th</sup> Step:** Let

$$\Delta_k = \Delta_{k-1} \oplus \text{span} \{[\tau_i, \tau_j] : \tau_i \in \Delta_{k-1}, \tau_j \in \Delta_{k-1} \text{ and } [\tau_i, \tau_j] \notin \Delta_{k-1}\}$$

thus we have:  $\Delta_{k-1} \subset \Delta_k$  and  $d_{\Delta_k} > d_{\Delta_{k-1}} = \dim \Delta_{k-1}$ .

**End Step:** Stop when

$$[\tau_i, \tau_j] \in \Delta_k, \forall \tau_i \in \Delta_k, \forall \tau_j \in \Delta_k$$

This integer  $k$  will be denoted  $k_{\text{end}}$  and in the following it is referred to this finite integer obtained by using this algorithm. Then  $\Delta_{k_{\text{end}}+1} = \Delta_{k_{\text{end}}}$  and  $\Delta_k = \Delta_{k_{\text{end}}}$  for  $k > k_{\text{end}}$ . The obtained distribution is involutive ( $\Delta_G$ ): the Lie bracket does not generate other “free directions”. It is the smallest involutive distribution containing  $\Delta_1$  and it is known to be the involutive closure of  $\Delta_1$ .

**Remark 61** Note that  $\Delta_G$ , the smallest involutive distribution containing  $\Delta_1$ , can be also obtained by

$$\Delta_G = \text{span} [Ad_{g_i}^k g_j(x) : i \in \{1..m\}, j \in \{1..m\}, k \in \{0..\infty\}]$$

$$\Delta_G = \text{span} [Ad_{g_i}^k g_j(x) : i \in \{1..m\}, j \in \{1..m\}, k \in \{0..k_{\text{end}}\}]$$

Let us consider the following system:

$$\frac{dx}{dt} = u \quad (7.15)$$

$$\frac{dy}{dt} = -u + v \quad (7.16)$$

$$\frac{dz}{dt} = xu - yv \quad (7.17)$$

The distribution  $\Delta_1 = \text{span} \{g_1, g_2\}$  with  $g_1 = (1, -1, x)^\top$  and  $g_2 = (0, 1, -y)^\top$  is not involutive since  $[g_1, g_2] = (0, 0, 1)^\top$ , but  $\Delta_G = \text{span} \{g_1, g_2, [g_1, g_2]\}$  is the smallest involutive closure of  $\Delta_1$  and thus we cannot find a diffeomorphism for (7.15) such that the inputs act only on two states.

### 7.3.2 Effect of perturbations on the regular form

Let us consider a perturbed nonlinear system defined as

$$(PM) \equiv \begin{cases} \dot{x} = f(x) + G(x)u + p(x) \\ y = h(x) \end{cases} \quad (7.18)$$

where  $p(x)$  is an additive perturbation. The problem is to see how the regular form (RF) (7.11) is affected by  $p(x)$ .

**Theorem 62** Let us suppose that

$$H1) p \in \Delta_G = \text{span} \{Ad_{g_i}^k g_j(x) : i \in \{1..m\}, j \in \{1..m\}, k \in \{0..\infty\}\}$$

H2)  $\Delta_G$  is nonsingular at  $x_0$  (i.e., of constant dimension  $\dim \Delta_G = d_{\Delta_G} \leq n$ )

Then there exists a neighborhood  $\mathcal{N}(x_0)$  of  $x_0$  and a local diffeomorphism  $z = \phi(x)$  defined on  $\mathcal{N}(x_0)$ , such that (PM) (7.18) is transformed into

$$(PRF) \equiv \begin{cases} \dot{z}_1 = f_1^R(z_1, z_2) + G^R(z_1, z_2)u + p^R(z) \\ \dot{z}_2 = f_2^R(z_1, z_2) \\ z_1 \in \mathbb{R}^d, z_2 \in \mathbb{R}^{(n-d)} \end{cases} \quad (7.19)$$

with  $d = d_{\Delta_G}$ .

**Remark 63** If  $\Delta_G = \text{span}\{g_1(x), \dots, g_m(x)\}$  is involutive, then H1) of the above theorem corresponds to the classical “matching condition” (see [8]). In that case it is well known that sliding modes are insensitive to such perturbations.

### A remark about the number of components involved in the sliding surface

A relation between the existence of sliding motions and the number of components involved in the sliding surface is given here. Let

$$s = [s_1(x), \dots, s_l(x)]^\top$$

a smooth vector field such that  $\{s = 0\}$  is an  $(n - l)$ -dimensional smooth manifold: this is the case if  $s(0) = 0$  and the Jacobi matrix  $\left(\frac{\partial s}{\partial x}\right)$  is of full rank  $l < n$ .

**Theorem 64** The existence of sliding motions on the surface  $\mathcal{S} = \{s(x) = 0\}$  is equivalent to:

$$\text{rank} \left( \frac{\partial s}{\partial x} G(x) \right) \leq m \quad (7.20)$$

$$\frac{\partial s}{\partial x} f(x) \in \text{span} \left\{ \text{col} \left( \frac{\partial s}{\partial x} G(x) \right)_i, i = 1, \dots, m \right\} \quad (7.21)$$

Note that  $l + \text{rank}(G) - n \leq \text{rank} \left( \frac{\partial s}{\partial x} G(x) \right) \leq \min[l, \text{rank}(G)]$  and thus  $l \leq \text{rank}(G)$ , implies (7.20).

**Remark 65** Note that for a single input system, (7.20) and (7.21) are equivalent to the classical condition  $\frac{\partial s}{\partial x} g(x) \neq 0$ , for the existence of the equivalent control (see [30, 32, 26, 27]). Obviously, if the classical condition “ $\frac{\partial s}{\partial x} G(x)$  is of full rank  $m$ ” holds (the matrix is invertible and the equivalent control is well defined see [30, 32, 26, 27]), then (7.20) and (7.21) hold.

## 7.4 Estimation of initial sliding domain

In this section, we assume that a regular form has been performed, with  $\Delta = \text{span}\{g_1(x), \dots, g_m(x)\}$ , and  $d = d_\Delta = m$ : the vector  $s = s(z)$  defining the surface  $\mathcal{S} = \{z \in \mathbb{R}^n : s(z) = 0\}$  is an  $m$ -vector such that  $\forall z \in \mathbb{R}^n$ :

$$\text{rank}\left(\frac{\partial s(z_1, z_2)}{\partial z_1}\right) = \text{rank}(b(z)) = m \quad (7.22)$$

$$b(z) = \frac{\partial s(z_1, z_2)}{\partial z_1} G^R(z_1, z_2) \quad (7.23)$$

Thus, the system may be rewritten in the form:

$$\begin{cases} \dot{s} = a(s, z_2) + b(s, z_2)u \\ \dot{z}_2 = f_2^R(s, z_2) \\ s \in \mathbb{R}^m, z_2 \in \mathbb{R}^{(n-m)} \\ a(s, z_2) = \frac{\partial s(z_1, z_2)}{\partial z_1} f_1^R(z_1, z_2) \end{cases} \quad (7.24)$$

In the example of the introduction, we pointed out that bounded discontinuous control law can achieve sliding mode. Thus in this section, we give sufficient conditions for a domain of the state space to be an initial domain of sliding motions, that is, a domain for which any initial trajectory starting in this domain will lead to a sliding motion.

### 7.4.1 Problem formulation

#### Definitions

Utkin has pointed out a tight connection between the study of sliding motions and the stability theory (see [30, 32]).

**Definition 66** *A domain  $D_s$  of dimension  $(n-m)$  included in the manifold  $\mathcal{S} = \{z \in \mathbb{R}^n : s(z) = 0\}$ , is a **sliding domain** for system (7.24) with the control law  $u$ , if assumption  $P$  holds:*

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \text{ s.t. } \forall z_0 \in \mathcal{N}(D_s, \delta) \\ z(t) \text{ can only leave } \mathcal{N}(D_s; \varepsilon) \text{ through } \mathcal{N}(\partial D_s; \varepsilon) \end{aligned} \quad (7.25)$$

This is not the original definition that excludes the case where motions may occur on  $D_s$  for the  $2^m$  continuous system adjacent to the manifold  $s = 0$  (compare with [32] p. 45): in the following work all motions evolving on the sliding surface are taken into account and not only “sliding motions”. As pointed out in [32, 22]:  $P$  is a stability-like definition and a natural question arises which is “find the estimation of initial state  $(x_0)$  such that assumption  $P$  is true”. This is the motivation of the following notion.

**Definition 67** A domain  $D_i(D_s)$  of dimension  $n$  included in the state space is the **initial domain of sliding motions** for system (7.24) with the control law  $u$ , if:

- 1)  $\forall \varepsilon > 0$ , let  $D_i(D_s; \varepsilon)$  be a neighborhood of  $D_s$  such that the solution reach  $\mathcal{N}(D_s; \varepsilon)$  and can only leave it through  $\mathcal{N}(\partial D_s; \varepsilon)$  if and only if  $x_0 \in D_i(D_s; \varepsilon)$
- 2)  $D_i(D_s) = \bigcup_{\varepsilon > 0} D_i(D_s; \varepsilon)$

This is to say, that  $D_i(D_s)$  is the largest set of initial state  $x_0$  such that assumption  $P$  is true.

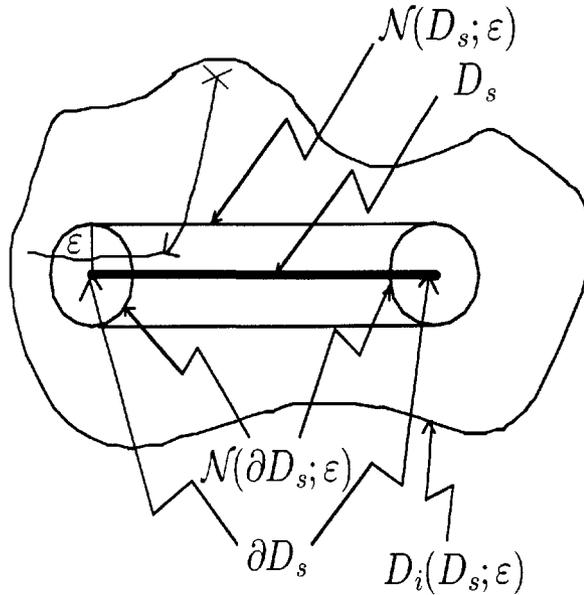


Figure 7.1: Definition 66 and 67

#### 7.4.2 Sliding domain and initial domain of sliding motion

The following result, based on the existence of two functions, gives sufficient conditions for the obtention of some estimates of  $D_s$  and  $D_i(D_s)$ .

**Theorem 68** [22] *Let us suppose that there exists two functions  $V_1(s)$  and  $V_2(z)$  satisfying conditions H1) and H2):*

H1)  $V_1(z)$  is a  $C^1(\mathbb{R}^m, \mathbb{R}_+)$  radially unbounded Lyapunov function with respect to  $s$  ( $V_1 : \mathbb{R}^m \rightarrow \mathbb{R}_+, V_1(s) = 0 \iff s = 0$ , and  $\lim_{\|s\| \rightarrow +\infty} V_1(s) = +\infty$ ),  $V_2(s)$  is a  $C^1(\mathbb{R}^n, \mathbb{R}_+)$  function with respect to  $z$  such that  $V_2 : \mathbb{R}^n \rightarrow \mathbb{R}_+$

H2) *There exists  $\alpha_1 > 0$  (finite or infinite) and  $\alpha_2 > 0$  such that:*

$$S_i(\alpha_i) = \{z \in \mathbb{R}^n : V_i(z) \leq \alpha_i\} \quad (7.26)$$

$$S_{12} = S_1(\alpha_1) \cap S_2(\alpha_2) \neq \emptyset \quad (7.27)$$

$$\overset{\circ}{S}_2(\alpha_2) \cap \{z \in \mathbb{R}^n : s(z) = 0\} \neq \emptyset \quad (7.28)$$

$$E(D_s) = S_2(\alpha_2) \cap \{z \in \mathbb{R}^n : s(z) = 0\} \quad (7.29)$$

$$\forall i \in \{1..m\}, S_i = \{z \in \mathbb{R}^n : s_i(z) = 0\}$$

$$\forall z \in \partial S_{12} - \{\cup_{i \in 1..m} \{S_i\} \cap \partial S_1\} : dV_2(z)\dot{z} < 0 \quad (7.30)$$

$$\forall z \in S_{12}(\alpha_1; \alpha_2) - \cup_{i \in 1..m} \{S_i\} : dV_1(s)\dot{s} \leq 0 \quad (7.31)$$

*Then:*

C1)  $E(D_s)$  is an estimate of the sliding domain for system (7.24)

C2)  $S_{12}(\alpha_1; \alpha_2)$  is an underestimate of  $D_i(D_s)$ : that is, for every initial condition  $z_0$  in  $S_{12}(\alpha_1; \alpha_2)$ , the solution  $z(t; t_0, z_0)$  tends to  $D_s$  and can only leave it through its boundary.

In [22] results were provided for single input systems, hypothesis on functions  $V_1$  and  $V_2$  were relaxed so that functions with discontinuous derivatives could be used.

**Theorem 69** *Assume that  $m = 1$  and*

$$\limsup_{\theta \rightarrow 0^+} \frac{|s(t+\theta)| - |s(t)|}{\theta} \underset{a.e.}{\leq} h(|s|) \quad (7.32)$$

*such that any solution of system*

$$\dot{x} = h(x), x \geq 0 \quad (7.33)$$

$$h(0) = 0 \quad (7.34)$$

starting in the set  $\{0 < x \leq \alpha_1\}$ , stays in this set and converges asymptotically to zero. Then defining

$$S_1(\alpha_1) = \{z \in \mathbb{R}^n : |s(z)| \leq \alpha_1\} \quad (7.35)$$

and under the same hypothesis as the above theorem for  $V_2$  and  $S_{12}$ ,  $S_{12}$  leads to a similar conclusion.

For multi-input systems the problem is more complex, but functions like  $|x|$  may be used under some restrictions.

### 7.4.3 Application

Let us consider two tanks in cascade (Figure 7.2). The model is

$$\begin{cases} \dot{x}_1 = \frac{1}{S}(-\beta_1\sqrt{x_1} + u_1) \\ \dot{x}_2 = \frac{1}{S}(-\beta_2\sqrt{x_2} + \beta_1\sqrt{x_1} + u_2) \end{cases} \quad (7.36)$$

where

- $S > 0$  is the section of the two tanks
- $\beta_i = s_i\sqrt{2g} > 0$ ,  $s_i$  is the section of the output of the  $i^{\text{th}}$  tank and  $g$  is the gravitational constant
- $x_1$  and  $x_2$  are the heights in the two tanks, and
- $u_1$  and  $u_2$  are the input flows in the two tanks.

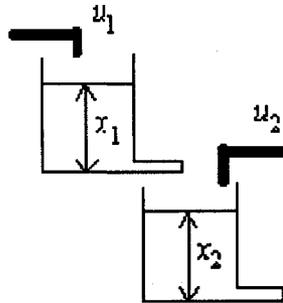


Figure 7.2: Two interconnected tanks

In the tanks, chemical reactions occur and the goal is to obtain a constant output flow (in each tank) which is equivalent to maintaining  $x_1$  and

$x_2$  to desired values, respectively,  $x_{1c} > 0$  and  $x_{2c} > 0$ . Note also that the model has sense if and only if  $x_1 \geq 0$  and  $x_2 \geq 0$ . The “degenerated sliding surface” is defined by the vector

$$s = \begin{pmatrix} s_1 = (x_1 - x_{1c}) \\ s_2 = (x_2 - x_{2c}) \end{pmatrix} \quad (7.37)$$

which has sense if and only if  $s_1 \geq -x_{1c}$  and  $s_2 \geq -x_{2c}$ . The nominal flows are:  $u_{1n} = \beta_1 \sqrt{x_{1c}}$  and  $u_{2n} = \beta_2 \sqrt{x_{2c}} - \beta_1 \sqrt{x_{1c}}$ . Thus, in order to stabilize the system with respect to bounded perturbations or parameter variations  $(\beta_1, \beta_2)$ , one can select a bounded control defined by

$$u_i = u_{in} - k_i \operatorname{sgn}(s_i), \forall i \in \{1, 2\} \quad (7.38)$$

Choosing the following function  $V_1 = \frac{S}{2}(s_1^2 + s_2^2) \geq 0$ , one obtains

$$\begin{aligned} \dot{V}_1 = & -\frac{\beta_1 s_1^2}{\sqrt{s_1 + x_{1c}} + \sqrt{x_{1c}}} - k_1 s_1 \operatorname{sgn}(s_1) + s_2 \left( \frac{\beta_1 s_1}{\sqrt{s_1 + x_{1c}} + \sqrt{x_{1c}}} \right. \\ & \left. - k_2 \operatorname{sgn}(s_2) \right) - \frac{\beta_2 s_2^2}{\sqrt{s_2 + x_{2c}} + \sqrt{x_{2c}}} \end{aligned} \quad (7.39)$$

if  $|s_1| \leq x_{1c}$ ,  $s_2 \geq -x_{2c}$ ,  $k_1 > 0$ ,  $k_2 = \frac{\beta_1 \sqrt{x_{1c}}}{2} > 0$ , then  $s_2 \left( \frac{\beta_1 s_1}{\sqrt{s_1 + x_{1c}} + \sqrt{x_{1c}}} - k_2 \operatorname{sgn}(s_2) \right) \leq 0$ , and thus

$$\dot{V}_1 < 0, \text{ for } s_1 \neq 0, s_2 \neq 0 \quad (7.40)$$

Thus using a **variant** of Theorem 68 [ $V_1 = V_2$ ,  $\alpha_1 = \alpha_2 = \frac{Sx_{1c}^2}{2}$ ] leads to  $D_s = \{s_1 = 0, s_2 = 0\}$  and is the origin (there is not really a “sliding” motion but the principle is the same) and  $D_i(D_s) = \{(s_1^2 + s_2^2) \leq x_{1c}^2\}$ . Note, that due to the signum function the system belongs to the class of variable structure systems and classical Lyapunov theory can not be applied directly. For the design of the controller one must take into account the physical limitations on the inputs:  $0 < u_i < u_{i \max} = 2u_{in}$ . Let us introduce two integrators and saturation functions before the physical inputs, then the system can be rewritten as

$$\begin{cases} \dot{x}_1 = \frac{1}{S} [-\beta_1 \sqrt{x_1} + \operatorname{sat}(\xi_1)] \\ \dot{x}_2 = \frac{1}{S} [-\beta_2 \sqrt{x_2} + \beta_1 \sqrt{x_1} + \operatorname{sat}(\xi_2)] \\ \dot{\xi}_1 = v_1 \\ \dot{\xi}_2 = v_2 \end{cases} \quad (7.41)$$

where  $\operatorname{sat}$  is a saturation function defined as

$$\operatorname{sat}(\xi_i) = \begin{cases} u_{i \max} & \text{if } \xi_i \geq u_{i \max} \\ \xi_i & \\ 0 & \text{if } \xi_i \leq 0 \end{cases} \quad (7.42)$$

where  $u_{i\max}$  are the maximum admissible inputs of the  $i^{\text{th}}$  tank. The sliding surface is then defined by the vector

$$s = \begin{pmatrix} -\beta_1\sqrt{x_1} + \text{sat}(\xi_1) + S(x_1 - x_{1c}) \\ -\beta_2\sqrt{x_2} + \beta_1\sqrt{x_1} + \text{sat}(\xi_2) + S(x_2 - x_{2c}) \end{pmatrix} \quad (7.43)$$

Let us assume that  $0 \leq \xi_i \leq u_{i\max}$ . Then the equivalent control leads to:

$$v_{eq1} = \frac{1}{S} \left( S - \frac{\beta_1}{2\sqrt{x_1}} \right) [\beta_1\sqrt{x_1} - \text{sat}(\xi_1)], \quad (7.44)$$

$$v_{eq2} = \left( 1 - \frac{\beta_2}{2S\sqrt{x_2}} \right) [\beta_2\sqrt{x_2} - \beta_1\sqrt{x_1} - \text{sat}(\xi_2)] \quad (7.45)$$

$$+ \frac{\beta_1}{2S\sqrt{x_1}} [\beta_1\sqrt{x_1} - \text{sat}(\xi_1)] \quad (7.46)$$

Thus, in order to stabilize the system with respect to bounded perturbations or parameter variations  $(\beta_1, \beta_2)$ , we select a bounded control defined by

$$v_i = v_{eqi} - k_i(\cdot) \text{sgn}(s_i), \forall i \in \{1, 2\} \quad (7.47)$$

where  $k_i(\cdot)$  are positive functions. If we choose  $V_1 = \frac{1}{2}(s_1^2 + s_2^2) \geq 0$ , and the constant " $\alpha_1 = (\min_i u_{i\max})^2$ " of Theorem 68, obviously  $\xi_i \leq u_{i\max}$  and  $\dot{V}_1 \leq 0$  (equality only on  $\mathcal{S}$ ). Choosing  $V_2 = V_1 + \frac{S}{2}((x_1 - x_{1c})^2 + (x_2 - x_{2c})^2) \geq 0$ , and the constant " $\alpha_2 = (\min_i u_{i\max})^2$ " of Theorem 68, obviously  $\xi_i \leq u_{i\max}$  and  $\dot{V}_2 = \dot{V}_1 + S((x_1 - x_{1c})\dot{x}_1 + (x_2 - x_{2c})\dot{x}_2)$  using (7.41) and (7.43) we obtain

$$S\dot{x}_i = s_i - S(x_i - x_{ic}), i = 1, 2 \quad (7.48)$$

$$\begin{aligned} \dot{V}_2 &= -k_1(\cdot) |s_1| - k_2(\cdot) |s_2| \\ &\quad - S((x_1 - x_{1c})^2 + (x_2 - x_{2c})^2) \\ &\quad - (x_1 - x_{1c})s_1 - (x_2 - x_{2c})s_2 \end{aligned} \quad (7.49)$$

Selecting  $k_i(\cdot) = 1 + |x_i - x_{ic}|$ , one obtains  $\dot{V}_2 \leq 0$ . Thus applying Theorem 68 we conclude that  $D_s = \{s = 0\} \cap \{\frac{1}{2}(s_1^2 + s_2^2) \leq (\min_i u_{i\max})^2\}$  and  $D_i(D_s) = \{\frac{1}{2}(s_1^2 + s_2^2) + \frac{S}{2}((x_1 - x_{1c})^2 + (x_2 - x_{2c})^2) \leq (\min_i u_{i\max})^2\}$ .

The following figures show the simulation with the following numerical values  $S = 1(\text{m}^2)$ ,  $\beta_1 = 0.8(\text{m}^{5/2}\text{s}^{-1})$ ,  $\beta_2 = 1(\text{m}^{5/2}\text{s}^{-1})$ ,  $x_{1c} = 0.6(\text{m})$ ,  $x_{2c} = 0.6(\text{m})$ ,  $u_{1\max} = 1.24(\text{m}^3\text{s}^{-1})$ , and  $u_{2\max} = 1.54(\text{m}^3\text{s}^{-1})$ , starting from the initial conditions  $x_{10} = 1(\text{m})$ ,  $x_{20} = 0.3(\text{m})$ , and  $\xi_{10} = \xi_{20} = 0$  (no input at time zero).

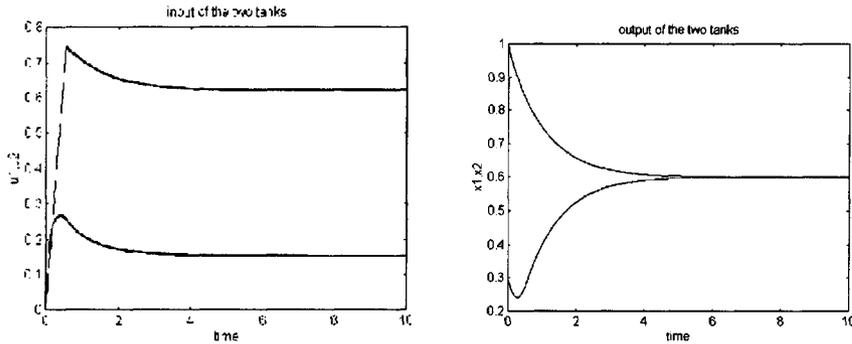


Figure 7.3: Stabilization of two interconnected tanks

## 7.5 Stabilization

To stabilize  $(M)$  (7.6), it is clear from  $(RF)$  (7.11) that one can design a sliding mode control in the following way:

- 1) let  $s = z_1 - a(z_2) \in \mathbb{R}^d$ , where the  $d$ -vector valued function  $a$  is not a priori known
- 2) design a sliding mode control such that a sliding regime occurs on the manifold  $s = 0$  of dimension  $(n - d)$ , and
- 3) because in sliding regime:  $\dot{z}_2 = f_2^R(a(z_2), z_2)$ , use the function  $a$  to stabilize the origin.

This idea was first used in [21] under the assumption of first  $d = m$ , and secondly  $G$  being of rank  $m$  and under an integrability condition (which is always fulfilled when  $m = 1$ ).

In the following, we will improve the stabilization when  $d = m$  (subsection 7.5.1) and give a nonlinear discontinuous control that reduces the chattering phenomenon. And lastly, in subsection 7.5.2, we will see an example of how stabilization is possible in the more general case  $d > m$ : for this integrators may be used.

### 7.5.1 Stabilization in the case $d = m$

This subsection considers the classical case  $d = m$  (see [21]). In the classical design of the controller, we propose a new design, that allows use more

general sliding surface: in [21], all trajectories (this means for all initial conditions) belonging to the surface must converge asymptotically. In our case, only local asymptotic stability is required. The sliding gain is calculated in such a way that the motions reach the sliding surface in its stable part.

**Theorem 70** *Let us suppose that*

*H1)  $k_{end}$  is finite (or equivalently  $\Delta_G$  exists: a regular form exists)*

*H2)  $m = \dim \Delta_G$*

*H3) the sliding surface is defined as:*

$$s = z_1 - a(z_2) \quad (7.50)$$

*with  $a(z_2) \in C^1(\mathbb{R}^{(n-m)}; \mathbb{R}^m)$ ,  $a(0_2) = 0_1 \in \mathbb{R}^m$*

*H4) the origin  $0_2 \in \mathbb{R}^{(n-m)}$  of system*

$$\dot{z}_2 = f_2^R[a(z_2), z_2] \quad (7.51)$$

*is locally asymptotically stable, and*

*H5)  $f_2^R$  is at least  $C^1(\mathbb{R}^n; \mathbb{R}^{(n-m)})$*

*Then:*

*C1) there exists a gain  $k(\cdot)$  providing a local asymptotic stabilization of the origin with respect to (M) (7.6), by means of the control*

$$u = (G_r^R(z))^{-1} (-f_1^R(z) + v) \quad (7.52)$$

$$v = -k(\cdot) \text{SGN}(s) + \frac{\partial a(z_2)}{\partial z_2} f_2^R(z) \quad (7.53)$$

*C2) if  $0_2 \in \mathbb{R}^{(n-m)}$  of (7.51), is globally asymptotically stable, then the origin of (M) (7.6) is globally asymptotically stable under the control  $u$  (7.52) and (7.53) defined with any nonzero constant gain  $k$ .*

*C3) Moreover, if we consider system (7.18) with a perturbation term  $p(x)$  satisfying hypothesis H1) of Theorem 62 and  $\|p^R(x)\| \leq \pi_{p,R}$ , for any norm on  $\mathbb{R}^m$ , then the above conclusions are still valid.*

**Proof :** see [25].

**Theorem 71** *If hypotheses H1) through H3) of Theorem 70 are fulfilled and H4) is replaced by H4') there exists a Lyapunov function  $V_2(z_2)$  and a constant  $\rho_2$  such that the set*

$$S_2(\rho_2) = \left\{ z_2 \in \mathbb{R}^{(n-m)} : V_2(z_2) \leq \rho_2 \right\}$$

*is an estimate of the domain of asymptotic stability of the origin  $0_2 \in \mathbb{R}^{(n-m)}$  of (7.51), then, control  $u$  is defined by (7.52) and (7.53) with the following gain*

$$k(\cdot) = k'(z) + \frac{L\rho_2}{\alpha} \left\| \frac{\partial V_2}{\partial z_2} \right\|_1, k'(z) > 0 \quad (7.54)$$

*and achieves asymptotic stability of the origin for system (M) (7.6) with the following estimate of the domain of asymptotic stability*

$$S(\rho_2) = \left\{ z \in \mathbb{R}^n : \frac{\alpha}{2} s^\top s + V_2(z_2) \leq \rho_2 \right\} \quad (7.55)$$

*with  $0 < \alpha$ . Moreover, if we consider system (7.18) with a perturbation term  $p(x)$  satisfying hypothesis H1) of Theorem 62 and  $\|p^R(x)\| \leq \pi_{p^R}$ , for any norm on  $\mathbb{R}^m$ , then the above conclusions are still valid provided that in (7.54):  $k'(z) > \sup_z \left( \frac{\|z\|_1}{\|z\|} \right) \pi_{p^R}$ .*

**Remark 72** *If, in addition,  $\lim_{z_2 \rightarrow 0} \left\| \frac{\partial V_2}{\partial z_2} \right\| = \lim_{z \rightarrow 0} k'(z) = 0$ , then “chattering” tends to zero as the motion approaches the origin. This condition is not very restrictive because, most of the time, Lyapunov functions are locally at least quadratic. And  $k'(\cdot)$  can be set to  $\frac{L\rho_2}{\alpha} \left\| \frac{\partial V_2}{\partial z_2} \right\|_1$  for example. Note that  $k'(\cdot)$  can also be set to any other sigmoid function zeroing at the origin: in [?, 29], the authors replaced the signum function by a saturation function in order to smooth the discontinuity. The use of sigmoid functions achieved the same result. Let us give a few examples ( $\phi$  permit to select the bandwidth):  $\frac{2}{\pi} \arctan\left(\frac{x}{\phi}\right)$ ,  $\frac{\left(\frac{x}{\phi}\right)^{2n} \text{sign}(x)}{\left(\frac{x}{\phi}\right)^{2n} + \alpha}$ ,  $\tanh\left(\frac{x}{\phi}\right)$ , etc... These sigmoid functions can be composed to any suitable function.*

Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1 \\ \dot{x}_2 &= x_1 + u_1 + u_2 \\ \dot{x}_3 &= x_1 x_3 + u_1 \end{aligned} \quad (7.56)$$

Distribution  $\Delta = \text{span}\{g_1, g_2\}$ ,  $g_1 = (1, 1, 1)^\top$ ,  $g_2 = (0, 1, 0)^\top$ , is a two-dimensional involutive distribution. In order to find  $\phi$ , one must find a basis of  $\Delta$ :

$$\begin{cases} d\lambda g_1 = 0 \\ d\lambda g_2 = 0 \end{cases} \iff \begin{cases} \frac{\partial \lambda}{\partial x_1} + \frac{\partial \lambda}{\partial x_2} + \frac{\partial \lambda}{\partial x_3} = 0 \\ \frac{\partial \lambda}{\partial x_2} = 0 \end{cases} \quad (7.57)$$

It has solution  $\lambda(x) = x_1 - x_3$ , thus  $z = \phi(x) = (x_1, x_2, x_1 - x_3)^\top$ , leads to (using Theorem 58):

$$\begin{aligned} \dot{z}_1 &= z_2 + u_1 \\ \dot{z}_2 &= z_1 + u_1 + u_2 \\ \dot{z}_3 &= z_2 - z_1(z_1 - z_3) \end{aligned} \quad (7.58)$$

Here  $\text{rank}(G) = 2$ , so according to Theorem 70, let  $u$  be defined as

$$\begin{aligned} u &= (G^R(z))^{-1}(-f_1^R(z) + v) \\ &= (-z_2 + v_1, z_2 - z_1 - v_1 + v_2)^\top \\ &= (-x_2 + v_1, x_2 - x_1 - v_1 + v_2)^\top \end{aligned} \quad (7.59)$$

and let  $s$  be defined as

$$s = (z_1 - z_3, z_2 + z_3)^\top = (x_3, x_1 + x_2 - x_3)^\top \quad (7.60)$$

Let  $v_1 = -\text{Sgn}(x_3) + x_2 - x_1 x_3$  and  $v_2 = -\text{Sgn}(x_1 + x_2 - x_3) - x_2 + x_1 x_3$ . Then, in sliding regime:  $\dot{z}_3 = -z_3$ , thus global asymptotic stability of the origin of (7.56) is achieved.

## 7.5.2 Stabilization in the case $d > m$

The idea is to use an augmented form in order to come back to the first case. This is explained in the following example.

Let us consider a monocycle (see Figure 7.4) with two controls:

- 1) the pedaling rolling action ( $u_1$ ), and
- 2) and the rotating action ( $\frac{d\theta}{dt} = u_2$ ).

The model is

$$\begin{aligned} \frac{dx}{dt} &= \sin(\theta)u_1 \\ \frac{dy}{dt} &= \cos(\theta)u_1 \\ \frac{d\theta}{dt} &= u_2 \end{aligned} \quad (7.61)$$

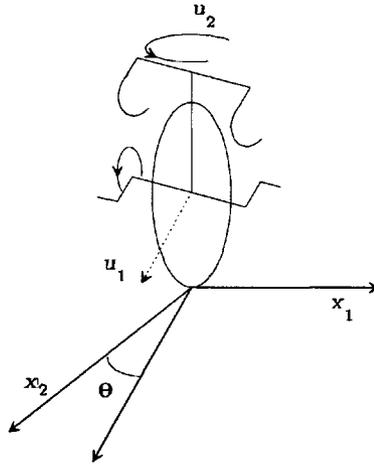


Figure 7.4: Monocycle

Note that (7.61) also models the dynamics of a two-wheel cart (see [18]). Let  $g_1 = (\sin(\theta), \cos(\theta), 0)^\top$ ,  $g_2 = (0, 0, 1)^\top$ . Then according to the algorithm proposed in Section 7.3, defining

$$\Delta_1 = \text{span} \left\{ g_1(x) \triangleq \tau_1(x), g_2(x) \triangleq \tau_2(x) \right\} \quad (7.62)$$

(a two-dimensional distribution), we have:

$$\tau_3(x) = [\tau_1, \tau_2] = (-\cos(\theta), \sin(\theta), 0)^\top \notin \Delta_1 \quad (7.63)$$

Let  $\Delta_2 = \Delta_1 \oplus \text{span} \{ \tau_3(x) \}$ , this is a three-dimensional involutive distribution (compute the Lie brackets  $[\tau_1, \tau_3]$  and  $[\tau_2, \tau_3]$ ). Note that  $\dim(\Delta_2) = 3$  implies that (7.61) is locally accessible. Thus the fact that distribution  $\Delta_G = \Delta_2$  implies that the only diffeomorphism transforming  $(M)$  (7.6) into  $(RF)$  (7.11) is the identity: this is  $d = n > m$ . If we consider an augmented state:

$$\begin{aligned} \frac{dx}{dt} &= \sin(\theta)\xi \\ \frac{dy}{dt} &= \cos(\theta)\xi \\ \frac{d\theta}{dt} &= u_2 = v_2 \\ \frac{d\xi}{dt} &= v_1, u_1 = \xi \end{aligned} \quad (7.64)$$

Then, according to the previous results, let  $s$  be defined as

$$s = \begin{pmatrix} \xi & \theta \end{pmatrix}^T - p(x, y) \quad (7.65)$$

The sliding mode controller is designed as follows

$$\frac{ds}{dt} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} \frac{\partial p_1}{\partial x} & \frac{\partial p_1}{\partial y} \\ \frac{\partial p_2}{\partial x} & \frac{\partial p_2}{\partial y} \end{pmatrix} \begin{pmatrix} \sin(\theta)\xi \\ \cos(\theta)\xi \end{pmatrix} = -\text{SGN}(s) \quad (7.66)$$

In the sliding regime

$$\begin{aligned} \frac{dx}{dt} &= \sin(p_2(x, y))p_1(x, y) \\ \frac{dy}{dt} &= \cos(p_2(x, y))p_1(x, y) \end{aligned} \quad (7.67)$$

which can be set respectively to the values  $-\alpha x$  and  $-\beta y$ , for any  $\alpha > 0$  and  $\beta > 0$ . Let  $p_1(x, y) = -\psi(y)\sqrt{(\alpha x)^2 + (\beta y)^2}$ , [ $\psi(0) = 1, \psi(y) = \text{Sgn}(y)$ ], and  $p_2(x, y) = \arctan\left(\frac{\alpha x}{\beta y}\right)$  be solutions of the following system:

$$\begin{aligned} \sin(p_2(x, y))p_1(x, y) &= -\alpha x \\ \cos(p_2(x, y))p_1(x, y) &= -\beta y \end{aligned}$$

Note that the function  $\psi$  is fundamental for the exact dynamics assignment because of the sign “-” in (7.66). Moreover in order to stabilize the origin of (7.61) to zero we need  $\theta$  to tend to zero which is achieved if

$$\lim_{t \rightarrow +\infty} \left( \arctan\left(\frac{\alpha x(t)}{\beta y(t)}\right) \right) = 0 \quad (7.68)$$

As  $x$  and  $y$  tend to zero asymptotically in sliding regime, one must select the rate of convergence ( $\alpha$ ) such that  $x$  tends faster to zero than  $y$ . And, as in the sliding regime,  $x(t) = O[\exp(-\alpha t)]$  and  $y(t) = O[\exp(-\beta t)]$ , this leads to the sufficient condition

$$(-\alpha + \beta) < 0 \quad (7.69)$$

So let us select:  $\alpha = 2$  and  $\beta = 1$ . Thus, global asymptotic stability of the origin of (7.61) is achieved using the control laws defined by:

$$\begin{aligned} u_1 &= \xi \\ \frac{d\xi}{dt} &= v_1 = -\text{Sgn}(s_1) - \psi(y)(2 \sin(\theta)x + \cos(\theta)y) \frac{\xi}{\sqrt{4x^2 + y^2}} \\ u_2 &= v_2 = -\text{Sgn}(s_2) + (\sin(\theta)y - \cos(\theta)x) \frac{2\xi}{4x^2 + y^2} \\ s &= \begin{pmatrix} \xi \\ \theta \end{pmatrix} - \begin{pmatrix} -\psi(y)\sqrt{4x^2 + y^2} \\ \arctan\left(\frac{2x}{y}\right) \end{pmatrix} \end{aligned} \quad (7.70)$$

See Gulden and Utkin [18] for an other approach, for which they imposed the cart (or monocycle) to approach the origin according to a “Lyapunov navigation function” (the tracked path is derived from this Lyapunov function). Note that here, using a different control, we also achieved the same result for the following “partial” Lyapunov function  $V(x, y) = \frac{1}{2}((\alpha x)^2 + (\alpha y)^2)$ . The following figure (Figure 7.4) illustrates the stabilization of the origin under the controls (7.70). The simulations were done using a 4/5 Runge–Kutta algorithm with the following initial conditions:  $x(0) = 1.2, y(0) = 2, \theta(0) = 0.2$  (rd), and  $\xi(0) = 0$  (no control at time zero). One can note that the first control  $u_1$  is rather smooth (no chat-

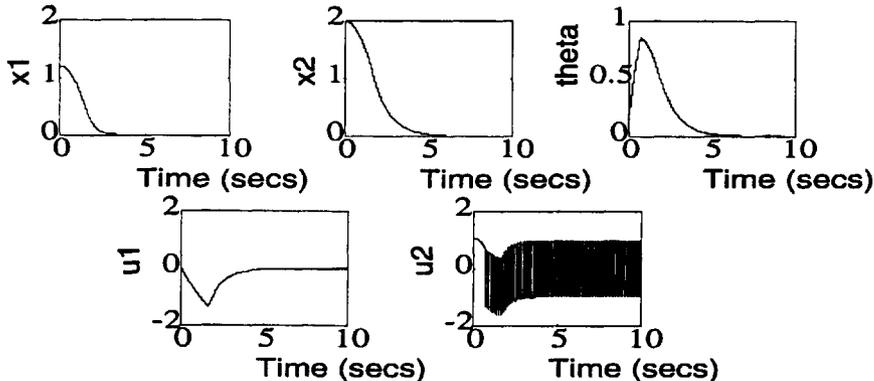


Figure 7.5: Stabilization using (7.70) with  $k = 1$

tering). This is due to the presence of an integrator before the physical actuator. For the second control  $u_2$ , there is some chattering which can be smoothed using different technics (see for example or the above mentioned sigmoïd functions). But, we can use a tracking gain as proposed in Theorem 71. Using the Lyapunov function  $V_2(z_2) = x^2 + \frac{y^2}{2}$  leads to choose the following gain:

$$k(\cdot) = 0.2\sqrt{4x^2 + y^2}. \quad (7.71)$$

This gain replace the gain “1” of the signum function “Sgn( $s_2$ )” in (7.70). thus we obtain the following simulations (Figure 7.5).

## 7.6 Conclusion

In this chapter, the problem of the estimation of the initial domain of sliding motions was investigated using the Lyapunov approach. The given

sufficient conditions should be improved by allowing the functions to be less regular (smooth). The last part concerned the stabilization using the regular form and nonlinear gain in the signum function, reducing the chattering phenomenon. One can enlarge this approach to design a higher-order sliding mode control as developed in Chapter 3.

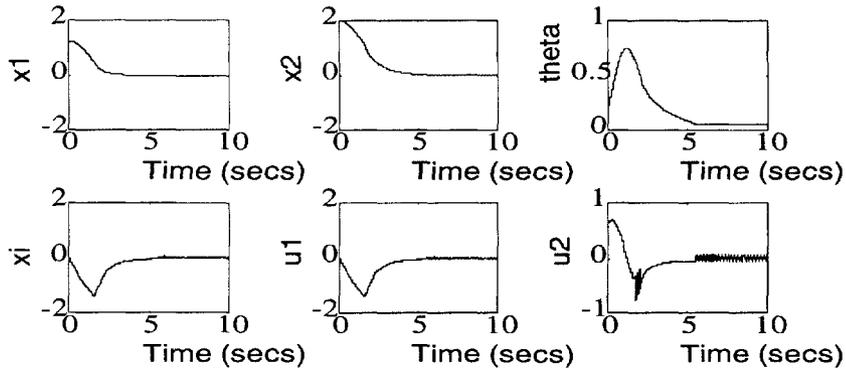


Figure 7.6: Stabilization using (7.70) with  $0.2\sqrt{4x^2 + y^2} \text{Sgn}(s_2)$

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# Chapter 8

## Discretization Issues

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### 8.1 Introduction

The classical sliding mode technique based on Fillipov's mathematical theory has as its objective to force the system to evolve on a "sliding surface", which represents a desired dynamics. This sliding surface is reached after a finite time using the fact that in the neighborhood of the sliding surface, the control gain is ideally infinite (when a true sign function is used) and has an infinite frequency value too.

For several reasons, such as the chattering phenomenon, different approaches were proposed (Emelyanov and Korovin 1981 [10]). If some dynamics is implanted, instead of an ordinary sign function (relay), a new sliding modes appear. Such modes are called the higher sliding modes order (Levantovsky 1993 [15], Fridman and Levant 1996 [12]). For these sliding modes, the system slides on the dynamics  $\sigma = 0$  ( $\sigma(x)$  is the sliding surface), but it almost verifies that the successive derivatives of  $\sigma$  vanish in a finite time. These enhancements, in comparison to the relay technique, will help us to define a discrete sliding mode approach based on the higher sliding mode techniques.

In fact, the practical design of the sliding mode controller is often done using computers and microcontrollers. After acquiring the output measures, it computes the control law value, which is maintained constant during the sampling time. This technique introduces a discrete element in the sliding control law and sets the well known chattering problem on the

state variables. In addition to the discretization of the control input, the chattering phenomenon is also due to the actuator's physical limitations.

In this chapter, it is important to recall some practical results about the discrete time sliding mode [1, 7, 13, 14, 19, 21, 23], in order to put in a prominent position some theoretical difficulties due to the discretization of the differential inclusions [6]. We lay great stress upon the exponential of Lie derivative [17, 2] as upon its first approximation, the scheme of Euler. Later, this will give us a control law of classical discrete-time sliding mode (under sampling), and this underlying the differences with the continuous case. At last, we will conclude our remarks showing what brings the high order sliding mode in the discretization problems [15].

## 8.2 Mathematical recalls

Let us consider the following system :

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x)\end{aligned}\tag{8.1}$$

where  $x \in \mathbb{R}^n$  denotes the state vector,  $u \in \mathbb{R}^p$  is the control input (for beginning the integration's period  $\delta$  is supposed constant) and  $f$  and  $h$  are analytic functions.

Then, the solution of system (8.1) at time  $t + \delta$  in function of initial conditions at time  $t$ , is written positively as follows :

$$\begin{aligned}x(t + \delta) &= e^{\delta L_{f(\cdot, u(t))}} I_d|_{x(t)} \\ y(t + \delta) &= e^{\delta L_{f(\cdot, u(t))}} h|_{x(t)}\end{aligned}\tag{8.2}$$

Where  $L_f := \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$  is the usual Lie derivative,  $|_{x(t)}$  signifies that all the function is evaluated at  $x(t)$ ,  $I_d$  is the identity function and  $e^{L_{f(\cdot, u(t))}} := \sum_{j=0}^{\infty} \frac{\delta^j}{j!} L_f^j(\cdot, u(t))$  with  $\frac{\delta^0}{0!} L_f^0(\cdot, u(t)) = 0$  is the usual Lie exponential.

The proof of this result is immediate, by deriving (8.2) and recalling that for  $\delta = 0$ , one obtains  $e^{\delta L_{f(\cdot, u(t))}} = I_d$ .

From the literal solution (8.2), it can be noted immediately that two cases of figures could then arise, one  $e^{\delta L_{f(\cdot, u(t))}} I_d$  admits a finite development<sup>1</sup> (i.e.  $\exists k$  such that  $\forall j \geq k \ L_f^j(\cdot, u) I_d = 0$ ) or the other case is to

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<sup>1</sup>We say then that the system is finitely discretisable

compute all the exponential terms. The latter contingency is not possible in practice and that is why we have recourse to some approximations. The most classical among them is the approximation in the first order in  $\delta$  called Euler's discretization scheme:

$$\begin{aligned}x(t_0 + \delta) &= x(t_0) + \delta f(x(t_0), u(t_0)) + O(\delta^2) \\y(t_0 + \delta) &= y(t_0) + \delta L_{f(\cdot, u(t_0))} h|_{x(t_0)} + O(\delta^2)\end{aligned}\tag{8.3}$$

where  $O(\delta^2)$  signifies that all the neglected terms are at least of second order in  $\delta$ .

**Remark 73**  $y(t_0 + \delta)$  is often computed according to  $x(t_0 + \delta)$ . Nevertheless, as  $h$  is an analytic function,  $x(t_0 + \delta)$  approximated at second order in  $\delta$  implies that  $y(t_0 + \delta)$  is also approximated at the same order.

Now, what about discretization if the input  $u$  is a sliding mode control? Obviously, this leads to a system characterized by a differential inclusion:

$$\begin{aligned}\dot{x} &= f(x, u) \\y &= h(x)\end{aligned}\tag{8.4}$$

where the function  $f(x, u)$  is discontinuous. It is then impossible to use the Lie exponential derivative<sup>2</sup>, but can its  $k$ -order Euler's sampling algorithm approaches an exact solution of the differential inclusion system when  $\delta$  tends to zero?

Let us consider the interval of time  $[t_0, t_0 + a]$  with  $a > 0$  and choose, to simplify<sup>3</sup>, a constant sampling time  $\delta$  such that  $a$  is a multiple of  $\delta$ .

We have then a temporal regular curve  $\{t_0, t_0 + \delta, \dots, t_0 + m\delta, \dots, t_0 + a\}$  and a solution curve  $\{x(t_0), x(t_0 + \delta), \dots, x(t_0 + m\delta), \dots, x(t_0 + a)\}$ , where the solutions are calculated as follows:

$$x(t_0 + m\delta) = x(t_0 + (m - 1)\delta) + \delta f(t_0 + (m - 1)\delta, u(x(t_0 + (m - 1)\delta)))$$

and where  $x(t + (m - 1)\delta)$  is the precedent sampling time solution.

The question is then to know if, for  $\delta \rightarrow 0$ , the Euler's solution approaches, at least at sampling times, an exact solution of differential inclusion. For this, let us take an example.

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<sup>2</sup>We will see afterwards that a development at second order in 2 could be revealed very useful.

<sup>3</sup>without loss of generality [6].

**Example 74** Consider the following scalar system

$$\dot{x} = -x + u$$

where the input  $u$  is defined by

$$u(t) = \begin{cases} 0 & \text{if } x(t) = e^{-t} \\ 1 & \text{if } x(t) > e^{-t} \\ -1 & \text{if } x(t) < e^{-t} \end{cases}$$

with, for initial conditions  $t_0 = 0$ ,  $x(0) = 1$  and  $a = 1$ . The exact solution is then  $x(t) = e^{-t}$  while the Euler's solution tends to  $x(t) = 2e^{-t} - 1$ .

The difficulty of this example comes from the fact that the exact solution is not attractive. Consequently, for all other initial conditions, the exact solution moves away from  $x(t) = e^{-t}$ .

**Example 75** Let us take again the previous system:

$$\dot{x} = -x + u$$

where the input  $u$  is defined by

$$u(t) = \begin{cases} 0 & \text{if } x(t) = e^{-t} \\ -1 & \text{if } x(t) > e^{-t} \\ +1 & \text{if } x(t) < e^{-t} \end{cases}$$

with, for initial conditions,  $t_0 = 0$ ,  $x(0) = 1$  and  $a = 1$ . The exact solution is  $x(t) = e^{-t}$  and the Euler's solution tends also to  $x(t) = e^{-t}$ .

In [6], other examples are given, in particular the example for which the Euler's solution with a temporal regular curve differs from the solutions obtained with the irregular temporal curves<sup>4</sup>.

The first question that can be set about the solutions obtained thank to the Euler's scheme is that of their existence and of their properties, without for the moment looking at if these solutions are close to the exact one.

**Theorem 76** Consider the system (8.4) and let us suppose that, for all  $u$ , the function  $f(x, u)$  verifies the following linear growth condition

$$\|f(x(t), u(t))\| \leq K\|x\| + C \quad \forall (t, x) \in [t_0, t_0 + \delta] \times \mathbb{R}^n \quad (8.5)$$

Then, there exists at least an Euler's solution<sup>5</sup> on the interval of time  $[t_0, t_0 + \delta]$  and all the Euler's solutions are Lipschitz and satisfy :

$$\|x(t) - x(t_0)\| \leq (t - t_0)e^{K(t-t_0)}(K\|x(t_0)\| + C) \quad \forall t \in [t_0, t_0 + \delta] \quad (8.6)$$

<sup>4</sup>We send back readers to this reference for more information.

<sup>5</sup>It can be not unique and can depend from the chosen temporal curve,

The proof of this theorem is done using simple inequality relations. Now, let us consider the system (8.1) (with an analytic input)

**Proposition 77** *Let us take the system (8.1) with  $f$  verifying the condition (8.5). Let  $x(\cdot)$  be an Euler's solution on the time interval  $[t_0, t_0 + \delta]$ ,  $T$  be an open set<sup>6</sup> containing  $x(t)$  for all  $t \in [t_0, t_0 + \delta]$ , then the trajectory  $x(\cdot)$  is attractive on  $T$ , if  $\forall z \in T$  we have :*

$$\langle f(z, u(z)), (z - p) \rangle \leq 0 \quad (8.7)$$

where  $p$  is the  $z$  projection on the trajectory  $x(\cdot)$ .

The proof of this theorem is done by considering at each time the "worse case" of distances in comparison with the trajectory  $x(\cdot)$ .

**Remark 78** *This condition of attractivity of the solution means that all initial and computation errors tend to vanish or at least are not amplified.*

To conclude this paragraph the condition of attractivity for the case of system (8.4) is given

**Proposition 79** *Consider the system (8.4) with  $f$  verifying the condition (8.5), let  $x(\cdot)$  be an Euler's solution on the time interval  $[t_0, t_0 + \delta]$ , let  $T$  be an open set containing  $x(t)$  for all  $t \in [t_0, t_0 + \delta]$ , Then the trajectory  $x(\cdot)$  is attractive on  $T$ , if  $\forall z \in T$  we have :*

$$\langle f(z, u(z)), (z - p) \rangle \leq 0 \quad (8.8)$$

where  $p$  is the  $z$  projection on the  $x(\cdot)$  trajectory. Moreover  $x$  is a Filippov's solution for the system (8.4).

The proof is identical to the previous one, except that  $x(\cdot)$  must be verify Filippov's theorem conditions<sup>7</sup>.

### 8.3 Classical sliding modes in discrete time

First of all, it is important to compare the continuous time case to the discrete one (or more exactly under sampling). For this, let us take a simple example:

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<sup>6</sup>generally a tube. Furthermore,  $T$  is an opened sets, such that the projection for each point would be unique, this imply that there would not be singular point on the trajectory.

<sup>7</sup>Or in an equivalent way, due to the form of our system, the "standing" hypothesis defined in [6] is only required.

**Example 80** (*Continuous time case*)

Consider the system

$$\dot{x} = x + u + di \quad (8.9)$$

where  $x \in [-C, C]$  is the state,  $u$  the input,  $di$  an unknown but bounded external perturbation such that  $\forall t \geq 0 \ |di(t)| < D$ . Stabilization of  $x$  at the origin is ensured by the following sliding mode control law :

$$u(t) = -\lambda \text{sign}(x)$$

with  $\lambda > C + D$ .

The perturbation rejection and the stabilization are guaranteed by the use of a high gain and without other knowledge of the bounds on the state and the perturbation.

**Example 81** (*Under sampling system case*)

Discretizing the system (8.9) leads to

$$x(k+1) = e^\delta x(k) + (e^\delta - 1)u(k) + \int_0^\delta e^{\delta-t} di(t+k\delta) dt \quad (8.10)$$

It immediately appears that the stabilization and the perturbation rejection do not require a very high amplitude control. This way of proceeding corresponds to sample and hold the continuous control, leading to chattering phenomena<sup>8</sup>. It is then possible to write

$$u(k) = (1 - e^\delta)^{-1} (e^\delta x(k) + \int_0^\delta e^{\delta-t} di(t+k\delta) dt)$$

to obtain a dead beat response (*i.e.* reach our objective in one step).

Unfortunately, it is known that this type of control is not robust under parametric uncertainties and moreover requires the knowledge of the perturbation. This kind of control is not therefore the discrete version of sliding mode. So what is the discrete “version” of sliding mode ?

Classically, in the literature, the proposed methods consist into using the properties of the system under sampling and to define new invariant manifolds.

Here is just given the simple case of a linear system with limited perturbation but without parametric uncertainties [13, 23] :

$$\dot{x} = Ax + Bu + Ddi \quad (8.11)$$

---

<sup>8</sup>A solution to eliminate the chattering will be given in next paragraph.

with  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $d \in \mathbb{R}^p$  and  $\|di(t)\| < C$  for all  $t \geq 0$  and  $A$ ,  $B$  and  $D$  are matrix of appropriate dimensions. Since Drazenovic works [8]<sup>9</sup>, it is known that the condition called “matching condition” (i.e. simply the condition for which the input  $u$  can instantly remove the perturbation) is :

$$\text{Rank}[B, D] = \text{Rank}[B] \quad (8.12)$$

But is this condition preserved after the system discretization ?

One can immediately see that, it is verified in the case of a one dimension of system (8.9). Now, integrating (8.11) on a constant sampling period equal to  $\delta$ , one obtains :

$$x(k+1) = A_d x(k) + B_d u(k) + \int_{t=0}^{\delta} e^{A(\delta-t)} D di(t+k\delta) dt \quad (8.13)$$

with  $A_d = e^{A\delta}$  and  $B_d = \int_{t=0}^{\delta} e^{A t} B dt$ . It follows that  $\int_{t=0}^{\delta} e^{A(\delta-t)} D di(t+k\delta) dt \in \text{span}\{B_d\}$  depends on the temporal behavior of the perturbation  $di$ <sup>10</sup>. Now, restraining the scheme of discretization to a simple Euler’s scheme and that making the hypothesis that  $di$  moves slightly during the sampling period leads to :

$$x(k+1) = A_e x(k) + B_e u(k) + D_e di(k) dt + O(\delta^2) \quad (8.14)$$

with  $A_e = I_d + \delta A$ ,  $B_e = \delta B$  and  $D_e = \delta D$ . Thus, if the condition (8.12) holds one has :

$$\text{Rank}[B_e, D_e] = \text{Rank}[B_e] \quad (8.15)$$

The Euler’s scheme is employed for the previously mentioned reason and also if we considered some coupling between the perturbation and the parametric uncertainties. However the hypothesis must be kept in mind :

- $d_i$  is a slow variable,
- $\delta$  is small compared to the time constants of the system.
- $u$  is bounded and small in front of  $\delta^{-1}$  (in order not to break the homogeneity of the limited development).

If all these hypothesis are verified, the sliding mode control can be defined on the basis of (8.14). Let us firstly define an attractive surface at the sampling times (note that the notion of sliding is here a little bit erroneous, since the trajectories between two sampling times are not constrained on

<sup>9</sup>see also [18] for recent extensions on the subject.

<sup>10</sup>Note that a constant perturbation on the sampling period preserves the “matching condition”. This hypothesis of constant perturbation during sampling period is always implicitly done in the classical sliding mode control.

the manifold. This aspect of the dynamic behavior between two sampling times is linked to the zero dynamics of sampled nonlinear systems [16, 3]).

$$s(k) = Sx(k) = 0 \quad \forall k > 0 \quad (8.16)$$

with  $S \in \mathbb{R}^{m \times n}$ . Let us suppose that the perturbation is identically zero. Then :

$$s(k+1) = Sx(k+1) = S(A_e x(k) + B_e u(k)) = 0$$

Assuming that  $SB_e$  is invertible (this can be done by a suitable choice of  $S$ ), a discrete equivalent control  $u_{eq}$  can be deduced :

$$u_{eq}(k) = (SB_e)^{-1}(-SA_e x(k)) \quad (8.17)$$

After a change of coordinates (see the Generalized Canonical Forms [18]) system (8.13) (with  $di = 0$ ) becomes

$$\begin{aligned} x_1(k+1) &= A_{e11}x_1(k) + A_{e12}x_2(k) \\ x_2(k+1) &= A_{e21}x_1(k) + A_{e22}x_2(k) + B_{e2}u(k) \end{aligned} \quad (8.18)$$

Moreover the decomposition <sup>11</sup> of (8.16) gives us

$$s(k) = S_1x_1(k) + S_2x_2(k) = 0 \quad (8.19)$$

The motion on the sliding manifold is then given by (8.18) and (8.19)

$$x_1(k+1) = A_{e11}x_1(k) - A_{e12}S_2^{-1}S_1x_1(k) \quad (8.20)$$

and  $S$  must be chosen so that the  $x_1$  dynamics be stable.

### Remark 82

- $SB_e$  invertible implies  $S_2$  invertible.
- In the expression (8.17) appears a term in  $(SB_e)^{-1}$  with  $B_e$  in  $O(\delta)$  consequently  $(SB_e)^{-1}$  is in  $\delta^{-1}$ . This can bring out an excessive amplitude on the input and destroy the homogeneity of Euler's scheme hence the introduction of the saturation on  $u$  :

$$u(k) = \begin{cases} u_{eq} & \text{if } \|u_{eq}\| \leq H \\ u_{eq} \frac{H}{\|u_{eq}\|} & \text{if } \|u_{eq}\| > H \end{cases} \quad (8.21)$$

with  $H \ll \delta^{-1}$ .

- The introduction of these saturations do not guarantee the stability out of the input's limits.
- In the dynamics (8.20), the eigenvalues have to be inside of a circle of radius  $1 - \alpha\delta^2$ .

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<sup>11</sup> $(S_1, S_2) = ST^{-1}$  where  $T$  is the matrix of the change of coordinates.

The previous remarks shown that the discrete sliding mode control differs fundamentally from the continuous one (cf. small gain). This will be emphasized again by the method of rejection of the perturbation. In fact, it is then out of the question to “crush”  $di$ . A “predictor” is used, recalling that  $di(k+1) - di(k) = O(\delta)$  (i.e. perturbation weakly variable). Moreover as  $D_e = \delta D$  the variation of the perturbation  $di$  has a negligible effect in Euler’s scheme. Thus the perturbation term satisfies

$$D_e di(k) = D_e di(k-1) = x(k) - A_e x(k-1) - B_e u(k-1) \quad (8.22)$$

And the equivalent control is the equal to :

$$u_{eq}(k) = (SB_e)^{-1}[-SA_e x(k) - (x(k) - A_e x(k-1) - B_e u(k-1))]$$

The rest of the control scheme and the method remain identical. The following chapter deals with if it is possible to improve the approximation order in the sampling period  $\delta$ .

## 8.4 Second-order sliding mode under sampling

It has been showed in the previous section that there exist in the literature lots of algorithms of second order sliding mode [15, 12, 5, 4]. Here will simply recall an ideal version of the “Twisting algorithm” (i.e. in continuous time and without any constraint on the dynamic of actuators). Afterwards, we will present a discrete (real) version of the “Twisting algorithm” for the same linear system than before (8.11), but with a scalar input <sup>12</sup>.

$$\dot{x} = Ax + Bu + Ddi \quad (8.23)$$

The chosen manifold is the set defined by

$$S = \{x \in \mathbb{R}^n / s(x) = Sx = 0\} \quad (8.24)$$

such that the origin of the system (8.11) is (asymptotically) stable on this manifold.

The necessary hypothesis to realize the “Twisting algorithm” are :

### **Hypothesis 1:**

There exist some constants  $s_0, K_M, K_m, C_0$  such that for all  $x, t \in \mathbb{R}^n \times \mathbb{R}$  satisfying  $|s(t, x)| < s_0$ , the system (8.11) verifies, in relation to the constraint surface  $s = 0$  the following inequalities

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<sup>12</sup>The nonlinear case is easily deduced by considering the worst case. On the other hand, the multi input case raises lots of problems.

- $0 < K_m \leq SB \leq K_M$
- $|SA(Ax + Bu + Ddi) + SD\dot{d}i| < C_0$

**Definition 83** Consider the system (8.23) and the sliding (constraint) manifold (8.24). The control law

$$\dot{u} = \begin{cases} -u & \text{if } |u| \geq 1 \\ -\alpha_M \text{sign}(s) & \text{if } s\dot{s} > 0, |u| \leq 1 \\ -\alpha_m \text{sign}(s) & \text{if } s\dot{s} \leq 0, |u| \leq 1 \end{cases} \quad (8.25)$$

with  $\alpha_M$  and  $\alpha_m$  verifying the following inequalities:

$$\begin{aligned} \alpha_m &> 4 \frac{K_M}{s_0} \\ \alpha_m &> \frac{C_0}{K_m^2} \\ \alpha_M &> \frac{2C_0}{K_m} + \frac{K_M \alpha_m}{K_m} \end{aligned} \quad (8.26)$$

is called the second-order "Twisting algorithm".

**Remark 84** In the algorithm, the constraints on  $|u|$  are taken be equal to 1 in order to simplify our statement. In the general case,  $\dot{u} = -u$  if  $|u| \geq Cst$ , ensures the second condition of hypothesis 1.

**Theorem 85** [15] Under Hypothesis 1 and the conditions (8.26), the "twisting algorithm" is an ideal second order sliding mode algorithm for the system (8.23).

**Remark 86** One recall that a second order sliding mode corresponds to  $s = \dot{s} = 0$  and this without having recourse to the Filippov's solutions [11].

**Proof :**

- If  $|u| \geq 1$ ,  $u$  tends to zero whatever the value of  $s$  and  $\dot{s}$ , so that there exists a finite time  $t_1$  such that  $|u| < 1$ . In accordance with the Russian literature, the region  $|u| < 1$  (or  $|s| < s_0$ ) is called the "linear" domain. Now, let us study the second derivative of the constraint surface.

$$\ddot{s} = SA(Ax + Bu + Ddi) + SB\dot{u} + SD\dot{d}i$$

or in a more synthetic way :

$$\ddot{s} = C(x, u, di, \dot{d}i) + SB\dot{u}$$

with by hypothesis for  $|u| < 1$

$$|C(x, u, di, \dot{d}i)| < C_0 \quad \text{and} \quad 0 < K_m \leq SB < K_M$$

Using the following notations  $s = x_1$ ,  $\dot{s} = x_2$ ,  $C \triangleq C(x, u, di, \dot{di})$  and  $K \triangleq SB$ , we have

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= C + K\dot{u}\end{aligned}\tag{8.27}$$

where  $\dot{u}$  is the “twisting algorithm”. Thus, for  $s\dot{s} > 0$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= C - K\alpha_M \text{sign}(x_1)\end{aligned}\tag{8.28}$$

The perturbation  $C$  has a direct effect on the input image vector. Assuming that  $\alpha_M > \alpha_m \gg \frac{C_0}{K_m}$ , (8.28) is equivalent to :

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -K\alpha_M \text{sign}(x_1)\end{aligned}$$

Then, one obtains

$$\begin{aligned}x_1(t) &= \int_0^t x_2(\tau) d\tau + x_1(0) \\ x_2(t) &= \int_0^t -K(\tau)\alpha_M d\tau + x_2(0)\end{aligned}\tag{8.29}$$

As the surfaces  $x_1 = 0$  ( $x_2 \in \mathbb{R}$ ) et  $x_2 = 0$  ( $x_1 \in \mathbb{R}$ ) are periodically crossed<sup>13</sup>, let us take as initial conditions  $x_1(0) = 0^+$  and  $x_2(0) = x_2(0) > 0$ .

Thus, as long as the control  $\dot{u}$  has not commuted

$$\begin{aligned}x_1(t) &= -\alpha_M K \frac{t^2}{2} + x_2(0)t \\ x_2(t) &= -\alpha_M K t + x_2(0)\end{aligned}\tag{8.30}$$

Equation (8.30) stays unchanged as long as the surface  $\dot{s} = 0$  is not reached, that is to say until time  $t_2$

$$t_2 = \frac{x_2(0)}{\alpha_M K}\tag{8.31}$$

and

$$x_1(t_2) = \frac{x_2(0)^2}{2\alpha_M K}$$

---

<sup>13</sup>This hypothesis is implicitly verified further in the proof because we will see below that the manifolds  $s = 0$  and  $\dot{s} = 0$  are crossed periodically.

Then, for  $t \geq t_2$ ,  $\dot{s} \leq 0$  and  $s > 0$ , which gives, according to (8.26)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -K\alpha_m \text{sign}(x_1) = -K\alpha_m\end{aligned}$$

The system reaches the manifold  $s = 0$  in a finite time  $t_3$ , computed as followed

$$\begin{aligned}x_1(t_3 + t_2) &= -\alpha_m K \frac{t_3^2}{2} + x_1(t_2) = 0 \\ x_2(t_3 + t_2) &= -\alpha_m K t_3\end{aligned}\tag{8.32}$$

so

$$t_3 = \frac{x_2(0)}{\sqrt{\alpha_M \alpha_m} K}\tag{8.33}$$

and  $\dot{s}$  at time  $t_3 + t_2$  is equal to

$$x_2(t_2 + t_3) = -\sqrt{\frac{\alpha_m}{\alpha_M}} x_2(0)$$

since  $\alpha_m < \alpha_M$ , one has

$$|x_2(t_2 + t_3)| < |x_2(0)|$$

Thus  $|x_2|$  is decreasing in geometric progression (it is also the case for  $x_1$ ), and it can be shown that (8.27) reaches the point  $x_1 = x_2 = 0$  (or  $s = \dot{s} = 0$ ) in a finite time ( $t_\infty$ ). Indeed

$$t_\infty = \sum_{i=0}^{\infty} \left( \sqrt{\frac{\alpha_m}{\alpha_M}} \right)^i \left( \frac{1}{\alpha_M K} + \frac{1}{K \sqrt{\alpha_M \alpha_m}} \right) x_2(0)$$

and since  $0 < \sqrt{\frac{\alpha_m}{\alpha_M}} < 1$ , one obtains

$$t_\infty = \frac{1}{1 - \sqrt{\frac{\alpha_m}{\alpha_M}}} \left( \frac{1}{K \alpha_M} + \frac{1}{K \sqrt{\alpha_M \alpha_m}} \right) x_2(0)$$

**Remark 87** *The inequalities (8.26) stems from the following conditions [15]*

- $s_0 > \frac{4K_M}{\alpha_m}$  ensures that  $s$  stays in the desired area ( $s < s_0$ );
- $\alpha_m > \frac{C_0}{K_m}$  ensures that  $|u| < 1$  is sufficient to render attractive the surfaces  $s = 0$  and  $\dot{s} = 0$ ;
- $K_m \alpha_M - C_0 > K_M \alpha_m + C_0$  ensures that the oscillations are constrained and consequently do not exceed the considered worst cases.

## 8.5 The sampled “twisting algorithm”

The algorithm that is presented here has the advantage that the knowledge of the manifold derivatives  $\dot{s}$  is not required and only takes into account the sampled character of the informations and control. This is exactly what occurs in practical applications. The unique drawback is that it is a real sliding mode algorithm, that is to say, that the sliding surface is not reached exactly. In spite of all, the sampled “twisting algorithm” gives a better solution than the one obtained by the classical real algorithm<sup>14</sup>. Here, for the sake of simplify, we consider that the informations and control are sampled simultaneously, thus we can define a real (sampled) version of the “Twisting algorithm”, in the following manner:

**Definition 88** [15] *Consider the system (8.23) and the sliding manifold  $s = 0$ . The control algorithm*

$$\dot{u} = \begin{cases} -u & \text{with } |u| \geq 1 \\ -\alpha_M \text{sign}(s) & \text{with } s\Delta_s > 0, |u| < 1 \\ -\alpha_m \text{sign}(s) & \text{with } s\Delta_s \leq 0, |u| < 1 \end{cases} \quad (8.34)$$

with  $\Delta_s \triangleq (s(k\delta) - s((k-1)\delta))$  and  $\alpha_M, \alpha_m$  verifying the conditions (8.26) is called the real second-order “Twisting algorithm”.

From the results of the two previous sections, one can immediately state the following proposition.

**Proposition 89** [15] *Under the hypothesis 1 and the conditions (8.26), the sampled real “twisting algorithm” is, for the system (8.11), a real second order sliding mode algorithm on the period of sampling  $\delta$  ( $\delta$  playing here the role of the function  $\gamma(\varepsilon)$ ).*

**Remark 90** *In accordance with the literature, a real second order sliding mode control on  $\delta$  is a control such that after a finite time  $t_1$ ,*

$$|s(t)| \leq O(\delta^2) \quad \forall t > t_1$$

**Proof :**

The proof is immediate, seeing that the equations stem from the proof of the theorem 85, in fact for  $s\Delta_s > 0$  and  $x_1(0) = 0^+$ , one gets

$$\begin{aligned} x_1(t) &= -\alpha_M K \frac{t^2}{2} + x_2(0)t \\ x_2(t) &= -\alpha_M K t + x_2(0) \end{aligned}$$

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<sup>14</sup>The errors on the surface are of order  $O(\delta^2)$  instead of  $O(\delta)$ .

Likewise for  $s\Delta_s \leq 0$  and  $x_2(0) = 0^-$ , one gets

$$\begin{aligned}x_1(t) &= -\alpha_m K \frac{t^2}{2} + x_1(0) \\x_2(t) &= -\alpha_m K t\end{aligned}$$

these equations are chosen according to the  $s$  and  $\Delta_s$  signum. We deduce that after a time  $t_k$ ,  $x_1$  is of order  $O(\delta)^2$  and  $x_2$  of order  $O(\delta)$ .  $\triangle$

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# Chapter 9

# Adaptive and Sliding Mode Control

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## 9.1 Introduction

Adaptive control allows the treatment of uncertain dynamic systems, linear and nonlinear, the uncertainties of which can be expressed as the product of an uncertain constant matrix and a vector of known time function

$$\Theta^* X(t)$$

that is,  $\Theta^*$  is an unknown constant matrix and  $X$  is a matrix the entries of which are known functions of the time, where  $X$  is defined as the regressor.

This situation is encountered in both identification and control problems. In particular for the control of dynamic systems, the techniques that can be applied to systems with uncertainties of this kind rely on two-step procedures. The first step consists of solving the control problem for the system where the matrix  $\Theta$  is regarded as known. The outcome of this step consists in a control law characterized by a specific parametrization that is

$$u = \Theta_u^* (\Theta^*) X_u (\Theta^*, X, t)$$

The class of problems which can be dealt with are those for which  $X_u$  does not depend on the parameter  $\Theta^*$ , but only on the available signal  $X$  that

is

$$u^* = \Theta_u^* (\Theta^*) X_u (X)$$

in the sequel  $u^* = \Theta_u^* X_u$ .

The second step is to use a control, in the uncertain  $\Theta^*$  case, which has the same parametrization of the ideal  $u$  but with time varying parameters  $\Theta_u$

$$u = \Theta_u (t) X_u$$

This actual control signal can be expressed as

$$u = \Theta_u^* X_u + \tilde{\Theta}_u (t) X_u$$

The uncertain term  $\tilde{\Theta}_u (t) X_u$  is usually called prediction error. This signal plays a fundamental role in the adaptation mechanism where the explicit identification of  $\Theta_u^*$  is required. The regressor vector  $X_u$  is usually constituted by known time functions derived from the available system states through linear operations like linear filtering and linear combination. In some cases the  $X_u$  components result to be known nonlinear functions of the state.

It is well known that the adaptive control scheme can be divided into two categories:

- Direct adaptive control schemes
- Indirect adaptive control schemes

Within the first category are found those control schemes which explicitly compare system state trajectories with that of a reference model, traducing the expected ideal behavior of the system, which is active on line during the control process. The control aim is that of forcing some, suitably defined, error function to zero despite the parametric uncertainties of the system. In principle, the attainment of this objective does not require the identification of the unknown parameters which, in this case, are the regulator's parameters.

The second category is based on the so called certainty equivalence principle which means that the system is identified through an adaptive procedure which yields, an estimation  $\hat{\Theta}$ , of the unknown plant parameter, which, as  $t \rightarrow \infty$  tends to  $\Theta^*$  and in the meantime the control law is modified according to

$$u = \Theta_u^* (\hat{\Theta}) X_u (X) \quad (9.1)$$

where  $\Theta_u^* (\hat{\Theta})$  means that the controller parameters are chosen at any instant as the  $\hat{\Theta}(t)$  parameters were the true system parameters  $\Theta^*$ . This

situation corresponds to the ideal one only if  $\hat{\Theta}(t) = \Theta^*$  and it is realistic only if  $t \rightarrow \infty$ . During the transient, the assumed separation between the identification process and the closed-loop control raises a certain number of sensible questions:

- Is the identification process relevant to the plant or to one of the possible closed-loop systems generated (feedback changes system dynamics) by the control  $u$ ?
- Is the overall nonlinear system, identification plus control loops, stable during the adaptation process?
- Is the ideal controlled plant the only equilibrium point of the procedure or are there other possible limiting behaviors which do not fit the stated control objectives?

A considerable amount of literature has been devoted to these types of problems and is out of the scope of this chapter to provide a deep insight to such a matter. Here we prefer to stress the fact that identification is an important issue for any control strategy, since identification means the possibility to predict the future evolution (at least on the basis of the actual and past system states) and prediction is a prerequisite for dealing with any form of optimization problem. Therefore even if we do not deal with certainty equivalence methods in a systematic manner, we consider a good way to start this chapter would be to describe an identification procedure for continuous linear systems.

## 9.2 Identification of continuous linear systems in I/O form

Consider a linear system described by

$$y^{(n)} = - \sum_{i=0}^{n-1} a_i y^{(i)} + \sum_{i=0}^{n-1} b_i u^{(i)} \quad (9.2)$$

where  $y^{(i)}$  and  $u^{(i)}$  are the  $i^{th}$  derivative of  $y$  and  $u$  respectively. The system can be written as

$$y^{(n)}(t) = \Theta^* X$$

with

$$\begin{aligned} \Theta^* &= [-a_0 \dots -a_{n-1} \ b_0 \dots b_{n-1}] \\ X^T &= [y \dots y^{(n-1)} \ u \dots u^{(n-1)}] \end{aligned}$$

if  $y^{(n)}$  and  $X$  were accessible and  $\Theta^*$  an unknown, it is possible to build an estimate  $\hat{y}^{(n)}(t) = \Theta(t)X$  so that

$$\hat{y}^{(n)}(t) - y^{(n)}(t) = \tilde{\Theta}(t)X = \Theta(t)X - \Theta^*X$$

This results in the prediction error. The first step in the identification procedure consists in generating a prediction error by means of data derived from the available signals, in particular from  $u(t)$  and  $y(t)$ .

Consider two filters

$$\begin{aligned} \dot{x}_{11} &= x_{12} & \dot{x}_{21} &= x_{22} \\ \vdots & & \vdots & \\ \dot{x}_{1n} &= -\sum_{i=0}^{n-1} d_i x_{1(i+1)} + u & \dot{x}_{2n} &= -\sum_{i=0}^{n-1} d_i x_{2(i+1)} + y \end{aligned}$$

Let  $x_{11} = u_F$  and  $x_{21} = y_F$  be the filtered values of  $u$  and  $y$ , all the first  $n^{th}$  derivatives, if  $u_F$  and  $y_F$  are available for measurement. The zero state equation relating  $u_F$  and  $y_F$  is

$$y_F^{(n)} = -\sum_{i=0}^{n-1} a_i y_F^{(i)} + \sum_{i=0}^{n-1} b_i u_F^{(i)} \quad (9.3)$$

where  $y_F^{(n)} = \dot{x}_{2n} = -\sum_{i=0}^{n-1} d_i x_{2(i+1)} + y$  is available. (9.3) can be rewritten as

$$y_F^{(n)}(t) = \Theta^* X_F$$

with

$$\begin{aligned} \Theta^* &= [-a_0 \dots -a_{n-1} \ b_0 \dots b_{n-1}] \\ X^T &= [y_F \dots y_F^{(n-1)} \ u_F \dots u_F^{(n-1)}] \end{aligned}$$

Note that in this case the components of the regressor  $X_F$  are available.

The identification procedure is based on the realization of an estimate  $\hat{y}_F^{(n)} = \Theta(t)X_F$  so that the prediction error  $e_p(t)$  is

$$e_p(t) = \hat{y}_F^{(n)} - y_F^{(n)} = \tilde{\Theta}(t)X_F$$

The adaptation is chosen to be

$$\dot{\Theta}^T(t) = \dot{\tilde{\Theta}}^T(t) = -\Gamma(t)X_F e_p(t)$$

where  $\Gamma(t)$  is chosen according to the least squares with forgetting factor criteria [11], that is

$$\dot{\Gamma}(X) = -\Gamma^T X_F X_F^T \Gamma + \lambda(t) [\Gamma^T \Gamma - \Gamma k_0]$$

$$\dot{\Gamma}^{-1}(X) = X_F X_F^T - \lambda(t) [\Gamma^{-1} - I/k_0]$$

which corresponds to the minimization [11] of

$$J = \int_0^t \exp^{-\int_s^t \lambda(\tau) d\tau} \|y_F(s) - \Theta(t) X_F(s)\|^2 ds$$

The convergence of the parametric error to zero is guaranteed by the following procedure.

Choose a scalar function  $V(\tilde{\Theta}, t) = \frac{1}{2} \tilde{\Theta} \dot{\Gamma}^{-1}(t) \tilde{\Theta}^T$ . Its time derivative is

$$\begin{aligned} \dot{V}(\tilde{\Theta}, t) &= -\tilde{\Theta} X_F e_p(t) + \frac{1}{2} \tilde{\Theta} \dot{\Gamma}^{-1}(t) \tilde{\Theta}^T \\ &= -\tilde{\Theta} X_F X_F^T \tilde{\Theta}^T + \frac{1}{2} \tilde{\Theta} X_F X_F^T \tilde{\Theta}^T - \frac{1}{2} \lambda \tilde{\Theta} \left[ \Gamma^{-1} - \frac{I}{k_0} \right] \tilde{\Theta}^T \quad (9.4) \end{aligned}$$

$$V(\tilde{\Theta}, t) \leq -\frac{1}{2} \lambda \tilde{\Theta} \left[ \Gamma^{-1} - \frac{I}{k_0} \right] \tilde{\Theta}^T$$

If  $\left[ \Gamma^{-1} - \frac{I}{k_0} \right]$  is positive definite, it is possible to apply Lyapunov stability criterion to state the exponential convergence of  $\tilde{\Theta}$  to zero.

A sufficient condition for the positive definiteness of  $\left[ \Gamma^{-1} - \frac{I}{k_0} \right]$  is represented by the so called Persistent Excitation (PE) condition which can be written as: there exist a time instant  $t^*$  and a time interval  $\delta$  so that for  $t \geq t^*$ ,

$$\int_t^{t+\delta} X_F(\tau) X_F^T(\tau) d\tau \geq \alpha_0 I$$

In [10] and [7], it was proved that such a property can be guaranteed if the plant input  $u$  has at least  $\frac{N}{2}$  spectral lines, with  $N$  being the dimension of the regressor.

A very important property of the adaptation mechanism, based on prediction error, is its robustness with respect to bounded disturbances. Assume that the prediction error is available with a bounded disturbance, that is  $\bar{e}_p = e_p + \eta$ ,  $|\eta| < \Delta$ , an adaptation mechanism

$$\dot{\tilde{\Theta}}(t) = \dot{\Theta}(t) = -\Gamma(t) X(e_p + \eta)$$

with

$$\dot{\Gamma}^{-1}(t) = X X^T + \lambda \left[ \Gamma^{-1}(t) - \frac{I}{k_0} \right]$$

is analyzed by means of the Lyapunov function

$$\begin{aligned}
 V[\tilde{\Theta}(t)] &= \frac{1}{2} \tilde{\Theta}(t) \Gamma^{-1}(t) \tilde{\Theta}^T(t) \\
 \dot{V} &= -\tilde{\Theta}(t) X X^T \tilde{\Theta}^T(t) - \tilde{\Theta}(t) X \Delta + \frac{1}{2} \tilde{\Theta}(t) X X^T \tilde{\Theta}^T(t) \\
 &\quad - \lambda \tilde{\Theta}(t) \left[ \Gamma^{-1}(t) - \frac{I}{k_0} \right] \tilde{\Theta}^T(t) \\
 \dot{V} &= -\frac{1}{2} \left( \tilde{\Theta}(t) X + \Delta \right)^2 + \frac{\Delta^2}{2} - \lambda \tilde{\Theta}(t) \left[ \Gamma^{-1}(t) - \frac{I}{k_0} \right] \tilde{\Theta}^T(t)
 \end{aligned}$$

This means that  $\|\tilde{\Theta}(t)\| \rightarrow O(\Delta)$ . This fact is a strong motivation to the use of sliding mode control effect in adaptive control schemes.

### 9.3 MRAC model reference adaptive control

Any model reference adaptive control is characterized by the following common feature :

- The tracking problem can be solved univocally, in the known parameter case, by means of a control strategy of the form

$$u^*(t) = \Theta_u^* X_u$$

where  $\Theta_u^*$  is the vector of the ideal controller parameters and  $X_u$  is a regressor vector of known function or available signals

If the actual system (with uncertain parameters), is controlled by a control with the same parametrization but with time-varying parameters

$$u(t) = \Theta_u(t) X_u$$

then a suitable reference model is put on line in parallel with the controlled plant. The resulting tracking error equation can be described in general by a differential equation

$$y_e = W_m(s) \tilde{\Theta}_u X_u(t)$$

where  $s = \frac{d}{dt}$ ,  $\tilde{\Theta}_u = \Theta_u(t) - \Theta_u^*$ , and  $W_m(s)$  represents a transfer function associated to the model.

Equation errors of this kind with an adaptation mechanism of the type

$$\dot{\Theta}_u = \dot{\tilde{\Theta}}_u = -\Gamma X y_e$$

have proved to give rise to an asymptotically vanishing error, if  $W_m(s)$  is positive real, that is

- $Re [W_m(s)] > 0$  when  $Re [s] > 0$
- $Re [W_m(j\omega)] > 0$
- Let  $(A, B, C)$  be a realization of  $W_m(s)$  then there exists positive definite matrices  $P$  and  $Q$  such that

$$\begin{aligned} A^T P + P A &= -Q \\ C &= B^T P \end{aligned} \quad (9.5)$$

### 9.3.1 MRAC with accessible states

In this section we consider the case of an adaptive control system for linear plants with measurable states. Consider the linear system

$$\dot{x} = Ax + Bu$$

with  $A, B$  uncertain,  $x \in \mathbb{R}^n$ ,  $u \in U \subset \mathbb{R}^1$ , and the model

$$\dot{x}_m = A_m x_m + B_m u_m$$

Define the state error as

$$\begin{aligned} e &= x - x_m \\ \dot{e} &= A_m e + [Bu - B_m u_m - (A_m - A)x] \end{aligned} \quad (9.6)$$

If  $A, B$  were known and  $rank [B] = rank [B|A_m - A|B_m]$  (matching conditions), the ideal control law

$$u^* = (BB^T)^{-1} B^T [(A_m - A)x + B_m u_m] = k_1 x + k_2 u_m$$

$$\begin{aligned} u^* &= \Theta^* X \\ x^T &= [x u_m] \end{aligned}$$

would cancel the terms within the brackets of the Equation (9.6) and as a result the state error converges exponentially to zero, therefore  $u = \Theta(t)X = \Theta^* X - \tilde{\Theta}(t)X$  and the state error equation is

$$\dot{e} = A_m e + B\tilde{\Theta}(t)X \quad (9.7)$$

To solve the problem, an artificial output  $\nu = D e$  must be chosen so that  $D(sI - A_m)^{-1}B$  is positive real. An adaptation mechanism

$$\dot{\tilde{\Theta}} = \dot{\Theta} = -GX\nu$$

guarantees that  $\lim_{t \rightarrow \infty} e(t) = 0$ .

Indeed choose

$$V = \frac{1}{2}e^T P e + \frac{1}{2}\tilde{\Theta}(t)G^{-1}\tilde{\Theta}^T(t)$$

then

$$\dot{V} = e^T P A e + e^T A^T P e + X^T \tilde{\Theta}^T B^T P e + e^T P B \tilde{\Theta} X$$

from (9.5)

$$\dot{V} = -e^T Q e$$

this means that  $\dot{V}$  is zero in a subset and therefore Lyapunov stability criterion cannot be applied. Nevertheless since  $V(e, \tilde{\Theta})$  is monotone positive and upper-bounded if  $\dot{V}$  is uniformly continuous,  $e \rightarrow 0$  asymptotically from Barbalat lemma.  $\dot{V}$  is uniformly continuous if  $\ddot{V}$  is bounded

$$\ddot{V} = -e^T Q A_m e - e^T Q B \dot{\tilde{\Theta}} X$$

$e$  is bounded,  $\tilde{\Theta}$  is bounded,  $X$  is bounded if  $x_m, u_m$  are bounded, and therefore  $\ddot{V}$  is bounded. As a result  $e(t) \rightarrow 0$  asymptotically.

For exponential stability it is required that  $\dot{V}$  results in a negative quadratic form of both  $e$  and  $\tilde{\Theta}$ . Exponential stability implies that the identification problem must also be considered.

If the adaptation gain matrix is time varying

$$V = \frac{1}{2}e^T P e + \frac{1}{2}\tilde{\Theta}(t)G^{-1}(t)\tilde{\Theta}^T(t)$$

$$\dot{V} = -e^T Q e + \frac{1}{2}\tilde{\Theta}(t)\dot{G}^{-1}(t)\tilde{\Theta}^T(t)$$

a term which is quadratic in  $\tilde{\Theta}(t)$  appears in the time derivative of the Lyapunov function. The problem is to identify  $G(t)$ , which guarantees that

$$\dot{V} \leq -e^T Q e - \tilde{\Theta}(t)G^* \tilde{\Theta}^T(t)$$

This problem could be solved in an analogy with the identification problem by means of a matrix  $G(t)$  derived from the least squares with the forgetting factor approach. But in the case under consideration, the prediction error  $\tilde{\Theta}(t)X$  is not available, only the tracking error is measurable. This fact deserves some attention because it is one point in which the combination with the sliding modes approach makes sense. One possible way to attain a prediction error in order to use a time varying adaptation gain matrix (within the adaptive control approach), is the following: consider the error equation

$$\dot{e} = A_m e + B \tilde{\Theta} X$$

$$\nu = De$$

$$\nu = D(sI - A_m)^{-1}B\tilde{\Theta}X$$

where  $D(sI - A_m)^{-1}B$  is positive real with relative degree one transfer function. Without affecting the generality of the problem,  $D(sI - A_m)^{-1}B$  can be reduced to

$$\frac{1}{s+a} \frac{P(s)}{P(s)}$$

where  $P(s)$  is a strictly Hurwitz polynomial, so that

$$\nu = \frac{1}{s+a} \tilde{\Theta}X$$

Consider  $\eta = \nu + \left[ \Theta(t)\varphi - \frac{1}{s+a}\Theta(t)X \right]$  where  $\dot{\varphi} = -a\varphi + X$ , i.e.  $\varphi = \frac{1}{s+a}X$  by exploiting the fact that

$$\Theta^* \frac{X}{s+a} = \frac{1}{s+a} \Theta^* X \quad (9.8)$$

It results in  $\eta = \tilde{\Theta}(t)\varphi$ , which is a prediction error. Now  $\eta = \tilde{\Theta}\varphi$  with  $\eta$  available is of the same form of the identification procedure previously presented. Therefore the following adaptation mechanism can be used.

$$\dot{\Theta} = \dot{\tilde{\Theta}}(t) = -G(t)\varphi\eta$$

$$\dot{G}^{-1}(t) = \varphi\varphi^T - \lambda \left[ G^{-1}(t) - \frac{I}{k_0} \right]$$

In this case, under persistent excitation of  $\varphi$ , the application of the same procedure (used for identification), guarantees that both  $\eta$  and  $\tilde{\Theta}(t)$  tend exponentially to zero.

### 9.3.2 Adaptive control for SISO plant in I/O form: an introductory example with relative degree equal to one

For more general, linear time-invariant systems, for which either matching conditions are not satisfied or the state is not completely available, it is necessary to develop a methodology pursuing the same objective, that is, on line tracking of a suitably chosen reference model. Given a system with transfer function

$$y = \frac{k_p(s+b)}{s^2 + a_1s + a_0} \cdot u$$

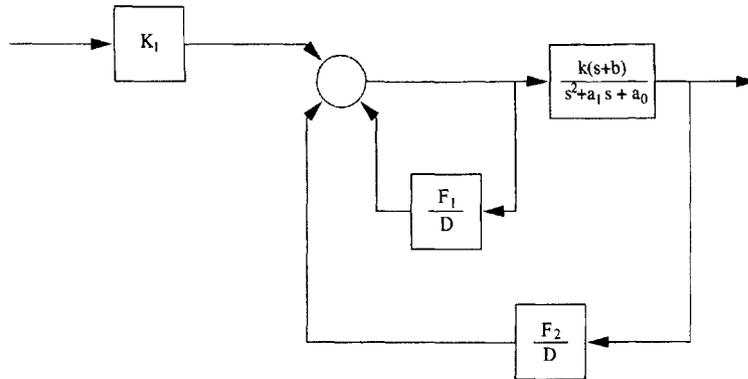
that is

$$\ddot{y} + a_1 \dot{y} + a_0 y = k_p \dot{u} + k_p b u$$

and  $k$ ,  $b$ ,  $a_1$  and  $a_0$  uncertain, find a control law  $u(t)$  so that  $y$  asymptotically tracks the output of a reference model

$$y_m = \frac{k_m(s + b_m)}{s^2 + a_{m1}s + a_{m0}} \cdot u_m \quad (9.9)$$

The first step is to find a control strategy that achieves the same objective in the case of a plant with known parameters. To this end, consider the following structure



where  $D = s + d$ ,  $F_1 = f_{10}$ ,  $F_2 = f_{21}s + f_{20}$  and  $u = k_1 u_m - y_1 - y_2$ . The transfer function is

$$y = \frac{k \cdot \frac{s+b}{s^2+a_1s+a_0}}{1 + \frac{f_{10}}{s+d} + \frac{k(s+b)}{s^2+a_1s+a_0} \cdot \frac{f_{21}s+f_{20}}{s+d}} \cdot k_1 u_m$$

$$y = \frac{k_1 k(s+b)(s+d)}{[(s+d) + f_{10}](s^2 + a_1s + a_0) + k(s+b)(f_{21}s + f_{10})} \cdot u_m \quad (9.10)$$

The pole-zero cancellation involving  $(s + b)$  requires that the system be at minimum phase if

$$k_1 = \frac{k_m}{k}, \quad d + f_{10} = b, \quad d = b_m$$

and

$$f_{21}s + f_{20} = \frac{1}{k} [(a_{m1} - a_1)s + a_{m0} - a_0]$$

$$y = \frac{k_m(s + b_m)}{s^2 + a_{m1}s + a_{m0}} \cdot u_m$$

The differential Equation (9.10) relating  $y$  to the model input  $u_m$ , coincides with that of model (9.9). This means that there exists a set of parameters of the control structure  $k_m, f_{12}, f_{21}$  and  $f_{20}$  which depends on model and system parameters, which satisfies the control objective.

The two regulators

$$\frac{F_1}{D} = \frac{f_{10}}{s+d} = \frac{y_1}{u} \quad (9.11)$$

$$\frac{F_2}{D} = \frac{f_{21}s + f_{20}}{s+d} = f_{21} + \frac{f_{20} - df_{21}}{s+d} + \frac{y_2}{y} \quad (9.12)$$

can be realized in state form

$$\begin{aligned} \dot{x}_{11} &= -dx_{11} + u \\ y_1 &= f_{11}x_{11} \end{aligned}$$

$$\begin{aligned} \dot{x}_{21} &= -dx_{21} + y \\ y_2 &= (f_{20} - df_{21})x_{21} + f_{21}y \end{aligned}$$

The resulting control law is

$$u = k_1 u_m - f_{11}x_{11} - f_{21}y - (f_{20} - df_{21})x_{21}$$

that is

$$u = [k_1 - f_{11} - f_{21} - (f_{20} - df_{21})] \begin{bmatrix} u_m \\ x_{11} \\ y \\ x_{12} \end{bmatrix}$$

and

$$u = \Theta^* X$$

where  $\Theta^{*T}$  is the regulator parameter vector.  $X$ , called the regressor, is the vector of measurable variables.

As in the previous case, in the adaptive case, the control is chosen as

$$u = \Theta(t)X$$

where  $X$  is the same regressor of the nonadaptive case.

This means that filters (9.11) and (9.12) are the same but the coefficients  $k_1, f_{11}, f_{21}$  and  $f_{20}$  are to be replaced by time-dependent signals, that is,

$$u = [\Theta_1(t), \Theta_2(t), \Theta_3(t), \Theta_4(t)] \begin{bmatrix} u_m \\ x_{11} \\ y \\ x_{12} \end{bmatrix}$$

If the reference model is put in parallel to the system it can be proven (as in the available state case) that the output error is described by the following differential equation

$$e_y = k_p \frac{s + b_m}{s^2 + a_{m1}s + a_{m0}} \tilde{\Theta} X = [\text{the transfer function of the model}] \cdot \tilde{\Theta} X$$

with

$$\tilde{\Theta} = \Theta(t) - \Theta^*$$

In this case, if the model is chosen positive real (passive), the constant gain matrix adaptation mechanism

$$\dot{\tilde{\Theta}}^T(t) = \dot{\Theta}^T(t) = -\Gamma X e_y$$

is sufficient to attain

$$\lim_{t \rightarrow \infty} e_y = 0$$

The procedure is more complex if the relative degree of the system were  $> 2$ .

It is easy to show that the relative degree of the model cannot be chosen differently from that of the plant. If, for example, the system were

$$y = \frac{k_p}{s^2 + a_1s + a_0}$$

then the model should be

$$y_m = \frac{k_m}{s^2 + a_{1m}s + a_{0m}}$$

The procedure to define  $e_y$  is the same. As a result

$$e_y = \frac{k_p}{s^2 + a_{1m}s + a_{0m}} \cdot \tilde{\Theta} X \quad (9.13)$$

The following points must be considered:

- A necessary condition for a transfer function to be positive real is that the relative degree is one, therefore, in (9.13) the model transfer function is not positive real.
- A nonpositive real transfer function can be made positive real if it is multiplied by an anticipatory operator (with a negative relative degree).

$$\frac{k_p}{s^2 + a_{1m}s + a_{0m}} \cdot (s + b_m)$$

is positive real as in the previous case.

The operator  $(s + b_m)$  is not physically realizable while its inverse  $1/(s + b_m)$  can be realized. Indeed, for any signal  $v(t)$

$$z(t) = \frac{1}{s + b_m} v(t)$$

means

$$\dot{z}(t) = -b_m z(t) + v(t)$$

If the following signal is added to  $e_y$

$$v(t) \frac{s + b_m}{s^2 + a_{m1}s + a_{m0}} k_1(t) \left[ \Theta(t) \frac{X}{s + b_m} - \frac{1}{s + b_m} \Theta(t) X \right]$$

with

$$k_1(t) = k_p + \tilde{k}(t)$$

$$\Theta(t) = \Theta^* + \tilde{\Theta}(t)$$

$$\Theta^* \frac{X}{s + b_m} = \frac{1}{s + b_m} \Theta^* X$$

there it results in

$$\eta = e_y + v(t)$$

$$\eta = \frac{s + b_m}{s^2 + a_{m1}s + a_{m0}} \left[ k_p \tilde{\Theta} X + \tilde{k}_1(t) E(t) \right]$$

$$E(t) = \Theta(t) \frac{X}{s + a} - \frac{1}{s + a} \Theta(t) X$$

$$\eta = \frac{s + b_m}{s^2 + a_{m1}s + a_{m0}} \tilde{\Theta}_1 X_1$$

$$X_1^T = \left[ \frac{X}{s + b_m} \mid E(t) \right]$$

$$\tilde{\Theta}_1 = \left[ k_p \tilde{\Theta} \mid \tilde{k}_1(t) \right]$$

At this point, by using  $\eta$  in place of  $e_y$  and  $\varphi = \frac{1}{s + b_m} X$  in place of  $X$ , the adaptative mechanism results

$$\dot{\tilde{\Theta}}^T(t) = -G\varphi\eta$$

### 9.3.3 Generalization to system of relative degree greater than one

The previous situation can be generalized to higher relative degree [6]. Given a plant

$$y = k_p W_p(s)u$$

and a model with the same relative degree

$$y_m = W_m(s)u_m$$

there exists a controller

$$u(t) = \Theta^* X$$

so that

$$e_y = W_m(s)k_p \tilde{\Theta} X$$

If an operator  $L(s)$  exists such that  $W_m(s)L(s)$  is positive real, by adding to  $e_y$

$$v(t) = W_m(s)L(s)k_1(t)[\Theta(t)L^{-1}(s)X - L^{-1}(s)\Theta(t)X]$$

the signal is related to the modified prediction error by:

$$\eta = W_m(s)L(s)k_p \tilde{\Theta}_1 X_1 + W_m(s)L(s)\tilde{k}_1(t) [\Theta(t)L^{-1}(s)X - L^{-1}(s)\Theta(t)X]$$

therefore with

$$\begin{aligned} \dot{\tilde{\Theta}}^T(t) &= \Theta(t)^T = -\Gamma X \eta \\ \dot{\tilde{k}}_1(t) &= \dot{k}_1(t) = -\Gamma [\Theta(t)L^{-1}(s)X - L^{-1}(s)\Theta(t)X] \eta \end{aligned}$$

$\eta \rightarrow 0$ , as  $t \rightarrow \infty$ .

If  $\eta \rightarrow 0$  what can be said about  $e_y$ ? Papers and books have been devoted to this problem. [8]

The result is that the following conjecture is true. When  $\eta \rightarrow 0$ ,  $\dot{\tilde{\Theta}} \rightarrow 0$ , then  $\tilde{\Theta}(t)$  and  $\Theta(t)$  tend to constants. The augmented error  $v$  is zero for  $\Theta(t) = \text{const}$  therefore

$$\eta \rightarrow 0; e_y \rightarrow \eta$$

Recent development of this approach, the so-called Morse scheme [9], tries to attain the same result by using a dynamic adaptive mechanism involving higher-order tuners. The result is a simpler scheme. The analysis of this scheme is out of the scope of this chapter.

## 9.4 Sliding mode and adaptive control

In this section the basic features of sliding mode control, which are suitable to introduce an approach within the framework of adaptive control, are considered.

### Result 1

Given

$$\begin{cases} \dot{x} = a(x) + b(x)u \\ \dot{y} = u \end{cases}$$

and given  $s(x)$  and  $u = -H(x)s \operatorname{sign}\{s\}$  with  $H(x)$  so that

$$s(x) [Ga(x) - Gb(x)H(x)s \operatorname{sign}\{s\}] < -k^2|s(x)|$$

after a finite time  $s(x) = 0$ . By the Filippov [3] solution method as the equivalent control method, the system is represented by

$$\begin{cases} \dot{x} = a(x) - b(x) [G(x)b(x)]^{-1} G(x)a(x) \\ \dot{y} = -[G(x)b(x)]^{-1} G(x)a(x) \end{cases} \quad (9.14)$$

where  $-[G(x)b(x)]^{-1} G(x)a(x)$  is the so-called equivalent control.

The representation is robust with respect to nonidealities, causing a motion close to the sliding manifold  $s(x) = 0$  if the system (9.14) is asymptotically stable.

### Result 2

Let  $s^*(x) = s(x) + \eta(t)$ , being  $\eta(t)$  a signal free to be selected by the designer. Consider

$$\dot{x}(t) = a(x) + b(x)u \quad (9.15)$$

let

$$u = -H(x, \dot{\eta}(t)) \operatorname{sign}\{s^*(t)\}$$

so that

$$s^*(t) [Ga(x) + \dot{\eta}(t) - GbH(x, \dot{\eta}, t) \operatorname{sign}\{s^*(t)\}] \leq -k^2|s^*(t)|$$

then after a finite time  $s \rightarrow 0$  and the Filippov representation of the system (9.15) is

$$\begin{cases} \dot{x}(t) = a(x) - b(x) [G(x)b(x)]^{-1} [Ga(x) + \dot{\eta}(t)] \\ s^*(x) = 0 \end{cases}$$

which is a reduced order continuous system evolving under the action of the control  $\eta(t)$ .

### Example 1

$$\begin{aligned}\dot{x}_1 &= x_2 + \Delta(t) \\ \dot{x}_2 &= f(x_1, x_2) + g(x)u\end{aligned}$$

with  $s = x_2 + cx_1$  the equivalent system on  $s = 0$  is  $\dot{x}_1 = -cx_1 + \Delta(t)$  and no counteraction to  $\Delta$  is possible. If

$$s^* = x_2 + cx_1 = \eta(t)$$

on  $s^*(x, t) = 0$ , the system is equivalent to

$$\dot{x}_1 = -cx_1 + \Delta(t) - \eta(t)$$

Through the choice of  $\eta(t)$  the disturbance  $\Delta(t)$  can be counteracted. The sliding manifold is chosen so that the reduced system evolving on it (the so-called zero-dynamics [5]) is characterized by a sufficient degree of freedom to attain the final control objective, e.g., stabilization disturbance rejection and counteraction of residual uncertainties.

### Result 3

Given a system

$$\dot{x} = a(x) + b(x)u$$

and  $s(x) = 0$ , assume that by the action of a control  $\tilde{u}(t)$  the system motion results confined in a boundary layer  $|s(x)| < \delta$ . How is it possible to achieve information relevant to the ideal equivalent control

$$u_{eq} = -[G(x)b(x)]^{-1}a(x)$$

Utkin proved [12] that under reasonable assumptions regarding system dynamics that if the control  $\tilde{u}(t)$  is filtered by an high-gain linear filter

$$\tau u_{av} = -u_{av} + \tilde{u}$$

then

$$|u_{av}(t) - u_{eq}(t)| < \mathcal{O}(\delta) + \mathcal{O}(\tau) + \mathcal{O}\left(\frac{\delta}{\tau}\right)$$

which tends to zero if  $\tau = \mathcal{O}(\sqrt{\delta})$  and  $\delta \rightarrow 0$ .

This result is extremely important when the problem of reducing the control amplitude is dealt with or when, as in the case of the combined adaptive sliding mode control, the equivalent control is explicitly used in evaluating the prediction error.

## 9.5 Combining sliding mode with adaptive control

From adaptive control.

$$\begin{aligned}\dot{e} &= A_m e + B\tilde{\Theta}X \\ \xi &= W_m \tilde{\Theta}X \\ \eta &= W_m \tilde{\Theta}X + W_m L \left( \tilde{\Theta}L^{-1} - L^{-1}\tilde{\Theta} \right) X = W_m L \tilde{\Theta}L^{-1}X \\ \dot{\Theta} &= -\Gamma(t)X(\nu, e, \eta)\end{aligned}$$

The same situation arises if a control  $u_d$  is introduced at suitable points of the controller structure by adding  $u_d + \tilde{\Theta}X$  in the relevant error equation. This could be the first link between the sliding mode control approach and adaptive control.

Every equation can be represented by means of an I/O relationship between an input  $\tilde{\Theta}X$  and an output:  $\nu, \xi, \eta$  respectively. This input/output is of relative degree 1.

In pure model-tracking problems, under the assumption that overestimation of the controller parameters of the type

$$\tilde{\Theta}_M > \max_i |\Theta_i^*|$$

it is possible to steer the output errors  $\nu, \xi, \eta$  to zero in finite time as follows

$$\begin{aligned}u &= \Theta(t)X - \left[ 2\tilde{\Theta}_M \sum |x_i| + h^2 \right] \text{sign}\{\nu, e, \eta\} \\ \dot{e} &= A_m e + B \left[ \tilde{\Theta}X - (2\tilde{\Theta}_M \sum |x_i| + h^2) \text{sign}\{\nu\} \right] \\ \nu &= De \\ \xi &= W_m \left[ \tilde{\Theta}X - (2\tilde{\Theta}_M \sum |x_i| + h^2) \text{sign}\{\xi\} \right] \\ \eta &= LW_m \left[ \tilde{\Theta}L^{-1}X - (2\tilde{\Theta}_M \sum |x_i| + h^2) \text{sign}\{\eta\} \right]\end{aligned}$$

As a result,  $\nu, \xi$  and  $\eta$  tend to zero in finite time.

Among the three considered cases, the third one, namely the case of I/O with relative degree greater than 1, deserves particular attention. While the attainment of the condition  $\nu = 0, \xi = 0$  are a direct expression of the control objective, the condition  $\eta = 0$  does not imply that the tracking error is zero.

Indeed  $\eta = 0$  implies that the equivalent control  $u_{d_{eq}}$  is equal to  $-\tilde{\Theta}L^{-1}X$  and nothing else:

$$u_{d_{eq}} = -\tilde{\Theta}L^{-1}X$$

The true tracking error is  $\xi = W_m \tilde{\Theta} X$  and it is not directly affected by  $u_d$ .

A way consists in modifying the adaptive mechanism as follows:

$$\dot{\tilde{\Theta}} = -\Gamma \eta + \Gamma(t) u_d$$

which in sliding mode is equivalent to

$$\dot{\tilde{\Theta}} = -\Gamma \varphi \eta - \Gamma(t) \varphi \varphi^T \tilde{\Theta}^T$$

where  $\Gamma(t)$  is the least square with bounded forgetting gain matrix satisfying

$$\frac{d\Gamma^{-1}(t)}{dt} = -\varphi \varphi^T - \lambda \left( \Gamma^{-1} - \frac{1}{k_0} \right)$$

The first term vanishes as  $\eta = 0$ . During sliding motion it is possible, as done previously, to choose a Lyapunov function for the equivalent system

$$U = \frac{1}{2} \tilde{\Theta} \Gamma^{-1}(t) \tilde{\Theta}^T$$

$$\dot{V} = -\tilde{\Theta} \varphi \varphi^T \tilde{\Theta}^T + \frac{1}{2} \tilde{\Theta} \varphi \varphi^T \tilde{\Theta}^T - \frac{1}{2} \lambda \tilde{\Theta} \left( \Gamma^{-1} - \frac{1}{k_0} \right) \tilde{\Theta}^T$$

If the regressor  $\varphi$  is persistently exciting,  $\Gamma^{-1} - \frac{1}{k_0}$  is positive definite and therefore  $\tilde{\Theta} \rightarrow 0$  exponentially. If  $\tilde{\Theta} \rightarrow 0$  also the tracking error  $e = W_m \tilde{\Theta} X$  tends to zero since  $W_m$  is an asymptotical stable I/O relationship.

This way is the most direct manner to couple MRAC with VSS control requires the persistent excitation of the regressor  $\varphi$ , which is obtained by filtering the regressor  $X$ . Sastry and Bodson [10] proved that the persistent excitation depends on the number of spectral lines contained in the model reference input.

Another way to deal with this problem exploits the second important property of the sliding mode control approach which is that the practical availability of the equivalent control at the output of an high-gain filter has discontinuous control at the input [4].

To clarify this procedure, we consider as a starting point the error equation

$$\xi = W_m \tilde{\Theta} X$$

with  $W_m$  having relative degree  $r$ . By multiplying for an operator

$$L = (s + a)^{r-1}$$

then  $LW_m$  has relative degree 1.

Adding to  $\xi$  the signal  $LW_m u_d$  we obtain

$$\xi = \xi + LW_m u_d = LW_m(L^{-1}\tilde{\Theta}X + u_d)$$

A sliding mode on  $\xi$  implies

$$u_{deq} = -L^{-1}\tilde{\Theta}X$$

Assume that this signal is available at the output of a filter  $F_1$

$$u_{deq} = -L^{-1}\tilde{\Theta}X = F_1(\cdot)u_d(t)$$

Consider a control  $u_{1d}(t)$  as follows. Let

$$\dot{y}_1 = -ay_1 + u_{1d}(t)$$

Also consider

$$z_1 = u_{deq} - y_1$$

$$z_1 = \frac{1}{s+a}u_{1d} - \frac{\tilde{\Theta}X}{(s+a)^{r-1}} = \frac{1}{s+a} \left( u_{1d} - \frac{\tilde{\Theta}X}{(s+a)^{r-2}} \right)$$

is attained by  $u_{1d} = -k_1 \text{sign}\{z_1\}$ . If  $z_1 = 0$

$$u_{1deq} = \frac{1}{(s+a)^{r-2}}\tilde{\Theta}X$$

and is available at the output of an high-gain filter.

Continuing this procedure,

$$z_2 = \frac{1}{s+a} \left( u_{2d} \pm \frac{\tilde{\Theta}X}{(s+a)^{r-3}} \right)$$

$$u_{2deq} = \pm \frac{\tilde{\Theta}X}{(s+a)^{r-3}}$$

$$z_{r-1} = \frac{1}{s+a} \left( u_{r-1d} \pm \tilde{\Theta}X \right)$$

$$u_{r-1deq} = \pm \tilde{\Theta}X$$

$u_{r-1deq}$  can be used in an adaptive mechanism as a prediction error. This procedure requires an adaptive mechanism which will take into account the errors accumulated by the sequence of filtering.

The following example has dealt with the proposed procedure

1) A plant with an uncertain parameter,

$$W_p(s) = \frac{1}{s^2 + 4}$$

2) Reference model

$$W_m(s) = \frac{2}{s^2 + 3s + 2}$$

3) Reference signal

$$u_m(t) = 1.5\sin(2.5t) - 1.8\sin(3.14t) + 1.6\sin(5.14t) + \sin(10t)$$

4) Monic polynomial

$$D(s) = (s + 1.5)$$

5) Equivalent controls

$$\begin{aligned} u_{d_{eq}} &= -4\text{sign}(\xi) \\ u_{1d_{eq}} &= -6\text{sign}(z_1) \end{aligned}$$

6) Operators

$$L(s) = \frac{s + 2}{2} \quad L_1(s) = 1$$

7) Sliding mode filter

$$\frac{1/\tau}{s + 1/\tau} = \frac{30}{s + 30}$$

8) Forgetting factor and upperbound matrix

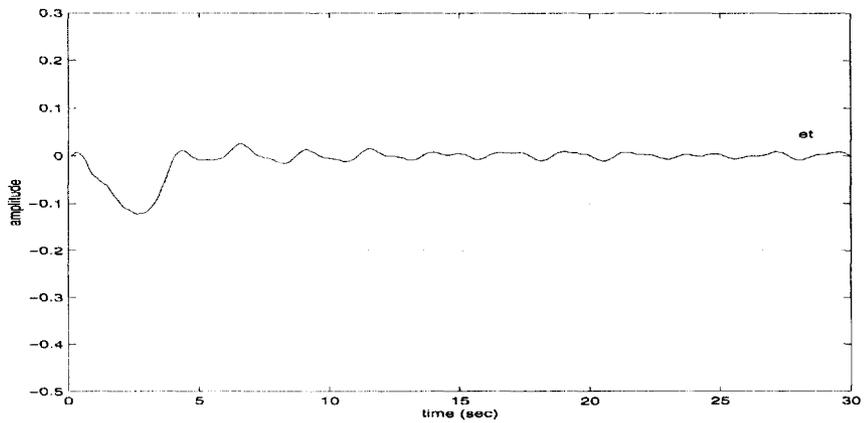
$$\lambda(t) = 10^{-3} \quad K_0 = \frac{I}{10}$$

The vector of the ideal controller parameters is

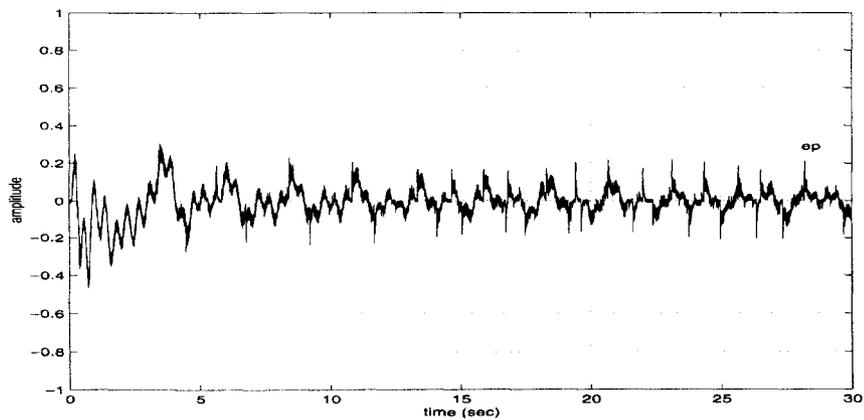
$$\Theta^* = [2, 3, -19.5, 2.5]$$

Figures 9.1, 9.2 and 9.3 show the simulink scheme and control performances.





(a)



(b)

Figure 9.2: Tracking error (a) and prediction error (b)

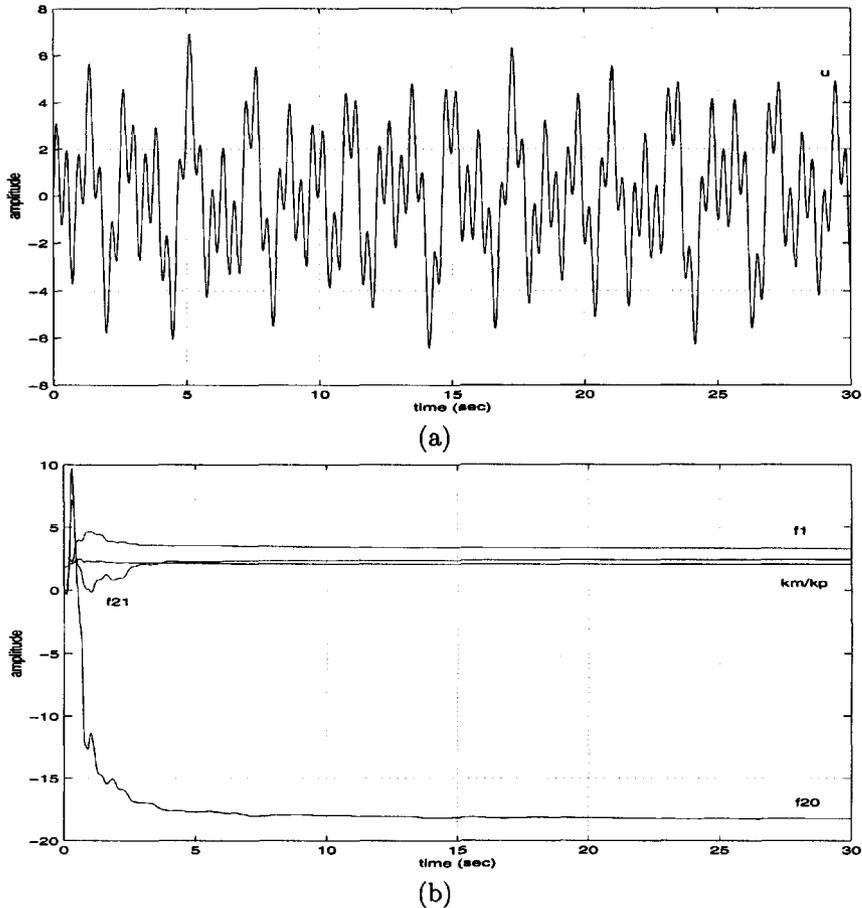


Figure 9.3: Control signal (a) and parameter estimate evolution (b).

Sliding mode requires filtering operations, but does not require the introduction of the complicated augmented error signal

$$W_m L (\Theta(t) L^{-1} X - L^{-1} \Theta(t) X)$$

The robustness of this procedure with respect to the propagation of the errors introduced by the successive filtering operations is due to the adaptive mechanism based on the prediction error (PE) as outlined in the previous section.

It is also possible to counteract disturbances with a direct control of the amplitude of the residual set by using a procedure described in [13].

Disturbance can be considered as external signals, as a residual error due to the sequence of operations described above or even as part of the prediction error, when for example, due to a lack of PE it is desired to identify some parameters neglecting the other.

This procedure starts from a prediction error of the type  $\tilde{\theta}X + d(t)$  and the aim is make  $\|\tilde{\theta}\| < \varepsilon_\theta$ , where  $\varepsilon_\theta$  is an arbitrarily small positive number. To this end the scalar  $\nu$ , defined by

$$\dot{\nu} = -a\nu + \tilde{\theta}X + d(t) + u_a(t) \quad (9.16)$$

where  $u_a(t)$  is an auxiliary signal to be defined in the sequel. Our aim is to force  $\hat{\theta}(t)$  to be less than any prespecified small constant.

To this end we prove that a signal

$$\hat{d} = \varepsilon(t) + \beta\nu(t) + \eta(t)\tilde{\theta}(t) \quad (9.17)$$

where  $\varepsilon$  and  $\eta$  are chosen according to

$$\begin{aligned} \dot{\varepsilon}(t) &= -\beta\varepsilon(t) + \beta a\nu(t) - \eta\dot{\tilde{\theta}} - \beta^2\nu \\ \dot{\eta}(t) &= -\beta\eta(t) - \beta X(t) \end{aligned} \quad (9.18)$$

This represents an estimation of  $d(t)$  such that

$$\lim_{t \rightarrow \infty} |d(t) - \hat{d}(t)| < h_1 \quad (9.19)$$

where  $h_1$  is any small positive constant. Note that  $\hat{d}$ , due to last addendum, is not available. Indeed choosing

$$\dot{V} = (\dot{d} - \hat{d})^2 = \frac{1}{2} (d - \varepsilon - \beta\nu - \eta\tilde{\theta})^2 \quad (9.20)$$

we have

$$\begin{aligned} \dot{V} &= (\dot{d} - \dot{\varepsilon} - \beta\dot{\nu} - \dot{\eta}\tilde{\theta} - \eta\dot{\tilde{\theta}}) (d - \varepsilon - \beta\nu - \eta\tilde{\theta}) \\ &= \left[ \dot{d} - \beta (d - \varepsilon - \beta\nu - \eta\tilde{\theta}) \right] (d - \varepsilon - \beta\nu - \eta\tilde{\theta}) \\ &= \dot{d} (d - \hat{d}) + \beta (d - \hat{d})^2 \\ &\leq -\frac{\beta}{2} (d - \hat{d})^2 - \frac{\beta}{2} \left( (d - \hat{d})^2 - 2 \frac{\dot{d} (d - \hat{d})}{\beta} + \frac{\dot{d}^2}{\beta^2} \right) + \frac{\dot{d}^2}{2\beta} \\ \dot{V} &\leq -\beta V + \frac{\bar{D}^2}{2\beta} \end{aligned} \quad (9.21)$$

This means that

$$(d - \hat{d})^2 \leq (d(0) - \hat{d}(0)) e^{-\beta t} + \frac{\bar{D}^2}{2\beta} \quad (9.22)$$

where  $|d - \hat{d}| < \frac{D}{\eta}$ . Now we show how it is possible to introduce  $\hat{d}$  in Equation (9.16). Choose  $u_a(t) = -\varepsilon\beta\nu$  so that  $\dot{\varepsilon} = \beta a\nu - \eta\dot{\theta}$  and (9.16) becomes

$$\begin{aligned} \dot{\nu} &= -a\nu + \tilde{\theta}X + d(t) - \varepsilon - \beta\nu \pm \tilde{\theta}\varphi \\ \dot{\nu} &= -a\nu + \tilde{\theta}(X + \varphi) + (d - \hat{d}) \end{aligned} \quad (9.23)$$

which can be interpreted as a new error equation with a prediction error that has a modified regressor and a disturbance which is arbitrarily small.

The second step is that of making available  $\tilde{\theta}(\eta + X) + (d - \hat{d})$ , to end we can introduce  $u_a = -\varepsilon - \beta\nu + u_d$  and as a result (9.16) becomes

$$\dot{\nu} = -a\nu + \tilde{\theta}(\eta + X) + (d - \hat{d}) + u_d \quad (9.24)$$

choose  $u_d = -k \text{sign}(\nu)$

$$k > \left| \tilde{\theta}(|\eta| + |X|) + \sqrt{[d(0) - \hat{d}(0)] e^{-\beta t} + \frac{\bar{D}^2}{\beta^2} + h^2} \right| \quad (9.25)$$

In finite time  $\nu \rightarrow 0$  and  $u_{deq} = -\tilde{\theta}(\eta + X) + (d - \hat{d})$ .

We now consider the following adaptative mechanism

$$\dot{\theta} = \dot{\tilde{\theta}} = -\gamma(t)(\eta + X)u_{deq} \quad (9.26)$$

where

$$\dot{\gamma}^{-1}(t) = (\eta + X)^2 - \lambda \left( \gamma^{-1} - \frac{1}{k_0} \right) \quad (9.27)$$

We shall prove that the estimate  $\theta(t)$  converges to a residual set arbitrarily small of  $\theta^*$ , which is the true unknown value under reduced persistent excitation condition. Just a single parameter needs to be identified.

Indeed using

$$V = \gamma(t)\tilde{\theta}^2 \quad (9.28)$$

$$\begin{aligned}
\dot{V} &= -\tilde{\theta}^2(\eta + X)^2 - \theta(\eta + X) \left( d - \hat{d} \right) \frac{1}{2} - \tilde{\theta}^2(\eta + X)^2 - \lambda \left( \gamma - \frac{1}{k_0} \right) \tilde{\theta}^2 \\
&= -\frac{1}{2} \left[ \theta(\eta + X) + \left( d - \hat{d} \right) \right]^2 \frac{\left( d - \hat{d} \right)^2}{2} - \lambda \left( \gamma - \frac{1}{k_0} \right) \tilde{\theta}^2 \\
&\leq - \left[ \lambda \left( \gamma - \frac{1}{k_0} \right) \gamma(t) \right] + \frac{\left( d - \hat{d} \right)^2}{2} \\
&\leq \delta V + \frac{\bar{D}^2}{\beta^2} \tag{9.29}
\end{aligned}$$

$$\tilde{\theta}^2 < \tilde{\theta}(0) e^{\delta t} + \frac{D^2}{\beta^2 \delta} \tag{9.30}$$

with  $\beta$  a project parameter which, not affecting any physical input of the plant but only the artificial systems  $\left( \dot{V} = \dots, \dot{\varepsilon} = \dots, \dot{\eta} = \dots \right)$ , can be made arbitrarily small. The parameters  $\delta = \left[ \lambda \left( \gamma - \frac{1}{k_0} \right) \gamma(t) \right]$  can be ensured to be positive by exploiting the natural excitation of the applied control since only one regressor is involved.

## Problems involving nonlinear systems

Consider a nonlinear system with the actuator

$$\begin{cases} \dot{x} = Ax + Bu \\ \dot{u} = f(x, u) + g(x)v \end{cases}$$

From adaptive control we know that there exists a control  $u^* = \Theta^*(x)$  so that  $\dot{x} = Ax + Bu^*$  tracks a reference model

$$\dot{x}_m = A_m x_m + B_m u_m$$

It is possible to define the following sliding manifold

$$s(x, t) = u(t) - \Theta(t)X$$

which can be forced to be zero in a finite time if the upper bounds of the uncertainties appearing in the actuator dynamic are available. After that time, the zero dynamic of the system is characterized by an error equation

$$e = A_m e + B \tilde{\Theta} X \quad \nu = D e$$

and  $e \rightarrow 0$  if  $\dot{\Theta}(t) = -\Gamma X e$ .

This first combination of sliding mode control and adaptive control can be generalized to single input system with nonmatching uncertainties of the type

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \Theta^* \Phi(x) \\ \dot{x}_n &= f(x) + g(x)u\end{aligned}$$

with  $|f(x)| < F(x)$   $g_1 \leq g(x) \leq g_2$ ,  $\Theta^*$  an uncertain constant parameter vector and  $\Phi(x)$  a regressor vector whose components are known nonlinear functions of the states.

$$s(x) = x_n + \Theta(t)\Phi(x) - \Theta_m X$$

$$\Theta_m = [a_{m_{n-2}} \cdots a_{m_0} | b_m] \quad X = [x_1 \cdots x_{n-1} | u_m]$$

with  $s = 0$  the zero dynamic is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= -\sum_{i=0}^{n-2} a_i x_{i+1} + b_m u_m + \tilde{\Theta}(t)\Phi(x)\end{aligned}$$

which can be dealt with by standard adaptive control. This situation can be generalized to systems in strict parametric or pure parametric form, for which the back-stepping procedure is applicable. The result is that with the use of the sliding mode control, a step in this rather cumbersome procedure can be saved.

## 9.6 Conclusions

In this chapter some basic features of control algorithms were derived from the suitable combination of sliding mode and adaptive control theory. we stressed the importance of extracting the prediction error from the equivalent control in order to cope with the problem of the controlling system with higher-order relative degree. Hints to the possibility of dealing with nonlinear systems through a suitable combination of the two approaches was also presented. The complementarity of the two approaches was based on the fact that with sliding mode it is possible to force system motion in a manifold of the state space so that the associated zero dynamics can be stabilized by adaptive control or equivalent passivity-based nonlinear algorithms.

These topics are related to the existing literature. The next step in this direction will be based on the introduction of new, recently developed tools like terminal control and higher-order sliding modes.

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# Chapter 10

## Steady Modes in Relay Systems with Delay

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### 10.1 Introduction

This chapter is devoted to relay control systems with a relatively big time delay in the control element. Relay control systems are widely used due to the following reasons:

- the relay control law is one of the simplest control algorithms;
- relay controllers are robust;
- there are control systems in which only the sign of variables is observable ([8, 22]); and
- sliding motions on a discontinuity surface, (a special kind of motions in discontinuous systems), are quite useful to design an efficient control.

On the other hand, time delay in control systems is usually present and must be taken into account. In practice, time delay is caused by the following:

- *Measuring devices have time delay.* An example of such a system is the controller of exhausted gas in the fuel-injector automotive control systems (see for example [8, 22]).

- *Actuators have a time delay.* An example of such a system is the controller for stabilization of the fingers of an underwater manipulator [4].

We distinguish between the two classes of relay control systems with delay:

- *Systems with time delay in the state.*
- *Systems with time delay in the input.*

The usual approach to the systems with delay in the state consists of two steps ([5, 6]):

- definition of the sliding equation; and
- application of the sliding mode technique.

We will concentrate on systems with time delay in the input and describe in detail what kind of stabilization can be achieved, though the standard sliding mode technique does not apply here.

The following simple example shows that the time delay in the relay control law does not allow the realization of an ideal sliding mode and underlines the meaning of the general results presented in the sequel.

## The simplest example of steady modes

The equation

$$\dot{x}(t) = -\text{sign}[x(t-1)] \quad (\text{SE})$$

has a 4-periodic solution

$$g_0(t) = \begin{cases} t, & \text{for } -1 \leq t \leq 1 \\ 2-t, & \text{for } 1 \leq t \leq 3 \end{cases}$$

$$g_0(t+4k) = g_0(t) \quad k \in \mathbb{Z}$$

Since

$$\dot{g}_0(t) = -\text{sign}[g_0(t-1-4n)]$$

We can substitute  $t$  for  $(4n+1)t$  and obtain

$$\frac{1}{4n+1} [g_0((4n+1)t)]' = -\text{sign} \left[ \frac{1}{4n+1} g_0((4n+1)t) \right]$$

Thus, a  $4/(4n+1)$ -periodic solution to (SE) is

$$g_n(t) = \frac{1}{4n+1} g_0((4n+1)t), \quad t \in \mathbb{R}$$

for each integer,  $n \geq 1$ . This means that there exists a countable set of periodic solutions, or the so-called *steady modes* (SM).

We will show later that any solution  $x(t) \not\equiv 0$ , of (SE) is equivalent to  $g_n(t + \alpha)$  for some  $n \geq 0$ ,  $\alpha \in \mathbb{R}$ ; moreover, a solution  $g_n(t)$  is stable for  $n = 0$ , and unstable for  $n \geq 1$ . These crucial features persist in more general situations.

## Statement of the problem

Consider the equation

$$\dot{x}(t) = -\text{sign}[x(t-1)] + F(x(t), t), \quad t \geq 0 \quad (10.1)$$

$$|F(x, t)| \leq p < 1, \quad F \in C^1(\mathbb{R}^2) \quad (10.2)$$

$$x(t) = \varphi(t), \quad t \in [-1; 0], \quad \varphi \in C[-1, 0] \quad (10.3)$$

Under condition (10.2), for any  $\varphi \in C[-1; 0]$ , there exists a unique continuous solution  $x_\varphi(t)$ ,  $t \in [-1; \infty)$ , of the problem (10.1), and (10.3) [21]. We will consider further only such solutions.

The time delay does not allow the realization of an ideal sliding mode, but implies oscillations, whose stability is determined by one discrete parameter called *oscillation frequency*, which is the number of zeroes on the time interval with length of delay preceding some zero of  $x_\varphi(t)$ . The basic property of the frequency, its monotone decrease, has been observed in other situations (see [23, 25]). A specific topic for discontinuous delay equations, infinite frequency oscillations, have been studied in [27, 1, 26, 9]. Some problems of the qualitative behavior of solutions of relay equations with delay was considered in [20]. The relay control algorithms for systems with delay have been suggested in [8, 22, 2]. We also show that any motion of system (10.1) turns into a steady mode, with a motion of constant frequency, as it happens in the case of usual sliding modes. At the same time this means that there are no asymptotically decreasing solutions.

All these observations are used in our approach to the following, main questions on relay controllers with delay:

1. What steady modes are stable?
2. How could relay controllers with delay be used for stabilization of unstable systems?
3. How could relay controllers with delay be used for stabilization of oscillations in a small neighborhood of constraints for stable systems, in which perturbations accumulate and take the system rather far from constraints?

## Organization of the material

Section 10.2.1 contains one of the main results, Theorem 98, which states that any solution of Equation (10.1) can be basically characterized by one discrete parameter, an average oscillation frequency, which is the number of zeroes on the time interval, preceding some zero of the solution, of length equal to the delay. A similar result was obtained for the smooth system in [23]. It was shown that each solution of Equation (10.1) is equivalent to steady mode, which is a *solution with a constant frequency*. That means we have a *finite time* of input in steady mode. Moreover, in the autonomous case, there exists a countable set of periodic SM generating all other SM by translations in  $t$ . Another important result consists in a description of classes of stable and unstable SM (Section 10.2.2). A multidimensional singularly perturbed relay system with time delay is studied in Section 10.3, where we prove the existence of slow stable periodic solutions, which is a generalization of a similar result for system (SE).

The algorithms of stabilization are presented in Section 10.4. After this section we discuss possible generalization and open problems. The proofs are presented in Section 10.6.

## 10.2 Steady modes and stability

### 10.2.1 Steady modes

The main object of this section is a special characteristic of a solution, its *oscillation frequency*. Our main result (Theorem 92) states that, for any solution, its frequency becomes constant after a period of time. Two solutions are called *equivalent* if they coincide after some time period. So, each solution is equivalent to some steady mode, a solution with a constant frequency.

Here we formulate and discuss the statements. The proofs are presented in the Appendix.

Let  $Z_\varphi$  denote a set of zeros of  $x_\varphi(t)$ . Put  $Z_\varphi^+ = Z_\varphi \cap [0; +\infty)$

**Lemma 91** *For any  $\varphi \in C[-1; 0]$  the set  $Z_\varphi$  is nonempty and unbounded.*

Thus, we can define the frequency function  $\nu_\varphi : Z_\varphi^+ \rightarrow \mathbb{N} \cup \{0\} \cup \{\infty\}$  by

$$\nu_\varphi(t) = \text{card} (Z_\varphi \cap (t - 1; t)) , \quad t \in Z_\varphi^+$$

**Theorem 92** *For any  $\varphi \in C[-1; 0]$  the function  $\nu_\varphi$  is nonincreasing, and thus there exists a limit*

$$N_\varphi \stackrel{\text{def}}{=} \lim_{\substack{t \rightarrow \infty \\ t \in Z_\varphi^+}} \nu_\varphi(t)$$

**Lemma 93** *If  $N_\varphi < \infty$  then  $N_\varphi$  is even, and  $C[-1; 0]$  is divided into sets*

$$\mathcal{U}_\infty = \{\varphi \in C[-1; 0] : N_\varphi = \infty\}$$

$$\mathcal{U}_n = \{\varphi \in C[-1; 0] : N_\varphi = 2n\}, \quad n \geq 0$$

Introduce the following subset of  $C[-1; 0]$ :

$$\mathcal{F} = \{\varphi \in C[-1; 0] : \varphi^{-1}(0) \text{ is finite}\}$$

It follows immediately from Theorem 92 that

$$\mathcal{F} \subset \bigcup_{0 \leq n < \infty} \mathcal{U}_n$$

**Definition 94** *A solution  $x_\varphi(t)$  with  $\nu_\varphi \equiv \text{const}$  is called steady mode (SM).*

The set of SM is represented naturally as the union of disjoint sets  $\mathcal{S}_n = \{x_\varphi(t) : \nu_\varphi \equiv 2n\}$ ,  $n \geq 0$ ,  $\mathcal{S}_\infty = \{x_\varphi(t) | \nu_\varphi \equiv \infty\}$ .

**Theorem 95** *For any integer  $n \geq 0$  and real  $T \geq 0$ , there exists  $g(t) \in \mathcal{S}_n$  such that*

$$g(T) = 0, \quad \dot{g}(T) > 0 \tag{10.4}$$

*If  $n = 0$  then such SM is unique.*

In the autonomous case, we give a more precise description of the SM set as follows

**Theorem 96** *In the autonomous case for any  $n \geq 0$ , the SM are unique in the following sense: there are periodic steady modes  $g_0, g_1 \dots g_n, \dots$  such that*

$$\mathcal{S}_n = \{g_n(t + \alpha) : \alpha \in \mathbb{R}, \quad n \geq 0\}$$

*and their periods satisfy inequalities*

$$\tau_0 > 2, \quad n^{-1} > \tau_n > (n+1)^{-1}, \quad n \geq 1 \tag{10.5}$$

**Remark 97** *In fact, in the autonomous case  $\mathcal{S}_\infty = \emptyset$  if  $F(0) \neq 0$ , and  $\mathcal{S}_\infty = \{0\}$  if  $F(0) = 0$ . This was recently proved by Akian, Bliman [1] and Nussbaum and Shustin [26]. For the non-autonomous case see [9, 27].*

As a consequence of the above statements we obtain

**Theorem 98** *Any solution  $x_\varphi(t)$  of the (10.1), (10.3) is equivalent to a suitable SM.*

## 10.2.2 Stability

Here we study the stability of solutions of our equation with respect to the standard metric in the space  $C[-1; 0]$  of initial functions. First we show that the zero steady frequency is stable, then from this we derive the non-asymptotic stability of zero-frequency SM in the autonomous case and give a condition of the closeness to the autonomous case, where the same type of stability is present. Finally, we establish that SM with positive frequency are unstable.

**Theorem 99** *The set  $\mathcal{U}_0$  has a nonempty interior. Moreover,  $\text{Int } \mathcal{U}_0$  contains the nonempty set*

$$\tilde{\mathcal{U}}_0 = \mathcal{U}_0 \cap \{\varphi \in C[-1; 0] : \text{mes}[\varphi^{-1}(0)] = 0\}$$

In particular, we get that the function  $N(\varphi) = N_\varphi = 0$  is stable if  $\text{mes}[\varphi^{-1}(0)] = 0$ .

**Corollary 100** *In the autonomous case, all of the solutions  $x_\varphi(t), \varphi \in \tilde{\mathcal{U}}_0$ , are nonasymptotically stable.*

**Theorem 101** *If*

$$\int_0^\infty \max_x \left| \frac{\partial F(x, t)}{\partial t} \right| dt < \infty \quad (10.6)$$

*then all solutions  $x_\varphi(t), \varphi \in \tilde{\mathcal{U}}_0$ , are nonasymptotically stable.*

We should underline that there are unstable solutions  $x_\varphi(t)$  with  $\varphi \in \mathcal{U}_0$ . For example, let  $\psi \in \mathcal{U}_n, n \geq 1$ , then  $\varphi(t) = \max\{0; \psi(t)\} \in \mathcal{U}_0$ , but  $\varphi_\tau(t) = \varphi(t) + \tau\psi(t) \in \mathcal{U}_n$ , for any  $\tau > 0$ .

**Theorem 102** *If*

$$\sup \left| \frac{\partial F}{\partial x} \right| = M_x < 2(1-p)^2(1+p)^{-3} \quad (10.7)$$

*or*

$$\sup \left| \frac{\partial F}{\partial t} \right| = M_t < 2(1-p)^2(1+p)^{-2} \quad (10.8)$$

*then all solutions  $x_\varphi(t)$  and  $\varphi \in \bigcup_{1 \leq n \leq \infty} \mathcal{U}_n$ , are unstable.*

Note that the conditions of Theorems 101 and 102 are fulfilled in the autonomous case.

## 10.3 Singular perturbation in relay systems with time delay

### 10.3.1 Existence of stable zero frequency periodic steady modes for a singularly perturbed multidimensional system

Here we study a multidimensional generalization of system (SE). Consider the system

$$\mu \frac{dz}{dt} = f(z, s, x, u), \quad \frac{ds}{dt} = g(z, s, x, u), \quad \frac{dx}{dt} = h(z, s, x, u) \quad (10.9)$$

where  $z \in \mathbb{R}^m, s \in \mathbb{R}, x \in \mathbb{R}^n, u(s) = \text{sign}[s(t-1)]; f, g, h \in C^2(\bar{Z}), Z \subset \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times [-1, 1];$  and  $\mu$  is a small parameter.

Ignoring additional dynamics, accepting  $\mu = 0$ , and expressing  $z_0$  from the equation

$$g(z_0, s, x, u(s)) = 0$$

we obtain from the formula  $z_0 = \varphi(s, x, u)$  that

$$\begin{aligned} \frac{ds}{dt} &= g(\varphi(s, x, u), s, x, u) = G(s, x, u) \\ \frac{dx}{dt} &= h(\varphi(s, x, u), s, x, u) = H(s, x, u) \end{aligned} \quad (10.10)$$

which satisfies the sufficient conditions for the existence of a zero frequency steady mode.

Suppose that

**C1:** The function  $z_0 = \varphi(s, x, u)$ , for all  $(s, x, u) \in \bar{S}; S \subset \mathbb{R} \times \mathbb{R}^n \times [-1, 1]$ , is a uniformly asymptotically stable isolated equilibrium point of system  $dz/d\tau = f(z, s, x, u)$ ; moreover, the matrix  $\frac{\partial f(z, s, x, u)}{\partial z}$  is stable at all  $(s, x, u) \in \bar{S}$ , and the inequality  $\text{Re Spec } \frac{\partial f(z, s, x, u)}{\partial z} < -\alpha < 0$  holds.

Under condition C1 we design the point mapping of surface  $s = 0$  into itself determined by system (10.10).

Namely, consider the solution to (10.10) for  $u = 1$ :

$$\frac{d\bar{s}_0^+}{dt} = G(\bar{s}_0^+, \bar{x}_0^+, 1), \quad \frac{d\bar{x}_0^+}{dt} = H(\bar{s}_0^+, \bar{x}_0^+, 1) \quad (10.10+)$$

with the initial conditions

$$\begin{aligned} \bar{s}_0^+(0) &= 0, \quad \bar{s}_0^+(t) < 0, \quad t \in [-1, 0); \quad \bar{x}_0^+(0) = \xi \\ \xi &\in V \subset S^+ = \{\xi : G(0, \xi, 1) > 0\} \end{aligned}$$

Suppose that, for  $t = 1$ , the relay control  $u$  changes its value from  $+1$  to  $-1$  so that the behavior of a solution to (10.10) is described by the system

$$\frac{d\bar{s}_0^-}{dt} = G(\bar{s}_0^-, \bar{x}_0^-, -1) \quad \frac{d\bar{x}_0^-}{dt} = H(\bar{s}_0^-, \bar{x}_0^-, -1) \quad (10.10-)$$

$$\bar{s}_0^-(1) = \bar{s}_0^+(1), \quad \bar{x}_0^-(1) = \bar{x}_0^+(1)$$

Suppose that, for all  $\xi \in V$ , there exists the smallest root of equation  $\bar{s}_0^-(\theta(\xi)) = 0$  such that  $G(0, \bar{x}_0^-(\theta(\xi)), -1) < 0$  and consequently for  $t = \theta(\xi) + 1$  the control  $u$  changes its value from  $-1$  to  $+1$ . Then the behavior of the solution of (10.10) and the behavior of the system for  $t > \theta(\xi) + 1$  is described by system (10.10+) with initial condition

$$\bar{s}_0^+(\theta(\xi) + 1) = \bar{s}_0^-(\theta(\xi) + 1), \quad \bar{x}_0^+(\theta(\xi) + 1) = \bar{x}_0^-(\theta(\xi) + 1)$$

Suppose also that, for all  $\xi \in V$ , there exists  $T(\xi)$ , the smallest root of the equation  $\bar{s}_0^+(T(\xi)) = 0$ , such that  $T(\xi) > \theta(\xi) + 1$  and  $G(0, \bar{x}_0^+(T(\xi)), 1) > 0$ . Then the point mapping  $\Psi(\xi) : \xi \rightarrow \bar{x}_0^+(T(\xi))$  is the point mapping of the domain  $V$  on the surface  $s = 0$  produced by system (10.10).

We now introduce the following assumptions:

**C2:** System (10.10) has an isolated zero frequency steady mode  $[s_0(t), x_0(t)]$ , which has exactly two intersection points with the surface  $s = 0$  such that

$$s_0(0) = 0, \quad \frac{ds_0}{dt}(0) > 0 \quad s_0(\theta_0) = 0, \quad \frac{ds_0}{dt}(\theta_0) < 0$$

**C3:** The point mapping  $\Psi(x)$  of the surface  $s = 0$  into itself, which was made by system (10.10), has the stable isolated equilibrium point  $x_0$  corresponding to  $[s_0(t), x_0(t)]$ , moreover

$$\left\| \frac{\partial \Psi(x_0)}{\partial x} \right\| < q < 1$$

**C4:** The points  $\varphi(s_0(1), x_0(1), -1)$  and  $\varphi(s_0(\theta_0+1), x_0(\theta_0+1), 1)$  are situated in the attractive domains of stable equilibrium points  $\varphi(s_0(1), x_0(1), 1)$  and  $\varphi(s_0(\theta_0 + 1), x_0(\theta_0 + 1), -1)$ , respectively.

**Theorem 103** *Under conditions C1-C4 system (10.9) has an orbitally asymptotically stable isolated periodic solution close to  $(s_0(t), x_0(t))$  with a period  $T(\mu)$  which tends to  $T$  as  $\mu \rightarrow 0$ , and the boundary layers close to  $t = 1$ ,  $t = \theta_0 + 1$ .*

**Remark 104** *An algorithm for the asymptotic representation of a zero frequency periodic steady mode [34], based on the boundary layer method is suggested in [17].*

### 10.3.2 Existence of stable zero frequency steady modes in systems of arbitrary order

Consider the system

$$\begin{aligned}\mu \frac{dz_1}{dt} &= -z_1 + u, \quad u(s) = -\text{sign}[s(t-1)] \\ \mu \frac{dz_2}{dt} &= z_1 - z_2, \dots, \mu \frac{dz_k}{dt} = z_{k-1} - z_k \\ \frac{ds}{dt} &= x, \quad \frac{dx}{dt} = -x + z_k\end{aligned}\tag{10.11}$$

where  $z_1, \dots, z_k, s, x \in R$  and  $\mu$  is the small parameter. It is obvious that system (10.11) is a system with relative degree  $(k+2)$  with respect to output variable  $s$ . Let's show that for system (10.11) the conditions of Theorem 103 hold and consequently system (10.11) has an orbitally asymptotically stable zero frequency steady mode at least for small  $\mu$ .

For  $\mu = 0$  system (10.11) has the form

$$\begin{aligned}\bar{z}_1 &= \bar{z}_2 = \dots = \bar{z}_k = u \\ \frac{d\bar{s}_0}{dt} &= \bar{x}_0, \quad \frac{d\bar{x}_0}{dt} = -\bar{x}_0 + u\end{aligned}\tag{10.12}$$

Then for the solution of (10.12) with initial conditions

$$\bar{x}_0^+(0) = \xi, \quad \bar{s}_0^+(0) = 0$$

$$\text{sign}[\bar{s}_0^+(t-1)] = -1, \quad u = 1 \quad \text{for } t \in [-1, 0]$$

we have

$$\bar{x}_0^+(t, \xi) = e^{-t}(\xi - 1) + 1; \quad \bar{s}_0^+(t, \xi) = (1 - e^{-t})(\xi - 1) + t$$

and consequently

$$\bar{x}_0^+(1, \xi) = e^{-1}(\xi - 1) + 1; \quad \bar{s}_0^+(1, \xi) = (1 - e^{-1})(\xi - 1) + 1$$

For  $t > 1, u = -1$  and, until the switching of  $\text{sign}(u)$ ,

$$\bar{x}_0^-(t, \xi) = e^{-(t-1)}(\bar{x}_0^+(1, \xi) + 1) - 1$$

$$\bar{s}_0^-(t, \xi) = (1 - e^{-(t-1)})(\bar{x}_0^+(1, \xi) + 1) - (t-1) + (1 - e^{-1})(\xi - 1) + 1$$

In this case the switching moment  $\theta(\xi)$  is defined by equation  $\bar{s}_0^-(\theta(\xi), \xi) = 0$ . Taking into account the symmetry of system (10.12) with respect to the point  $s = x = 0$ , we can conclude that the semi-period of the desired

periodic solution  $\theta_0$  and the fixed point  $\xi_0$  of the point mapping  $\Psi(\xi)$  are described by equation

$$\bar{s}_0^-(\theta_0, \xi_0) = 0, \quad \bar{x}_0^-(\theta_0, \xi_0) = -\xi_0$$

Thus,

$$\xi_0 = 1 - 2 \frac{e^{-\theta_0+1}}{1 + e^{-\theta_0}}; \quad \theta_0 = 4 - 4 \frac{e^{-\theta_0+1}}{1 + e^{-\theta_0}}$$

This system has the solution  $\theta_0 \approx 3,75$  and  $\xi_0 \approx 0,87$ . Here  $\xi_0$  is the fixed point of point mapping  $\Psi(\xi)$ , corresponding to the  $2\theta_0$ - periodic solution of (10.12) determined by the equations

$$(\bar{s}_0(t), \bar{x}_0(t)) = \begin{cases} (\bar{s}_0^+(t, \xi_0), \bar{x}_0^+(t, \xi_0)), & \text{for } -\theta_0 + 1 \leq t \leq 1, \\ (\bar{s}_0^-(t, \xi_0), \bar{x}_0^-(t, \xi_0)), & \text{for } 1 \leq t \leq \theta_0 + 1. \end{cases}$$

Moreover,

$$\begin{aligned} \frac{d\Psi}{d\xi}(\xi_0) &= \left( \frac{dx^-(\theta(\xi), \xi)}{d\xi}(\theta_0, \xi_0) \right)^2 = \left( e^{-\theta_0} - 2 \frac{e^{-\theta_0+1}(1 - e^{-\theta_0})}{e^{-\theta_0} + 1 - 2e^{-\theta_0+1}} \right)^2 \\ &\approx 0,0144. \end{aligned}$$

Then the conditions of Theorem 103 hold for system (10.11), therefore system (10.11) has an orbitally asymptotically stable periodic zero frequency steady mode at least for the small  $\mu$ . This means that for any  $k$  there exists at least one orbitally asymptotically stable zero frequency periodic steady mode of  $(k + 2)$ -th order.

## 10.4 Design of delay controllers of relay type

### 10.4.1 Stabilization of the simplest unstable system

Consider the stabilization problem for the simplest unstable system

$$\dot{x} = kx, \quad (x \in \mathbb{R}, k > 0) \tag{US}$$

by means of a delay relay control law of the form  $u = -\text{sign}[x(t - \gamma)]$ , where  $\gamma$  is time delay. In this case the equation for the control system has the form

$$\dot{x}(t) = -\text{sign}[x(t - \gamma)] + kx \tag{CS}$$

Let us compute the constant  $A > 0$  for which the system (CS) with initial function

$$\varphi(t) = A, \quad t \in [-\gamma, 0] \tag{IF}$$

has a stable periodic solution for  $t > 0$ .

Before the switching moment we have

$$x(t) = \frac{1}{k} + \left(A - \frac{1}{k}\right) e^{kt}$$

The function  $x(t)$  could change its sign if and only if the condition

$$A - \frac{1}{k} < 0$$

holds. In this case we can rewrite equation for  $\tau$ -, which is the root of equation  $x(\tau) = 0$  in form  $e^{k\tau} = \frac{1}{1-kA}$ . From periodicity of  $x(t)$  we have the equation for the switching moment of the control law in the form  $x(\tau + \gamma) = -A$  Then

$$\frac{1}{k} + \left(A - \frac{1}{k}\right) e^{k\tau} e^{k\gamma} = -A,$$

and consequently  $A = (e^{k\gamma} - 1)/k$ . This means that sufficient condition for existence of the periodic solution has the form

$$k\gamma < \log 2 \tag{SC}$$

This implies that for any positive feedback coefficient  $k$  we can choose the time delay  $\gamma$ , for which there exists a zero frequency stable periodic steady mode of (CS). Moreover the equation (CS) has a countable set of steady modes in the interior of the strip  $|x| < (e^{k\gamma} - 1)/k$ . System (CS) has unstable solutions  $x = \pm 1/k$ , and unbounded solutions in the regions  $|x| > 1/k$ .

This means that the Cauchy problem (CS), (10.3) has a bounded solution if for any  $t \in [0, \gamma]$ ,  $k|x_\varphi(t)| < 1$ . This means that if  $\varphi(0) > 0$ , then

$$k|x_\varphi(t)| = |-1 + (k\varphi(0) + 1)e^{kt}| < 1$$

This implies:

**Theorem 105** *If condition (CS) holds and  $|\varphi(0)| < \frac{2-e^{k\gamma}}{ke^{k\gamma}}$ , then the solution  $x_\varphi(t)$  of (CS) and (10.3) is bounded.*

## 10.4.2 Stable systems with bounded perturbation and relay controllers with delay

Consider the simplest stable system with bounded perturbations

$$\dot{x} = -kx + F(t, x), \quad (x \in \mathbb{R}, k > 0) \tag{PS}$$

Here  $|F(t, x)| \leq \varepsilon$  is bounded perturbation. Suppose that we can use the relay control with delay  $\gamma$  in form  $u(s) = -\lambda \cdot \text{sign}[x(t - \gamma)]$ ,  $\lambda > \varepsilon$ . The behavior of the control system is described by equation

$$\dot{x} = -kx + F(t, x) - \lambda \cdot \text{sign}[x(t - \gamma)] \quad (\text{CPS})$$

Then for the amplitude we have the following estimation

$$|x(\gamma)| \leq \int_0^\gamma e^{-k(\gamma-\tau)} (|\lambda| + |F(\tau, x(\tau))|) d\tau \leq \frac{\lambda + \varepsilon}{k} (1 - e^{-k\gamma}) \leq \gamma(\lambda + \varepsilon)$$

It allows us to conclude that the motions in stable systems are in the  $O(\varepsilon)$  neighborhood from constraints. If we are using the relay control with delay, the amplitude of oscillation is  $O((\lambda + \varepsilon)\gamma)$ . It is important in the case of sufficiently small  $\lambda, \varepsilon$  and  $\gamma$ .

### 10.4.3 Statement of the adaptive control problem

Consider the system

$$\dot{x}(t) = F(x, t) + u(t) \quad (10.13)$$

$$u(t) = \alpha(t) \cdot \text{sign}[x(t - 1)]$$

A real controller operates with an unavoidable time delay. Here we develop the direct adaptive delay control of relay type  $u(t) = -\alpha \cdot \text{sign}[x(t - 1)]$  with a step function  $\alpha$  dependent only on the information of the time interval  $(-1, t - 1)$  and provides exponentially decreasing oscillations even in the presence of disturbances. Here we restrict ourselves to those systems satisfying [20]

$$F(0, t) \equiv 0$$

and everywhere below we suppose this equality.

Note that here we lose the restriction (10.2), and solutions may be unbounded and inextensible to the infinite interval. On the other hand, there are SM with sufficiently big frequency and small amplitude. It turns out that the existence of stable SM with *zero frequency* implies the existence of a wide class of bounded solutions. Namely,

**Lemma 106** *Let*

$$F(0, t) \equiv 0 \quad (10.14)$$

$$\frac{\partial F}{\partial x}(x, t) \leq k < \ln 2, t \in \mathbb{R}, |x| < \alpha/k \quad (10.15)$$

*Then all the solutions of equation*

$$\dot{x}(t) = F(x, t) + \alpha \cdot \text{sign}[x(t - 1)]$$

with initial condition (10.3), where

$$|x(0)| = |\varphi(0)| < \alpha(2 \exp(-k) - 1)/k \quad (10.16)$$

are extensible to the interval  $(-1; \infty)$  and satisfy inequalities

$$|x_\varphi(t)| \leq \frac{\alpha}{k}(e^k - 1), \quad |\dot{x}_\varphi(t)| \leq \alpha e^k \quad (10.17)$$

These solutions hold all properties mentioned in Section 10.2.

In a non-ideal situation we'll use the following simple estimate

**Proposition 107** *Under the conditions of Lemma 106, let  $T$  be a zero of some solution  $x(t)$  of (10.1), and  $|T^* - T| < \delta$ . Then*

$$|x(T^*)| < \alpha(e^{k\delta} - 1)/k$$

#### 10.4.4 The case of definite systems

Assume  $F(x, t)$  holds (10.15), and we have complete information on  $F(x, t)$  and have the observer, which indicates zeroes of  $x(t)$  and signs of  $x(t)$  with the delay 1. We design the desired control by means of the following algorithm.

Let (10.16) hold with some constant  $\alpha = \alpha_0$ . Put  $\alpha(t) = \alpha_0$ ,  $t \geq 0$ , and consider the equation

$$\dot{x}(t) = -\alpha_0 \cdot \text{sign}[x(t-1)] + F(x(t), t), \quad t \geq 0$$

We fix a time moment  $t_1 + 1$ , when the observer indicates the first zero  $t_1$  of  $x(t)$  greater than 1. Using the distribution of zeroes and signs of  $x(t)$  on the segment  $[0; t_1]$ , we extrapolate  $x(t)$  on the interval  $t > t_1$  and compute the first zero  $t_2$  of  $x(t)$  greater than  $t_1 + 1$ . Now in the ideal situation we can put

$$\alpha(t) = \alpha_1, \quad t \geq t_2$$

where  $\alpha_1$  is small positive constant and, according to (10.17), we obtain a solution  $x(t)$  close to zero.

Assume we compute  $t_2$  with error  $\delta$ . Let  $\delta$  satisfy the condition

$$\rho \stackrel{\text{def}}{=} \frac{e^{k\delta} - 1}{2e^{-k} - 1} < 1 \iff \delta < \frac{\ln 2}{k} - 1 \quad (10.18)$$

From Proposition 107 it follows immediately that property (10.16) at  $t_2$  with the constant  $\alpha = \alpha_0 \rho$ . Now we put  $\alpha(t) = \alpha_0 \rho$ ,  $t \geq t_2$  and repeat our algorithm from the beginning. After  $m$  steps we get, from (10.17),

$$|x(t)| \leq \frac{e^k - 1}{k} \alpha_0 \rho^m = \frac{k^{m-1}(e^k - 1)}{(2e^{-k} - 1)^m} \delta^m + O(\delta^{m+1}) \quad (10.19)$$

The left side of (10.19) tends to zero for  $m \rightarrow \infty$ .

### 10.4.5 The case of indefinite systems

Having the error  $\delta_0$  of the observer and the property (10.15) as the only information on  $F(x, t)$ , we must solve a single problem in using the previous algorithm: to construct a zeroes sequence on an interval  $(t; \infty)$  having a zeroes sequence on  $(-1; t - 1)$ .

In the autonomous case Theorem 102 provides (with the probability 1), turning any bounded solution of the equation

$$\dot{x}(t) = -\alpha \cdot \text{sign}[x(t - 1)] + F(x(t))$$

into some zero frequency SM. Assume that by the time moment  $t_{2n} + 1$  our observer indicated consequent zeroes  $t_0, t_1, \dots, t_{2n}$  such that  $t_i + 1 < t_{i+1}$ ,  $i = 0, \dots, 2n - 1$ . According to the periodicity of SM (see Theorem 98), the following zero equals  $t_{2n+1} = t_{2n-1} + (t_{2n} - t_0)/n > t_{2n} + 1$  with error  $\delta = \delta_0(1 + 2/n)$ . If  $\delta$  satisfies (10.18) then, repeating such steps, we stabilize the zero solution as above.

## 10.5 Generalizations and open problems

### 10.5.1 The case when $|F(x)| > 1$ for some $x$

In [29] it was shown that the results of Section 10.2 for system (10.1) hold for the case when for some  $x$  the function  $F(x)$  has values out of  $[-1, 1]$ , but satisfies the following conditions:

$$(i) x_{-1}^+ \leq x_1^+ \text{ or } \int_0^{x_1^+} \frac{dx}{1+F(x)} > 1, \text{ and}$$

$$(ii) x_1^- \geq x_{-1}^- \text{ or } \int_{x_{-1}^-}^0 \frac{dx}{1-F(x)} > 1,$$

where

$$x_1^+ = \inf\{x > 0 : F(x) = 1\}, \quad x_{-1}^+ = \inf\{x > 0 : F(x) = -1\}$$

$$x_1^- = \inf\{x < 0 : F(x) = 1\}, \quad x_{-1}^- = \inf\{x < 0 : F(x) = -1\}$$

*Systems with different delays* are more complicated [3]. This is a very interesting subject for study.

### 10.5.2 Systems and steady modes of the second order

Relay control systems with delay of second order are considered in the form

$$\frac{d^2x}{dt^2} = -\frac{dx}{dt} + F(x) - \text{sign}[x(t - 1)], \quad (\text{SONRCSD})$$

$$x(t) = \varphi(t), t \in [-1, 0], \varepsilon = \text{const} > 0, \dot{x}(0) = x_0$$

$$|F(x, t)| < p < 1, F \in C^1(\mathbb{R}^2)$$

This was considered in [18]. It is shown that if the frequency  $\nu$  of solution of (SONRCSD) is even, then  $\nu$  does not increase. If the frequency  $\nu$  is odd it could increase by 1. This allows us to introduce the notion of frequency for the second-order relay control systems with delay in the form  $\psi = [(\nu + 1)/2]$  (here  $[\cdot]$  is the entire part), which is a nonincreasing function. It is shown that for each solution of (SONRCSD), there exists the limit value of frequency  $N = \lim_{t \rightarrow \infty} \psi$ . It was proved that in the case when  $F$  is autonomous for any integer  $\psi > 0$  there exists a periodic steady mode.

Second-order linear relay control system with delay

$$\varepsilon \frac{d^2 x}{dt^2} = -\frac{dx}{dt} + kx - \text{sign}[x(t - 1)] \quad (\text{SOLRCSD})$$

was considered in [15, 16, 18, 19, 33]. For such system there have been conditions found providing that

- the frequency  $\psi$  is non-increasing;
- there exists a countable set of periodic steady modes for any integer with nonnegative value of  $\psi$ ; and
- the zero frequency periodic steady modes are orbitally asymptotically stable.

The natural sufficient conditions for orbital asymptotic stability of zero frequency steady modes for (SOLDRCS) was found in [15, 16, 18, 19, 33]. For second-order relay control systems with delay, the problem of instability of steady modes with nonzero frequency is still open.

### 10.5.3 Stability and instability of steady modes for multidimensional case

In [14] the multidimensional relay control systems with delay in form was considered

$$\begin{cases} \dot{s}(t) = -\text{sign}[s(t - 1)] + F[s(t), x(t)] \\ \dot{x}(t) = As(t) + Bx(t), \end{cases} \quad (\text{MDRCSD})$$

$$s \in \mathbb{R}, x \in \mathbb{R}^n, |F(s, x)| < p < 1$$

It was shown that if  $B$  is a stable matrix then, for any even value of frequency, there exists a periodic steady mode. In [14] the problem of stability of zero frequency steady modes of (MDRDCS) is reduced to the problem of contractibility of point mapping of the surface  $s = 0$  into itself made by the original system. In fact, it is practically impossible to check this property of (MDRCSD) and the problem of stability is open. As in the previous case, the problem of instability is open too.

## 10.6 Conclusions

1. The notions of frequency and steady modes were introduced. The existence of steady modes for any even frequency were established.
2. The steady modes possess properties similar to that of sliding modes:
  - (i) the set of switches of any steady mode is unbounded, thus a steady mode is not equivalent to any solution of a continuous part of the given equation;
  - (ii) for any solution there exists a finite time preceding its input into a steady mode;
  - (iii) the shift operator is not invertible; and
  - (iv) the properties (i)-(iii) are invariant with respect to bounded perturbations which satisfy condition (10.2).
3. Stability criteria for steady modes with zero frequency were established.
4. It was proved that all steady modes with the positive frequency were unstable under some mild conditions.
5. The existence of a slow stable periodic solution of the multidimensional singularly perturbed relay system with time delay, which corresponds to the stable zero frequency steady mode, was proved.
6. A direct, adaptive control of relay type with time delay that extinguished parasite auto-oscillations in this model was designed.

## 10.7 Appendix: proofs

Lemma 91 is obvious.

**Proof of Theorem 92.** If  $t_1 < t_2$ ,  $t_1, t_2 \in Z_\varphi^+$ , then, according to Rolle's theorem and (10.1), (10.2), there exists  $\xi \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$ . Therefore

$$\text{card}(Z_\varphi \cap (t_1 - 1; t_2 - 1)) \geq \text{card}(Z_\varphi^+ \cap (t_1; t_2)) + 1$$

thus

$$\nu_\varphi(t_1) = \text{card}(Z_\varphi \cap (t_1 - 1; t_1)) \geq \text{card}(Z_\varphi \cap (t_2 - 1; t_2)) = \nu_\varphi(t_2)$$

**Proof of Lemma 93.** Let  $\nu_\varphi(t) = N_\varphi < \infty$ , when  $t \geq T$ . Then  $x_\varphi(t)$  changes its sign at every point  $t \in Z_\varphi \cap [T; +\infty)$ . Indeed, if  $t_1 < t_2$  are neighboring points from  $Z_\varphi \cap [T + 1; \infty)$  then, according to above assumption, there is a unique  $z \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$ , and hence  $x_\varphi(t)$

changes its sign at  $z$ . Let us suppose, for example, that  $z \in Z_\varphi^+$  and  $x_\varphi(z)$  change its sign at some point from plus to minus. Hence  $\dot{x}(z)$  is negative. This means that  $x_\varphi(z-1)$  is positive. This is possible only in case when the number of switches is even.

**Proof of Theorem 95.** In the case  $N = 0$ , the desired statement is obvious. Fix even  $N > 0$ . Put

$$\Sigma = \{(a_0, \dots, a_N) \in R^{N+1} : a_0 \geq 0, \dots, a_N \geq 0, a_0 + \dots + a_N = 1\}$$

Let  $Z_\varphi \cap [T; +\infty)$  be locally finite, and

$$T = t_1 < t_2 < t_3 < \dots$$

be all zeroes of  $x_\varphi(t)$  in  $[T; +\infty)$ . Let us define the operators of "step forward" and "step back". Assume that  $\nu_\varphi(t_k) = \nu_\varphi(t_{k+1}) = N$ . Define the following vectors of sign changes:  $\bar{a} = (a_0, \dots, a_N), \bar{b} = (b_0, \dots, b_N) \in \Sigma$ , where

$$a_0 = t_k - t_{k-1}, \quad a_1 = t_{k-1} - t_{k-2}, \dots, a_{N-1} = t_{k-N+1} - t_{k-N}$$

$$a_N = t_{k-N} - (t_k - 1)$$

$$b_0 = t_{k+1} - t_k, b_1 = t_k - t_{k-1}, \dots, b_{N-1} = t_{k-N+2} - t_{k-N+1}$$

$$b_N = t_{k-N+1} - (t_{k+1} - 1)$$

Thus we obtain a correspondence

$$\Gamma : (\bar{a}, \alpha, \varepsilon) \rightarrow (\bar{b}, \beta, -\varepsilon)$$

where  $\alpha = t_k, \beta = t_{k+1}, \varepsilon = \text{sign } \dot{x}_\varphi(t_k)$

**Proposition 108** *For a fixed  $\varepsilon$ , the correspondence inverse to  $\Gamma$ , is a smooth map*

$$M_\varepsilon : \Sigma \times R \rightarrow \Sigma \times R$$

**Proof.** Denote by  $x_\varepsilon(t_0, x_0, a)$ ,  $\varepsilon = \pm 1$ , the solution of the Cauchy problem

$$\frac{dx}{da} = \varepsilon + F(x, t_0 + a), \quad x(0) = x_0$$

Define functions  $T = \lambda_\varepsilon(t, a)$ ,  $\varepsilon = \pm 1$ , by equations

$$x_{-\varepsilon}(t+a, x_\varepsilon(t, 0, a), b) = 0, \quad T = t + a + b \quad (10.20)$$

It is easy to see that for a fixed  $t_0$ , the function  $\lambda_\pm(t_0, a)$  increases strictly, and  $\lambda_\pm(t_0, a) > a$  if  $a > 0$ . Therefore, for a fixed  $t_0$ , we can define positive functions of  $b > 0$ :

- $\rho_\varepsilon(t_0, b)$  inverse to  $b = \lambda_\varepsilon(t_0, \rho_\varepsilon)$ ; and
- $\sigma_\varepsilon(t_0, b) = b - \rho_\varepsilon(t_0, b)$ .

Thus  $(\bar{a}, \alpha) = M_\varepsilon(\bar{b}, \beta)$  can be defined as

$$\begin{aligned} a_0 &= b_1, & a_1 &= b_2, \dots, a_{N-2} = b_{N-1} \\ a_{N-1} &= b_N + \sigma_\varepsilon(\beta - b_0, b_0), & a_N &= \rho_\varepsilon(\beta - b_0, b_0) \\ \alpha &= \beta - b_0 \end{aligned} \quad (10.21)$$

Thereby Proposition 108 defines the operator of step back with a constant frequency independently from initial assumption  $\nu_\varphi(t_k) = \nu_\varphi(t_{k+1}) = N$ .

So, given a triple  $(\bar{a}, \alpha, \varepsilon)$ , we can construct a solution of (10.1) for  $t \geq \alpha$ , and using maps  $M_\pm$  we can extend this solution on the interval  $(-\infty, \alpha)$  with a constant frequency function. Now let us introduce the decreasing sequence of closed connected sets

$$\Pi_0 = \Sigma \times R, \quad \Pi_{n+1} = (M_- M_+)(\Pi_n), n \geq 0$$

The set  $\Pi = \Pi_0 \cap \Pi_1 \cap \Pi_2 \cap \dots$  is an invariant set of operator step back. The statement of Theorem 95 is equivalent to  $\Pi \cap (\Sigma \times \{\alpha\}) \neq \emptyset$  for any  $\alpha \in R$ . It is obvious that, for any  $k > 0$ ,

$$\Pi_k \cap (\Sigma \times \{\beta\}) \neq \emptyset$$

for  $\beta$  both big and small, because the time decrease in one step is absolutely bounded. Then (10.21) is fulfilled for any  $k \geq 0, \beta \in R$ , because  $\Pi_k$ , for  $k \geq 0$ , are connected. Thus,  $\Pi \cap (\Sigma \times \{\alpha\}) \neq \emptyset$ , because  $\Pi_k \cap (\Sigma \times \{\alpha\}) \neq \emptyset$ ,  $k \geq 0$ , are nonempty compacts.

**Proof of Theorem 96.** We shall prove that, for any  $n \geq 1$  and a fixed  $T \in R$ , there is a unique  $g_{n,T} \in \mathcal{S}_n$  with property (10.4). Since  $M_\varepsilon$ , defined by (10.21), does not depend on  $\beta$ , we get a map  $M_\varepsilon : \Sigma \rightarrow \Sigma$  such that

$$\bar{a} = M_\varepsilon(\bar{b}); \quad \bar{a}, \bar{b} \in \Sigma$$

$$\begin{aligned} a_0 &= b_1, & a_1 &= b_2, \dots, a_{N-2} = b_{N-1} \\ a_{N-1} &= b_N + \sigma_\varepsilon(b_0), & a_N &= \rho_\varepsilon(b_0) \end{aligned} \quad (10.22)$$

where  $N = 2n$  and according to the definition of  $\rho_\varepsilon, \sigma_\varepsilon$  (see Proposition 108) and (10.2)

$$\frac{1-p}{2} \leq \rho'_\varepsilon(b) \leq \frac{1+p}{2}, \quad \frac{1-p}{2} \leq \sigma'_\varepsilon(b) \leq \frac{1+p}{2} \quad (10.23)$$

We have to show that the intersection of a decreasing sequence of compacts

$$(M_- \circ M_+)^k(\Sigma), \quad k \geq 0$$

is one point.

**Proposition 109** *For the metric*

$$\|\bar{a} - \bar{b}\| = \sum_{i=0}^N |a_i - b_i|$$

the operator

$$M = (M_- \circ M_+)^{N^2-1} : \Sigma \longrightarrow \Sigma$$

is a contraction with a coefficient  $1 - \gamma$ , where

$$\gamma = \frac{1}{N} \left( \frac{1-p}{2} \right)^{N^2-1}$$

**Proof.** If  $\bar{a}, \bar{b} \in \Sigma$  then the vector  $\bar{a} - \bar{b}$  has at least one pair of coordinates with different signs. Let

$$a_j - b_j = \max_i \{a_i - b_i\} > 0, \quad a_k - b_k = \min_i \{a_i - b_i\} < 0$$

It is easy to see that

$$a_j - b_j \geq \frac{\|\bar{a} - \bar{b}\|}{2N}, \quad b_k - a_k \geq \frac{\|\bar{a} - \bar{b}\|}{2N}$$

According to (10.22),  $\bar{c} = M_\varepsilon(\bar{a}) - M_\varepsilon(\bar{b})$  can be defined by

$$c_0 = \rho'_\varepsilon(\theta) \cdot (a_0 - b_0), c_1 = a_1 - b_1, \dots, c_{N-1} = a_{N-1} - b_{N-1}$$

$$c_N = a_N - b_N + \sigma'_\varepsilon(\theta) \cdot (a_0 - b_0)$$

Thus, the transformation  $\bar{a} - \bar{b} \mapsto \bar{c}$  can be described as a multiplication by a matrix  $\{\alpha_{ij}\}$  (depending on  $\bar{a}, \bar{b}$ ), where according to (10.23)

$$\alpha_{ij} \geq 0, \quad 0 \leq i, j \leq N \quad \sum_{i=0}^N \alpha_{ij} = 1, \quad j = 0, \dots, N \quad (10.24)$$

with

$$\min\{\alpha_{ij} : \alpha_{ij} > 0\} \geq \frac{1-p}{2} \quad (10.25)$$

A product of matrices of type (10.24) is of the same type. Also it is not difficult to see that the product of  $N + 1$  matrices of type (10.24) does not contain zeroes on the principal diagonal and on the next upper diagonal. Hence the product of the  $N^2 - 1$  matrices of type (10.24) contains the first string with, by (10.25),

$$\min_{k=0, \dots, N} \{m_{0k}\} \geq \left( \frac{1-p}{2} \right)^{N^2-1} = N\gamma$$

This implies immediately that

$$\begin{aligned}
\|M(\bar{a}) - M(\bar{b})\| &= \sum_{i=0}^N \left| \sum_{q=0}^N m_{iq}(a_q - b_q) \right| \leq \\
&\leq \left( \sum_{q=0}^N m_{0q}|a_q - b_q| - 2N\gamma \cdot \frac{\|\bar{a} - \bar{b}\|}{2N} \right) + \sum_{i=1}^N \sum_{q=0}^N m_{iq}\|a_q - b_q\| \leq \\
&< \|\bar{a} - \bar{b}\| - \gamma \cdot \|\bar{a} - \bar{b}\| = (1 - \gamma)\|\bar{a} - \bar{b}\|
\end{aligned}$$

This uniqueness and the autonomy imply the equality  $g_{n,T}(t) = g_{n,0}(t-T)$ ,  $t, T \in R$ , as well as the periodicity of  $g_{n,0}$ . Inequalities (10.5) follow from the frequency of  $g_{n,0}$  which is equal to  $2n$ .

**Proof of Theorem 98.** It is easy to deduce from the proof of Proposition 108 that every solution  $g(t)$ ,  $t \geq T$ , of (10.1) with a constant finite frequency can be extended on  $[-1; \infty)$  with the same frequency. That finishes the proof according to Lemmas 91, 93, and Theorem 95.

**Proof of Theorem 99 and Corollary 100.** The set  $\tilde{\mathcal{U}}_0$  is nonempty, because it contains  $\mathcal{S}_0 \neq \emptyset$ . Now let  $\varphi \in \mathcal{U}_0$ , and  $\text{mes}(\varphi^{-1}(0)) = 0$ . Then  $x_\varphi(t) = g_{0,T}(t)$ ,  $t \geq T$ , for a relevant  $T \in R$ . That means

$$x_\varphi(T) = 0, \quad \dot{x}_\varphi(t) > 0, \quad t \in \left( T; T + \frac{2}{1+p} \right)$$

If  $\psi \in C[-1; 0]$  is close to  $\varphi$ , then  $\psi^{-1}(0)$  is contained in a sufficiently small neighborhood of  $\varphi^{-1}(0)$ , and

$$\text{mes}(\{\varphi > 0\} \circ \{\psi > 0\}), \quad \text{mes}(\{\varphi < 0\} \circ \{\psi < 0\})$$

are small enough, where  $A \circ B$  denotes  $(A \setminus B) \cup (B \setminus A)$ . Hence  $Z_\psi \cap [0; T+2]$  is contained in a sufficiently small neighborhood of  $Z_\varphi \cap [0; T+2]$ . Therefore

$$x_\psi(t) > 0, \quad t \in \left( T + \delta; T + \frac{2}{1+p} - \delta \right), \quad 2\delta < \frac{2}{1+p} - 1$$

and implies  $\psi \in \mathcal{U}_0$ . The statement of Corollary 100 follows from this immediately.

**Proof of Theorem 101.** Let  $\varphi \in \tilde{\mathcal{U}}_0$ , and  $x_\varphi(t) = g_{0\alpha}(t)$ ,  $t \geq T$ . We have just shown that if  $\psi$  is sufficiently close to  $\varphi$  then  $x_\psi(t) = g_{0\beta}(t)$ ,  $t \geq T$ , where  $|\beta - \alpha|$  is small enough. Let

$$\alpha = t_1 < t_2 < \dots, \quad \beta = t'_1 < t'_2 < \dots$$

be all zeroes of the functions  $g_{0\alpha}, g_{0\beta}$ , respectively, in the interval  $[T; \infty)$ . It is enough to prove that

$$C_1 \cdot |\beta - \alpha| < |t_k - t'_k| < C_2 \cdot |\beta - \alpha|, \quad C_1, C_2 = \text{const}, \quad k = 1, 2, \dots$$

According to the definition of the functions  $\lambda_{\pm}(t_0, a)$

$$t_{k+1} = \lambda_{\pm}(t_k, 1), \quad t'_{k+1} = \lambda_{\pm}(t'_k, 1)$$

Thus

$$t'_{k+1} - t_{k+1} = \frac{\partial \lambda_{\pm}(\theta_k, 1)}{\partial t} \cdot (t'_k - t_k), \quad |\theta_k - t_k| < |t'_k - t_k|, \quad k \geq 1$$

$$t'_n - t_n = \prod_{k=1}^{n-1} \frac{\partial \lambda_{\pm}(\theta_k, 1)}{\partial t} \cdot (\beta - \alpha)$$

The desired statement follows from:

**Proposition 110** *Under condition (10.6), the product*

$$\prod_{k=1}^{\infty} \frac{\partial \lambda_{\pm}}{\partial t}(\theta_k, 1)$$

*converges uniformly when*

$$\theta_{k+1} \geq \theta_k + 1, \quad k = 1, 2, 3, \dots \quad (10.26)$$

**Proof.** We will show that the series

$$\sum_{k=1}^{\infty} \left( \frac{\partial \lambda_{\pm}}{\partial t}(\theta_k, 1) - 1 \right)$$

converges uniformly. Put

$$\mu(t) = \max_x \left| \frac{\partial F}{\partial t}(x, t) \right|, \quad t \geq 0$$

It follows from (10.20) and well-known formulae for the derivatives of solutions with respect to initial data that

$$\begin{aligned} \frac{\partial \lambda_{\varepsilon}}{\partial t}(t, a) &= 1 - (-\varepsilon + F(0, \tau))^{-1} \cdot \exp \int_{t+a}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt \\ &\times \left( \int_{t+a}^{\tau} \frac{\partial F}{\partial t}(x_{-\varepsilon}, t) dt + \int_t^{t+a} \frac{\partial F}{\partial t}(x_{\varepsilon}, t) dt \cdot \exp \int_t^{t+a} \frac{\partial F}{\partial x}(x_{\varepsilon}, t) dt \right) \end{aligned}$$

where  $\tau = \lambda_\varepsilon(t, a)$ . Thus,

$$\begin{aligned} \left| \frac{\partial \lambda_\varepsilon}{\partial t}(\theta, 1) - 1 \right| &\leq \frac{1}{1-p} \exp \int_{\theta+1}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt \\ &\times \left( \int_{\theta+1}^{\tau} \mu(t) dt + \int_{\theta}^{\theta+1} \mu(t) dt \cdot \exp \int_{\theta}^{\theta+1} \frac{\partial F}{\partial x}(x_\varepsilon, t) dt \right) \end{aligned} \quad (10.27)$$

According to (10.26), one may admit

$$\theta \gg 0, \quad \int_{\theta}^{\infty} \mu(t) dt \leq 1$$

Then

$$\begin{aligned} \int_{\theta+1}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt &= \int_{\theta+1}^{\tau} \frac{dF}{dt} \cdot (-\varepsilon + F(x_{-\varepsilon}, t))^{-1} dt \\ - \int_{\theta+1}^{\tau} \frac{\partial F}{\partial t} \cdot (-\varepsilon + F(x_{-\varepsilon}, t))^{-1} dt &\leq \log \frac{1+p}{1-p} + \frac{1}{1-p} \cdot \int_{\theta+1}^{\tau} \mu(t) dt \\ &\leq \log \frac{1+p}{1-p} + \frac{1}{1-p} \\ \int_{\theta}^{\theta+1} \frac{\partial F}{\partial x}(x_\varepsilon, t) dt &\leq \log \frac{1+p}{1-p} + \frac{1}{1-p} \end{aligned}$$

Put  $q = \exp(2p+1/(1-p))$ ,  $N = [(1+p)/(1-p)] + 1$ . Then (10.27) implies

$$\begin{aligned} \left| \frac{\partial \lambda_\varepsilon}{\partial t}(\theta, 1) - 1 \right| &\leq \frac{q^2}{1-p} \int_{\theta}^{\tau} \mu(t) dt \\ \sum_{\theta_i > \theta} \left| \frac{\partial \lambda_{\pm}}{\partial t}(\theta_i, 1) - 1 \right| &\leq \frac{q^2 N}{1-p} \int_{\theta}^{\infty} \mu(t) dt \xrightarrow{\theta \rightarrow \infty} 0 \end{aligned}$$

because  $\tau \leq \theta + (1+p)/(1-p)$  according to (10.2), and that completes the proof of Theorem 101.

**Proof of Theorem 102.** We shall use the two following propositions.

**Proposition 111** *If*

$$a \leq \frac{1+p}{2} \quad (10.28)$$

*and one of (10.7) or (10.8) is fulfilled, then*

$$\frac{\partial \lambda_\varepsilon}{\partial a}(t, a) \geq q > 1, \quad \varepsilon = \pm 1 \quad (10.29)$$

**Proof.** It is not difficult to derive that

$$\begin{aligned} \frac{\partial \lambda_\varepsilon}{\partial a}(t, a) &= 1 + (1 - \varepsilon F(0, T))^{-1} \exp \int_{t+a}^T \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt \\ &\times \left( 1 + \varepsilon F(x_\varepsilon(t, 0, t+a), t+a) + \varepsilon \int_{t+a}^T \frac{\partial F}{\partial t}(x_{-\varepsilon}, t) dt \right) \end{aligned}$$

where  $T = \lambda_\varepsilon(t, a)$ . Therefore (10.8) implies

$$\begin{aligned} &1 + \varepsilon F(x_\varepsilon(t, 0, t+a), t+a) + \varepsilon \int_{t+a}^T \frac{\partial F}{\partial t}(x_{-\varepsilon}, t) dt \\ &> 1 - p - (T - t - a)M_t \geq 1 - p - aM_t \frac{1+p}{1-p} > 0 \\ &\int_{t+a}^T \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt = \int_{t+a}^T \left( \frac{dF}{dt} - \frac{\partial F}{\partial t} \right) \cdot (\dot{x}_{-\varepsilon})^{-1} dt \\ &= \int_{t+a}^T \frac{dF}{dt} \cdot (-\varepsilon + F(x_{-\varepsilon}, t))^{-1} dt - \int_{t+a}^T \frac{\partial F}{\partial t} \cdot (-\varepsilon + F(x_{-\varepsilon}, t))^{-1} dt \\ &\geq -\log \frac{1+p}{1-p} - M_t \frac{T-t-a}{1-p} \geq -\log \frac{1+p}{1-p} - M_t a \frac{1+p}{(1-p)^2} \\ &\geq -\log \frac{1+p}{1-p} - M_t \frac{(1+p)^2}{2(1-p)^2} \end{aligned}$$

and that implies (10.29). Analogously (10.7) implies

$$\begin{aligned} &1 + \varepsilon F(x_\varepsilon(t, 0, t+a), t+a) + \varepsilon \int_{t+a}^T \frac{\partial F}{\partial t}(x_{-\varepsilon}, t) dt \\ &= 1 + \varepsilon F(x_\varepsilon(t, 0, t+a), t+a) + \varepsilon \int_{t+a}^T \frac{dF}{dt} dt - \varepsilon \int_{t+a}^T \frac{\partial F}{\partial x} \cdot \dot{x}_{-\varepsilon} dt \\ &\geq 1 + \varepsilon \cdot F(0, T) - M_x(1+p)(T-t-a) \geq 1 - p - M_x a \frac{(1+p)^2}{1-p} > 0 \\ &\int_{t_0}^{t_0+a} \frac{\partial F}{\partial x}(x_\varepsilon, t) dt \geq -M_x a \geq -M_x(1+p)/2 \end{aligned}$$

and that implies (10.29).

**Proposition 112** *Under conditions of Theorem 102 the measure of the set  $\Pi$  from the proof of Theorem 95 is zero.*

**Proof.** First we show that any  $\bar{a} = (a_0, \dots, a_N) = M_\varepsilon(\bar{b}), \bar{b} \in \Sigma$ , satisfies  $a_N \leq (1+p)/2$ . Indeed, we have  $a_N \leq a_{N-1}(1+p)/(1-p)$ , that implies the above inequality.

Now from (10.21) the Jacobian  $|M'_\varepsilon|$  of the map  $M_\varepsilon$  is equal to

$$\left. \frac{\partial \rho_\varepsilon}{\partial b}(t, b) \right|_{t=\alpha, b=b_0} = \left( \left. \frac{\partial \lambda_\varepsilon}{\partial a}(t, a) \right|_{t=\alpha, a=a_N} \right)^{-1} \leq \frac{1}{q} < 1$$

according to Proposition 111. Then

$$|(M_- \circ M_+)'| \leq q^{-2} < 1 \quad (10.30)$$

Fix  $A \in R$  and  $T > A$ . Then

$$\Pi \cap (\Sigma \times (-\infty; A]) \subset \bigcup_{k \geq n} (M_- \circ M_+)^k (\Sigma \times [T; T+1])$$

where  $n$  might be chosen big enough, because  $T > A$  arbitrarily. Thus, we obtain from (10.30)

$$\text{mes}(\Pi \cap (\Sigma \times (-\infty; A])) \leq q^{-2(n-1)} \frac{\text{mes}(\Sigma)}{q^2 - 1} \xrightarrow{n \rightarrow \infty} 0$$

and that completes the proof.

Now we can finish the proof of Theorem 102. Now fix  $\varphi \in \mathcal{U}_n$  and a neighborhood  $V$  of  $\varphi$  in  $C[-1; 0]$ . The set  $\mathcal{F}$  is dense in  $C[-1; 0]$ , evidently. Put

$$m = \min\{k : \mathcal{F} \cap \mathcal{U}_k \cap V \neq \emptyset\}$$

Assume  $m \geq 1$ , and  $\psi \in \mathcal{F} \cap \mathcal{U}_m \cap V$ . Then there is  $\xi \in \mathcal{S}_m$  such that  $x_\psi(t) = \xi(t)$ ,  $t \geq T$ ,  $\xi(T) = 0$ . Let  $2k$  be a number of sign changes of  $\psi$  in  $[-1; 0]$ , and  $\bar{a} \in \Sigma_k \subset R^{2k+1}$  be a vector of sign changes of  $\psi$ , constructed as in the proof of Theorem 95, as well as  $\bar{b} \in \Sigma_m \subset R^{2m+1}$  be a vector of sign changes of  $\xi$  in  $(T-1; T)$ . Suppose  $\bar{c} \in \Sigma_t, \bar{d} \in \Sigma_s$  are vectors of sign changes of  $x_\psi(t)$  in intervals  $(t_n - 1; t_n)$  and  $(t_{n+1} - 1; t_{n+1})$ , respectively. If  $r = s$  then, according to the proof of Theorem 95, the Equation (10.1) generates a diffeomorphism of neighborhoods of  $(\bar{c}, t_n), (\bar{d}, t_{n+1})$  in  $\Sigma_r \times R$ . If  $r < s$  then it is possible to deduce, following arguments from the proof of Theorem 95,

$$c_0 = d_1, \dots, c_{2s-1} = d_{2s}, \quad c_{2r} = \lambda(d_0, c_{2s}, \dots, c_{2r-2}, t_{n+1})$$

$$c_{2r-1} = 1 - c_0 - \dots - c_{2r-2} - c_{2r}, \quad t_n = t_{n+1} - d_0$$

where  $\lambda$  is some smooth function. Thus an inverse image of  $(\bar{d}, t_{n+1})$  in a neighborhood of  $(\bar{c}, t_n)$  in  $\Sigma_r \times R$  has the codimension  $2s + 1$ . That

implies the measure of an inverse image of  $\Pi \cap (\Sigma_m \times R)$  in  $\Sigma_k \times R$  is zero. Therefore, after a suitable small variation of  $(\bar{a}, 0)$  in  $\Sigma_k \times R$  an image of  $(\bar{a}, 0)$  in  $\Sigma_m \times R$  leaves  $\Pi$ , i.e. a limit frequency of the changed solution is less than  $2m$ , which contradicts the definition of  $m$ , and hence to our assumption  $m > 0$ .

Thus, we get that  $\mathcal{U}_0 \cap \mathcal{F}$  is dense in  $\mathcal{F}$ , and also in  $C[-1; 0]$ , because  $\mathcal{F}$  is dense in  $C[-1; 0]$ . According to Theorem 99, it means that  $\mathcal{U}_\infty \cup \bigcup_{k \geq 1} \mathcal{U}_k$  is dense nowhere in  $C[-1; 0]$ .

**Proof of Lemma 106.** From (10.15) we deduce that

$$\frac{F(x, t)}{x} \leq k, x \neq 0 \quad (10.31)$$

In particular, that means if  $x(t)$  is a solution of (10.1), then for  $x(T) \geq 0, x(t) \leq \omega(t), t \geq T$ , where  $\omega(t) = ((\alpha + kx(T))e^{k(t-T)} - \alpha)/k$  is the solution of Cauchy problem

$$\dot{\omega}(t) = \alpha + k\omega(t), \quad \omega(T) = x(T)$$

and, for  $x(T) \leq 0, x(t) \geq \omega(t), t \geq T$ , where  $\omega(t) = [(-\alpha + kx(T))e^{k(t-T)} + \alpha]/k$  is the solution of Cauchy problem

$$\dot{\omega}(t) = -\alpha + k\omega(t), \quad t \geq T$$

Those inequalities and (10.31) imply that  $|F(x, t, t)| < \alpha$  when  $t \in [0, 1]$  and  $x(0) = \varphi(0)$  satisfies (10.16), and that  $x(t)$  satisfies (10.17) when  $t \in [T, T + 1], x(T) = 0$ , and secondly,  $x(t)$  does not leave the strip  $|x| \leq \alpha(e^k - 1)/k$  for  $t \leq T$ .

**Proof of Theorem 103.** Let us study the point mapping  $\Phi(z, x, \mu)$  of the switching surface  $s = 0$  into itself induced by the full-order system (10.9). First we show that under the conditions of Theorem 103 there exists a neighborhood of the point  $(\varphi(0, x_0, 1), 0, x_0)$  in the  $z, x$  space on the surface  $s = 0$  mapped into itself.

It follows from the continuous dependence of the solutions to differential equations on the parameters and initial conditions that there exists  $\bar{U}(\alpha)$ , the closed ball with the center at the point  $x_0$  and radius  $\alpha$  on the surface  $s = 0$  in the  $x$  space such that for some  $q'$  for all  $x' \in U(\alpha)$

- the point  $\varphi(s_0^-(1), x_0^-(1), -1)$  is situated in the interior of the attractive domain of the equilibrium point  $\varphi(s_0^-(1), x_0^-(1), 1)$ , where  $[\bar{s}_0^-(t), \bar{x}_0^-(t)]$  is the solution of system (10.10) for  $u = -1$  with the initial conditions  $\bar{s}_0^-(0) = 0, \bar{x}_0^-(0) = x^0, \bar{s}_0^-(t) < 0, t \in [-1, 0]$ ;
- there exists the smallest root  $\theta^0$  of the equation  $\bar{s}_0^+(\theta^0) = 0$  such that  $d\bar{s}_0^+(\theta^0)/dt < 0$ ; here  $(\bar{s}_0^+(t), \bar{x}_0^+(t))$  is the solution of system (10.10) for  $u = 1$  with the initial conditions  $\bar{s}_0^+(1) = \bar{s}_0^-(1), \bar{x}_0^+(1) = \bar{x}_0^-(1)$ ;

- the point  $\varphi(s_0^+(\theta_0 + 1), x_0^+(\theta_0 + 1), 1)$  is situated in the interior of the attractive domain of the equilibrium point  $\varphi(s_0^+(\theta_0 + 1), x_0^+(\theta_0 + 1), -1)$ ;
- there exists the smallest root  $T(x^0)$  of the equation  $\bar{s}_0^-(T(x^0)) = 0$  such that  $T(x^0) > \theta + 1$ ,  $d\bar{s}_0^-(T(x^0))/dt > 0$ ; here  $(s_0^-(t), x_0^-(t))$  is the solution of system (10.10) with  $u = -1$  and initial conditions  $[s_0^-(\theta_0 + 1), x_0^-(\theta_0 + 1)] = [s_0^+(\theta_0 + 1), x_0^+(\theta_0 + 1)]$ ; and
- $\|\partial\Psi(x^0)/\partial x\| < q' < 1$ .

Consider the set  $A = \text{co}[\varphi(0, \bar{U}(\alpha), -1)] \times \bar{U}(\alpha)$  and arbitrary  $(z^0, x^0) \in A$ . Then according to Tichonov's theorem [34] and the implicit function theorem, there exists  $\mu(z^0, x^0)$  such that, for all  $\mu \in [0, \mu(z^0, x^0)]$ ,

- there exists the unique solution  $[z^-(t, \mu), s^-(t, \mu), x^-(t, \mu)]$  of system (10.9) for  $u = -1$  on  $[0, 1]$  with the initial conditions
 
$$z^-(0, \mu) = z^0, s^-(0, \mu) = 0, x^-(0, \mu) = x^0, s^-(t, \mu) < 0, t \in [-1, 0];$$
- the point  $z^-(t, \mu)$  is situated in the interior of the attractive domain of the equilibrium point  $\varphi[s_0^-(1), x_0^-(1), 1]$
- there exists the smallest root  $\theta(\mu, z^0, x^0)$  of the following relations

$$s^+(\theta(\mu, z^0, x^0), \mu) = 0, \quad \frac{ds^+(\theta(\mu, z^0, x^0), \mu)}{dt} < 0$$

where  $[z^+(t, \mu), s^+(t, \mu), x^+(t, \mu)]$  is the solution of system (10.9) for  $u = 1$  with the initial conditions

$$z^+(1, \mu) = z^-(1, \mu), \quad s^+(1, \mu) = s^-(1, \mu), \quad x^+(1, \mu) = x^-(1, \mu);$$

- the point  $z^+(\theta(\mu, z^0, x^0) + 1, \mu)$  is situated in the interior of the attractive domain of the equilibrium point  $\varphi(\bar{s}_0^+(\theta_0 + 1), \bar{x}_0^+(\theta_0 + 1), -1)$ ;
- there exists the smallest root  $T(\mu, z^0, x^0)$  of the following relations

$$s^-(T(\mu, z^0, x^0), \mu) = 0, \quad \frac{ds^-(T(\mu, z^0, x^0), \mu)}{dt} > 0$$

$$T(\mu, z^0, x^0) > \theta(\mu, z^0, x^0) + 1$$

where  $[z^-(t, \mu), s^-(t, \mu), x^-(t, \mu)]$  is the solution of system (10.9) with  $u = -1$  and the initial conditions

$$\begin{aligned} & (z^-(\theta(\mu, z^0, x^0) + 1, \mu), s^-(\theta(\mu, z^0, x^0) + 1, \mu), x^-(\theta(\mu, z^0, x^0) + 1, \mu)) \\ & = (z^+(\theta(\mu, z^0, x^0) + 1, \mu), s^+(\theta(\mu, z^0, x^0) + 1, \mu), x^+(\theta(\mu, z^0, x^0) + 1, \mu)); \end{aligned}$$

and

- at last,

$$(z^-(T(\mu, z^0, x^0), \mu), x^-(T(\mu, z^0, x^0), \mu)) \in \\ \{\varphi(0, \bar{U}((1+q')\alpha/2), -1), \bar{U}((1+q')\alpha/2)\} \subset A$$

This means that the image of the set  $A$  by the point mapping

$$\Phi(z^0, x^0, \mu) = [\Phi_1(z^0, x^0, \mu), \Phi_2(z^0, x^0, \mu)] \\ = [z^-(T(\mu, z^0, x^0), \mu), x^-(T(\mu, z^0, x^0), \mu)]$$

induced by system (10.9) for all  $\mu \in [0, \mu(z^0, x^0)]$  is a subset of the interior of  $A$ . Moreover, for all  $\mu \in [0, \mu(z^0, x^0)]$ ,

$$\Phi(z^0, x^0, 0) = \lim_{\mu \rightarrow 0} \Phi(z^0, x^0, \mu) = \{\varphi(0, x^-[T(x^0)], -1), x^-[T(x^0)]\} \quad (10.32)$$

and

$$\Phi(\varphi(0, x_0, -1), x_0, 0) = (\varphi(0, x_0, -1), x_0)$$

This means that the point mapping  $\Phi$  is continuous on  $A \times [0, \mu']$ ,  $\mu' > 0$ , and at all  $\mu \in [0, \mu']$  and has a fixed point which corresponds to a periodic solution of system (10.9) close to  $[s_0(t), x_0(t)]$ . Let us show that this periodic solution is stable and unique. The derivative of the point mapping  $\Phi$  is a smooth function of the derivatives of the functions  $\theta(\mu, x^0, z^0)$ ,  $T(\mu, x^0, z^0)$ ,  $z^-(1, \mu)$ ,  $x^-(1, \mu)$ ,  $z^-(T(\mu, x^0, z^0), \mu)$ ,  $x^-(T(\mu, x^0, z^0), \mu)$ ,  $z^+(\theta(\mu, x^0, z^0) + 1, \mu)$  and  $x^+(\theta(\mu, x^0, z^0) + 1, \mu)$ , thus the derivatives of  $\Phi$  exist and they are continuous.

Let us consider the new variable  $\eta = z^0 - \varphi(0, x^-(T(x^0)), -1)$  and the auxiliary point mapping

$$\Xi(\eta, x^0, \mu) = [\Xi_1(\eta, x^0, \mu), \Xi_2(\eta, x^0, \mu)] \\ = (\Phi_1(\eta + \varphi(0, x^-[T(x^0)], -1), x^0, \mu) - \varphi(0, x^-[T(x^0)], -1), \\ \Phi_2[\eta + \varphi(0, x^-[T(x^0)], -1), x^0, \mu]).$$

The point  $(0, x^0)$  is a fixed point of  $\Xi$  for  $\mu = 0$ . For a sufficiently small  $\mu > 0$ , the point mapping takes the set

$$B(\beta, \alpha, \mu^n) = \{(\eta, x, \mu) : \|\eta\| \leq \beta, x \in \bar{U}(\alpha), \mu \in [0, \mu^n]\}$$

into itself.

It follows from (10.32) that the value of  $\Xi(\eta, x^0, 0)$  does not depend on  $\eta$ . Then

$$\frac{\partial \Xi}{\partial(\eta, x)} = \begin{pmatrix} O(\mu) & O(\mu) \\ O(\mu) & \partial \Psi / \partial x(x_0) + O(\mu) \end{pmatrix}$$

This means that for some  $q_1 < 1$

$$\sup_{B(\alpha, \beta, \mu'')} \left\| \frac{\partial \Xi}{\partial(\eta, x)} \right\| < q_1 < 1$$

Consequently the point mapping  $\Xi$  is a contraction and  $\Phi$  has a unique fixed point, thus the desired periodic solution of system (10.9) is unique and orbitally asymptotically stable.

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# Chapter 11

## Sliding Mode Control for Systems with Time Delay

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### 11.1 Introduction

Sliding mode control has a deep historical background: one of the reasons is that many physical systems naturally present discontinuity in their dynamics, as mechanical systems with Coulomb friction [49] or electrical systems with ideal relays. This has led control theorists to begin (mostly in Eastern countries) with the study of relay-based control systems. This kind of research was the starting point of the variable structure system theory. In particular, the *sliding mode control* (SMC) approach [49] provides an efficient way to tackle challenging robust stabilization problems<sup>1</sup> for finite-dimensional dynamic systems (then, without delay). For instance, it is known that if a complex system can be stated with a normal form (see [19, 29]) as equation (11.1), then an appropriate sliding mode strategy

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<sup>1</sup>In variable structure systems, the control commutates between  $d$  different values in order to force the system flow to behave as “a nonsmooth contracting map”, which means the motions converge to the origin with some discontinuity in the time-derivatives of the state variables. In the development of sliding mode control, which is a particular case of variable structure system control ( $d = 2$ ), many authors (see Andronov [5]) introduced in the switching device some nonlinear terms depending on a small parameter  $\varepsilon$  in order to obtain *real* qualitative behavior. Then, one makes  $\varepsilon$  tend to zero in order to derive results in sliding regime, viewed as an *ideal* behavior (see [3, 4]). Based on such theory, many different control schemes have been developed (see [10, 16, 17, 36, 42, 43, 48, 49]).

can achieve stabilization because the nonlinear terms are “dominated” (see [7, 46, 47]). We shall refer to such a form in this chapter.

However, SMC should also be considered for *systems with aftereffect*<sup>2</sup>. Time delays are natural phenomena in numerous engineering devices [30, 31] and the modeling phase cannot neglect them when aiming for increased dynamic performance. Delays still constitute a classical source of control problems: they are reputed to cause oscillations and to deteriorate the stability of feedback systems<sup>3</sup>. Consequently, specific models, analysis and controllers (see a survey in [44]) must take into account the *infinite dimensional* nature of such systems. Even for linear models, the design of controllers is not obvious, mainly because applying the existing necessary and sufficient stability conditions is very tricky.

Concerning robust stabilization of linear time-delay systems with either constant or time-varying *parameter uncertainties*, the methods are mainly based on the time-domain of Krasovskii’s approach (the results are then expressed in terms of Riccati equations [15, 33] or, equivalently, of LMIs [15, 32]) or on the comparison approach (results in terms of matrix norms and measures [15, 27]). Both allow one to deal with time-varying delays, whereas the frequency-domain and complex-plane methods (generally leading to diophantine polynomial equations) need the delays to be constant. The resulting control laws are of the continuous (often memoryless) feedback type.

The results concerning robustness with respect to *external disturbances* rely on  $H_\infty$  design (also leading to Riccati equations and LMIs [35, 38]), or on generalizations of the structural approaches, such as disturbance decoupling using models over rings [11]. In the first case, all results exclude input delays and many of them even need the stability of the open loop; in the second case, the corresponding control laws can be more complex and powerful, but parametric robustness or practical realization still needs to be studied. Thus, regarding its properties in finite dimension, sliding mode control appears as an attractive alternative.

Although the SMC has been extended to infinite-dimensional systems [40, 41], the combination of delay phenomenon with relay actuators makes the situation much more complex [23] and the concrete results are scarce [2, 8, 13, 25, 26].

Section 11.2 investigates a case study of sliding mode under such *relay-delay* effects. In [21, 22, 23]<sup>4</sup>, the question of the periods of induced oscillation was studied for first and second order systems. Here, conditions for the estimation of amplitude of oscillating solutions will be given for more

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<sup>2</sup>This means one must take into account an irreducible influence of the past.

<sup>3</sup>In some particular cases, they may improve it (see [1]; also recalled in [44]).

<sup>4</sup>See also Chapter 10 by Fridman in this book.

general orders. Moreover, through an example, simulations confirm the obtained results and suggest that some bifurcations can occur.

In a more general study, the third section develops sliding mode controllers for linear systems with state delay but with instantaneous input effect<sup>5</sup>. This very basic idea uses the classical “small values” principle: if a system without delay is asymptotically stable, then the system stability is still in force for a *sufficiently small* delay [24]. This qualitative result, of course, is to be completed with quantitative computation for *admissible delays*. Subsection 11.3.1 provides a straightforward extension of regular form-based algorithms [36, 43] for the sliding mode control of linear systems with delay. Along with the result given in [24], subsection 11.3.2 provides complementary upper bounds of admissible delays (i.e., keeping the stability property). Subsection 11.3.3 applies these modeling and stability results to the control, leading to two discontinuous, unit stabilizing controllers. The first controller, with constant gain, may yield some undesired dynamics caused by some chattering, whereas the second nonlinear-gain controller reduces this phenomenon. The proposed control laws are shown to stabilize the system for all values of the delay that do not exceed an explicitly calculated upper bound. The general design procedure is summarized as follows:

- 1) The original system is transformed into a regular form (i.e., a two-subsystem decomposition).
- 2) The delay is temporarily neglected and a stabilizing feedback gain is constructed for a subsystem.
- 3) This gain, in turn, determines a discontinuity sliding manifold that leads to the sliding mode controller for the overall system with delay. Both constant-gain and nonlinear-gain controllers are shown to impose useful robustness properties to the closed-loop system.

Then, in Section 11.4, the results are extended to the problem of linear systems with input delay. Such a case is often more realistic, since many actuators or sensors introduce time lags in the feedback loops. Note that a mixed case involving both instantaneous and delayed inputs could be considered as well, with some stronger hypothesis. For the sake of conciseness, it will be omitted as well as the case of multiple delays.

A simulation example concludes the chapter, illustrating the effectiveness of the proposed method. The sliding mode strategy is straightforward, easy to implement and of standard complexity since the stabilization problem has been reduced to two subproblems of lower dimensions: the first problem is to design a linear manifold of dimension  $n - m$  and the second

---

<sup>5</sup>such “inner delay” phenomenon appears in several cases as in chemical transformations (reaction lags), epidemiology (germ incubation time), population dynamics (average life duration)...

problem is to synthesize a discontinuous unit controller, which steers all the trajectories of the closed-loop system to this manifold.

## NOTATIONS

Throughout this chapter the following notations are used: the octahedral norm  $\|e\| = \sum_{i=1}^{i=n} |e_i|$  and the corresponding matrix norm  $\|A\| = \sup_{\|e\|=1} \frac{\|Ae\|}{\|e\|}$

stand for the vector  $e = (e_1, \dots, e_n)^T \in \mathbb{R}^n$  and for the matrix  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ , respectively. The vector function  $sign(s(z))$  is defined as  $sign(s(z)) = [sign(s_1(z)), \dots, sign(s_r(z))]^T$  and  $diag(\lambda_i)$  stands for the diagonal matrix  $(\lambda_{ij}) \in \mathbb{R}^{n \times n}$  with  $\lambda_{ij} = 0$  for  $i \neq j$  and  $\lambda_{ij} = \lambda_i$  for  $i = j$ .

## 11.2 SMC under delay effect: a case study

Let us consider the following system which is in regular form

$$\begin{cases} \frac{dx_i}{dt} = x_{i+1}, & \forall i = 1, \dots, (n-1) \\ \frac{dx_n}{dt} = f(t, x) + g(t, x)u \\ y = x_1 \end{cases} \quad (11.1)$$

Then, one can design a classical sliding mode controller achieving the asymptotic stability of the overall system. A practical question arises, however linked with robustness purpose: “*What are the qualitative behavioral changes of system (11.1) with a sliding mode control under delay effects?*” For instance, if the output sensors cannot provide instantaneous informations of the state, then in general

$$y(t) = h[x(t - \tau)] \quad (11.2)$$

In the following, we assume that  $\tau$  is a *constant* delay and that a reconstruction of  $x(t - \tau)$  is available via  $y(t)$ : such an estimation is possible either *via* a numeric approximation (see [6] for systems without delay) or *via* an observer for which separation principle or finite-time convergence is valid (see [9, 14] for systems without delay).

### 11.2.1 Problem formulation

In the following, we assume that in (11.1) the gain function  $g(t, x)$  is constant, equal to  $g$ <sup>6</sup> and that

$$|f(t, x)| < M_f \quad (11.3)$$

---

<sup>6</sup>This assumption of  $g(t, x)$  can be relaxed, as we shall see in Equation (11.24).

Thus, selecting a linear sliding manifold  $\mathcal{S}$  described by the equation

$$\text{manifold } \mathcal{S} : \quad s(x) = \sum_{i=1}^n a_i x_i, \quad a_n = 1 \quad (11.4)$$

with the  $a_i$  coefficients determined in such a way that  $a_0 + a_1 x + \dots + x^n$  is an Hurwitz polynomial. We apply a sliding mode control (if  $\langle \nabla s, g \rangle \neq 0$ )

$$u(t) = u_{\text{eq}}(t, x(t)) - \frac{k}{g} \text{sign}[s(t)] \quad (11.5)$$

$$u_{\text{eq}}(t, x(t)) = -\frac{1}{g} \left( \sum_{i=1}^{n-1} a_i x_{i+1}(t) + f(t, x(t)) \right) \quad (11.6)$$

so that  $\dot{s} = -k \text{sign}(s)$ , where  $k > 0$ .

Now, if the applied control is delayed, then (11.5) becomes

$$u(t) = u_{\text{eq}}(t, x(t - \tau)) - \frac{k}{g} \text{sign}[s(t - \tau)] \quad (11.7)$$

One can conjecture that motions of (11.1) with control (11.7) will present additional oscillations, the amplitude of which will increase with the delay  $\tau$ , the gain  $k$ , and the speed of change of the control near the sliding surface.

## 11.2.2 A case study

### Attractivity of a neighborhood $\mathcal{R}_\infty$ of the manifold $\mathcal{S}$

Let us consider  $V(x(t)) = \frac{1}{2} s^2[x(t)]$ , which derivative will be studied in relation with the convergence of  $x(t)$  to the manifold  $\mathcal{S}$ . The function

$$\begin{aligned} \dot{s}(t) &= \sum_{i=1}^{n-1} a_i x_{i+1}(t) - \sum_{i=1}^{n-1} a_i x_{i+1}(t - \tau) \\ &\quad + f(t, x(t)) - f[t, x(t - \tau)] \\ &\quad - k \text{sign}[s(t - \tau)] \end{aligned} \quad (11.8)$$

is Lebesgue-integrable, thus  $s(t) = s(t - \tau) + \int_{t-\tau}^t \dot{s}(w) dw$  holds. Now, using (11.6),

$$\dot{s}(t) = g \Delta_t^{(t-\tau)}(u_{\text{eq}}) - k \text{sign}[s(t - \tau)] \quad (11.9)$$

$$\Delta_t^{(t-\tau)}(u_{\text{eq}}) = \{u_{\text{eq}}[t, x(t - \tau)] - u_{\text{eq}}[t, x(t)]\} \quad (11.10)$$

$$\begin{aligned}
\dot{V}(x(t)) &= \left( s(t-\tau) + \int_{t-\tau}^t \dot{s}(w)dw \right) \times \\
&\quad \left( g\Delta_t^{(t-\tau)}(u_{\text{eq}}) - k \text{sign}\{s[x(t-\tau)]\} \right) \\
&= -k|s(t-\tau)| + g\Delta_t^{(t-\tau)}(u_{\text{eq}})s(t-\tau) \\
&\quad + k^2 \int_{t-\tau}^t \{\text{sign}[s(t-\tau)] \text{sign}[s(w-\tau)]\}dw \\
&\quad + g^2 \int_{t-\tau}^t \Delta_t^{(t-\tau)}(u_{\text{eq}})\Delta_w^{(w-\tau)}(u_{\text{eq}})dw \tag{11.11}
\end{aligned}$$

Take into account that

$$\int_{t-\tau}^t \{\text{sign}[s(t-\tau)] \text{sign}[s(w-\tau)]\}dw \leq \tau \tag{11.12}$$

and assume<sup>7</sup> that  $|\Delta_t^{(t-\tau)}(u_{\text{eq}})| < M\tau$ . Then (see Remark 114),

$$\dot{V}(x(t)) < (gM\tau - k)\sqrt{V(x(t-\tau))} + \tau(k^2 + g^2M^2) \tag{11.13}$$

It is straightforward to see that

$$gM\tau < k \tag{11.14}$$

is a necessary condition: it will be assumed throughout the rest of this section. Denote by  $V = v_\infty^2$  the following equilibrium of (11.13):

$$v_\infty^2 = \frac{\tau^2(k^2 + g^2M^2)^2}{(k - gM\tau)^2}$$

Under assumption (11.14), notation  $V = y + v_\infty^2$  leads to

$$\begin{aligned}
\dot{y}(t) &< -\frac{\alpha}{2v_\infty}y(t-\tau) \\
&\quad + \frac{\alpha y^2(t-\tau)}{2v_\infty \left( v_\infty + \sqrt{v_\infty^2 + y(t)} \right)^2} \tag{11.15}
\end{aligned}$$

$$\alpha = k - gM\tau \tag{11.16}$$

---

<sup>7</sup>This assumption is verified, for instance, if  $u_{\text{eq}}(t, x)$  is at least locally Lipschitz in its second argument,  $\tau$  is small, and the dynamics are bounded.

Using a linearized equation leads to  $\frac{\alpha\tau}{v_\infty} < \pi$  and then,

$$\sqrt{\pi}(k^2 + g^2M^2) > (k - gM\tau) > 0 \quad (11.17)$$

Condition (11.17) ensures the following neighborhood  $\mathcal{R}_\infty$  of the manifold  $S$  to be (locally) attractive

$$\mathcal{R}_\infty = \{x \in \mathbb{R}^n : s^2(x(t)) < 2v_\infty\} \quad (11.18)$$

### Estimation of the attractivity domain $\mathcal{I}_0$ of $\mathcal{R}_\infty$

Throughout, *locally* means that solutions will reach  $\mathcal{R}_\infty$  only for initial values sufficiently closed to it: at the price of stronger conditions, one can obtain an estimate  $\mathcal{I}_0$  of the set of initial conditions for which solutions tend to  $\mathcal{R}_\infty$ . For this, using  $y(t) = y(t - \tau) + \int_{t-\tau}^t \dot{y}(w)dw$  leads to

$$\begin{aligned} \dot{y}(t) &\leq -\frac{\alpha}{2v_\infty}y(t) + \frac{\alpha}{2v_\infty^3}y^2(t - \tau) \\ &\quad + \frac{\alpha}{2v_\infty} \int_{t-\tau}^t \dot{y}(w)dw \end{aligned} \quad (11.19)$$

$$\begin{aligned} \int_{t-\tau}^t \dot{y}(w)dw &< \int_{t-\tau}^t -\frac{\alpha}{2v_\infty}y(w - \tau) \\ &\quad + \frac{\alpha}{2v_\infty^3}y^2(w - \tau)dw \end{aligned} \quad (11.20)$$

Following the Lyapunov - Razumikhin's theory [44], we assume that  $|y(t + s)| < q|y(t)|$ ,  $\forall s < 0$  for some  $q > 1$ . Then,

$$\begin{aligned} |\dot{y}(t)| &< -\frac{\alpha}{4v_\infty^2}(2v_\infty - \alpha\tau q)|y(t)| \\ &\quad + \frac{\alpha}{4v_\infty^4}q^2(2v_\infty + \alpha\tau)y^2(t) \end{aligned} \quad (11.21)$$

This leads to the asymptotic stability condition  $(2v_\infty - \alpha\tau) > 0$  (with  $\alpha > 0$ ), which is

$$\sqrt{2}(k^2 + g^2M^2) > (k - gM\tau) > 0 \quad (11.22)$$

and ensures convergence for any initial condition in the domain

$$\mathcal{I}_0 = \left\{x \in \mathbb{R}^n : |s^2(x) - 2v_\infty| < v_\infty^2 \frac{2v_\infty - \alpha\tau}{2v_\infty + \alpha\tau}\right\} \quad (11.23)$$

**Remark 113** Note that condition (11.22) is more restrictive than the previous one (11.17), since (11.22) $\Rightarrow$ (11.17).

**Remark 114** Even if the gain  $g(t, x)$  of system (11.1) is not assumed to be constant, an inequality similar to (11.13) may be obtained, say

$$\dot{V}(x(t)) < \omega\sqrt{V[x(t-\tau)]} + \beta \quad (11.24)$$

Thus, dealing with (11.13) seems reasonable.

### 11.2.3 An example with simulation

Consider

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_1(t)x_2(t) + u(t). \end{cases} \quad (11.25)$$

Defining  $s(x) = x_2 + 2x_1$  leads to the classical SMC:

$$u(t) = -x_1(t)x_2(t) - 2x_2(t) - k \operatorname{sign}[s(t)] \quad (11.26)$$

First, if we set  $k$  to 10 and suppose that a 0.1s time delay exists but has been neglected in the control design procedure, then one can check (11.3) for  $M \approx 30$ , for the given initial conditions,  $v_\infty = \frac{100}{7} \approx 14.3$ . Moreover, as condition (11.22) is valid :  $\sqrt{2}(k^2 + g^2M^2) > (k - gM\tau) > 0$ ,  $1000\sqrt{2} > 7 > 0$  (so, from Remark 113, (11.17) is valid), then the previous results ensure that solutions starting in the set

$$\mathcal{I}_0 = \left\{ x \in \mathbb{R}^n : \left| s^2(x) - \frac{200}{7} \right| < 193 \right\} \quad (11.27)$$

(as  $M$  does not increase!) reach the set

$$\mathcal{R}_\infty = \{ x \in \mathbb{R}^n : s^2(x(t)) < 28.6 \} \quad (11.28)$$

This conclusion is confirmed by the simulation (Figure 11.1).

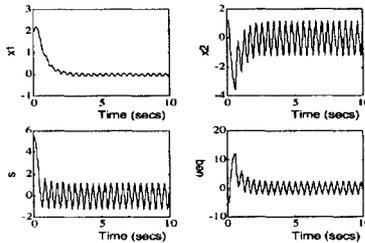


Figure 11.1: System (11.25) with control (11.26)  $k = 10$ , computed without care of a delay  $\tau = 0.1$

It is interesting to note that  $x_1$  and  $x_2$  have one oscillation frequency which leads to a limit cycle (see Figure 11.2). For different values of  $k$  and  $\tau$  the motion converges (see Figure 11.3) into a band around the sliding surface, corresponding to the neighborhood  $\mathcal{R}_\infty$ .

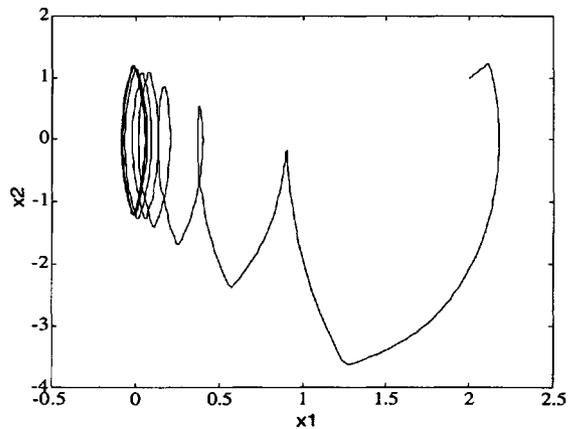


Figure 11.2: Phase portrait of system (11.25) with control (11.26),  $k = 10$  and delay  $\tau = 0.1$  with convergence to a simple limit cycle

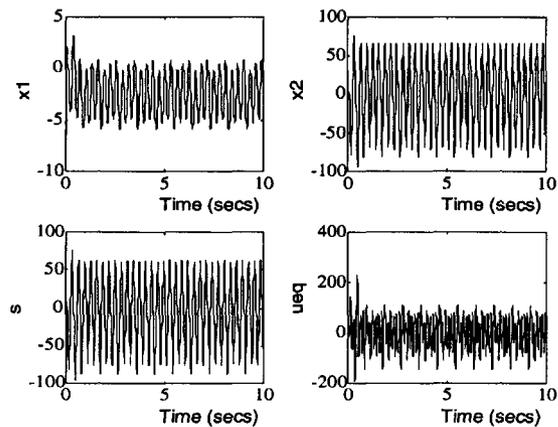


Figure 11.3: System (11.25) with control (11.26),  $k = 1000$ , delayed by  $\tau = 0.08$

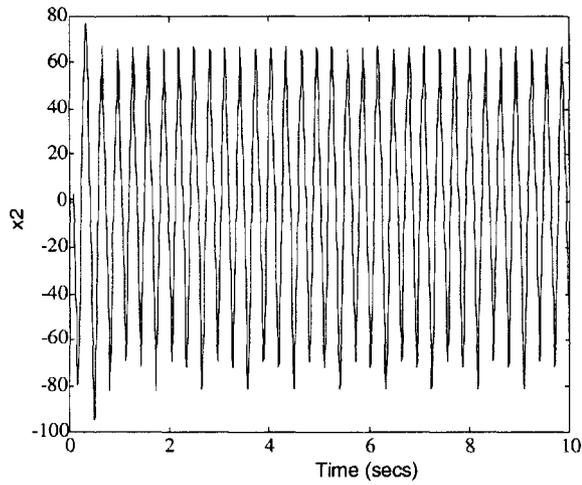


Figure 11.4: Zoom of Fig. 11.3 showing the 3-oscillation period

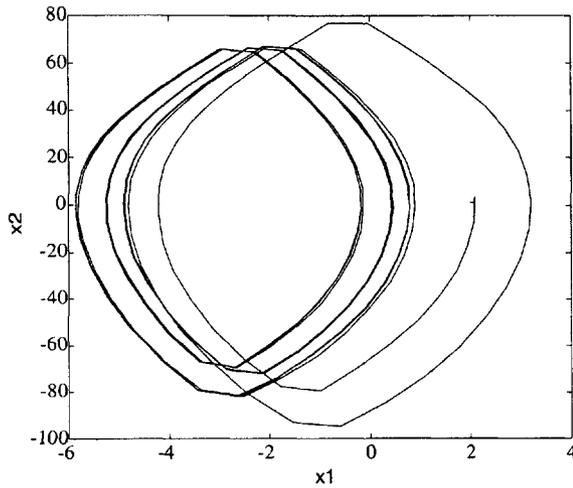


Figure 11.5: Phase portrait of system (11.25) with control (11.26),  $k = 1000$ , and delay  $\tau = 0.08$  with convergence to an asymptotic set composed of 3 loops

It is useful to stress that as the parameters are varying, bifurcations occur: for example, with  $k = 1000$  and  $\tau = 0.08$ ,  $x_1$  and  $x_2$  have three oscillatory frequencies (see Figure 11.4), leading to an asymptotic limit set with three loops (see Figure 11.5). Note that the parameters  $k = 10$  and  $\tau = 0.1$  lead to divergent motions. Lastly, from an engineering point of view, one would like to derive a sliding mode control that is less sensitive to time delay effect. For instance control (11.26) with the nonlinear gain  $k(t) = 4x_1(t)(3x_1(t) + x_1^2(t))$  and  $\tau = 0.1$  leads to applied control

$$u(t) = -x_1(t-\tau)x_2(t-\tau) - 2x_2(t-\tau) - k(t-\tau) \operatorname{sign}[s(t-\tau)] \quad (11.29)$$

$$k(t-\tau) = 4x_1(t-\tau)[3x_1(t-\tau) + x_1^2(t-\tau)] \quad (11.30)$$

$$s(t-\tau) = x_2(t-\tau) + 2x_1(t-\tau) \quad (11.31)$$

and motions converge asymptotically to the origin (see Figure 11.6).

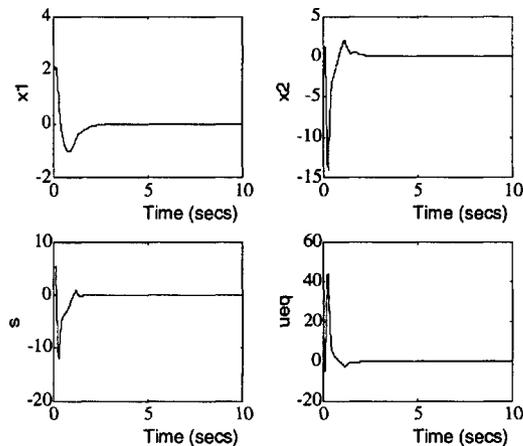


Figure 11.6: System (11.25) with control (11.26) with a nonlinear gain and delay  $\tau = 0.1$

This preliminary study has concerned the sensitivity of SMC with respect to time-delay effect. It was shown that, under some conditions (11.17) or (11.22), motions will reach an asymptotic limit set  $\mathcal{R}_\infty$  given by (11.18) around the sliding surface. Moreover, we obtained some information about the set of initial conditions (estimated by  $\mathcal{I}_0$ ), ensuring the motions to reach and stay in  $\mathcal{R}_\infty$ .

## 11.3 A SMC design for linear time delay systems

This section deals with the design of discontinuous controllers for linear time delay systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau) + Du(t - h), & t > 0, \\ x(t) = \phi(t), & t \in [-\tau, 0] \end{cases} \quad (11.32)$$

where  $x(t) \in \mathbb{R}^n$ ;  $A, B$  are constant  $n \times n$  matrices;  $D$  is a  $n \times m$  matrix;  $u \in \mathbb{R}^m$  is the input vector;  $\tau \geq 0$  is a constant delay; and  $h = 0$  or  $\tau$ .

The first result concerns the case ( $h = 0$ ) involving a memoryless input. In order to design a stabilizing controller, we first convert system (11.32) into a *regular form with delay* and then, *provisionally neglecting the time delay*, we design a *sliding mode controller*, which is stabilizing for sufficiently small delays as well, provided they belong to a *computable set*  $[0, \tau_{\max}]$  of *admissible delays*.

The second result concerns the extension to the case of systems with input delay ( $h = \tau$ )<sup>8</sup>.

### 11.3.1 Regular form

We assume that

- A1)  $\text{rank}(D) = r \leq m$ ,
- A2)  $(A + B, D)$  is controllable

The aim of the next three lemmas is to transform the original system into a well-known, appropriate form for sliding mode control (called regular form [36]).

**Lemma 115** *If A1) holds [25], then there exists new inputs  $v \in \mathbb{R}^m$ ,  $u = Wv$ , where  $W \in \mathbb{R}^{m \times n}$  is nonsingular, such that*

$$DW = \begin{pmatrix} D' & 0 \\ N & 0 \end{pmatrix} \quad (11.33)$$

with  $D' \in \mathbb{R}^{r \times r}$  of full rank,  $N \in \mathbb{R}^{(n-r) \times r}$ , and  $v = (v_1, \dots, v_m)^T$ .

**Lemma 116** [25] *If A1) holds, then there exists a nonsingular coordinate transformation  $T \in \mathbb{R}^{n \times n}$  such that system (11.32), written in the*

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<sup>8</sup>Note that enlarging these two cases to multiple delays (thus, different values of delay on the various inputs) should be possible, but needs some additional computing effort that will not be included here.

new variables  $z(t) = (z_1, z_2)^T = Tx(t)$ ,  $z_1 \in \mathbb{R}^{(n-r)}$ ,  $z_2 \in \mathbb{R}^r$ , takes the following **regular form**:

$$\begin{cases} \frac{dz_1(t)}{dt} = \sum_{i=1}^2 [A_{1i}z_i(t) + B_{1i}z_i(t - \tau)] \\ \frac{dz_2(t)}{dt} = \sum_{i=1}^2 [A_{2i}z_i(t) + B_{2i}z_i(t - \tau)] + D_2\tilde{v} \end{cases} \quad (11.34)$$

with  $D_2 \in \mathbb{R}^{r \times r}$  nonsingular and  $\tilde{v} = (v_1, \dots, v_r)^T$ .

**Lemma 117** System (11.34) subject to  $\tau = 0$  [36], that is (with  $E_{ij} = A_{ij} + B_{ij}$ ,  $(i, j) \in \{1, 2\}^2$ ),

$$\begin{cases} \frac{dz_1(t)}{dt} = E_{11}z_1(t) + E_{12}z_2(t) \\ \frac{dz_2(t)}{dt} = E_{21}z_1(t) + E_{22}z_2(t) + D_2\tilde{v} \end{cases} \quad (11.35)$$

is such that  $(E_{11}, E_{12})$  is controllable.

### 11.3.2 Asymptotic stability of systems with small delays

The aim of this subsection is to provide a time delay with upper bound  $\tau_{\max}$  such that the asymptotic stability of system

$$\frac{dz(t)}{dt} = Ez(t), \quad E = A + B \quad (11.36)$$

ensures the asymptotic stability of the corresponding delay system

$$\frac{dz(t)}{dt} = Az(t) + Bz(t - \tau) \quad (11.37)$$

for any  $\tau \in [0, \tau_{\max}]$ .

If  $E = A + B$  is supposed to be asymptotically stable, then for any symmetric, positive-definite matrix  $Q \in \mathbb{R}^{n \times n}$ , there is symmetric, positive-definite matrix  $P$  solution of the Lyapunov equation

$$(A + B)^T P + P(A + B) = -Q \quad (11.38)$$

Let  $Q_1$  be the root square of the matrix  $Q$ ,

$$Q_1^T Q_1 = Q \quad (11.39)$$

**Theorem 118** *Let system (11.36) be asymptotically stable. Then (11.37) is asymptotically stable for all  $\tau \leq \tau_{\max}$ , where*

$$\tau_{\max} = \frac{1}{2\sqrt{\lambda_{\max}(Q_1^{-T}E^T P B Q^{-1}B^T P E Q_1^{-1})}} \quad (11.40)$$

**Proof** The system (11.37) is rewritten as

$$h(t) = \left[ z(t) + \int_{t-\tau}^t Bz(w)dw \right] \quad (11.41)$$

$$\frac{dh(t)}{dt} = Ez(t) \quad (11.42)$$

Then, considering the Lyapunov - Krasovskii functional

$$\begin{aligned} V &= V_1 + V_2 \\ V_1 &= \alpha h(t)^T P h(t) \\ V_2 &= \int_{t-\tau}^t \left[ \int_w^t z^T(v) Q z(v) dv \right] dw \end{aligned} \quad (11.43)$$

some overvaluations of the time-derivative of  $V$  lead to the result (the complete proof is in [26]).

**Remark 119** *In ([24] p. 214), another upper bound*

$$\tau_0 = \frac{\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}}{2 \|PB\| (\|A\| + \|B\|)} \quad (11.44)$$

*was given in terms of the smallest and the largest eigenvalues  $\lambda_{\min}(P)$ ,  $\lambda_{\max}(P)$  of the real symmetric positive definite matrix solution of (11.38) in the particular case  $Q = I$ .*

### 11.3.3 Sliding mode controller synthesis

This subsection proposes two kinds of controllers stabilizing (11.32). The first one has a constant gain whereas the second controller has a nonlinear gain. One of the main drawbacks of the SMC (11.46) with constant gain  $g$  is a possible, undesired chattering phenomenon. The second controller reduces this phenomenon, but may slow the convergence when far from the sliding manifold. From a practical point of view, when implementing the control on a plant, the first control (constant  $g$ ) should be used far from the surface, then be switched to the nonlinear gain (11.52) when approaching the target.

In the case of a delayed state with memoryless input ( $h = 0$ ), constant gain  $g$  is designed in Theorem 120 in the unperturbed case, and in Corollary 123 when additive perturbations are considered. The nonlinear gain is proposed in Theorem 121 in the unperturbed case, and in Corollary 124 for additive perturbations.

Using some additional integrators, the case of delayed inputs will also be considered in the same way (Theorem 125: constant gain ; Theorem 126: nonlinear gain).

### Sliding system for systems with state delay

To begin, introduce the function

$$s(t) = z_2(t) + K z_1(t) \quad (11.45)$$

where the matrix  $K \in \mathbb{R}^{r \times (n-r)}$  makes the matrix  $(E_{11} - E_{12}K)$  (notation of Lemma 117) be Hurwitz [this is possible due to the controllability of  $(E_{11}, E_{12})$ ].

**Theorem 120** *The control law*

$$\begin{aligned} \tilde{v} = -D_2^{-1} \{ & \sum_{i=1}^2 [A_{2i}z_i(t) + B_{2i}z_i(t - \tau) + \\ & K(A_{1i}z_i(t) + B_{1i}z_i(t - \tau))] + g \operatorname{sign}(s) \} \end{aligned} \quad (11.46)$$

with constant gain  $g > 0$ , makes the manifold  $s(z) = 0$  attractive in finite time and positively invariant. The system  $\dot{x}(t) = Ax(t) + Bx(t - \tau) + Du(t - h)$ , (11.32) with  $h = 0$  and feedback (11.46) is asymptotically stable for all  $\tau \in [0, \tau_{\max}]$  with  $\tau_{\max}$  defined by (11.40) from the scheme of Theorem 118 (11.38) and (11.39) applied to  $E = (E_{11} - E_{12}K)$ .

**Proof** Consider the positive, semi-definite function

$$V = \frac{s^T(t)s(t)}{2} \quad (11.47)$$

Differentiating  $V$  along the solutions of (11.34) yields

$$\dot{V} = s^T(t)\dot{s}(t) \quad (11.48)$$

$$= s(t) \left( \sum_{i=1}^2 [A_{2i}z_i(t) + B_{2i}z_i(t - \tau) + \right. \quad (11.49)$$

$$\left. K(A_{1i}z_i(t) + B_{1i}z_i(t - \tau))] + D_2\tilde{v} \right) \quad (11.50)$$

$$= -g \|s(t)\| \leq -g\sqrt{V(t)} \quad (11.51)$$

The latter inequality is known to guarantee the convergence of system trajectories onto the surface  $s(z) = 0$  in finite time. According to Theorem 118, the subsystem  $\frac{dz_1(t)}{dt} = (A_{11} - A_{12}K)z_1(t) + (B_{11} - B_{12}K)z_1(t - \tau)$  is then asymptotically stable for  $\tau \leq \tau_{\max}$ .  $\triangleleft$

Define now  $\tilde{A} = (A_{11} - A_{12}K)$ ,  $\tilde{B} = (B_{11} - B_{12}K)$ ,  $\tilde{E} = \tilde{A} + \tilde{B}$ ,  $G = Q_1^{-T} \tilde{E}^T P \tilde{B} Q_1^{-1} \tilde{B}^T P \tilde{E} Q_1^{-1}$ ,  $H = B_{12}^T P \tilde{E} Q_1^{-1} \tilde{E}^T P B_{12}$ , and

$$g > \|2\alpha\varepsilon(A_{12} + B_{12})^T Ph(t) + \tau\alpha Hs(t)\| \quad (11.52)$$

$$h(t) = z_1(t) + \int_{t-\tau}^t (\tilde{B}z_1(w)dw) + \int_{t-\tau}^t B_{12}s[x(w)]dw \quad (11.53)$$

$$\varepsilon > 0 \quad (11.54)$$

$$\alpha = \frac{1}{\sqrt{\lambda_{\max}(G)}} \quad (11.55)$$

**Theorem 121** *The control law (11.46) with nonlinear gain (11.52) stabilizes asymptotically the system (11.32) with  $h = 0$  for all  $\tau \in [0, \tau_{\max}(\varepsilon)]$*

*with  $\tau_{\max}(\varepsilon) = \frac{1}{2\sqrt{\lambda_{\max}(G)} + \varepsilon}$  and for any  $\varepsilon > 0$ .*

**Proof** Let  $P$  be the real, symmetric, positive definite-matrix solution of the Lyapunov equation  $\tilde{E}^T P + P \tilde{E} = -Q$ . Rewrite the system (11.34) as

$$\begin{cases} \frac{dz_1(t)}{dt} = \tilde{A}z_1(t) + \tilde{B}z_1(t - \tau) + A_{12}s(x) + B_{12}s(x(t - \tau)) \\ \frac{dz_2(t)}{dt} = \sum_{i=1}^2 (A_{2i}z_i(t) + B_{2i}z_i(t - \tau)) + D_2\tilde{v} \end{cases}$$

Consider the following Lyapunov - Krasovskii functional

$$\begin{aligned}
V &= V_5 + V_6 + V_7 \\
V_5 &= \frac{\gamma}{2} s^T(t) s(t), \gamma > 0 \\
V_6 &= V_{61} + V_{62} \\
V_{61} &= \alpha h(t)^T P h(t) \\
h(t) &= z_1(t) + \int_{t-\tau}^t (\tilde{E} z_1(w) dw) + \int_{t-\tau}^t B_{12} s(z(w)) dw \\
V_{62} &= \int_{t-\tau}^t \left\{ \int_w^t z_1^T(v) Q z_1(v) dv \right\} dw \\
V_7 &= \frac{\alpha}{\varepsilon} \int_{t-\tau}^t \left\{ \int_w^t s^T(v) H s(v) dv \right\} dw
\end{aligned} \tag{11.56}$$

We have

$$\dot{h}(t) = \tilde{E} z_1(t) + (A_{12} + B_{12}) s(z).$$

Let us differentiate the equation (11.56) along the solution of (11.34)

$$\dot{V}_5(t) = \gamma s^T(t) \dot{s}(t) = -(g\gamma) s^T(t) \text{sign}[s(t)]$$

$$\begin{aligned}
\dot{V}_{61}(t) &= -z_1^T(t) (\alpha Q) z_1(t) + 2\alpha \int_{t-\tau}^t [z_1^T(t) \tilde{E} P \tilde{E} z_1(w)] dw \\
&\quad + 2\alpha \int_{t-\tau}^t z_1^T(t) \tilde{E}^T P B_{12} s(w) dw \\
&\quad + 2\alpha h^T(t) P (A_{12} + B_{12}) s(t)
\end{aligned} \tag{11.57}$$

$$\dot{V}_{62}(t) = - \int_{t-\tau}^t z_1(v)^T Q z_1(v) dv + \tau z_1(t)^T Q z_1(t) \tag{11.58}$$

$$\dot{V}_7(t) = -\frac{\alpha}{\varepsilon} \int_{t-\tau}^t s(v)^T H s(v) dv + \frac{\tau\alpha}{\varepsilon} s(t)^T H s(t) \tag{11.59}$$

Some majorations give

$$\dot{V}_{61}(t) = L_{61}(t) + N_{61}(t) \tag{11.60}$$

$$\begin{aligned}
L_{61}(t) &= -z_1^T(t) (\alpha Q) z_1(t) \\
&\quad + 2\alpha \int_{t-\tau}^t (z_1^T(t) \tilde{E} P \tilde{E} z_1(w)) dw
\end{aligned} \tag{11.61}$$

$$\begin{aligned}
N_{61}(t) &= 2\alpha \int_{t-\tau}^t z_1^T(t) \tilde{E}^T P B_{12} s(w) dw \\
&\quad + 2\alpha h^T(t) P (A_{12} + B_{12}) s(t)
\end{aligned} \tag{11.62}$$

$$\begin{aligned}
N_{61}(t) &\leq \alpha\epsilon\tau z_1^T(t)Qz_1(t) + \frac{\alpha}{\epsilon} \int_{t-\tau}^t s(w)^T Hs(w)dw \\
&\quad + 2\alpha h^T(t)P(A_{12} + B_{12})s(t)
\end{aligned} \tag{11.63}$$

According to Theorem 118,

$$L_{61}(t) + \dot{V}_{62}(t) < z_1^T(t)Mz_1(t), \tag{11.64}$$

$$M = -\alpha Q + \tau[Q + \alpha^2 \tilde{E}^T P \tilde{B} Q^{-1} \tilde{B}^T P \tilde{E}] \tag{11.65}$$

Thus one obtains,

$$\begin{aligned}
\dot{V}(t) &\leq z_1^T(t)Mz_1(t) \\
&\quad + \alpha\epsilon\tau z_1^T(t)Qz_1(t) \\
&\quad + 2\alpha h^T(t)P(A_{12} + B_{12})s(z) \\
&\quad + \frac{\tau\alpha}{\epsilon} s(t)^T Hs(t) \\
&\quad - (g\gamma) s^T(t) \text{sign}[s(t)]
\end{aligned} \tag{11.66}$$

Choosing  $\gamma = \frac{1}{\epsilon}$ , inequality (11.66) becomes

$$\begin{aligned}
\dot{V}(t) &\leq z_1^T(t)(M + \alpha\epsilon\tau Q)z_1(t) \\
&\quad + \frac{1}{\epsilon} s^T[z(t)]\{2\alpha\epsilon(A_{12} + B_{12})^T Ph(t) + \tau\alpha Hs(t) - g \text{sign}[s(z(t))]\} \\
M + \alpha\epsilon\tau Q &= -\alpha Q + \tau[Q(1 + \alpha\epsilon) + \alpha^2 \tilde{E}^T P \tilde{B} Q^{-1} \tilde{B}^T P \tilde{E}]
\end{aligned} \tag{11.67}$$

If we choose  $g$  so that 11.52 holds and  $\tau < \tau_{\max}(\epsilon) = \frac{1}{2\sqrt{\lambda_{\max}(G)} + \epsilon}$ , then  $\dot{V}(t) \leq 0$  along the trajectories of (11.34). The control (11.46) stabilizes the system (11.32) for  $\tau \leq \tau_{\max}(\epsilon)$ .

**Remark 122** Note that  $\tau_{\max} > \tau_{\max}(\epsilon)$  and  $\tau_{\max}(\epsilon) \rightarrow \tau_{\max}$  when  $\epsilon \rightarrow 0$ . By choosing a sufficiently small  $\epsilon > 0$ , an asymptotic stabilizing controller can be designed for any  $\tau < \tau_{\max}$ . Indeed, if  $\epsilon\tau_{\max} \ll 1$ , then  $\tau_{\max}(\epsilon) \cong \tau_{\max}(1 - \epsilon\tau_{\max})$ .

### Perturbation effect and stabilization

Now consider the system (11.32) submitted to additive perturbations,

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) + Du(t) + p(t, x) \tag{11.68}$$

where  $p : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth vector field. We assume that

- A3)  $\|p(t, x)\| \leq p_{ub} + M \|x(t)\|$ , where  $p_{ub}$  and  $M \in \mathbb{R}_+$ ; and  
A4)  $p \in \text{span}\{d_1, \dots, d_m\}$ , where  $d_i$  ( $i \in \{1 \dots m\}$ ) are column vectors of  $D$ .

Using Lemmas 115 and 116, system (11.68) is equivalent to

$$\begin{cases} \frac{dz_1(t)}{dt} = \sum_{i=1}^2 [A_{1i}z_i(t) + B_{1i}z_i(t - \tau)] \\ \frac{dz_2(t)}{dt} = \sum_{i=1}^2 [A_{2i}z_i(t) + B_{2i}z_i(t - \tau)] + p_2(t, z) + D_2\tilde{v} \end{cases} \quad (11.69)$$

where  $z_1$  and  $z_2$  are defined as previously and  $p_2$  is the perturbation field expressed in the new basis. Then,  $p_2(t, z)$  is bounded as follows

$$\|p_2(t, z)\| \leq \|T\| p_{ub} + M \|T\| \|T^{-1}\| \|z(t)\| = p'_{ub} + M' \|z(t)\| \quad (11.70)$$

**Corollary 123** *The control law (11.46)*

$$\begin{aligned} \tilde{v} = & -D_2^{-1} \left\{ \sum_{i=1}^2 [A_{2i}z_i(t) + B_{2i}z_i(t - \tau)] + \right. \\ & \left. K(A_{1i}z_i(t) + B_{1i}z_i(t - \tau)) \right\} + g \text{ sign}(s) \end{aligned}$$

where  $g > p'_{ub} + M' \|z(t)\|$  (from 11.70), makes the manifold  $s(z) = 0$  attractive in finite time and positively invariant. System (11.32) with  $h = 0$  and feedback (11.46) is asymptotically stable for all  $\tau \in [0, \tau_{\max}]$  with  $\tau_{\max}$  defined by (11.40) from the scheme of Theorem 118 (11.38) and (11.39) applied to  $E = (E_{11} - E_{12}K)$ .

As stated previously, one can switch the constant gain near the sliding manifold to a nonlinear one.

**Corollary 124** *If assumptions A1, A2, A3, and A4 hold, the control law (11.46) with nonlinear gain  $g$*

$$g > \|2\alpha\varepsilon(A_{12} + B_{12})^T Ph(t) + \tau\alpha Hs(z(t))\| + p'_{ub} + M' \|z(t)\| \quad (11.71)$$

$$\varepsilon > 0 \quad (11.72)$$

$$h(t) = z_1(t) + \int_{t-\tau}^t (\tilde{B}z_1(w)dw) + \int_{t-\tau}^t B_{12}s(x(w))dw \quad (11.73)$$

$$\alpha = \frac{1}{\sqrt{\lambda_{\max}(G)}} \quad (11.74)$$

stabilizes system (11.32) asymptotically with  $h = 0$  for all  $\tau \in [0, \tau_{\max}(\varepsilon)]$

with  $\tau_{\max}(\varepsilon) = \frac{1}{2\sqrt{\lambda_{\max}(G)} + \varepsilon}$  and for any  $\varepsilon > 0$ .

**Proof** The proof is based on the Lyapunov - Krasovski functional

$$V = V_5 + V_6 + V_7 \quad (11.75)$$

where  $V_5, V_6, V_7$  are defined in the proof of Theorem 6. A modification of the computation of  $\dot{V}(t)$

$$\dot{V}_5 = \gamma s^T(z(t))(p(t, z(t)) - g \operatorname{sign}[s(z(t))]) \quad (11.76)$$

combined with the previous calculus leads to the result.

### Sliding system with state and input delay

We now consider system (11.32) with  $h = \tau \geq 0$  and extend the previous results to the corresponding system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau) + Du(t - \tau), & t > 0 \\ x(t) = \phi(t), & t \in [-\tau, 0] \\ u(t) = \psi(t), & t \in [-\tau, 0] \end{cases} \quad (11.77)$$

Throughout this section we assume that:

A'2)  $(A + B, D)$  is controllable.

We propose to add integrators on each of the  $m$  inputs, in such a way that the system is transformed as follows

$$\begin{cases} \frac{dz_1(t)}{dt} = Az_1(t) + Bz_1(t - \tau) + Dz_2(t - \tau) \\ \frac{dz_2(t)}{dt} = \tilde{v}, \quad \tilde{v} \in \mathbb{R}^m \end{cases} \quad (11.78)$$

This system is in its regular form and the scheme (regular form  $\rightarrow$  delay cancellation  $\rightarrow$  SMC design  $\rightarrow$  stability study with delay) is the same as previously noted. Thus the above results can be extended to the case of systems with input delays.

**Theorem 125** *If assumption A'2) holds, then the control law*

$$\tilde{v}(t) = -K(Az_1(t) + Bz_1(t - \tau) + Dz_2(t - \tau)) - g \operatorname{sign}(s) \quad (11.79)$$

(with  $g \in \mathbb{R}_+$ ) makes the manifold  $s(z) = 0$  attractive in finite time and positively invariant. System (11.77) with feedback (11.79) is asymptotically stable for all  $\tau \in [0, \tau_{\max}]$  with  $\tau_{\max}$  defined by

$$\tau_{\max} = \frac{1}{2\sqrt{\lambda_{\max}(Q_1^{-T} \tilde{E}^T P \tilde{B} Q^{-1} \tilde{B}^T P \tilde{E} Q_1^{-1})}} \quad (11.80)$$

where  $\tilde{B} = B - DK$ ,  $\tilde{E} = A + \tilde{B}$ , and  $P, Q$  are symmetric, positive-definite matrices verifying the Lyapunov equation  $\tilde{E}^T P + P \tilde{E} = -Q$ .

The second theorem of this section developed a second controller with a nonlinear gain  $g$ , for which the chattering was reduced.

**Theorem 126** *If assumption A'2) holds, the control law (11.79) with nonlinear gain (11.52) stabilizes system (11.77) asymptotically for all  $\tau \in [0, \tau_{\max}(\varepsilon)]$  with  $\tau_{\max}(\varepsilon) = \frac{1}{2\sqrt{\lambda_{\max}(Q_1^{-T}\tilde{E}^T P \tilde{B} Q^{-1} \tilde{B}^T P \tilde{E} Q_1^{-1})} + \varepsilon}$  and for any  $\varepsilon > 0$ .*

### 11.3.4 Example: delay in the state

Consider the following model

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bz(t - \tau) + D\tilde{v} \\ A &= \begin{pmatrix} 2.3 & 0 & 1 \\ -4.9 & 3 & -3 \\ -1 & 0 & 1 \end{pmatrix} \\ B &= \begin{pmatrix} 0.2 & -1 & 1 \\ -0.9 & -1 & 0 \\ -0.2 & 0.1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned} \quad (11.81)$$

Conditions A1) and A2) hold:  $\text{rank}(D) = 1$  and  $(A+B, D)$  is controllable. This system has the regular form: by choosing  $z = (z_1; z_2)^T$ ,  $z_1 \in \mathbb{R}^2$ , and  $z_2 \in \mathbb{R}$ , it is decomposed into

$$\begin{aligned} \dot{z}_1(t) &= \begin{pmatrix} 2.3 & 0 \\ -4.9 & 3 \end{pmatrix} z_1(t) + \begin{pmatrix} 1 \\ -3 \end{pmatrix} z_2(t) \\ &+ \begin{pmatrix} 0.2 & -1 \\ -0.9 & -1 \end{pmatrix} z_1(t - \tau) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} z_2(t - \tau) \\ \dot{z}_2(t) &= (-1 \ 0) z_1(t) + z_2(t) \\ &+ (-0.2 \ 0.1) z_1(t - \tau) + \tilde{v} \end{aligned} \quad (11.82)$$

Letting  $\tau = 0$ , we design a feedback gain  $K = (1.6, -1)$  so that the following subsystem 1

$$\dot{z}_1(t) = \left\{ \begin{pmatrix} 2.5 & -1 \\ -5.8 & 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} K \right\} z_1(t) \quad (11.83)$$

is asymptotically stable. Then we derive the sliding manifold, which leads to the sliding mode control law

$$s(z) = z_2 + (1.6 - 1) z_1 \quad (11.84)$$

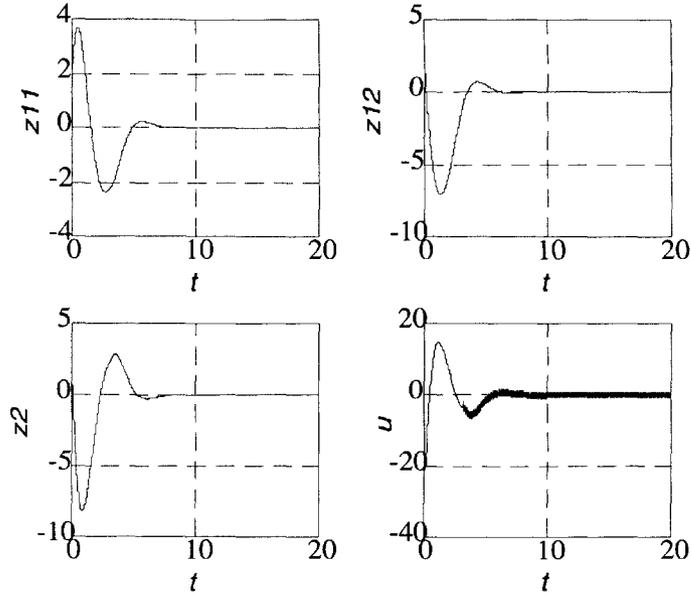


Figure 11.7: Stabilization of (11.85) using (11.45) with constant gain  $g = 1$

$$\dot{z}_1(t) = \begin{pmatrix} -0.7 & 1 \\ -1 & -1 \end{pmatrix} z_1(t) \quad (11.85)$$

On the sliding manifold, the eigenvalues of the reduced system with  $\tau = 0$  is  $-0.85 \pm j$ . The time response of the system is about 0.47.

Applying the control (11.46), and using Theorem 120, system (11.82) is asymptotically stable for  $\tau \leq \tau_{\max}$ . We find the following upper values of delay

$$\tau_{\max} = 0.430 \quad (11.86)$$

The first control (11.46), with  $g = 1$ , gives the following simulation using a first-order integration scheme of 0.01 step: all initial conditions have been set to 2 (Figure 11.7). Note that there are oscillations of amplitude 2 in the control  $u$  and the time response is close to the one of the linear system with  $\tau = 0$ .

Now, when applying control (11.46) to system (11.82), Theorem 6 ensures that the closed-loop system is asymptotically stable for  $\tau \leq 0.43$ . In order to compare this control with the first one, we select a delay  $\tau = 0.1$ ,  $g = \frac{\|h(t)\|}{2} + 0.4 \|s(t)\|$  and using the control (11.46), we obtain the results

of Figure 11.8.

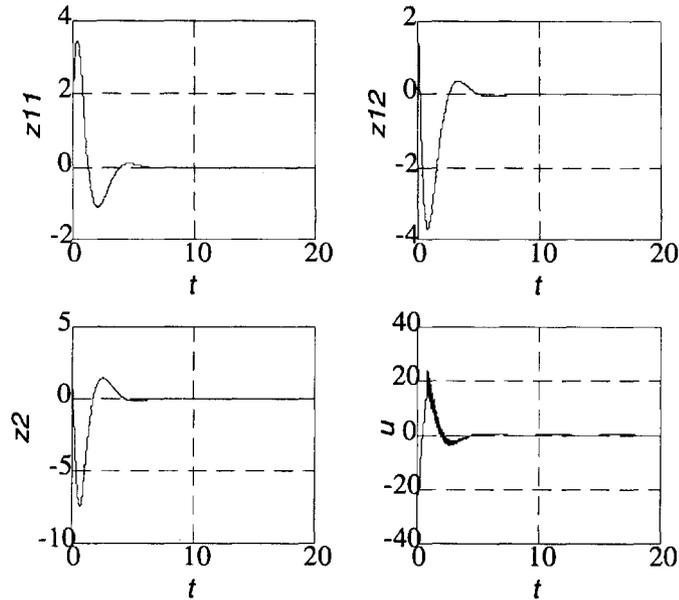


Figure 11.8: Stabilization of (11.89) using (11.45) with nonlinear gain  $g = \frac{\|h(t)\|}{2} + 0.4 \|s(t)\|$

The chattering amplitude of the control with constant gain  $g = 1$  is about 2, whereas using the control with nonlinear gain  $g = \frac{\|h(t)\|}{2} + 0.4 \|s(t)\|$ , and chattering can be neglected. This shows efficiency of the nonlinear gain. Nevertheless, note that chattering reappears if the delay approaches the calculated upper bound  $\tau_{\max} = 0.43$  (this makes us think that this calculated bound is near the real one). Indeed, increasing the delay to  $\tau = 0.35$  and keeping  $g = \frac{\|h(t)\|}{2} + 0.4 \|s(t)\|$ , control (11.46) leads to the results of Figure 11.9.

## 11.4 Conclusion

The presence of delay within a sliding mode control can induce oscillations around the design surface. The opening case study pointed out possible

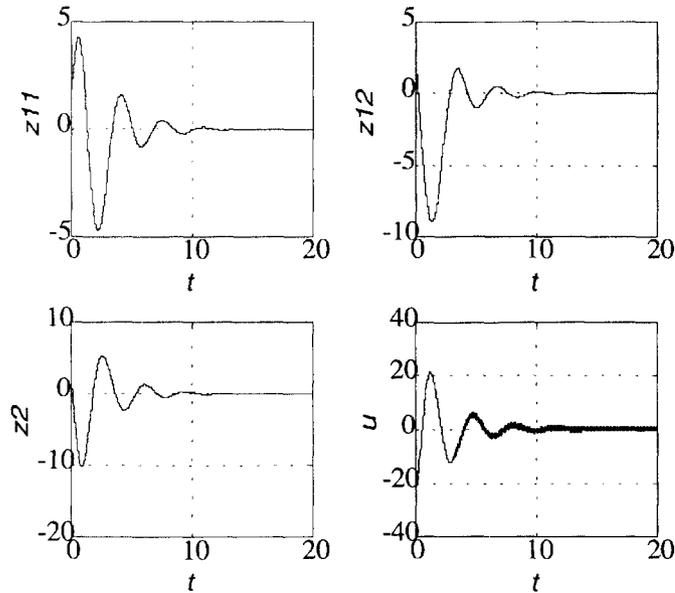


Figure 11.9: Stabilization of (11.89) with delay  $\tau = 0.35$  using (11.45) with nonlinear gain  $g = \frac{\|h(t)\|}{2} + 0.4 \|s(t)\|$ .

behavioral changes (bifurcations) arising in such relay/delay systems. This motivated the study of specific SMC design for systems with state and/or input aftereffect.

Here, the main contribution lays in the analysis of delay/relay motions (amplitude of the possible oscillations around the sliding manifold and admissible initial conditions<sup>9</sup>) and the design of a SMC for systems with both input and state delay. In addition, new stability results were provided in Theorem 4 [condition (11.40)]. Note that these results allowed consideration of the presence of a delay affecting sensors [observation of  $x(t - \tau)$  instead of  $x(t)$ ] or actuators [control  $u(t - \tau)$  instead of  $u(t)$ ]. Calculable control laws were provided, together with upper bounds of the delay values, while preserving asymptotic stability.

The control implementation was rather simple, even if the proofs appeared complex. The control law assured the *existence* of a Lyapunov -

<sup>9</sup>More precisely, it was the determination of an attracting neighborhood of the sliding manifold and estimation of its asymptotic stability domain.

Krasovskii functional. The chattering phenomenon was avoided by using nonlinear gains (as in the simulation of Fig. 8).

In the case of input delays, it was remarkable that the inner computation of the controller algorithm was discontinuous (it may be chattering), but the actual control  $u(t)$  remained smooth because of the input integrators ( $z_2$  variable). Furthermore, a difference with the nondelayed case can be noted: the feedback is not computed in a space of dimension  $(n - m)$  but on the entire system  $(n)$ , which is a bit more complicated.

Let us sum up the several cases and solutions which have been considered for system (11.32):

- *delayed state, memoryless input ( $h = 0$ ), unperturbed case*: design of a constant gain (Theorem 120) and of a nonlinear one (Theorem 121);
- *delayed state, memoryless input ( $h = 0$ ), additive perturbations*: design of a constant gain (Corollary 123) and a nonlinear one (Corollary 123); and
- *delayed input and state ( $h = \tau$ )*: design of a constant gain (Theorem 125) and of a nonlinear one (Theorem 126).

Some extensions of these results are possible:

1) The present modeling assumptions demand a linear model (even if the controller is not). Relaxing this constraint seems possible, as well as introducing parameter uncertainties.

2) The study could be extended to multiple delays (for instance,  $\tau \neq h$  in 11.32) with some additional computing effort.

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# Chapter 12

# Sliding Mode Control of Infinite-Dimensional Systems

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## 12.1 Introduction

Many important plants, such as flexible manipulators and structures as well as heat transfer processes, combustion, and fluid mechanical systems, are governed by partial differential equations and are often described by models with a significant degree of uncertainty. The existing results [8, 13] on feedback control of distributed parameter systems (DPS), operating under uncertainty, extend the finite-dimensional results [6] for the standard  $H_\infty$  control problem and may be viewed as a disturbance attenuation problem in the class of square integrable external disturbances. Relating to  $H_2$ -design framework, this approach admits generalization to the class of external disturbances of impulsive type, whereas the persistent excitation case of uniformly bounded disturbances, which also often occurs in practice, calls for a separate investigation. Thus, it is of interest to develop consistent methods that are capable of utilizing distributed parameter models and provide the desired system performance in spite of the significant uniformly bounded model uncertainties.

Sliding mode control of finite-dimensional systems is known to guarantee a certain degree of robustness with respect to uniformly bounded unmod-

eled dynamics. Since the sliding mode equation is control-independent, the approach based on the deliberate introduction of sliding motions into the control system splits the control problem into two independent problems of lower dimensions. We will design, firstly, a discontinuity manifold with prescribed dynamic properties of the sliding motion and, secondly, a discontinuous control that ensures the sliding motion on this manifold. Apart from decoupling of the original control problem, the sliding mode approach makes the closed-loop system insensitive with respect to matched disturbances. Due to these advantages and simplicity of implementation, sliding mode controllers are widely used in various applications. An overview of finite-dimensional sliding mode control theory and applications can be found in [29].

The first papers [18, 25] on the application of sliding mode control algorithms to DPS corroborated the utility of their use for infinite-dimensional systems as well and motivated further theoretical investigations [26, 32], which were confined, however, to semilinear parabolic systems with a finite horizon. In order to describe sliding modes in these systems, the sliding mode equation was shown in [26] to be well-posed via relating the discontinuous control law to the continuous one. The conditions for the infinite-dimensional sliding mode to exist were obtained in [32] through finite-dimensional Faedo - Galerkin approximations of the original discontinuous control system.

Later, a set of sliding mode control algorithms was proposed for distributed parameter plants governed by uncertain partial differential equations (see [30, 31] and the references quoted therein). All the algorithms, however, followed the conventional finite-dimensional approach, which implies that each component of a control action undergoes discontinuities on "its own" surface and as a result a sliding mode is enforced in their intersection. In general, this component-wise design idea proves to be inapplicable in infinite-dimensional setting because neither control input nor sliding manifold are representable in the component form.

In this chapter, mathematical tools for discontinuous infinite-dimensional systems, viewed over an infinite time interval, and general sliding mode control algorithms for various DPS, are developed.

The chapter is organized as follows. Section 2 demonstrates some attractive features of discontinuous control systems in a Hilbert space and motivates the subsequent theoretical development. We present an infinite-dimensional system driven by a discontinuous control along the discontinuity manifold for an infinite time interval. The discontinuous control law results from the Lyapunov min-max approach, the origins of which may be found in [10]. Based on this approach, the control is synthesized to guarantee that the time-derivative of a Lyapunov function, selected for a

nominal system, is negative on the trajectories of the perturbed system. The approach brings us to the control action, referred to as a unit control [21, 27], the norm of which is equal to 1 everywhere but on the discontinuity manifold. The closed-loop system enforced by the unit control is shown to be exponentially stable and robust with respect to matched disturbances. However, allowing discontinuous feedback leads to a major problem, which concerns the precise meaning of solution to the state differential equation with discontinuous right-hand side.

In Section 3, the mathematical tools of discontinuous control systems in a Hilbert space are developed. In order to describe the system behavior in the manifold where the control input undergoes discontinuities, a special regularization-based technique is involved because conventional theorems on the existence and uniqueness of the solution are inapplicable to differential equations with discontinuous right-hand side. As usual, regularization of discontinuous dynamic systems implies that the original system is replaced by a related system, the solution of which exists in the conventional sense. A sliding mode equation is then obtained by making the characteristics of the new system approach those of the original one.

All types of regularization of finite-dimensional systems, the right-hand side of which satisfies the Lipschitz condition beyond the discontinuity manifold, were shown in [29] to result in the same sliding mode equation whenever this equation was uniquely derived by the equivalent control method. Recall that according to the equivalent control method, the sliding mode equation describes the system dynamics under appropriate initial conditions in the discontinuity manifold and suitable continuous control function maintaining this system within the manifold. In terms of minimum phase systems [4], this equation would be called a zero dynamics equation and has also become standard in the literature.

Section 3 extends the equivalent control method to infinite-dimensional control systems governed by a semilinear differential equation in a Hilbert space with the infinitesimal operator, generating a strongly continuous semigroup. This extension, however, is hindered by major difficulties because the Lipschitz condition on the right-hand side of the plant equation, which is quite natural for finite-dimensional systems, is generally invalid for the infinite-dimensional systems due to unboundedness of the infinitesimal operator in the plant equation. Relating the discontinuous control law to the continuous one, the main result stated here legitimates the infinite time sliding mode equation for a wide class of infinite-dimensional semilinear systems.

Section 4 presents synthesis of a discontinuous control law, which imposes the desired dynamic properties as well as robustness with respect to matched disturbances on the closed-loop system. If the undisturbed motion

of the system contains two components, one of them is stable and another one belongs to a finite-dimensional subspace, then the control synthesis is split into two independent synthesis procedures. The first procedure uses the standard finite-dimensional setting while the second procedure is carried out within the infinite-dimensional subspace of the exponentially stable internal dynamics. The latter procedure is developed in Section 2 of the present work. As an illustration of the capabilities of the procedure, a scalar unit controller for a minimum phase system of finite relative degree is constructed.

Section 5 presents the conclusions.

## Notation

The notation is fairly standard. For any Hilbert space  $H$ , the inner product and norm are denoted by  $(\cdot, \cdot)_H$  and  $\|\cdot\|_H$ , respectively. Subscripts of the norm and inner product are often omitted if any confusion does not arise. Symbol  $\text{span}\{x_i\}_{i=1}^r$  stands for the linear space spanned by the vectors  $x_i \in H$ ,  $i = 1, \dots, r$ . By  $L_\infty(a, b; H)$  we denote the set of  $H$ -valued functions  $f(t)$  such that  $(f(\cdot), x)$  is Lebesgue measurable for all  $x \in H$  and  $\text{ess max}_{t \in (a, b)} \|f(t)\| < \infty$ . For Hilbert spaces  $H, U$ , and  $L(U, H)$  denotes the Hilbert space of bounded linear operators from  $U$  to  $H$ ;  $L(H) := L(H, H)$ .

A strongly continuous semigroup on a Hilbert space, generated by the infinitesimal operator  $A$ , is denoted by  $T_A(t)$ ;  $A^*$  stands for the adjoint operator of  $A$ . Recall that (see, e.g., [17] for details): a) an operator family  $\{T(t) \in L(H)\}_{t \geq 0}$  forms a strongly continuous semigroup on  $H$  if the identity  $T(t + \tau) = T(t)T(\tau)$  is satisfied for all  $t, \tau \geq 0$  and functions  $T(t)x$  are continuous with respect to  $t \geq 0$  for all  $x \in H$ ; b) the induced operator norm  $\|T(t)\|$  of the semigroup satisfies the inequality  $\|T(t)\| \leq \omega e^{\beta t}$ ,  $t \geq 0$  with some growth bound  $\beta$  and some  $\omega > 0$ ; and c) if the growth bound  $\beta$  is negative then  $T(t)$  is exponentially stable.

The domain of  $A$  forms the Hilbert space  $\mathcal{D}(A)$  with the graph inner product defined as follows:

$$(x, y)_{\mathcal{D}(A)} = (x, y)_H + (Ax, Ay)_H, \quad x, y \in \mathcal{D}(A)$$

If  $\beta$  is the growth bound of the semigroup, then given  $\lambda > \beta$ , there holds  $(A - \lambda I)^{-1}H = \mathcal{D}(A)$  where  $I$  is the identity operator, and the norm of  $x \in \mathcal{D}(A)$  given by  $\|(A - \lambda I)x\|_H$  is equivalent to the graph norm  $\|x\|_{\mathcal{D}(A)}$  of  $\mathcal{D}(A)$ . In particular,  $\|x\|_{\mathcal{D}(A)} = \|Ax\|_H$  if  $A$  generates an exponentially stable semigroup. It should be noted that  $\mathcal{D}(A) \hookrightarrow H$ , i.e.  $\mathcal{D}(A) \subset H$ ,  $\mathcal{D}(A)$  is dense in  $H$  and the inequality  $\|x\|_H \leq \omega_0 \|x\|_{\mathcal{D}(A)}$  holds for all  $x \in \mathcal{D}(A)$  and some constant  $\omega_0 > 0$ .

## 12.2 Motivation: disturbance rejection in Hilbert space

We now discuss some attractive capabilities of sliding mode controllers in addressing infinite-dimensional systems. To begin with, let us consider a dynamical system

$$\dot{x} = e(x), \quad x(0) = x^0 \in H \quad (12.1)$$

in a real Hilbert space  $H$  enforced by the unit control

$$e(x) = -x/\|x\|$$

which undergoes discontinuities in the trivial manifold  $x = 0$ . The example, although extremely simple, illustrates the fact that discontinuous infinite-dimensional systems can be driven along discontinuity manifolds.

Since the norm  $\|x\| = \sqrt{(x, x)}$  in the Hilbert space is defined via the inner product  $(\cdot, \cdot)$ , then

$$\frac{1}{2} \frac{d\|x\|^2}{dt} = (x(t), \dot{x}(t)) = -\|x(t)\|^2$$

and therefore  $\|x(t)\| = (\|x^0\| - t)$  for  $t \leq \|x^0\|$ . Hence, in the infinite-dimensional system (12.1) starting from the time moment  $t = \|x^0\|$ , there appears a sliding mode in the discontinuity manifold  $x = 0$ . Clearly, the sliding mode is unambiguously constituted by the manifold equation  $x = 0$  regardless of uniformly bounded additive dynamic nonidealities  $h(x, t)$  such that  $\|h(x, t)\|_H < 1$  for all  $t \geq 0$ ,  $x \in H$ , which are rejected by the unit control (in this case the sign of the time derivative of the Lyapunov functional along the trajectories of the perturbed system  $\dot{x} = e + h$  remains negative). However, in general, neither the unit control belongs to the state space nor the discontinuity manifold is trivial, so that their synthesis presents a formidable problem.

According to the unit feedback approach, developed in this chapter, a linear discontinuity manifold  $cx = 0$  in the control space  $U$ , differed from the state space  $H$ , may be constructed in compliance with some performance criterion, particularly, according to the Lyapunov min-max approach, whereas a sliding mode in the manifold is enforced by the corresponding unit control  $M(x, t)e(cx) = -M(x, t)cx/\|cx\|_U$ , possibly with a nonunit gain  $M(x, t) \neq 1$ . This design idea is now illustrated for an uncertain dynamic system governed by a differential equation

$$\dot{x} = Ax + f(x, t) + bu(x, t), \quad x(0) = x^0 \in \mathcal{D}(A) \quad (12.2)$$

where the state  $x(t)$  and control signal  $u(x, t)$  are abstract functions with values in Hilbert spaces  $H$  and  $U$ , respectively;  $A$  is the infinitesimal generator of an exponentially stable semigroup  $T_A(t)$  on  $H$ ; and  $b \in L(U, H)$ . The operator function  $f(x, t)$  with values in  $H$  represents the system uncertainties, whose influence on the control process should be rejected. This function is assumed to be continuously differentiable in all arguments and satisfy the matching condition

$$f(x, t) = bh(x, t) \quad (12.3)$$

where the uncertain function  $h(x, t)$  has an a priori-known upper scalar estimate  $N(x) \in C^1$ , i.e.,

$$\|h(x, t)\|_U < N(x) \text{ for all } x \in H, t \geq 0 \quad (12.4)$$

In order to apply the afore-mentioned Lyapunov approach to the infinite-dimensional system (12.2), let us note that the positive definite solution  $W_A = \int_0^\infty T_A^*(t)T_A(t)dt$  to the Lyapunov equation  $W_AA + A^*W_A = -I$  assigns the quadratic Lyapunov functional  $V(x) = (W_Ax, x)$  for the nominal system  $\dot{x} = Ax$ . Then, taking into account (12.3) and differentiating the Lyapunov functional with respect to  $t$  along the trajectories of the perturbed system (12.2), we obtain

$$\begin{aligned} dV/dt &= (W_A\dot{x}(t), x(t)) + (W_Ax(t), \dot{x}(t)) = -(x, x) + \\ 2(W_Ax(t), b(u + h)) &= -(x, x) + 2(b^*W_Ax(t), u + h) \end{aligned} \quad (12.5)$$

A straightforward application of the Lyapunov min-max approach, which requires to minimize the right-hand side of (12.5) under the control constraint  $\|u(\cdot)\|_U \leq M = \text{const}$ , results in the unit control

$$u(x) = -Me(b^*W_Ax) = -M \frac{b^*W_Ax}{\|b^*W_Ax\|_U} \quad (12.6)$$

Given the state-dependent gain  $M = N(x)$ , the time derivative of the Lyapunov functional along the trajectories of the perturbed system (12.2) driven by the unit control (12.6) becomes negative

$$dV/dt \leq -(x, x) \leq -\frac{1}{\|W_A\|} (W_Ax, x) = -\frac{1}{\|W_A\|} V(x) \quad (12.7)$$

for all  $x \in H$  (including the discontinuity manifold!), regardless of the admissible plant perturbations  $f(x, t)$ . It follows exponential stability of the closed-loop system. Thus, the unit control (12.6) with the gain  $M = N(x)$  rejects any admissible perturbation  $f(x, t)$  and imposes desired dynamic

and robustness properties on the uncertain system (12.2). Along with relative simplicity of the implementation of unit control signals (cf. that of [8, 13]), these properties make attractive the use of unit controllers in the infinite-dimensional case.

Unfortunately, applying the discontinuous control law to the infinite-dimensional system results in an immediate difficulty: *how should one define the meaning of the solution of the differential equation with discontinuous right-hand side?* In the subsequent presentation, this fundamental problem is studied for a wide class of semilinear both stable and unstable dynamic systems and after that the unit feedback approach is developed to this class of infinite-dimensional systems.

## 12.3 Mathematical description of sliding modes in Hilbert space

It is well-known [5, 11], that a wide class of real-life control problems including those, mentioned in the Introduction, may be described by differential equations in Hilbert spaces. In this section, the preliminary machinery needed to define the concept of sliding mode in a Hilbert space is presented. To describe behavior of a discontinuous semilinear system in its discontinuity manifold a sliding mode equation is introduced. The validity of the sliding mode equation is shown by means of the regularization principle.

### 12.3.1 Semilinear differential equation

We will study infinite-dimensional dynamic systems

$$\dot{x} = Ax + f(x, t) + bu(x, t), \quad t > 0, \quad x(0) = x^0 \in \mathcal{D}(A) \quad (12.8)$$

driven in the Hilbert space  $H$  by a discontinuous control action  $u$ . From now on the infinitesimal operator  $A$  is assumed to generate a strongly continuous semigroup  $T_A(t)$  on  $H$ , rather than an exponentially stable semigroup, whereas  $f(x, t)$  and  $b$  are the same as before. All the assumptions made above guarantee that the unforced initial-value problem (12.8) subject to  $u(x, t) \equiv 0$  locally have a unique strong solution  $x(t)$ , which is defined as follows (see [11, 17] for details).

**Definition 127** *A continuous function  $x(t)$ , defined on  $[0, T)$ , is a strong solution of the differential equation (12.8) under  $u = 0$  iff  $\lim_{t \downarrow 0} \|x(t) - x^0\|_H = 0$ , and  $x(t)$  is continuously differentiable and satisfies the equation for  $t \in (0, T)$ .*

Thus, our development is confined to the investigation of the strong solution of the initial-value problem, although all the results seem to admit generalization to the case when the solution of the problem is defined in a mild sense as a solution to a corresponding integral equation. However, such a relaxation of the solution concept, which is known [15] to guarantee the existence and uniqueness of the mild solution to (12.8) under  $u = 0$  even for integrable in  $t$  functions  $f(x, t)$ , is not trivial and is beyond the scope of this chapter.

We conclude this subsection with examples of differential generators of strongly continuous semigroups to be used subsequently.

**Example 128** *The differential operator (see [5] for details)*

$$\mathcal{A} = \rho^{-1}(y)\{\partial[k(y)(\partial/\partial y)]/\partial y - q(y)\}$$

*with continuously differentiable everywhere positive functions  $\rho(y), k(y)$ , continuous nonnegative function  $q(y)$  and domain*

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \{\xi(y) \in L_2(0, 1) : \partial^2 \xi(y)/\partial y^2 \in L_2(0, 1) \\ \mu_0^2 \xi(0) - \nu_0^2 \partial \xi(0)/\partial y &= \mu_1^2 \xi(1) + \nu_1^2 \partial \xi(1)/\partial y = 0\} \end{aligned}$$

*where  $\mu_i^2 + \nu_i^2 \neq 0, i = 0, 1$ , generates a strongly continuous semigroup  $T_{\mathcal{A}}(t)$  on the Hilbert space  $L_2(0, 1)$  and has compact resolvent. If  $q(y) > 0$  for all  $y \in [0, 1]$  then  $T_{\mathcal{A}}(t)$  is exponentially stable.*

**Example 129** *The differential operator (see [5] for details)*

$$\hat{\mathcal{A}} = \begin{bmatrix} 0 & I \\ \mathcal{A} & -\alpha(y) \end{bmatrix}$$

*with the same operator  $\mathcal{A}$  as before and a continuous nonnegative function  $\alpha(y)$  generates a strongly continuous semigroup  $T_{\hat{\mathcal{A}}}(t)$  on the Hilbert space  $L_2(0, 1) \oplus L_2(0, 1)$ . If  $q(y)$  and  $\alpha(y)$  are positive for all  $y \in [0, 1]$ , then  $T_{\hat{\mathcal{A}}}(t)$  is exponentially stable.*

### 12.3.2 Discontinuous control input and sliding mode equation

Throughout the chapter we deal with discontinuous unit-wise control functions  $u(x, t)$ , which are continuously differentiable in all arguments everywhere but a linear discontinuity manifold

$$cx = 0 \tag{12.9}$$

with  $c \in L(H, S)$  and  $S$  being a Hilbert space. Since arbitrary subspace of a Hilbert space is complementable [14], then

$$H_1 = \ker c = \{x_1 \in H : cx_1 = 0\} \subseteq H$$

is complementable as well, i.e., there exists  $H_2 \subseteq H$  such that

$$H = H_1 \oplus H_2$$

To describe sliding motion in the infinite-dimensional system let us rewrite equation (12.8) in terms of variables  $x_1(t) \in H_1$  and  $x_2(t) \in H_2$  :

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + P_1f(x_1, x_2, t) + P_1bu(x_1, x_2, t), \quad t \geq 0 \\ x_1(0) &= x_1^0 \end{aligned} \quad (12.10)$$

$$\begin{aligned} \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + P_2f(x_1, x_2, t) + P_2bu(x_1, x_2, t), \quad t \geq 0 \\ x_2(0) &= x_2^0 \end{aligned} \quad (12.11)$$

Here  $x_1(t) \oplus x_2(t) = x(t)$ ,  $x_1^0 \oplus x_2^0 = x^0$ ,  $P_i$  is the projector on the subspace  $H_i$ ,  $A_{ij} = P_iA_j$  is the operator from  $H_j$  to  $H_i$ , and  $A_j = A|_{H_j}$  is the operator restriction on  $H_j$ ,  $i, j = 1, 2$ . Clearly, the discontinuity manifold (12.9), written through the new coordinates, takes the form  $x_2 = 0$ .

We will assume the following assumptions

- 1) the operator  $P_2b$  from  $U$  to  $H_2$  is boundedly invertible, i.e. the operator  $(P_2b)^{-1}$  from  $H_2$  to  $U$  is bounded (the operator  $G = P_1b(P_2b)^{-1}$  from  $H_2$  to  $H_1$  is then bounded as well);
- 2) the operator  $\tilde{A} = A_{11} - GA_{21}$  generates an exponentially stable semi-group  $T_{\tilde{A}}(t)$  on  $H_1$ , i.e.  $\|T_{\tilde{A}}(t)\| \leq \omega e^{-\beta t}$ ,  $\omega, \beta > 0$ ; and
- 3) the operator  $G_0 = \tilde{A}G$  from  $H_2$  to  $H_1$  is governed by  $A_{12}$  in the sense that  $\mathcal{D}(G_0) \subseteq \mathcal{D}(A_{12})$  and  $\|G_0y\| \leq k\|A_{12}y\|$  for all  $y \in \mathcal{D}(G_0)$  and some  $k > 0$ .

According to the equivalent control method, the sliding mode equation

$$\dot{x}_1 = \tilde{A}x_1 + [P_1 - GP_2]f(x_1, 0, t) \quad (12.12)$$

in the discontinuity manifold  $x_2 = 0$  is derived by substituting the continuous solution

$$u_{eq}(x, t) = -(P_2b)^{-1}[A_{21}x_1 + P_2f(x_1, 0, t)]$$

of the equation  $\dot{x}_2 = 0$  into (12.10) for  $u(x_1, x_2, t)$ . Since the external disturbance satisfies the matching condition, then  $[P_1 - GP_2]f(x, t) = [P_1 -$

$P_1 b(P_2 b)^{-1} P_2] b h(x, t) = 0$  and hence the sliding mode equation (12.12) becomes disturbance-independent:

$$\dot{x}_1 = \tilde{A} x_1 \quad (12.13)$$

The validity of the sliding mode equation (12.13) for the class of infinite-dimensional systems (12.8) is guaranteed by the following theorem.

**Theorem 130** *Let the above assumptions 1)-3) be satisfied and let, for all  $\delta$  in some interval  $(0, \delta_0)$  the discontinuous control function  $u(x, t)$  in system (12.10), and (12.11), be replaced in the  $\delta$ -vicinity  $\|x_2\|_{\mathcal{D}(A_2)} \leq \delta$  of the manifold  $x_2 = 0$  by a signal  $u^\delta(x, t)$  such that there exists a unique globally-defined strong solution  $x^\delta(t) = [x_1^\delta(t), x_2^\delta(t)]$  of the system under  $u(x, t) = u^\delta(x, t)$ , which belongs to the boundary layer  $\|x_2^\delta(t)\|_{\mathcal{D}(A_2)} \leq \delta$  for all  $t \geq 0$ . Then*

$$\lim_{\delta \rightarrow 0} \|x_1^\delta(t) - x_1(t)\| = 0 \text{ uniformly in } t \geq 0 \quad (12.14)$$

where  $x_1(t)$  is the solution of the sliding mode equation (12.13) with the initial value satisfying the condition  $\|x_1^\delta(0) - x_1(0)\| \leq \delta$ .

The proof of Theorem 130 is given in [21].

In order to describe sliding motions in infinite-dimensional systems, Theorem 130 utilizes the following regularization scheme. In the  $\delta$ -vicinity of the discontinuity manifold (12.9) the original system (12.8) is replaced by a new one, which takes into account all possible imperfections in the new control  $u^\delta(x, t)$  (e.g. delay, hysteresis, saturation, etc.) and for which there exists a globally-defined strong solution. The types of imperfections are not specified and the entire class of functions  $u^\delta(x, t)$  leading the system motion in the  $\delta$ -vicinity of the discontinuity manifold is considered. Whichever imperfections appear in a real life system, usually they do not violate the assumptions made on  $u^\delta(x, t)$ . Theorem 130 states that all types of regularization of the infinite-dimensional system lead to the same limiting motion as the value of  $\delta$  tends to zero. This motion is governed by the sliding mode equation, which is a result of the equivalent control method.

Assumption 1, inherited from the finite-dimensional case, ensures uniqueness of the sliding mode (see [29], p. 16). Assumption 2, caused by the complexities of the limiting process for an infinite time interval rather than by the specifics of our problem, complies with the typical requirement for any automatic control system to be asymptotically stable. Assumption 3 governs the operator  $G_0$  by the operator  $A_{22}$ . If these operators were differential, then Assumption 3 means that the order of the former is less or

equivalent to that of the latter. This assumption is intrinsic for infinite-dimensional systems: if it fails to hold then the sliding mode becomes ill-posed in the sense that different imperfections may result in different sliding modes.

Another assumption that requires the external disturbance to satisfy the matching condition (12.3), can be replaced by the weakened assumption

$$[P_1 - GP_2]f(x_1, 0, t) = F(t) \quad (12.15)$$

which means the function  $[P_1 - GP_2]f(x_1, 0, t)$  is state-independent. The following extension of Theorem 130 is introduced to be used subsequently.

**Theorem 131** *Let along with assumptions 1)-3), condition (12.15) (rather than (12.3)) be satisfied. Then all the regularization of the discontinuous system (12.8) lead to the same sliding mode equation (12.12).*

Hopefully, assumption (12.15) admits further relaxation, which is, however, beyond the scope of this chapter.

## 12.4 Unit control synthesis for uncertain systems with a finite-dimensional unstable part

According to the synthesis procedure for finite-dimensional systems, proposed in [29], the design of sliding mode controllers consists of two steps. First, a sliding mode is designed to have the prescribed properties by a proper choice of a discontinuity manifold. And second, a discontinuous control is constructed to guarantee existence of the sliding motion along the manifold. Since the sliding mode equation is control-independent, this approach leads to decoupling of the original design problem into two independent problems and enables us to construct a control system which is insensitive to matched disturbances.

Clearly, the above procedure can be used in the infinite-dimensional setting (12.8) as well: the control problem is split into a selection of a discontinuity manifold (12.9) with the desired zero dynamics (12.13) and design of a discontinuous control, which ensures the motion of the system along this manifold. This idea was exemplified in Section 2 by the unit control-based disturbance rejection in the uncertain system with exponentially stable internal dynamics. Applicability of the discontinuous unit control law to the infinite-dimensional system, questioned there, is now resolved by Theorem 130.

Later, we will give conditions that allow us to reduce the infinite-dimensional control problem and use well-known synthesis procedures for finite-dimensional systems. Such a situation appears if the undisturbed motion of (12.8) under  $u(x, t) \equiv f(x, t) \equiv 0$  contains two partial components: one of them is stable and does not require to be corrected, and another one belongs to a finite-dimensional subspace.

### 12.4.1 Disturbance rejection in exponentially stabilizable systems

The aim of this subsection is to demonstrate how the uncertainties (12.3), and (12.4) in the exponentially stabilizable system (12.8) with finite-dimensional unstable part can be rejected by means of a unit stabilizing controller.

So, throughout this subsection we assume that the pair  $\{A, b\}$  is exponentially stabilizable and the spectrum  $\sigma(A) = \sigma_1(A) \cup \sigma_2(A)$  of the infinitesimal operator  $A$  consists of two parts: one of them,  $\sigma_1(A) = \{\lambda \in \sigma(A) : \operatorname{Re} \lambda \geq 0\}$ , is finite-dimensional and another one,  $\sigma_2(A) = \{\lambda \in \sigma(A) : \operatorname{Re} \lambda < 0\}$ , is in the open left half-plane. Recall that the pair  $\{A, B\}$  is said to be exponentially stabilizable if there exists  $D \in L(X, U)$  such that the operator  $A + BD$  generates an exponentially stable semigroup (we refer to [5] for details).

Let  $P_1$  and  $P_2$  be projectors corresponding to the spectral sets  $\sigma_1(A), \sigma_2(A)$ , respectively, and  $H_j = P_j H$ ,  $j = 1, 2$ . Then (see [11, Section 1.5])

- 1)  $H = H_1 \oplus H_2$ ,  $H_j$  are invariant with respect to  $A$ , i.e.,  $AH_j \subset H_j$ ,  $j = 1, 2$ ; and
- 2) the operator  $A_1 = A|_{H_1}$  is finite-dimensional, i.e.,  $H_1 = \mathbb{R}^n$ ;
- 3) the operator  $A_2 = A|_{H_2}$  generates an exponentially stable semigroup  $T_{A_2}(t)$  with some negative growth bound  $-\beta$ , i.e.,

$$\|T_{A_2}(t)\| \leq \omega e^{-\beta t}, \omega > 0. \quad (12.16)$$

If the operator  $A$  has compact resolvent then its spectrum  $\sigma(A) = \{\lambda_i\}_{i=1}^{\infty}$  would be discrete, and for any  $\beta > 0$  there would exist a number  $l$  such that  $\sigma_2(A) = \{\lambda_i\}_{i=l}^{\infty} < -\beta$ , and hence the growth bound of the semigroup  $T_{A_2}(t)$  could be arbitrarily prescribed.

The above properties of the operator  $A$  admit representation of system

(12.10), and (12.11) in the form

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + P_1 f(x_1, x_2, t) + P_1 b u(x_1, x_2, t), \quad t \geq 0 \\ x_1(0) &= x_1^0 \end{aligned} \quad (12.17)$$

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + P_2 f(x_1, x_2, t) + P_2 b u(x_1, x_2, t), \quad t \geq 0 \\ x_2(0) &= x_2^0 \end{aligned} \quad (12.18)$$

It should be pointed out that by virtue of  $P_1 b \in L(U, \mathbb{R}^n)$ , the subspace  $U_2 = \ker P_1 b = \{u \in U : P_1 b u = 0\}$  has the finite codimension  $l$ . Hence, there exists a finite-dimensional subspace  $U_1 = \mathbb{R}^m$  such that

$$U = U_1 \oplus \ker P_1 b$$

and due to (12.3) the finite-dimensional subsystem (12.17) takes the form

$$\dot{x}_1 = A_1 x_1 + B_1 [h_1(x, t) + u_1(x, t)] \quad (12.19)$$

where the partition

$$\begin{aligned} h(x, t) &= h_1(x, t) + h_2(x, t) \\ u(x, t) &= u_1(x, t) + u_2(x, t) \end{aligned} \quad (12.20)$$

of the exogenous signals  $h_1(x, t) \in U_1$ ,  $h_2(x, t) \in U_2$ ,  $u_1(x, t) \in U_1$ ,  $u_2(x, t) \in U_2$  is used;  $B_1 = P_1 b|_{U_1}$  and the matrix pair  $\{A_1, B_1\}$  is controllable, because the pair  $\{A, b\}$ , otherwise, it would not be exponentially stabilizable.

The solution to the afore-mentioned rejection problem is based on the deliberate introduction of sliding modes into the closed-loop system. Following the design procedure for controllable finite-dimensional systems proposed in [29, Chapter 10] we select such a discontinuity manifold

$$C x_1 = 0, C \in \mathbb{R}^{m \times l}, \det C B_1 \neq 0$$

that ensures exponential stability

$$\|x_1(t)\| \leq \omega \|x_1(T)\| e^{-\alpha t}, \quad t \geq T \quad (12.21)$$

of the sliding mode which arises, starting from some time moment  $T > 0$ , in the finite-dimensional system (12.19) under the control law

$$u_1(x) = -[N(x) + L \|x_1\|] \frac{C x_1}{\|C x_1\|} \quad (12.22)$$

where  $\alpha, \omega, L$  are positive constants and  $\alpha$  may be as large as desired. Formally, in order to specify the matrix  $C$  and constant  $L$  in an appropriate

manner, one should represent system (12.19) in the canonical form where the choice of  $C$  and  $L$  is straightforward and particularly given in Subsection 4.B. The sliding motion in (12.19) is then governed by the disturbance-independent equation

$$\dot{x}_1 = [A_1 - B_1(CB_1)^{-1}CA_1]x_1 \quad (12.23)$$

obtained through the equivalent control method by substituting the continuous solution

$$u_{1eq}(x, t) = -(CB_1)^{-1}CA_1x_1 - h_1(x, t)$$

of the equation  $C\dot{x}_1 = 0$  into (12.19) for  $u_1$ . Equation (12.18) is respectively rewritten as follows

$$\dot{x}_2 = A_2x_2 - B_{21}(CB_1)^{-1}CA_1x_1 + B_2[u_2(x, t) + h_2(x, t)], \quad t \geq T \quad (12.24)$$

where  $B_{21} = P_2b|_{U_1}$ ,  $B_2 = P_2b|_{U_2}$ . Due to (12.16), and (12.21), the unforced system (12.23), and (12.24) under the zero exogenous inputs  $u = f = 0$  is exponentially stable, and it remains to employ the results of Section 2 to reject the external disturbance  $h_2(x, t)$ . Putting  $W_{A_2} = \int_0^\infty T_{A_2}^*(t)T_{A_2}(t)dt$ , we design the second component

$$u_2(x, t) = -N(x) \frac{B_2^*W_{A_2}x_2}{\|B_2^*W_{A_2}x_2\|} \quad (12.25)$$

of the control  $u(x, t)$  in the unit form (12.16) that imposes the desired robustness property on the closed-loop system.

If the operator  $A$  has compact resolvent then, as mentioned earlier, the growth bound  $-\beta$  of the semigroup  $T_{A_2}(t)$  and, consequently, the value of  $\|W_0\|^{-1}$  could be specified arbitrarily large in magnitude. Combining with (12.7), and (12.21), this would guarantee the desired decay rate of the closed-loop system (12.17), (12.18), (12.22), and (12.25) whenever the pair  $\{A, b\}$  is approximately controllable. For convenience of the reader we recall (see, e.g., [5]):

**Definition 132** *The pair  $\{A, b\}$  is approximately controllable if and only if the reachability domain*

$$R = \{x : x = \int_0^t e^{A(t-\tau)}bu(\tau)d\tau, u \in L_\infty(0, \infty; U), t > 0\}$$

*is dense in the state space  $X$ .*

Summarizing, the following theorem has been proved.

**Theorem 133** *Let the unstable part of the unforced dynamics of (12.8) under  $u = f = 0$  be finite-dimensional and let the pair  $\{A, B\}$  be exponentially stabilizable. Then the uncertain infinite-dimensional system (12.8) is exponentially stabilizable by the discontinuous unit controller (12.20), (12.22), and (12.25) which imposes the robustness property with respect to admissible perturbations (12.3), and (12.4) on the closed-loop system. Furthermore, if  $A$  has compact resolvent and the pair  $\{A, b\}$  is approximately controllable, then the decay rate of the closed-loop system may be specified as large as desired.*

**Example 134** *To exemplify the above result let us consider a distributed parameter system described by the parabolic partial differential equation*

$$\begin{aligned} \partial Q(y, t)/\partial t &= \partial^2 Q(y, t)/\partial y^2 + b(y)[u(Q, t) + h(Q, t)] \quad t > 0 \\ Q(y, 0) &= Q_0(y), \quad 0 \leq y \leq 1 \end{aligned} \quad (12.26)$$

*with Dirichlet boundary conditions*

$$Q(0, t) = Q(1, t) = 0, \quad t \geq 0 \quad (12.27)$$

*Here  $Q_0(y)$  is a scalar twice continuously differentiable initial distribution, which satisfies the boundary conditions (12.27);  $b(y)$  is a scalar quadratically integrable function, all Fourier coefficients of which are nonzero;  $u(Q, t)$  is a scalar control function; and  $h(Q, t)$  is a scalar unknown disturbance to be rejected, an upper estimate  $N(Q) \in C^1$ , which is known a priori.*

*It is required to design a feedback control law that imposes the desired decay rate  $-\alpha$  as well as robustness with respect to matched disturbances on the closed-loop system.*

*If along with the operator  $b$  of the multiplication by the function  $b(y) \in L_2(0, 1)$  we introduce the operator  $A = -\partial^2/\partial y^2$  of double differentiation with the dense domain  $\mathcal{D}(A) = \{\xi(y) \in L_2(0, 1) : \partial^2 \xi(y)/\partial y^2 \in L_2(0, 1), \xi(0) = \xi(1) = 0\}$ , then the boundary-value problem (12.26), (12.27) can be rewritten as the differential Equation (12.8) in the Hilbert space  $L_2(0, 1)$ . According to Example 128, the operator  $A$  generates a strongly continuous semigroup and has compact resolvent. Furthermore, the pair  $\{A, b\}$  is approximately controllable by virtue of the assumption on the function  $b(y)$  [5, p. 63]. Hence Theorem 133 is applicable to system (12.26), and (12.27).*

*In order to design a unit control-based solution to the stabilization problem stated above, let us select such a number  $n > 1$  that  $\pi^2(n+1)^2 \geq \alpha$  and decouple the spectrum  $\{-(\pi i)^2\}_{i=1}^{\infty} = \{-(\pi i)^2\}_{i=1}^n + \{-(\pi i)^2\}_{i=n+1}^{\infty}$  of*

*A into two parts. Then*

$$\begin{aligned} H_1 &= \text{span}\{\sin \pi iy\}_{i=1}^n, H_2 = \{\sin \pi iy\}_{i=n+1}^\infty, U_1 = \mathbb{R}^1, U_2 = \{0\} \\ A_1 &= \text{diag}\{-(\pi i)^2\} \in \mathbb{R}^{n \times n}, B_1 = [P^1 b(\cdot), \dots, P^n b(\cdot)]^T \\ P^i b &= 2 \int_0^1 b(y) \sin \pi iy dy, \quad i = 1, \dots, n \\ x_1(t) &= [P^1 Q(\cdot, t), \dots, P^n Q(\cdot, t)]^T \in H_2 \end{aligned}$$

*By virtue of the special choice of the row  $C = (C_1, \dots, C_n)$  and constant  $L$  in (12.22) the desired decay rate (12.21), and robustness with respect to the matched disturbances are imposed on the closed-loop system (12.22), (12.26), and (12.27) in the finite-dimensional subspace  $H_1$ . Since the internal dynamics in  $H_2$  is of the desired decay rate by construction then, due to triviality of the subspace  $U_2$  the resulting control law (12.20) consists of the first component (12.22) only. Thus, the unit controller*

$$u(Q) = -[N(Q) + L\sqrt{\sum_{i=1}^n (P^i Q)^2}] \text{sign}(\sum_{i=1}^n C_i P^i Q)$$

*gives a solution to the stated stabilization problem.*

*We conclude the example with a remark that the closed-loop system driven by the unit control signal is of the desired decay rate even in the case when a point-wise action  $b(y)u(Q) = \delta(y - y_0)u(Q)$ ,  $y_0 \in (0, 1)$  is applied. To interpret the state equation in a rigorous manner, one should involve another "extended" Hilbert space (e.g., a Sobolev space) where multiplication by the Dirac function is a bounded operator (see [13] for details).*

## 12.4.2 Disturbance rejection in minimum phase systems

The problems considered in this subsection are to make the output

$$z(t) = [s, x(t)], \quad s \in H \tag{12.28}$$

of the uncertain system (12.8) tend to zero as closely as desired and to ascertain conditions that ensure exponential stability of the closed-loop system. For the sake of simplicity, the development is confined to the scalar output, however, the extension to the systems with arbitrary finite-dimensional output is straightforward.

Throughout this subsection we assume that system (12.8), and (12.28) has a finite relative degree

$$r := \min\{i = 1, 2, \dots : b^* A^{*i-1} s \neq 0\}$$

and  $s \in \mathcal{D}(A^{*r})$ . It follows

$$\begin{aligned} z_i(t) &:= z^{(i-1)}(t) = (A^{*^{i-1}} s, x(t)), \quad i = 1, 2, \dots, r \\ z^{(r)}(t) &= [A^{*r} s, x(t)] + (A^{*^{r-1}} s, f(x(t), t) + bu[x(t), t]) \end{aligned} \quad (12.29)$$

Partitioning the state space as

$$\begin{aligned} H &= H_1 \oplus H_2, \quad H_1 = \text{span}\{A^{*^{i-1}} s\}_{i=1}^r \\ H_2 &= \{x \in H : (A^{*^{i-1}} s, x) = 0, \quad i = 1, 2, \dots, r\} \end{aligned}$$

and exploiting (12.29), let us represent the original system (12.8), subjected to the matched disturbance (12.3), in terms of  $x_1(t) = \sum_{i=1}^r z_i(t) A^{*^{i-1}} s \in H_1$ ,  $x_2(t) \in H_2$ :

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \dots, \dot{z}_{r-1}(t) = z_r(t) \\ \dot{z}_r(t) &= [A^{*r} s, x(t)] + (b^* A^{*^{r-1}} s, u_1(x, t) + h_1(x, t), t) \end{aligned} \quad (12.30)$$

$$\begin{aligned} \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_{21}[u_1(x, t) + h_1(x, t)] + \\ &\quad B_{22}[u_2(x, t) + h_2(x, t)] \end{aligned} \quad (12.31)$$

where  $U = U_1 \oplus U_2$ ,  $U_1 = \text{span}\{b^* A^{*^{r-1}} s\}$ ,  $U_2 = \{u \in U : (b^* A^{*^{r-1}} s, u) = 0\}$ ,  $u_i, h_i \in U_i$ ,  $B_{2i} = P_2 b|_{U_i}$ ,  $A_{2i} = P_2 A|_{H_i}$ ,  $i = 1, 2$ , and  $P_2$  is the projector on  $H_2$ . It should be noted that the operator  $A_{21}$ , defined everywhere in  $H_1$  ( $A_{21}$  is a densely defined operator on the finite-dimensional space  $H_1$  and hence  $\mathcal{D}(A_{21}) = H_1$ ), is bounded.

The solution to the above problem is based on the deliberate introduction of sliding modes in the manifold

$$cx = \sum_{i=1}^r c_i (A^{*^{i-1}} s, x) = 0 \quad (12.32)$$

where parameters

$$c_r = 1, \quad c_{r-1} = -\sum_{i=1}^{r-1} \mu_i, \quad c_{r-2} = \sum_{i < k} \mu_i \mu_k, \dots, \quad c_1 = (-1)^{r-1} \prod_{i=1}^{r-1} \mu_i$$

are specified in such a manner to place the roots of the characteristic polynomial of the equation

$$\sum_{i=1}^r c_i z_i(t) = \sum_{i=1}^r c_i z^{(i-1)}(t) = 0 \quad (12.33)$$

in the open left-half plane at the desired locations  $\mu_i$ ,  $i = 1, \dots, r-1$ . We demonstrate that the discontinuous unit control law

$$u_1(x) = -M(x) \frac{b^* A^{*^{r-1}} s}{\|b^* A^{*^{r-1}} s\|} \text{sign}(cx) \quad (12.34)$$

with

$$M(x) = \gamma + N(x) + \frac{\|A^{*r} s\|}{\|b^* A^{*r-1} s\|} \|x(t)\| + \frac{\sum_{i=1}^{r-1} |c_i| |(A^{*i} s, x)|}{\|b^* A^{*r-1} s\|}, \quad \gamma > 0 \quad (12.35)$$

drives system (12.8) to the discontinuity manifold (12.32) for a finite time moment. Indeed, differentiating the functional  $V(t) = \frac{1}{2}[cx(t)]^2$  along the trajectories of (12.30) and utilizing (12.4), (12.30), (12.34), and (12.35), we obtain

$$\dot{V}(t) = cx(t)c\dot{x}(t) = cx(t)\{\sum_{i=1}^{r-1} c_i z_{i+1}(t) + \langle A^{*r} s, x(t) \rangle + \langle b^* A^{*r-1} s, h_1(x(t), t) \rangle - \|b^* A^{*r-1} s\| M(x) \text{sign}(cx(t))\} \leq -2\gamma\sqrt{V(t)}$$

that gives rise to (12.33) for  $t \in [T, \infty)$ , where  $T = \gamma^{-1}\sqrt{V(0)}$ . In order to reproduce this conclusion, one should note that for all  $t \geq 0$  arbitrary solution  $V(t)$  to the latter inequality is majored  $V(t) \leq V_0(t)$  by the solution to the differential equation  $\dot{V}_0(t) = -2\gamma\sqrt{V_0(t)}$ , initialized with the same initial condition  $V_0(0) = V(0)$ . Since  $V_0(t) = 0$  for all  $t \geq T$ , then  $V(t)$  vanishes after the finite time moment  $T$ .

Thus, starting from the time moment  $T = \gamma^{-1}\sqrt{V(0)}$ , in the finite-dimensional system (12.30), driven by the unit control signal (12.34), there appears the sliding mode (12.33), which results in the desired decay rate  $-\alpha = \max_{1 \leq i \leq r} \text{Re } \mu_i$  of the variable  $x_1(t) = \sum_{i=1}^r z_i(t) A^{*i-1} s \in H_1$ :

$$\|x_1(t)\| \leq \omega \|x_1(T)\| e^{-\alpha t}, \quad t \geq T, \quad \omega = \text{const.} \quad (12.36)$$

In order to derive the sliding mode equation (12.33) one needs to substitute the continuous solution

$$u_{1eq}(x, t) = -\frac{\sum_{i=1}^r c_i z_{i+1} + (A^{*r} s, x)}{\|b^* A^{*r-1} s\|} - h_1(x, t)$$

of the equation  $c\dot{x}(t) = 0$  into (12.30) for  $u_1$ . By the same substitution, Equation (12.31) is rewritten as follows:

$$\begin{aligned} \dot{x}_2 &= \tilde{A}x_2 + A_{21}x_1 - \\ &\frac{B_{21}[\sum_{i=1}^r c_i z_{i+1} + (A^{*r} s, x_1)]}{\|b^* A^{*r-1} s\|} + B_{22}[u_2(x, t) + h_2(x, t)], \quad t \geq T \end{aligned} \quad (12.37)$$

where  $\tilde{A}x_2 = A_{22}x_2 - \frac{B_{21}(A^{*r} s, x_2)}{\|b^* A^{*r-1} s\|}$ .

If the operator  $\tilde{A}$  generates an exponentially stable semigroup, then due to (12.36) and boundedness of  $A_{21}$ , the second control component

$$u_2(x, t) = -N(x) \frac{B_{22}^* \int_0^\infty T_{\tilde{A}}^*(t) T_{\tilde{A}}(t) dt x_2}{\|B_{22}^* \int_0^\infty T_{\tilde{A}}^*(t) T_{\tilde{A}}(t) dt x_2\|} \quad (12.38)$$

similar to (12.25), rejects the external disturbance  $h_2(x, t)$  and ensures the exponential stability of the closed-loop system with the same line of reasoning as in Subsection 4.A. In fact,  $\tilde{A}$  generates an exponentially stable semigroup if and only if the input-output system (12.8), and (12.28) is exponentially minimum phase. In analogy to the finite-dimensional theory (see, e.g., [4]), we define

**Definition 135** *Systems (12.8), and (12.28) is said to be exponentially minimum phase if its zero dynamics, subject to appropriate initial conditions and a suitable control producing output (12.28) identically zero, is exponentially stable.*

So, the following result has been shown.

**Theorem 136** *Let  $s \in \mathcal{D}(A^{*r})$  and let systems (12.8), and (12.28) be exponentially minimum phase and of the finite relative degree  $r$ . Then the uncertain system (12.8) is exponentially stabilizable by the composition  $u(x) = u_1(x) + u_2(x)$  of the unit controllers (12.34), and (12.38) and the closed-loop system is robust with respect to external disturbances (12.3), and (12.4).*

**Example 137** *To exemplify the constructive abilities of the above theorem let us modify Example 134 and replace the boundary conditions by the appropriate Neumann boundary conditions*

$$\partial Q(0, t) / \partial y = \partial Q(1, t) / \partial y = 0, \quad t \geq 0 \quad (12.39)$$

The Fourier coefficients of the function  $b(y)$  are no longer assumed to be nonzero, with the only exception being  $\int_0^1 b(y) dy$ . The operator  $A = -\partial^2 / \partial y^2$  of double differentiation is now defined in  $\mathcal{D}(A) = \{\xi(y) \in L_2(0, 1) : \partial^2 \xi(y) / \partial y^2 \in L_2(0, 1), \partial \xi(0) / \partial y = \partial \xi(1) / \partial y = 0\}$  and the boundary-value problems (12.26), and (12.39) are still represented as the differential Equation (12.8) in the Hilbert space  $L_2(0, 1)$ .

Specifying the system output (12.28) as follows

$$z(t) = \int_0^1 Q(y, t) dy \quad (12.40)$$

one can check that the input-output systems (12.26), (12.39), and (12.40) satisfy all the assumptions of Theorem 133. Indeed,  $s = 1$  and hence  $s \in \mathcal{D}(A^l)$  for the self-adjoint operator  $A$  and arbitrary integer  $l$ . Furthermore, differentiating (12.40) with respect to  $t$  along the solutions of (12.26), and employing integration by parts and applying the boundary conditions (12.39), yields

$$\dot{z}(t) = (u + h) \int_0^1 b(y) dy$$

with  $\int_0^1 b(y) dy \neq 0$ , which proves that systems (12.8), and (12.28) are of the relative degree  $r = 1$ . Finally, representing the solution

$$Q(y, t) = \int_0^1 G(y, \xi, t) Q_0(\xi) d\xi + \int_0^t \int_0^1 G(y, \xi, t - \tau) b(\xi) d\xi [v(Q, \tau) + \varphi(Q, \tau)] d\tau$$

of the Neumann boundary-value problems (12.26), and (12.39), via the Green function

$$G(y, \xi, t) = \sum_{j=0}^{\infty} \exp\{-k(\pi j)^2 t\} \cos \pi j y \cos \pi j \xi$$

one can show the exponential stability of the zero dynamics

$$\Theta(y, t) = \sum_{j=1}^{\infty} \int_0^1 \cos \pi j \xi Q_0(\xi) d\xi \exp\{-k(\pi j)^2 t\} \cos \pi j y$$

of (12.26), (12.39), and (12.40), written under appropriate initial conditions such that  $\int_0^1 Q(y, 0) dy = 0$ , and the suitable control signal  $u(Q, t) = -h(Q, t)$ , producing the system output identically zero.

Thus, Theorem 136 is applicable to systems (12.26), (12.39), and (12.40). According to the theorem, the output controller

$$u(Q) = -\frac{N(Q)}{\int_0^1 b(y) dy} \text{sign } z(t) \quad (12.41)$$

imposes a sliding mode along the manifold  $z = 0$  so that the closed-loop system is exponentially stable and robust with respect to the matched disturbances.

It is plausible that the output feedback

$$u(z) = -\frac{M}{\int_0^1 b(y) dy} \text{sign } z(t)$$

with a sufficiently large constant  $M > 0$  still drives the system to the discontinuity manifold  $z = 0$  and, consequently, imposes the desired dynamic property as well as robustness with respect to the matched disturbances on the closed-loop system. Generally, in order to construct the output controller in the case of the higher relative degree, the asymptotical observer of the time output derivatives, proposed in [19], could be utilized.

**Example 138** Let a distributed parameter system be governed by the hyperbolic partial differential equation

$$\begin{aligned} \partial^2 Q / \partial t^2 &= \partial^2 Q / \partial y^2 - 2\partial Q / \partial t + \\ &+ b(y)[u(Q, t) + h(Q, t)], 0 < y < 1, t > 0 \\ Q(y, 0) &= Q_0(y), \partial Q(y, 0) / \partial t = Q_1(y), 0 \leq y \leq 1 \end{aligned} \quad (12.42)$$

subject to the Neumann boundary conditions (12.39). The boundary-value problem (12.39), and (12.42) describe the oscillations of a homogeneous string, insulated at both ends, where the state vector consists of the deflection  $Q(y, t)$  of the string and its velocity  $\dot{Q}(y, t)$  at time moment  $t \geq 0$  and location  $y$  along the string. The initial distributions  $Q_0(y), Q_1(y)$  are twice continuously differentiable functions which satisfy the boundary conditions (12.39);  $b$ , and  $u$ , and  $h$  as well as the operator  $\mathcal{A}$  and the output  $z$ , utilized below, are the same as in Example 129.

If we introduce the operator

$$\tilde{\mathcal{A}} = \begin{bmatrix} 0 & I \\ \mathcal{A} & -2 \end{bmatrix}$$

then the boundary-value problems (12.39), and (12.42) can be represented as the differential Equation (12.8) in the Hilbert space  $L_2(0, 1) \oplus L_2(0, 1)$ . Due to [5], the operator  $\tilde{\mathcal{A}}$  generates a strongly continuous semigroup, however, the spectrum  $\sigma(\tilde{\mathcal{A}}) = \{-(\pi j)^2\}_{j=0}^{\infty}$  of  $\tilde{\mathcal{A}}$  contains zero eigenvalue and, therefore, the unforced systems (12.39), and (12.42) under  $u = h = 0$  are not asymptotically stable.

Verification of the assumptions of Theorem 136 are similar to that of Example 129, except that the input-output system (12.39), (12.40), and (12.42) are of the relative degree  $r = 2$ , since  $\dot{z}(t) = \int_0^1 \dot{Q}(y, t) dy$ ,

$$\ddot{z}(t) = (u + h) \int_0^1 b(y) dy$$

where  $\int_0^1 b(y) dy \neq 0$ . Thus, by applying Theorem 136 to (12.39), (12.40),

and (12.42), the control law

$$u(Q) = -\frac{N_1(Q)}{\int_0^1 b(y)dy} \text{sign} \{ \dot{z}(t) + z(t) \}$$
$$N_1(Q) > N(Q) + |\dot{z}(\cdot)| \quad (12.43)$$

which imposes a sliding mode along the manifold  $\dot{z} + z = 0$  (thereby yielding  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$ ), makes the closed-loop system (12.40), (12.39), (12.42), and (12.43) exponentially stable and robust to the matched disturbances.

## 12.5 Conclusions

Mathematical tools for discontinuous infinite-dimensional systems were developed. Sliding modes, appearing in the discontinuity manifold, were shown to be governed by the equation derivable by means of the equivalent control method. Based on the sliding mode equation the procedure of synthesis of both a manifold in the state space, such that if confined to this manifold the system has desired properties, and a discontinuous unit control law, which makes this manifold an area of attraction for the closed-loop system, was proposed. The controller, generated by the procedure, ensures the strong robustness properties against the matched disturbances. As an illustration of the capabilities of the proposed procedure, a scalar unit controller of an uncertain exponentially minimum phase dynamic system was constructed and applied to heat processes and distributed oscillators. Recent applications of the theory to adaptive control and identification of time-delay and distributed parameter systems may be found in [2, 22].

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# Chapter 13

## Application of Sliding Mode Control to Robotic Systems

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### 13.1 Introduction

Most research and applications in robot control deal with electrically actuated robots (this is generally due to ease of their use and low cost) but use of pneumatic and hydraulic systems is increasing. The latter are used in robotic systems when large forces and direct drive possibilities with high capabilities are required. This justifies their extensive use in industrial applications. This chapter deals with the application of sliding mode control and passive feedback systems for mechanical systems encountered in robotics involving pneumatic and hydraulic actuators. We show that sliding mode control is a viable approach for such systems. Simple examples are studied first in order to introduce a methodology for this kind of control techniques.

The mathematical model of a hydraulically or pneumatically actuated system is highly nonlinear and time-varying. Several energy conversions are present (electro-mechanical to hydraulic or pneumatic pressure and then to mechanical motion). Generally the control of such systems has been first based on classical or PID feedback approaches [1, 2]. Next the intent was to enhance the control by use of state space design and adaptive control

[3, 4, 5, 6]. Standard or linearization-based control design methods have some drawbacks for pneumatic and hydraulic systems; this is due to the lack of knowledge of the model and parameters. The approximation by locally linear models is not applicable [7, 8]. Consequently the well known control methods like the computed torque or classic controllers are not directly applicable.

Recent applications in robotics involve complex systems with regard to nonlinearities, time variations, and performance requirements [9, 10]. Linearization-based methods or computed torque have been suggested in robotics as an effective way of using the nonlinear model of the system in the control law [11]. However, the dynamic parameters used in the control law must match the real ones [12]. In practical cases, neglected dynamics remain after modeling (nonlinear frictions, thermodynamic or hydraulic effects and parameter variations). The inability to consider the total dynamic model is "penalizing" for decoupling and compensation. These problems are caused by the fact that thermodynamics parameters depend on initial conditions, on temperature, pressure, added to offsets and cable effects. Anyway, this kind of control is rather sophisticated and remains complex to be implemented in real time (for fast motion) [13]. Robotic applications revealed the need for further investigation in order to enhance control robustness and reduce the implementation complexity [14]. In those cases we found that it was more efficient to use passivity-based controllers and sliding mode approach which enhanced the robustness of control by exploiting the system's robotic properties [15, 16]. They provided good performances whatever the robot configuration and desired speed.

For such system the control structure requires robustness of the feedback controller to parameter changes and disturbances. These performances can be obtained by sliding mode control [17]. Many applications of variable structure control in robotics have been reported [18, 19, 20]. Exact modeling is not necessary, since the control is based only on knowledge of uncertainties or variation bounds of the system model[9].

The main objective of this chapter is the design of a robust control law by use of a sliding mode approach. The considered problem for sliding mode control design can be stated as follows: given a desired sliding manifold function of the system's states ( $s(x) = 0$ ), which can be nonlinear or time varying, determine a control (or input  $u$ ) such that sliding mode occurs on this sliding surface. Then the desired performance can be achieved by an involved reduced-order dynamics in the sliding regime. We show that it gives a viable alternative for high performance tasks in industrial applications. The control stability can be studied by use of Lyapunov theory and the method can be shown to obey passivity property.

The organization of this chapter is as follows. First we present some ba-

sis features on modeling of mechanical systems and the involved properties of frictions and inertia effects and actuator limits. Then we recall some key points on passive systems and hyperstability theory, in order to make comprehensive the approach of sliding mode control design based on system hyperstability. The passivity property is illustrated for different mechanical systems. Then the sliding mode control design by this approach is applied for some chosen examples and stability analysis is presented to emphasize the robustness and control parameters effects.

## 13.2 Modeling and properties of robotic systems

### 13.2.1 Dynamics of mechanical systems

#### Euler-Lagrange systems

For mechanical systems, the dynamic equations or the model can be formulated by means of energy quantities [21]. The model of a rigid mechanical system, with  $n$  degrees of freedom ( $n$  DOF), can be obtained by use of the Lagrange method [22, 23]:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \Upsilon \quad (13.1)$$

where  $L = E_c - E_p$  is the Lagrangian function,  $E_c = \frac{1}{2} \dot{q}^T M(q) \dot{q}$  is the kinetic energy, and  $E_p$  potential energy and  $\Upsilon$  are the applied external forces / torques.

Recall that  $q$ ,  $\dot{q}$ ,  $\ddot{q}$ ,  $\tau$  denote respectively the  $(n \times 1)$  vectors of joint positions, speeds, accelerations, and torques.  $M(q)$  is the  $(n \times n)$  generalized inertia matrix,  $G(q)$  is the  $(n \times 1)$  vector of gravitational forces. The matrix  $C(q, \dot{q})$  (Centripetal and Coriolis effects) is commonly obtained by use of Christoffel symbols and the matrix  $\frac{1}{2} \dot{M}(q) - C(q, \dot{q})$  is skew symmetric [24]. This leads to the general equation form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + F_v(\dot{q}) = \tau \quad (13.2)$$

where the term  $F_v(\dot{q})$  has as components  $F_{v_i}(\dot{q}_i)$  the friction and disturbance torques (all the friction effects, angular, linear and nonlinear terms and disturbances) for each joint.

## Physical properties

The following physical properties of the rigid robots (with revolute joints) can be used for control [25, 26].

- 1)  $\exists \alpha_0, \alpha_1 \in \mathbb{R}$  such that  $\alpha_0 I_n < M(q) < \alpha_1 I_n, \forall q$
- 2)  $\exists \alpha_2 \in \mathbb{R}$  such that  $\|C(q, z)\| < \alpha_2 \|z\|, \forall q, \forall z$
- 3)  $\exists \alpha_3 \in \mathbb{R}$  such that  $\|G(q)\| < \alpha_3, \forall q$
- 4) Frictions and load disturbance torques are bounded by [27]:  $\exists \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$  such that  $\|F_v(\dot{q})\| < \alpha_4 + \alpha_5 \|\dot{q}\|, \forall \dot{q}$ .
- 5) Real systems have limited velocities and accelerations, then we have  $\|\dot{q}\| < \dot{q}_{\max}$  and  $\|\ddot{q}\| < \ddot{q}_{\max}$ .

This limitation is introduced to take into account real physical limits of actuators and of power systems and perturbations.

The model equation (13.2) can be written in state space form with state components:  $x_1 = q, x_2 = \dot{q}$  ( $x = (x_1^T, x_2^T)^T$ ) and measurable output  $y = q = x_1$ :

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -M^{-1}(x_1) [C(x_1, x_2)x_2 + G(x_1) - \tau] \\ y = x_1 \end{cases} \quad (13.3)$$

### 13.2.2 Control design approach

Variable structure control systems design can be realized in several ways. For our point of view, passivity-based design or the hyperstability approach is the most straightforward and practical method. It allows us to exploit the physical properties of the system for the control design. Then the obtained control law is more suited to the system and easier to tune. In what follows, we recall some features of passive systems theory.

#### Passive systems and hyperstability

It is not our objective, in this section, to give a detailed presentation of passivity, but only to introduce the needed notions for control design (see details in [28, 29, 21, 30, 31, 32, 33]).

**Proposition 139** *A passive system verifies the following property [34]:*

$$E(t_1) = E(0) + E_s(0, t_1) - E_L(0, t_1)$$

where  $E(t_1)$  and  $E(0)$  represent, respectively, the system energy at time  $t_1$  and its initial value at  $t = 0$ ;  $E_s(0, t_1)$  is the supplied energy during  $[0, t_1]$ , and  $E_L(0, t_1)$  represents the lost energy dissipated in frictions during  $[0, t_1]$ .

For passive systems (with input  $u$  and output  $y$ ), the following property is always verified (the Popov inequality) [34]:

$$\exists \gamma : \gamma_0^2 < \infty, \quad \int_{t_0}^{t_1} y^T u dt \geq -\gamma_0^2 \quad (13.4)$$

For system (13.2) we can write the expression of time derivative of its kinetic energy:

$$\frac{1}{2} \frac{d}{dt} (\dot{q}^T M(q) \dot{q}) = \dot{q}^T [\tau - G(q)] \quad (13.5)$$

we obtain by integration [ $q_o = q(0)$ ]:

$$\int_0^t \dot{q}^T (\tau - G(q)) dt = \frac{1}{2} (\dot{q}^T(t_1) M(q(t_1)) \dot{q}(t_1)) - \frac{1}{2} (\dot{q}_o^T M(q_o) \dot{q}_o) \quad (13.6)$$

If we consider as the robot input  $u = [\tau - G(q)]$  and as output  $y = \dot{q}$  then the system transfer  $u = \tau - G(q) \rightarrow y = \dot{q}$  is passive.

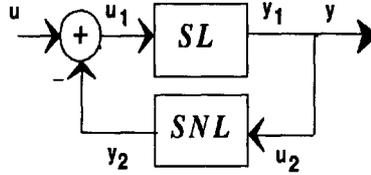


Figure 13.1: Linear system with nonlinear feedback

In order to introduce the control approach let us consider the dynamic system of Figure (13.1), where the transfer function of the linear system (block SL) is  $H(p)$  and we assume that  $(A, B)$  is completely controllable and  $(A, C)$  is completely observable for a minimal realization  $(A, B, C, D)$ . The input  $u$  is assumed equal to zero.

$$H(p) = \frac{y_1(p)}{u_1(p)} = D + C[pI - A]^{-1} B \quad (13.7)$$

$$\dot{x} = Ax + Bu_1 = Ax - B(y_2 - u) \quad (13.8)$$

$$y_1 = Cx + Du_1 = Cx - D(y_2 - u) \quad (13.9)$$

The feedback block (SNL), may be generally a nonlinear time variant subsystem:  $y_2 = f(u_2, t, \tau)$  with  $\tau \leq t$ . It is assumed to verify the sector

condition (see Figure 13.2)

$$\phi(0) = 0 \text{ and } k_1 u_2^2 \leq \phi(u_2) \cdot u_2 \leq k_2 u_2^2 \quad \forall u \in \mathbb{R} \quad (13.10)$$

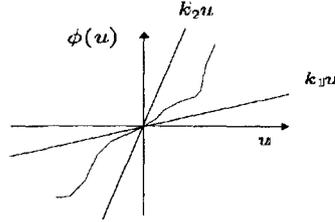


Figure 13.2: Conicity property for a nonlinear block ( $k_1 u_2^2 \leq \phi(u_2) \cdot u_2 \leq k_2 u_2^2 \quad \forall u_2 \in \mathbb{R}$ )

**Definition 140** *The nonlinear block (SNL) is passive if it verifies the Popov inequality (for all  $t_1 > 0$ ).*

$$\int_{t_0}^{t_1} y_2(\tau)^T \cdot u_2(\tau) d\tau \geq -\gamma_o^2, \text{ with } \gamma_o^2 < \infty \quad \forall t \geq 0 \quad (13.11)$$

**Definition 141** *A transfer function  $H(p)$  with  $p$  complex  $p = \sigma + j\omega$  is strictly positive real (SPR) if:*

- 1) *The poles of  $H(p)$  are in the half plane  $\text{Re}(p) < 0$ ,*
- 2)  *$H(j\omega) + H^T(-j\omega)$  is positive definite Hermitian for all real  $\omega$ .*

The linear system (block SL), can be characterized also by use of its minimal state space representation  $(A, B, C, D)$ , by means of the following lemma (**positive real lemma**).

**Lemma 142 positive real lemma:** *Let  $H(p)$  be an  $(m \times m)$  matrix of real rational function of the complex variable  $p$ , with  $H(\infty) < \infty$ , and  $(A, B, C, D)$  the minimal state space realization of  $H(p)$  (assumed controllable and observable). Then  $H(p)$  is positive real if and only if there exist real matrices  $P, L$ , and  $W$  with  $P$  symmetric positive definite such that:*

$$A^T P + PA = -LL^T, \quad PB = C^T - LW, \quad (13.12)$$

$$\text{and } W^T W = D + D^T \quad (13.13)$$

For the stability analysis of the class of nonlinear systems that can be represented in the form of Figure (13.1), the following theorems are very useful.

**Theorem 143 *Hyperstability:*** *The system of Figure (13.1) is hyperstable if and only if the transfer function of the linear block  $H(p)$  is positive real (PR) and the nonlinear time-varying feedback block is passive; Every solution  $x(x(0), t)$  of the system satisfies the following property:*

$$\|x(t)\| < \delta (\|x(0)\| + \gamma_o), \quad \delta > 0, \gamma_o > 0, \forall t > 0 \quad (13.14)$$

**Theorem 144 *Asymptotic hyperstability:*** *The system of Figure (13.1) is asymptotically hyperstable if and only if the transfer function of the linear block  $H(p)$  is strictly positive real (SPR) and the nonlinear time-varying feedback block is passive. Every solution  $x(x(0), t)$  of the system satisfies the property (13.14) with  $\lim_{t \rightarrow \infty} x(t) = 0$ , for any bounded input  $u_1(t)$ .*

### System decomposition and design objective

For control design, the problem is to find an equivalent feedback system which obviates passive parts of the system dynamics and allows us to find linear and nonlinear control terms in order to ensure asymptotic hyperstability [35]. In what follows we emphasize some interesting properties for connections of passive systems. We can note explicitly that by connecting hyperstable systems in parallel or in feedback, as shown by Figure (13.3), we obtain a hyper-stable system, i.e. hyperstability is preserved by connections in feedback or in parallel. This property is not valid for serial or cascade connections.

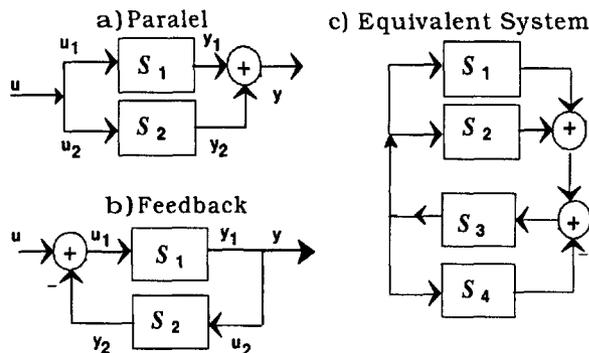


Figure 13.3: Passive systems associations

Connecting the passive blocks  $S_1$  and  $S_2$  in parallel as in Figure (13.3a), leads to:  $u = u_1 = u_2$  and  $y = y_1 + y_2$  and then we have:  $\int_0^{t_1} y^T u dt =$

$$\int_0^{t_1} y_1^T u_1 dt + \int_0^{t_1} y_2^T u_2 dt.$$

The same considerations can be made for the feedback connection of two passive blocks  $S_1$  and  $S_2$ , as in Figure (13.3b). We have  $u = u_1 + y_2$  and  $u_2 = y_1 = y$ , and then  $\int_0^{t_1} y^T u dt = \int_0^{t_1} y^T (u_1 + y_2) dt = \int_0^{t_1} y_1^T u_1 dt + \int_0^{t_1} y_2^T u_2 dt$ . We can then state the following lemma.

**Lemma 145** *Combination of two passive blocks in parallel gives a passive system.*

**Lemma 146** *Feedback connection of two passive blocks in parallel gives a passive system.*

Then for complex passive systems, often by choosing a new system state vector, we can find a passive equivalent feedback system as a combination of parallel and feedback connections of  $n$  passive subsystems. These properties are very important for stability analysis and control design of complex systems (Figure 13.3c).

For mechanical system as robot manipulators or legged robots, knowledge of physical properties allows us to point out passive subsystems in the system modeling. These passive blocks can be used to find an equivalent system and then complete this scheme by appropriate control terms in order to ensure asymptotic hyperstability.

### 13.2.3 Examples

#### Mass and spring systems

**A second-order example** Let us consider as an example a mass  $m$  attached to the ground through a spring with stiffness  $k$  (Figure 13.4).

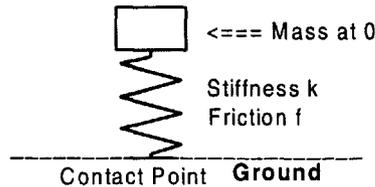


Figure 13.4: Mass and spring system

The damping coefficient (friction) is denoted  $f$ . The system equation can be written  $m\ddot{x} + f(\dot{x}) + k(x) = u$  (the input is either the gravity force  $u = -mg$  or zero if we assume its compensation by a spring initial compression) for a linear case we have  $f(\dot{x}) = f.\dot{x}$  and  $k(x) = k.x$ .

This system, in free motion, starting from an initial position with some initial velocity has a behavior like the trajectory represented in the phase space and versus time by figure (13.5). This behavior is function of the system parameters.

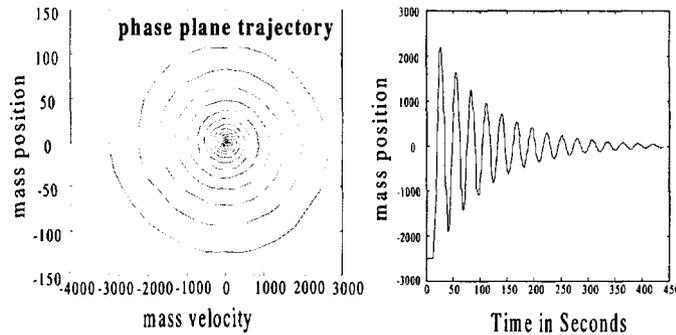


Figure 13.5: Trajectories in the phase plane and versus time

Let us now assume that as input  $u$ , we can modify the spring length or position of its attachment to the mass (or simply the spring force by an additional term  $u$ ):  $\ddot{x} + \frac{1}{m}f.\dot{x} + \frac{1}{m}k.x = \frac{1}{m}u$ . Note that for this system, if we consider as input  $u$  and as output the position  $x$  ( $H(p) = \frac{1}{mp^2 + fp + k}$ ), the system is not passive because it has a relative degree greater than unity.

**Remark 147** We can consider as output  $y$  a function of the velocity  $\dot{x}$  and the position  $x$ , such as the transfer from  $u$  to  $y$  to be SPR. For example, we can take  $y = \dot{x} + \lambda x$ , so we have as transfer function  $H(p) = \frac{p + \lambda}{mp^2 + fp + k}$  and then this system is SPR if some conditions on the transfer function parameters are respected (all coefficients positive and  $\lambda < \frac{f}{m}$ ). This key point shows that use of an auxiliary signal  $y$  is important for two main features: 1) its use allows us to involve, for the control performance, the passivity characteristic of the system (we will see later that the system dynamics appears as a feedback block in the equivalent feedback system); 2) this choice must respect the dynamic of the system:  $\lambda < \frac{f}{m}$  means that the introduced zero, in order to render the transfer function SPR, must be compatible with the system damping ratio.

For stabilization (at  $x = 0$ ), PD control can be applied to this system. Let us consider the following control law  $u = -k_0(\dot{x} + \lambda_0 x)$ . In closed-loop, we obtain  $\ddot{x} + \frac{1}{m}(f.\dot{x} + k_0\dot{x}) + \frac{1}{m}(k.x + k_0\lambda_0 x) = 0$ .

Stability is ensured if  $\phi_1(x)\dot{x} = (f.\dot{x} + k_0\dot{x})\dot{x} > 0$  and  $\phi_2(x)x = (k.x + k_0\lambda_0 x) > 0 \forall x, \dot{x} \in \mathbb{R} \times \mathbb{R}$ . This can be easily proved by use of the Lyapunov

method with as Lyapunov candidate function  $V = \frac{1}{2}m.\dot{x}^2 + \frac{1}{2} \int_0^x (k.y + k_0\lambda y)dy$ . We obtain  $\dot{V} = -\dot{x}(f.\dot{x} + k_0\dot{x})$ . We can see clearly that  $V$  and  $-\dot{V}$  are positive if and only if stiffness and damping function verify the positivity property  $v.\phi_i(v) > 0$ .

The behavior of the system in closed-loop remains sensitive to the parameters of the system and the control ( $\ddot{x} + \frac{1}{m}(f.\dot{x} + k_0\dot{x}) + \frac{1}{m}(k.x + k_0\lambda x) = 0$ ).

For stabilization by use of sliding mode control, we can also define a commutation surface by  $s = \dot{x} + \lambda x$  [see the previous remark for SPR linear transfer function  $H(p)$ ] and apply as control law  $u = -k_0 \text{sign}(s)$  or  $u = -k_0 \text{sat}(s)$  (passive element) with the saturation function defined (for some small constant  $\varepsilon$ ) by:

$$\text{sat}(s_i) = \begin{cases} 1 & s_i > \varepsilon \\ s_i/\varepsilon & |s_i| \leq \varepsilon \\ -1 & s_i < -\varepsilon \end{cases} \quad (13.15)$$

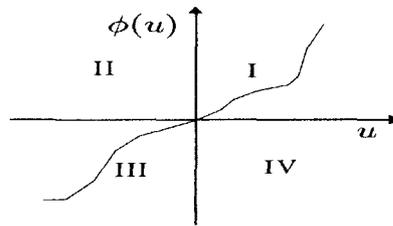


Figure 13.6: Memoryless passive element characteristic

The  $\text{sat}$  function verifies the passivity condition represented by Figure (13.6). It can be easily verified that this leads us to the scheme of Figure (13.1), with, as linear SPR transfer block  $H(p) = \frac{S(p)}{U(p)} = \frac{p+\lambda}{mp^2+fp+k}$  and nonlinear feedback block  $y_2 = f(u_2, t, \tau) = k_0 \text{sat}(s)$  which is passive. Time response and phase space trajectories obtained by this control are represented in Figure (13.7).

**Remark 148** *Sliding mode control allows high speed responses independently from the system damping and parameters. This performance is easy to obtain with respect of the key points of the previous remark.*

*If  $f$  is too small or negative (unstable system), a preliminary feedback can be used to reinforce the system dissipation and simplify the choice of commutation surface  $s$ . With PD precompensation we have  $f' = f + k_0$  and*

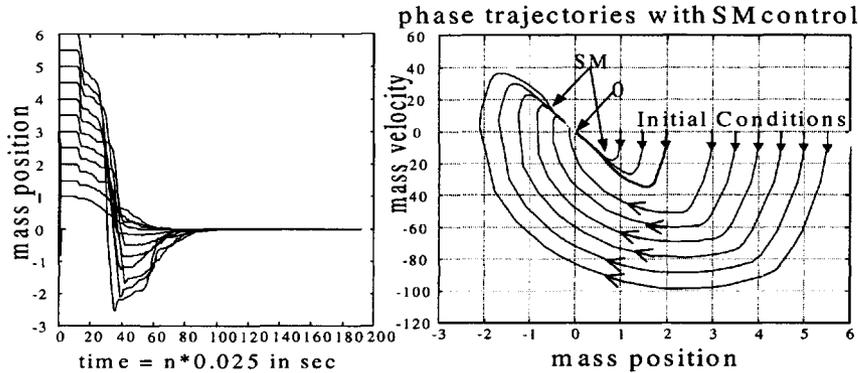


Figure 13.7: Behavior of the mass with passive SM-control

$k' = k + k_0 \lambda_0$  instead of  $f$  and  $k$ , respectively, and then  $\frac{S(p)}{U(p)} = \frac{p+\lambda}{mp^2+f'p+k'}$ .  
The new constraint becomes  $\lambda < \frac{f'}{m} = \frac{f+k_0}{m}$ .

**Mechanical impedance** In robotic applications involving manipulations, very often contacts between the robot and its environment appear. An example is represented by Figure (13.8). It can be modeled by:

$$M\ddot{x} + B\dot{x} + Kx = F \quad (13.16)$$

where  $M$  is the moving mass,  $B$  the damping coefficient or friction, and  $K$  the stiffness of contact (robot + environment). The applied forces are noted  $F$  and  $x$  is the cartesian position. The mechanical impedance [a transfer between force and velocity  $F = Z(\dot{x})$ ] is defined, in linear cases ( $M$ ,  $B$ , and  $K$  are three constants), by a symmetric positive definite transfer matrix:

$$Z(p) = \frac{Mp^2 + Bp + K}{p}, \text{ with } F(p) = Z(p)\dot{x}(p) \quad (13.17)$$

It corresponds to the energy function defined by

$$W(t) = \frac{1}{2} \{ \dot{x}^T M \dot{x} + x^T(t) K x(t) \} \quad (13.18)$$

Variation of energy is equal to the one supplied by  $F$  minus the friction loss:  $\dot{W}(t) = \dot{x}^T M \ddot{x} + x^T(t) K \dot{x}(t)$ ; using equation (13.16) we obtain:

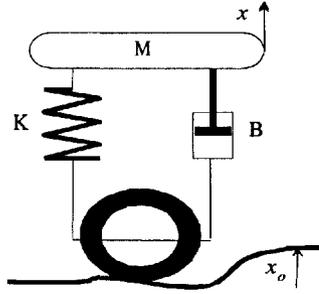


Figure 13.8: Suspension system

$\dot{W}(t) = \dot{x}^T (F - B\dot{x} - Kx + Kx) = \dot{x}^T F - \dot{x}^T B\dot{x}$ . Then it can be shown that the transfer matrix  $Z(p)$  is positive real and satisfies the passivity property:

$$\int_0^t \dot{x}^T F d\tau = W(t) - W(0) + \int_0^t V(\tau) d\tau \geq -\gamma_0^2 \quad (13.19)$$

$$\text{with } \gamma_0^2 = W(0) \text{ and } V = \dot{x}^T B\dot{x} \quad (13.20)$$

This feature can be exploited for control of the behavior in case of contact between the system and its environment; this is the case for vehicles and legged robots [36]. Note that this system has the same properties as the previous simple example and then the sliding mode control is able to give the same type of results and the previously pointed out particularities can be physically interpreted. For example, preliminary PD feedback is not necessary if the system is damped enough. This is the case of the vehicle suspensions.

### Pneumatic actuated systems

Figure (13.9) represents a pneumatic robot leg with two rigid links and two rotational joints. This robot leg is an experimental platform at the LRP. Each joint is actuated by a pneumatic cylinder (double effects linear jacks) driven by an electro-pneumatic servo-valve. The obtention of the dynamic actuator's model [1, 2, 4, 8, 11] is based on the study of the flow stage supplied with fixed pressure  $P_a$  and energy conversions. This system has the following dynamic model (see appendix):

$$\begin{cases} (ml + M(q))\ddot{q} + C'(q, \dot{q})\dot{q} + G'(q) = \tau \\ \dot{\tau} = Ji - B\tau - E\dot{q} \text{ with } \tau = K(\Delta P_p - \Delta P_n) \end{cases} \quad (13.21)$$

where  $m$  is the mass of the cable and of the piston that is negligible compared with the inertia of the segment,  $l$  is the radius of the pulley, and  $i$  is the servo-valve input current.

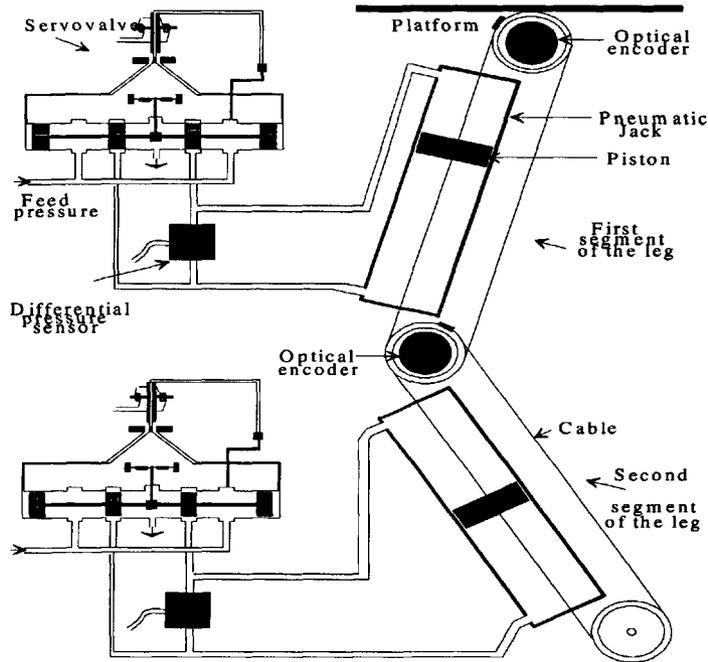


Figure 13.9: Pneumatic robot leg

The output  $\tau$  can be obtained from the measure of the differential pressure applied to the piston.  $J$ ,  $B$  and  $E$  are  $(2 \times 2)$  diagonal matrices called the thermodynamics parameters depending on the temperature gas characteristics and initial conditions of pressures and chambers volume.

Note that the first equation of the system (13.21) verifies the Popov inequality (13.4) and then corresponds to a passive transfer between  $u = \tau - G(q)$  and  $\dot{q}$ . The second equation also involves an SPR transfer function ( $H(p) = \frac{E}{B+p}$ ). And then the system can be represented as in Figure (13.1) by a linear block ( $H(p) = \frac{E}{B+p}$ , SPR) in feedback connection with a nonlinear passive block such as:  $M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau - G(q)$  and  $\dot{\tau} + B\tau = -E(\dot{q} - \frac{J}{E}i)$ . Note that parameters  $J$ ,  $B$ , and  $E$  are assumed constant for

this representation.

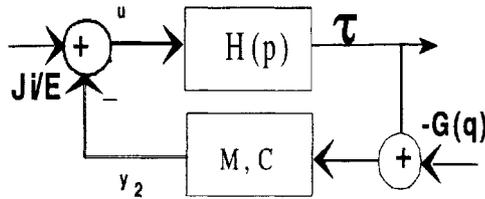


Figure 13.10: Passive equivalent feedback system of the pneumatic robot

It is then obvious that the sliding mode control will be designed in order to give a passive equivalent feedback system and impose the desired dynamics.

### Hydraulic robot manipulator

In this section, we show that a hydraulic robot manipulator designed for underwater applications (Figure 13.11 shows how a joint is actuated), has several similarities in its features with a pneumatic leg. Thus the same control approach can be applied in order to obtain good performances and robustness. From the robot dynamic equations (see for details [5, 6, 37]) and introducing a term  $F_v$  comprising all the friction effects (angular, linear and nonlinear terms and disturbances), we can obtain the complete dynamic model of the hydraulic actuated manipulator.

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + F_v = \tau \quad (13.22)$$

$$\dot{\tau} + B\tau + E\dot{q} = Jk_o\dot{i} \quad (13.23)$$

Note that  $J$ ,  $k_o$ ,  $B$ , and  $E$  are  $(n \times n)$  nonlinear diagonal matrices and depend on the actuator variables [6]. We note that the model is expressed in the same form as the pneumatic robot, so the same kind of equivalent feedback representation can be found.

Let us consider in what follows another way of representation, assuming that pressure or torque is not available for measurement. For control, we can consider as state variables the positions, velocities, and accelerations. Then the robot model can be rewritten in one stage to eliminate torque from model equations. The torque derivative is obtained analytically by



disturbances. This approach has attracted intense research interest in the past decade for robot manipulators [38, 39, 19]. The sliding mode control is designed by means of the passive systems approach [34]. This simplifies the design and allows us to exploit the physical system properties in a direct way. For boundedness of the derivative of the inertia matrix and the Coriolis and centrifugal terms we need the following lemma [40].

**Lemma 149 (Bernstein Lemma).** *If  $x(t)$  is a bounded signal ( $|x(t)| \leq M$ ) and has its frequency spectrum bounded by pulsation  $\omega_m$ , then all its derivatives are bounded such as  $|\frac{d^n}{dt^n} x(t)| \leq (\omega_m)^n M$ .*

### 13.3.1 Sliding mode control for a pneumatic system

#### Sliding mode control design

Let us consider the dynamic model equation (13.21) of the robot leg and rewrite it as follows (arguments are dropped for ease of notation):

$$M\ddot{q} + C\dot{q} + H(q, \dot{q}, \ddot{q}) = Ji \quad (13.27)$$

$$H(q, \dot{q}, \ddot{q}) = (\dot{M} + BM)\ddot{q} + (\dot{C} + E + BC)\dot{q} + (\dot{G} + BG)$$

Usually the components of the matrices  $M$ ,  $C$ ,  $G$ ,  $J$ ,  $B$ , and  $E$  are not well-known but only the estimated terms  $\widehat{M}$ ,  $\widehat{C}$ ,  $\widehat{G}$ ,  $\widehat{J}$ ,  $\widehat{B}$ , and  $\widehat{E}$  and bounds of the errors  $\widetilde{M}$ ,  $\widetilde{C}$ ,  $\widetilde{G}$ ,  $\widetilde{J}$ ,  $\widetilde{B}$ , and  $\widetilde{E}$  can be available:

$$\widetilde{M} = \widehat{M} - M, \widetilde{C} = \widehat{C} - C, \widetilde{G} = \widehat{G} - G, \widetilde{J} = \widehat{J} - J, \widetilde{B} = \widehat{B} - B, \widetilde{E} = \widehat{E} - E$$

For ease of presentation we will assume  $J$  is well-known. If this is not the case this approach can be easily extended. Note that, for sliding mode control, the estimates can be chosen constant ( $\widehat{M} = M_0 = cst$ ,  $\widehat{C} = C_0 = cst$ , ...). Robustness toward structure errors is guaranteed by the sliding mode control [17]. Let us define the tracking error vector as

$$\begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} q - q^d \\ \dot{q} - \dot{q}^d \\ \ddot{q} - \ddot{q}^d \end{bmatrix} \quad (13.28)$$

Let  $\ddot{q}_r$  be called the reference acceleration and defined as a function of the desired trajectories  $q^d, \dot{q}^d, \ddot{q}^d$  and errors:

$$\ddot{q}_r = \ddot{q}^d - \Lambda_1 (\dot{q} - \dot{q}^d) - \Lambda_2 (q - q^d) \quad (13.29)$$

Then the chosen switching surface  $s$  (function of the output trajectory error) is given by:

$$s = \ddot{e} + \Lambda_1 \dot{e} + \Lambda_2 e = \ddot{q} - \ddot{q}_r \quad (13.30)$$

$\Lambda_1 = \text{diag}(\lambda_1^1, \dots, \lambda_n^1)$   $\Lambda_2 = \text{diag}(\lambda_1^2, \dots, \lambda_n^2)$  are diagonal matrices with strictly positive components.

We can apply a partial feedback using the estimated nominal model  $Ji = Ji_{eq} + v$ ,  $v = K \text{sat}(s)$  (the *sign* function can be also used but chattering may occur due to measurement noise). Choosing  $Ji_{eq}$  as follows gives us in closed-loop the passive equivalent system of Figure (13.12).

$$Ji_{eq} = \widehat{M}q_r^{(3)} + \widehat{C}\ddot{q}_r + \widehat{H}(q, \dot{q}, \ddot{q}) - f_v s \quad (13.31)$$

$$\widehat{H}(q, \dot{q}, \ddot{q}) = (\widehat{M} + \widehat{B}\widehat{M})\ddot{q} + (\widehat{C} + \widehat{E} + \widehat{B}\widehat{C})\dot{q} + (\widehat{G} + \widehat{B}\widehat{G})$$

The closed-loop system can be expressed:

$$Ms + Cs = -f_v s + v - \delta \quad (13.32)$$

$$\delta = (\widetilde{M}q_r^{(3)} + \widetilde{C}\ddot{q}_r + H - \widehat{H}) \quad (13.33)$$

where  $f_v$  is a positive diagonal matrix.

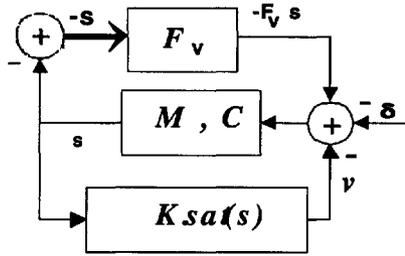


Figure 13.12: Equivalent feedback system for the SM-controlled robot

The equivalent feedback system is composed of a linear block with gain  $f_v$ , a nonlinear feedback block composed by the mechanical dynamic parts and then, in feedback with this one, a nonlinear block for control commutation. The input  $\delta$  represent a perturbation due to the modeling error. We can see clearly that this perturbation can be tackled (matched uncertainty) by use of the sliding mode control term  $v = K \text{sat}(s)$  to enhance the robustness of the controlled system.

### Stability analysis

In order to prove the stability of the system in closed loop and see how to adjust the control parameters, we can consider as a Lyapunov function

candidate

$$V(s, t) = \frac{1}{2} s^T M s \quad (13.34)$$

Differentiating (13.34) with respect to time yields

$$\begin{aligned} \dot{V}(s, t) &= \frac{1}{2} s^T \dot{M} s + s^T [J\dot{i} - M\dot{q}_r^{(3)} - C\ddot{q} - (\dot{M} + BM)\dot{q} \\ &\quad - (\dot{C} + E + BC)\dot{q} - (\dot{G} + BG)] \end{aligned}$$

Owing to the preceding equations, the derivative  $\dot{V}(s, t)$  becomes

$$\dot{V}(s, t) = -s^T f_v s - s^T K \text{sat}(s) - s^T (\widetilde{M} \dot{q}_r^{(3)} + \widetilde{C} \ddot{q}_r + \widetilde{H})$$

with

$$K \geq \left| \widetilde{M} \dot{q}_r^{(3)} + \widetilde{C} \ddot{q}_r + \widetilde{H} \right|$$

To ensure  $\dot{V} < 0$ , we choose  $K = \Delta M \left| \dot{q}_r^{(3)} \right| + \Delta C |\ddot{q}_r| + \Delta H$ , where these bounds are

$$\Delta M \geq \left| M - \widehat{M} \right| ; \Delta C \geq \left| C - \widehat{C} \right| ; \Delta H \geq \left| H - \widehat{H} \right|$$

Using the Bernstein lemma [40], we can prove the boundedness of the estimation errors appearing above in the Lyapunov function.

**Lemma 150** *Using the rigid robot properties and Equations (13.27) and (13.31), it can be proved that*

$$\begin{aligned} i) & \left\| \widetilde{H} \right\| \leq \kappa_0 + \kappa_1 \|\dot{q}\| + \kappa_2 \|\dot{q}\|^2 + \kappa_3 \|\ddot{q}\| ; \text{ans} \\ ii) & \left\| \widetilde{M} \dot{q}_r^{(3)} + \widetilde{C} \ddot{q}_r + \widetilde{H} \right\| \leq \kappa_0 + \kappa_1 \|\dot{q}\| + \kappa_2 \|\dot{q}\|^2 + \kappa_3 \|\ddot{q}\| + \alpha_2 \|\ddot{q}_r\| + \alpha_1 \|\ddot{q}_r^{(3)}\| = \\ & \eta(\ddot{q}_r, \dot{q}_r, \dot{q}, \ddot{q}) \end{aligned}$$

This, when applied to the Lyapunov derivative, leads to  $\dot{V}(s) \leq -s^T (f_v s - K \text{sat}(s)) + \eta \|s\|$ , with a certain bound  $\eta$  positive valued function or constant depending of the maximum velocity, acceleration and desired jerk trajectory. In order to ensure  $\dot{V}(s) < 0$ , in the presence of model uncertainties, we see from the previous equation that we can choose  $K \geq \eta_o \geq \eta(\ddot{q}_r, \dot{q}_r, \dot{q}, \ddot{q})$ , which give us

$$\dot{V}(s) \leq -s^T f_v s - (K - \eta_o) \|s\| < 0 \quad (13.36)$$

Then we can conclude to the stability of the system:  $s$  converges to a neighborhood of zero. Once the system state trajectory reaches a neighborhood of the switching surface, subsequent motion of the state trajectories involves the sliding of the trajectories on the surface.

### 13.3.2 Sliding mode control of a hydraulic robot

Let us rewrite the dynamic model of the hydraulic underwater manipulator as follows<sup>1</sup>:

$$Jk_o i = M\ddot{q} + C\dot{q} + H(q, \dot{q}, \ddot{q}) \quad (13.37)$$

$$H(q, \dot{q}, \ddot{q}) = \beta\ddot{q} + \gamma\dot{q} + \delta \quad (13.38)$$

The matrices  $M$ ,  $C$ ,  $G$ ,  $J$ ,  $B$ , and  $E$ , defined above, are not known but only estimates can be available for  $M_o$ ,  $C_o$ ,  $G_o$ ,  $J_o$ ,  $B_o$ , and  $E_o$ . The estimated terms are chosen constant ( $M_o = cst, \dots$ ). We assume that bounds on the estimation errors are known. The tracking error vector is defined by Equation (13.28) and  $\ddot{q}_r$  the acceleration reference is defined in Equation (13.29) with  $\Lambda_i = \text{diag}(\lambda_1^i, \dots, \lambda_n^i)$  diagonal matrices with strictly positive components. We choose as switching surface  $s$  function of the state error given by

$$s = \ddot{q} - \ddot{q}_r = \ddot{e} + \Lambda_1 \dot{e} + \Lambda_2 e \quad (13.39)$$

Assume  $k_o$  to be known, for simplicity, and choose the control  $i$  (based on the nominal model of the robot) as follows [37]:

$$J_o k_o i = M_o \ddot{q}_r + C_o \dot{q}_r + H_o(q, \dot{q}, \ddot{q}) - f_v s - K \text{sat}(s) \quad (13.40)$$

where  $f_v$  is a positive diagonal matrix chosen for transient duration adjustment and  $H_o(q, \dot{q}, \ddot{q}) = (\dot{M}_o + B_o M_o)\dot{q} + (\dot{C}_o + E_o + B_o C_o)\dot{q} + \dot{G}_o + \dot{F}_{vo} + B_o G_o + B_o F_{vo}$ .

Let us denote the parameter estimation errors (for control) as follows:  $\tilde{M} = J_o J_o^{-1} M_o - M$ ,  $\tilde{C} = J_o J_o^{-1} C_o - C$ ,  $\tilde{H} = J_o J_o^{-1} H_o - H$ .

This leads us to the same equivalent feedback scheme as for the pneumatic robot leg (see Figure 13.12), which emphasizes how the passive dynamics of the robot are involved in the control design. In this way the control involves a term [ $v = K \text{sat}(s)$ ] used to tackle uncertainties on the model and the involved perturbation (which, in this manner, verify the matching condition).

Stability analysis can also be considered in the same line as for the pneumatic case by use of the Lyapunov function candidate:  $V(s) = \frac{1}{2} s^T M s$ .

Differentiating  $V(s)$  with respect to time, using the model equations and the system's passivity property ( $\frac{1}{2} \dot{M} - C$  is skew symmetric), we obtain

$$\dot{V} = -s^T (f_v s + K \text{sat}(s) - \tilde{M} \ddot{q}_r - \tilde{C} \dot{q}_r - \tilde{H}) \quad (13.41)$$

We can also prove here the boundedness of the estimation errors appearing above in the Lyapunov function.

<sup>1</sup>This work was done in collaboration with the LIRMM.

**Lemma 151** *Using the rigid robot properties and Equations (13.37) and (13.40), it can be proved that*

$$\begin{aligned} i) \quad & \left\| \tilde{H} \right\| \leq \kappa_0 + \kappa_1 \|\dot{q}\| + \kappa_2 \|\dot{q}\|^2 + \kappa_3 \|\ddot{q}\|; \text{ and} \\ ii) \quad & \left\| \tilde{M}\ddot{q}_r + \tilde{C}\dot{q}_r + \tilde{H} \right\| \leq \kappa_0 + \kappa_1 \|\dot{q}\| + \kappa_2 \|\dot{q}\|^2 + \kappa_3 \|\ddot{q}\| + \alpha_2 \|\ddot{q}_r\| + \alpha_1 \|\ddot{q}_r\| = \\ & \eta(\ddot{q}_r, \dot{q}_r, \ddot{q}, \dot{q}) \end{aligned}$$

This, when applied to the Lyapunov derivative, leads to  $\dot{V}(s) \leq -s^T f_v s - s^T K \text{sat}(s) + \eta \|s\|$ , with certain bound  $\eta$  positive valued function or constant depending of the maximum velocity, acceleration, and desired jerk trajectory. Recall that accelerations and velocities are limited for real systems ( $\|\dot{q}\| < \dot{q}_{\max}$  and  $\|\ddot{q}\| < \ddot{q}_{\max}$  then  $\|\dot{q}\|^2 < \dot{q}_{\max} \|\ddot{q}\|$ ). In order to ensure  $\dot{V}(s) < 0$ , in the presence of model uncertainties and system limitations, we see from the previous equation and (13.41) that we can choose  $K \geq \eta_o \geq \eta(\ddot{q}_r, \dot{q}_r, \ddot{q}, \dot{q})$ . It leads us to

$$\dot{V}(s) \leq -s^T f_v s - (K - \eta_o) \|s\| \leq 0 \quad (13.42)$$

Then we can conclude to the stability of the system and  $s$  converges to a neighborhood of zero.

### 13.3.3 Simulation results

#### Results for the hydraulic underwater manipulator

Hereafter, we will illustrate the behavior of the proposed control for 2 DOF of an underwater manipulator (Slingsby TA9)<sup>2</sup> with hydraulic actuators [5, 6, 37]. Sliding mode control has been applied in joint space. The robot is simulated using Matlab - Simulink software packages. In simulations, we must take into account the physical positions and pressure limits and choose appropriate a priori estimates for the control. We take as control  $J_o k_o i = M_o \ddot{q}_r + C_o \dot{q}_r - f_v s - K \text{sign}(s)$  with  $H_o = 0$  and  $M_o, C_o$  equal to their median values. Control is implemented in discrete time with  $T_s = 1ms$  as sampling time. The results presented in Figures (13.13) and (13.14), have been obtained with  $f_v = K = 2500$ .

With these values the input current remains less than 1 volt, in the admissible zone, without saturation. The desired positions are 0.15 and 0.2 rad. The components  $M_o \ddot{q}_r$  and  $C_o \dot{q}_r$  anticipate the main dynamics effects. The feedback adjusts compensation by use of weights on position, velocity and acceleration errors. These two components contribute for perturbation

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<sup>2</sup>This work has been done in collaboration with the LIRMM.

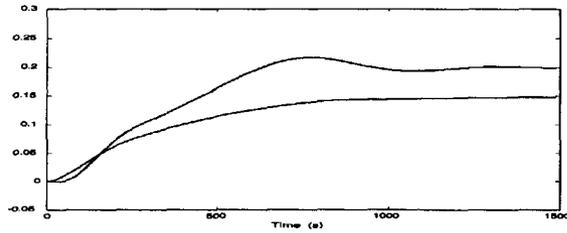


Figure 13.13: Positions in radians, versus time

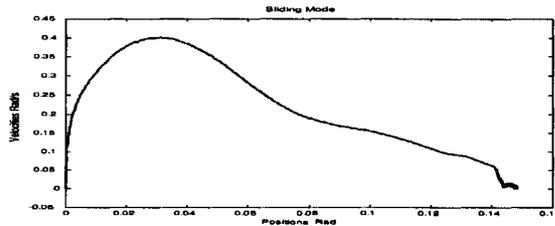


Figure 13.14: Phase plane for joint 1 (velocity versus position)

damping and rejection. The increase of  $f_v$  allows us to enhance the time response and damp the acceleration, velocity, and position errors, during the transient, depending on weights by means of  $\Lambda_1$  and  $\Lambda_2$  (recall that  $s = \ddot{e} + \Lambda_1 \dot{e} + \Lambda_2 e$ ). The commutation in control enhances the robustness versus modeling errors and perturbations like friction variation  $F_v$  and torque disturbance. With  $K = 0$ , we can obtain, in the regulation case, a stable behavior with small oscillations of positions. The chattering may be eliminated by use of the commutation function (13.15). These features emphasize the effectiveness of the sliding control scheme for underwater manipulators.

### Simulation results for the pneumatic robot

The following results show the obtained position velocity and acceleration errors. We can also see that no chattering appears on the control. Simulation results emphasize the robustness of the sliding mode control and the ease of its adjustment.

The sliding mode control design by use of passivity approach leads to efficient and robust controllers for pneumatic systems and manipulators with hydraulic actuators. The control law involves an acceleration reference

model (Equation 13.29), a passive feedback ( $f_v, s$  in 13.40), and a commutation term to enhance robustness property. This feedback term accelerates the transient behavior as can be seen in the expression of the Lyapunov function derivative (13.42 and 13.36). The obtained robustness comes from sliding mode control properties and from exploitation of the passivity of the robot (skew symmetry property and passive equivalent feedback scheme). This was confirmed by simulation results. For implementation of this control, positions, velocities, and accelerations were needed; observers based on sliding mode approach have been developed [41, 42, 43, 44, 45] for pneumatic robot leg.

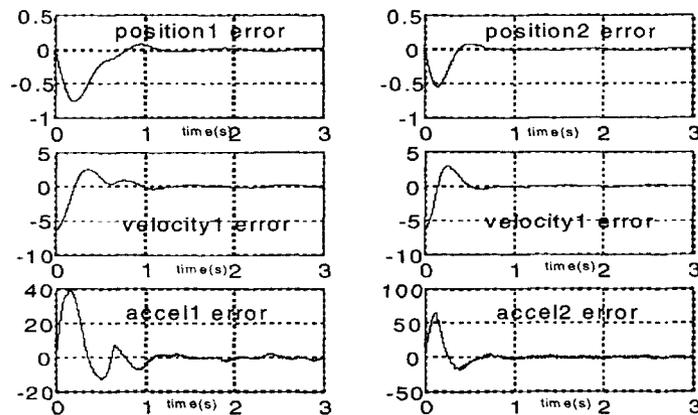


Figure 13.15: Position, velocity and acceleration errors

## 13.4 SM observers based control

Several structures are possible [46, 20] for joint observation and control. We present, briefly, one of them in this section, which is an extension of the sliding observer described in [39, 49], for the pneumatic robot leg. We suppose the parameters of the model unknown and design a sliding mode observer and sliding mode controller, and then prove the closed-loop stability.

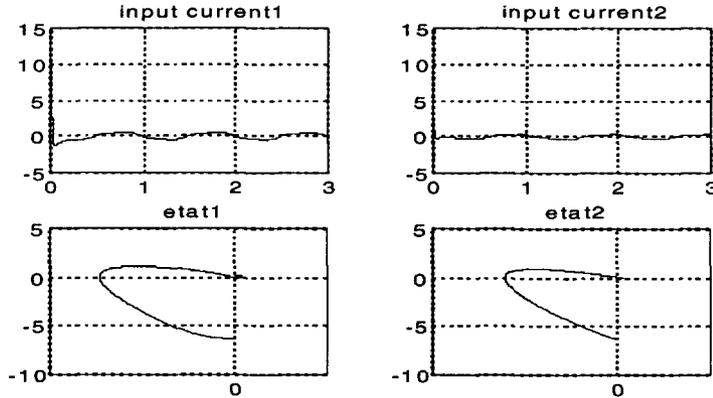


Figure 13.16: Input currents and phase plane errors  $\dot{e}$  versus  $e$

### 13.4.1 Observer design

Introducing the state vector  $x = (x_1^T, x_2^T, x_3^T)^T$  ( $x_1 = q, x_2 = \dot{q}, x_3 = \ddot{q}$ ), the model (13.27) can be rewritten in the following state-space representation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = f_4(x, i) \\ y = x_1 = q \end{cases} \quad (13.43)$$

with

$$f_4(x, i) = -M^{-1}[(\dot{M} + C + BM)x_3 + (\dot{C} + BC + E)x_2 + (BG + \dot{G}) - Ji]$$

This state-space representation is observable if only joint positions are measured [47]. In order to estimate the complete state  $\hat{x} = (\hat{x}_1^T, \hat{x}_2^T, \hat{x}_3^T)^T$  (joint positions, velocities, and accelerations) used in the control law, a nonlinear sliding structure for the state observer is considered [46, 47, 48, 19, 20].

$$\begin{cases} \dot{\hat{x}}_1 = -\Gamma_1 \tilde{x}_1 + \hat{x}_2 - \Lambda_1 \text{sgn}(\tilde{x}_1) \\ \dot{\hat{x}}_2 = -\Gamma_2 \tilde{x}_1 + \hat{x}_3 - \Lambda_2 \text{sgn}(\tilde{x}_1) \\ \dot{\hat{x}}_3 = -\Gamma_3 \tilde{x}_1 + f_4(\hat{x}, i) - \Lambda_3 \text{sgn}(\tilde{x}_1) + v \end{cases} \quad (13.45)$$

The term  $v$  is added in order to guarantee the stability of the observer error dynamics against the parameters uncertainties. Actually, this term is needed to account for the interaction between controller and observer.  $\Gamma_1, \Gamma_2, \Gamma_3$  are positive diagonal matrices.  $\Gamma_1 = \text{diag}(\gamma_{11}, \gamma_{21})$ ,

$\Gamma_2 = \text{diag}(\gamma_{12}, \gamma_{22})$ , and  $\Gamma_3 = \text{diag}(\gamma_{13}, \gamma_{23})$ ; they are chosen such that the linear part of the system is asymptotically stable. The matrices  $\Lambda_1, \Lambda_2$  are chosen positive diagonal.  $\Lambda_1 = \text{diag}(\lambda_{11}, \lambda_{21})$ ,  $\Lambda_2 = \text{diag}(\lambda_{12}, \lambda_{22})$ , and the nonlinear matrix  $\Lambda_3(\cdot)$  will be defined later.

The dynamics of the observation error  $\tilde{x} = \hat{x} - x$  is then

$$\begin{cases} \dot{\tilde{x}}_1 = -\Gamma_1 \tilde{x}_1 + \tilde{x}_2 - \Lambda_1 \text{sgn}(\tilde{x}_1) \\ \dot{\tilde{x}}_2 = -\Gamma_2 \tilde{x}_1 + \tilde{x}_3 - \Lambda_2 \text{sgn}(\tilde{x}_1) \\ \dot{\tilde{x}}_3 = -\Gamma_3 \tilde{x}_1 + f_4(\hat{x}, i) - f_4(x, i) \\ \quad - \Lambda_3 \text{sgn}(\tilde{x}_1) + v \end{cases} \quad (13.46)$$

In order to obtain the system's dynamics behavior inside the attractive region [49], we proceed step by step. First, we show that  $\tilde{x}_1 = 0$  has an attractive region under some conditions on velocities with the Lyapunov function  $V_1 = \frac{1}{2} \tilde{x}_1^T \tilde{x}_1$  [39]. It is obvious that  $\dot{V}_1 < 0$  under the conditions

$$|\tilde{x}_2^i| < +\lambda_{i1} + \gamma_{i1} |\tilde{x}_1^i| \quad i \in \{1, 2\} \quad (13.47)$$

Thus, the domains defined above and the hyperplane  $\tilde{x}_1 = 0$  are attractive. On the intersection of the region (13.47) and  $\tilde{x}_1 = 0$ , we consider the obtained reduced dynamics of the observation error. That is, in the mean average [50], the behavior is described by<sup>3</sup>

$$\dot{\tilde{x}}_1 = \tilde{x}_2 - \Lambda_1 \overline{\text{sgn}}(\tilde{x}_1) = 0 \quad (13.48)$$

$$\dot{\tilde{x}}_2 = \tilde{x}_3 - \Lambda_2 \Lambda_1^{-1} \tilde{x}_2 \quad (13.49)$$

In a second step, to show that  $\tilde{x}_2$  tends toward zero (existence of an attraction region), let us consider a second Lyapunov function defined by:  $V_2 = \frac{1}{2} \tilde{x}_2^T \tilde{x}_2$ . Then,

$$\dot{V}_2 = \tilde{x}_2^T \dot{\tilde{x}}_2 = \tilde{x}_2^T (\tilde{x}_3 - \Lambda_2 \Lambda_1^{-1} \tilde{x}_2)$$

$$\dot{V}_2 = \tilde{x}_2^T \tilde{x}_3 - \tilde{x}_2^T \Lambda_2 \Lambda_1^{-1} \tilde{x}_2$$

So  $\dot{V}_2 < 0$  under the conditions [39]

$$\|\tilde{x}_3\| < \lambda_{\min} \{ \Lambda_2 \Lambda_1^{-1} \} \|\tilde{x}_2\| \quad (13.50)$$

The domains defined above and the hyperplane  $\tilde{x}_2 = 0$  are then attractive.

If we consider only one link, the intersection of the regions (13.50), and the hyperplanes  $\tilde{x}_1 = 0$ ,  $\tilde{x}_2 = 0$  is reduced to a segment. On this segment,

<sup>3</sup> $\overline{\text{sgn}}$  denotes the function equivalent, in the mean, to the  $\text{sgn}$  effect.

the reduced dynamics of the observation error can be written as

$$\begin{cases} \dot{\tilde{x}}_2 = \tilde{x}_3 - \Lambda_2 \overline{\text{sgn}}(\tilde{x}_1) = 0 \\ \overline{\text{sgn}}(\tilde{x}_1) = \Lambda_2^{-1} \tilde{x}_3 \end{cases} \quad (13.51)$$

That is

$$\dot{\tilde{x}}_3 = f_4(\hat{x}, i) - f_4(x, i) - \Lambda_3 \Lambda_2^{-1} \tilde{x}_3 + v \quad (13.52)$$

with

$$\begin{aligned} f_4(\hat{x}, i) = & -\widehat{M}^{-1} \left( \dot{\widehat{M}} + \widehat{C} + \widehat{B}\widehat{M} \right) \hat{x}_3 \\ & + \left( \dot{\widehat{C}} + \widehat{B} + \widehat{C}\widehat{E} \right) x_2 + \left( \widehat{B}\widehat{G} + \dot{\widehat{G}} \right) - \widehat{J}i \end{aligned}$$

We can write

$$\begin{aligned} \dot{\tilde{x}}_3 = & -M^{-1} \left( \dot{M} + C + BM + M\Lambda_3\Lambda_2^{-1} \right) \tilde{x}_3 \\ & - \overline{M^{-1}C} \hat{x}_3 + \overline{M^{-1}J}i - H_1 + v \end{aligned} \quad (13.53)$$

with

$$\begin{aligned} H_1 = & \left( \overline{M^{-1}\dot{M}} + \overline{M^{-1}BM} \right) \hat{x}_3 + \left( \overline{M^{-1}C} + \overline{M^{-1}BC} + \overline{M^{-1}E} \right) x_2 \\ & + \left( \overline{M^{-1}BG} + \overline{M^{-1}\dot{G}} \right) \end{aligned}$$

where the notation corresponds to:

$$\begin{cases} \overline{AB} = \widehat{A}\widehat{B} - AB \\ \overline{ABC} = \widehat{A}\widehat{B}\widehat{C} - ABC \end{cases}$$

$(A, B, C)$  are three matrices, members of the set

$$\left\{ M^{-1}, \dot{M}, C, \dot{C}, G, \dot{G}, B, E, J \right\}$$

In this section we have finally obtained the reduced dynamics of the observation error. Next, the convergence condition on  $\tilde{x}_3$  will be studied in closed-loop, with the equation of tracking error.

### 13.4.2 Tracking error equation: observer and control

The control objective is to track the desired position, velocity, and acceleration  $\{x_1^d, x_2^d, x_3^d\}$ , time-varying trajectories. The sliding mode controller as developed previously [48] has the following structure:

$$i = \hat{J}^{-1} \left( \widehat{M}\dot{x}_3^r + \widehat{C}x_3^r + \widehat{H} - F_v s - k \operatorname{sign}(s) \right)$$

where

$$\begin{aligned} s &= \ddot{e} + \delta_1 \dot{e} + \delta_2 e = (\widehat{x}_3 - x_3^d) + \delta_1 \dot{e} + \delta_2 e = \widehat{x}_3 - x_3^r \\ \dot{s} &= \dot{e}^{(3)} + \delta_1 \ddot{e} + \delta_2 \dot{e} = (\widehat{\dot{x}}_3 - \dot{x}_3^d) + \delta_1 \ddot{e} + \delta_2 \dot{e} = \widehat{\dot{x}}_3 - \dot{x}_3^r \end{aligned}$$

$x_3^r$  is the acceleration reference signal and

$$\widehat{H} = \left( \widehat{M} + \widehat{B}\widehat{M} \right) \widehat{x}_3 + \left( \widehat{C} + \widehat{B}\widehat{C} + \widehat{E} \right) x_2 + \widehat{G} + \widehat{B}\widehat{G}$$

If we apply this control law to the system, we obtain

$$\begin{aligned} \dot{s} &= JM^{-1} \left[ - (F + J^{-1}C) s - K \operatorname{sign}(s) + J^{-1}M\dot{\tilde{x}}_3 \right. \\ &\quad \left. + J^{-1} \left( \dot{M} + C + BM \right) \tilde{x}_3 + \overline{J^{-1}M}\dot{x}_3^r + \overline{J^{-1}C}x_3^r + H_2 \right] \end{aligned} \quad (13.54)$$

where  $F = \widehat{J}^{-1}F_v$ ,  $K = \widehat{J}^{-1}k$ ,  $F_v$  and  $k$  are positive diagonal matrices, and

$$\begin{aligned} H_2 &= \left( \overline{J^{-1}\dot{M}} + \overline{J^{-1}BM} \right) \widehat{x}_3 + \left( \overline{J^{-1}\dot{C}} + \overline{J^{-1}BC} + \overline{J^{-1}E} \right) x_2 \\ &\quad + \overline{J^{-1}\dot{G}} + \overline{J^{-1}BG} \end{aligned}$$

### 13.4.3 Stability of observer based control

The closed-loop analysis is performed on the basis of the reduced order manifold dynamics (13.53) and the tracking error dynamics (13.54). Then we define the augmented state vector  $z(t) = (s^T; \tilde{x}_3^T)^T$  with as equation:

$$\begin{cases} \dot{s} = JM^{-1} \left[ - (F + J^{-1}C) s - K \operatorname{sign}(s) \right. \\ \quad \left. + J^{-1}M\dot{\tilde{x}}_3 + J^{-1} \left( \dot{M} + C + BM \right) \tilde{x}_3 \right. \\ \quad \left. + \overline{J^{-1}M}\dot{x}_3^r + \overline{J^{-1}C}x_3^r + H_2 \right] \\ \dot{\tilde{x}}_3 = -M^{-1} \left( \dot{M} + C + BM + M\Lambda_3\Lambda_2^{-1} \right) \tilde{x}_3 \\ \quad - \overline{M^{-1}C}\widehat{x}_3 + \overline{M^{-1}J}i - H_1 + v \end{cases}$$

The following Lyapunov function is used.

$$V(s, \tilde{x}_3) = \frac{1}{2} s^T J^{-1} M s + \frac{1}{2} \tilde{x}_3^T \tilde{x}_3$$

The time derivative of  $V$  is given by

$$\begin{aligned} \dot{V} &= s^T J^{-1} \dot{M} s + \frac{1}{2} s^T J^{-1} \dot{M} s + \tilde{x}_3^T \dot{\tilde{x}}_3 \\ &= -s^T F s - \tilde{x}_3^T Q \tilde{x}_3 + s^T (\beta_1 - K \operatorname{sgn}(s)) - \tilde{x}_3^T \beta_2 + \tilde{x}_3^T v \end{aligned}$$

where

$$\begin{aligned} Q &= \widehat{M}^{-1} \dot{\widehat{M}} + \widehat{B} + \widehat{C} + \Lambda_3 \Lambda_2^{-1} \\ \beta_1 &= J^{-1} M \dot{\tilde{x}}_3 + J^{-1} (\dot{M} + C + BM) \tilde{x}_3 + \overline{J^{-1} M} \dot{x}_3^r + \overline{J^{-1} C} x_3^r + H_2 \\ \beta_2 &= \left( \overline{M^{-1} \dot{M}} + \overline{M^{-1} B M} + \widetilde{C} \right) \tilde{x}_3 + \overline{M^{-1} C} \hat{x}_3 + H_1 - \overline{M^{-1} J} i \end{aligned}$$

The stability of the closed-loop system is studied under the physical assumption of bounded system states ( $\|x(t)\| < \infty, \forall t \geq 0$ ) and the following assumptions:

$$\begin{aligned} \|M(x_1)\| &\leq \alpha_1; & \|\widehat{M}(x_1)\| &\leq \alpha'_1; \\ \|\dot{M}(x_1)\| &\leq \alpha_1 w_m; & \|\dot{\widehat{M}}(x_1)\| &\leq \alpha'_1 w_m; \\ \|\dot{M}(x_1)^{-1}\| &\leq \alpha''_1; & \|\dot{\widehat{M}}(x_1)^{-1}\| &\leq \alpha''_1; \\ \|C(x_1, x_2)\| &\leq \alpha_2 \|x_2\|; & \|\widehat{C}(x_1)\| &\leq \alpha'_2 \|x_2\|; \\ \|\dot{C}(x_1, x_2)\| &\leq \alpha_2 w_m \|x_2\|; \\ \|G(x_1)\| &\leq g_0; & \|\dot{G}(x_1)\| &\leq g_0 w_m; \\ \|J\| &\leq J_m; & \|\widehat{J}\| &\leq J'_m; \\ \|J^{-1}\| &\leq J''_m; & \|B\| &\leq b_m; \end{aligned}$$

These assumptions are compatible with the real system, and come from mechanical properties of the system and limited bandwidth of the real signals. The following constants  $f_0, q_0, \beta_{01}$ , and  $\beta_{02}$  are defined by  $f_0 =$

$\lambda_{\min} \{F\}$ ,  $q_0 = \lambda_{\min} \{Q\}$ , and

$$\begin{aligned}\beta_{01} &= \sup_{0 \leq \tau \leq t} \|\beta_1(\tau)\| \\ &= \sup_{0 \leq \tau \leq t} [\eta_1 \|\dot{\tilde{x}}_3\| + (\eta_2 + \eta_3 \|x_2\|) \|\tilde{x}_3\| \\ &\quad + \eta_4 \|\dot{x}_3^r\| + \eta_5 \|x_2\| + k_0 + k_1 \|x_2\| \\ &\quad + k_2 \|x_2\|^2 + k_3 \|\hat{x}_3\|]\end{aligned}$$

$$\begin{aligned}\beta_{02} &= \sup_{0 \leq \tau \leq t} \|\beta_2(\tau)\| \\ &= \sup_{0 \leq \tau \leq t} [(\eta'_1 + \eta'_2 \|x_2\|) \|\tilde{x}_3\| + \eta'_3 \|x_2\| \|\hat{x}_3\| \\ &\quad + \eta'_4 \|i\| + k'_0 + k'_1 \|x_2\| + k'_2 \|x_2\|^2 \\ &\quad + k'_3 \|\hat{x}_3\|]\end{aligned}$$

where  $\eta_i, \eta'_i, k_i, k'_i$  are positive constants. The signal  $v$  can then be defined as [49]

$$v = \begin{cases} -\beta_{02} \frac{\tilde{x}_3}{\|\tilde{x}_3\|} & \text{if } \|\tilde{x}_3\| \neq 0 \\ 0 & \text{if } \|\tilde{x}_3\| = 0 \end{cases}$$

Note that on the sliding surface,  $x_3 = \hat{x}_3 - \Lambda_2 \text{sgn}(\tilde{x}_1)$ , then  $v$  can be rewritten for implementation  $v = \frac{-\beta_{02}}{\|\Lambda_2\|} \Lambda_2 \text{sgn}(\tilde{x}_1)$ . The time derivative of  $V(s, \tilde{x}_3)$  can then be bounded as follows.

$$\dot{V} \leq -f_0 \|s\|^2 - q_0 \|\tilde{x}_3\| + \|s\| (\beta_{01} - K) + \|\tilde{x}_3\| \beta_{02} - \frac{\tilde{x}_3^T \tilde{x}_3}{\|\tilde{x}_3\|} \beta_{02}$$

$$\dot{V} \leq -f_0 \|s\|^2 - q_0 \|\tilde{x}_3\| + \|s\| (\beta_{01} - K)$$

where  $Q$  and  $F$  are diagonal positive definite matrices, and  $K$  and  $\Lambda_3$  are chosen such as:

$$\begin{aligned}K &\geq \beta_{01} \\ \Lambda_3 &= \Lambda_2 \left( Q - \widehat{M}^{-1} \dot{\widehat{M}} - \widehat{M}^{-1} \widehat{B} \widehat{M} - \widehat{C} \right)\end{aligned}$$

So,  $\dot{V}$  is strictly negative and thus  $\tilde{x}_3$  and  $e$  tend asymptotically to zero. The system is thus asymptotically stable.

### 13.4.4 Simulation results

In our simulations, we considered the sliding mode control law developed previously. Errors on the structure of system was taken into account. We took the matrix  $\widehat{C}(x_1, x_2) = 0$  and the inertia matrix  $\widehat{M}(x_1)$  constant diagonal, and we considered an error of 30% on the remaining parameters ( $J, B, E$ ).

The results obtained show that the complete system, controller and ob-

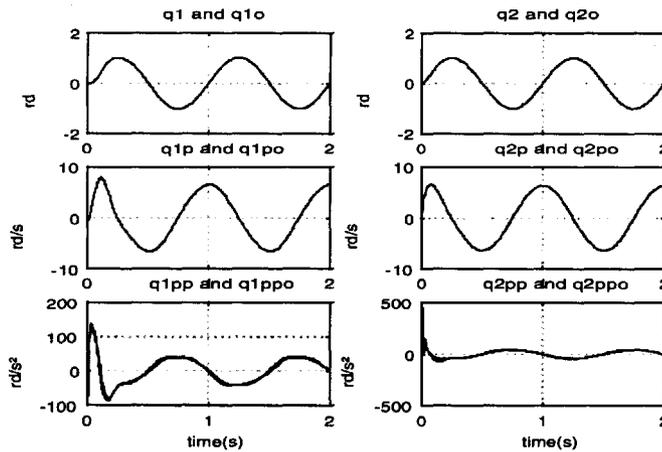


Figure 13.17: Actual and observed state variables

server, remained stable and showed a good estimation of the velocities and accelerations. For simulation, the following gains were used:  $\Lambda_1 = \text{diag}(40, 90)$ ,  $\Lambda_2 = 10\Lambda_1$ .

Note that we have also taken into account the limitations (saturation) of the servo-valve current of  $\pm 20mA$ . For this simulation, we showed the convergence of the observed state toward the real one [Figure(13.17)]. We note that the convergence needs less than 0.25s and that the observer error is about 0.1% to 1% [Figure (13.18)]. The tracking position error is about 1% to 2%, as shown by Figure (13.19).

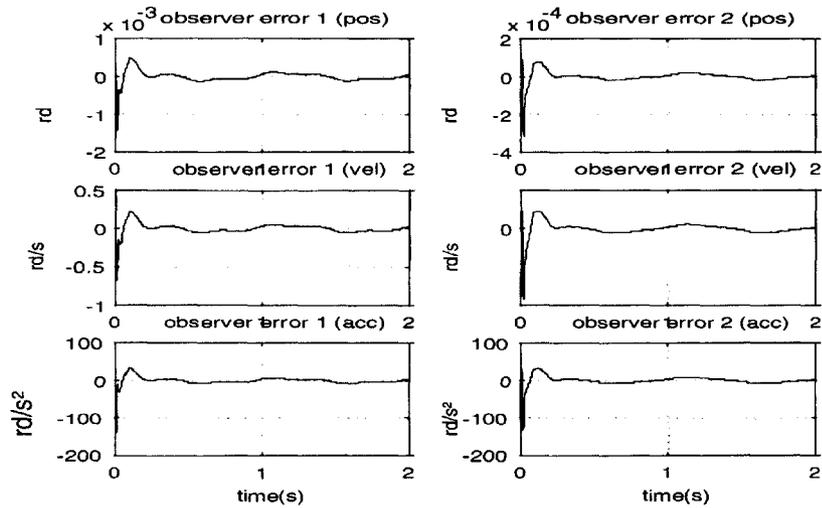


Figure 13.18: Observation error

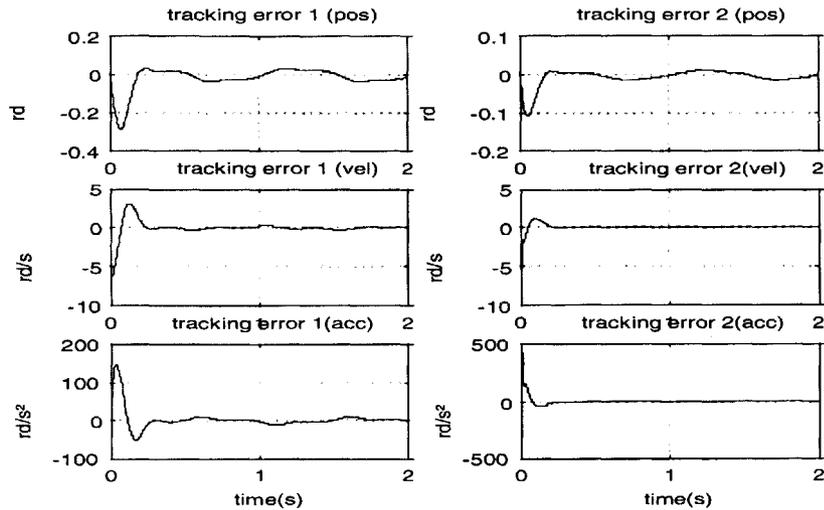


Figure 13.19: Tracking error

### 13.4.5 Conclusion

In this chapter we showed how sliding mode controllers can be designed using passive systems approach and coupled with a sliding mode observer

for state estimation. The given sliding mode control design approach was systematic and physically well-suited with simple controllers. The passivity property of the nonlinear robot dynamics was involved to simplify the design and allowed high performance. Its use allowed us to directly choose a well-suited commutation surface and when this one was reached, in the sliding regime, the system behaved like a linear-time invariant system (with a reduced order) and was robust against modeling error, parameters variation and environment perturbations. Simulation results was presented to emphasize these features for a pneumatic robot leg and a hydraulic underwater manipulator. An observer can be used to produce estimation of velocities and accelerations for a pneumatic leg of a robot when there is a lack of knowledge of the system parameters model and structure and only angular positions are measured. The stability analysis, convergence, and simulation results emphasized robustness versus uncertainties on the model parameters and efficiency of the sliding mode observers and controllers.

## 13.5 Appendix

### 13.5.1 Pneumatic actuators model

Each joint is actuated by an electropneumatic servo-valve with double effects linear jacks. It is made up of a current motor and of two pneumatic stages. It is composed of a set slide-sleeve and of four variable restrictions modulated by the slide position. The variations of the flow rate of air drives the jacks. Figure 13.20 illustrates the flow stage of a servo-valve. The determination of the dynamic actuator's model [1]- [11] is based on the study of the flow stage supplied with fixed pressure  $P_a$ . The valve controls the air flow which is converted in pressure supply for the two chambers of the cylinder via four restrictions. The application of thermodynamic relationships has the following assumptions.

**Assumptions A:** The fluid is an ideal gas. Potential and kinetic energy within the fluid are neglected. No leakage exists between the two piston chambers.

**Assumption B:** The piston's displacement is due to small slide's variations around its central position. The pneumatic system is symmetric. The pressure equations can be linearized around the initial position.

The mass flow rate depends on the current  $i$ , the jack motion, and the chamber's pressure. Pneumatic actuator equations are derived from the thermodynamic study of the system.

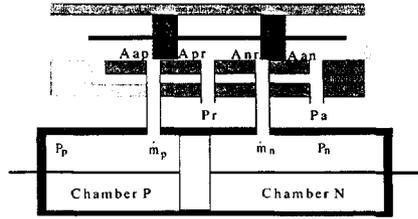


Figure 13.20: Flow stage of a servo-valve

Let us define the following parameters:  $m_{ij}$  denotes the mass flow rate in restriction  $ij$

$P_p, P_n$ : pressures in chambers P and N, respectively

$P_a, P_r$ : supply and output pressures, respectively

$V_p, V_n$ : volumes in chambers P and N, respectively

$\gamma = 1.405$ : ratio of specific heat

$r = 286 \text{ J kg}^{-1} \text{ K}^{-1}$ : ideal gas constant

$T$ : temperature in Kelvin

$l$ : the radius of the pulley

$i$ : the motor input current

$i_0$ : initial offset current (servovalve) and  $\beta = \text{constant} \geq 0$

$S$ : denotes the cross-section area of the piston

$y$ : piston's displacement

$x$  the jack displacement

Table 1 – Thermodynamic Equations

Current action on the jack:	$i = f(x) + i_0 = \beta \cdot x + i_0$
Chambers fluid flow:	$\frac{dm_{ij}}{dt} = f_{ij}(x, P_{ij})$
Actuator fluid flow:	$\begin{cases} \frac{dP_p}{dt} = -\frac{\gamma \cdot P_p}{V_p} \frac{dV_p}{dt} + \frac{rT\gamma}{V_p} \Delta \dot{m}_p \\ \frac{dP_n}{dt} = -\frac{\gamma \cdot P_n}{V_n} \frac{dV_n}{dt} + \frac{rT\gamma}{V_n} \Delta \dot{m}_n \end{cases}$
Piston relations:	$\begin{cases} V_p = V - V_n = S \cdot y \quad \tau = l \cdot F_r \\ \frac{dV_p}{dt} = -\frac{dV_n}{dt} = S \dot{y} \quad y = l \cdot q \end{cases}$
Piston dynamic:	$\begin{cases} V(y, \dot{y}) = b_v(y, \dot{y}) \dot{y} + b_c(y, \dot{y}) \text{sign}(\dot{y}) \\ m \frac{d\dot{y}}{dt} = S(P_p - P_n) - F_v(y, \dot{y}) - F_r \end{cases}$

The system parameters are:

$$M(q) = \begin{bmatrix} A + 4m_2l_1l_2c_2 & I_2 + 2m_2l_1l_2c_2 \\ I_2 + 2m_2l_1l_2c_2 & I_2 \end{bmatrix}$$

$$C(q, \dot{q}) = \begin{bmatrix} -2m_2l_1l_2\dot{q}_2s_2 & -2m_2l_1l_2(\dot{q}_1, \dot{q}_2) \\ 2m_2l_1l_2\dot{q}_1s_2 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} m_2gl_2s_{12} + m_1gl_1s_1 \\ m_2gl_2s_{12} \end{bmatrix}$$

where  $m_i$ ,  $l_i$ ,  $I_i$  are respectively the mass, length, and the segment inertia, with  $g$  the gravitation.  $A = I_1 + I_2 + 4m_2l_1^2$ ,  $c_i = \cos(q_i)$ ,  $s_i = \sin(q_i)$ ,  $c_{ij} = \cos(q_{ij})$ ,  $s_{ij} = \sin(q_{ij})$

### 13.5.2 Hydraulic manipulator model

The robot considered here is the Slingsby TA9. A detailed description and modeling of this manipulator can be found in [5]. The robot model is composed of two stages: a dynamic one for the mechanical part and a second one which will be called the hydrodynamic stage according to the hydraulic part.

The pressure, for each robot link, is converted into a force ( $F_p$  is the force applied by the piston and  $F_{v_i}$  the friction and disturbance torques) and then into a torque by use of actuator geometric transmission and its Jacobian  $J_{p_i}(q_i)$  where  $q_i$ , angular displacement of the link  $i$ .

$$J_{p_i}(q_i) = \frac{l_a l_b \sin(q_i)}{\sqrt{l_a^2 + l_b^2 - 2l_a l_b \cos(q_i)}} \quad (13.55)$$

Lengths  $l_a$  and  $l_b$  are the geometric characteristics of the attachments of links and the actuator's cylinder. For each joint we can write, if  $y$  is the piston's displacement and  $A_2$  the cross-section area of the piston,

$$J_{p_i}(q_i)F_p = J_{p_i}(q_i)A_2P = \tau + m \frac{dy}{dt} + J_{p_i}(q_i)F_{v_i} \quad (13.56)$$

Joint torques and their derivatives can be expressed as a function of force as follows.

$$\tau = J_{p_i}(q_i)F_p = J_{p_i}(q_i)A_2P \quad (13.57)$$

The current action on the position of the valve's jack  $x$  is defined by  $x = f(i - i_o) = K_i \cdot (i - i_o)$  with  $K_i$  the valve displacement gain and  $i_o$  the initial offset current of the servo-valve. This offset will now be neglected assuming that it is experimentally compensated ( $x = K_i i$ ).

Let us define the following parameters:

$P_s = 175 \times 10^5 N.m^{-2}$ : supply pressure,

$\rho = 870 kg.m^{-3}$ : oil density coefficient,

$\beta_o = 7 \times 10^8 N.m^{-2}$ : Bulk modulus,

$d_1$ : Spool diameter,

$k_f$ : leakage coefficient,

$V_t$ : total oil volume in chambers and connection tubes, and

$C_d$ : constant factor taking into account turbulent flow across the orifice.

Let  $P$  denote the differential pressures in the chambers and  $x$  the valve spool displacement. Let us take the flow out of the valve to be positive for output ports and by use of the square root law for turbulent flow across an orifice, assuming energy conservation and if heat exchange is neglected, we can write the following expression where  $Q_s$  is the bidirectional flow across orifices,

$$Q_s = K_v x (P_s - \text{sign}(x)P)^{\frac{1}{2}} \text{ with } K_v = \frac{C_d \pi d_1}{\sqrt{\rho}}$$

$$\text{and } Q_s = A_2 \dot{y} + k_f P + \frac{V_t}{4\beta} \dot{P}.$$

This leads to the differential pressure evolution equation

$$\dot{P} + \frac{4\beta_o k_f}{V_t} P + \frac{4\beta_o A_2}{V_t} \dot{y} = \frac{4\beta_o K_v \sqrt{P_s - \text{sign}(x)P}}{V_t} x \quad (13.58)$$

The force applied by the hydraulic actuator is a function of the cross-section area of the piston  $A_2$ , ( $F_p = A_2 P$ ) and obeys the following differential equation

$$\dot{F}_p + \frac{4\beta_o k_f}{V_t} F_p + \frac{4\beta_o A_2^2}{V_t} \dot{y} = \frac{4\beta_o K_v A_2 \sqrt{P_s - \text{sign}(x)P}}{V_t} K_i i$$

The pressure differential equation can be written as

$$\dot{F}_p + B_1 F_p + E_1 \dot{q} = J_1 k_o i \quad (13.59)$$

Expressions of the parameters  $B_1$ ,  $E_1$ ,  $J_1$  and  $k_o$  can be obtained from the two preceding equations. We are rather interested in modeling and control, by expression of torques. Using relations (13.56) and (13.57), we can obtain, from the above equation, a differential equation for torque dynamics

$$\begin{aligned} \dot{\tau} + \left( \frac{4\beta_o k_f}{V_t} - J_{p_i}(q_i) J_{p_i}(q_i)^{-1} \right) \tau + \frac{4\beta_o A_2^2}{V_t} J_{p_i}(q_i) \dot{q} \\ = \frac{4\beta_o K_v A_2}{V_t} J_{p_i}(q_i) \sqrt{P_s - \text{sign}(x)P} K_i i \end{aligned}$$

$$\dot{\tau} + B \tau + E \dot{q} = J k_o i \quad (13.60)$$

$$\text{with } B = B(q) = \frac{4\beta_o k_f}{V_t} - J_{p_i}(q_i) J_{p_i}(q_i)^{-1}, E = \frac{4\beta_o A_2^2}{V_t} J_{p_i}(q_i), k_o = \frac{4\beta_o K_v A_2}{V_t} K_i \text{ and } J = J(x, P) = J_{p_i}(q_i) \sqrt{P_s - \text{sign}(x)P}.$$

Defining  $J_p(q) = \text{diag}(J_{p_1}(q_1), \dots, J_{p_n}(q_n))$  (positive diagonal matrix) for gain transfer between force and torque, we can now generalize this equation for an  $n$  link manipulator. Then  $J$ ,  $k_o$ ,  $B$ , and  $E$  become  $(n \times n)$  diagonal matrices called the actuator's parameters depending on the temperature, oil characteristics, positions and initial conditions. Note that  $J$  and  $E$  are nonlinear and depend on the actuators variables. From the robot dynamic equations and introducing a term  $F_v$  comprising all the friction effects (angular, linear and nonlinear terms and disturbances), we can obtain the complete dynamic model of the hydraulic actuated manipulator.

$$\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + F_v \quad (13.61)$$

$$\dot{\tau} + B.\tau + E\dot{q} = J.k_o.i \quad (13.62)$$

### 13.5.3 Proof of lemma

We can prove the boundedness of derivative of the inertia matrix  $\dot{M}(q)$  and the term  $\dot{C}(q, \dot{q})$ ,  $\dot{F}_v(q, \dot{q})$ . Recall that from the robot physical properties we have

$$\|M(q)\| \leq \alpha_1, \|C(q, \dot{q})\| \leq \alpha_2 \|\dot{q}\|, \text{ and } \|F_v(\dot{q})\| \leq \alpha_4 + \alpha_5 \|\dot{q}\|$$

and suppose that  $\omega_m$  is the highest frequency of  $(q, \dot{q})$  the trajectory components (positions and velocities), then from  $M \leq \alpha_1 I_n$  it can be shown that  $\|\dot{M}(q)\| \leq \alpha_1 \omega_m$ . The same conclusion can be obtained for  $\|\dot{C}(q, \dot{q})\| \leq \alpha_2 \omega_m \|\dot{q}\|$ , and for frictions  $\|\dot{F}_v(q, \dot{q})\| \leq \alpha_4 \omega_m + \alpha_5 \omega_m \|\dot{q}\|$ .

Proof of lemma 2: Let us start by boundedness of  $J.J_o^{-1}$  and consider  $\kappa \in R$  such as  $J = J_{p_i}(q_i)\sqrt{P_s - \text{sign}(x)P} \leq \max(J_{p_i}(q_i))\sqrt{P_s} = \kappa$  and take  $J_o = \kappa \implies J.J_o^{-1} \leq 1$ . Then we can conclude that  $\tilde{M} = J.J_o^{-1}M_o - M \leq \alpha_1 I_n$ , and  $\|\tilde{C}\| = \|J.J_o^{-1}C_o - C\| \leq \alpha_2 \|\dot{q}\|$ .

Friction disturbances and gravitation effects are bounded:

$$\|F_v(q, \dot{q})\| \leq \alpha_4 + \alpha_5 \|\dot{q}\|, \text{ and } \|G(q)\| < \alpha_3.$$

Let us consider bounds for the hydraulic parameters:

$$\|k_o\| = \left\| \frac{4\beta K_v A_2}{V_t} K_i \right\| \leq k_m \text{ and } \|E\| = \left\| \frac{4\beta A_2^2}{V_t} \right\| \leq e_m,$$

and for the position dependant parameter, we can write:

$$\|B\| = \|B(q)\| = \left\| \frac{4\beta k_f}{V_t} - J_{p_i}(q_i)J_{p_i}(q_i)^{-1} \right\| \leq b_m.$$

Proof of (i): By considering the expression  $H(q, \dot{q}, \ddot{q}) = (\dot{M} + BM)\ddot{q} + (\dot{C} + E + BC)\dot{q} + \dot{G} + \dot{F}_v + BG + BF_v$  and  $\tilde{H} = J.J_o^{-1}H_o - H$  we can write

$$\begin{aligned} \|\tilde{H}\| &\leq (\alpha_1 \omega_m + \alpha_1) \|\ddot{q}\| + (\alpha_2 \omega_m \|\dot{q}\| + e_m + b_m \alpha_2 \|\dot{q}\|) \|\dot{q}\| + \\ &+ \alpha_3 \omega_m + \alpha_4 \omega_m + \alpha_5 \omega_m \|\dot{q}\| + b_m \alpha_3 + b_m (\alpha_4 \omega_m + \alpha_5 \omega_m \|\dot{q}\|) \end{aligned}$$

This leads to  $\|\tilde{H}\| \leq (\alpha_1\omega_m + \alpha_1)\|\dot{q}\| + (e_m + \alpha_5\omega_m + \alpha_5b_m\omega_m)\|\dot{q}\|$   
 $+ (\alpha_2\omega_m + b_m\alpha_2)\|\dot{q}\|^2 + \alpha_3\omega_m + \alpha_4\omega_m + b_m\alpha_3 + b_m\alpha_4\omega_m$

Then we obtain:  $\|\tilde{H}\| \leq \kappa_0 + \kappa_1\|\dot{q}\| + \kappa_2\|\dot{q}\|^2 + \kappa_3\|\dot{q}\|$

Proof of (ii): By considering the inequalities:  $\|\tilde{M}\| \leq \alpha_1$ ,  $\|\tilde{C}\| \leq \alpha_2\|\dot{q}\|$

and the previous result we have:  $\|\tilde{M}\ddot{q}_r + \tilde{C}\dot{q}_r + \tilde{H}\| \leq \alpha_1\|\ddot{q}_r\| + \alpha_2\|\dot{q}_r\| +$   
 $\kappa_0 + \kappa_1\|\dot{q}\| + \kappa_2\|\dot{q}\|^2 + \kappa_3\|\dot{q}\|$ .

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## Chapter 14

# Sliding Modes Control of the Induction Motor: a Benchmark Experimental Test

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### 14.1 Introduction

It is well known that the induction motor has very interesting characteristics because of its squirrel cage rotor and of the absence of a brush-collector device. These characteristics are highly appreciated in industry for a number of reasons among which is certainly the traditional robustness of this electromechanical device. The induction motor is also cheap to buy and maintain relative to other types of electric motors. However, speed control (or position) is particularly complex for this type of motor.

For many decades, the Direct Current (D.C.) motor constituted the principal electromechanical actuator for variable speed applications because of its simplicity of control. The induction motor has benefited from recent

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advances in the field of nonlinear control combined with the evolution of microprocessor technology. Power electronics also makes it possible to implement powerful nonlinear control laws. This allows users to apply new methods [10, 2], derived from the so-called differential-geometric approach [6, 2], from passivity-based-controllers [7], or from the differential-algebraic approach [1]. However, a major difficulty is probably worth some research effort: the robustness of control law with respect to parametric uncertainties or disturbances. Also, the accurate observation of the rotor variables that are inaccessible for direct measurement is a difficulty inherent to the design of induction motor control. It is precisely in this context that the control technique based on sliding modes finds its best justification since it is able to cope with model uncertainties.

We think the main contribution of this chapter is to present the sliding mode control technique for the induction motor and to show its applicability to one significant benchmark <sup>2</sup>, for both simulation and experimentation conditions. The general principles of the sliding modes control theory by a nonlinear approach are briefly developed in the second section of the chapter. The special case of the speed and flux control of the induction motor is studied in the third section. The “horizontal handling” benchmark of electric motors as well as the national hardware setup located at IRCCyN ([www.ircyn.ec-nantes.fr/Banc.Essai](http://www.ircyn.ec-nantes.fr/Banc.Essai)) are described in the fourth section. The fifth section describes the experimental results obtained from the platform. Finally some conclusions and research perspectives are developed.

## 14.2 Sliding modes control

This section briefly presents a control technique using sliding modes. For more details, see [3, 4, 5, 8, 9, 11] and the previous chapters of this book.

Consider the multivariable nonlinear system described by the equations:

$$\begin{aligned}\dot{x} &= f(x, t) + g(x, t)u \\ y &= h(x, t)\end{aligned}\tag{14.1}$$

where  $x \in \mathbb{R}^n$  is the state vector of the system,  $u \in \mathbb{R}^m$  is the control vector, and  $y \in \mathbb{R}^m$  is the output vector. One technique of control by sliding modes can be defined as:

- 1) finding a sliding surface  $S(x, t) = 0 \in \mathbb{R}^m$  that yields the convergence of the output  $y \in \mathbb{R}^m$  for the desired references; and
- 2) finding a control law in terms of a new input discontinues  $u_n(x, t)$ :

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<sup>2</sup>This benchmark was defined by the French national program “Commande de Machines” of the CNRS and French Minister of Education and Research, MENSUR.

$$u_n(x, t) = \begin{cases} u_n^+ & \text{if } S(x, t) > 0 \\ u_n^- & \text{if } S(x, t) < 0 \end{cases} \quad (14.2)$$

to attract the trajectory of the system towards surface  $S(x, t) = 0$  in a finite time. The design of  $u_n$  was obtained from a particular Lyapunov function and will be defined latter. The sliding surfaces are designed to impose a trajectory tracking of the output  $y$  with respect to a reference  $y_{ref}$ . Thus, for each component of  $S(x, t)$ , one may choose:

$$S_j(x, t) = \sum_{i=0}^{r_j-1} l_{ji}(y_{jref} - y_j)^{(i)} \quad \text{with } j = 1, \dots, m \quad (14.3)$$

where  $r_j$  is the relative degree of the output  $y_j(t)$  [6]. The value of  $r_j$  implies the  $u$ -dependence of  $\dot{S}$ . The sliding surface (14.3) was designed as a linear dynamics of tracking error  $(y_{ref} - y)$ , and it is possible to guarantee by an adequate choice of the coefficients  $l_{ji}$  that if the system is constrained to remain in surface  $S(x, t) = 0$ , it slides towards the origin, i.e., the error  $(y_{ref} - y)$  tends toward zero with trajectory dynamics constrained by the choice of  $l_{ji}$ .  $\dot{S}(x, u, t)$  reads as:

$$\dot{S}(x, u, t) = \frac{dS}{dt} = \frac{\partial S}{\partial x} \frac{dx}{dt} + \frac{\partial S}{\partial t}$$

with for  $j = 1, \dots, m$

$$\frac{\partial S_j}{\partial t} := c_j(t) = \sum_{i=0}^{r_j-1} \frac{\partial S_j}{\partial y_{ref}^{(i)}} \frac{\partial y_{ref}^{(i)}}{\partial t}$$

For example,

$$\dot{S}(x, u, t) = \frac{\partial S}{\partial x} (f(x, t) + g(x, t)u) + c(t) \quad (14.4)$$

Then, we can write (14.4) in the following way:

$$\dot{S}(x, u, t) = a(x, t) + b(x, t)u \quad (14.5)$$

The control law for (14.5) is defined as

$$u = \frac{-a(x, t) + u_n}{b(x, t)} \quad (14.6)$$

in order to linearize and to decouple the dynamics of the error of each output. The result of the application of this control is

$$\dot{S}(x, u, t) = u_n \quad (14.7)$$

The design of  $u_n(x, t)$  is based on the concept of stability according to Lyapunov theory. Choosing as Lyapunov function:

$$V(x, t) = \frac{1}{2} S^T S \geq 0 \quad (14.8)$$

which is definite positive semi-definite, we compute the time derivative

$$\dot{V}(x, u, t) = S^T u_n \quad (14.9)$$

To guarantee the negativity of  $\dot{V}(x, u, t)$  and thus the stability of the system toward the origin of  $S(x, t)$ , it is sufficient that

$$u_n = -k \text{sign}(S) \quad (14.10)$$

with  $k := [k_1, \dots, k_m]$  where the  $k_j$  are strictly positive. The control by sliding modes is thus written:

$$u = \frac{1}{b(x, t)} [-a(x, t) - k \text{sign}(S)] \quad (14.11)$$

Figure 14.1 represents the dynamics of the system after feedback for each component of  $S(x, t)$ .

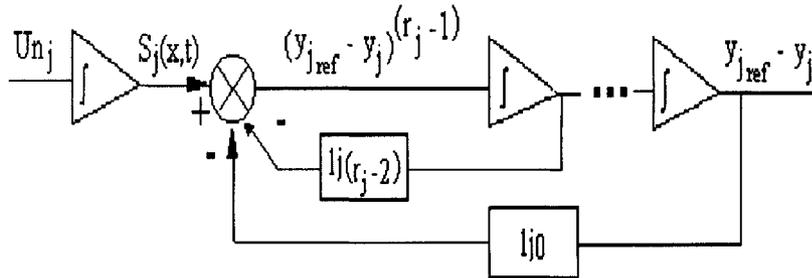


Figure 14.1: Dynamics of the error ( $y_{ref} - y$ ) after feedback

We obtain a dynamic equation of  $(y_{ref} - y)$ , which is autonomous for  $u_{n,j} = 0$ . The dynamics of the feedback system is such that there is convergence toward surface  $S(x, t) = 0$  and then sliding mode along this surface. In the case of uncertainties or disturbances, the control known as equivalent control  $(-a(x, t)/b(x, t))$  is not able to guarantee  $S(x, t) = 0$  continuously [i.e., it does not carry out the exact input-output linearization and the decoupling of the virtual output  $S(x, t)$ ]. The trajectories of the system leave the surface instead of remaining (or sliding) on it. It is the role

of the control  $u_n$ , to force the trajectories of the system to return to the sliding surface. Because of the presence of the discontinuous term  $u_n$ , the control can present a succession of commutations. This is the phenomenon of chattering, as represented in Figure 14.2.

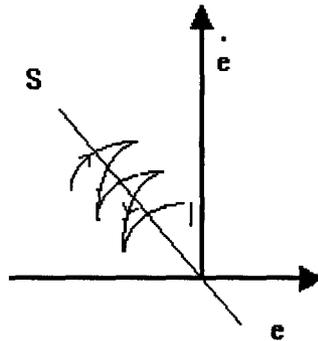


Figure 14.2: Phenomenon of chattering

This phenomenon sometimes limits the application of the sliding modes control to physical systems. This problem leads to a high number of oscillations of the system trajectory around the sliding surface, and thus the excessive use of the actuators. To reduce the frequency of the oscillations, the control is modified so that the response is slower during the sign change of  $S(x, t)$ . We applied a continuous and "smooth" law of switching as in [4, 9]. Other possibilities for the smoothing of the control can be found and are also presented in the previous chapters of this book. A possible way to design the switching function is to use one dead zone and two linear zones to smooth the control. The thickness of the "boundary layer" designed by  $\varepsilon_1$  and  $\varepsilon_2$  is a compromise between the reduction of the phenomenon of chattering and the precision of the tracking trajectory. Figure 14.3 shows the areas in the plan of phases, which corresponds to the various types of action of the control. The variation of  $\varepsilon_2$  has been defined by

$$\varepsilon_2 = \max \{ \varepsilon_{\min}, \varepsilon(e) \} \quad (14.12)$$

where the function  $\varepsilon_2$  is proportional to the absolute value of the error  $e$ . The idea behind this technique is to impose a variable speed of convergence for the sliding surface, which is slower when the linear dynamics represented by (14.3) is away from the origin and increases when it comes closer to the origin. This way, when the dynamics of the variation is some distance

away the origin, the control is softer and chattering is reduced. This can be represented by a cone effect (see Figure 14.3) in the phase map.

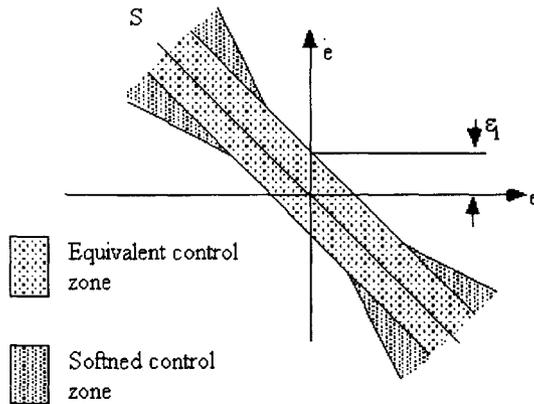


Figure 14.3: Softened control by a variable function “Sign” ( $l_{j1} = 1$ )

We thus obtain the precision given by the classical Sign function and the required attenuation of the chattering.

### 14.3 Application to the induction motor

For an induction motor under the classical assumptions of sinusoidal distribution of magnetic induction in the air-gap, with no saturation of the magnetic circuit, the diphasic model  $\alpha\beta$  [2] is

$$\dot{x} = f(x) + gu + \xi \quad (14.13)$$

where

$$x = [\Omega, \Phi_{r\alpha}, \Phi_{r\beta}, i_{s\alpha}, i_{s\beta}]^T, \quad u = [u_{s\alpha}, u_{s\beta}]^T \quad (14.14)$$

and  $\xi$  is a disturbance input (load torque) and

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \end{bmatrix} = \begin{bmatrix} (pM_{sr}/JL_r) (\Phi_{r\alpha}i_{s\beta} - \Phi_{r\beta}i_{s\alpha}) - (f_v/J) \Omega \\ -(R_r/L_r) \Phi_{r\alpha} - p\Omega\Phi_{r\beta} + (R_r/L_r) M_{sr}i_{s\alpha} \\ p\Omega\Phi_{r\alpha} - (R_r/L_r) \Phi_{r\beta} + (R_r/L_r) M_{sr}i_{s\beta} \\ (M_{sr}/\sigma L_s L_r) ((R_r/L_r) \Phi_{r\alpha} + p\Omega\Phi_{r\beta}) - \gamma i_{s\alpha} \\ (M_{sr}/\sigma L_s L_r) ((R_r/L_r) \Phi_{r\beta} - p\Omega\Phi_{r\alpha}) - \gamma i_{s\beta} \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 0 & 0 & 1/\sigma L_s & 0 \\ 0 & 0 & 0 & 0 & 1/\sigma L_s \end{bmatrix}^T, \quad \xi = \begin{bmatrix} -T_l/J & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$R_s$  and  $R_r$  are the resistances of the stator and the rotor.  $L_s$  and  $L_r$  are the self-inductances of stator and of rotor,  $M_{sr}$  is the mutual inductance between the stator and rotor windings,  $J$  is the inertia of the system (motor and load),  $p$  is the number of pole-pair,  $f_v$  the coefficient of viscous damping and  $T_l$  is the load torque. The parameters  $\sigma$  and  $\gamma$  are defined by:

$$\sigma := 1 - \frac{M_{sr}^2}{L_s L_r}, \quad \gamma := \frac{L_r^2 R_s + M_{sr}^2 R_r}{\sigma L_s L_r^2}$$

As defined by (14.14), the states of the system are the mechanical speed, and the two components of the rotor flux and of the stator current. The inputs are the stator voltages. The load is considered as a nonmeasured disturbance.

## Design of the control by sliding modes

The outputs  $y_1$  and  $y_2$  are the speed  $\Omega$  and the square of the rotor flux  $\Phi^2 = \Phi_{r\alpha}^2 + \Phi_{r\beta}^2$ . The goal is to force these outputs to track a given trajectory. According to the technique presented in Section 14.2, the sliding surfaces selected are

$$S_1 = (\dot{y}_{1ref} - \dot{y}_1) - l_1(y_{1ref} - y_1) = (\dot{\Omega}_{ref} - \dot{\Omega}) - l_1(\Omega_{ref} - \Omega) \quad (14.15)$$

$$S_2 = (\dot{y}_{2ref} - \dot{y}_2) - l_2(y_{2ref} - y_2) = (\dot{\Phi}_{ref}^2 - \dot{\Phi}^2) - l_2(\Phi_{ref}^2 - \Phi^2) \quad (14.16)$$

These functions can be regarded as virtual outputs. Then the objective is to force these outputs to zero to obtain a sliding mode. The dynamic equation of  $S_1(x, t)$  is

$$\dot{S}_1(x, u, t) = (\ddot{\Omega}_{ref} - \ddot{\Omega}) - l_1(\dot{\Omega}_{ref} - \dot{\Omega}) \quad (14.17)$$

If we do not take into account the load disturbance, Equation (14.17) becomes

$$\begin{aligned}\dot{S}_1(x, u, t) &= \ddot{\Omega}_{ref} - \dot{f}_1(x, u) - l_1 \left[ \dot{\Omega}_{ref} - f_1(x) \right] \\ &:= a_1(x, t) + b_{11}(x)u_{s\alpha} + b_{12}(x)u_{s\beta}\end{aligned}\quad (14.18)$$

The dynamics of the second virtual output  $S_2(x, t)$  is

$$\begin{aligned}\dot{S}_2(x, u, t) &= \ddot{\Phi}_{ref}^2 - 2(\Phi_{r\alpha}\dot{f}_2(x, u) + [f_2(x)]^2 + \Phi_{r\beta}\dot{f}_3(x, u) + [f_3(x)]^2) \\ &\quad - l_2 \left\{ \dot{\Phi}_{ref}^2 - 2[\Phi_{r\alpha}f_2(x) + \Phi_{r\beta}f_3(x)] \right\} \\ &:= a_2(x, t) + b_{21}(x)u_{s\alpha} + b_{22}(x)u_{s\beta}\end{aligned}\quad (14.19)$$

Thus the control is written as

$$u = \begin{bmatrix} u_{s\alpha} \\ u_{s\beta} \end{bmatrix} = - \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix}^{-1} \begin{bmatrix} a_1(x, t) \\ a_2(x, t) \end{bmatrix} + \begin{bmatrix} k_1 \text{sign}(S_1) \\ k_2 \text{sign}(S_2) \end{bmatrix}\quad (14.20)$$

where  $k_j$  are the gains of the switching control.

To decrease the high frequency oscillations (chattering), the discontinuous control is softened by means of variable Sign function (see Figure 14.3). The choice of the parameters  $l_j$  determines the slope of the sliding surface, i.e., the convergence speed of the error dynamics when the system is in sliding mode. In the following section, this technique will be validated on a specific benchmark.

## 14.4 Benchmark “horizontal handling”

The objective of benchmark “horizontal handling” implemented on the experimental set-up located at IRCyN ([www.ircyn.ec-nantes.fr/Banc\\_Essai](http://www.ircyn.ec-nantes.fr/Banc_Essai)) is to allow the study of problems arising within the framework of horizontal handling, mainly:

- conveyer belt with (at nominal speed) abrupt constraints of load; and
- travelling crane with constraints of controlled accelerations and emergency stop.

This benchmark is checked here with speed sensor and without flux sensor. It can be extended to the same applications without flux and velocity sensors.

### 14.4.1 Speed and flux references and load disturbance

The flux reference is a constant value computed from the plate of the manufacturer (value of peak of the rotor flux  $\Phi_r \text{ ref} = 0,595 \text{ Wb}$ ).

The speed reference and the load disturbance are in accordance with the curves shown in Figure 14.4.

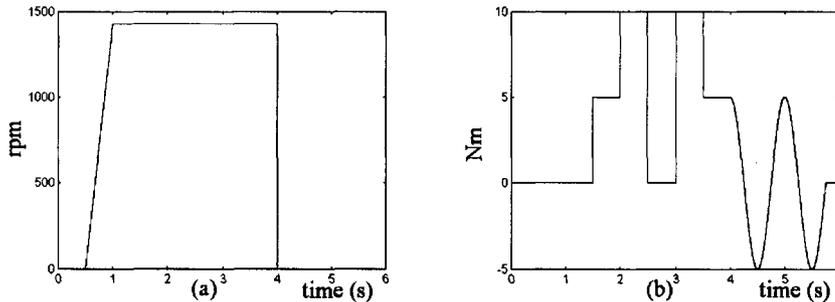


Figure 14.4: (a) Speed reference and (b) load disturbance

Figure 14.4(a) shows that the reference corresponds to a constant acceleration up to the nominal speed. Then, various torques of disturbance corresponding to loading and unloading a conveyor belt are applied. For a time longer than 4s, an emergency stop (electric braking) is applied with a sinusoidal torque corresponding to the swing of the load on a travelling crane.

### 14.4.2 Induction motor parameters (squirrel cage rotor)

Normal rated power: 1,5 kW

Number of pole pairs:  $p = 2$

Nominal speed: 1430 rpm

Nominal voltage: 220V

Nominal intensity: 6,1 A

Stator resistance:  $R_s = 1,47 \text{ Ohm}$

Rotor resistance:  $R_r = 0,79 \text{ Ohm}$

Stator self-inductance:  $L_s = 0,105 \text{ H}$

Rotor self-inductance:  $L_r = 0,094 \text{ H}$

Mutual inductance:  $M_{sr} = 0,094 \text{ H}$

(i.e.,  $\tau_r = 0,119 \text{ s}$ ,  $\sigma = 0,105$ )

Inertia (motor and load):  $J = 25,6 \times 10^{-3} \text{ Nm/rad/s}^2$

Viscous damping coefficient:  $f_v = 2,9 \times 10^{-3}$  Nm/rad/s.  
Constant torque friction:  $C_s = 0,38$  Nm

### 14.4.3 Variations of the parameters for robustness test

In order to keep the robustness property three cases are considered:

- 1) Increase of resistances ( $\Delta R_s = \Delta R_r = +50\%$ );
- 2) Decrease of resistances ( $\Delta R_s = \Delta R_r = -50\%$ ); and
- 3) Increase of inductances ( $\Delta L_s = \Delta L_r = \Delta M_{sr} = +20\%$ ).

## 14.5 Simulation and experimentation results

In this section, some experimental results were obtained using the control laws for speed and flux proposed in Section 14.3. All the experimental tests were made on the motor set-up of IRCyN, at Nantes, with the use of the "horizontal handling" benchmark described in Section 14.4. The results correspond to the three studied cases. The first case was when the control law was based on the nominal values of the system parameters. A second case was when the control was based on stator and rotor resistance value deviations of -50% with respect to the nominal values. In this case, we tried to evaluate the robustness of the control despite resistance variation due to the motor's internal temperature variation. A third and last case was when the stator, rotor, and mutual inductance, values were deviated by -20% with respect to the nominal values. In this case the goal was to evaluate the control robustness with respect to the parameter errors due to the magnetic saturation. It is important to note that no torque observer was necessary to apply the sliding control.

The rotor flux component necessary for the control, was obtained by an observer similar to that in [12]. The choice of the surfaces parameters (14.3) gave the error trajectories when the sliding mode occurred. This choice was made so that it would not saturate the equivalent controls to allow the application of the discontinuous control  $u_n$ . The tuning of the coefficients  $\varepsilon_1$  and  $\varepsilon_2$  of the switching function was a compromise between the precision and the attenuation of chattering as studied in [4, 9].

### 14.5.1 Results of simulations

In this subsection, we show the results of four different cases of simulation: the simulation of the nominal system and three other simulations corresponding to the robustness tests described in subsection 14.4.3:

- Nominal system

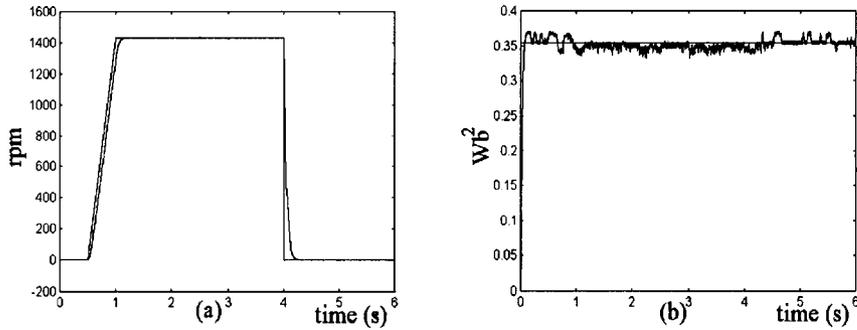


Figure 14.5: (a) Speed motor and reference; (b) flux (square) and reference

Figure 14.5(a) shows the correct response for a speed trajectory tracking in spite of a very significant load disturbance (nominal load 10 Nm). The delay in speed that appears, was introduced by a filter on the references of flux and speed in order to smooth and limit the amplitude of control and current in transient.

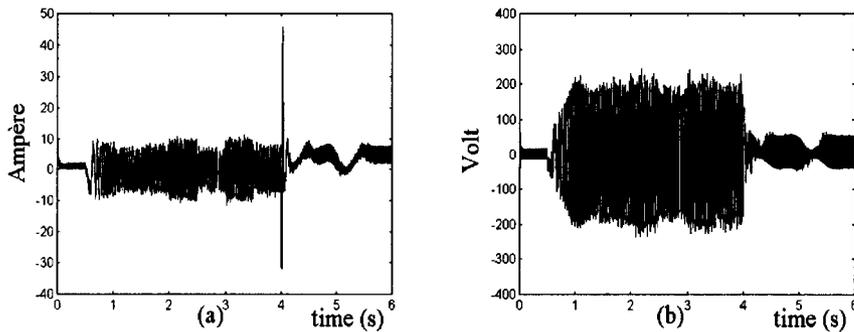


Figure 14.6: (a) Stator current and (b) voltage

Figure 14.5(b) shows the response of flux (square value : 0,35 Wb<sup>2</sup>).

We can see in Figure 14.6(a) a peak of stator current that appeared right at the time of the motor stopped operating ( $\Omega_{ref} = 0$ ). If needed, these overcurrents can be decreased either by adjusting the filters or by

decreasing the gains of the discontinuous control. Figure 14.6(b) shows that the stator voltage was not saturated.

- Tests of robustness

First case:  $\Delta R_s = \Delta R_r = +50\%$

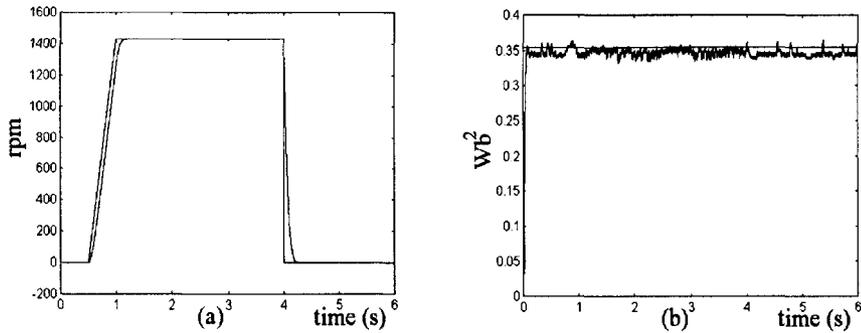


Figure 14.7: (a) Speed motor and reference; (b) flux (square) and reference

Second case:  $\Delta R_s = \Delta R_r = -50\%$

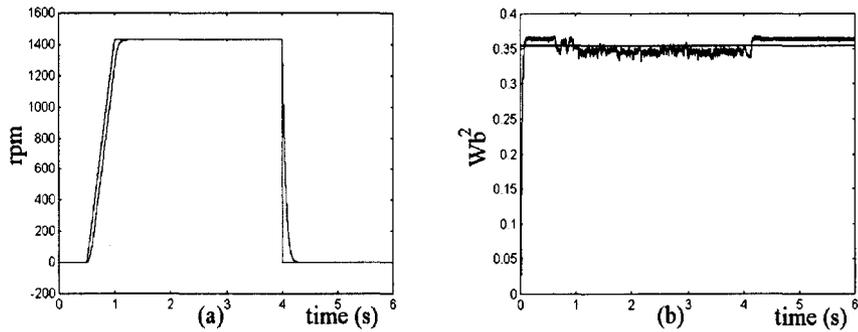


Figure 14.8: (a) Speed motor and reference; (b) flux (square) and reference

Third case:  $\Delta L_s = \Delta L_r = \Delta M_{sr} = +20\%$

From Figures 14.7, 14.8, and 14.9, we can deduce that the sliding mode control of the induction motor is very robust with respect to parametric uncertainties as well as with the load disturbance without its measurement

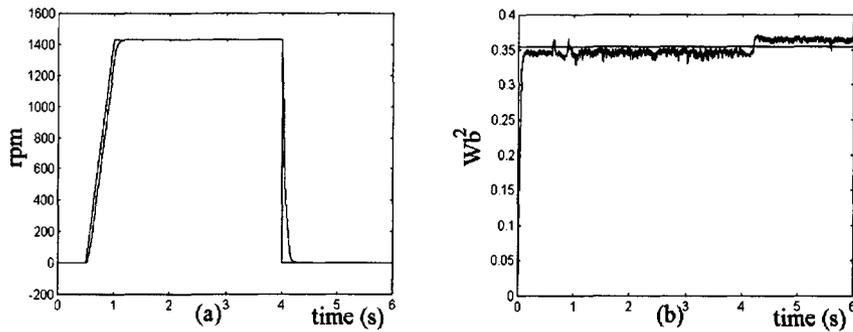


Figure 14.9: (a) Speed motor and reference; (b) flux (square) and reference

or estimation. For all the studied cases, there is no sensible difference on the speed trajectories. Nevertheless, for the flux control, there are some differences for the three parameters variation cases. This result is a way to understand the sensitivity of this kind of control for future developments.

### 14.5.2 Experimental results

This subsection gives some experimental results obtained with the use of the benchmark described in Section 14.4.

Figure 14.10 shows the response to the control of the speed and the flux.

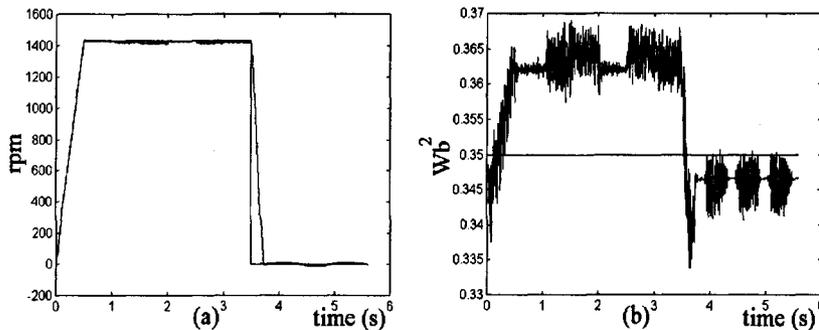


Figure 14.10: (a) Speed motor and reference; (b) flux (square) and reference

We can see in Figure 14.10(a) that the dynamic behavior for tracking the

speed reference is good: in nominal speed case, the application of the load disturbance causes a transient error of 1% in the speed tracking which is immediately corrected by the control. When the speed reference is zero, we perceive that the sinusoidal torque of disturbance causes a small angular oscillation despite the reaction of the motor control. This result can be improved by a more precise adjustment of the coefficients of surface speed.

From Figure 14.10(b), we can verify that the flow error is about 1.5% when the motor reaches its nominal speed. For the situation where the speed reference is zero the flow error is approximately 1.3%.

The Figure 14.11 shows the measurement of the torque applied to the rotor of the induction motor.

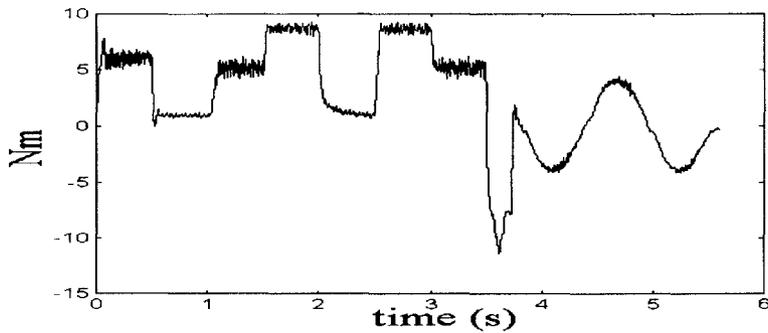


Figure 14.11: Measured torque

The measured torque is the sum of the disturbance torque, and the inertial and friction rotor forces. This can be particularly observed for the transient part of the response.

Figure 14.12 shows the measured stator current and the voltage applied to the motor.

On these experimental measurements, we can verify that there is no saturation in the control (nor peaks in the absorbed current) by the induction motor that could make impracticable the implementation of sliding mode control for a industrial converter.

## 14.6 Conclusion

This chapter investigated the induction motor sliding modes control. Its main objective was to verify the applicability of this technique to the speed and flux controls of the induction motor. This was carried out using a

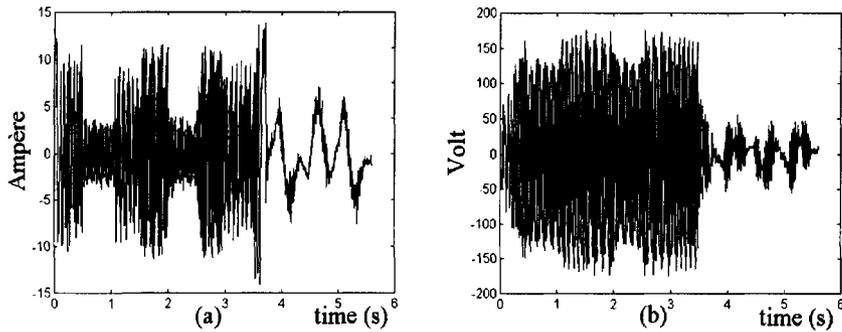


Figure 14.12: (a) Stator current and (b) voltage

significant benchmark related to the “horizontal handling” problem. The design of a control for nonlinear control method was used in terms of nonlinear decoupling control. The sliding surfaces were chosen so that the dynamics of the speed and flux references tracking errors were linear. The control carried out the decoupling and the accurate linearization of the nominal system error dynamics. In case the system was badly identified or disturbed, one term of discontinuous control designed with a particular Lyapunov function guaranteed the convergence of the system to the sliding surface. The phenomenon of chattering was limited using a particular smooth function  $\text{Sign}$ .

Many simulations and experimental results certified the applicability of the technique in the context of the tasks related to the “horizontal handling” problem. By simulation, several results certified the robustness of the technique used with relation to the parameters errors of the model.

The results of this work showed the necessity for better experimental evaluation of the robustness in relation to the parameters errors and the observation quality. This prospect will be studied more thoroughly and applied on our experimental set-up.

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