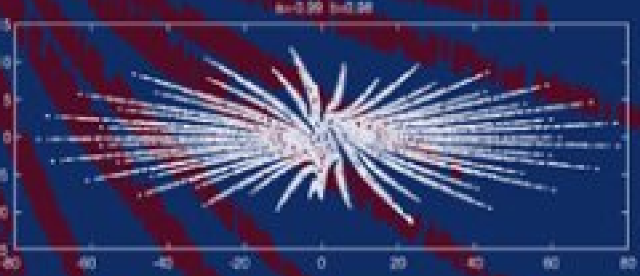
ELEMENTARY MATHEMATICAL and COMPUTATIONAL TOOLS for ELECTRICAL and COMPUTER ENGINEERS USING MATLAB®



Jamal T. Manassah

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About the Author

Jamal T. Manassah, has been Professor of Electrical Engineering at the City College of New York since 1981. He received his B.Sc. degree in Physics from the American University of Beirut, and his M.A. and Ph.D. in Theoretical Physics from Columbia University. Dr. Manassah was a Member of the Institute for Advanced Study. His current research interests are in theoretical and computational quantum and nonlinear optics, and in photonics.

Introduction

This book is mostly based on a series of notes for a primer course in electrical and computer engineering that I taught at the City College of New York School of Engineering. Each week, the class met for an hour of lecture and a three-hour computer laboratory session where students were divided into small groups of 12 to 15 students each. The students met in an informal learning community setting, a computer laboratory, where each student had the exclusive use of a PC. The small size of the groups permitted a great deal of individualized instruction, which was a key ingredient to cater successfully to the needs of students with heterogeneous high school backgrounds.

A student usually takes this course in the second semester of his or her freshman year. Typically, the student would have completed one semester of college calculus, and would be enrolled in the second course of the college calculus sequence and in the first course of the physics sequence for students in the physical sciences and engineering.

My purpose in developing this book is to help bring the beginner engineering student's analytical and computational skills to a level of competency that would permit him or her to participate, enjoy, and succeed in subsequent electrical and computer engineering courses. My experience indicates that the lack of mastery of fundamental quantitative tools is the main impediment to a student's progress in engineering studies.

The specific goals of this book are

1. To make you more comfortable applying the mathematics and physics that you learned in high school or in college courses, through interactive activities.
2. To introduce you, through examples, to many new practical tools of mathematics, including discrete variables material that are essential to your success in future electrical engineering courses.
3. To instruct you in the use of a powerful computer program, MATLAB®*, which was designed to be simultaneously user-friendly and powerful in tackling efficiently the most demanding problems of engineering and sciences.
4. To give you, through the applications and examples covered, glimpses of some of the fascinating problems that an electrical or

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computer engineer solves in the course of completing many of his or her design projects.

My experience indicates that you can achieve the above goals through the following work habits that I usually recommend to my own students:

- Read carefully the material from this book that is assigned to you by your instructor for the upcoming week, and make sure to solve the suggested preparatory exercises in advance of the weekly lecture.
- Attend the lecture and follow closely the material presented, in particular the solutions to the more difficult preparatory exercises and the demonstrations.
- Following the lecture, make a list of questions on the preparatory material to which you still seek answers, and ask your instructor for help and clarification on these questions, preferably in the first 30 minutes of your computer lab session.
- Complete the in-class exercises during the computer lab session. If you have not finished solving all in-class exercises, make sure you complete them on your own, when the lab is open, or at home if you own a computer, and certainly before the next class session, along with the problems designated in the book as homework problems and assigned to you by your instructor.

In managing this course, I found it helpful for both students and instructors to require each student to solve all problems in a bound notebook. The advantage to the student is to have easy access to his or her previous work, personal notes, and reminders that he or she made as the course progressed. The advantage to the instructor is to enhance his or her ability to assess, more easily and readily, an individual student's progress as the semester progresses.

This book may be used for self-study by readers with perhaps a little more mathematical maturity acquired through a second semester of college calculus. The advanced reader of this book who is familiar with numerical methods will note that, in some instances, I did not follow the canonical order for the sequence of presentation of certain algorithms, thus sacrificing some optimality in the structure of some of the elementary programs included. This was necessitated by the goal I set for this book, which is to introduce both analytical and computational tools simultaneously.

The sections of this book that are marked with asterisks include material that I assigned as projects to students with either strong theoretical interest or more mathematical maturity than a typical second semester freshman student. Although incorporated in the text, they can be skipped in a first reading. I hope that, by their inclusion, I will facilitate to the interested reader a smooth transition to some new mathematical concepts and computational tools that are of particular interest to electrical engineers.

This text greatly benefited from course material previously prepared by my colleagues in the departments of electrical engineering and computer science at City College of the City University of New York, in particular, P. Combettes, I. Gladkova, B. Gross, and F. Thau. They provided either the starting point for my subsequent efforts in this course, or the peer critique for the early versions of this manuscript. I owe them many thanks and, of course, do not hold them responsible for any of the remaining imperfections in the text.

The preparation of this book also owes a lot to my students. Their questions and interest in the material contributed to many modifications in the order and in the presentation of the different chapters. Their desire for working out more applications led me to expand the scope of the examples and exercises included in the text. To all of them, I am grateful.

I am also grateful to Erwin Cohen, who introduced me to the fine team at CRC Press, and to Jerry Papke whose stewardship of the project from start to end at CRC Press was most supportive and pleasant. The editorial and production teams at CRC in particular, Samar Haddad, the project editor, deserve credit for the quality of the final product rendering. Naomi Fernandes and her colleagues at The MathWorks Inc. kindly provided me with a copy of the new release of MATLAB for which I am grateful.

I dedicate this book to Azza, Tala, and Nigh whose support and love always made difficult tasks a lot easier.

Jamal T. Manassah
New York, January 2001

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*The asterisk indicates more advanced material that may be skipped in a first reading.

1

Introduction to MATLAB® and Its Graphics Capabilities

1.1 Getting Started

MATLAB can be thought of as a library of programs that will prove very useful in solving many electrical engineering computational problems. MATLAB is an ideal tool for numerically assisting you in obtaining answers, which is a major goal of engineering analysis and design. This program is very useful in circuit analysis, device design, signal processing, filter design, control system analysis, antenna design, microwave engineering, photonics engineering, computer engineering, and all other sub-fields of electrical engineering. It is also a powerful graphic and visualization tool.

The first step in using MATLAB is to know how to call it. It is important to remember that although the front-end and the interfacing for machines with different operating systems are sometimes different, once you are inside MATLAB, all programs and routines are written in the same manner. Only those few commands that are for file management and for interfacing with external devices such as printers may be different for different operating systems.

After entering MATLAB, you should see the prompt **`>>`**, which means the program interpreter is waiting for you to enter instructions. (Remember to press the Return key at the end of each line that you enter.)

Now type **`clf`**. This command creates a graph window (if one does not already exist) or clears an existing graph window.

Because it is impossible to explain the function of every MATLAB command within this text, how would you get information on a certain command syntax? The MATLAB program has extensive help documentation available with simple commands. For example, if you wanted help on a function called **`roots`** (we will use this function often), you would type **`help roots`**.

Note that the help facility cross-references other functions that may have related uses. This requires that you know the function name. If you want an idea of the available help files in MATLAB, type **`help`**. This gives you a list of topics included in MATLAB. To get help on a particular topic such as the Optimization Toolbox, type **`help toolbox/optim`**. This gives you a list of all relevant functions pertaining to that area. Now you may type **help** for any function listed. For example, try **help fmin**.

1.2 Basic Algebraic Operations and Functions

The MATLAB environment can be used, on the most elementary level, as a tool to perform simple algebraic manipulations and function evaluations.

Example 1.1

Exploring the calculator functions of MATLAB. The purpose of this example is to show how to manually enter data and how to use basic MATLAB algebraic operations. Note that the statements will be executed immediately after they are typed and entered (no equal sign is required).

Type and enter the text that follows the **>>** prompt to find out the MATLAB responses to the following:

```
2+2
5^2
2*sin(pi/4)
```

The last command gave the sine of $\pi/4$. Note that the argument of the function was enclosed in parentheses directly following the name of the function. Therefore, if you wanted to find $\sin^3(\pi/4)$, the proper MATLAB syntax would be

```
sin(pi/4)^3
```

To facilitate its widespread use, MATLAB has all the standard elementary mathematical functions as built-in functions. Type **help elfun**, which is indexed in the main help menu to get a listing of some of these functions. Remember that this is just a small sampling of the available functions.

```
help elfun
```

The response to the last command will give you a large list of these elementary functions, some of which may be new to you, but all of which will be used in your future engineering studies, and explored in later chapters of this book.

Example 1.2

Assigning and calling values of parameters. In addition to inputting data directly to the screen, you can assign a symbolic constant or constants to represent data and perform manipulations on them. For example, enter and note the answer to each of the following:

```
a=2
b=3
c=a+b
d=a*b
e=a/b
f=a^3/b^2
g=a+3*b^2
```

Question: From the above, can you deduce the order in which MATLAB performs the basic operations?

In-Class Exercise

Pb. 1.1 Using the above values of a and b, find the values of:

a. $h = \sin(a)\ \sin(b)$

b. $i = a^{1/3}b^{3/7}$

c. $j = \sin^{-1}(a/b) = \arcsin(a/b)$

1.3 Plotting Points

In this chapter section, you will learn how to use some simple MATLAB graphics commands to plot points. We use these graphics commands later in the text for plotting functions and for visualizing their properties. To view all the functions connected with 2-dimensional graphics, type:

```
help plot
```

All graphics functions connected with 3-dimensional graphics can be looked up by typing

```
help plot3
```

A point P in the x-y plane is specified by two coordinates. The x-coordinate measures the horizontal distance of the point from the y-axis, while the y-coordinate measures the vertical distance above the x-axis. These coordinates are called Cartesian coordinates, and any point in the plane can be described in this manner. We write for the point, P(x, y).

Other representations can also be used to locate a point with respect to a particular set of axes. For example, in the polar representation, the point is specified by an r-coordinate that measures the distance of the point from the origin, while the θ-coordinate measures the angle which the line passing through the origin and this point makes with the x-axis.

The purpose of the following two examples is to learn how to represent points in a plane and to plot them using MATLAB.

Example 1.3

Plot the point P(3, 4).

Solution: Enter the following:

```
x1=3;
y1=4;
plot(x1,y1,'*')
```

Note that the semicolon is used in the above commands to suppress the echoing of the values of the inputs. The **`'*'`** is used to mark the point that we are plotting. Other authorized symbols for point displays include **`'o'`**, **`'+'`**, **`'x'`**, … the use of which is detailed in **`help plot`**.

Example 1.4

Plot the second point, R(2.5, 4) on the graph while keeping point P of the previous example on the graph.

Solution: If we went ahead, defined the coordinates of R, and attempted to plot the point R through the following commands:

```
x2=2.5;
y2=4;
plot(x2,y2,'o')
```

we would find that the last plot command erases the previous plot output.

Thus, what should we do if we want both points plotted on the same graph? The answer is to use the **`hold on`** command after the first plot.

The following illustrates the steps that you should have taken instead of the above:

```
hold on
x2=2.5;
```

```
y2=4;
plot(x2,y2,'o')
hold off
```

The **hold off** turns off the **hold on** feature.

NOTES

1. There is no limit to the number of plot commands you can type before the hold is turned off.
2. An alternative method for viewing multiple points on the same graph is available: we may instead, following the entering of the values of **x1, y1, x2, y2**, enter:

```
plot(x1,y1,'*',x2,y2,'o')
```

This has the advantage, in MATLAB, of assigning automatically a different color to each point.

1.3.1 Axes Commands

You may have noticed that MATLAB automatically adjusts the scale on a graph to accommodate the coordinates of the points being plotted. The axis scaling can be manually enforced by using the command **axis([xmin xmax ymin ymax])**. Make sure that the minimum axis value is less than the maximum axis value or an error will result.

In addition to being able to adjust the scale of a graph, you can also change the aspect ratio of the graphics window. This is useful when you wish to see the correct x to y scaling. For example, without this command, a circle will look more like an ellipse.

Example 1.5

Plot the vertices of a square, keeping the geometric proportions unaltered.

Solution: Enter the following:

```
x1=-1;y1=-1;x2=1;y2=-1;x3=-1;y3=1;x4=1;y4=1;
plot(x1,y1,'o',x2,y2,'o',x3,y3,'o',x4,y4,'o')
axis([-2 2 -2 2])
axis square                %square shape
```

Note that prior to the **axis square** command, the square looked like a rectangle. If you want to go back to the default aspect ratio, type **axis normal**. The **%** symbol is used so that you can type comments in your program. Comments following the **%** symbol are ignored by the MATLAB interpreter.

1.3.2 Labeling a Graph

To add labels to your graph, the functions **`xlabel`**, **`ylabel`**, and **`title`** can be used as follows:

```
xlabel('x-axis')
ylabel('y-axis')
title('points in a plane')
```

If you desire to add a caption anywhere in the graph, you can use the MATLAB command **`gtext('caption')`** and place it at the location of your choice, on the graph, by clicking the mouse when the crosshair is properly centered there.

1.3.3 Plotting a Point in 3-D

In addition to being able to plot points on a plane (2-D space), MATLAB is also able to plot points in a three-dimensional space (3-D space). For this, we utilize the **`plot3`** function.

Example 1.6

Plot the point P(3, 4, 5).

Solution: Enter the following commands:

```
x1=3; y1=4; z1=5;
plot3(x1,y1,z1,'*')
```

You can also plot multiple points in a 3-D space in exactly the same way as you did on a plane. Axis adjustment can still be used, but the vector input into the **`axis`** command must now have six entries, as follows:

```
axis([xmin xmax ymin ymax zmin zmax])
```

You can similarly label your 3-D figure using **`xlabel`**, **`ylabel`**, **`zlabel`**, `and` **`title`**.

1.4 M-files

In the last section, we found that to complete a figure with a caption, we had to enter several commands one by one in the command window. Typing errors will be time-consuming to fix because if you are working in the command window, you need to retype all or part of the program. Even if you do not make any mistakes (!), all of your work may be lost if you inadvertently quit MATLAB and have not taken the necessary steps to save the contents of the important program that you just finished developing. To preserve large sets of commands, you can store them in a special type of file called an *M-file*.

MATLAB supports two types of *M-file*s: *script* and *function M-file*s. To hold a large collection of commands, we use a *script M-file*. The *function M-file* is discussed in Chapter 3. To make a *script M-file*, you need to open a file using the built-in MATLAB editor. For both Macs and PCs, first select New from the file menu. Then select the *M-file* entry from the pull-down menu. After typing the *M-file* contents, you need to save the file:

For Macs and PCs, select the **`save as`** command from the file window. A field will pop up in which you can type in the name you have chosen for this file (make sure that you do not name a file by a mathematical abbreviation, the name of a mathematical function, or a number). Also make sure that the file name has a **`.m`** extension added at the end of its name.

For Macs, save the file in a user's designated volume.

For PCs, save the file in the default (bin) subdirectory.

To run your *script M-file*, just type the filename (omitting the **`.m`** extension at its end) at the MATLAB prompt.

Example 1.7

For practice, go to your file edit window to create the following file that you name **`myfile.m`**.

```
clear, clf
x1=1;y1=.5;x2=2;y2=1.5;x3=3;y3=2;
plot(x1,y1,'o',x2,y2,'+',x3,y3,'*')
axis([0 4 0 4])
xlabel('xaxis')
ylabel('yaxis')
title('3points in a plane')
```

After creating and saving **`myfile.m`**, go to the MATLAB command window and enter **`myfile`**. MATLAB will execute the instructions in the order of the statements stored in your **`myfile.m`** file.

1.5 MATLAB Simple Programming

1.5.1 Iterative Loops

The power of computers lies in their ability to perform a large number of repetitive calculations. To do this without entering the value of a parameter or variable each time that these are changed, all computer languages have control structures that allow commands to be performed and controlled by counter variables, and MATLAB is no different. For example, the MATLAB "**`for`**" loop allows a statement or a group of statements to be repeated.

Example 1.8

Generate the square of the first ten integers.

Solution: Edit and execute the the following *script M-file:*

```
for m=1:10
x(m)=m^2;
end;
```

In this case, the number of repetitions is controlled by the index variable **`m`**, which takes on the values **`m`** = 1 through **`m`** = 10 in intervals of 1. Therefore, ten assignments were made. What the above loop is doing is sequentially assigning the different values of **`m^2`** (i.e., m^2) in each element of the "**`x`**-array." An array is just a data structure that can hold multiple entries. An array can be 1-D such as in a vector, or 2-D such as in a matrix. More will be said about vectors and matrices in subsequent chapters. At this time, think of the 1-D and 2-D arrays as pigeonholes with numbers or ordered pair of numbers respectively assigned to them.

To find the value of a particular slot of the array, such as slot 3, enter:

```
x(3)
```

To read all the values stored in the array, type:

```
x
```

Question: What do you get if you enter **`m`**?

1.5.2 If-Else-End Structures

If a sequence of commands must be conditionally evaluated based on a relational test, the programming of this logical relationship is executed with some variation of an **`if-else-end`** structure.

A. The simplest form of this structure is:

```
if expression
   commands evaluated if expression is True
else
   commands evaluated if expression is False
end
```

NOTES

1. The commands between the **if** and **else** statements are evaluated if all elements in the expression are true.
2. The conditional expression uses the Boolean logical symbols **&** (and), **|** (or), and **~** (not) to connect different propositions.

Example 1.9

Find for integer $0 < a \leq 10$, the values of C, defined as follows:

$$C = \begin{cases} ab & \text{for } a > 5 \\ \frac{3}{2}ab & \text{for } a \leq 5 \end{cases}$$

and $b = 15$.

Solution: Edit and execute the following *script M-file:*

```
for a=1:10
b=15;
  if a>5
    C(a)=a*b;
  else
    C(a)=(a*b)*(3/2);
  end
end
```

Check that the values of C that you obtain by typing **C** are:

```
22.5 45 67.5 90 112.50 90 105 120 135 150
```

B. When there are three or more alternatives, the **if-else-end** structure takes the form:

```
if expression 1
   Commands 1 evaluated if expression 1 is True
```

```
elseif expression 2
   Commands 2 evaluated if expression 2 is True
elseif  expression 3
   Commands 3 evaluated if expression 3 is True
…
else
   Commands evaluated if no other expression is True
end
```

In this form, only the commands associated with the first True expression encountered are evaluated; ensuing relational expressions are not tested.

1.5.2.1 Alternative Syntax to the `if` *Statement*

As an alternative to the **if** syntax, we can use, in certain instances, Boolean expressions to specify an expression in different domains. For example, **`(x>=l)`** has the value 1 if **`x`** is larger than or equal to 1 and zero otherwise; and **`(x<=h)`** is equal to 1 when **`x`** is smaller than or equal to **`h`**, and zero otherwise.

The relational operations allowed inside the parentheses are: **`==`**, **`<=`**, **`>=`**, **`~=`**, **`<`**, **`>`**.

Homework Problem

Pb. 1.2 For the values of integer a going from 1 to 10, using separately the methods of the **`if`** syntax and the Boolean alternative expressions, find the values of C if:

$$\begin{array}{ll} C = a^2 & \text{for } \ a < 3 \\ C = a + 5 & \text{for } \ 3 \le a < 7 \\ C = a & \text{for } \ a \ge 7 \end{array}$$

Use the **`stem`** command to graphically show C.

1.6 Array Operations

In the above examples, we used **`for`** loops repeatedly. However, this kind of loop-programming is very inefficient and must be avoided as much as possible in MATLAB. In fact, ideally, a good MATLAB program will always minimize the use of loops because MATLAB is an interpreted language — not a compiled one. As a result, any looping process is very inefficient. Nevertheless, at times we use the **for** loops, when necessitated by pedagogical reasons.

To understand array operations more clearly, consider the following:

```
a=1:3 % a starts at 1, goes to 3 in increments of 1.
```

If the increment is not 1, you must specify the increment; for example:

```
b=2:2:6 % b starts at 2, goes to 6 in increments of 2
```

To distinguish arrays operations from either operations on scalars or on matrices, the symbol for multiplication becomes **.***, that of division **./**, and that of exponentiation **.^**. Thus, for example:

```
c=a.*b % takes every element of a and multiplies
  % it by the element of b in the same array location
```

Similarly, for exponentiation and division:

```
d=a.^b
e=a./b
```

If you try to use the regular scalar operations symbols, you will get an error message.

Note that array operations such as the above require that the two arrays have the same length (i.e., the same number of elements). To verify that two arrays have the same number of elements (dimension), use the **length** command. Thus, to find the length of **a** and **b**, enter:

```
length(a)
length(b)
```

NOTE The expression **x=linspace(0,10,200)** is also the generator for an x-array with first element equal to 0, a last element equal to 10, and having 200 equally spaced points between 0 and 100. Here, the number of points rather than the increment is specified; that is, **length(x)=200**.

1.7 Curve and Surface Plotting

Review the sections of the Supplement pertaining to lines, quadratic functions, and trigonometric functions before proceeding further.

1.7.1 *x-y* Parametric Plot

Now edit another *M-file* called **`myline.m`** as follows and execute it.

```
N=10;
for m=1:N
  x(m)=m;
  y(m)=2*m+3;
end
plot(x,y)
```

After executing the *M-file* using **`myline`**, you should see a straight line connecting the points (1, 5) and (10, 23). This demonstration shows the basic construct for creating two arrays and plotting the points with their *x*-coordinate from a particular location in one array and their *y*-coordinate from the same location in the second array. We say that the **`plot`** command here plotted the *y*-array vs. the *x*-array.

We note that the points are connected by a continuous line making a smooth curve; we say that the program graphically interpolated the discrete points into a continuous curve. If we desire to see additionally the individual points corresponding to the values of the arrays, the last command should be changed to:

```
plot(x,y,x,y,'o')
```

Example 1.10

Plot the two curves $y_1 = 2x + 3$ and $y_2 = 4x + 3$ on the same graph.

Solution: Edit and execute the following *script M-file:*

```
for m=1:10                           m=1:10;
  x(m)=m;                            x=m;
  y1(m)=2*m+3;        or better      y1=2*m+3;
  y2(m)=4*m+3;                       y2=4*m+3;
end                                  plot(x,y1,x,y2)
plot(x,y1,x,y2)
```

Finally, note that you can separate graphs in one figure window. This is done using the **`subplot`** function in MATLAB. The arguments of the subplot function are **`subplot(m,n,p)`**, where m is the number of rows partitioning the graph, n is the number of columns, and p is the particular subgraph chosen (enumerated through the left to right, top to bottom convention).

1.7.1.1 Demonstration: Plotting Multiple Figures within a Figure Window

Using the data obtained in the previous example, observe the difference in the partition of the page in the following two sets of commands:

```
subplot(2,1,1)
plot(x,y1)
subplot(2,1,2)
plot(x,y2)
```

and

```
clf
subplot(1,2,1)
plot(x,y1)
subplot(1,2,2)
plot(x,y2)
```

1.7.2 More on Parametric Plots in 2-D

In the preceding subsection, we generated the x- and y-arrays by first writing the x-variable as a linear function of a parameter, and then expressed the dependent variable y as a function of that same parameter. What we did is that, instead of thinking of a function as a relation between an independent variable x and a dependent variable y, we thought of both x and y as being dependent functions of a third independent parameter. This method of curve representation, known as the parametric representation, is described by ($x(t)$, $y(t)$), where the parameter t varies over some finite domain (t_{min}, t_{max}). Note, however, that in the general case, unlike the examples in the previous chapter subsection, the independent variable x need not be linear in the parameter, nor is the process of parametrization unique.

Example 1.11

Plot the trigonometric circle.

Solution: Recalling that the x-coordinate of any point on the trigonometric circle has the cosine as x-component and the sine as y-component, the generation of the trigonometric circle is immediate:

```
th=linspace(0,2*pi,101)
x=cos(th);
y=sin(th);
```

```
plot(x,y)
axis square
```

The parametric representation of many common curves is our next topic of interest. The parametric representation is defined such that if x and y are continuous functions of t over the interval I, we can describe a curve in the x-y plane by specifying:

$$C: x = x(t),\ y = y(t),\ \text{and } t \in I$$

More Examples:

In the following examples, we want to identify the curves $f(x, y) = 0$ corresponding to each of the given parametrizations.

Example 1.12

C: $x = 2t - 1$, $y = t + 1$, and $0 < t < 2$. The initial point is at $x = -1$, $y = 1$, and the final point is at $x = 3$, $y = 3$.

Solution: The curve $f(x, y) = 0$ form can be obtained by noting that:

$$2t - 1 = x \Rightarrow t = (x + 1)/2$$

Substitution into the expression for y results in:

$$y = \frac{x}{2} + \frac{3}{2}$$

This describes a line with slope 1/2 crossing the x-axis at x = –3.

Question: Where does this line cross the y-axis?

Example 1.13

C: $x = 3 + 3\cos(t)$, $y = 2 + 2\sin(t)$, and $0 < t < 2\pi$. The initial point is at $x = 6$, $y = 2$, and the final point is at $x = 6$, $y = 2$.

Solution: The curve $f(x, y) = 0$ can be obtained by noting that:

$$\sin(t) = \frac{y-2}{2} \quad \text{and} \quad \cos(t) = \frac{x-3}{3}$$

Using the trigonometric identity $\cos^2(t) + \sin^2(t) = 1$, we deduce the following equation:

$$\frac{(y-2)^2}{2^2}+\frac{(x-3)^2}{3^2}=1$$

This is the equation of an ellipse centered at $x = 3$, $y = 2$ and having major and minor radii equal to 3 and 2, respectively.

Question 1: What are the coordinates of the foci of this ellipse?

Question 2: Compare the above curve with the curve defined through:

$$x = 3 + 3\cos(2t),\ y = 2 + 2\sin(2t),\text{ and } 0 < t < 2\pi$$

What conclusions can you draw from your answer?

In-Class Exercises

Pb. 1.3 Show that the following parametric equations:

$$x = h + a\sec(t),\ y = k + b\tan(t),\text{ and } -\pi/2 < t < \pi/2$$

are those of the hyperbola also represented by the equation:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Pb. 1.4 Plot the hyperbola represented by the parametric equations of **Pb. 1.3**, with $h = 2$, $k = 2$, $a = 1$, $b = 2$. Find the coordinates of the vertices and the foci. (*Hint:* One branch of the hyperbola is traced for $-\pi/2 < t < \pi/2$, while the other branch is traced when $\pi/2 < t < 3\pi/2$.)

Pb. 1.5 The parametric equations of the cycloid are given by:

$$x = R\omega t + R\sin(\omega t),\ y = R + R\cos(\omega t),\text{ and } 0 < t$$

Show how this parametric equation can be obtained by following the kinematics of a point attached to the outer rim of a wheel that is uniformly rolling, without slippage, on a flat surface. Relate the above parameters to the linear speed and the radius of the wheel.

Pb. 1.6 Sketch the curve C defined through the following parametric equations:

$$x(t)=\begin{cases} t+2 & \text{for } -3\le t\le -1 \\ +1-\dfrac{1}{\sqrt{3}}\tan\left(\dfrac{\pi}{3}(1-t^2)\right) & \text{for } -1<t<0 \\ -1+\dfrac{1}{\sqrt{3}}\tan\left(\dfrac{\pi}{3}(1-t^2)\right) & \text{for } 0<t<1 \end{cases}$$

$$y(t)=\begin{cases} 0 & \text{for } -3\le t\le -1 \\ \dfrac{1}{\sqrt{3}}\tan\left(\dfrac{\pi}{3}(1-t^2)\right) & \text{for } -1<t<0 \\ \dfrac{1}{\sqrt{3}}\tan\left(\dfrac{\pi}{3}(1-t^2)\right) & \text{for } 0<t<1 \end{cases}$$

Homework Problems

The following set of problems provides the mathematical basis for understanding the graphical display on the screen of an oscilloscope, when in the *x-y* mode.

Pb. 1.7 To put the quadratic expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

in standard form (i.e., to eliminate the *x-y* mixed term), make the transformation

$$x = x'\cos(\theta) - y'\sin(\theta)$$

$$y = x'\sin(\theta) + y'\cos(\theta)$$

Show that the mixed term is eliminated if $\cot(2\theta) \equiv \dfrac{(A-C)}{B}$.

Pb. 1.8 Consider the parametric equations

$$C\colon x = a\cos(t),\ y = b\sin(t+\varphi),\ \text{and } 0 < t < 2\pi$$

where the initial point is at $x = a$, $y = b\sin(\varphi)$, and the final point is at $x = a$, $y = b\sin(\varphi)$.

a. Obtain the equation of the curve in the form $f(x, y) = 0$.

b. Using the results of **Pb. 1.7**, prove that the ellipse inclination angle is given by:

$$\cot(2\theta) \equiv \frac{(a^2-b^2)}{2ab\sin(\varphi)}$$

Pb. 1.9 If the parametric equations of a curve are given by:

$$C\colon x = \cos(t),\ y = \sin(2t),\ \text{and } 0 < t < 2\pi$$

where the initial point is at $x = 1$, $y = 0$, and the final point is at $x = 1$, $y = 0$.

The curve so obtained is called a Lissajous figure. It has the shape of a figure 8 with two nodes in the x-direction and only one node in the y-direction.

What do you think the parametric equations should be if we wanted m nodes on the x-axis and n nodes on the y-axis? Test your hypothesis by plotting the results.

1.7.3 Plotting a 3-D Curve

Our next area of exploration is plotting 3-D curves.

Example 1.14

Plot the helix.

Solution: To plot a helical curve, we can imagine initially that a point is revolving at a uniform speed around the perimeter of a circle. Now imagine that as the circular motion is continuing, the point is moving away from the x-y plane at some constant linear speed. The parametric representation of this motion can be implemented in MATLAB through the following:

```
for m=1:201
  th(m)=2*pi*.01*(m-1);
  x(m)=cos(th(m));
  y(m)=sin(th(m));
  z(m)=th(m);
end
plot3(x,y,z)
```

In-Class Exercises

Pb. 1.10 In the helix of Example 1.14, what is the vertical distance (the pitch) between two consecutive helical turns. How can you control this distance? Find two methods of implementation.

Pb. 1.11 If instead of a circle in 2-D, as in the helix, the particle describes in 2-D a Lissajous pattern having two nodes in the y-direction and three nodes in the x-direction, assuming that the z-parametric equation remains the same, show the resulting 3-D trajectory.

Pb. 1.12 What if $z(t)$ is periodic in t? For example, $z(t) = \cos(t)$ or $z(t) = \cos(2t)$, while the 2-D motion is still circular. Show the 3-D trajectory.

In Example 1.14, we used the **for** loop to generate the dependent arrays for the helix; but as pointed out previously, a more efficient method to program the helix is in the array notation, as follows:

```
th=[0:.01:2]*2*pi;
x=cos(th);
y=sin(th);
z=th;
plot3(x,y,z)
```

1.7.4 Plotting a 3-D Surface

We now explore the two different techniques for rendering, in MATLAB, 3-D surface graphics: the mesh and the contour representations.

- A function of two variables $z = f(x, y)$ represents a surface in 3-D geometry; for example:

$$z = ax + by + c$$

  represents a plane that crosses the vertical axis (z-axis) at c.
- There are essentially two main techniques in MATLAB for viewing surfaces: the **mesh** function and the **contour** function.
- In both techniques, we must first create a 2-D array structure (like a checkerboard) with the appropriate x- and y-values. To implement this, we use the MATLAB **meshgrid** function.
- The z-component is then expressed in the variables assigned to implement the **meshgrid** command.
- We then plot the function with either the **mesh** command or the **contour** command. The **mesh** command gives a 3-D rendering of the surface, while the **contour** command gives contour lines, wherein each contour represents the locus of points on the surface having the same height above the x-y plane. This last rendering technique is that used by mapmakers to represent the topography of a terrain.

1.7.4.1 Surface Rendering

Example 1.15

Plot the sinc function whose equation is given by:

$$z = \frac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}$$

over the domain $-8 < x < 8$ and $-8 < y < 8$.

Solution: The implementation of the mesh rendering follows:

```
x=[-8:.1:8];
y=[-8:.1:8];
[X,Y]=meshgrid(x,y);
R=sqrt(X.^2+Y.^2)+eps;
Z=sin(R)./R;
mesh(X,Y,Z)
```

The variable **eps** is a tolerance number = 2^{-52} used for determining expressions near apparent singularities, to avoid numerical division by zero.

To generate a contour plot, we replace the last command in the above by:

```
contour(X,Y,Z,50)  % The fourth argument specifies
  % the number of contour lines to be shown
```

If we are interested only in a particular contour level, for example, the one with elevation Z_0, we use the contour function with an option, as follows:

```
contour(X,Y,Z,[Zo Zo])
```

Occasionally, we might be interested in displaying simultaneously the mesh and contour rendering of a surface. This is possible through the use of the command **meshc.** It is the same as the **mesh** command except that a contour plot is drawn beneath the mesh.

Preparatory Activity: Look in your calculus book for some surfaces equations, such as those of the hyperbolic paraboloid and the elliptic paraboloid and others of your choice for the purpose of completing **Pb. 1.16** of the next in-class activity.

In-Class Exercises

Pb. 1.13 Use the **`contour`** function to graphically find the locus of points on the above sinc surface that are 1/2 units above the *x-y* plane (i.e., the surface intersection with the $z = 1/2$ plane).

Pb. 1.14 Find the *x-y* plane intersection with the following two surfaces:

$$z_1 = 3 + x + y$$

$$z_2 = 4 - 2x - 4y$$

Pb. 1.15 Verify your answers to **Pb. 1.14** with that which you would obtain analytically for the shape of the intersection curves of the surfaces with the *x-y* plane. Also, compute the coordinates of the point of intersection of the two obtained curves. Verify your results graphically.

Pb. 1.16 Plot the surfaces that you have selected in your preparatory activity. Look in the help folder for the **`view`** command to learn how to view these surfaces from different angles.

1.8 Polar Plots

MATLAB can also display polar plots. In the first example, we draw an ellipse of the form $r = 1 + \varepsilon \cos(\theta)$ in a polar plot; other shapes are given in the other examples.

Example 1.16

Plot the ellipse in a polar plot.

Solution: The following sequence of commands plot the polar plot of an ellipse with $\varepsilon = 0.2$:

```
th=0:2*pi/100:2*pi;
rho=1+.2*cos(th);
polar(th,rho)
```

The shape you obtain may be unfamiliar; but to verify that this is indeed an ellipse, view the curve in a Cartesian graph. For that, you can use the MATLAB polar to Cartesian converter **`pol2cart`**, as follows:

```
[x,y]=pol2cart(th,rho);
plot(x,y)
axis equal
```

Example 1.17

Graph the polar plot of a spiral.

Solution: The equation of the spiral is given by:

$$r = a\theta$$

Its polar plot can be viewed by executing the following *script M-file* ($a = 3$):

```
th=0:2*pi/100:2*pi;
rho=3*th;
polar(th,rho)
```

In-Class Exercises

Pb. 1.17 Prove that the polar equation $r = 1 + \varepsilon\cos(\theta)$, where ε is always between –1 and 1, results in an ellipse. (*Hint:* Relate ε to the ratio between the semi-major and semi-minor axis.) It is worth noting that the planetary orbits are usually described in this manner in most astronomy books.

Pb. 1.18 Plot the three curves described by the following polar equations:

$$r = 2 - 2\sin(\theta),\;\; r = 1 - \sqrt{2}\,\sin(\theta),\;\; r = \sqrt{2\sin(2\theta)}$$

Pb. 1.19 Plot:

$$r = \sin(2\theta)\,\cos(2\theta)$$

The above gives a flower-type curve with eight petals. How would you make a flower with 16 petals?

Pb. 1.20 Plot:

$$r = \sin^2(\theta)$$

This two-lobed structure shows the power distribution of a simple dipole antenna. Note the directed nature of the radiation. Can you increase the directivity further?

Pb. 1.21 Acquaint yourself with the polar plots of the following curves: (choose first $a = 1$, then experiment with other values).

a. Straight lines: $r = \dfrac{1}{\cos(\theta) + a\sin(\theta)}$ for $0 \le \theta \le \dfrac{\pi}{2}$

b. Cissoid of Diocles: $r = a\dfrac{\sin^2(\theta)}{\cos(\theta)}$ for $-\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{3}$

c. Strophoid: $r = \dfrac{a\cos(2\theta)}{\cos(\theta)}$ for $-\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{3}$

d. Folium of Descartes: $r = \dfrac{3a\sin(\theta)\cos(\theta)}{\sin^3(\theta) + \cos^3(\theta)}$ for $-\dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2}$

1.9 Animation

A very powerful feature of MATLAB is its ability to render an animation. For example, suppose that we want to visualize the oscillations of an ordinary spring. What are the necessary steps to implement this objective?

1. Determine the parametric equations that describe the curve at a fixed time. In this instance, it is the helix parametric equations as given earlier in Example 1.14.
2. Introduce the time dependence in the appropriate curve parameters. In this instance, make the helix pitch to be oscillatory in time.
3. Generate 3-D plots of the curve at different times. Make sure that your axis definition includes all cases.
4. Use the **`movie`** commands to display consecutively the different frames obtained in step 3.

The following *script M-file* implements the above workplan:

```
th=0:pi/60:32*pi;
a=1;
A=0.25;
w=2*pi/15;
M=moviein(16);
  for t=1:16;
  x=a*cos(th);
```

```
    y=a*sin(th);
    z=(1+A*cos(w*(t-1)))*th;
    plot3(x,y,z,'r');
    axis([-2 2 -2 2 0 40*pi]);
    M(:,t)=getframe;
    end
  movie(M,15)
```

The statement **`M=moviein(16)`** creates the 2-D structure that stores in each column the data corresponding to a frame at a specific time. The frames themselves are generated within the **`for`** loop. The **`getframe`** function returns a pixel image of the image of the different frames. The last command plays the movie *n*-times (15, in this instance).

1.10 Histograms

The most convenient representation for data collected from experiments is in the form of histograms. Typically, you collect data and want to sort it out in different bins; the MATLAB command for this operation is **`hist`**. But prior to getting to this point, let us introduce some array-related definitions and learn the use of the MATLAB commands that compute them.

Let $\{y_n\}$ be a data set; it can be represented in MATLAB by an array. The largest element of this array is obtained through the command **`max(y)`**, and the smallest element is obtained through the command **`min(y)`**.

The mean value of the elements of the array is obtained through the command **`mean(y)`**, and the standard deviation is obtained through the command **`std(y)`**.

The definitions of the mean and of the standard deviation are, respectively, given by:

$$\bar{y} = \frac{\sum_{i=1}^{N} y(i)}{N}$$

$$\sigma_y = \sqrt{\frac{N\sum_{i=1}^{N}(y(i))^2 - \left(\sum_{i=1}^{N} y(i)\right)^2}{N(N-1)}}$$

where N is the dimension of the array.

The data (i.e., the array) can be organized into a number of bins (n_b) and exhibited through the command **[n,y]=hist(y,nb)**; the array n in the output will be the number of elements in each of the bins.

Example 1.18

Find the mean and the standard deviation and draw the histogram, with 20 bins, for an array whose 10,000 elements are chosen from the MATLAB built-in normal distribution with zero mean and standard deviation 1.

Solution: Edit and execute the following *script M-file:*

```
y=randn(1,10000);
meany=mean(y)
stdy=std(y)
nb=20;
hist(y,nb)
```

You will notice that the results obtained for the mean and the standard deviation vary slightly from the theoretical results. This is due to the finite number of elements chosen for the array and the intrinsic limit in the built-in algorithm used for generating random numbers.

NOTE The MATLAB command for generating an N-elements array of random numbers generated uniformly from the interval [0, 1] is **rand(1,N)**.

1.11 Printing and Saving Work in MATLAB

Printing a figure: Use the MATLAB **print** function to print a displayed figure directly to your printer. Notice that the printed figure does not take up the entire page. This is because the default orientation of the graph is in portrait mode. To change these settings, try the following commands on an already generated graphic window:

```
orient('landscape') %full horizontal layout
orient('tall')      %full vertical layout
```

Printing a program file (*script M-file*): For both the Mac and PC, open the *M-file* that you want to print. Go to the **File** pull-down menu, and select **Print**.

Saving and loading variables (data): You can use the MATLAB **save** function to either save a particular variable or the entire MATLAB workspace. To do this, follow the following example:

```
x=1;y=2;
save 'user volume:x'
save 'user volume:workspace'
```

The first **save** command saved the variable x into a file **x.mat**. You can change the name of the **.mat** file so it does not match the variable name, but that would be confusing. The second command saves all variables (*x* and *y*) in the workspace into **workspace.mat**.

To load **x.mat** and **workspace.mat**, enter MATLAB and use the MATLAB load functions; note what you obtain if you entered the following commands:

```
load 'user volume:x'
x
load 'user volume:workspace'
y
```

After loading the variables, you can see a list of all the variables in your workplace if you enter the MATLAB **who** command.

What would you obtain if you had typed and entered the **who** command at this point?

Now, to clear the workspace of some or all variables, use the MATLAB **clear** function.

```
clear x %clears variable x from the workspace
clear   %clears all variables from workspace
```

1.12 MATLAB Commands Review

axis Sets the axis limits for both 2-D and 3-D plots. Axis supports the arguments equal and square, which makes the current graphs aspect ratio 1.

contour Plots contour lines of a surface.

clear Clears all variables from the workspace.

clf Clears figure.

for Runs a sequence of commands a given number of times.

getframe Returns the pixel image of a movie frame.

help Online help.

hold on(off) Holds the plot axis with existing graphics on, so that multiple figures can be plotted on the same graph (release the hold of the axes).

if Conditional evaluation.

length Gives the length of an array.

load Loads data or variable values from previous sessions into current MATLAB session.

linspace Generates an array with a specified number of points between two values.

meshgrid Makes a 2-D array of coordinate squares suitable for plotting surface meshes.

mesh Plots a mesh surface of a surface stored in a matrix.

meshc The same as mesh, but also plots in the same figure the contour plot.

min Finds the smallest element of an array.

max Finds the largest element of an array.

mean Finds the mean of the elements of an array.

moviein Creates the matrix that contains the frames of an animation.

movie Plays the movie described by a matrix M.

orient Orients the current graph to your needs.

plot Plots points or pairs of arrays on a 2-D graph.

plot3 Plots points or array triples on a 3-D graph.

polar Plots a polar plot on a polar grid.

pol2cart Polar to Cartesian conversion.

print Prints a figure to the default printer.

quit or **exit** Leave MATLAB program.

rand Generates an array with elements randomly chosen from the uniform distribution over the interval $[0, 1]$.

randn Generates an array with elements randomly chosen from the normal distribution function with zero mean and standard deviation 1.

subplot Partitions the graphics window into sub-windows.

save Saves MATLAB variables.

std Finds the standard deviation of the elements of an array.

stem Plots the data sequence as stems from the x-axis terminated with circles for the data value.

view Views 3-D graphics from different perspectives.

who Lists all variables in the workspace.

xlabel, ylabel, zlabel, title Labels the appropriate axes with text and title.

(x>=x1) Boolean function that is equal to 1 when the condition inside the parenthesis is satisfied, and zero otherwise.

2

Difference Equations

This chapter introduces difference equations and examines some simple but important cases of their applications. We develop simple algorithms for their numerical solutions and apply these techniques to the solution of some problems of interest to the engineering professional. In particular, it illustrates each type of difference equation that is of widespread interest.

2.1 Simple Linear Forms

The following components are needed to define and solve a difference equation:

1. An ordered array defining an index for the sequence of elements
2. An equation connecting the value of an element having a certain index with the values of some of the elements having lower indices (the order of the equation being defined by the number of lower indices terms appearing in the difference equation)
3. A sufficient number of the values of the elements at the lowest indices to act as seeds in the recursive generation of the higher indexed elements.

For example, the Fibonacci numbers are defined as follows:

1. The ordered array is the set of positive integers
2. The defining difference equation is of second order and is given by:

$$F(k+2) = F(k+1) + F(k) \tag{2.1}$$

3. The initial conditions are $F(1) = F(2) = 1$ (note that the required number of initial conditions should be the same as the order of the equation).

From the above, it is then straightforward to compute the first few Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Example 2.1

Write a program for finding the first 20 Fibonacci numbers.

Solution: The following program fulfills this task:

```
N=18;
F(1)=1;
F(2)=1;
for k=1:N
  F(k+2)=F(k)+F(k+1);
end
F
```

It should be noted that the value of the different elements of the sequence depends on the values of the initial conditions, as illustrated in **Pb. 2.1**, which follows.

In-Class Exercises

Pb. 2.1 Find the first 20 elements of the sequence that obeys the same recursion relation as that of the Fibonacci numbers, but with the following initial conditions:

$$F(1) = 0.5 \quad \text{and} \quad F(2) = 1$$

Pb. 2.2 Find the first 20 elements of the sequence generated by the following difference equation:

$$F(k + 3) = F(k) + F(k + 1) + F(k + 2)$$

with the following boundary conditions:

$$F(1) = 1, \quad F(2) = 2, \quad \text{and} \quad F(3) = 3$$

Why do we need to specify three initial conditions?

2.2 Amortization

In this application of difference equations, we examine simple problems of finance that are of major importance to every engineer, on both the personal and professional levels. When the purchase of any capital equipment or real estate is made on credit, the assumed debt is normally paid for by means of a process known as amortization. Under this plan, a debt is repaid in a sequence of periodic payments where a portion of each payment reduces the outstanding principal, while the remaining portion is for interest on the loan.

Suppose that the original debt to be paid is C and that interest charges are compounded at the rate r per payment period. Let $y(k)$ be the outstanding principal after the k^{th} payment, and $u(k)$ the amount of the k^{th} payment.

After the k^{th} payment period, the outstanding debt increased by the interest due on the previous principal $y(k-1)$, and decreased by the amount of payment $u(k)$, this relation can be written in the following difference equation form:

$$y(k) = (1 + r)\, y(k - 1) - u(k) \tag{2.2}$$

We can simplify the problem and assume here that the bank wants its money back in equal amounts over N periods (this can be in days, weeks, months, or years; note, however, that whatever unit is used here should be the same as used for the assignment of the value of the interest rate r). Therefore, let

$$u(k) = p \quad \text{for} \quad k = 1, 2, 3, \ldots, N \tag{2.3}$$

Now, using Eq. (2.2), let us iterate the first few terms of the difference equation:

$$y(1) = (1 + r)y(0) - p = (1 + r)C - p \tag{2.4}$$

Since C is the original capital borrowed;

At $k = 2$, using Eq. (2.2) and Eq. (2.4), we obtain:

$$y(2) = (1 + r)y(1) - p = (1 + r)^2C - p(1 + r) - p \tag{2.5}$$

At $k = 3$, using Eq. (2.2), (2.4), and (2.5), we obtain:

$$y(3) = (1 + r)y(2) - p = (1 + r)^3C - p(1 + r)^2 - p(1 + r) - p \tag{2.6}$$

etc. …

and for an arbitrary k, we can write, by induction, the general expression:

$$y(k) = (1+r)^k C - p \sum_{i=0}^{k-1} (1+r)^i \tag{2.7}$$

Using the expression for the sum of a geometric series, from the appendix, the expression for $y(k)$ then reduces to:

$$y(k) = (1+r)^k C - p\left[\frac{(1+r)^k - 1}{r}\right] \tag{2.8}$$

At $k = N$, the debt is paid off and the bank is owed no further payment; therefore:

$$y(N) = 0 = (1+r)^N C - p\left[\frac{(1+r)^N - 1}{r}\right] \tag{2.9}$$

From this equation, we can determine the amount of each of the (equal) payments:

$$p = \frac{r(1+r)^N}{(1+r)^N - 1} C \tag{2.10}$$

Question: What percentage of the first payment is going into retiring the principal?

In-Class Exercises

Pb. 2.3 Given the principal, the number of periods and the interest rate, use Eq. (2.10) to write a MATLAB program to find the amount of payment per period, assuming the payment per period is the same for all periods.

Pb. 2.4 Use the same reasoning as for the amortization problem to write the difference equation for an individual's savings plan. Let $y(k)$ be the savings balance on the first day of the k^{th} year and $u(k)$ the amount of deposit made in the k^{th} year.

Write a MATLAB program to compute $y(k)$ if the sequence $u(k)$ and the interest rate r are given. Specialize to the case where you deposit an amount that increases by the rate of inflation i. Compute and plot the total value of the savings as a function of k if the deposit in the first year is $1000, the yearly interest rate is 6%, and the yearly rate of inflation is 3%. (*Hint:* For simplicity, assume that the deposits are made on December 31 of each year, and that the balance statement is issued on January 1 of each year.)

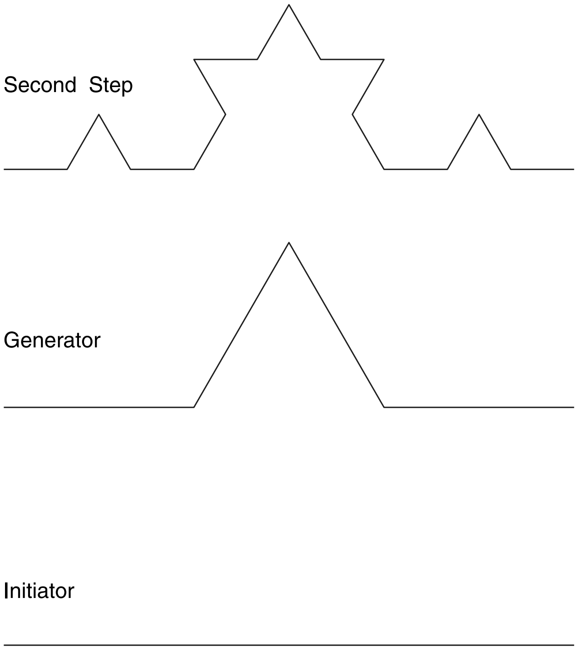

FIGURE 2.1
The first few steps in the construction of the Koch curve.

2.3 An Iterative Geometric Construct: The Koch Curve

In your previous studies of 2-D geometry, you encountered classical geometric objects such as the circle, the triangle, the square, different polygons, etc. These shapes only approximate the shapes that you observe in nature (e.g., the shapes of clouds, mountain ranges, rivers, coastlines, etc.). In a successful effort to address the limitations of classical geometry, mathematicians have developed, over the last century and more intensely over the last three decades, a new geometry called fractal geometry. This geometry defines the geometrical object through an iterative transformation applied an infinite number of times on an initial simple geometrical object. We illustrate this new concept in geometry by considering the Koch curve (see Figure 2.1).

The Koch curve has the following simple geometrical construction. Begin with a straight line of length L. This initial object is called the initiator. Now partition it into three equal parts. Then replace the middle line segment by an equilateral triangle (the segment you removed is its base). This completes the basic construction, which transformed the line segment into four non-colinear smaller parts. This constructional prescription is called the generator. We now repeat the transformation, taking each of the resulting line segments, partitioning them into three equal parts, removing the middle section, etc.

This process is repeated indefinitely. Figure 2.1 the first two steps of this construction. It is interesting to observe that the Koch curve is an example of a curve where there is no way to fit a tangent to any of its points. In a sense, it is an example of a curve that is made out of corners everywhere.

The detailed study of these objects is covered in courses in fractal geometry, chaos, dynamic systems, etc. We limit ourselves here to the simple problems of determining the number of segments, the length of each segment, the length of the curve, and the area bounded by the curve and the horizontal axis, following the k^{th} step:

1. After the first step, we are left with a curve made up of four line segments of equal length; after the second step, we have (4 × 4) segments; and the number of segments after k steps, is

$$n(k) = 4^k \tag{2.11}$$

2. If the initiator had length L, the length of the segment after the first step is $L/3$, $L/(3)^2$, after the second step and after k steps:

$$s(k) = L/(3)^k \tag{2.12}$$

3. Combining the results of Eqs. (2.11) and (2.12), we deduce that the length of the curve after k steps:

$$P(k) = L \times \left(\frac{4}{3}\right)^k \tag{2.13}$$

4. The number of vertices in this curve, denoted by $u(k)$, is equal to the number of segments plus one:

$$u(k) = 4^k + 1 \tag{2.14}$$

5. The area enclosed by the Koch curve and the horizontal line can be deduced from solving a difference equation: the area enclosed after the k^{th} step is equal to the area enclosed in the $(k-1)^{\text{th}}$ step plus the number of the added triangles multiplied by their individual area:

$$\text{Number of new triangles} = \left(\frac{u(k) - u(k-1)}{3}\right) \tag{2.15}$$

$$\text{Area of the new equilateral triangle} = \frac{\sqrt{3}}{4} s^2(k) = \frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^{2k} L^2 \tag{2.16}$$

from which the difference equation for the area can be deduced:

$$\begin{aligned} A(k) &= A(k-1) + \left[\frac{u(k)-u(k-1)}{3}\right]\frac{\sqrt{3}}{4}\frac{L^2}{3^{2k}} \\ &= A(k-1) + \frac{\sqrt{3}}{24}\left(\frac{2}{3}\right)^{2k-1} L^2 \end{aligned} \tag{2.17}$$

The initial condition for this difference equation is:

$$A(1) = \frac{\sqrt{3}}{4}\frac{L^2}{9} \tag{2.18}$$

Clearly, the solution of the above difference equation is the sum of a geometric series, and can therefore be written analytically. For $k \to \infty$, this area has the limit:

$$A(k \to \infty) = \frac{\sqrt{3}}{20} L^2 \tag{2.19}$$

However, if you did not notice the relationship of the above difference equation with the sum of a geometric series, you can still solve this equation numerically, using the following routine and assuming $L = 1$:

```
N=25;
A=zeros(N,1); %preallocating size of array speeds
  % computation
m=1:N;
A(1)=(sqrt(3)/24)*(2/3);
  for k=2:N
  A(k)=A(k-1)+(sqrt(3)/24)*((2/3)^(2*k-1));
  end
stem(m,A,'*')
```

The above plot shows the value of the area on the first 20 iterations of the function, and as can be verified, the numerical limit of this area has the same value as the analytical expression given in Eq. (2.19).

Before leaving the Koch curve, we note that although the area of the curve goes to a finite limit as the index increases, the value of the length of the curve [Eq. (2.13)] continues to increase. This is a feature not encountered in the classical geometric objects with which you are most familiar.

In-Class Exercise

Pb. 2.5 Write a program to draw the Koch curve at the k^{th} step. (*Hint:* Starting with the farthest left vertex and going clockwise, write a difference equation relating the coordinates of a vertex with those of the preceding vertex, the length of the segment, and the angle that the line connecting the two consecutive vertices makes with the x-axis.)

2.4 Solution of Linear Constant Coefficients Difference Equations

In Section 2.1, we explored the general numerical techniques for solving difference equations. In this section, we consider, some special techniques for obtaining the analytical solutions for the class of linear constant coefficients difference equations. The related physical problem is to determine, for a linear system, the output $y(k)$, $k > 0$, given a specific input $u(k)$ and a specific set of initial conditions. We discuss, at this stage, the so-called direct method.

The general expression for this class of difference equation is given by:

$$\sum_{j=0}^{N} a_j y(k-j) = \sum_{m=0}^{M} b_m u(k-m) \tag{2.20}$$

The direct method assumes that the total solution of a linear difference equation is the sum of two parts — the homogeneous solution and the particular solution:

$$y(k) = y_{\text{homog.}}(k) + y_{\text{partic.}}(k) \tag{2.21}$$

The homogeneous solution is independent of the input $u(k)$, and the RHS of the difference equation is equated to zero; that is,

$$\sum_{j=0}^{N} a_j y(k-j) = 0 \tag{2.22}$$

2.4.1 Homogeneous Solution

Assume that the solution is of the form:

$$y_{\text{homog.}}(k) = \lambda^k \tag{2.23}$$

Substituting in the homogeneous equation, we obtain the following algebraic equation:

$$\sum_{j=0}^{N} a_j \lambda^{k-j} = 0 \tag{2.24}$$

or

$$\lambda^{k-N}(a_0\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_{N-1}\lambda + a_N) = 0 \tag{2.25}$$

The polynomial in parentheses is called the characteristic polynomial of the system. The roots can be obtained analytically for all polynomials up to order 4; otherwise, they are obtained numerically. In MATLAB, they can be obtained graphically when they are all real, or through the **`roots`** command in the most general case. We introduce this command in Chapter 5. In all the following examples in this chapter, we restrict ourselves to cases for which the roots can be obtained analytically.

If we assume that the roots are all distinct, the general solution to the homogeneous difference equation is:

$$y_{\text{homog.}}(k) = C_1\lambda_1^k + C_2\lambda_2^k + \ldots + C_N\lambda_N^k \tag{2.26}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$ are the roots of the characteristic polynomial.

Example 2.2

Find the homogeneous solution of the difference equation

$$y(k) - 3y(k-1) - 4y(k-2) = 0$$

Solution: The characteristic polynomial associated with this equation leads to the quadratic equation:

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of this equation are –1 and 4, respectively. Therefore, the solution of the homogeneous equation is:

$$y_{\text{homog.}}(k) = C_1(-1)^k + C_2(4)^k$$

The constants C_1 and C_2 are determined from the initial conditions $y(1)$ and $y(2)$. Substituting, we obtain:

$$C_1 = -\frac{4}{5}y(1) + \frac{y(2)}{5} \quad \text{and} \quad C_2 = \frac{y(1)+y(2)}{20}$$

NOTE If the characteristic polynomial has roots of multiplicity m, then the portion of the homogeneous solution corresponding to that root can be written, instead of $C_1\lambda^k$, as:

$$C_1^{(1)}\lambda^k + C_1^{(2)}k\lambda^k + \ldots + C_1^{(m)}k^{m-1}\lambda^k$$

In-Class Exercises

Pb. 2.6 Find the homogeneous solution of the following second-order difference equation:

$$y(k) = 3y(k-1) - 2y(k-2)$$

with the initial conditions: $y(0) = 1$ and $y(1) = 2$. Then check your results numerically.

Pb. 2.7 Find the homogeneous solution of the following second-order difference equation:

$$y(k) = [2\cos(\theta)]y(k-1) - y(k-2)$$

with the initial conditions: $y(-2) = 0$ and $y(-1) = 1$. Check your results numerically.

2.4.2 *Particular Solution*

The particular solution depends on the form of the input signal. The following table summarizes the form of the particular solution of a linear equation for some simple input functions:

Input Signal	Particular Solution
A (constant)	B (constant)
AM^k	BM^k
Ak^M	$B_0k^M + B_1k^{M-1} + \ldots + B_M$
$\{A\cos(\omega_0 k), A\sin(\omega_0 k)\}$	$B_1\cos(\omega_0 k) + B_2\sin(\omega_0 k)$

For more complicated input signals, the z-transform technique provides the simplest solution method. This technique is discussed in great detail in courses on linear systems.

In-Class Exercise

Pb. 2.8 Find the particular solution of the following second-order difference equation:

$$y(k) - 3y(k-1) + 2y(k-2) = (3)^k \quad \text{for } k > 0$$

2.4.3 General Solution

The general solution of a linear difference equation is the sum of its homogeneous solution and its particular solution, with the constants adjusted, so as to satisfy the initial conditions. We illustrate this general prescription with an example.

Example 2.3

Find the complete solution of the first-order difference equation:

$$y(k+1) + y(k) = k$$

with the initial condition $y(0) = 0$.

Solution: First, solve the homogeneous equation $y(k + 1) + y(k) = 0$. The characteristic polynomial is $\lambda + 1 = 0$; therefore,

$$y_{\text{homog.}} = C(-1)^k$$

The particular solution can be obtained from the above table. Noting that the input signal has the functional form k^M, with $M = 1$, then the particular solution is of the form:

$$y_{\text{partic.}} = B_0 k + B_1 \tag{2.27}$$

Substituting back into the original equation, and grouping the different powers of k, we deduce that:

$$B_0 = 1/2 \quad \text{and} \quad B_1 = -1/4$$

The complete solution of the difference equation is then:

$$y(k) = C(-1)^k + \frac{2k-1}{4}$$

The constant C is determined from the initial condition:

$$y(0) = 0 = C(-1)^0 + \frac{(-1)}{4}$$

giving for the constant C the value 1/4.

In-Class Exercises

Pb. 2.9 Use the following program to model Example 2.3:

```
    N=19;
    y(1)=0;
    for k=1:N
      y(k+1)=k-y(k);
    end
  y
```

Verify the closed-form answer.

Pb. 2.10 Find, for $k \geq 2$, the general solution of the second-order difference equation:

$$y(k) - 3y(k-1) - 4y(k-2) = 4^k + 2 \times 4^{k-1}$$

with the initial conditions $y(0) = 1$ and $y(1) = 9$. (*Hint:* When the functional form of the homogeneous and particular solutions are the same, use the same functional form for the solutions as in the case of multiple roots for the characteristic polynomial.)

Answer: $y(k) = \left[-\frac{1}{25}(-1)^k + \frac{26}{25}(4)^k\right]\left(\frac{6}{5}k4^k\right)$

Homework Problems

Pb. 2.11 Given the general geometric series $y(k)$, where:

$$y(k) = 1 + a + a^2 + \ldots + a^k$$

show that $y(k)$ obeys the first-order equation:

$$y(k) = y(k-1) + a^k$$

Pb. 2.12 Show that the response of the system:

$$y(k) = (1-a)u(k) + a\,y(k-1)$$

to a step signal of amplitude c; that is, $u(k) = c$ for all positive k, is given by:

$$y(k) = c(1 - a^{k+1}) \quad \text{for } k = 0, 1, 2, \ldots$$

where the initial condition $y(-1) = 0$.

Pb. 2.13 Given the first-order difference equation:

$$y(k) = u(k) + y(k-1) \quad \text{for } k = 0, 1, 2, \ldots$$

with the input signal $u(k) = k$, and the initial condition $y(-1) = 0$. Verify that its solution also satisfies the second-order difference equation

$$y(k) = 2y(k-1) - y(k-2) + 1$$

with the initial conditions $y(0) = 0$ and $y(-1) = 0$.

Pb. 2.14 Verify that the response of the system governed by the first-order difference equation:

$$y(k) = bu(k) + a\,y(k-1)$$

to the alternating input: $u(k) = (-1)^k$ for $k = 0, 1, 2, 3, \ldots$ is given by:

$$y(k) = \frac{b}{1+a}[(-1)^k + a^{k+1}] \quad \text{for} \quad k = 0,1,2,3,\ldots$$

if the initial condition is: $y(-1) = 0$.

Pb. 2.15 The impulse response of a system is the output from this system when excited by an input signal $\delta(k)$ that is zero everywhere, except at $k = 0$, where it is equal to 1. Using this definition and the general form of the solution of a difference equation, write the output of a linear system described by:

$$y(k) - 3y(k-1) - 4y(k-2) = \delta(k) + 2\delta(k-1)$$

The initial conditions are: $y(-2) = y(-1) = 0$.

Answer: $y(k) = \left[-\frac{1}{5}(-1)^k + \frac{6}{5}(4)^k\right] \quad \text{for} \quad k > 0$

Pb. 2.16 The expression for the National Income is given by:

$$y(k) = c(k) + i(k) + g(k)$$

where c is consumer expenditure, i is the induced private investment, g is the government expenditure, and k is the accounting period, typically corresponding to a particular quarter. Samuelson theory, introduced to many engineers in Cadzow's classic *Discrete Time Systems* (see reference list), assumes the following properties for the above three components of the National Income:

1. Consumer expenditure in any period k is proportional to the National Income at the previous period:

$$c(k) = ay(k-1)$$

2. Induced private investment in any period k is proportional to the increase in consumer expenditure from the preceding period:

$$i(k) = b[c(k) - c(k-1)] = ab[y(k-1) - y(k-2)]$$

3. Government expenditure is the same for all accounting periods:

$$g(k) = g$$

Combining the above equations, the National Income obeys the second-order difference equation:

$$y(k) = g + a(1+b)\,y(k-1) - aby(k-2) \quad \text{for } k = 1, 2, 3, \ldots$$

The initial conditions $y(-1)$ and $y(0)$ are to be specified.

Plot the National Income for the first 40 quarters of a new national entity, assuming that: $a = 1/6$, $b = 1$, $g = \$10{,}000{,}000$, $y(-1) = \$20{,}000{,}000$, $y(0) = \$30{,}000{,}000$.

How would the National Income curve change if the marginal propensity to consume (i.e., the constant a) is decreased to $1/8$?

2.5 Convolution-Summation of a First-Order System with Constant Coefficients

The amortization problem in Section 2.2 was solved by obtaining the present output, $y(k)$, as a linear combination of the present and all past inputs, ($u(k)$,

$u(k-1), u(k-2), \ldots$). This solution technique is referred to as the convolution-summation representation:

$$y(k) = \sum_{i=0}^{\infty} w(i)\, u(k-i) \tag{2.28}$$

where the $w(i)$ is the weighting function (or weight). Usually, the infinite sum is reduced to a finite sum because the inputs with negative indexes are usually assumed to be zeros.

On the other hand, in the difference equation formulation of this class of problems, the present output $y(k)$ is expressed as a linear combination of the present and m most recent inputs and of the n most recent outputs, specifically:

$$\begin{aligned} y(k) &= b_0u(k) + b_1u(k-1) + \ldots + b_mu(k-m) \\ &\quad - a_1y(k-1) - a_2y(k-2) - \ldots - a_ny(k-n) \end{aligned} \tag{2.29}$$

where, of course, n is the order of the difference equation. Elementary techniques for solving this class of equations were introduced in Section 2.4. However, the most powerful technique to directly solve the linear difference equation with constant coefficients is, as pointed out earlier, the z-transform technique.

Each of the above formulations of the input-output problem has distinct advantages in different circumstances. The direct difference equation formulation is the most amenable to numerical computations because of lower computer memory requirements, while the convolution-summation technique has the advantage of being suitable for developing mathematical proofs and finding general features for the difference equation.

Relating the parameters of the two formulations of this problem is usually cumbersome without the z-transform technique. However, for first-order difference equations, this task is rather simple.

Example 2.4

Relate, for a first-order difference equation with constant coefficients, the sets $\{a_n\}$ and $\{b_n\}$ with $\{w_n\}$.

Solution: The first-order difference equation is given by:

$$y(k) = b_0u(k) + b_1u(k-1) - a_1y(k-1)$$

where $u(k) = 0$ for all k negative. From the difference equation and the initial conditions, we can directly write:

$$y(0) = b_0u(0)$$

$$\text{for}\quad k=1,\quad \begin{cases} y(1)=b_0u(1)+b_1u(0)-a_1y(0) \\ \qquad =b_0u(1)+b_1u(0)-a_1b_0u(0) \\ \qquad =b_0u(1)+(b_1-a_1b_0)u(0) \end{cases}$$

Similarly,

$$y(2)=b_0u(2)+(b_1-a_1b_0)u(1)-a_1(b_1-a_1b_0)u(0)$$

$$y(3)=b_0u(3)+(b_1-a_1b_0)u(2)-a_1(b_1-a_1b_0)u(1)+a_1^2(b_1-a_1b_0)u(0)$$

or, more generally, if:

$$y(k) = w(0)u(k) + w(1)u(k-1) + \dots + w(k)u(0)$$

then,

$$w(0)=b_0$$

$$w(i)=(-a_1)^{i-1}(b_1-a_1b_0)\quad \text{for}\quad i=1,2,3,\dots$$

In-Class Exercises

Pb. 2.17 Using the convolution-summation technique, find the closed form solution for:

$$y(k)=u(k)-\frac{1}{3}u(k-1)+\frac{1}{2}y(k-1)$$

and the input function given by: $\begin{cases} u(k)=0 & \text{for } k \text{ negative} \\ u(k)=1 & \text{otherwise} \end{cases}$

Compare your analytical answer with the numerical solution.

Pb. 2.18 Show that the resultant weight functions for two systems are, respectively:

$$w(k) = w_1(k) + w_2(k) \quad \text{if connected in parallel}$$

$$w(k)=\sum_{i=0}^{k} w_2(i)w_1(k-i) \quad \text{if connected in cascade}$$

2.6 General First-Order Linear Difference Equations*

Thus far, we have considered difference equations with constant coefficients. Now we consider first-order difference equations with arbitrary functions as coefficients:

$$y(k+1) + A(k)y(k) = B(k) \tag{2.30}$$

The homogeneous equation corresponding to this form satisfies the following equation:

$$l(k+1) + A(k)l(k) = 0 \tag{2.31}$$

Its expression can be easily found:

$$\begin{aligned} l(k+1) &= -A(k)l(k) = A(k)A(k-1)l(k-1) = \ldots = \\ &= (-1)^{k+1}A(k)A(k-1)\ldots A(0)l(0) = \left\{\prod_{i=0}^{k}[-A(i)]\right\}l(0) \end{aligned} \tag{2.32}$$

Assuming that the general solution is of the form:

$$y(k) = l(k)v(k) \tag{2.33}$$

let us find $v(k)$. Substituting the above trial solution in the difference equation, we obtain:

$$l(k+1)v(k+1) + A(k)l(k)v(k) = B(k) \tag{2.34}$$

Further, assuming that

$$v(k+1) = v(k) + \Delta v(k) \tag{2.35}$$

substituting in the difference equation, and recalling that $l(k)$ is the solution of the homogeneous equation, we obtain:

$$\Delta v(k) = \frac{B(k)}{l(k+1)} \tag{2.36}$$

Summing this over the variable k from 0 to k, we deduce that:

$$v(k+1)=\sum_{j=0}^{k}\frac{B(j)}{l(j+1)}+C \tag{2.37}$$

where C is a constant.

Example 2.5

Find the general solution of the following first-order difference equation:

$$y(k+1)-k^2y(k)=0$$

with $y(1)=1$.

Solution:

$$\begin{aligned}y(k+1)&=k^2y(k)=k^2(k-1)^2y(k-1)=k^2(k-1)^2(k-2)^2y(k-2)\\&=k^2(k-1)^2(k-2)^2(k-3)^2y(k-3)=\dots\\&=k^2(k-1)^2(k-2)^2(k-3)^2\dots(2)^2(1)^2y(1)=(k!)^2\end{aligned}$$

Example 2.6

Find the general solution of the following first-order difference equation:

$$(k+1)y(k+1)-ky(k)=k^2$$

with $y(1)=1$.

Solution: Reducing this equation to the standard form, we have:

$$A(k)=-\frac{k}{k+1}\quad\text{and}\quad B(k)=\frac{k^2}{k+1}$$

The homogeneous solution is given by:

$$l(k+1)=\frac{k!}{(k+1)!}=\frac{1}{(k+1)}$$

The particular solution is given by:

$$v(k+1)=\sum_{j=1}^{k}\frac{j^2}{(j+1)}(j+1)+C=\sum_{j=1}^{k}j^2+C=\frac{(k+1)(2k+1)k}{6}+C$$

where we used the expression for the sum of the square of integers (see Appendix).

The general solution is then:

$$y(k+1) = \frac{(2k+1)k}{6} + \frac{C}{(k+1)}$$

From the initial condition $y(1) = 1$, we deduce that: $C = 1$.

In-Class Exercise

Pb. 2.19 Find the general solutions for the following difference equations, assuming that $y(1) = 1$.

a. $y(k + 1) - 3ky(k) = 3^k$.

b. $y(k + 1) - ky(k) = k$.

2.7 Nonlinear Difference Equations

In this and the following chapter section, we explore a number of nonlinear difference equations that exhibit some general features typical of certain classes of solutions and observe other instances with novel qualitative features. Our exploration is purely experimental, in the sense that we restrict our treatment to guided computer runs. The underlying theories of most of the models presented are the subject of more advanced courses; however, many educators, including this author, believe that there is virtue in exposing students qualitatively early on to these fascinating and generally new developments in mathematics.

2.7.1 Computing Irrational Numbers

In this model, we want to exhibit an example of a nonlinear difference equation whose solution is a sequence that approaches a specific limit, irrespective, within reasonable constraints, of the initial condition imposed on it. This type of difference equation has been used to compute a class of irrational numbers. For example, a well-defined approximation for computing $\sqrt{A}$ is the feedback process:

$$y(k+1) = \frac{1}{2}\left[y(k) + \frac{A}{y(k)}\right] \tag{2.38}$$

This equation's main features are explored in the following exercise.

In-Class Exercise

Pb. 2.20 Using the difference equation given by Eq. (2.38):

a. Write down a routine to compute $\sqrt{2}$. As an initial guess, take the initial value to be successively: 1, 1.5, 2; even consider 5, 10, and 20. What is the limit of each of the obtained sequences?

b. How many iterations are required to obtain $\sqrt{2}$ accurate to four digits for each of the above initial conditions?

c. Would any of the above properties be different for a different choice of A.

Now, having established that the above sequence goes to a limit, let us prove that this limit is indeed $\sqrt{A}$. To prove the above assertion, let this limit be denoted by y_{lim}; that is, for large k, both $y(k)$ and $y(k + 1) \Rightarrow y_{\text{lim}}$, and the above difference equation goes in the limit to:

$$y_{\text{lim}} = \frac{1}{2}\left[y_{\text{lim}} + \frac{A}{y_{\text{lim}}}\right] \tag{2.39}$$

Solving this equation, we obtain:

$$y_{\text{lim}} = \sqrt{A} \tag{2.40}$$

It should be noted that the above derivation is meaningful only when a limit exists and is in the domain of definition of the sequence (in this case, the real numbers). In Section 2.7.2, we encounter a sequence where, for some values of the parameters, there is no limit.

2.7.2 The Logistic Equation

Section 2.7.1 illustrated the case in which the solution of a nonlinear difference equation converges to a single limit for large values of the iteration index. In this chapter subsection, we consider the case in which a succession of iterates (called orbits) bifurcate, yielding orbits of period length 2, 4, 8, 16, *ad infinitum,* ending in what is called a “chaotic” orbit of infinite period length. We illustrate the prototype for this class of difference equations by exploring the logistic difference equation.

The logistic equation was introduced by Verhulst to model the growth of populations limited by finite resources (the name logistic was coined by the French army under Napoleon when this equation was used for the planning of “logement” of troops in camps). In more modern settings of ecology, the above model is used to simulate a population growth model. Specifically, in an ecological or growth process, the normalized measure $y(k + 1)$ of the next generation of a specie (the number of animals, for example) is a linear function of the present measure $y(k)$; that is,

$$y(k + 1) = ry(k) \tag{2.41}$$

where r is the growth parameter. If unchecked, the growth of the specie follows a geometric series, which for $r > 1$ grows to infinity. But growth is often limited by finite resources. In other words, the larger $y(k)$, the smaller the growth factor. The simplest way to model this decline in the growth factor is to replace r by $r(1 - y(k))$, so that as $y(k)$ approaches the theoretical limit (1 in this case), the effective growth factor goes to zero. The difference equation goes to:

$$y(k + 1) = r(1 - y(k))y(k) \tag{2.42}$$

which is the standard form for the logistic equation.

In the next series of exercises, we explore the solution of Eq. (2.42) as we vary the value of r. We find that qualitatively different classes of solutions may appear for different values of r.

We start by writing the simple subroutine that models Eq. (2.42):

```
N=127; r= ; y(1)= ;
m=1:N+1;
  for k=1:N
  y(k+1)= r*(1-y(k))*y(k);
  end
plot(m,y,'*')
x
```

The values of r and $y(1)$ are to be keyed in for each of the specific cases under consideration.

In-Class Exercises

In the following two problems, we take in the logistic equation $r > 1$ and $y(1) < 1$.

Pb. 2.21 Consider the case that $1 < r < 3$ and $y(1) = 0.5$.

a. Show that by running the above program for different values of r and $y(1)$ that the iteration of the logistic equation leads to the limit $y(N >> 1) = \left(\frac{r-1}{r}\right)$.

b. Does the value of this limit change if the value of $y(1)$ is modified, while r is kept fixed?

Pb. 2.22 Find the iterates of the logistic equation for the following values of r: 3.1, 3.236068, 3.3, 3.498561699, 3.566667, and 3.569946, assuming the following three initial conditions:

$$y(1) = 0.2, \quad y(1) = 0.5, \quad y(1) = 0.7$$

In particular, specify for each case:

a. The period of the orbit for large N, and the values of each of the iterates.

b. Whether the orbit is super-stable (i.e., the periodicity is present for all values of N).

This section provided a quick glimpse of two types of nonlinear difference equations, one of which may not necessarily converge to one value. We discovered that a great number of classes of solutions may exist for different values of the equation's parameters. In Section 2.8 we generalize to 2-D. Section 2.8 illustrates nonlinear difference equations in 2-D geometry. The study of these equations has led in the last few decades to various mathematical discoveries in the branches of mathematics called Symbolic Dynamical theory, Fractal Geometry, and Chaos theory, which have far-reaching implications in many fields of engineering. The interested student/reader is encouraged to consult the References section of this book for a deeper understanding of this subject.

2.8 Fractals and Computer Art

In Section 2.4, we introduced a fractal type having *a priori* well-defined and apparent spatial symmetries, namely, the Koch curve. In Section 2.7, we discovered that a certain type of 1-D nonlinear difference equation may lead, for a certain range of parameters, to a sequence that may have different orbits. Section 2.8.1 explores examples of 2-D fractals, generated by coupled difference equations, whose solution morphology can also be quite distinct due solely to a minor change in one of the parameters of the difference equations. Section 2.8.2 illustrates another possible feature observed in some types of fractals. We show how the 2-D orbit representing the solution of a particular nonlinear difference equation can also be substantially changed through a minor variation in the initial conditions of the equation.

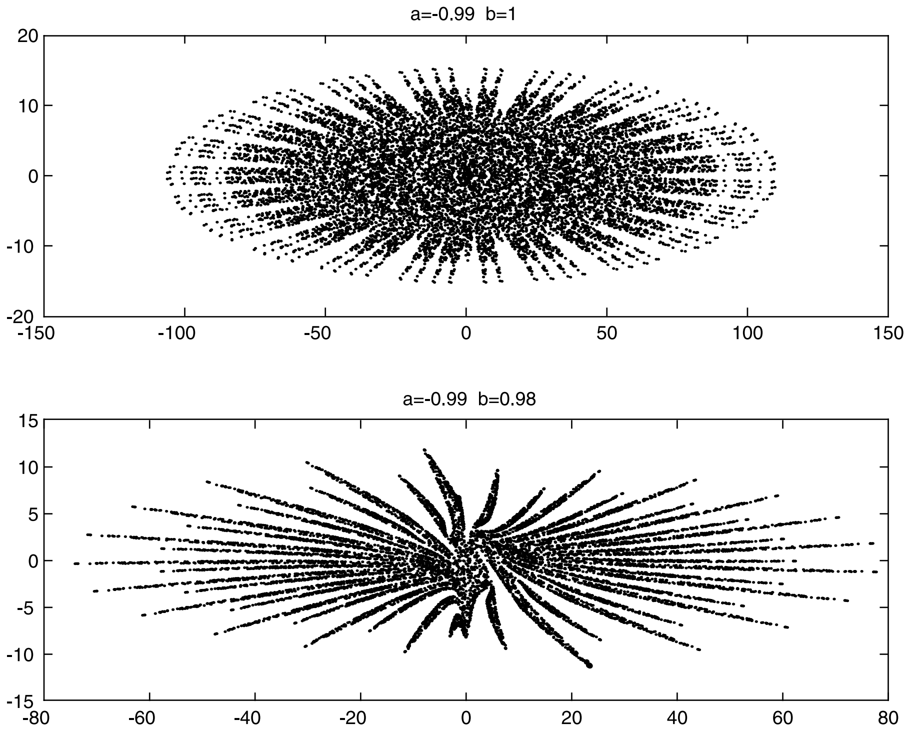

FIGURE 2.2
Plot of the Mira curve for a = 0.99. The starting point coordinates are (4, 0). Top panel: b = 1, bottom panel: b = 0.98.

2.8.1 Mira's Model

The coordinates of the points on the Mira curve are generated iteratively through the following system of nonlinear difference equations:

$$\begin{aligned} x(k+1) &= by(k) + F(x(k)) \\ y(k+1) &= -x(k) + F((x(k+1))) \end{aligned} \tag{2.43}$$

where

$$F(x) = ax + \frac{2(1-a)x^2}{1+x^2} \tag{2.44}$$

We illustrate the different morphologies of the solutions in two different cases, and leave other cases as exercises for your fun and exploration.

Case 1 Here, $a = -0.99$, and we consider the cases $b = 1$ and $b = 0.98$. The starting point coordinates are (4, 0). See Figure 2.2. This case can be viewed by editing and executing the following *script M-file:*

```
for n=1:12000
a=-0.99;b1=1;b2=0.98;
x1(1)=4;y1(1)=0;x2(1)=4;y2(1)=0;
x1(n+1)=b1*y1(n)+a*x1(n)+2*(1-a)*(x1(n))^2/(1+
   (x1(n)^2));
y1(n+1)=-x1(n)+a*x1(n+1)+2*(1-a)*(x1(n+1)^2)/(1+
   (x1(n+1)^2));
x2(n+1)=b2*y2(n)+a*x2(n)+2*(1-a)*(x2(n))^2/(1+
   (x2(n)^2));
y2(n+1)=-x2(n)+a*x2(n+1)+2*(1-a)*(x2(n+1)^2)/(1+
   (x2(n+1)^2));
end
subplot(2,1,1); plot(x1,y1,'.')
title('a=-0.99 b=1')
subplot(2,1,2); plot(x2,y2,'.')
title('a=-0.99 b=0.98')
```

Case 2 Here, $a = 0.7$, and we consider the cases $b = 1$ and $b = 0.9998$. The starting point coordinates are (0, 12.1). See Figure 2.3.

In-Class Exercise

Pb. 2.23 Manifest the computer artist inside yourself. Generate new geometrical morphologies, in Mira's model, by new choices of the parameters ($-1 < a < 1$ and $b \approx 1$) and of the starting point. You can start with:

a	b_1	b_2	(x_1, y_1)
–0.48	1	0.93	(4,0)
–0.25	1	0.99	(3,0)
0.1	1	0.99	(3,0)
0.5	1	0.9998	(3,0)
0.99	1	0.9998	(0,12)

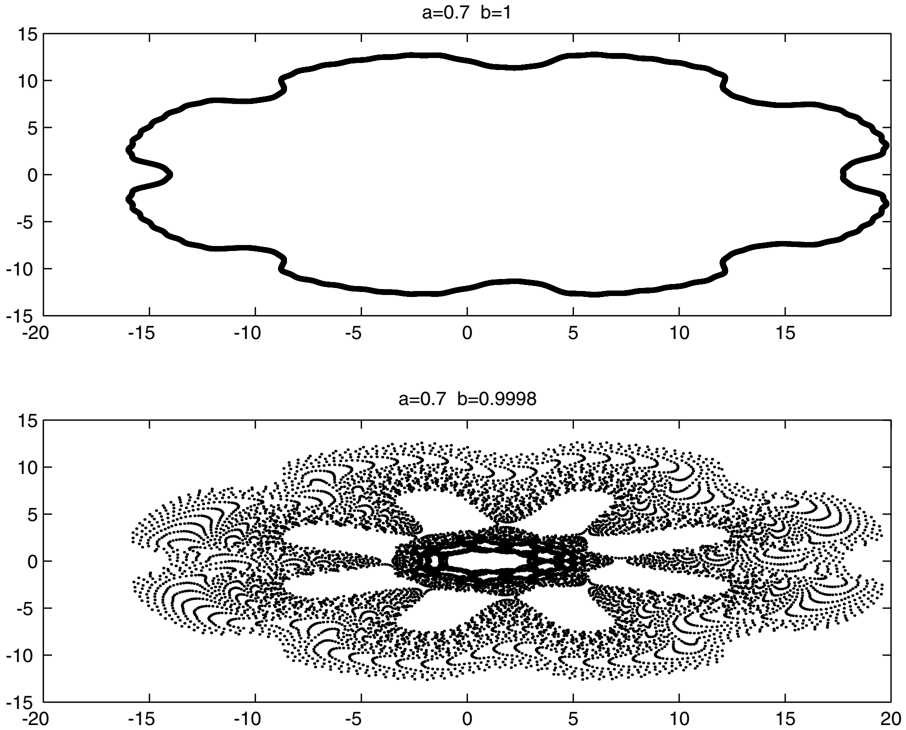

FIGURE 2.3
Plot of the Mira curve for a = 0.7. The starting point coordinates are (0, 12.1). Top panel: b = 1, bottom panel: b = 0.9998.

2.8.2 Hénon's Model

The coordinates of the Hénon's orbits are generated iteratively through the following system of nonlinear difference equations:

$$
\begin{aligned}
x(k+1) &= ax(k+1) - b(y(k) - (x(k))^2) \\
y(k+1) &= bx(k+1) + a(y(k) - (x(k))^2)
\end{aligned}
\tag{2.45}
$$

where $|a| \leq 1$ and $b = \sqrt{1-a^2}$.

Executing the following *script M-file* illustrates the possibility of generating two distinct orbits if the starting points of the iteration are slightly different (here, a = 0.24), and the starting points are slightly different from each other. The two cases initial point coordinates are given, respectively, by (0.5696, 0.1622) and (0.5650, 0.1650). See Figure 2.4.

```
a=0.24;
b=0.9708;
```

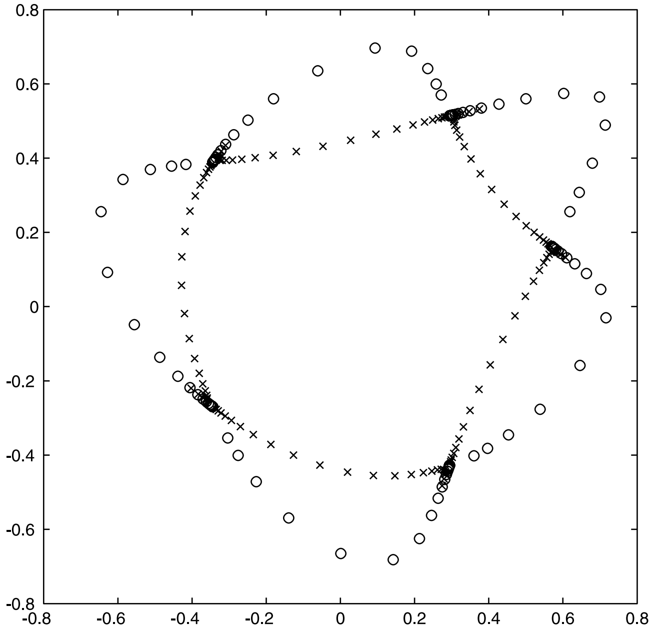

FIGURE 2.4
Plot of two Hénon orbits having the same a = 0.25 but different starting points. (o) corresponds to the orbit with starting point (0.5696, 0.1622), (x) corresponds to the orbit with starting point (0.5650, 0.1650).

```
x1(1)=0.5696;y1(1)=0.1622;
x2(1)=0.5650;y2(1)=0.1650;
for n=1:120
  x1(n+1)=a*x1(n)-b*(y1(n)-(x1(n))^2);
  y1(n+1)=b*x1(n)+a*(y1(n)-(x1(n))^2);
  x2(n+1)=a*x2(n)-b*(y2(n)-(x2(n))^2);
  y2(n+1)=b*x2(n)+a*(y2(n)-(x2(n))^2);
end
plot(x1,y1,'ro',x2,y2,'bx')
```

2.8.2.1 Demonstration

Different orbits for Hénon's model can be plotted if different starting points are randomly chosen. Executing the following *script M-file* illustrates the a = 0.24 case, with random initial conditions. See Figure 2.5.

```
a=0.24;
b=sqrt(1-a^2);
rx=rand(1,40);
ry=rand(1,40);
```

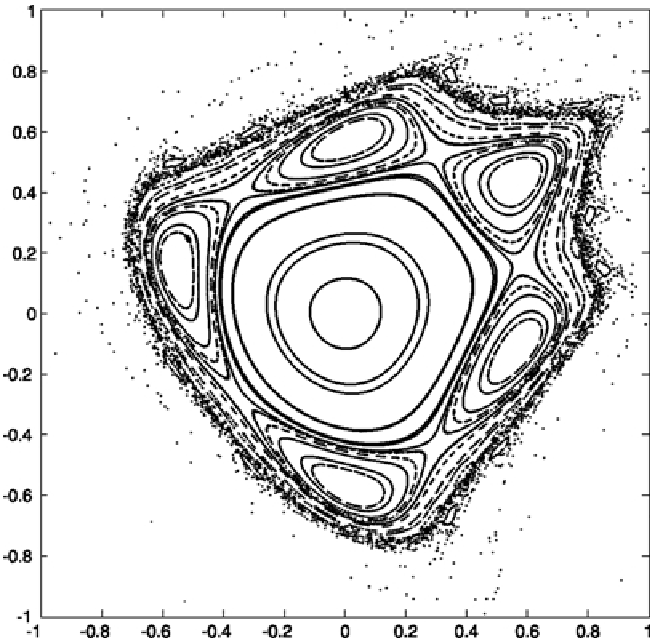

FIGURE 2.5
Plot of multiple Hénon orbits having the same a = 0.25 but random starting points.

```
for n=1:1500
  for m=1:40
  x(1,m)=-0.99+2*rx(m);
  y(1,m)=-0.99+2*ry(m);

  x(n+1,m)=a*x(n,m)-b*(y(n,m)-(x(n,m))^2);
  y(n+1,m)=b*x(n,m)+a*(y(n,m)-(x(n,m))^2);
  end
end
plot(x,y,'r.')
axis([-1 1 -1 1])
axis square
```

2.9 Generation of Special Functions from Their Recursion Relations*

In this section, we go back to more classical mathematics. We consider the case of the special functions of mathematical physics. In this case, we need to define the iterated quantities by two indices: the order of the function and the value of the argument of the function.

In many electrical engineering problems, it is convenient to use a class of polynomials called the orthogonal polynomials. For example, in filter design, the set of Chebyshev polynomials are of particular interest.

The Chebyshev polynomials can be defined through recursion relations, which are similar to difference equations and relate the value of a polynomial of a certain order at a particular point to the values of the polynomials of lower orders at the same point. These are defined through the following recursion relation:

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \tag{2.46}$$

Now, instead of giving two values for the initial conditions as we would have in difference equations, we need to give the explicit functions for two of the lower-order polynomials. For example, the first- and second-order Chebyshev polynomials are

$$T_1(x) = x \tag{2.47}$$

$$T_2(x) = 2x^2 - 1 \tag{2.48}$$

Example 2.7

Plot over the interval $0 \leq x \leq 1$, the fifth-order Chebyshev polynomial.

Solution: The strategy to solve this problem is to build an array to represent the x-interval, and then use the difference equation routine to find the value of the Chebyshev polynomial at each value of the array, remembering that the indexing should always be a positive integer.

The following program implements the above strategy:

```
N=5;
x1=1:101;
x=(x1-1)/100;
T(1,x1)=x;
T(2,x1)=2*x.^2-1;
  for k=3:N
  T(k,x1)=2.*x.*T(k-1,x1)-T(k-2,x1);
  end
y=T(N,x1);
plot(x,y)
```

In-Class Exercise

Pb. 2.24 By comparing their plots, verify that the above definition for the Chebyshev polynomial gives the same graph as that obtained from the closed-form expression:

$$T_N(x) = \cos(N \cos^{-1}(x)) \quad \text{for } 0 \leq x \leq 1$$

In addition to the Chebyshev polynomials, you will encounter other orthogonal polynomials in your engineering studies. In particular, the solutions of a number of problems in electromagnetic theory and in quantum mechanics (QM) call on the Legendre, Hermite, Laguerre polynomials, etc. In the following exercises, we explore, in a preliminary manner, some of these polynomials. We also explore another important type of the special functions: the spherical Bessel function.

Homework Problems

Pb. 2.25 Plot the function y defined, in each case:

a. Legendre polynomials:

$$(m+2)P_{m+2}(x) = (2m+3)xP_{m+1}(x) - (m+1)P_m(x)$$

$$P_1(x) = x \quad \text{and} \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

For $0 \leq x \leq 1$, plot $y = P_5(x)$

These polynomials describe the electric field distribution from a nonspherical charge distribution.

b. Hermite polynomials:

$$\begin{cases} H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x) \\ H_1(x) = 2x \quad \text{and} \quad H_2(x) = 4x^2 - 2 \end{cases}$$

For $0 \leq x \leq 6$, plot $y = A_5 H_5(x)\exp(-x^2/2)$, where $A_m = (2^m m!\sqrt{\pi})^{-1/2}$

The function y describes the QM wave-function of the harmonic oscillator.

c. Laguerre polynomials:

$$\begin{cases} L_{m+2}(x) = [(3+2m+-x)L_{m+1}(x) - (m+1)^2 L_m(x)]/(m+2) \\ L_1(x) = 1 - x \quad \text{and} \quad L_2(x) = (1 - 2x + x^2/2) \end{cases}$$

For $0 \leq x \leq 6$, plot $y = \exp(-x/2)L_5(x)$

The Laguerre polynomials figure in the solutions of the QM problem of atoms and molecules.

Pb. 2.26 The recursion relations can, in addition to defining orthogonal polynomials, also define some special functions of mathematical physics. For example, the spherical Bessel functions that play an important role in defining the modes of spherical cavities in electrodynamics and scattering amplitudes in both classical and quantum physics are defined through the following recursion relation:

$$j_{m+2}(x) = \left(\frac{3+2m}{x}\right) j_{m+1}(x) - j_m(x)$$

With

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \quad \text{and} \quad j_2(x) = \left[\frac{3}{x^3} - \frac{1}{x}\right]\sin(x) - \frac{3\cos(x)}{x^2}$$

Plot j_5(x) over the interval $0 < x < 15$.

3

Elementary Functions and Some of Their Uses

The purpose of this chapter is to illustrate and build some practice in the use of elementary functions in selected basic electrical engineering problems. We also construct some simple signal functions that you will encounter in future engineering analysis and design problems.

NOTE It is essential to review the Supplement at the end of this book in case you want to refresh your memory on the particular elementary functions covered in the different chapter sections.

3.1 Function Files

To analyze and graph functions using MATLAB, we have to be able to construct functions that can be called from within the MATLAB environment. In MATLAB, functions are made and stored in *function M-files*. We already used one kind of *M-file* (script file) to store various executable commands in a routine. *Function M-files* differ from *script M-files* in that they have designated input(s) and output(s).

The following is an example of a function. Type and save the following function in a file named **aline.m**:

```
function y=aline(x)
% (x,y) is a point on a line that has slope 3
% and y-intercept -5
y=3*x-5;
```

NOTES

1. The word **function** at the beginning of the file makes it a function rather than a script file.
2. The function name, **aline**, that appears in the first line of this file should match the name that we assign to this file name when saving it (i.e., **aline.m**).

Having created a *function M-file* in your user volume, move to the command window to learn how to call this function. There are two basic ways to use a function file:

1. To evaluate the function for a specified value **x=x1**, enter **aline(x1)** to get the function value at this point; that is, $y_1 = 3x_1 - 5$.
2. To plot $y_1 = 3x_1 - 5$ for a range of x values, say [–2, 7], enter:

```
fplot('aline',[-2,7])
```

NOTE The above example illustrates a function with one input and one output. The construction of a *function M-file* of a function having n inputs and m outputs starts with:

```
function  [y1,y2,...,ym]=funname(x1,x2,...,xn)
```

Above, using a *function M-file,* we showed a method to plot the defined function **aline** on the interval (–2, 7) using the **fplot** command. An alternative method is, of course, to use arrays, in the manner specified in Chapter 1. Specifically, we could have plotted the **'aline'** function in the following alternate method:

```
x=-2:.01:7;
y=3*x-5;
plot(x,y)
```

To compare the two methods, we note that:

1. **plot** requires a user-supplied x-array (abscissa points) and a constructed y-array (ordinate points), while **fplot** only requires the name of the function file, defined previously and stored in a *function M-file* and the endpoints of the interval.
2. The **fplot** automatically creates a sampled domain that is used to plot the function, taking into account the type of function being plotted and using enough points to make the display appear continuous. On the other hand, **plot** requires that you choose the array length yourself.

Both methods, therefore, have their own advantages and it depends on the particular problem whether to use **plot** or **fplot**.

We are now in position to explore the use of some of the most familiar functions.

3.2 Examples with Affine Functions

The equation of an affine function is given by:

$$y(x) = ax + b \tag{3.1}$$

In-Class Exercises

Pb. 3.1 Generate four *function M-files* for the following four functions:

$$y_1(x) = 3x + 2;\ \ y_2(x) = 3x + 5;\ \ y_3(x) = -\frac{x}{3} + 3;\ \ y_4(x) = -\frac{x}{3} + 4$$

Pb. 3.2 Sketch the functions of **Pb. 3.1** on the interval $-5 < x < 5$. What can you say about the angle between each of the two lines' pairs. (Did you remember to make your aspect ratio = 1?)

Pb. 3.3 Read off the graphs the coordinates of the points of intersection of the lines in **Pb. 3.1**. (Become familiar with the use and syntax of the **`zoom`** and **`ginput`** commands for a more accurate reading of the coordinates of a point.)

Pb. 3.4 Write a *function M-file* for the line passing through a given point and intersecting another given line at a given angle.

$$\left(\textit{Hint:}\ \ \tan(a + b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}\right)$$

Application to a Simple Circuit

The purpose of this application is to show that:

1. The solution to a simple circuit problem can be viewed as the simultaneous solution of two affine equations, or, equivalently, as the intersection of two straight lines.
2. The variations in the circuit performance can be studied through a knowledge of the affine functions, relating the voltages and the current.

Consider the simple circuit shown in Figure 3.1. In the terminology of the circuit engineer, the voltage source V_S is called the input to the circuit, and the current I and the voltage V are called the circuit outputs. Thus, this is an example of a system with one input and two outputs. As you may have studied in high school physics courses, all of circuit analysis with resistors as elements can be accomplished using Kirchhoff's current law, Kirchoff's voltage law, and Ohm's law.

- Kirchoff's voltage law: The sum of all voltage drops around a closed loop is balanced by the sum of all voltage sources around the same loop.

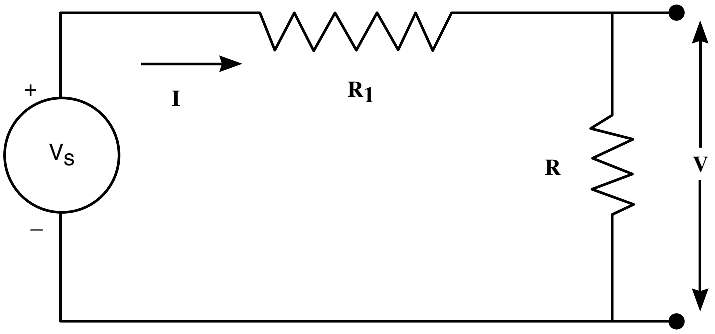

FIGURE 3.1
A simple resistor circuit.

- Kirchoff's current law: The algebraic sum of all currents entering (exiting) a circuit node must be zero. (Assign the + sign to those currents that are entering the node, and the – sign to those current exiting the node.)
- Ohm's law: The ratio of the voltage drop across a resistor to the current passing through the resistor is a constant, defined as the resistance of the element; that is, $R = \frac{V}{I}$

The quantities we are looking for include (1) the current *I* through the circuit, and (2) the voltage *V* across the load resistor *R*.

Using Kirchoff's voltage law and Ohm's law for resistance R_1, we obtain:

$$V_s = V + V_1 = V + IR_1 \tag{3.2}$$

while applying Ohm's law for the load resistor gives:

$$V = IR \tag{3.3}$$

These two equations can be rewritten in the form of affine functions of *I* as functions of *V*:

$$L_1:\ I = \frac{(V_s - V)}{R_1} \tag{3.4}$$

$$L_2:\ I = \frac{V}{R} \tag{3.5}$$

If we know the value of V_s, R, and R_1, then Eqs. (3.4) and (3.5) can be represented as lines drawn on a plane with ordinate I and abscissa V.

Suppose we are interested in finding the value of the current I and the voltage V when $R_1 = 100\Omega$, $R = 100\Omega$, and $V_s = 5$ V. To solve this problem graphically, we plot each of the L_1 and L_2 functions on the same graph and find their point of intersection.

The functions L_1 and L_2 are programmed as follows:

```
function I=L1(V)
R1=100;
R=100;
Vs=5;
I=(Vs-V)/R1;

function I=L2(V)
R1=100;
R=100;
Vs=5;
I=V/R;
```

Because the voltage V is smaller than the source potential, due to losses in the resistor, a suitable domain for V would be [0, 5]. We now plot the two lines on the same graph:

```
fplot('L1',[0,5])
hold on
fplot('L2',[0,5])
hold off
```

In-Class Exercise

Pb. 3.5 Verify that the two lines L_1 and L_2 intersect at the point: ($I = 0.025$, $V = 2.5$).

In the above analysis, we had to declare the numerical values of the parameters R_1 and R in the definition of each of the two functions. This can, at best, be tedious if you are dealing with more than two *function M-files* or two parameters; or worse, can lead to errors if you overlook changing the values of the parameters in any of the relevant *function M-files* when you decide to modify them. To avoid these types of problems, it is good practice to call all functions from a single *script M-file* and link the parameters' values together so that you only need to edit the calling *script M-file*. To link the values of parameters to all functions in use, you can use the MATLAB **global** command. To see how this works, rewrite the above *function M-files* as follows:

```
function I=L1(V)
global R1 R             % global statement
Vs=5;
I=(Vs-V)/R1;

function I=L2(V)
global R1 R             % global statement
Vs=5;
I=V/R;
```

The calling *script M-file* now reads:

```
global R1 R             %global statement
R1=100;                 %set global resistance values
R=100;
V=0:.01:5;              %set the voltage range
I1=L1(V);               %evaluate I1
I2=L2(V);               %evaluate I2
plot(V,I1,V,I2,'-')     %plot the two curves
```

In-Class Exercise

Pb. 3.6 In the above *script M-file,* we used arrays and the **plot** command. Rewrite this script file such that you make use of the **fplot** command.

Further Consideration of Figure 3.1

Calculating the circuit values for fixed resistor values is important, but we can also ask about the behavior of the circuit as we vary the resistor values. Suppose we keep $R_1 = 100\Omega$ and $V_s = 5\ V$ fixed, but vary the value that R can take. To this end, an analytic solution would be useful because it would give us the circuit responses for a range of values of the circuit parameters R_1, R, V_s. However, a plot of the lines L_1 and L_2 for different values of R can also provide a great deal of qualitative information regarding how the simultaneous solution to L_1 and L_2 changes as the value of R changes.

The following problem serves to give you a better qualitative idea as to how the circuit outputs vary as different values are chosen for the resistor *R*.

In-Class Exercise

Pb. 3.7 This problem still refers to the circuit of Figure 3.1.

a. Redraw the lines L_1 and L_2, using the previous values for the circuit parameters.

b. Holding the graph for the case $R = 100\Omega$, sketch L_1 and L_2 again for $R = 50\Omega$ and $R = 500\Omega$. How do the values of the voltage and the current change as *R* increases; and decreases?

c. Determine the largest values of the current and voltage that can exist in this circuit when *R* varies over non-negative values.

d. The usual nomenclature for the circuit conditions is as follows: the circuit is called an open circuit when $R = \infty$, while it is called a short circuit when $R = 0$. What are the (*V, I*) solutions for these two cases? Can you generalize your statement?

Now, to validate the qualitative results obtained in **Pb. 3.7**, let us solve analytically the L_1 and L_2 system. Solving this system of two linear equations in two unknowns gives, for the current and the voltage, the following expressions:

$$V(R) = \left(\frac{R}{R + R_1}\right) V_s \tag{3.6}$$

$$I(R) = \left(\frac{1}{R + R_1}\right) V_s \tag{3.7}$$

Note that the above analytic expressions for *V* and *I* are neither linear nor affine functions in the value of the resistance.

In-Class Exercise

Pb. 3.8 This problem still refers to the circuit of Figure 3.1.

a. Keeping the values of V_s and R_1 fixed, sketch the functions $V(R)$ and $I(R)$ for this circuit, and verify that the solutions you found previously in **Pbs. 3.7** and **3.8**, for the various values of *R*, agree with those found here.

b. Given that the power lost in a resistive element is the product of the voltage across the resistor multiplied by the current through the resistor, plot the power through the variable resistor as a function of R.

c. Determine the value of R such that the power lost in this resistor is maximized.

d. Find, in general, the relation between R and R_1 that ensures that the power lost in the load resistance is maximized. (This general result is called Thevenin's theorem.)

3.3 Examples with Quadratic Functions

A quadratic function is of the form:

$$y(x) = ax^2 + bx + c \tag{3.8}$$

Preparatory Exercises

Pb. 3.9 Find the coordinates of the vertex of the parabola described by Eq. (3.8) as functions of the a, b, c parameters.

Pb. 3.10 If $a = 1$, show that the quadratic Eq. (3.8) can be factored as:

$$y(x) = (x - x_+)(x - x_-)$$

where $x_\pm$ are the roots of the quadratic equation. Further, show that, for arbitrary a, the product of the roots is $\frac{c}{a}$, and their sum is $\frac{-b}{a}$.

In-Class Exercises

Pb. 3.11 Develop a *function M-file* that inputs the two real roots of a second-degree equation and returns the value of this function for an arbitrary x. Is this function unique?

Pb. 3.12 In your elementary mechanics course, you learned that the trajectory of a projectile in a gravitational field (oriented in the $-y$ direction) with an initial velocity $v_{0,x}$ in the x-direction and $v_{0,y}$ in the y-direction satisfies the following parametric equations:

$$x = v_{0,x}t \text{ and } y = -\frac{1}{2}gt^2 + v_{0,y}t$$

where t is time and the origin of the axis was chosen to correspond to the position of the particle at $t = 0$ and $g = 9.8\ \text{ms}^{-2}$

a. By eliminating the time t, show that the projectile trajectory $y(x)$ is a parabola.

b. Noting that the components of the initial velocity can be written as function of the projectile initial speed and its angle of inclination:

$$v_{0,y} = v_0 \sin(\phi) \quad \text{and} \quad v_{0,x} = v_0 \cos(\phi)$$

show that, for a given initial speed, the maximum range for the projectile is achieved when the inclination angle of the initial velocity is 45°.

c. Plot the range for a fixed inclination angle as a function of the initial speed.

3.4 Examples with Polynomial Functions

As pointed out in the Supplement, a polynomial function is an expression of the form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \tag{3.9}$$

where $a_n \neq 0$ for an n^{th}-degree polynomial. In MATLAB, we can represent the polynomial function as an array:

$$p = [a_n a_{n-1} \ldots a_0] \tag{3.10}$$

Example 3.1

You are given the array of coefficients of the polynomial. Write a *function M-file* for this polynomial using array operations. Let $p = [1\quad 3\quad 2\quad 1\quad 0\quad 3]$:

Solution:

```
function y=polfct(x)
p=[1 3 2 1 0 3];
```

```
L=length(p);
v=x.^[(L-1):-1:0];
y=sum(p.*v);
```

In-Class Exercises

Pb. 3.13 Show that, for the polynomial p defined by Eq. (3.9), the product of the roots is $(-1)^n \frac{a_0}{a_n}$, and the sum of the roots is $-\frac{a_{n-1}}{a_n}$.

Pb. 3.14 Find graphically the real roots of the polynomial p = [1 3 2 1 0 3].

3.5 Examples with the Trigonometric Functions

A time-dependent cosine function of the form:

$$x = a\cos(\omega t + \phi) \tag{3.11}$$

appears often in many applications of electrical engineering: a is called the amplitude, ω the angular frequency, and ϕ the phase. Note that we do not have to have a separate discussion of the sine function because the sine function, as shown in the Supplement, differs from the cosine function by a constant phase. Therefore, by suitably changing only the value of the phase parameter, it is possible to transform the sine function into a cosine function.

In the following example, we examine the period of the different powers of the cosine function; your preparatory task is to predict analytically the relationship between the periods of the two curves given in Example 3.2 and then verify your answer numerically.

Example 3.2

Plot simultaneously, $x_1(t) = \cos^3(t)$ and $x_2 = \cos(t)$ on t ∈ [0, 6π].

Solution: To implement this task, edit and execute the following *script M-file*:

```
t=0:.2:6*pi;          % t-array
a=1;w=1;              % desired parameters
x1=a*(cos(w*t))^3;    % x1-array constructed
```

```
x2=a*cos(w*t);          % x2-array constructed
plot(t,x1,t,x2,'--')
```

In-Class Exercises

Pb. 3.15 Determine the phase relation between the sine and cosine functions of the same argument.

Pb. 3.16 The meaning of amplitude, angular frequency, and phase can be better understood using MATLAB to obtain graphs of the cosine function for a family of *a* values, ω values, and ϕ values.

a. With ω = 1 and ϕ = π/3, plot the cosine curves corresponding to *a* = 1:0.1:2.

b. With *a* = 1 and ω = 1, plot the cosine curves corresponding to ϕ = 0:π/10:π.

c. With *a* = 1 and ϕ = π/4, plot the cosine curves corresponding to ω = 1:0.1:2.

Homework Problem

Pb. 3.17 Find the period of the function obtained by summing the following three cosine functions:

$$x_1 = 3\cos(t/3 + \pi/3),\ \ x_2 = \cos(t + \pi),\ \ x_3 = \frac{1}{3}\cos\left(\frac{3}{2}(t + \pi)\right)$$

Verify your result graphically.

3.6 Examples with the Logarithmic Function

3.6.1 Ideal Coaxial Capacitor

An ideal capacitor can be loosely defined as two metallic plates separated by an insulator. If a potential is established between the plates, for example through the means of connecting the two plates to the different terminals of a battery, the plates will be charged by equal and opposite charges, with the battery serving as a pump to move the charges around. The capacitance of a capacitor is defined as the ratio of the magnitude of the charge accumulated on either of the plates divided by the potential difference across the plates.

Using the Gauss law of electrostatics, it can be shown that the capacitance per unit length of an infinitely long coaxial cable is:

$$\frac{C}{l} = \frac{2\pi\varepsilon}{\ln(b/a)} \tag{3.12}$$

where a and b are the radius of the internal and external conductors, respectively, and ε is the permittivity of the dielectric material sandwiched between the conductors. (The permittivity of vacuum is approximately $\varepsilon_0 = 8.85 \times 10^{-12}$, while that of oil, polystyrene, glass, quartz, bakelite, and mica are, respectively, 2.1, 2.6, 4.5–10, 3.8–5, 5, and 5.4-6 larger.)

In-Class Exercise

Pb. 3.18 Find the ratio of the capacitance of two coaxial cables with the same dielectric material for, respectively: $b/a = 5$ and 50.

3.6.2 The Decibel Scale

In the SI units used by electrical engineers, the unit of power is the Watt. However, in a number of applications, it is convenient to express the power as a ratio of its value to a reference value. Because the value of this ratio can vary over several orders of magnitude, it is often more convenient to represent this ratio on a logarithmic scale, called the decibel scale:

$$G[\text{dB}] = 10\log\left(\frac{P}{P_{ref}}\right) \tag{3.13}$$

where the function log is the logarithm to base 10. The table below converts the power ratio to its value in decibels (dB):

P/P_{ref} (10^n)	dB values ($10\,n$)
4	6
2	3
1	0
0.5	–3
0.25	–6
0.1	–10
10^{-3}	–30

In-Class Exercise

Pb. 3.19 In a measurement of two power values, P_1 and P_2, it was determined that:

$$G_1 = 9 \text{ dB} \quad \text{and} \quad G_2 = -11 \text{ dB}$$

Using the above table, determine the value of the ratio P_1/P_2.

3.6.3 Entropy

Given a random variable X (such as the number of spots on the face of a thrown die) whose possible outcomes are x_1, x_2, x_3, …, and such that the probability for each outcome is, respectively, $p(x_1)$, $p(x_2)$, $p(x_3)$, … then, the entropy for this system described by the outcome of one random variable is defined by:

$$H(X) = -\sum_{i=1}^{N} p(x_i)\log_2(p(x_i)) \tag{3.14}$$

where N is the number of possible outcomes, and the logarithm is to base 2.

The entropy is a measure of the uncertainty in the value of the random variable. In Information Theory, it will be shown that the entropy, so defined, is the number of bits, on average, required to describe the random variable X.

In-Class Exercises

Pb. 3.20 In each of the following cases, find the entropy:

a. $N = 32$ and $p(x_i) = \frac{1}{32}$ for all i

b. $N = 8$ and $p = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}\right]$

c. $N = 4$ and $p = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right]$

d. $N = 4$ and $p = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right]$

Pb. 3.21 Assume that you have two dices (die), one red and the other blue. Tabulate all possible outcomes that you can obtain by throwing these die together. Now assume that all you care about is the sum of spots on the two die. Find the entropy of the outcome.

Homework Problem

Pb. 3.22 A so-called A-law compander (compressor followed by an expander) uses a compressor that relates output to input voltages by:

$$y = \pm \frac{A|x|}{1+\log(A)} \qquad \text{for } |x| \leq 1/A$$

$$y = \pm \frac{1+\log(A|x|)}{1+\log(A)} \qquad \text{for } \frac{1}{A} \leq |x| \leq 1$$

Here, the + sign applies when x is positive and the – sign when x is negative. $x = v_i/V$ and $y = v_0/V$, where v_i and v_o are the input and output voltages. The range of allowable voltages is $-V$ to V. The parameter A determines the degree of compression.

For a value of $A = 87.6$, plot y vs. x in the interval [–1, 1].

3.7 Examples with the Exponential Function

Take a few minutes to review the section on the exponential function in the Supplement before proceeding further.

(Recall that exp(1) = e.)

In-Class Exercises

Pb. 3.23 Plot the function $y(x) = (x^{13} + x^9 + x^5 + x^2 + 1)\exp(-4x)$ over the interval [0,10].

Pb. 3.24 Plot the function $y(x) = \cos(5x)\exp(-x/2))$ over the interval [0, 10].

Pb. 3.25 From the results of **Pbs. 3.23** and **3.24**, what can you deduce about the behavior of a function at infinity if one of its factors is an exponentially decreasing function of x, while the other factor is a polynomial or trigonometric function of x? What modification to the curve is observed if the degree of the polynomial is increased?

Application to a Simple RC Circuit

The solution giving the voltage across the capacitor in Figure 3.2 following the closing of the switch can be written in the following form:

$$V_c(t) = V_c(0)\exp\left[-\frac{t}{RC}\right] + V_s\left[1-\exp\left[-\frac{t}{RC}\right]\right] \tag{3.15}$$

$V_c(t)$ is called the time response of the RC circuit, or the circuit output resulting from the constant input V_s. The time constant RC of the circuit has the units of seconds and, as you will observe in the present analysis and other problems in subsequent chapters, its ratio to the characteristic time of a given input potential determines qualitatively the output of the system.

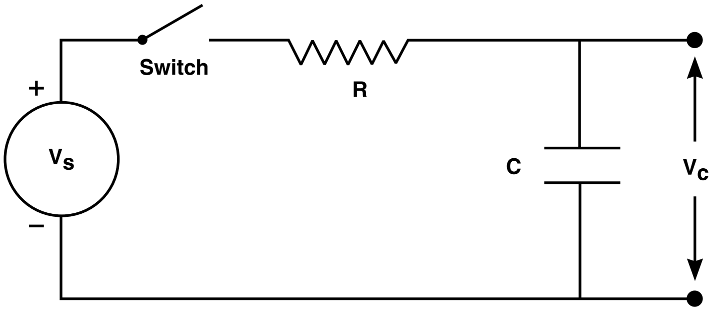

FIGURE 3.2
The circuit used in charging a capacitor.

In-Class Exercise

Pb. 3.26 A circuit designer can produce outputs of various shapes by selecting specific values for the circuit time constant RC. In the following simulations, you can examine the influence of this time constant on the response of the circuit of Figure 3.2.

Using $V_c(0) = 3$ volts, $V_s = 10$ volts (capacitor charging process), and $RC = 1$ s:

a. Sketch a graph of $V_c(t)$. What is the asymptotic value of the solution? How long does it take the capacitor voltage to reach the value of 9 volts?

b. Produce an *M-file* that will plot several curves of $V_c(t)$ corresponding to:

(i) $RC = 1$

(ii) $RC = 5$

(iii) $RC = 10$

Which of these time constants results in the fastest approach of $V_c(t)$ toward V_s?

c. Repeat the above simulations for the case $V_s = 0$ (capacitor discharge)?

d. What would you expect to occur if $V_c(0) = V_s$?

Homework Problem

Pb. 3.27 The Fermi-Dirac distribution, which gives the average population of electrons in a state with energy ε, neglecting the electron spin for the moment, is given by:

$$f(\varepsilon) = \frac{1}{\exp[(\varepsilon - \mu)/\Theta] + 1}$$

where μ is the Fermi (or chemical) potential and Θ is proportional to the absolute (or Kelvin) temperature.

a. Plot the function $f(\varepsilon)$ as function of ε, for the following cases:

(i) $\mu = 1$ and $\Theta = 0.002$

(ii) $\mu = 0.03$ and $\Theta = 0.025$

(iii) $\mu = 0.01$ and $\Theta = 0.025$

(iv) $\mu = 0.001$ and $\Theta = 0.001$

b. What is the value of $f(\varepsilon)$ when $\varepsilon = \mu$?

c. Determine the condition under which we can approximate the Fermi-Dirac distribution function by:

$$f(\varepsilon) \approx \exp[(\mu - \varepsilon)/\Theta]$$

3.8 Examples with the Hyperbolic Functions and Their Inverses

3.8.1 Capacitance of Two Parallel Wires

The capacitance per unit length of two parallel wires, each of radius *a* and having their axis separated by distance *D*, is given by:

$$\frac{C}{l} = \frac{\pi\varepsilon_0}{\cosh^{-1}\left(\frac{D}{2a}\right)} \tag{3.16}$$

where ε_0 is the permittivity of air (taken to be that of vacuum) = 8.854×10^{-12} Farad/m.

Question: Write this expression in a different form using the logarithmic function.

In-Class Exercises

Pb. 3.28 Find the capacitance per unit length of two wires of radii 1 cm separated by a distance of 1 m. Express your answer using the most appropriate of the following sub-units:

mF = 10^{-3} F (milli-Farad);	μF = 10^{-6} F (micro-Farad);
nF = 10^{-9} F (nano-Farad);	pF = 10^{-12} F (pico-Farad);
fF = 10^{-15} F (femto-Farad);	aF = 10^{-18} F (atto-Farad);

Pb. 3.29 Assume that you have two capacitors, one consisting of a coaxial cable (radii *a* and *b*) and the other of two parallel wires, separated by the distance *D*. Further assume that the radius of the wires is equal to the radius of the inner cylinder of the coaxial cable. Plot the ratio $\frac{D}{a}$ as a function of $\frac{b}{a}$, if we desire the two geometrical configurations for the capacitor to end up having the same value for the capacitance. $\left(\text{Take } \frac{\varepsilon}{\varepsilon_0} = 2.6.\right)$

3.9 Commonly Used Signal Processing Functions

In studying signals and systems, you will also encounter, *inter alia,* the following functions (or variation thereof), in addition to the functions discussed previously in this chapter:

- Unit step function
- Unit slope ramp function

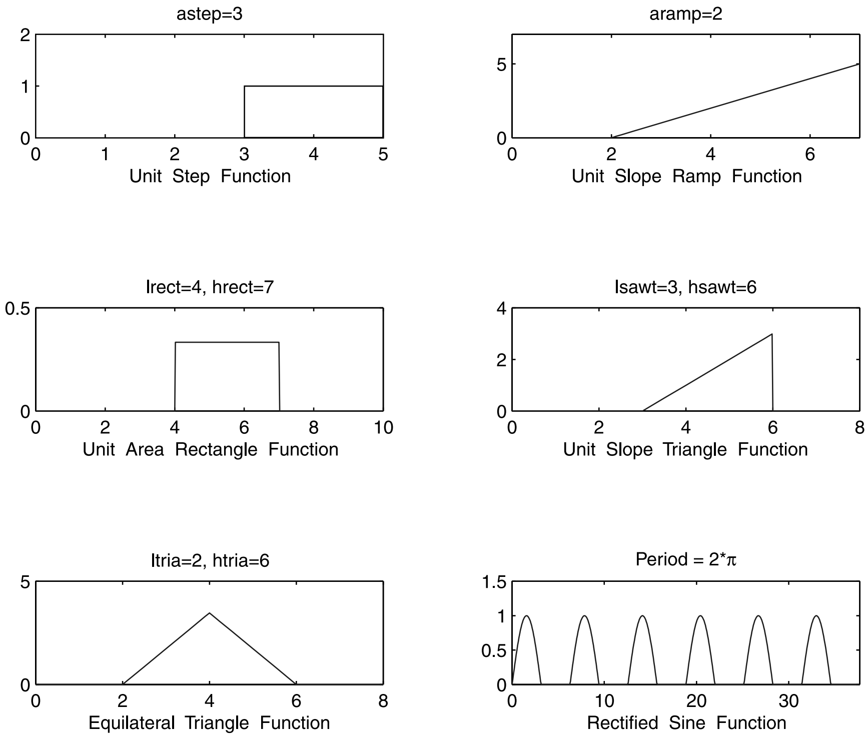

FIGURE 3.3
Various useful signal processing functions.

- Unit area rectangle pulse
- Unit slope right angle triangle function
- Equilateral triangle function
- Periodic traces

These functions are plotted in Figure 3.3, and the corresponding *function M-files* are (*x* is everywhere a scalar):

A. Unit Step function

```
function y=stepf(x)
global astep
  if x<astep
  y=0;
  else
  y=1;
  end
```

B. Unit Slope Ramp function

```
function y=rampf(x)
global aramp
  if x<aramp
  y=0;
  else
  y=x-aramp;
  end
```

C. Unit Area Rectangle function

```
function y=rectf(x)
global lrect hrect
  if x<lrect
  y=0
  elseif lrect<x & x<hrect
  y=1/(hrect-lrect);
  else
  y=0;
  end
```

D. Unit Slope Right Angle Triangle function

```
function y=sawtf(x)
global lsawt hsawt
  if x<lsawt
  y=0;
  elseif lsawt<x & x<hsawt
  y=x-lsawt;
  else
  y=0;
  end
```

E. Equilateral Triangle function

```
function y=triaf(x)
global ltria htria
  if x<ltria
  y=0;
  elseif ltra<x & x<(ltria+htria)/2
```

```
    y=sqrt(3)*(x-ltria);
    elseif (ltria+htria)/2=<x & x<htria
    y=sqrt(3)*(-x+htria);
    else
    y=0
    end
```

F. Periodic functions

It is often necessary to represent a periodic signal train where the elementary representation on one cycle can easily be written. The technique is to use the modulo arithmetic to map the whole of the *x*-axis over a finite domain. This is, of course, possible because the function is periodic. For example, consider the rectified sine function train. Its *function M-file* is

```
function y=psinef(x)
s=rem(x,2*pi)
  if s>0 & s=<pi
  y=sin(s);
  elseif s>pi & s=<2*pi
  y=0;
  else
  y=0
  end
```

In-Class Exercises

Pb. 3.30 In the above definition of all the special shape functions, we used the if-else-end form. Write each of the *function M-files* to define these same functions using only Boolean expressions.

Pb. 3.31 An adder is a device that adds the input signals to give an output signal equal to the sum of the inputs. Using the functions previously obtained in this section, write the *function M-file* for the signal in Figure 3.4.

Pb. 3.32 A multiplier is a device that multiplies two inputs. Find the product of the inputs given in Figures 3.5 and 3.6.

Homework Problems

The first three problems in this set are a brief introduction to the different analog modulation schemes of communication theory.

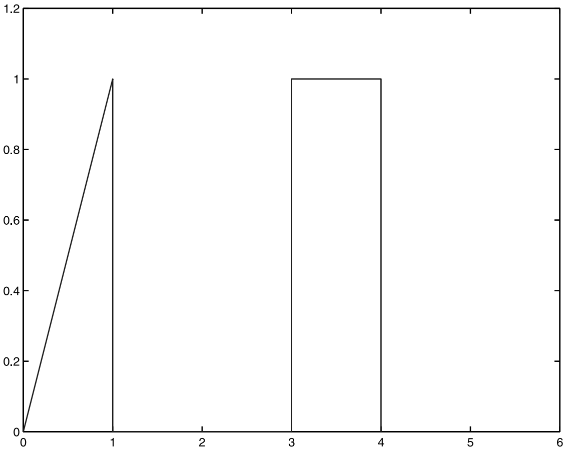

FIGURE 3.4
Profile of the signal of Pb. 3.31.

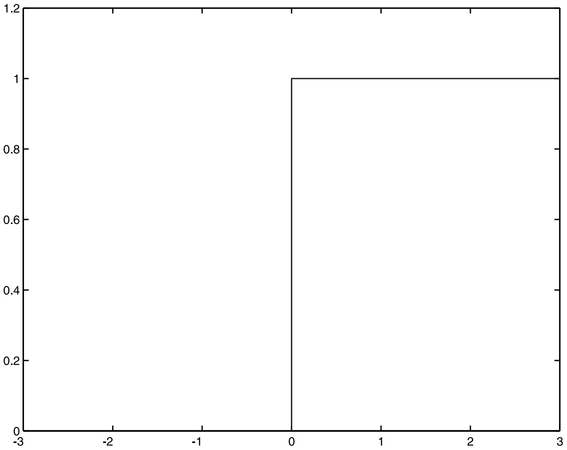

FIGURE 3.5
Profile of the first input to Pb. 3.32.

Pb. 3.33 In DSB-AM (double-sideband amplitude modulation), the amplitude of the modulated signal is proportional to the message signal, which means that the time domain representation of the modulated signal is given by:

$$u_{DSB}(t) = A_c m(t)\cos(2\pi f_c t)$$

where the carrier-wave shape is

$$c(t) = A_c \cos(2\pi f_c t)$$

and the message signal is $m(t)$.

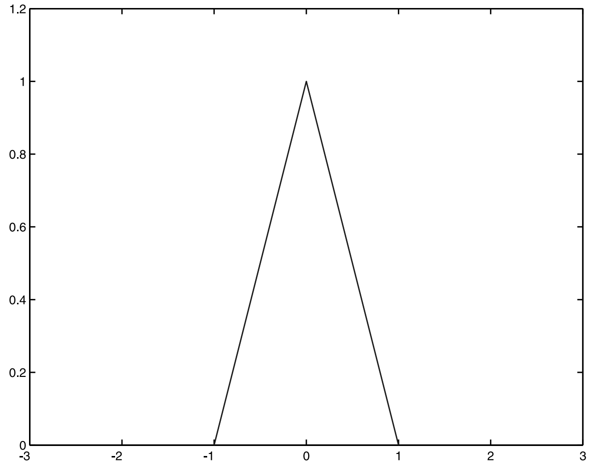

FIGURE 3.6
Profile of the second input to Pb. 3.32.

For a message signal given by:

$$m(t) = \begin{cases} 1 & 0 \le t \le t_0 / 3 \\ -3 & t_0 / 3 < t \le 2t_0 / 3 \\ 0 & \text{otherwise} \end{cases}$$

- **a.** Write the expression for the modulated signal using the unit area rectangle and the trigonometric functions.
- **b.** Plot the modulated signal as function of time. (Let f_c = 200 and t_0 = 0.01.)

Pb. 3.34 In conventional AM, $m(t)$ in the DSB-AM expression for the modulated signal is replaced by $[1 + am_n(t)]$, where $m_n(t)$ is the normalized message signal (i.e., $m_n(t) = \frac{m(t)}{\max(m(t))}$ and a is the index of modulation ($0 \le a \le 1$). The modulated signal expression is then given by:

$$u_{AM}(t) = A_c[1 + am_n(t)]\cos(2\pi f_c t)$$

For the same message as that of **Pb. 3.33** and the same carrier frequency, and assuming the modulation index a = 0.85:

- **a.** Write the expression for the modulated signal.
- **b.** Plot the modulated signal.

Pb. 3.35 The angle modulation scheme, which includes frequency modulation (FM) and phase modulation (PM), has the modulated signal given by:

$$u_{PM}(t) = A_c \cos(2\pi f_c t + k_p m(t))$$

$$u_{FM}(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right)$$

Assuming the same message as in **Pb. 3.33**:

a. Write the expression for the modulated signal in both schemes.

b. Plot the modulated signal in both schemes. Let $k_p = k_f = 100$.

Pb. 3.36 If $f(x) = f(-x)$ for all x, then the graph of $f(x)$ is symmetric with respect to the y-axis, and the function $f(x)$ is called an even function. If $f(x) = -f(-x)$ for all x, the graph of $f(x)$ is anti-symmetric with respect to the origin, and we call such a function an odd function.

a. Show that any function can be written as the sum of an odd function plus an even function. List as many even and odd functions as you can.

b. State what conditions must be true for a polynomial to be even, or to be odd.

c. Show that the product of two even functions is even; the product of two odd functions is even; and the product of an odd and even function is odd.

d. Replace in c above the word product by either quotient or power and deduce the parity of the resulting function.

e. Deduce from the above results that the sign/parity of a function follows algebraic rules.

f. Find the even and odd parts of the following functions:

(i) $f(x) = x^7 + 3x^4 + 6x + 2$

(ii) $f(x) = (\sin(x) + 3)\sinh^2(x)\exp(-x^2)$

Pb. 3.37 Decompose the signal shown in Figure 3.7 into its even and odd parts:

Pb. 3.38 Plot the function y defined through:

$$y(x) = \begin{cases} x^2 + 4x + 4 & \text{for } -2 \le x < -1 \\ 0.16x^2 - 0.48x & \text{for } -1 < x < 1.5 \\ 0 & \text{elsewhere} \end{cases}$$

and find its even and odd parts.

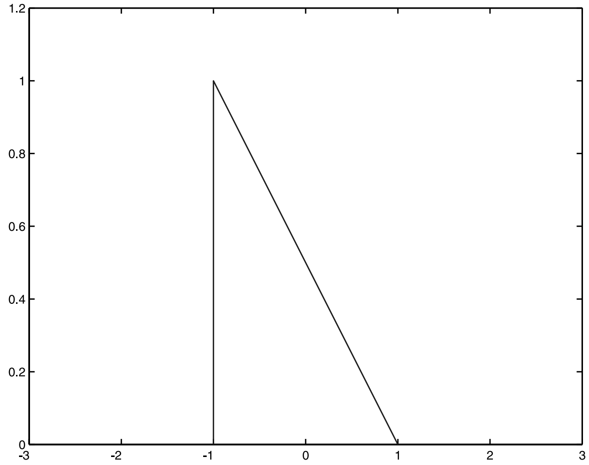

FIGURE 3.7
Profile of the signal of Pb. 3.37.

3.10 Animation of a Moving Rectangular Pulse

You might often want to plot the time development of one of the above signal processing functions if its defining parameters are changing in time. Take, for example, a theatrical spotlight of constant intensity density across its cross-section, but assume that its position varies with time. The light spot size can be represented by a rectangular pulse (e.g., of width 2 m and height 1 m) that is moving to the right with a constant speed of 1 m/s. Assume that the center of the spot is originally at $x = 1$ m, and that its final position is at $x = 8$ m. We want to write a program that will illustrate its time development, and then play the resulting movie.

To illustrate the use of other commands not often utilized in this chapter, we can, instead of the **if-else-end** syntax used in the previous section, use the Boolean syntax, and define the array by the `linspace` command.

Edit and execute the following *script M-file*:

```
lrect=0;hrect=2;
x=linspace(0,10,200);
t=linspace(0,8,40);
M=moviein(40);
  for m=1:40
  y=(x>=lrect+t(m)).*(x<=hrect+t(m));
  plot(x,y,'r')
```

```
    axis([-2 12 0 1.2]);
    M(:,m)=getframe;
    end
  movie(M,3)
```

Question: How would you modify the above program if the speed of the light beam is not 1?

3.11 MATLAB Commands Review

`fplot`	Plots a specified function over a specified interval.
`ginput`	Mouse-controlling command to read off coordinates of a point in a graph.
`global`	Allows variables to share their values in multiple programs.
`zoom`	Zooms in and out on a 2-D plot.

4

Numerical Differentiation, Integration, and Solutions of Ordinary Differential Equations

This chapter discusses the basic methods for numerically finding the value of the limit of an indeterminate form, the value of a derivative, the value of a convergent infinite sum, and the value of a definite integral. Using an improved form of the differentiator, we also present first-order iterator techniques for solving ordinary first-order and second-order linear differential equations. The Runge-Kutta technique for solving ordinary differential equations (ODE) is briefly discussed. The mode of use of some of the MATLAB packages to perform each of the previous tasks is also described in each instance of interest.

4.1 Limits of Indeterminate Forms

DEFINITION If $\lim_{x \to x_0} u(x) = \lim_{x \to x_0} v(x) = 0,$ the quotient $u(x)/v(x)$ is said to have an indeterminate form of the $0/0$ kind.

- If $\lim_{x \to x_0} u(x) = \lim_{x \to x_0} v(x) = \infty,$ the quotient $u(x)/v(x)$ is said to have an indeterminate form of the ∞/∞ kind.

In your elementary calculus course, you learned that the standard technique for solving this kind of problem is through the use of *L'Hopital's Rule,* which states that:

if:

$$\lim_{x \to x_0} \frac{u'(x)}{v'(x)} = C \tag{4.1}$$

then:

$$\lim_{x \to x_0} \frac{u(x)}{v(x)} = C \tag{4.2}$$

In this section, we discuss a simple algorithm to obtain this limit using MATLAB. The method consists of the following steps:

1. Construct a sequence of points whose limit is x_0. In the examples below, consider the sequence $\left\{ x_n = x_0 - \left(\frac{1}{2}\right)^n \right\}$. Recall in this regard that as $n \to \infty$, the n^{th} power of any number whose magnitude is smaller than one goes to zero.
2. Construct the sequence of function values corresponding to the x-sequence, and find its limit.

Example 4.1

Compute numerically the $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Solution: Enter the following instructions in your MATLAB command window:

```
N=20;  n=1:N;
x0=0;
dxn=-(1/2).^n;
xn=x0+dxn;
yn=sin(xn)./xn;
plot(xn,yn)
```

The limit of the **yn** sequence is clearly equal to 1. The deviation of the sequence of the **yn** from the value of the limit can be obtained by entering:

```
dyn=yn-1;
semilogy(n,dyn)
```

The last command plots the curve with the ordinate y expressed logarithmically. This mode of display is the most convenient in this case because the ordinate spans many decades of values.

In-Class Exercises

Find the limits of the following functions at the indicated points:

Pb. 4.1 $\frac{(x^2 - 2x - 3)}{(x-3)}$ at $x \to 3$

Pb. 4.2 $\left(\frac{1+\sin(x)}{x} - \frac{1}{\sin(x)} \right)$ at $x \to 0$

Pb. 4.3 $(x\cot(x))$ at $x \to 0$

Pb. 4.4 $\dfrac{(1-\cos(2x))}{x^2}$ at $x \to 0$

Pb. 4.5 $\sin(2x)\cot(3x)$ at $x \to 0$

4.2 Derivative of a Function

DEFINITION The derivative of a certain function at a particular point is defined as:

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \tag{4.3}$$

Numerically, the derivative is computed at the point x_0 as follows:

1. Construct an x-sequence that approaches x_0.
2. Compute a sequence of the function values corresponding to the x-sequence.
3. Evaluate the sequence of the ratio, appearing in the definition of the derivative in Eq. (4.3).
4. Read off the limit of this ratio sequence. This will be the value of the derivative at the point x_0.

Example 4.2

Find numerically the derivative of the function ln(1 + x) at x = 0.

Solution: Edit and execute the following *script M-file:*

```
N=20;n=1:N;
x0=0;
dxn=(1/2).^[1:N];
xn=x0+dxn;
yn=log(1+xn);
dyn=yn-log(1+x0);
```

```
deryn=dyn./dxn;
plot(n,deryn)
```

The limit of the **deryn**'s sequence is clearly equal to 1, the value of this function derivative at 0.

NOTE The choice of **N** should always be such that **dxn** is larger than the machine precision; that is, **N** < 53, since $(1/2)^{53} \approx 10^{-16}$.

In-Class Exercises

Find numerically, to one part per 10,000 accuracy, the derivatives of the following functions at the indicated points:

Pb. 4.6 $x^4(\cos^3(x) - \sin(2x))$ at $x \to \pi$

Pb. 4.7 $\dfrac{\exp(x^2+3)}{(2+\cos^2(x))}$ at $x \to 0$

Pb. 4.8 $\dfrac{(1+\sin^2(x))}{(2-\cos^3(x))}$ at $x \to \pi/2$

Pb. 4.9 $\ln\left(\dfrac{x-1/2}{x+1}\right)$ at $x \to 1$

Pb. 4.10 $\tan^{-1}(x^2+3)$ at $x \to 0$

Example 4.3

Plot the derivative of the function $x^2 \sin(x)$ over the interval $0 \le x \le 2\pi$.

Solution: Edit and execute the following *script M-file:*

```
dx=10^(-4);
x=0:dx:2*pi+dx;
df=diff(sin(x).*x.^2)/dx;
plot(0:dx:2+pi,df)
```

where **diff** is a MATLAB command, which when acting on an array X, gives the new array [X(2) – X(1)X(3) – X(2) … X(n) – X(n – 1)], whose length is one unit shorter than the array X.

The accuracy of the above algorithm depends on the choice of **dx**. Ideally, the smaller it is, the more accurate the result. However, using any computer, we should always choose a **dx** that is larger than the machine precision, while still much smaller than the value of the variation of **x** over which the function changes appreciably.

For a systematic method to choose an upper limit on **dx**, you might want to follow these simple steps:

1. Plot the function on the given interval and identify the point where the derivative is largest.
2. Compute the derivative at that point using the sequence method of Example 4.2, and determine the **dx** that would satisfy the desired tolerance; then go ahead and use this value of **dx** in the above routine to evaluate the derivative throughout the given interval.

In-Class Exercises

Plot the derivatives of the following functions on the indicated intervals:

Pb. 4.11 $\ln\left|\frac{x-1}{x+1}\right|$ on $2 < x < 3$

Pb. 4.12 $\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right|$ on $1 < x < 2$

Pb. 4.13 $\ln|\tanh(x/2)|$ on $1 < x < 5$

Pb. 4.14 $\tan^{-1}|\sinh(x)|$ on $0 < x < 10$

Pb. 4.15 $\ln|\csc(x)+\tan(x)|$ on $0 < x < \pi/2$

4.3 Infinite Sums

An infinite series is denoted by the symbol $\sum_{n=1}^{\infty} a_n$. It is important not to confuse the series with the sequence $\{a_n\}$. The sequence is a list of terms, while the series is a sum of these terms. A sequence is convergent if the term a_n approaches a finite limit; however, convergence of a series requires that the sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ approaches a finite limit. There are cases where the sequence may approach a limit, while the series is divergent. The classical example is that of the sequence $\left\{\frac{1}{n}\right\}$; this sequence approaches the limit zero, while the corresponding series is divergent.

In any numerical calculation, we cannot perform the operation of adding an infinite number of terms. We can only add a finite number of terms. The infinite sum of a convergent series is the limit of the partial sums S_N.

You will study in your calculus course the different tests for checking the convergence of a series. We summarize below the most useful of these tests.

- The Ratio Test, which is very useful for series with terms that contain factorials and/or n^{th} power of a constant, states that:

$$\text{for } a_n > 0, \text{ the series} \sum_{n=1}^{\infty} a_n \text{ is convergent if } \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) < 1$$

- The Root Test stipulates that for $a_n > 0$, the series $\sum_{n=1}^{\infty} a_n$ is convergent if

$$\lim_{n\to\infty}(a_n)^{1/n} < 1$$

- For an alternating series, the series is convergent if it satisfies the conditions that

$$\lim_{n\to\infty}|a_n| = 0 \quad \text{and} \quad |a_{n+1}| < |a_n|$$

Now look at the numerical routines for evaluating the limit of the partial sums when they exist.

Example 4.4

Compute the sum of the geometrical series $S_N = \sum_{n=1}^{N}\left(\frac{1}{2}\right)^n$.

Solution: Edit and execute the following *script M-file*:

```
for N=1:20
n=N:-1:1;
fn=(1/2).^n;
Sn(N)=sum(fn);
end
NN=1:20;
plot(NN,Sn)
```

You will observe that this partial sum converges to 1.

NOTE The above summation was performed backwards because this scheme will ensure a more accurate result and will keep all the significant digits of the smallest term of the sum.

In-Class Exercises

Compute the following infinite sums:

Pb. 4.16 $\sum_{k=1}^{\infty} \frac{1}{(2k-1)2^{2k-1}}$

Pb. 4.17 $\sum_{k=1}^{\infty} \frac{\sin(2k-1)}{(2k-1)}$

Pb. 4.18 $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^4}$

Pb. 4.19 $\sum_{k=1}^{\infty} \frac{\sin(k/2)}{k^3}$

Pb. 4.20 $\sum_{k=1}^{\infty} \frac{1}{2^k} \sin(k)$

4.4 Numerical Integration

The algorithm for integration discussed in this section is the second simplest available (the trapezoid rule being the simplest, beyond the trivial, is given at the end of this section as a problem). It has been generalized to become more accurate and efficient through other approximations, including Simpson's rule, the Newton-Cotes rule, the Gaussian-Laguerre rule, etc. Simpson's rule is derived in Section 4.6, while other advanced techniques are left to more advanced numerical methods courses.

Here, we perform numerical integration through the means of a Rieman sum: we subdivide the interval of integration into many subintervals. Then we take the area of each strip to be the value of the function at the midpoint of the subinterval multiplied by the length of the subinterval, and we add the strip areas to obtain the value of the integral. This technique is referred to as the midpoint rule.

We can justify the above algorithm by recalling the Mean Value Theorem of Calculus, which states that:

$$\int_a^b f(x)dx = (b-a)f(c) \tag{4.4}$$

where $c \in [a, b]$. Thus, if we divide the interval of integration into narrow subintervals, then the total integral can be written as the sum of the integrals over the subintervals, and we approximate the location of c in a particular subinterval by the midpoint between its boundaries.

Example 4.5

Use the above algorithm to compute the value of the definite integral of the function sin(x) from 0 to π.

Solution: Edit and execute the following program:

```
dx=pi/200;
x=0:dx:pi-dx;
xshift=x+dx/2;
yshift=sin(xshift);
Int=dx*sum(yshift)
```

You get for the above integral a result that is within 1/1000 error from the analytical result.

In-Class Exercises

Find numerically, to a 1/10,000 accuracy, the values of the following definite integrals:

Pb. 4.21 $\int_0^\infty \frac{1}{x^2+1}dx$

Pb. 4.22 $\int_0^\infty \exp(-x^2)\cos(2x)dx$

Pb. 4.23 $\int_0^{\pi/2} \sin^6(x)\cos^7(x)dx$

Pb. 4.24 $\displaystyle\int_0^{\pi} \frac{2}{1+\cos^2(x)}\,dx$

Example 4.6

Plot the value of the indefinite integral $\int_0^x f(x)dx$ as a function of x, where $f(x)$ is the function $\sin(x)$ over the interval $[0, \pi]$.

Solution: We solve this problem for the general function $f(x)$ by noting that:

$$\int_0^x f(x)dx \approx \int_0^{x-\Delta x} f(x)dx + f(x - \Delta x + \Delta x / 2)\Delta x \tag{4.5}$$

where we are dividing the x-interval into subintervals and discretizing x to correspond to the coordinates of the boundaries of these subintervals. An array $\{x_k\}$ represents these discrete points, and the above equation is then reduced to a difference equation:

$$\text{Integral}(x_k) = \text{Integral}(x_{k-1}) + f(\text{Shifted}(x_{k-1}))\Delta x \tag{4.6}$$

where

$$\text{Shifted}(x_{k-1}) = x_{k-1} + \Delta x/2 \tag{4.7}$$

and the initial condition is $\text{Integral}(x_1) = 0$.

The above algorithm can then be programmed, for the above specific function, as follows:

```
a=0;
b=pi;
dx=0.001;
x=a:dx:b-dx;
N=length(x);
xshift=x+dx/2;
yshift=sin(xshift);
Int=zeros(1,N+1);
Int(1)=0;
  for k=2:N+1
  Int(k)=Int(k-1)+yshift(k-1)*dx;
```

```
    end
  plot([x b],Int)
```

It may be useful to remind the reader, at this point, that the algorithm in Example 4.6 can be generalized to any arbitrary function. However, it should be noted that the key to the numerical calculation accuracy is a good choice for the increment **`dx`**. A very rough prescription for the estimation of this quantity, for an oscillating function, can be obtained as follows:

1. Plot the function inside the integral (i.e., the integrand) over the desired interval domain.
2. Verify that the function does not blow-out (i.e., goes to infinity) anywhere inside this interval.
3. Choose **`dx`** conservatively, such that at least 30 subintervals are included in any period of oscillation of the function (see Section 6.8 for more details).

In-Class Exercises

Plot the following indefinite integrals as function of x over the indicated interval:

Pb. 4.25 $\int_0^x \left(\frac{\cos(x)}{\sqrt{1+\sin(x)}} \right) dx \quad 0 < x < \pi/2$

Pb. 4.26 $\int_1^x \frac{(1+x^{2/3})^6}{x^{1/3}} dx \quad 1 < x < 8$

Pb. 4.27 $\int_0^x \left[\frac{(x+2)}{(x^2+2x+4)^2} \right] dx \quad 0 < x < 1$

Pb. 4.28 $\int_0^x x^2 \sin(x^3) dx \quad 0 < x < \pi/2$

Pb. 4.29 $\int_0^x \sqrt{\tan(x)} \sec^2(x) dx \quad 0 < x < \pi/4$

Homework Problem

Pb. 4.30 Another simpler algorithm than the midpoint rule for evaluating a definite integral is the Trapezoid rule: the area of the slice is approximated by the area of the trapezoid with vertices having the following coordinates: ($x(k)$, 0); ($x(k + 1)$, 0); ($x(k + 1)$, $y(k + 1)$); ($x(k)$, $y(k)$); giving for this trapezoid area the value:

$$\frac{1}{2}[x(k+1)-x(k)][y(k+1)+y(k)] = \frac{\Delta x}{2}[y(k+1)+y(k)]$$

thus leading to the following iterative expression for the Trapezoid integrator:

$$I_T(k+1) = I_T(k) + \frac{\Delta x}{2}[y(k+1)+y(k)]$$

The initial condition is: $I_T(1) = 0$.

a. Evaluate the integrals of **Pbs. 4.25** through **4.29** using the Trapezoid rule.

b. Compare for the same values of Δx, the accuracy of the Trapezoid rule with that of the midpoint rule.

c. Give a geometrical interpretation for the difference in accuracy obtained using the two integration schemes.

NOTE MATLAB has a built-in command for evaluating the integral by the Trapezoid rule. If the sequence of the sampling points and of the function values are given, **`trapz(x,y)`** gives the desired result.

4.5 A Better Numerical Differentiator

In Section 4.2, for the numerical differentiator, we used the simple expression:

$$d(k) = \frac{1}{\Delta x}(y(k) - y(k-1)) \tag{4.8}$$

Our goal in this section is to find a more accurate expression for the differentiator. We shall use the difference equation for the Trapezoid rule to derive this improved differentiator, which we shall denote by $D(k)$.

The derivation of the difference equation for $D(k)$ hinges on the basic observation that differentiating the integral of a function gives back the original function. We say that the numerical differentiator is the inverse of the numerical integrator. We shall use the convolution-summation representation of the solution of a difference equation to find the iterative expression for $D(k)$.

Denoting the weighting sequence representations of the identity operation, the numerical integrator, and the numerical differentiator by $\{w\}$, $\{w_1\}$, and

$\{w_2\}$, respectively, and using the notation and results of Section 2.5, we have for the identity operation the following weights:

$$w(0) = 1 \tag{4.9a}$$

$$w(i) = 0 \quad \text{for i} = 1, 2, 3, \ldots \tag{4.9b}$$

The Trapezoid numerical integrator, as given in **Pb. 4.25**, is a first-order system with the following parameters:

$$b_0^{(1)} = \frac{\Delta x}{2} \tag{4.10a}$$

$$b_1^{(1)} = \frac{\Delta x}{2} \tag{4.10b}$$

$$a_1^{(1)} = -1 \tag{4.10c}$$

giving for its weight sequence, as per Example 2.4, the values:

$$w_1(0) = \frac{\Delta x}{2} \tag{4.11a}$$

$$w_1(i) = \Delta x \quad \text{for} \quad i = 1, 2, 3, \ldots \tag{4.11b}$$

The improved numerical differentiator's weight sequence can now be directly obtained by noting, as noted above, that if we successively cascade integration with differentiation, we are back to the original function. Using the results of **Pb. 2.18**, we can write:

$$w(k) = \sum_{i=0}^{k} w_2(i) w_1(k-i) \tag{4.12}$$

Combining the above values for $w(k)$ and $w_1(k)$, we can deduce the following equalities:

$$w(0) = 1 = \frac{\Delta x}{2} w_2(0) \tag{4.13a}$$

$$w(1) = 0 = \Delta x \left[\frac{1}{2} w_2(1) + w_2(0) \right] \tag{4.13b}$$

$$w(2) = 0 = \Delta x\left[\frac{1}{2}w_2(2) + w_2(1) + w_2(0)\right] \tag{4.13c}$$

etc. …

from which we can directly deduce the following expressions for the weighting sequence $\{w_2\}$:

$$w_2(0) = \frac{2}{\Delta x} \tag{4.14a}$$

$$w_2(i) = \frac{4}{\Delta x}(-1)^i \quad \text{for} \quad i = 1, 2, 3, \ldots \tag{4.14b}$$

From these weights we can compute, as per the results of Example 2.4, the parameters of the difference equation for the improved numerical differentiator, namely:

$$b_0^{(2)} = \frac{2}{\Delta x} \tag{4.15a}$$

$$b_1^{(2)} = -\frac{2}{\Delta x} \tag{4.15b}$$

$$a_1^{(2)} = 1 \tag{4.15c}$$

giving for $D(k)$ the following defining difference equation:

$$D(k) = \frac{2}{\Delta t}[y(k) - y(k-1)] - D(k-1) \tag{4.16}$$

In **Pb. 4.32** and in other cases, you can verify that indeed this is an improved numerical differentiator. We shall, later in the chapter, use the above expression for $D(k)$ in the numerical solution of ordinary differential equations.

In-Class Exercises

Pb. 4.31 Find the inverse system corresponding to the discrete system governed by the difference equation:

$$y(k) = u(k) - \frac{1}{2}u(k-1) + \frac{1}{3}y(k-1)$$

Pb. 4.32 Compute numerically the derivative of the function

$$y = x^3 + 2x^2 + 5 \quad \text{in the interval} \quad 0 \le x \le 1$$

using the difference equations for both $d(k)$ and $D(k)$ for different values of Δx. Comparing the numerical results with the analytic results, compute the errors in both methods.

Application

In this application, we make use of the improved differentiator and corresponding integrator (Trapezoid rule) for modeling FM modulation and demodulation. The goal is to show that we retrieve back a good copy of the original message, using the first-order iterators, thus validating the use of these expressions in other communication engineering problems, where reliable numerical algorithms for differentiation and integration are needed in the simulation of different modulation-demodulation schemes.

As pointed out in **Pb. 3.35**, the FM modulated signal is given by:

$$u_{FM}(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) \tag{4.17}$$

The following *script M-file* details the steps in the FM modulation, if the signal in some normalized unit is given by the expression:

$$m(t) = \text{sinc}(10t) \tag{4.18}$$

Assuming that in the same units, we have $f_c = k_f = 25$.

The second part of the program follows the demodulation process: the phase of the modulated signal is unwrapped, and the demodulated signal is obtained by differentiating this phase, while subtracting the carrier phase, which is linear in time.

```
fc=25;kf=25;tlowb=-1;tupb=1;
t=tlowb:0.0001:tupb;
p=length(t);
dt=(tupb-tlowb)/(p-1);

m=sinc(10*t);
subplot(2,2,1)
plot(t,m)
title('Message')
```

```
intm=zeros(1,p);
  for k=1:p-1
  intm(k+1)=intm(k)+0.5*dt*(m(k+1)+m(k));
  end
subplot(2,2,2)
plot(t,intm)
title('Modulation Phase')

uc=exp(j*(2*pi*fc*t+2*pi*kf*intm));
u=real(uc);
phase=unwrap(angle(uc))-2*pi*fc*t;
subplot(2,2,3)
plot(t,u)
axis([-0.15 0.15 -1 1])
title('Modulated Signal')

Dphase(1)=0;
  for k=1:p-1
  Dphase(k+1)=(2/dt)*(phase(k+1)-phase(k))-...
  Dphase(k);
  end
md=Dphase/(2*pi*kf);
subplot(2,2,4)
plot(t,md)
title('Reconstructed Message')
```

As can be observed by examining Figure 4.1, the results of the simulation are very good, giving confidence in the expressions of the iterators used.

4.6 A Better Numerical Integrator: Simpson's Rule

Prior to discussing Simpson's rule for integration, we shall derive, for a simple case, an important geometrical result.

THEOREM
The area of a parabolic segment is equal to 2/3 of the area of the circumscribed parallelogram.

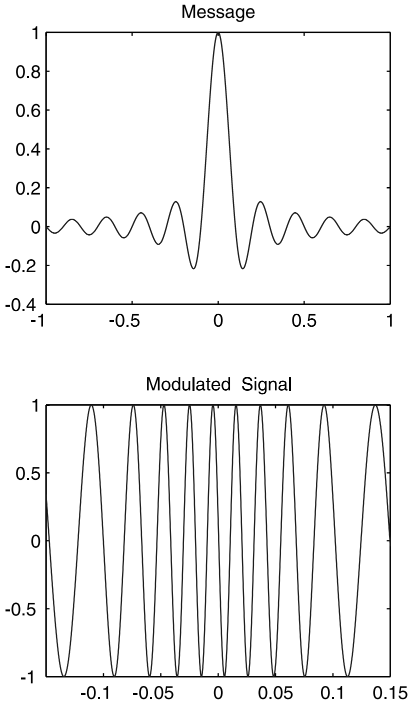

FIGURE 4.1
Simulation of the modulation and demodulation of an FM signal.

PROOF We prove this general theorem in a specialized case, for the purpose of making the derivation simple; however, the result is true for the most general case. Referring to Figure 4.2, we want to show that the area bounded by the x-axis and the parabola is equal to 2/3 the area of the ABCD rectangle. Now the details:

The parabola in Figure 4.2 is described by the equation:

$$y = ax^2 + b \tag{4.19}$$

It intersects the x-axis at the points $(-(-b/a)^{1/2}, 0)$ and $((-b/a)^{1/2}, 0)$, and the y-axis at the point $(0, b)$. The area bounded by the x-axis and the parabola is then simply the following integral:

$$\int_{-(-b/a)^{1/2}}^{(-b/a)^{1/2}} (ax^2 + b)dx = \frac{4}{3}\frac{b^{3/2}}{(-a)^{1/2}} \tag{4.20}$$

The area of the ABCD rectangle is: $b(2(-b/a)^{1/2}) = \dfrac{2b^{3/2}}{(-a)^{1/2}}$, which establishes the theorem.

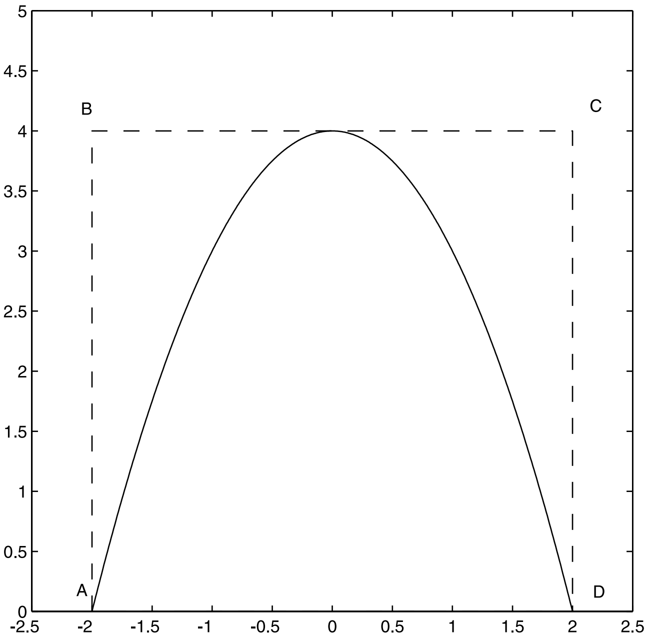

FIGURE 4.2
A parabolic segment and its circumscribed parallelogram.

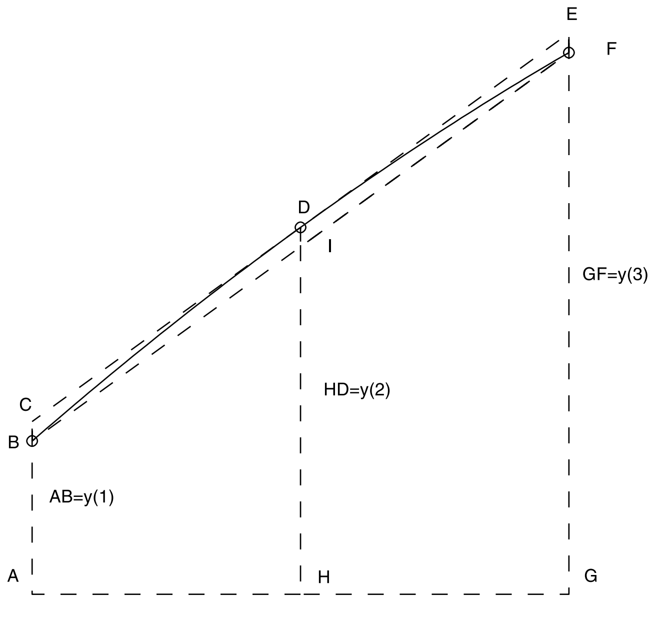

FIGURE 4.3
The first two slices in the Simpson's rule construct. AH = HG = Δx.

Simpson's Algorithm: We shall assume that the interval of integration is sampled at an odd number of points ($2N + 1$), so that we have an even number of intervals. The algorithm groups the intervals in pairs.

Referring to Figure 4.3, the points A, H, and G are the first three points in the sampled x-interval. The assumption underlying Simpson's rule is that the curve passing through the points B, D, and F, on the curve of the integrand, can have their locations approximated by a parabola. The line CDE is tangent to this parabola at the point D.

Under the above approximation, the value of the integral of the y-function between the points A and G is then simply the sum of the area of the trapezoid ABFG plus 2/3 the area of the parallelogram BCEF, namely:

$$\begin{aligned}\text{Area of the first two slices} &= \Delta x(y(1)+y(3)) + \frac{4\Delta x}{3}\left(y(2)-\left(\frac{y(1)+y(3)}{2}\right)\right) \\ &= \frac{\Delta x}{3}(y(1)+4y(2)+y(3))\end{aligned} \tag{4.21}$$

In a similar fashion, we can find the area of the third and fourth slices,

$$\text{Area of the third and fourth slices} = \frac{\Delta x}{3}(y(3)+4y(4)+y(5)) \tag{4.22}$$

Continuing for each successive pair of slices, we obtain for the total integral, or total area of all slices, the expression:

$$\text{Total area of all slices} = \frac{\Delta x}{3}\begin{pmatrix} y(1)+4y(2)+2y(3)+4y(4)+2y(5)+\dots \\ \dots \quad +\dots+4y(2N)+y(2N+1)\end{pmatrix} \tag{4.23}$$

that is, the weights are equal to 1 for the first and last elements, equal to 4 for even elements, and equal to 2 for odd elements.

Example 4.7

Using Simpson's rule, compute the integral of $\sin(x)$ over the interval $0 \le x \le \pi$.

Solution: Edit and execute the following *script M-file*:

```
a=0;b=pi;N=4;
x=linspace(a,b,2*N+1);
y=sin(x);
  for k=1:2*N+1
    if k==1 | k==2*N+1
    w(k)=1;
    elseif rem(k,2)==0
    w(k)=4;
    else
    w(k)=2;
    end
  end
Intsimp=((b-a)/(3*(length(x)-1)))*sum(y.*w)
```

Now compare the above answer with the one you obtain if you use the Trapezoid rule, by entering the command: `Inttrapz=trapz(x,y)`.

In-Class Exercise

Pb. 4.33 In the above derivation of Simpson's method, we constructed the algorithm by determining the weights sequence. Reformulate this algorithm into an equivalent iterator format.

Homework Problems

In this chapter, we surveyed three numerical techniques for computing the integral of a function. We observed that the different methods lead to different levels of accuracy. In Section 6.8, we derive formulas for estimating the accuracy of the different methods discussed here. However, and as noted previously, more accurate techniques than those presented here exist for calculating integrals numerically; many of these are in the MATLAB library and are covered in numerical analysis courses. In particular, familiarize yourself, using the help folder, with the commands **quad** and **quad8**.

Pb. 4.34 The goal of this problem, using the **quad8** command, is to develop a *function M-file* for the Gaussian distribution function of probability theory.

The Gaussian probability density function is given by:

$$f_X(x) = \frac{1}{(2\pi)^{1/2}\sigma_X}\exp\left[-\frac{(x-a_X)^2}{2\sigma_X^2}\right]$$

where $-\infty < a_X < \infty$, $0 < \sigma_X$ are constants, and are equal to the mean and the square root of the variance of x, respectively.

The Gaussian probability distribution function is defined as:

$$F_X(x) \equiv \int_{-\infty}^{x} f_X(\zeta)d\zeta$$

Through a change of variable (specify it!), the Gaussian probability distribution function can be written as a function of the normalized distribution function,

$$F_X(x) = F\left(\frac{x - a_X}{\sigma_X}\right)$$

where

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{\xi^2}{2}\right) d\xi$$

a. Develop the *function M-file* for the normal distribution function.

b. Show that for negative values of x, we have:

$$F(-x) = 1 - F(x)$$

c. Plot the normalized distribution function for values of x in the interval $0 \le x \le 5$.

Pb. 4.35 The computation of the arc length of a curve can be reduced to a one-dimensional integration. Specifically, if the curve is described parametrically, then the arc length between the adjacent points $(x(t), y(t), z(t))$ and the point $(x(t + \Delta t), y(t + \Delta t), z(t + \Delta t))$ is given by:

$$\Delta s = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,\Delta t$$

giving immediately for the arc length from t_0 to t_1, the expression:

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

a. Calculate the arc length of the curve described by: $x = t^2$ and $y = t^3$ between the points: $t = 0$ and $t = 3$.

b. Assuming that a 2-D curve is given in polar coordinates by $r = f(\theta)$, and then noting that:

$$x = f(\theta)\cos(\theta) \quad \text{and} \quad y = f(\theta)\sin(\theta)$$

use the above expression for the arc length (here the parameter is θ) to derive the formula for the arc length in polar coordinates to be

$$s = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

c. Use the result of (**b**) above to derive the length of the cardioid $r = a(1 + \cos(\theta))$ between the angles 0 and π.

Pb. 4.36 In **Pb. 3.27**, you plotted the Fermi-Dirac distribution. This curve represents the average population of fermions in a state with energy ε (ignore for the moment the internal quantum numbers of the fermions). As you would have noticed, this quantity is always smaller or equal to one. This is a manifestation of Pauli's exclusion principle, which states that no two fermions can be simultaneously in the same state. This, of course, means that even at zero absolute temperature, the momentum of almost all fermions is not zero; that is, we cannot freeze the thermal motion of all electrons at absolute zero. This fact is fundamental to our understanding of metals and semiconductors, and will be the subject of detailed studies in courses on physical electronics.

In nature, on the other hand, there is another family of particles that behaves quite the opposite; they are called Bosons. These particles are not averse to occupying the same state; moreover, they have a strong affinity, under the proper conditions, to aggregate in the lowest energy state available. When this happens, we say that the particles formed a Bose condensate. This phenomenon has been predicted theoretically to occur both on the laboratory scale and in some astrophysical objects (called neutron stars). The phenomena of superconductivity, superfluidity, and pion condensation, which occur in condensed or supercondensed matter, are manifestations of Bose condensates; however, it was only recently that this phenomenon has been observed to also occur experimentally in gaseous systems of atoms that were cooled in a process called laser cooling. The details of the cooling mechanism do not concern us at the moment, but what we seek to achieve in this problem is an understanding of the fashion in which the number density (i.e., the number per unit volume) of the condensate can become macroscopic. To achieve this goal, we shall use the skills that you have developed in numerically integrating and differentiating functions.

The starting point of the analysis is a formula that you will derive in future courses in statistical physics; it states that the number of particles in the condensate (i.e., the atoms in the gas that have momentum zero) can be written, for a noninteracting Bosons system, as:

$$n_{condensate} = n - \frac{1}{\lambda_T^3} g_{3/2}(z)$$

where λ_T is a quantity proportional to $T^{-1/2}$, n is the total number density, and the second term on the RHS of the equation represents the number density of the particles not in the condensate (i.e., those particles whose momentum is not zero). The function $g_{3/2}(z)$ is defined such that:

$$g_{3/2}(z) = z\frac{\partial}{\partial z}g_{5/2}(z)$$

where

$$g_{5/2}(z) = -\frac{4}{\sqrt{\pi}}\int_0^\infty dxx^2 \ln(1 - z\exp(-x^2))$$

and z, for physical reasons, always remains in the interval $0 < z \leq 1$.

a. Plot $g_{5/2}(z)$ as a function of z over the interval $0 < z \leq 1$.

b. Plot $g_{3/2}(z)$ over the same interval and find its maximum value.

c. As n increases or T decreases, the second term on the rhs of the population equation keeps adjusting the value of z so that the two terms on the RHS cancel each other, thus keeping $n_{condensate} = 0$. However, at some point, z reaches the value 1, which is its maximum value and the second term on the RHS cannot increase further. At this point, $n_{condensate}$ starts building up with any increase in the total number density. The value of the total density at which this starts happening is called the threshold value for the condensate formation. Prove that this threshold is given by: $n^{threshold}\lambda_T^3 = 2.612$.

4.7 Numerical Solutions of Ordinary Differential Equations

Ordinary linear differential equations are of the form:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1(t)\frac{dy}{dt} + a_0(t)y = u(t) \tag{4.24}$$

The a's are called the coefficients and $u(t)$ is called the source (or input) term.

Ordinary differential equations (ODEs) show up in many problems of electrical engineering, particularly in circuit problems where, depending on the circuit element, the potential across it may depend on the deposited charge, the current (which is the time derivative of the charge), or the derivative of the current (i.e., the second time derivative of the charge); that is, in the same equation, we may have a function and its first- and second-order derivatives. To focus this discussion, let us start by writing the potential difference across the passive elements of circuit theory. Specifically, the voltage drops across a resistor, capacitor, or inductor are given as follows:

1. The voltage across a resistor is given, using Ohm's law, by:

$$V_R(t) = RI(t) \tag{4.25}$$

   where R is the resistance, I is the current, and where R is measured in Ohms.

2. The voltage across a capacitor is proportional to the magnitude of the charge that accumulates on either plate, that is:

$$V_C(t) = \frac{Q(t)}{C} = \frac{\int I(t)dt}{C} \tag{4.26}$$

   The second equality reflects the relation of the current to the charge. C is the capacitance and, as previously pointed out, is measured in Farads.

3. The voltage across an inductor can be deduced from Lenz's law, which stipulates that the voltage across an inductor is proportional to the time derivative of the current going through it:

$$V_L(t) = L\frac{dI(t)}{dt} \tag{4.27}$$

   where L is the inductance and is measured in Henrys.

From these expressions for the voltage drop across each of the passive elements in a circuit, and using the Kirchoff voltage law, it is then an easy matter to write down the differential equations describing, for example, a series RC or an RLC circuit.

RC Circuit: Referring to the RC circuit diagram in Figure 4.4, the differential equation describing the voltage across the capacitor is given by:

$$RC\frac{dV_C}{dt} + V_C = V_s(t) \tag{4.28}$$

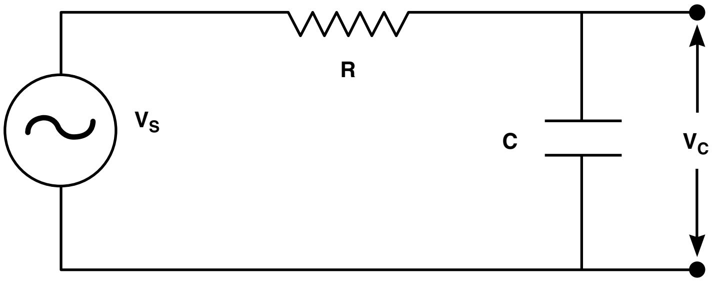

FIGURE 4.4
RC circuit with an ac source.

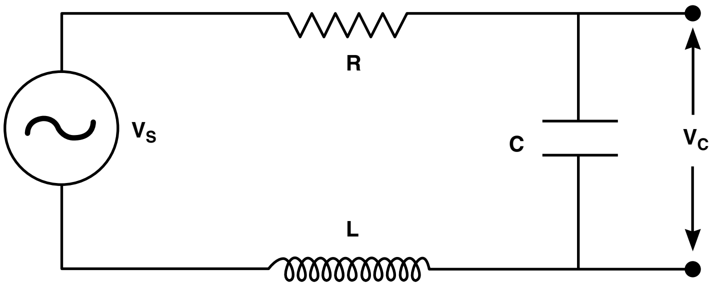

FIGURE 4.5
RLC circuit with ac source.

RLC Circuit: Referring to the *RLC* circuit in Figure 4.5, the voltage across the capacitor is described by the ODE:

$$LC\frac{d^2V_c}{dt^2}+RC\frac{dV_C}{dt}+V_C=V_s(t) \tag{4.29}$$

Numerically solving these and other types of ODEs will be the subject of the remainder of this section. In Section 4.7.1, we consider first-order iterators to represent the different-order derivatives, apply this algorithm to solve the above types of problems, and conclude by pointing out some of the limitations of this algorithm. In Section 4.7.2, we discuss higher-order iterators, particularly the Runge-Kutta technique. In Section 4.7.3, we familiarize ourselves with the use of standard MATLAB solvers for ODEs.

4.7.1 First-Order Iterator

In Section 4.5, we found an improved expression for the numerical differentiator, $D(k)$:

$$D(k)=\frac{2}{\Delta t}[y(k)-y(k-1)]-D(k-1) \tag{4.16}$$

which functionally corresponded to the inverse of the Trapezoid rule for integration. (Note that the independent variable here is *t*, and not *x*.)

Applying this first-order differentiator in cascade leads to an expression for the second-order differentiator, namely:

$$\begin{aligned}D2(k)&=\frac{2}{\Delta t}[D(k)-D(k-1)]-D2(k-1)\\&=\frac{4}{(\Delta t)^2}[y(k)-y(k-1)]-\frac{4}{\Delta t}D(k-1)-D2(k-1)\end{aligned} \tag{4.30}$$

Example 4.8

Find the first-order iterative scheme to solve the first-order differential equation given by:

$$a(t)\frac{dy}{dt} + b(t)y = u(t) \tag{4.31}$$

with the initial condition $y(t_1)$ specified.

Solution: Substituting Eq. (4.16) for the numerical differentiator in the differential equation, we deduce the following first-order difference equation for $y(k)$:

$$y(k) = \left[\frac{2a(k)}{\Delta t} + b(k)\right]^{-1}\left[\frac{2a(k)y(k-1)}{\Delta t} + a(k)D(k-1) + u(k)\right] \tag{4.32}$$

to which we should add, in the numerical subroutine, the expression for the first-order differentiator $D(k)$ as given by Eq. (4.16). The initial condition for the function at the origin of time, specify the first elements of the y and D arrays:

$$y(1) = y(t = t_1)$$
$$D(1) = (1/a(1))[u(1) - b(1)y(1)]$$

Application

To illustrate the use of the above algorithm, let us solve, over the interval $0 \leq t \leq 6$, for the potential across the capacitor in an RC circuit with an ac source; that is,

$$a\frac{dy}{dt} + y = \sin(2\pi t) \tag{4.33}$$

where $a = RC$ and $y(t = 0) = 0$.

Solution: Edit and execute the following *script M-file*, for $a = 1/(2\pi)$:

```
tin=0;
tfin=6;
t=linspace(tin,tfin,3000);
N=length(t);
y=zeros(1,N);
```

```
dt=(tfin-tin)/(N-1);
u=sin(t);
a=(1/(2*pi))*ones(1,N);
b=ones(1,N);
y(1)=0;
D(1)=(1/a(1))*(u(1)-b(1)*y(1));

  for k=2:N
  y(k)=((2*a(k)/dt+b(k))^(-1))*...
  (2*a(k)*y(k-1)/dt+a(k)D(k-1)+u(k));
  D(k)=(2/dt)*(y(k)-y(k-1))-D(k-1);
  end

plot(t,y,t,u,'--')
```

In-Class Exercise

Pb. 4.37 Plot the amplitude of *y*, and its dephasing from *u*, as a function of *a* for large *t*.

Example 4.9

Find the first-order iterative scheme to solve the second-order differential equation given by:

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = u(t) \tag{4.34}$$

with initial conditions $y(t = 0)$ and $\left.\frac{dy}{dt}\right|_{t=0}$ given.

Solution: Substituting the above first-order expression of the iterators for the first-order and second-order numerical differentiators [respectively Eqs. (4.16) and (4.30), into Eq. (4.34)], we deduce the following iterative equation for $y(k)$:

$$\begin{gathered} y(k) = \left\{4\frac{a(k)}{(\Delta t)^2} + 2\frac{b(k)}{\Delta t} + c(k)\right\}^{-1} \times \\ \left\{y(k-1)\left[4\frac{a(k)}{(\Delta t)^2} + 2\frac{b(k)}{\Delta t}\right] + D(k-1)\left[4\frac{a(k)}{\Delta t} + b(k)\right] + a(k)D2(k-1) + u(k)\right\} \end{gathered} \tag{4.35}$$

This difference equation will be supplemented in the ODE numerical solver routine with the iterative equations for $D(k)$ and $D2(k)$, as given respectively by Eqs. (4.16) and (4.30), and with the initial conditions for the function and its derivative. The first elements for the y, D, and $D2$ arrays are given by:

$$y(1) = y(t = 0)$$

$$D(1) = \left.\frac{dy}{dt}\right|_{t=0}$$

$$D2(1) = (1 / a(1))(-b(1)D(1) - c(1)y(1) + u(1))$$

Application 1

To illustrate the use of the first-order iterator algorithm in solving a second-order ordinary differential equation, let us find, over the interval $0 \leq t \leq 16\pi$, the voltage across the capacitance in an RLC circuit, with an ac voltage source. This reduces to solve the following ODE:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \sin(\omega t) \tag{4.36}$$

where $a = LC$, $b = RC$, $c = 1$. Choose in some normalized units, $a = 1$, $b = 3$, $\omega = 1$, and let $y(t = 0) = y'(t = 0) = 0$.

Solution: Edit and execute the following *script M-file*:

```
tin=0;
tfin=16*pi;
t=linspace(tin,tfin,2000);
a=1;
b=3;
c=1;
w=1;
N=length(t);
y=zeros(1,N);
dt=(tfin-tin)/(N-1);
u=sin(w*t);
y(1)=0;
D(1)=0;
D2(1)=(1/a)*(-b*D(1)-c*y(1)+u(1));

  for k=2:N
```

```
    y(k)=((4*a/dt^2+2*b/dt+c)^(-1))*...
    (y(k-1)*(4*a/dt^2+2*b/dt)+D(k-1)*(4*a/dt+b)+...
    +a*D2(k-1)+u(k));

    D(k)=(2/dt)*(y(k)-y(k-1))-D(k-1);

    D2(k)=(4/dt^2)*(y(k)-y(k-1))-(4/dt)*D(k-1)-D2
    (k-1);

    end
 plot(t,y,t,u,'--')
```

The dashed curve is the temporal profile of the source term.

In-Class Exercise

Pb. 4.38 Plot the amplitude of y and its dephasing from u as function of a for large t, for $0.1 < a < 5$.

Application 2

Solve, over the interval $0 < t < 1$, the following second-order differential equation:

$$(1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 20y = 0 \tag{4.37}$$

with the initial conditions: $y(t = 0) = 3/8$ and $y'(t = 0) = 0$.

Then, compare your numerical result with the analytical solution to this problem:

$$y = \frac{1}{8}(35t^4 - 30t^2 + 3) \tag{4.38}$$

Solution: Edit and execute the following *script M-file*:

```
tin=0;
tfin=1;
t=linspace(tin,tfin,2000);
N=length(t);
a=1-t.^2;
```

```
b=-2*t;
c=20*ones(1,N);
y=zeros(1,N);
D=zeros(1,N);
dt=(tfin-tin)/(N-1);
u=zeros(1,N);
y(1)=3/8;
D(1)=0;
D2(1)=(1/a(1))*(-b(1)*D(1)-c(1)*y(1)+u(1));

  for k=2:N
  y(k)=((4*a(k)/dt^2+2*b(k)/dt+c(k))^(-1))*...
  (y(k-1)*(4*a(k)/dt^2+2*b(k)/dt)+D(k-1)...
  *(4*a(k)/dt+b(k))+a(k)*D2(k-1)+u(k));
  D(k)=(2/dt)*(y(k)-y(k-1))-D(k-1);
  D2(k)=(4/dt^2)*(y(k)-y(k-1))-(4/dt)*D(k-1)-...
  D2(k-1);
  end

yanal=(35*t.^4-30*t.^2+3)/8;

plot(t,y,t,yanal,'--')
```

As you will observe upon running this program, the numerical solution and the analytical solution agree very well.

NOTE The above ODE is that of the Legendre polynomial of order $l = 4$, encountered earlier in Chapter 2, in **Pb. 2.25**.

$$(1-t^2)\frac{d^2P_l}{dt^2} - 2t\frac{dP_l}{dt} + l(l+1)P_l = 0 \tag{4.39}$$

where

$$P_l(-t) = (-1)^l P_l(t) \tag{4.40}$$

Homework Problem

Pb. 4.39 The above algorithms assume that the source function is continuous. If it is not, we may encounter problems upon applying this algorithm over a transition region, as will be illustrated in the following problem.

Solve, over the interval $0 \le t \le 20$, the following first-order differential equation for $a = 2$ and $a = 0.5$:

$$a\frac{dy}{dt} + y = 1$$

where $y(0) = 0$. (Physically, this would correspond to the charging of a capacitor from a dc source connected suddenly to the battery at time zero. Here, y is the voltage across the capacitor, and $a = RC$.)

NOTE The analytic solution to this problem is $y = 1 - \exp(-t/a)$.

4.7.2 Higher-Order Iterators: The Runge-Kutta Method*

In this subsection, we want to explore the possibility that if we sampled the function n-times per step, we will obtain a more accurate solution to the ODE than that obtained from the first-order iterator for the same value of Δt.

To focus the discussion, consider the ODE:

$$y'(t) = f(t, y(t)) \tag{4.41}$$

Higher-order ODEs can be reduced, as will be shown at the end of the subsection, to a system of equations having the same functional form as Eq. (4.41). The derivation of a technique using higher-order iterators will be shown below in detail for two evaluations per step. Higher-order recipes can be found in most books on numerical methods for ODE.

The key to the Runge-Kutta method is to properly arrange each of the evaluations in a particular step to depend on the previous evaluations in the same step.

In the second-order model:

if: $$k_1 = f(t(n), y(t(n)))(\Delta t) \tag{4.42}$$

then: $$k_2 = f(t(n) + \alpha\Delta t, y(t(n)) + \beta k_1)(\Delta t) \tag{4.43}$$

and $$y(t(n+1)) = y(t(n)) + ak_1 + bk_2 \tag{4.44}$$

where a, b, α, and β are unknown parameters to be determined. They should be chosen such that Eq. (4.44) is correct to order $(\Delta t)^3$.

To find a, b, α, and β, let us compute $y(t(n + 1))$ in two different ways. First, Taylor expanding the function $y(t(n + 1))$ to order $(\Delta t)^2$, we obtain:

$$y(t(n+1)) = y(t(n)) + \frac{dy(t(n))}{dt}(\Delta t) + \frac{d^2y(t(n))}{dt^2}\frac{(\Delta t)^2}{2} \tag{4.45}$$

Recalling Eq. (4.41) and the total derivative expression of a function in two variables as function of the partial derivatives, we have:

$$\frac{dy(t(n))}{dt} = f(t(n), y(t(n))) \tag{4.46}$$

$$\begin{aligned}\frac{d^2y(t(n))}{dt^2} &= \frac{d}{dt}\left(\frac{dy(t(n))}{dt}\right) \\ &= \frac{\partial f(t(n), y(t(n)))}{\partial t} + \frac{\partial f(t(n), y(t(n)))}{\partial y} f(t(n), y(t(n)))\end{aligned} \tag{4.47}$$

Combining Eqs. (4.45) to (4.47), it follows that to second order in (Δt):

$$\begin{aligned}&y(t(n+1)) = y(t(n)) + f(t(n), y(t(n)))(\Delta t) + \\ &+\left[\frac{\partial f(t(n), y(t(n)))}{\partial t} + \frac{\partial f(t(n), y(t(n)))}{\partial y} f(t(n), y(t(n)))\right]\frac{(\Delta t)^2}{2}\end{aligned} \tag{4.48}$$

Next, let us Taylor expand k_2 to second order in (Δt). This results in:

$$\begin{aligned}&k_2 = f(t(n) + \alpha\Delta t, y(t(n)) + \beta k_1)(\Delta t) = \\ &\left[f(t(n), y(t(n))) + \alpha(\Delta t)\frac{\partial f(t(n), y(t(n)))}{\partial t} + (\beta k_1)\frac{\partial f(t(n), y(t(n)))}{\partial y}\right](\Delta t)\end{aligned} \tag{4.49}$$

Combining Eqs. (4.42), (4.44), and (4.49), we get the other expression for $y(t(n + 1))$, correct to second order in (Δt):

$$\begin{aligned}&y(t(n+1)) = y(t(n)) + (a+b)f(t(n), y(t(n)))(\Delta t) + \\ &+\alpha b\frac{\partial f(t(n), y(t(n)))}{\partial t}(\Delta t)^2 + b\beta\frac{\partial f(t(n), y(t(n)))}{\partial y} f(t(n), y(t(n)))(\Delta t)^2\end{aligned} \tag{4.50}$$

Now, comparing Eqs. (4.48) and (4.50), we obtain the following equalities:

$$a + b = 1; \quad \alpha b = 1/2; \quad b\beta = 1 \tag{4.51}$$

We have three equations in four unknowns; the usual convention is to fix *a* = 1/2, giving for the other quantities:

$$b = 1/2; \quad \alpha = 1; \quad \beta = 1 \tag{4.52}$$

finally leading to the following expressions for the second-order iterator and its parameters:

$$k_1 = f(t(n), y(t(n)))(\Delta t) \tag{4.53a}$$

$$k_2 = f(t(n) + \Delta t, y(t(n)) + k_1)(\Delta t) \tag{4.53b}$$

$$y(t(n+1)) = y(t(n)) + \frac{k_1 + k_2}{2} \tag{4.53c}$$

Next, we give, without proof, the famous fourth-order iterator Runge-Kutta expression, one of the most widely used algorithms for solving ODEs in the different fields of science and engineering:

$$k_1 = f(t(n),\ y(n))(\Delta t) \tag{4.54a}$$

$$k_2 = f(t(n) + \Delta t / 2, y(t(n)) + k_1 / 2)(\Delta t) \tag{4.54b}$$

$$k_3 = f(t(n) + \Delta t / 2, y(t(n)) + k_2 / 2)(\Delta t) \tag{4.54c}$$

$$k_4 = f(t(n) + \Delta t, y(t(n)) + k_3)(\Delta t) \tag{4.54d}$$

$$y(t(n+1)) = y(t(n)) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \tag{4.54e}$$

The last point that we need to address before leaving this subsection is what to do in case we have an ODE with higher derivatives than the first. The answer is that we reduce the n^{th}-order ODE to a system of n first-order ODEs.

Example 4.10

Reduce the following second-order differential equation into two first-order differential equations:

$$ay'' + by' + cy = \sin(t) \tag{4.55}$$

with the initial conditions: $y(t = 0) = 0$ and $y'(t = 0) = 0$

(where the prime and double primes superscripted functions refer, respectively, to the first and second derivative of this function).

Solution: Introduce the two-dimensional array z, and define

$$z(1) = y \tag{4.56a}$$

$$z(2) = y' \tag{4.56b}$$

The system of first-order equations now reads:

$$z'(1) = z(2) \tag{4.57a}$$

$$z'(2) = (1/a)(\sin(t) - bz(2) - cz(1)) \tag{4.57b}$$

Example 4.11

Using the fourth-order Runge-Kutta iterator, numerically solve the same problem as in Application 1 following Example 4.9.

Solution: Edit and save the following *function M-files:*

```
function zp=zprime(t,z)
a=1; b=3; c=1;
zp(1,1)=z(2,1);
zp(2,1)=(1/a)*(sin(t)-b*z(2,1)-c*z(1,1));
```

The above file specifies the system of ODE that we are trying to solve.

Next, in another *function M-file*, we edit and save the fourth-order Runge-Kutta algorithm, specifically:

```
function zn=prk4(t,z,dt)
k1=dt*zprime(t,z);
k2=dt*zprime(t+dt/2,z+k1/2);
k3=dt*zprime(t+dt/2,z+k2/2);
k4=dt*zprime(t+dt,z+k3);
zn=z+(k1+2*k2+2*k3+k4)/6;
```

Finally, edit and execute the following *script M-file*:

```
yinit=0;
ypinit=0;
z=[yinit;ypinit];
```

```
tinit=0;
tfin=16*pi;
N=1001;
t=linspace(tinit,tfin,N);
dt=(tfin-tinit)/(N-1);

  for k=1:N-1
  z(:,k+1)=prk4(t(k),z(:,k),dt);
  end

plot(t,z(1,:),t,sin(t),'--')
```

In the above plot, we are comparing the temporal profiles of the voltage difference across the capacitor with that of the source voltage.

4.7.3 MATLAB ODE Solvers

MATLAB has many ODE solvers, **`ODE23`** and **`ODE45`** being most commonly used. **`ODE23`** is based on a pair of second-order and third-order Runge-Kutta methods running simultaneously to solve the ODE. The program automatically corrects for the step size if the answers from the two methods have a discrepancy at any stage of the calculation that will lead to a larger error than the allowed tolerance.

To use this solver, we start by creating a *function M-file* that includes the system of equations under consideration. This function is then called from the command window with the **`ODE23`** or **`ODE45`** command.

Example 4.12

Using the MATLAB ODE solver, find the voltage across the capacitor in the *RLC* circuit of Example 4.11, and compare it to the source potential time-profile.

Solution: Edit and save the following *function M-file:*

```
function zp=RLC11(t,z)
a=1;
b=3;
c=1;
zp(1,1)=z(2,1);
zp(2,1)=(1/a)*(sin(t)-b*z(2,1)-c*z(1,1));
```

Next, edit and execute the following *script M-file:*

```
tspan=[0 16*pi];
```

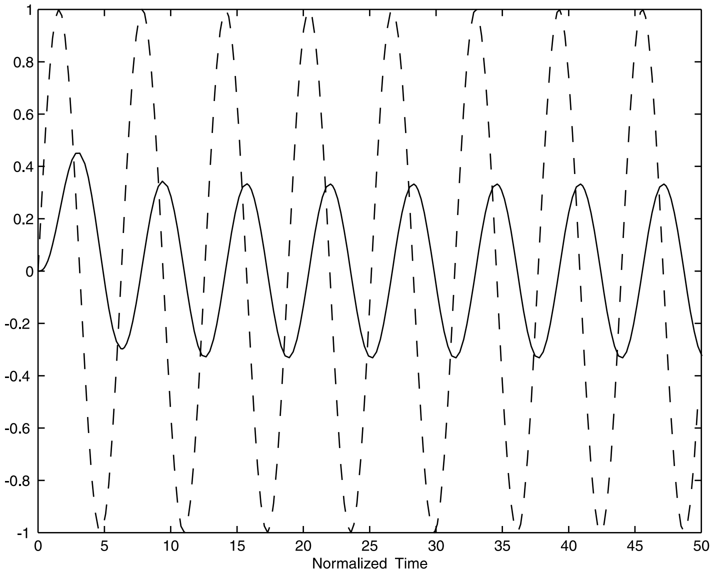

FIGURE 4.6
The potential differences across the source (dashed line) and the capacitor (solid line) in an RLC circuit with an ac source. [LC = 1, RC = 3, and $V_s = \sin(t)$].

```
zin=[0;0];
[t,z]=ode23('RLC11',tspan,zin);
plot(t,z(:,1),t,sin(t))
xlabel('Normalized Time')
```

The results are plotted in Figure 4.6. Note the phase shift between the two potential differences.

Example 4.13

Using the MATLAB ODE solver, solve the problem of relaxation oscillations in lasers.

Solution: Because many readers may not be familiar with the statement of the problem, let us first introduce the physical background to the problem.

A simple gas laser consists of two parallel mirrors sandwiching a tube with a gas, selected for having two excitation levels separated in energy by an amount equal to the energy of the photon quantum that we are attempting to have the laser system produce. In a laser (light amplification by stimulated emission radiation), a pumping mechanism takes the atom to the upper excited level. However, the atom does not stay in this level; it decays to lower levels, including the lower excited level, which is of interest for two reasons: (1) the finite lifetime of all excited states of atoms; and (2) stimulated emission, a quantum mechanical phenomenon, associated with the statistics of the photons (photons are bosons), which predicts that in the presence of an electromagnetic field having a frequency close to that of the frequency of the photon emitted in the transition between the upper excited and lower excited state, the atom emission rate is enhanced and this enhancement is larger, the more photons that are present in its vicinity. On the other hand, the rate of change of the number of photons is equal to the rate generated from the decay of the atoms due to stimulated emission, minus the decay due to the finite lifetime of the photon in the resonating cavity. Putting all this together, one is led, in the simplest approximation, to write what are called the rate equations for the number of atoms in the excited state and for the photon numbers in the cavity. These coupled equations, in their simplest forms, are given by:

$$\frac{dN}{dt} = P - \frac{N}{\tau_{decay}} - BnN \tag{4.58}$$

$$\frac{dn}{dt} = -\frac{n}{\tau_{cavity}} + BnN \tag{4.59}$$

where N is the normalized number of atoms in the atom's upper excited state, n is the normalized number of photons present, P is the pumping rate, τ_{decay} is the atomic decay time from the upper excited state, due to all effects except that of stimulated emission, τ_{cavity} is the lifetime of the photon in the resonant cavity, and B is the Einstein coefficient for stimulated emission.

These nonlinear differential equations describe the dynamics of laser operation. Now come back to relaxation oscillations in lasers, which is the problem at hand. Physically, this is an interplay between N and n. An increase in the photon number causes an increase in stimulated emission, which causes a decrease in the population of the higher excited level. This, in turn, causes a reduction in the photon gain, which tends to decrease the number of photons present, and in turn, decreases stimulated emission. This leads to the build-up of the higher excited state population, which increases the rate of change of photons, with the cycle resuming but such that at each new cycle the amplitude of the oscillations is dampened as compared with the cycle just before it, until finally the system reaches a steady state.

To compute the dynamics of the problem, we proceed into two steps. First, we generate the *function M-file* that contains the rate equations, and then proceed to solve these ODEs by calling the MATLAB ODE solver. We use typical numbers for gas lasers.

Specifically the *function M-file* representing the laser rate equations is given by:

```
function yp=laser1(t,y)
p=30;                          %pumping rate
gamma=10^(-2);                 %inverse natural lifetime
```

```
B=3;          %stimulated emission coefficient
c=30;         %inverse lifetime of photon in cavity

yp(1,1)=p-gamma*y(1,1)-B*y(1,1)*y(2,1);
yp(2,1)=-c*y(2,1)+B*y(1,1)*y(2,1);
```

The *script M-file* to compute the laser dynamics and thus simulate the relaxation oscillations is:

```
tspan=[0 3];
yin=[1 1];
[t,y]=ode23('laser1',tspan,yin);

subplot(3,1,1)
plot(t,y(:,1))
xlabel('Normalized Time')
ylabel('N')

subplot(3,1,2);
plot(t,y(:,2))
xlabel('Normalized Time')
ylabel('n')

subplot(3,1,3);
plot(y(:,1),y(:,2))
xlabel('N')
ylabel('n')
```

As can be observed in Figure 4.7, the oscillations, as predicted, damp-out after a while and the dynamical variables reach a steady state. The phase diagram, shown in the bottom panel, is an alternate method to show how the population of the atomic higher excited state and the photon number density reach the steady state.

Question: Compute analytically from Eqs. (4.58) and (4.59), the steady-state values for the higher excited state population and for the photon number, and compare with the numerically obtained asymptotic values.

In-Class Exercise

Pb. 4.40 By changing the values of the appropriate parameters in the above programs, find separately the effects of increasing or decreasing the value of

t_{cavity}, and the effect of the pumping rate on the magnitude and the persistence of the oscillation.

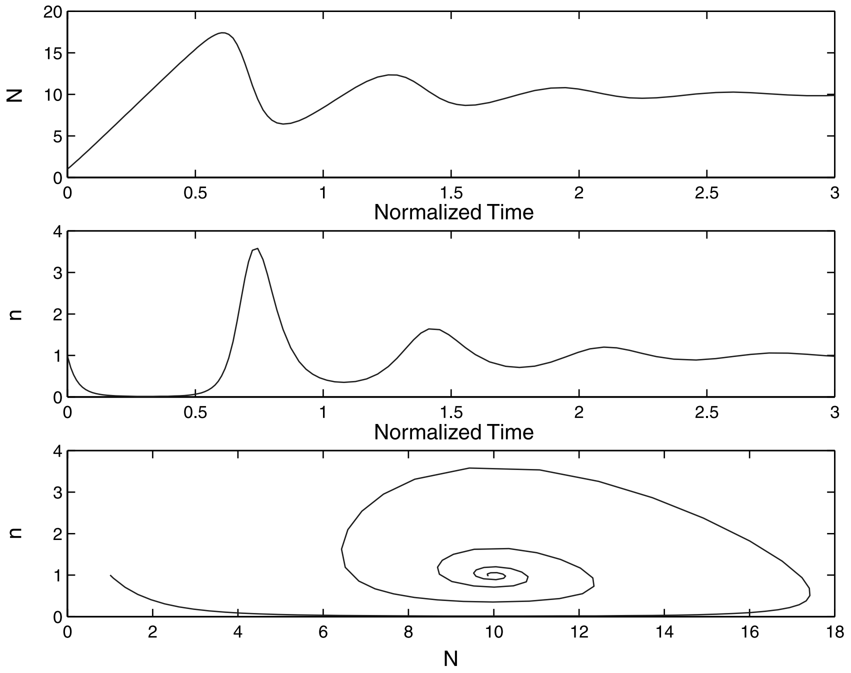

FIGURE 4.7
The dynamics of a laser in the relaxation oscillations regime. Top panel: Plot of the higher excited level atoms population as a function of the normalized time. Middle panel: Plot of the number of photons as a function of the normalized time. Bottom panel: Phase diagram of the photons number vs. the higher excited level atoms population.

Example 4.14

Using the rate equations developed in Example 4.13, simulate the Q-switching of a laser.

Solution: First, an explanation of the statement of the problem. In Example 4.13, we showed how, following an initial transient period whereby one observes relaxation oscillations, a laser, in the presence of steady pumping, reaches steady-state operation after a while. This is referred to as continuous wave (cw) operation of the laser. In this example, we shall explore the other mode of laser operation, the so-called pulsed mode. In this regime, the experimentalist, through a temporary modification in the absorption properties of the laser resonator, prevents the laser from oscillating, thus leading the higher excited state of the atom to keep building up its population to a very high level before it is allowed to emit any photons. Then, at the desired

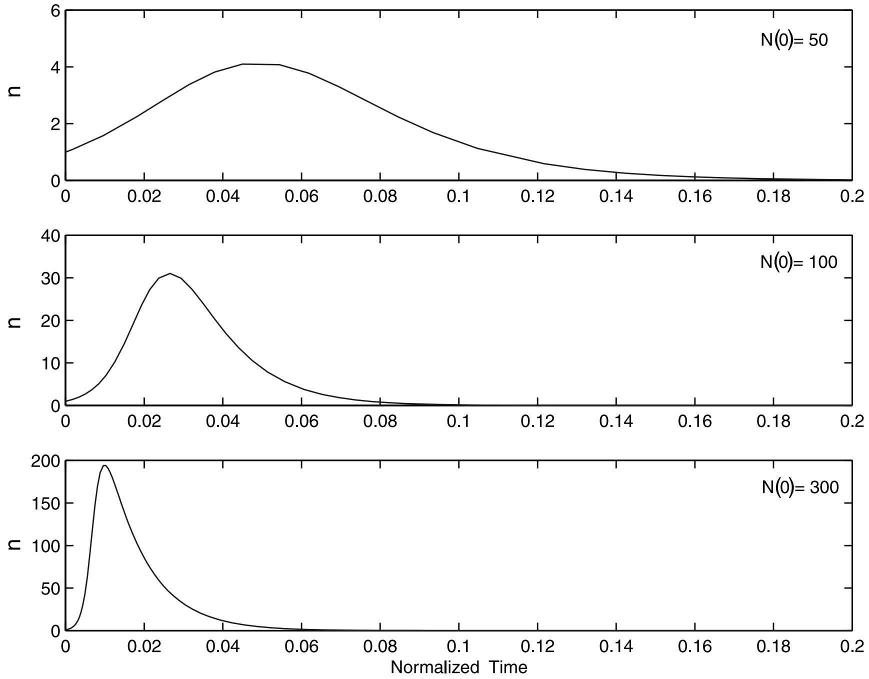

FIGURE 4.8
The temporal profile of the photon burst emitted in a Q-switched laser for different initial values of the excited level atoms population. Top panel: N(0) = 50. Middle panel: N(0) = 100. Botton panel: N(0) = 300.

moment, the laser resonator is allowed back to a state where the optical losses of the resonator are small, thus triggering the excited atoms to dump their stored energy into a short burst of photons. It is this regime that we propose to study in this example.

The laser dynamics are, of course, still described by the rate equations [i.e., Eqs. (4.58) and (4.59)]. What we need to modify from the previous problem are the initial conditions for the system of coupled ODE. At the origin of time [i.e., $t = 0$ or the triggering time, $N(0)$], the initial value of the population of the higher excited state of the atom is in this instance (because of the induced build-up) much larger than that of the corresponding photon population $n(0)$. Figure 4.8 shows the ensuing dynamics for the photon population for different values of $N(0)$. We assumed in these simulations the following values for the parameters in the laser1 *function M-file* **`(p=0; B=3; c=100; gamma=0.01)`**.

In examining Figure 4.8, we observe that as $N(0)$ increases, the pulse's total energy increases — as it should since more energy is stored in the excited atoms. Furthermore, the duration of the produced pulse (i.e., the width of the pulse temporal profile) narrows, and the delay in the position of its peak from the trigger time gets to be smaller as the number of the initial higher excited level atoms increases.

In-Class Exercise

Pb. 4.41 Investigate how changes in the values of τ_{cavity} and τ_{decay} modify the duration of the produced pulse. Plot the Q-switched pulse duration as function of each of these variables.

4.8 MATLAB Commands Review

`diff` Takes the difference between consecutive elements in an array.

`ode23` and **`ode45`** Ordinary Differential Equations solvers.

`prod` Finds the product of all the elements belonging to an array.

`quad` and **`quad8`** Integrate a function between fixed limits.

`semilogy` Plot a graph with the abscissa in linear scale, while the ordinate is in a logarithmic scale.

`sum` Sums all the elements of an array.

`trapz` Finds the integral using the Trapezoid rule.

5

Root Solving and Optimization Methods

In this chapter, we first learn some elementary numerical techniques and the use of the **fsolve** and **fzero** commands from the MATLAB library to obtain the real roots (or zeros) of an arbitrary function. Then, we discuss the use of the MATLAB command **roots** for finding all roots of a polynomial. Following this, we consider the Golden Section method and the **fmin** and **fmins** MATLAB commands for optimizing (finding the minimum or maximum value of a function) over an interval. Our discussions pertain exclusively to problems with one and two variables (input) and do not include the important problem of optimization with constraints.

5.1 Finding the Real Roots of a Function

This section explores the different categories of techniques for finding the real roots (zeros) of an arbitrary function. We outline the required steps for computing the zeros using the graphical commands, the numerical techniques known as the Direct Iterative and the Newton-Raphson methods, and the built-in **fsolve** and **fzero** functions of MATLAB.

5.1.1 Graphical Method

In the graphical method, we find the zeros of a single variable function by implementing the following steps:

1. Plot the particular function over a suitable domain.
2. Identify the neighborhoods where the curve crosses the x-axis (there may be more than one point); and at each such point, the following steps should be independently implemented.
3. Zoom in on the neighborhood of each intersection point by repeated application of the MATLAB **axis** or **zoom** commands.

4. Use the crosshair of the `ginput` command to read the coordinates of the intersection.

In problems where we desire to find the zeros of a function that depends on two input variables, we follow (conceptually) the same steps above, but use 3-D graphics.

In-Class Exercises

Pb. 5.1 Find graphically the two points in the *x-y* plane where the two surfaces, given below, intersect:

$$z_1 = 7 - \sqrt{25 + x^2 + y^2}$$

$$z_2 = 4 - 2x - 4y$$

(*Hint:* Use the techniques of surface and contour renderings, detailed in Chapter 1, to plot the zero height contours for both surfaces; then read off the intersections of the resulting curves.)

Pb. 5.2 Verify your graphical answer to **Pb. 5.1** with that you would obtain analytically.

5.1.2 Numerical Methods

This chapter subsection briefly discusses two techniques for finding the zeros of a function in one variable, namely the Direct Iterative and the Newton-Raphson techniques. We do not concern ourselves too much, at this point, with an optimization of the routine execution time, nor with the inherent limits of each of the methods, except in the most general way. Furthermore, to avoid the inherent limits of these techniques in some pathological cases, we assume that we plot each function under consideration, verify that it crosses the x-axis, and satisfy ourselves in an empirical way that there does not seem to be any pathology around the intersection point before we embark on the application of the following algorithms. These statements will be made more rigorous to you in future courses in numerical analysis.

5.1.2.1 The Direct Iterative Method

This is a particularly useful technique when the equation $f(x) = 0$ can be cast in the form:

$$x = F(x) \tag{5.1}$$

$F(x)$ is then called an iteration function, and it can be used for the generation of the sequence:

$$x_{k+1} = F(x_k) \tag{5.2}$$

To guarantee that this method gives accurate results in a specific case, the function should be continuous and it should satisfy the contraction condition:

$$\left|F(x_n) - F(x_m)\right| \leq s\left|x_n - x_m\right| \tag{5.3}$$

where $0 \leq s < 1$; that is, the changes in the value of the function are smaller than the changes in the value of the arguments. To prove that under these conditions, the iterative function possesses a fixed point (i.e., that ultimately the difference between two successive iterations can be arbitrarily small) that can be immediately obtained from the above contraction condition [Eq. (5.3)].

PROOF Let the x_{guess} be the first term in the iteration, then:

$$\left|F(x_1) - F(x_{guess})\right| \leq s\left|x_1 - x_{guess}\right| \tag{5.4}$$

but since

$$F(x_{guess}) = x_1 \quad \text{and} \quad F(x_1) = x_2 \tag{5.5}$$

then

$$\left|x_2 - x_1\right| \leq s\left|x_1 - x_{guess}\right| \tag{5.6}$$

Similarly,

$$\left|F(x_2) - F(x_1)\right| \leq s\left|x_2 - x_1\right| \tag{5.7}$$

translates into

$$\left|x_3 - x_2\right| \leq s\left|x_2 - x_1\right| \leq s^2\left|x_1 - x_{guess}\right| \tag{5.8}$$

The argument can be extended to the $(m + 1)$-iteration, where we can assert that:

$$\left|x_{m+1} - x_m\right| \leq s^m\left|x_1 - x_{guess}\right| \tag{5.9}$$

but, because *s* is a non-negative number smaller than 1, the right-hand-side of the inequality in Eq. (5.9) can be made, for large enough value of *m*, arbitrarily small, and the above iterative procedure does indeed converge to a fixed point.

Example 5.1

Find the zero of the function:

$$y = x - \sin(x) - 1 \tag{5.10}$$

Solution: At the zero, the iterative form can be written as:

$$x(k) = \sin(x(k-1)) + 1 \tag{5.11}$$

The contraction property, required for the application of this method, is valid in this case because the difference between two sines is always smaller than the difference between their arguments. The fixed point can then be obtained by the following MATLAB program:

```
x(1)=1;                %value of the initial guess
for k=2:20
x(k)=sin(x(k-1))+1;
end
```

If we display the successive values of **x**, we obtain:

```
x
  Ans
    1.0000 1.8415 1.9636 1.9238 1.9383 1.9332 1.9350
    1.9344 1.9346 1.9345 1.9346 1.9346 1.9346 1.9346
    1.9346 1.9346 1.9346 1.9346 1.9346 1.9346
```

As can be noticed from the above printout, about 11 iterations were required to get the value of the fixed point accurate to one part per 10,000.

NOTE A more efficient technique to find the answer within a proscribed error tolerance is to write the program with the **while** command, where we can specify the tolerance level desired.

5.1.2.2 The Newton-Raphson Method

This method requires a knowledge of both the function and its derivative. The method makes use of the geometrical interpretation of the derivative being the tangent at a particular point, and that the tangent is the limit of the chord between two close points on the curve. It is based on the fact that if $f(x_1)$ and $f(x_2)$ have opposite signs and the function *f* is continuous on the interval

$[x_1, x_2]$, we know from the Intermediate Value theorem of calculus that there is at least one value x_c between x_1 and x_2, such that $f(x_c) = 0$. A sufficient condition for this method to work is that $f'(x)$ and $f''(x)$ have constant sign on an open interval that contains the solution $f(x) = 0$; in that case, any starting point that is close enough to the solution will give successive Newton's approximations that converge to the solution.

Let x_{guess} and x have the same meaning as in the iterative method; therefore, $f(x) = 0$, and the definition of the derivative results in the equation:

$$x = x_{guess} - \frac{f(x_{guess})}{f'(x_{guess})} \tag{5.12}$$

This relation can now be the basis of an iterative function given by:

$$x(k) = x(k-1) - \frac{f(x(k-1))}{f'(x(k-1))} \tag{5.13}$$

The fixed point can be obtained, in general, for the same initial guess and tolerance, in a smaller number of iterations in the Newton-Raphson method than in the Direct Iteration method.

In-Class Exercise

Pb. 5.3 Write a routine to find the zero of the function $y = x - \sin(x) - 1$ using the Newton-Raphson algorithm.

Pb. 5.4 Compare the answers from the present algorithm with that of the Direct Iterative method, at each iteration step, in the search for the zeros of the function $y = x - \sin(x) - 1$, and comment on which of the two methods appears to be more effective and converges faster.

Example 5.2

Apply the Newton-Raphson method to find the voltage-current relation in a diode circuit with an ac voltage source.

Solution: The diode is a nonlinear semiconductor electronics device with a voltage current curve that is described, for voltage values larger than the reverse breakdown potential (a negative quantity), by:

$$i = I_s(e^{v/k'T} - 1) \tag{5.14}$$

where I_s is the reverse saturation current (which is typically on the order of 10^{-6} mA), and kT is the average thermal energy of an electron divided by its

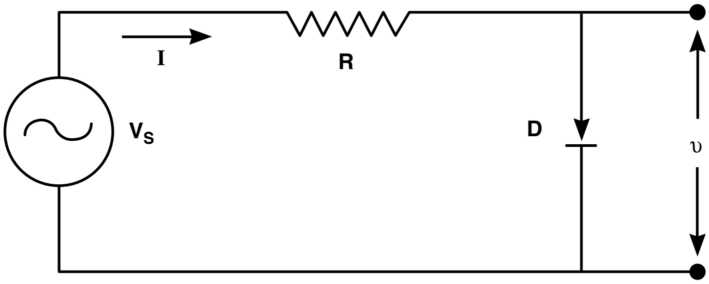

FIGURE 5.1
The diode semi-rectifier circuit.

charge at the diode operating temperature (equal to 1/40 V at room temperature). An important application of this device is to use it as a rectifier (a device that passes the current in one direction only). (Can you think of a practical application for this device?)

The problem we want to solve is to find the current through the circuit (shown in Figure 5.1) as a function of time if we are given a sinusoidal time-dependent source potential.

The other equation, in addition to Eq. (5.14) that we need in order to set the problem, is Ohm's law across *R*. This law, as previously noted, states that the current passing through a resistor is equal to the potential difference across the resistor, divided by the value of the resistance:

$$i = \frac{V_s - v}{R} \tag{5.15}$$

Eliminating the current from Eqs. (5.14) and (5.15), we obtain a nonlinear equation in the potential across the diode. Solving this problem is then reduced to finding the roots of the function *f* defined as:

$$f(v) = I_s[\exp(v / k'T) - 1] - \left(\frac{V_s - v}{R}\right) \tag{5.16}$$

where the potential across the diode is the unknown.

In the Newton-Raphson method, we also need for our iteration the derivative of this function:

$$f'(v) = \left(\frac{1}{k'T}\right) I_s \exp(v / k'T) + 1 / R \tag{5.17}$$

For a particular value of V_s, we need to determine v and, from this value of the potential across the diode, we can determine the current in the circuit. However, because we are interested in obtaining the current through the diode for a source potential that is a function of time, we need to repeat the Newton-Raphson iteration for each of the different source voltage values at the different times. The sequence of the computation would proceed as follows:

1. Generate the time array.
2. Generate the source potential for the different elements in the time array.
3. For each time array entry, find the potential across the diode using the Newton-Raphson method.
4. Obtain the current array from the potential array.
5. Plot the source potential and the current arrays as a function of the time array.

Assuming that the source potential is given by:

$$V_s = V_0 \sin(2\pi f t) \tag{5.18}$$

and that $f = 60$ Hz, $V_0 = 5$ V, $kT = 0.025$ V, $R = 500\ \Omega$, and the saturation current I_s is 10^{-6} mA; the following *script M-file* finds the current in this circuit:

```
Is=10^(-9);
R=500;
kT=1/40;
f=60;
V0=5;
t=linspace(0,2/f,600);
L=length(t);
K=200;
Vs=(V0*sin(2*pi*t*f))'*ones(1,K);
v=zeros(L,K);
i=zeros(L,K);

  for k=1:K-1
  v(:,k+1)=v(:,k)-(Is*(exp((1/kT)*v(:,k))-1)-...
    (1/R)*(Vs(:,k)-v(:,k)))./...
    ((1/kT)*Is*exp((1/kT)*v(:,k))+1/R);
  i(:,k+1)=(Vs(:,k+1)-v(:,k+1))/R;
  end

plot(t,1000*i(:,K),'b',t,Vs(:,K),'g')
```

The current (expressed in mA) and the voltage (in V) of the source will appear in your graph window when you execute this program.

Homework Problem

Pb. 5.5 The apparent simplicity of the Newton-Raphson method is very misleading, suffice it to say that some of the original work on fractals started with examples from this model.

a. State, to the best of your ability, the conditions that the function, its derivative, and/or the original guess should satisfy so that this iterate converges to the correct limit. Supplement your arguments with geometric sketches that illustrate each of the pathologies.

b. Show that the Newton-Raphson method iterates cannot find the zero of the function:

$$y = \sqrt{x - 3}$$

c. Illustrate, with a simple sketch, the reason that this method does not work in part (**b**).

5.1.3 MATLAB fsolve and fzero Built-in Functions

Next, we investigate the use of the MATLAB command **fsolve** for finding the zeros of any function. We start with a function of one variable.

The recommended sequence of steps for finding the zeros of a function is as follows:

1. Edit a *function M-file* for the function under consideration.
2. Plot the curve of the function over the appropriate domain, and estimate the values of the zeros.
3. Using each of the estimates found in (2) above as an initial “guess,” use the command **fsolve** to accurately find each of the roots. The syntax is as follows:

```
xroot=fsolve('funname',xguess)
```

NOTE Actually, the MATLAB command **fzero** is quite suitable for finding the zero of a function of one variable. However, we used **fsolve** in the text above because it can only be used for the two-variables problem.

In the following application, we use the command **`fzero`** to find the zeros of a Bessel function, and learn in the process some important facts about this often-used special function of applied mathematics.

Application

Bessel functions are solutions to Bessel's differential equations of order n, given by:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x - n)y = 0 \tag{5.19}$$

There are special types of Bessel functions referred to as "of the first, second, and third kinds." Bessel functions of integer order appear, *inter alia*, in the expression of the radiation field in cylindrically shaped resonant cavities, and in light diffraction from a circular hole. Bessel functions of half-integer indices (see **Pb. 2.26**) appear in problems of spherical cavities and scattering of electromagnetic waves. Airy functions, a member of the Bessel functions family, appear in a number of important problems of optics and quantum mechanics.

The recursion formula that relates the Bessel function of any kind of a certain order with those of the same kind of adjacent orders is

$$2nZ_n(x) = xZ_{n-1}(x) + xZ_{n+1}(x) \tag{5.20}$$

where $Z_n(x)$ is the generic designation for all kinds of Bessel functions.

In this application, we concern ourselves only with the Bessel function of the first kind, usually denoted by $J_n(x)$. Its MATLAB call command is **`besselj(n,x)`**. In the present problem, we are interested in the root structure of the Bessel function of the first kind and of zero order.

In the program that follows, we call the Bessel function from the MATLAB library; however, we could have generated it ourselves using the techniques of Section 4.7 because we know the ODE that it satisfies, and its value and that of its derivative at $x = 0$, namely:

$$J_0(x = 0) = 1 \quad \text{and} \quad J_0'(x = 0) = 0$$

The problem that we want to solve is to find the zeros of $J_0(x)$ and compare to these exact values those obtained from the approximate expression:

$$x_{0,k} \approx \frac{\pi}{4}(4k-1) + \frac{1}{2\pi(4k-1)} - \frac{31}{6\pi^3(4k-1)^3} + \frac{3779}{15\pi^5(4k-1)^5} + \ldots \tag{5.21}$$

To implement this task, edit and execute the following *script M-file*:

```
for k=1:10
p(k)=4*k-1;
```

```
x0(k)=fzero('besselj(0,x)',(pi/4)*p(k));
x0approx(k)=(pi/4)*p(k)+(1/(2*pi))*(p(k)^(-1))-...
  (31/6)*(1/pi^3)*(p(k)^(-3))+...
  (3779/15)*(1/pi^5)*(p(k)^(-5));
end
kk=1:10;
subplot(2,1,1);
plot(kk,x0,'o')
title('Zeros of Zero Order BesselJ Function')
subplot(2,1,2);
semilogy(kk,x0-x0approx,'o')
title('Error in Approximate Values of the Zeros')
```

As you can easily observe by examining Figure 5.2, the approximate series is suitable for calculating all (except the smallest) zeros of the function $J_0(x)$ correctly to at least five digits.

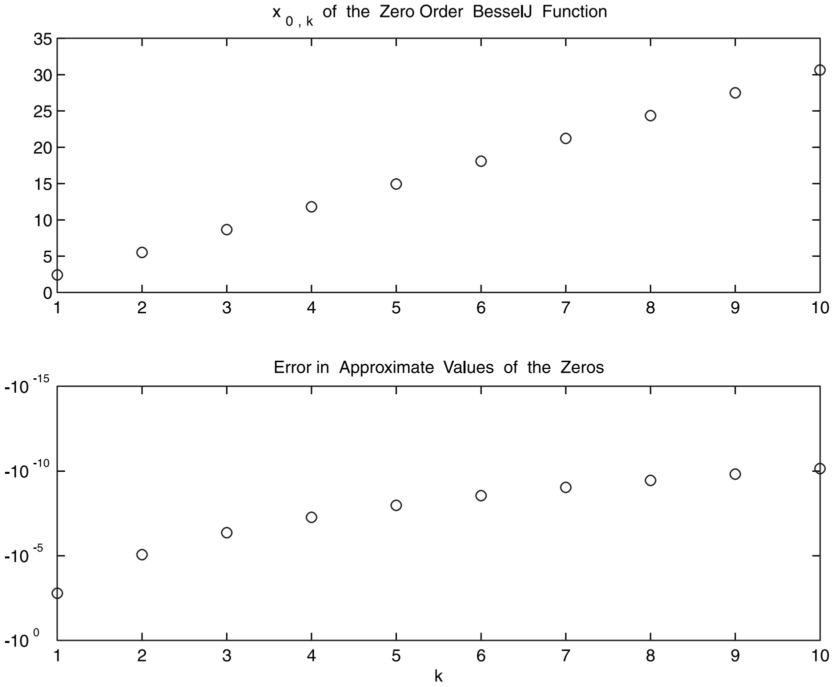

FIGURE 5.2
The first ten zeros of the Bessel function $J_0(x)$. Top panel: The values of the successive zeros (roots) of $J_0(x)$. Bottom panel: Deviation in the values of these zeros between their exact expressions and their approximate values as given in Eq. (5.21).

In-Class Exercises

In each of the following problems, find the zeros of the following functions over the interval [0, 5].

Pb. 5.6 $f(x) = x^2 + 1$. (Alert: Applying **`fsolve`** blindly could lead you into trouble!)

Pb. 5.7 $f(x) = \sin^2(x) - 1/2$. Compare your answer with the analytical result.

Pb. 5.8 $f(x) = 2\sin^2(x) - x^2$

Pb. 5.9 $f(x) = x - \tan(x)$

Zeros of a Function in Two Variables

As previously noted, the power of the MATLAB **`fsolve`** function really shines in evaluating the roots of multivariable functions.

Example 5.3

Find the intersection in the *x-y* plane of the parabaloid and the plane given in **Pb. 5.1**.

Solution: We follow these steps:

1. Use the **`contour`** command to estimate the coordinates of the points of intersection of the surfaces in the *x-y* plane.
2. Construct the *function M-file* for two functions (z_1, z_2) having two inputs (x, y):

```
function farray=funname(array)
x=array(1);
y=array(2);
farray(1)=7-sqrt(25+x.^2+y.^2);
farray(2)=4-2*x-4*y;
```

3. Use the approximate value found in step 1 as the value for the guess array; for example:

```
xyguess=[4 -1];
```

4. Finally, use the **`fsolve`** command to accurately find the root. The syntax is:

```
xyroots=fsolve('funname',xyguess)
xyroots =

  4.7081          -1.3541
```

5. To find the second root, use the second value of **`xyguess`**, which is the estimate of the other root, obtained from an examination of the contour plot in step 1 of the **`fsolve`** command:

```
xyguess=[-4 2];
xyroots=fsolve('funname',xyguess)

xyroots =

  -3.9081          2.9541
```

This method can be extended to any number of variables and nonlinear equations, but the estimate of the roots becomes much more difficult and we will not go into further details here.

In-Class Exercises

Find the values of x and y that simultaneously satisfy each pair of the following equations:

Pb. 5.10 $$\begin{cases} z_1 = 0 = x^3 + 2y - 3 \\ z_2 = 0 = x^2 + 3y^2 - 4 \end{cases}$$

Pb. 5.11 $$\begin{cases} z_1 = 0 = \sin^3(x^2 + y) + y^2 - x - 27/4 \\ z_2 = 0 = x^2 + 3y^2 - 31 \end{cases}$$

Pb. 5.12 $$\begin{cases} z_1 = 0 = x^{3/2} + (y-3)^2 - 12 \\ z_2 = 0 = x + y - 9 \end{cases}$$

Pb. 5.13 $$\begin{cases} z_1 = 0 = \tan(x) - \sqrt{y} \\ z_2 = 0 = \sin^2(x) - \dfrac{y}{4} - \dfrac{1}{4} \end{cases}$$

5.2 Roots of a Polynomial

While the analytical expressions for the roots of quadratic, cubic, and quartic equations are known, in general, the roots of higher-order polynomials cannot be found analytically. MATLAB has a built-in command that finds all the roots (real and complex) for any polynomial equation. As previously noted, the MATLAB command for finding the polynomial roots is **`roots`**:

```
r=roots(p)
```

In interpreting the results from this command, recall the Fundamental Theorem of Algebra, which states the root properties of a polynomial of degree n with real coefficients:

1. The n^{th} polynomial admits n complex roots.
2. Complex roots come in conjugate pairs. [If you are not familiar with complex numbers and with the term complex conjugate (the latter term should pique your curiosity), be a little patient. Help is on the way; Chapter 6 covers the topic of complex numbers].

Inversely, knowing the roots, we can reassemble the polynomial. The command is **`poly`**.

```
poly(r)
```

In-Class Exercise

Pb. 5.14 Find the roots of the polynomial $p = [1 \quad 3 \quad 2 \quad 1 \quad 0 \quad 3]$, and compute their sum and product.

Pb. 5.15 Consider the two polynomials:

$$p_1 = [1 \quad 3 \quad 2 \quad 1 \quad 0 \quad 3] \quad \text{and} \quad p_2 = [3 \quad 2 \quad 1]$$

Find the value(s) of x at which the curves representing these polynomials would intersect.

Pb. 5.16 Find the constants $A, B, C, D,$ and a, b, c, d that permits the following expansion in partial fractions:

$$\frac{1}{x^4 - 25x^2 + 144} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} + \frac{D}{(x-d)}$$

5.3 Optimization Methods

Many design problems call for the maximization or minimization (optimization) of a particular function belonging to a particular domain. (Recall the resistor circuit [Figure 3.1] in which we wanted to find the maximum power delivered to a load resistor.) In this section, we will learn the simple Golden Section rule and the use of the **fmin** command to solve the simplest forms of this problem. The important class of problems related to optimizing a function, while satisfying a number of constraints, will be left to more advanced courses.

Let us start by reminding ourselves of some terms definitions: The *domain* is the set of elements to which a function assigns values. The *range* is the set of values thus obtained.

DEFINITION Let I, the domain of the function $f(x)$, contain the point c. We say that:

1. $f(c)$ is the maximum value of the function on I if $f(c) \geq f(x)$ for all $x \in I$.
2. $f(c)$ is the minimum value of the function on I if $f(c) \leq f(x)$ for all $x \in I$.
3. An extremum is the common designation for either the maximum value or the minimum value.

Using the above definitions, we note that the maximum (minimum) may appear at an endpoint of the interval I, or possibly in the interior of the interval:

- If a maximum (minimum) appears at an endpoint, we describe this extreme point as an endpoint maximum (minimum).
- If a maximum (minimum) appears in the interior of the interval, we describe this extreme point as a local maximum (minimum).
- The largest (smallest) value among the maximum (minimum) values (either endpoint or local) is called the global maximum (minimum) and is the object of our search.

We note, in passing, the equivalence of finding the local extremum of a function with finding the zeros of the derivative of this function. The following methods are suitable when this direct method is not suitable due to a number of practical complications.

As with finding the zeros of a function, in this instance we will also explore the graphical method, the simple numerical method, and the MATLAB built-in commands for finding the extremum.

5.3.1 Graphical Method

In the graphical method, in steps very similar to those described in Section 5.1.1 for finding the zeros of a single variable function, we follow these steps:

1. Plot the particular function over the defined domain.
2. Examine the plot to determine whether the extremum is an endpoint extremum or a local extremum.
3. Zoom in on the neighborhood of the so-identified extremum by repeated application of the MATLAB **`axis`** or **`zoom`** commands.
4. Use the cross hair of the **`ginput`** command to read the coordinates of the extremum. [Be especially careful here. Extra caution is prompted by the fact that the curve is flat (its tangent is parallel to the x-axis) at a local extremum; thus, you may need to re-plot the curve in the neighborhood of this extremum to find, through visual means, accurate results for the coordinates of the extremum. There may be too few points in the original plot for the zooming technique to provide more than a very rough approximation.]

In-Class Exercises

Find, graphically, for each of the following exercises, the coordinates of the global maximum and the global minimum for the following curves in the indicated intervals. Specify the nature of the extremum.

Pb. 5.17 $y = f(x) = \exp(-x^2)$ on $-4 < x < 4$

Pb. 5.18 $y = f(x) = \exp(-x^2)\sin^2(x)$ on $-4 < x < 4$

Pb. 5.19 $y = f(x) = \exp(-x^2)\,[x^3 + 2x + 3]$ on $-4 < x < 4$

Pb. 5.20 $y = f(x) = 2\sin(x) - x$ on $0 < x < 2\pi$

Pb. 5.21 $y = f(x) = \sqrt{1+\sin(x)}$ on $0 < x < 2\pi$

5.3.2 Numerical Methods

We discuss now the Golden Section method for evaluating the position of the local minimum of a function and its value at this minimum. We assume that we have plotted the function and have established that such a local minimum exists. Our goal at this point is to accurately pinpoint the position and value of this minimum. We detail the derivation of an elementary technique for this search: the Golden Section method. More accurate and efficient techniques for this task have been developed. These are incorporated in the built-in command **`fmin`**; the mode of use is discussed in Section 5.3.3.

5.3.2.1 Golden Section Method

Assume that, by examining the graph of the function under consideration, you have established the local minimum $x_{\min} \in [a, b]$. This means that the curve of the function is strictly decreasing in the interval $[a, x_{\min}]$ and is strictly increasing in the interval $[x_{\min}, b]$. Next, choose a number $r < 1/2$, but whose precise value will be determined later, and define the internal points c and d such that:

$$c = a + r(b - a) \tag{5.22}$$

$$d = a + (1 - r)(b - a) \tag{5.23}$$

and such that $a < c < d < b$. Next, evaluate the values of the function at c and d. If we find that $f(c) \geq f(d)$, we can assert that $x_{\min} \in [c, b]$; that is, we narrowed the external bounds of the interval. (If the inequality was in the other sense, we could have instead narrowed the outer limit from the right.) If in the second iteration, we fix the new internal points such that the new value of c is the old value of d, then all we have to compute at this step is the new value of d. If we repeat the same iteration k-times, until the separation between c and d is smaller than the desired tolerance, then at that point we can assert that:

$$x_{\min} = \frac{c(k) + d(k)}{2} \tag{5.24}$$

Now, let us determine the value of r that will allow the above iteration to proceed as described. Translating the above statements into equations, we desire that:

$$\begin{aligned} & c(2) = d(1) \\ & \quad \Rightarrow c(2) = a(2) + r(b(2) - a(2)) = a(1) + (1 - r)(b(1) - a(1)) \end{aligned} \tag{5.25}$$

$$a(2) = c(1) = a(1) + r(b(1) - a(1)) \tag{5.26}$$

$$b(2) = b(1) \tag{5.27}$$

Now, replacing the values of $a(2)$ and $b(2)$ from Eqs. (5.26) and (5.27) into Eq. (5.25), we are led to a second-degree equation in r:

$$r^2 - 3r + 1 = 0 \tag{5.28}$$

The desired root is the value of the Golden ratio:

$$r = \frac{3 - \sqrt{5}}{2} \tag{5.29}$$

and hence, the name of the method.

The following *function M-file* implements the above algorithm:

```
function [xmin,ymin]=goldensection(funname,a,b,
  tolerance)
r=(3-sqrt(5))/2;
c=a+r*(b-a);
fc=feval(funname,c);
d=a+(1-r)*(b-a);
fd=feval(funname,d);

while d-c>tolerance
  if fc>=fd
  dnew=c+(1-r)*(b-c);
  a=c;
  c=d;
  fc=fd;
  d=dnew;
  fd=feval(funname,dnew);

  else
  cnew=a+r*(d-a);
  b=d;
  d=c;
  fd=fc;
  c=cnew;
  fc=feval(funname,cnew);
  end
end

xmin=(c+d)/2;
ymin=feval(funname,xmin);
```

For example, if we wanted to find the position of the minimum of the cosine function and its value in the interval $3 < x < 3.5$, accurate to 10^{-4}, we would enter in the command window, after having saved the above *function M-file*, the following command:

```
[xmin,ymin]=goldensection('cos',3,3.5,10^(-4))
```

5.3.3 MATLAB fmin and fmins Built-in Function

Following methodically the same steps using **fzero** to find the zeros of any function, we can use the **fmin** command to find the minimum of a function of one variable on a given interval. The recommended sequence of steps is as follows:

1. Edit a *function M-file* for the function under consideration.
2. Plot the curve of the function over the desired domain, to overview the function shape and have an estimate of the position of the minimum.
3. Use the command **fmin** to accurately find the minimum. The syntax is as follows:

```
xmin=fmin('funname',a,b) % [a,b] is the interval
```

The local maximum of a function $f(x)$ on an interval can be computed by noting that this quantity can be deduced from knowing the values of the coordinates of the local minimum of $-f(x)$. The implementation of this task consists of creating a file for the negative of this function (call it **n-funname**) and entering the following commands in the command window:

```
xmax=fmin('n-funname',xi,xf)
fmax=-1*feval('n-funname',xmax)
```

Homework Problems

Pb. 5.22 We have two posts of height 6 m and 8 m, and separated by a distance of 21 m. A line is to run from the top of one post to the ground between the posts and then to the top of the other post (Figure 5.3). Find the configuration that minimizes the length of the line.

Pb. 5.23 Fermat's principle states that light going from Point A to Point B selects the path which requires the least amount of travel time. Consider the situation in which an engineer in a submarine wants to communicate, using a laser-like pointer, with a detector at the top of the mast of another boat. At what angle θ to the vertical should he point his beam? Assume that the detector is 50 ft above the water surface, the submarine transmitter is 30 ft under the surface, the horizontal distance separating the boat from the submarine is 100 ft, and the velocity of light in water is 3/4 of its velocity in air (Figure 5.4).

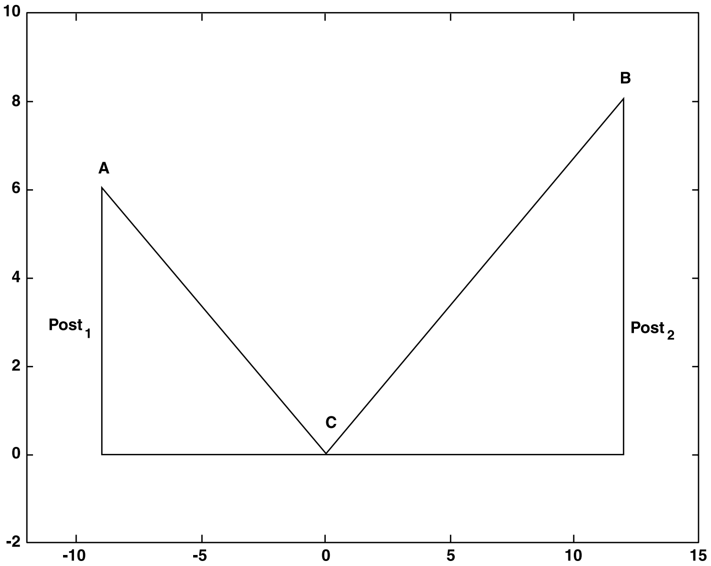

FIGURE 5.3
Schematics for Pb. 5.22. (ACB is the line whose length we want to minimize.)

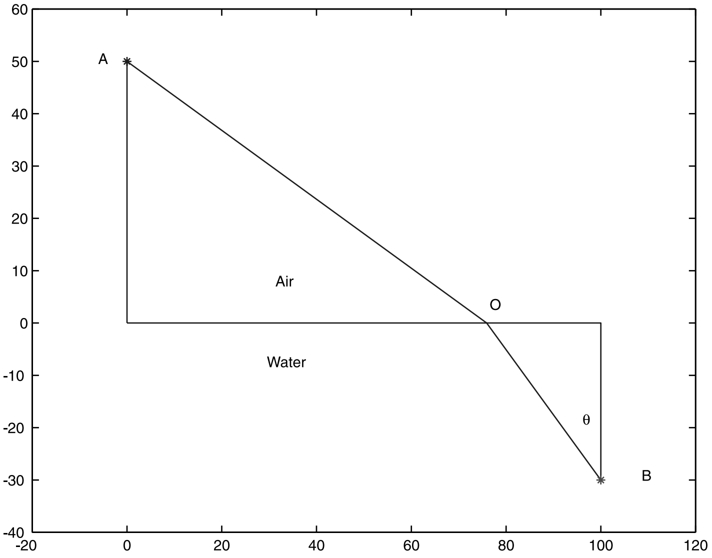

FIGURE 5.4
Schematics for Pb. 5.23. A is the location of the detector at the top of the mast, B is the location of the emitter in the submarine, and BOA is the optical path of the ray of light.

Minimum of a Function of Two Variables

To find the local minimum of a multivariable function, we use the MATLAB **`fmins`** function. Finding the maximum can be handled by the same technique as outlined for the one variable case.

Example 5.4

Find the position of the minimum of the surface $f(x, y) = x^2 + y^2$.

Solution:

1. First, make a function file and save it as **fname.m**.

```
function f=fname(array)

x=array(1); % x is stored in first element of array
y=array(2); % y is stored in second element of
%array
f=x.^2+y.^2; % function stored in f
```

2. Graph the contour plot for the surface; and from it, estimate the coordinates of the minimum:

```
arrayguess=[.1 .1];
```

The **`arrayguess`** holds the initial guess for both coordinates at the minimum. That is,

```
arrayguess=[xguess yguess];
```

3. The coordinates of the minimum are then obtained by entering the following commands in the command window:

```
arraymin=fmins('fname',arrayguess)
fmin=feval('fname',arraymin)
```

Homework Problem

Pb. 5.24 In this problem we propose to apply the above optimization techniques to the important problem of the optical narrow band transmission filter. This filter, in very wide use in optics, consists of two parallel semi-reflective surfaces (i.e., mirrors) with reflection coatings R_1 and R_2 and separated by a distance L. Assuming that the material between the mirrors has an index of refraction n and that the incoming beam of light has frequency ω and is making an angle θ_i with the normal to the semi-reflective surfaces, then the ratio of the transmitted light intensity to the incident intensity is

$$T = \frac{I_{transm.}}{I_{incid.}} = \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1\ R_2})^2 + 4\sqrt{R_1 R_2}\sin^2\left(\pi\frac{\omega}{\omega_0}\right)}$$

where $\omega_0 = \frac{\pi c}{nL\cos(\theta_t)}$, $\sin(\theta_i) = n\sin(\theta_t)$, and θ_t is the angle that the transmitted light makes with the normal to the mirror surfaces.

In the following activities, we want to understand how the above transmission filter responds as a function of the specified parameters. Choose the following parameters:

$$R_1 = R_2 = 0.8$$

$$0 \le \omega \le 4\omega_0$$

a. Plot T vs. ω/ω_0 for the above frequency range.

b. At what frequencies does the transmission reach a maximum? A minimum?

c. Devise two methods by which you can tune the filter so that the maximum of the filter transmission is centered around a particular physical frequency.

d. How sharp is the filter? By sharp, we mean: what is the width of the transmission band that allows through at least 50% of the incident light? Define the width relative to ω_0.

e. Answer question (**d**) with the values of the reflection coatings given now by:

$$R_1 = R_2 = 0.9$$

$$0 \le \omega \le 4\omega_0$$

Does the sharpness of the filter increase or decrease with an increase of the reflection coefficients of the coating surfaces for the two mirrors?

f. Choosing $\omega = \omega_0$, plot a 3-D mesh of T as a function of the reflection coefficients R_1 and R_2. Show, both graphically and numerically, that the best performance occurs when the reflection coatings are the same.

g. Plot the contrast function defined as $C = \frac{T_{\min}}{T_{\max}}$ as a function of the reflection coefficients R_1 and R_2. How should you choose your mirrors for maximum contrast?

h. For $\omega = \omega_0$, plot the variation of the transmission coefficient as function of θ_i.

i. Repeat (**h**), but now investigate the variation in the transmission coefficient as a function of L.

5.4 MATLAB Commands Review

`besselj` The built-in BesselJ function.

`fmin` Finds the minimum value of a single variable function or a restricted domain.

`fmins` Finds the local minimum of a multivariable function.

`fsolve` Finds a root to a system of nonlinear equations assuming an initial guess.

`fzero` Finds the zero of a single variable function assuming an initial guess.

`roots` Finds the roots of a polynomial if the polynomial coefficients are given.

`poly` Assembles a polynomial from its roots.

`zoom` Zooms in and out on a 2-D plot.

6

Complex Numbers

6.1 Introduction

Since $x^2 > 0$ for all real numbers x, the equation $x^2 = -1$ admits no real number as a solution. To deal with this problem, mathematicians in the 18th century introduced the imaginary number $i = \sqrt{-1} = j$. (So as not to confuse the usual symbol for a current with this quantity, electrical engineers prefer the use of the j symbol. MATLAB accepts either symbol, but always gives the answer with the symbol i).

Expressions of the form:

$$z = a + jb \tag{6.1}$$

where a and b are real numbers called complex numbers. As illustrated in Section 6.2, this representation has properties similar to that of an ordered pair (a, b), which is represented by a point in the 2-D plane.

The real number a is called the real part of z, and the real number b is called the imaginary part of z. These numbers are referred to by the symbols $a = \text{Re}(z)$ and $b = \text{Im}(z)$.

When complex numbers are represented geometrically in the x-y coordinate system, the x-axis is called the real axis, the y-axis is called the imaginary axis, and the plane is called the complex plane.

6.2 The Basics

In this section, you will learn how, using MATLAB, you can represent a complex number in the complex plane. It also shows how the addition (or subtraction) of two complex numbers, or the multiplication of a complex number by a real number or by j, can be interpreted geometrically.

Example 6.1

Plot in the complex plane, the three points (P_1, P_2, P_3) representing the complex numbers: $z_1 = 1$, $z_2 = j$, $z_3 = -1$.

Solution: Enter and execute the following commands in the command window:

```
z1=1;
z2=j;
z3=-1;
plot(z1,'*')
axis([-2 2 -2 2])
axis('square')
hold on
plot(z2,'o')
plot(z3,'*')
hold off
```

that is, a complex number in the **`plot`** command is interpreted by MATLAB to mean: take the real part of the complex number to be the *x*-coordinate and the imaginary part of the complex number to be the *y*-coordinate.

6.2.1 Addition

Next, we define addition for complex numbers. The rule can be directly deduced from analogy of addition of two vectors in a plane: the *x*-component of the sum of two vectors is the sum of the *x*-components of each of the vectors, and similarly for the *y*-component. Therefore:

If: $$z_1 = a_1 + jb_1 \tag{6.2}$$

and $$z_2 = a_2 + jb_2 \tag{6.3}$$

Then: $$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{6.4}$$

The addition or subtraction rules for complex numbers are geometrically translated through the parallelogram rules for the addition and subtraction of vectors.

Example 6.2

Find the sum and difference of the complex numbers

$$z_1 = 1 + 2j \quad \text{and} \quad z_2 = 2 + j$$

Solution: Grouping the real and imaginary parts separately, we obtain:

$$z_1 + z_2 = + 3j$$

and

$$z_1 - z_2 = -1 + j$$

Preparatory Exercise

Pb. 6.1 Given the complex numbers z_1, z_2, and z_3 corresponding to the vertices P_1, P_2, and P_3 of a parallelogram, find z_4 corresponding to the fourth vertex P_4. (Assume that P_4 and P_2 are opposite vertices of the parallelogram). Verify your answer graphically for the case:

$$z_1 = 2 + j, \quad z_2 = 1 + 2j, \quad z_3 = 4 + 3j$$

6.2.2 Multiplication by a Real or Imaginary Number

If we multiply the complex number $z = a + jb$ by a real number k, the resultant complex number is given by:

$$k \times z = k \times (a + jb) = ka + jkb \tag{6.5}$$

What happens when we multiply by j?

Let us, for a moment, return to Example 6.1. We note the following properties for the three points P_1, P_2, and P_3:

1. The three points are equally distant from the origin of the axis.
2. The point P_2 is obtained from the point P_1 by a $\pi/2$ counterclockwise rotation.
3. The point P_3 is obtained from the point P_2 through another $\pi/2$ counterclockwise rotation.

We also note, by examining the algebraic forms of z_1, z_2, z_3 that:

$$z_2 = jz_1 \quad \text{and} \quad z_3 = jz_2 = j^2 z_1 = -z_1$$

That is, multiplying by j is geometrically equivalent to a counterclockwise rotation by an angle of $\pi/2$.

6.2.3 Multiplication of Two Complex Numbers

The multiplication of two complex numbers follows the same rules of algebra for real numbers, but considers $j^2 = -1$. This yields:

$$z_1 = a_1 + jb_1 \quad \text{and} \quad z_2 = a_2 + jb_2$$

If:

$$\Rightarrow \quad z_1 z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \tag{6.6}$$

Preparatory Exercises

Solve the following problems analytically.

Pb. 6.2 Find $z_1 z_2, z_1^2, z_2^2$ for the following pairs:

a. $z_1 = 3j; \quad z_2 = 1 - j$

b. $z_1 = 4 + 6j; \quad z_2 = 2 - 3j$

c. $z_1 = \frac{1}{3}(2 + 4j); \quad z_2 = \frac{1}{2}(1 - 5j)$

d. $z_1 = \frac{1}{3}(2 - 4j); \quad z_2 = \frac{1}{2}(1 + 5j)$

Pb. 6.3 Find the real quantities m and n in each of the following equations:

a. $mj + n(1 + j) = 3 - 2\text{j}$

b. $m(2 + 3j) + n(1 - 4j) = 7 + 5j$

(*Hint:* Two complex numbers are equal if separately the real and imaginary parts are equal.)

Pb. 6.4 Write the answers in standard form: (i.e., $a + jb$)

a. $(3 - 2j)^2 - (3 + 2j)^2$

b. $(7 + 14j)^7$

c. $\left[(2 + j)\left(\frac{1}{2} + 2j\right)\right]^2$

d. $j(1 + 7j) - 3j(4 + 2j)$

Pb. 6.5 Show that for all complex numbers z_1, z_2, z_3, we have the following properties:

$$z_1 z_2 = z_2 z_1 \quad \text{(commutativity property)}$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad \text{(distributivity property)}$$

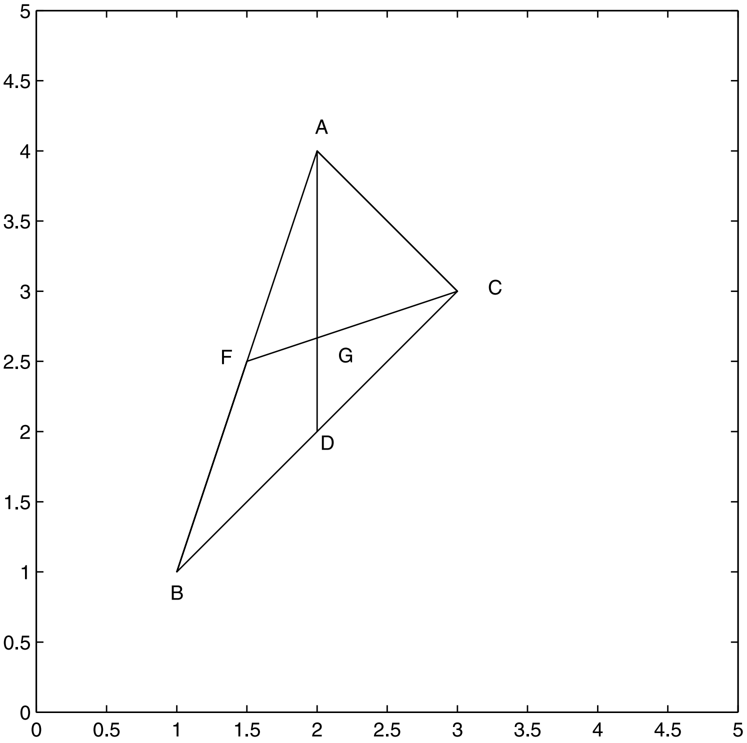

FIGURE 6.1
The center of mass of a triangle. (Refer to Pb. 6.6).

Pb. 6.6 Consider the triangle $\Delta(ABC)$, in which D is the midpoint of the BC segment, and let the point G be defined such that $(GD) = \frac{1}{3}(AD)$. Assuming that z_A, z_B, z_C are the complex numbers representing the points (A, B, C):

a. Find the complex number z_G that represents the point G.

b. Show that $(CG) = \frac{2}{3}(CF)$ and that F is the midpoint of the segment (AB).

6.3 Complex Conjugation and Division

DEFINITION The complex conjugate of a complex number z, which is denoted by $\bar{z}$, is given by:

$$\bar{z} = a - jb \quad \text{if} \quad z = a + jb \tag{6.7}$$

That is, $\bar{z}$ is obtained from z by reversing the sign of Im(z). Geometrically, z and $\bar{z}$ form a pair of symmetric points with respect to the real axis (x-axis) in the complex plane.

In MATLAB, complex conjugation is written as **`conj(z)`**.

DEFINITION The modulus of a complex number $z = a + jb$, denoted by $|z|$, is given by:

$$|z| = \sqrt{a^2 + b^2} \tag{6.8}$$

Geometrically, it represents the distance between the origin and the point representing the complex number z in the complex plane, which by Pythagorean theorem is given by the same quantity.

In MATLAB, the modulus of z is denoted by **`abs(z)`**.

THEOREM

For any complex number z, we have the result that:

$$|z|^2 = \bar{z}z \tag{6.9}$$

PROOF Using the above two definitions for the complex conjugate and the norm, we can write:

$$\bar{z}z = (a - jb)(a + jb) = a^2 + b^2 = |z|^2$$

In-Class Exercise

Solve the problem analytically, and then use MATLAB to verify your answers.

Pb. 6.7 Let $z = 3 + 4j$. Find $|z|, \bar{z}$, and $z\bar{z}$. Verify the above theorem.

6.3.1 Division

Using the above definitions and theorem, we now want to define the inverse of a complex number with respect to the multiplication operation. We write the results in standard form.

$$z^{-1} = \frac{1}{z} = \frac{1}{(a+jb)}\left(\frac{a-jb}{a-jb}\right) = \frac{a-jb}{a^2+b^2} = \frac{\bar{z}}{|z|^2} \tag{6.10}$$

from which we deduce that:

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{\mathrm{Re}(z)}{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2} \tag{6.11}$$

and

$$\mathrm{Im}\left(\frac{1}{z}\right) = \frac{-\mathrm{Im}(z)}{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2} \tag{6.12}$$

To summarize the above results, and to help you build your syntax for the quantities defined in this section, edit the following *script M-file* and execute it:

```
z=3+4*j
zbar=conj(z)
modulz=abs(z)
modul2z=z*conj(z)
invz=1/z
reinvz=real(1/z)
iminvz=imag(1/z)
```

In-Class Exercises

Pb. 6.8 Analytically and numerically, obtain in the standard form an expression for each of the following quantities:

a. $\frac{3+4j}{2+5j}$ **b.** $\frac{\sqrt{3}+j}{(1-j)(3+j)}$ **c.** $\left[\frac{1-2j}{2+3j} - \frac{3+j}{2j}\right]$

Pb. 6.9 For any pair of complex numbers z_1 and z_2, show that:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{z_1\, z_2} = \bar{z}_1\, \bar{z}_2$$

$$\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$$

$$\bar{\bar{z}} = z$$

6.4 Polar Form of Complex Numbers

If we use polar coordinates, we can write the real and imaginary parts of a complex number $z = a + jb$ in terms of the modulus of z and the polar angle θ:

$$a = r\cos(\theta) = |z|\cos(\theta) \tag{6.13}$$

$$b = r\sin(\theta) = |z|\sin(\theta) \tag{6.14}$$

and the complex number z can then be written in polar form as:

$$z = |z|\cos(\theta) + j|z|\sin(\theta) = |z|(\cos(\theta) + j\sin(\theta)) \tag{6.15}$$

The angle θ is called the argument of z and is usually evaluated in the interval $-\pi \leq \theta \leq \pi$. However, we still have the same complex number if we added to the value of θ an integer multiple of 2π.

$$\begin{aligned} \theta &= \arg(z) \\ \tan(\theta) &= \frac{b}{a} \end{aligned} \tag{6.16}$$

From the above results, it is obvious that the argument of the complex conjugate of a complex number is equal to minus the argument of this complex number.

In MATLAB, the convention for arg(z) is **`angle(z)`**.

In-Class Exercise

Pb. 6.10 Find the modulus and argument for each of the following complex numbers:

$$z_1 = 1 + 2j;\quad z_2 = 2 + j;\quad z_3 = 1 - 2j;\quad z_4 = -1 + 2j;\quad z_5 = -1 - 2j$$

Plot these points. Can you detect any geometrical pattern? Generalize.

The main advantage of writing complex numbers in polar form is that it makes the multiplication and division operations more transparent, and provides a simple geometric interpretation to these operations, as shown below.

6.4.1 New Insights into Multiplication and Division of Complex Numbers

Consider the two complex numbers z_1 and z_2 written in polar form:

$$z_1 = |z_1|(\cos(\theta_1) + j\sin(\theta_1)) \tag{6.17}$$

$$z_2 = |z_2|(\cos(\theta_2) + j\sin(\theta_2)) \tag{6.18}$$

Their product z_1z_2 is given by:

$$z_1z_2 = |z_1||z_2|\begin{bmatrix}(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) \\ + j(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))\end{bmatrix} \tag{6.19}$$

But using the trigonometric identities for the sine and cosine of the sum of two angles:

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \tag{6.20}$$

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \tag{6.21}$$

the product of two complex numbers can then be written in the simpler form:

$$z_1z_2 = |z_1||z_2|[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)] \tag{6.22}$$

That is, when multiplying two complex numbers, the modulus of the product is the product of the moduli, while the argument is the sum of arguments:

$$|z_1z_2| = |z_1||z_2| \tag{6.23}$$

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2) \tag{6.24}$$

The above result can be generalized to the product of n complex numbers and the result is:

$$|z_1z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n| \tag{6.25}$$

$$\arg(z_1z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + (z_n) \tag{6.26}$$

A particular form of this expression is the De Moivre theorem, which states that:

$$(\cos(\theta) + j\sin(\theta))^n = \cos(n\theta) + j\sin(n\theta) \tag{6.27}$$

The above results suggest that the polar form of a complex number may be written as a function of an exponential function because of the additivity of the arguments upon multiplication. We revisit this issue later.

In-Class Exercises

Pb. 6.11 Show that $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$.

Pb. 6.12 Explain, using the above results, why multiplication of any complex number by j is equivalent to a rotation of the point representing this number in the complex plane by $\pi/2$.

Pb. 6.13 By what angle must we rotate the point $P(3, 4)$ to transform it to the point $P'(4, 3)$?

Pb. 6.14 The points $z_1 = 1 + 2j$ and $z_2 = 2 + j$ are adjacent vertices of a regular hexagon. Find the vertex z_3 that is also a vertex of the same hexagon and that is adjacent to z_2 ($z_3 \neq z_1$).

Pb. 6.15 Show that the points A, B, C representing the complex numbers z_A, z_B, z_C in the complex plane lie on the same straight line if and only if:

$$\frac{z_A - z_c}{z_B - z_c} \quad \text{is real.}$$

Pb. 6.16 Determine the coordinates of the P' point obtained from the point $P(2, 4)$ through a reflection around the line $y = \frac{x}{2} + 2$.

Pb. 6.17 Consider two points A and B representing, in the complex plane, the complex numbers z_1 and $1/\bar{z}_1$. Let P be any point on the circle of radius 1 and centered at the origin (the unit circle). Show that the ratio of the length of the line segments PA and PB is the same, regardless of the position of point P on the unit circle.

Pb. 6.18 Find the polar form of each of the following quantities:

$$\frac{(1+j)^{15}}{(1-j)^9}, \quad \sqrt{(-1+j)}(j+2), \quad (1+j+j^2+j^3)^{99}$$

6.4.2 Roots of Complex Numbers

Given the value of the complex number z, we are interested here in finding the solutions of the equation:

$$v^n = z \tag{6.28}$$

Let us write both the solutions and z in polar forms,

$$v = \rho(\cos(\alpha) + j\sin(\alpha)) \tag{6.29}$$

$$z = r(\cos(\theta) + j\sin(\theta)) \tag{6.30}$$

From the De Moivre theorem, the expression for $v^n = z$ can be written as:

$$\rho^n(\cos(n\alpha) + j\sin(n\alpha)) = r(\cos(\theta) + j\sin(\theta)) \tag{6.31}$$

Comparing the moduli of both sides, we deduce by inspection that:

$$\rho = \sqrt[n]{r} \tag{6.32}$$

The treatment of the argument should be done with great care. Recalling that two angles have the same cosine and sine if they are equal or differ from each other by an integer multiple of 2π, we can then deduce that:

$$n\alpha = \theta + 2k\pi \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{6.33}$$

Therefore, the general expression for the roots is:

$$\begin{gathered} z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + j\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \\ \text{with } k = 0, 1, 2, \ldots, (n-1) \end{gathered} \tag{6.34}$$

Note that the roots reproduce themselves outside the range: $k = 0, 1, 2, \ldots, (n-1)$.

In-Class Exercises

Pb. 6.19 Calculate the roots of the equation $z^5 - 32 = 0$, and plot them in the complex plane.

a. What geometric shape does the polygon with the solutions as vertices form?

b. What is the sum of these roots? (Derive your answer both algebraically and geometrically.)

6.4.3 The Function $y = e^{j\theta}$

As alluded to previously, the expression $\cos(\theta) + j\sin(\theta)$ behaves very much as if it was an exponential; because of the additivity of the arguments of each term in the argument of the product, we denote this quantity by:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{6.35}$$

PROOF Compute the Taylor expansion for both sides of the above equation. The series expansion for $e^{j\theta}$ is obtained by evaluating Taylor's formula at $x = j\theta$, giving (see appendix):

$$e^{j\theta} = \sum_{n=0}^{\infty} \frac{1}{n!}(j\theta)^n \tag{6.36}$$

When this series expansion for $e^{j\theta}$ is written in terms of its even part and odd part, we have the result:

$$e^{j\theta} = \sum_{m=0}^{\infty} \frac{1}{(2m)!}(j\theta)^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}(j\theta)^{2m+1} \tag{6.37}$$

However, since $j^2 = -1$, this last equation can also be written as:

$$e^{j\theta} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}(\theta)^{2m} + j\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}(\theta)^{2m+1} \tag{6.38}$$

which, by inspection, can be verified to be the sum of the Taylor expansions for the cosine and sine functions.

In this notation, the product of two complex numbers z_1 and z_2 is: $r_1 r_2 e^{j(\theta_1+\theta_2)}$. It is then a simple matter to show that:

If: $$z = r\exp(j\theta) \tag{6.39}$$

Then: $$\bar{z} = r\exp(-j\theta) \tag{6.40}$$

and

$$z^{-1} = \frac{1}{r}\exp(-j\theta) \tag{6.41}$$

from which we can deduce Euler's equations:

$$\cos(\theta) = \frac{\exp(j\theta) + \exp(-j\theta)}{2} \tag{6.42}$$

and

$$\sin(\theta) = \frac{\exp(j\theta) - \exp(-j\theta)}{2j} \tag{6.43}$$

Example 6.3

Use MATLAB to generate the graph of the unit circle in the complex plane.

Solution: Because all points on the unit circle are equidistant from the origin and their distance to the origin (their modulus) is equal to 1, we can generate the circle by plotting the N-roots of unity, taking a very large value for N. This can be implemented by executing the following *script M-file*.

```
N=720;
z=exp(j*2*pi*[1:N]./N);
plot(z)
axis square
```

In-Class Exercises

Pb. 6.20 Using the exponential form of the n-roots of unity, and the expression for the sum of a geometric series (given in the appendix), show that the sum of these roots is zero.

Pb. 6.21 Compute the following sums:

a. $1 + \cos(x) + \cos(2x) + \ldots + \cos(nx)$

b. $\sin(x) + \sin(2x) + \ldots + \sin(nx)$

c. $\cos(\alpha) + \cos(\alpha + \beta) + \ldots + \cos(\alpha + n\beta)$

d. $\sin(\alpha) + \sin(\alpha + \beta) + \ldots + \sin(\alpha + n\beta)$

Pb. 6.22 Verify numerically that for $z = x + jy$:

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \exp(x)(\cos(y) + j\sin(y))$$

For what values of y is this quantity pure imaginary?

Homework Problems

Pb. 6.23 Plot the curves determined by the following parametric representations:

a. $z = 1 - jt \qquad 0 \le t \le 2$

b. $z = t + jt^2 \qquad -\infty < t < \infty$

c. $z = 2(\cos(t) + j\sin(t)) \qquad \frac{\pi}{2} < t < \frac{3\pi}{2}$

d. $z = 3(t + j - j\exp(-jt)) \qquad 0 < t < \infty$

Pb. 6.24 Find the expression $y = f(x)$ and plot the families of curves defined by each of the corresponding equations:

a. $\mathrm{Re}\left(\frac{1}{z}\right) = 2$ **b.** $\mathrm{Im}\left(\frac{1}{z}\right) = 2$

c. $\mathrm{Re}(z^2) = 4$ **d.** $\mathrm{Im}(z^2) = 4$

e. $\left|\frac{z-3}{z+3}\right| = 5$ **f.** $\arg\left(\frac{z-3}{z+3}\right) = \frac{\pi}{4}$

g. $|z^2 - 1| = 3$ **h.** $|z| = \mathrm{Im}(z) + 4$

Pb. 6.25 Find the image of the line $\mathrm{Re}(z) = 1$ upon the transformation $z' = z^2 + z$. (First obtain the result analytically, and then verify it graphically.)

Pb. 6.26 Consider the following bilinear transformation: $z' = \frac{az + b}{cz + d}$

Show how with proper choices of the constants a, b, c, d, we can generate all transformations of planar geometry (i.e., scaling, rotation, translation, and inversion).

Pb. 6.27 Plot the curves C' generated by the points P' that are the images of points on the circle centered at (3, 4) and of radius 5 under the transformation of the preceding problem, with the following parameters:

Case 1: $a = \exp(j\pi/4)$, $b = 0$, $c = 0$, $d = 1$

Case 2: $a = 1$, $b = 3$, $c = 0$, $d = 1$

Case 3: $a = 0$, $b = 1$, $c = 1$, $d = 0$

6.5 Analytical Solutions of Constant Coefficients ODE

Finding the solutions of an ODE with constant coefficients is conceptually very similar to solving the linear difference equation with constant coefficients. We repeat the exercise here for its pedagogical benefits and to bring out some of the finer technical details peculiar to the ODEs of particular interest for later discussions.

The linear differential equation of interest is given by:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = u(t) \tag{6.44}$$

In this section, we find the solutions of this ODE for the cases that $u(t) = 0$ and $u(t) = A\cos(\omega t)$.

The solutions for the first case are referred to as the homogeneous solutions. By substitution, it is a trivial matter to verify that if $y_1(t)$ and $y_2(t)$ are solutions, then $c_1 y_1(t) + c_2 y_2(t)$, where c_1 and c_2 are constants, is also a solution. This is, as previously mentioned, referred to as the superposition principle for linear systems.

If $u(t) \neq 0$, the general solution of the ODE will be the sum of the corresponding homogeneous solution and the particular solution peculiar to the specific details of $u(t)$. Furthermore, by inspection, it is clear that if the source can be decomposed into many components, then the particular solution can be written as the sum of the particular solutions for the different components and with the same weights as in the source. This property characterizes a linear system.

DEFINITION A system L is considered linear if:

$$L(c_1 u_1(t) + c_2 u_2(t) + \ldots + c_n u_n(t)) = c_1 L(u_1(t)) + c_2 L(u_2(t)) + \ldots + c_n L(u_n(t)) \tag{6.45}$$

where the c's are constants and the u's are time-dependent source signals.

6.5.1 Transient Solutions

To obtain the homogeneous solutions, we set $u(t) = 0$. We guess that the solution to this homogeneous differential equation is $y = \exp(st)$. You may wonder why we made this guess; the secret is in the property of the exponential function, whose derivative is proportional to the function itself. That is:

$$\frac{d(\exp(st))}{dt} = s\exp(st) \tag{6.46}$$

Through this substitution, the above ODE reduces to an algebraic equation, and the solution of this algebraic equation then reduces to finding the roots of the polynomial:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0 \tag{6.47}$$

We learned in Chapter 5 the MATLAB command for finding these roots, when needed. Now, using the superposition principle, and assuming all the roots are distinct, the general solution of the homogeneous differential equation is given by:

$$y_{\text{homog.}} = c_1 \exp(s_1 t) + c_2 \exp(s_2 t) + \ldots + c_n \exp(s_n t) \tag{6.48}$$

where $s_1, s_2, \ldots, s_n$ are the above roots and $c_1, c_2, \ldots, c_n$ are constant determined from the initial conditions of the solution and all its derivatives to order $n - 1$.

NOTE In the case that two or more of the roots are equal, it is easy to verify that the solution of the homogeneous ODE includes, instead of a constant multiplied by the exponential term corresponding to that root, a polynomial multiplying the exponential function. The degree of this polynomial is $(m - 1)$ if m is the degeneracy of the root in question.

Example 6.4

Find the transient solutions to the second-order differential equation.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{6.49}$$

Solution: The characteristic polynomial associated with this ODE is the second-degree equation given by:

$$as^2 + bs + c = 0 \tag{6.50}$$

The roots of this equation are $s_{\pm} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The nature of the solutions is very dependent on the sign of the descriminant $(b^2 - 4ac)$:

- If $b^2 - 4ac > 0$, the two roots are distinct and real. Call these roots α_1 and α_2; the solution is then:

$$y_{\text{homog.}} = c_1 \exp(\alpha_1 t) + c_2 \exp(\alpha_2 t) \tag{6.51}$$

In many physical problems of interest, we desire solutions that are zero at infinity, that is, decay over a finite time. This requires that both α_1 and α_2 be negative; or if only one of them is negative, that the c coefficient of the exponentially increasing solution be zero. This class of solutions is called the over-damped class.

- If $b^2 - 4ac = 0$, the two roots are equal, and we call this root $\alpha_{\text{degen.}}$. The solution to the differential equation is

$$y_{\text{homog.}} = (c_1 + c_2\, t)\exp(\alpha_{\text{degen.}} t) \tag{6.52}$$

The polynomial, multiplying the exponential function, is of degree one here because the degeneracy of the root is of degree two. This class of solutions is referred to as the critically damped class.

- If $b^2 - 4ac < 0$, the two roots are complex conjugates of each other, and their real part is negative for physically interesting cases. If we denote these roots by $s_\pm = -\alpha \pm j\beta$, the solutions to the homogeneous differential equations take the form:

$$y_{\text{homog.}} = \exp(-\alpha t)(c_1 \cos(\beta t) + c_2 \sin(\beta t)) \tag{6.53}$$

This class of solutions is referred to as the under-damped class.

In-Class Exercises

Find and plot the transient solutions to the following homogeneous equations, using the indicated initial conditions:

Pb. 6.28 $a = 1, b = 3, c = 2 \quad y(t = 0) = 1 \quad y'(t = 0) = -3/2$

Pb. 6.29 $a = 1, b = 2, c = 1 \quad y(t = 0) = 1 \quad y'(t = 0) = 2$

Pb. 6.30 $a = 1, b = 5, c = 6 \quad y(t = 0) = 1 \quad y'(t = 0) = 0$

6.5.2 Steady-State Solutions

In this subsection, we find the particular solutions of the ODEs when the driving force is a single-term sinusoidal.

As pointed out previously, because of the superposition principle, it is also possible to write the steady-state solution for any combination of such inputs. This, combined with the Fourier series techniques (briefly discussed in Chapter 7), will also allow you to write the solution for any periodic function.

We discuss in detail the particular solution for the first-order and the second-order differential equations because these represent, as previously shown in Section 4.7, important cases in circuit analysis.

Example 6.5

Find the particular solution to the first-order differential equation:

$$a\frac{dy}{dt} + by = A\cos(\omega t) \tag{6.54}$$

Solution: We guess that the particular solution of this ODE is a sinusoidal of the form:

$$\begin{aligned} y_{\text{partic.}}(t) &= B\cos(\omega t - \phi) = B[\cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega t)] \\ &= B_c\cos(\omega t) + B_s\sin(\omega t) \end{aligned} \tag{6.55}$$

Our task now is to find B_c and B_s that would force Eq. (6.55) to be the solution of Eq. (6.54). Therefore, we substitute this trial solution in the differential equation and require that, separately, the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$ terms match on both sides of the resulting equation. These requirements are necessary for the trial solution to be valid at all times. The resulting conditions are

$$B_s = \frac{a\omega}{b}B_c \qquad B_c = \frac{Ab}{a^2\omega^2 + b^2} \tag{6.56}$$

from which we can also deduce the polar form of the solution, giving:

$$B^2 = \frac{A^2}{a^2\omega^2 + b^2} \qquad \tan(\phi) = \frac{a\omega}{b} \tag{6.57}$$

Example 6.6

Find the particular solution to the second-order differential equation:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = A\cos(\omega t) \tag{6.58}$$

Solution: Again, take the trial particular solution to be of the form:

$$
\begin{aligned}
y_{\text{partic.}}(t) &= B\cos(\omega t - \phi) = B[\cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega t)] \\
&= B_c\cos(\omega t) + B_s\sin(\omega t)
\end{aligned}
\tag{6.59}
$$

Repeating the same steps as in Example 6.5, we find:

$$
B_s = \frac{b\omega}{(c - a\omega^2)^2 + \omega^2 b^2} A \qquad B_c = \frac{(c - a\omega^2)}{(c - a\omega^2)^2 + \omega^2 b^2} A \tag{6.60}
$$

$$
B^2 = \frac{A^2}{(c - a\omega^2)^2 + \omega^2 b^2} \qquad \tan(\phi) = \frac{b\omega}{c - a\omega^2} \tag{6.61}
$$

6.5.3 Applications to Circuit Analysis

An important application of the above forms for the particular solutions is in circuit analysis with inductors, resistors, and capacitors as elements. We describe later a more efficient analytical method (phasor representation) for solving this kind of problem; however, we believe that it is important that you also become familiar with the present technique.

6.5.3.1 RC Circuit

Referring to the *RC* circuit shown in Figure 4.4, we derived the differential equation that the potential difference across the capacitor must satisfy; namely:

$$
RC\frac{dV_C}{dt} + V_C = V_0\cos(\omega t) \tag{6.62}
$$

This is a first-order differential equation, the particular solution of which is given in Example 6.5 if we were to identify the coefficients in the ODE as follows: $a = RC$, $b = 1$, $A = V_0$.

6.5.3.2 RLC Circuit

Referring to the circuit, shown in Figure 4.5, the voltage across the capacitor satisfies the following ODE:

$$
LC\frac{d^2V_c}{dt^2} + RC\frac{dV_C}{dt} + V_C = V_0\cos(\omega t) \tag{6.63}
$$

This equation can be identified with that given in Example 6.6 if the ODE coefficients are specified as follows: $a = LC$, $b = RC$, $c = 1$, $A = V_0$.

In-Class Exercises

Pb. 6.31 This problem pertains to the *RC* circuit:

a. Write the output signal V_C in the amplitude-phase representation.

b. Plot the gain response as a function of a normalized frequency that you will have to select. (The gain of a circuit is defined as the ratio of the amplitude of the output signal over the amplitude of the input signal.)

c. Determine the phase response of the system (i.e., the relative phase of the output signal to that of the input signal as function of the frequency) also as function of the normalized frequency.

d. Can this circuit be used as a filter (i.e., a device that lets through only a specified frequency band)? Specify the parameters of this band.

Pb. 6.32 This problem pertains to the *RLC* circuit:

a. Write the output signal V_C in the amplitude-phase representation.

b. Defining the resonance frequency of this circuit as: $\omega_0 = \frac{1}{\sqrt{LC}}$, find at which frequency the gain is maximum, and find the width of the gain curve.

c. Plot the gain curve and the phase curve for the following cases: $\frac{\omega_0 L}{R} = 0.1,\ 1,\ 10.$

d. Can you think of a possible application for this circuit?

Pb. 6.33 Can you think of a mechanical analog to the *RLC* circuit? Identify in that case the physical parameters in the corresponding ODE.

Pb. 6.34 Assume that the source potential in the *RLC* circuit has five frequency components at ω, 2ω, …, 5ω of equal amplitude. Plot the input and output potentials as a function of time over the interval $0 < \omega t < 2\pi$. Assume that $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ and $\frac{\omega_0 L}{R} = 1.$

6.6 Phasors

A technique in widespread use to compute the steady-state solutions of systems with sinusoidal input is the method of phasors. In this and the following two chapter sections, we define phasors, learn how to use them to add two or more signals having the same frequency, and how to find the particular solution of an ODE with a sinusoidal driving function.

There are two key ideas behind the phasor representation of a signal:

1. A real, sinusoidal time-varying signal may be represented by a complex time-varying signal.
2. This complex signal can be represented as the product of a complex number that is independent of time and a complex signal that is dependent on time.

Example 6.7

Decompose the signal $V = A\cos(\omega t + \phi)$ according to the above prescription.

Solution: This signal can, using the polar representation of complex numbers, also be written as:

$$V = A\cos(\omega t + \phi) = \mathrm{Re}[A\exp(j(\omega t + \phi))] = \mathrm{Re}[Ae^{j\phi}e^{j\omega t}] \tag{6.64}$$

where the phasor, denoted with a tilde on top of its corresponding signal symbol, is given by:

$$\tilde{V} = Ae^{j\phi} \tag{6.65}$$

(*Warning:* Do not mix the tilde symbol that we use here, to indicate a phasor, with the overbar that denotes complex conjugation.)

Having achieved the above goal of separating the time-independent part of the complex number from its time-dependent part, we now learn how to manipulate these objects. A lot of insight can be immediately gained if we note that this form of the phasor is exactly in the polar form of a complex number, with clear geometric interpretation for its magnitude and phase.

6.6.1 Phasor of Two Added Signals

The sum of two signals with common frequencies but different amplitudes and phases is

$$V_{tot.} = A_{tot.}\cos(\omega t + \phi_{tot.}) = A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2) \tag{6.66}$$

To write the above result in phasor notation, note that the above sum can also be written as follows:

$$\begin{aligned} V_{tot.} &= \mathrm{Re}[A_1\exp(j(\omega t + \phi_1)) + A_2\exp(j(\omega t + \phi_2))] \\ &= \mathrm{Re}[(A_1e^{j\phi_1} + A_2e^{j\phi_2})e^{j\omega t}] \end{aligned} \tag{6.67}$$

and where

$$\tilde{V}_{tot.} = A_{tot.} e^{j\phi_{tot.}} = \tilde{V}_1 + \tilde{V}_2 \tag{6.68}$$

Preparatory Exercise

Pb. 6.35 Write the analytical expression for $A_{tot.}$ and $\phi_{tot.}$ in Eq. (6.68) as functions of the amplitudes and phases of signals 1 and 2.

The above result can, of course, be generalized to the sum of many signals; specifically:

$$\begin{aligned} V_{tot.} &= A_{tot.} \cos(\omega t + \phi_{tot.}) = \sum_{n=1}^{N} A_n \cos(\omega t + \phi_n) \\ &= \mathrm{Re}\left[\sum_{n=1}^{N} A_n \exp(j\omega t + j\phi_n)\right] = \mathrm{Re}\left[e^{j\omega t} \sum_{n=1}^{N} A_n e^{j\phi_n}\right] \end{aligned} \tag{6.69}$$

and

$$\tilde{V}_{tot.} = \sum_{n=1}^{N} \tilde{V}_n \tag{6.70}$$

$$\Rightarrow A_{tot.} = \left|\tilde{V}_{tot.}\right| \tag{6.71}$$

$$\phi_{tot.} = \arg(\tilde{V}_{tot.}) \tag{6.72}$$

That is, the resultant field can be obtained through the simple operation of adding all the complex numbers (phasors) that represent each of the individual signals.

Example 6.8

Given ten signals, the phasor of each of the form $A_n e^{j\phi_n}$, where the amplitude and phase for each have the functional forms $A_n = \dfrac{1}{n}$ and $\phi_n = n^2$, write a MATLAB program to compute the resultant sum phasor.

Solution: Edit and execute the following *script M-file:*

```
N=10;
n=1:N;
amplituden=1./n;
phasen=n.^2;
phasorn=amplituden.*exp(j.*phasen);
phasortot=sum(phasorn);
amplitudetot=abs(phasortot)
phasetot=angle(phasortot)
```

In-Class Exercises

Pb. 6.36 Could you have estimated the answer to Example 6.8? Justify your reasoning.

Pb. 6.37 Show that if you add N signals with the same magnitude and frequency but with phases equally distributed over the $[0, 2\pi]$ interval, the resultant phasor will be zero. (*Hint:* Remember the result for the sum of the roots of unity.)

Pb. 6.38 Show that the resultant signal from adding N signals having the same frequency has the largest amplitude when all the individual signals are in phase (this situation is referred to as maximal constructive interference).

Pb. 6.39 In this problem, we consider what happens if the frequency and amplitude of N different signals are still equal, but the different phases of the signals are randomly distributed over the $[0, 2\pi]$ interval. Find the amplitude of the resultant signal if $N = 1000$, and compare it with the maximal constructive interference result. (*Hint:* Recall that the **`rand(1,N)`** command generates a 1-D array of N random numbers from the interval [0, 1].)

Pb. 6.40 The service provided to your home by the electric utility company is a two-phase service. This means that two 110-V/60-Hz hot lines plus a neutral (ground) line terminate in your panel. The hot lines are π out of phase.

a. Which signal would you use to drive your clock radio or your toaster?

b. What configuration will you use to drive your oven or your dryer?

Pb. 6.41 In most industrial environments, electric power is delivered in what is called a three-phase service. This consists of three 110-V/60-Hz lines with phases given by $(0, 2\pi/3, 4\pi/3)$. What is the maximum voltage that you can obtain from any combination of two of these signals?

Pb. 6.42 Two- and three-phase power can be extended to N-phase power. In such a scheme, the N-110-V/60-Hz signals are given by:

$$V_n = 110\cos\left(120t + \frac{2\pi n}{N}\right) \quad \text{and} \quad n = 0, 1, \ldots, N-1$$

While the sum of the voltage of all the lines is zero, the instantaneous power is not. Find the total power, assuming that the power from each line is proportional to the square of its time-dependent expression. (*Hint:* Use the double angle formula for the cosine function.)

$$p_n(t) = A^2\cos^2\left(\omega t + \frac{2\pi n}{N}\right) \quad \text{and} \quad P = \sum_{n=0}^{N-1} p_n$$

NOTE Another designation in use for a 110-V line is an rms value of 110, and not the value of the maximum amplitude as used above.

6.7 Interference and Diffraction of Electromagnetic Waves

6.7.1 The Electromagnetic Wave

Electromagnetic waves (em waves) are manifest as radio and TV broadcast signals, microwave communication signals, light of any color, X-rays, γ-rays, etc. While these waves have different sources and methods of generation and require different kinds of detectors, they do share some general characteristics. They differ from each other only in the value of their frequencies. Indeed, it was one of the greatest intellectual achievements of the 19th century when Maxwell developed the system of equations, now named in his honor, to describe these waves' commonality. The most important of these properties is that they all travel in a vacuum with, what is called, the speed of light c ($c = 3 \times 10^8$ m/s). The detailed study of these waves is the subject of many electrophysics subspecialties.

Electromagnetic waves are traveling waves. To understand their mathematical nature, consider a typical expression for the electric field associated with such waves:

$$E(z, t) = E_0 \cos[kz - \omega t] \tag{6.73}$$

Here, E_0 is the amplitude of the wave, z is the spatial coordinate parallel to the direction of propagation of the wave, and k is the wavenumber.

Note that if we plot the field for a fixed time, for example, at $t = 0$, the field takes the shape of a sinusoidal function in space:

$$E(z, t = 0) = E_0 \cos[kz] \tag{6.74}$$

From the above equation, one deduces that the wavenumber $k = 2\pi/\lambda$, where λ is the wavelength of the wave (i.e., the length after which the wave shape reproduces itself).

Now let us look at the field when an observer, located at $z = 0$, would measure it as a function of time. Then:

$$E(z = 0, t) = E_0 \cos[\omega t] \tag{6.75}$$

The temporal period, that is, the time after which the wave shape reproduces itself, is $T = \frac{2\pi}{\omega}$, where ω is the angular frequency of the wave.

Next, we want to relate the wavenumber to the angular frequency. To do that, consider an observer located at $z = 0$. The observer measures the field at $t = 0$ to be E_0. At time Δt later, he should measure the same field, whether he uses Eq. (6.74) or (6.75) if he takes $\Delta z = c\Delta t$, the distance that the wave crest has moved, and where c is the speed of propagation of the wave. From this, one deduces that the wavenumber and the angular frequency are related by $kc = \omega$. This relation holds true for all electromagnetic waves; that is, as the frequency increases, the wavelength decreases.

If two traveling waves have the same amplitude and frequency, but one is traveling to the right while the other is traveling to the left, the result is a standing wave. The following program permits visualization of this standing wave.

```
x=0:0.01:5;
a=1;
k=2*pi;
w=2*pi;
t=0:0.05:2;
M=moviein(41);
  for m=1:41;
  z1=cos(k*x-w*t(m));
  z2=cos(k*x+w*t(m));
  z=z1+z2;
  plot(x,z,'r');
  axis([0 5 -3 3]);
```

```
    M(:,m)=getframe;
    end
  movie(M,20)
```

Compare the spatio-temporal profile of the resultant to that for a single wave (i.e., set **x2** = 0).

6.7.2 Addition of Two Electromagnetic Waves

In many practical instances, we are faced with the problem that two em waves originating from the same source, but following different spatial paths, meet again at a certain position. We want to find the total field at this position resulting from adding the two waves. We first note that, in the simplest case where the amplitude of the two fields are kept equal, the effect of the different paths is only to dephase one of the waves from the other by an amount: $\Delta\phi = k\Delta l$, where Δl is the path difference. In effect, the total field is given by:

$$E_{tot.}(t) = E_0 \cos[\omega t + \phi_1] + E_0 \cos[\omega t + \phi_2] \tag{6.76}$$

where $\Delta\phi = \phi_1 - \phi_2$. This form is similar to those studied in the addition of two phasors and we will hence describe the problem in this language.

The resultant phasor is

$$\tilde{E}_{tot.} = \tilde{E}_1 + \tilde{E}_2 \tag{6.77}$$

Preparatory Exercise

Pb. 6.43 Find the modulus and the argument of the resultant phasor given in Eq. (6.74) as a function of E_0 and $\Delta\phi$. From this expression, deduce the relation that relates the path difference corresponding to when the resultant phasor has maximum magnitude and that when its magnitude is a minimum. The curve describing the modulus square of the resultant phasor is what is commonly referred to as the interference pattern of two waves.

6.7.3 Generalization to N-waves

The addition of electromagnetic waves can be generalized to N-waves.

Example 6.9

Find the resultant field of equal-amplitude N-waves, each phase-shifted from the preceding by the same $\Delta\phi$.

Solution: The problem consists of computing an expression of the following kind:

$$\tilde{E}_{tot.} = \tilde{E}_1 + \tilde{E}_2 + \ldots + \tilde{E}_n = E_0(1 + e^{j\Delta\phi} + e^{j2\Delta\phi} + \ldots + e^{j(N-1)\Delta\phi}) \tag{6.78}$$

We have encountered such an expression previously. This sum is that corresponding to the sum of a geometric series. Computing this sum, the modulus square of the resultant phasor is

$$\begin{aligned} \left|\tilde{E}_{tot.}\right|^2 &= E_0^2 \frac{(1 - e^{jN\Delta\phi})}{(1 - e^{j\Delta\phi})} \frac{(1 - e^{-jN\Delta\phi})}{(1 - e^{-j\Delta\phi})} \\ &= E_0^2 \left(\frac{1 - \cos(N\Delta\phi)}{1 - \cos(\Delta\phi)} \right) = E_0^2 \left(\frac{\sin^2(N\Delta\phi / 2)}{\sin^2(\Delta\phi / 2)} \right) \end{aligned} \tag{6.79}$$

Because the source is the same for each of the components, the modulus of each phasor is related to the source amplitude by $E_0 = E_{source}/N$. It is usually as function of the source field that the results are expressed.

In-Class Exercises

Pb. 6.44 Plot the normalized square modulus of the resultant of N-waves as a function of $\Delta\phi$ for different values of N (5, 50, and 500) over the interval $-\pi < \Delta\phi < \pi$.

Pb. 6.45 Find the dependence of the central peak value of Eq. (6.79) on N.

Pb. 6.46 Find the phase shift that corresponds to the position of the first minimum of Eq. (6.79).

Pb. 6.47 Find in Eq. (6.79) the relative height of the first maximum (i.e., the one following the central maximum) to that of the central maximum as a function of N.

Pb. 6.48 In an antenna array with the field representing N aligned, equally spaced individual antennae excited by the same source is given by Eq. (6.78). If the line connecting the point of observation to the center of the array is making an angle θ with the antenna array, the phase shift is $\Delta\phi = \frac{2\pi}{\lambda} d\cos(\theta)$, where λ is the wavelength of radiation and d is the spacing between two consecutive antennae. Draw the polar plot of the total intensity as function of the angle θ for a spacing $d = \lambda/2$ for different values of N (2, 4, 6, and 10).

Pb. 6.49 Do the results of **Pb. 6.48** suggest to you a strategy for designing a multi-antenna system with sharp directivity? Can you think of a method, short of moving the antennae around, that permits this array to sweep a range of angles with maximum directivity?

Pb. 6.50 The following program simulates a 25-element array-swept radar beam.

```
th=0:0.01:pi;
t=-0.5*sqrt(3):0.05*sqrt(3):0.5*sqrt(3);
N=25;
M=moviein(21);
  for m=1:21;
  I=(1/N^2)*(sin(N*((pi/4)*cos(th)+(pi/4)*t(m)))...
  ^2)./((sin((pi/4)*cos(th)+(pi/4)*t(m))).^2);
  polar(th,I);
  M(:,m)=getframe;
  end
movie(M,10)
```

a. Determine the range of the sweeping angle.

b. Can you think of an electronic method for implementing this task?

6.8 Solving ac Circuits with Phasors: The Impedance Method

In Section 6.5, we examined the conventional technique for solving some simple ac circuits problems. We suggested that using phasors may speed up the determination of the solution. This is the subject of this chapter section.

We will treat, using this technique, the simple *RLC* circuit already solved through other means in order to give you a measure of the simplifications that can be achieved in circuit analysis through this technique. We then proceed to use the phasor technique to investigate another circuit configuration: the infinite *LC* ladder. The power of the phasor technique will also be put to use when we, topologically, solve much more difficult circuit problems than the one-loop category encountered thus far. Essentially, a straightforward algebraic technique can give the voltages and currents for any circuit. We illustrate this latter case in Chapter 8.

Recalling that the voltage drops across resistors, inductors, and capacitors can all be expressed as function of the current, its derivative, and its integral, our goal is to find a technique to replace these operators by simple algebraic operations. The key to achieving this goal is to realize that:

If: $$I = I_0\cos(\omega t + \phi) = \text{Re}[e^{j\omega t}(I_0 e^{j\phi})] \tag{6.80}$$

Then: $$\frac{dI}{dt} = -I_0\omega\sin(\omega t + \phi) = \text{Re}[e^{j\omega t}(I_0(j\omega)e^{j\phi})] \tag{6.81}$$

and

$$\int I dt = \frac{I_0}{\omega}\sin(\omega t + \phi) = \text{Re}\left[e^{j\omega t}\left(I_0\left(\frac{1}{j\omega}\right)e^{j\phi}\right)\right] \tag{6.82}$$

From Eqs. (4.25) to (4.27) and Eqs. (6.80) to (6.82), we can deduce that the phasors representing the voltages across resistors, inductors, and capacitors can be written as follows:

$$\tilde{V}_R = \tilde{I}R = \tilde{I}Z_R \tag{6.83}$$

$$\tilde{V}_L = \tilde{I}(j\omega L) = \tilde{I}Z_L \tag{6.84}$$

$$\tilde{V}_C = \frac{\tilde{I}}{(j\omega C)} = \tilde{I}Z_C \tag{6.85}$$

The terms multiplying the current phasor on the RHS of each of the above equations are called the resistor, the inductor, and the capacitor impedances, respectively.

6.8.1 *RLC* Circuit Phasor Analysis

Let us revisit this problem first discussed in Section 4.7. Using Kirchoff's voltage law and Eqs. (6.83) to (6.85), we can write the following relation between the phasor of the current and that of the source potential:

$$\tilde{V}_s = \tilde{I}R + \tilde{I}(j\omega L) + \frac{\tilde{I}}{(j\omega C)} = \tilde{I}\left[R + j\omega L + \frac{1}{j\omega C}\right] \tag{6.86}$$

That is, we can immediately compute the modulus and the argument of the phasor of the current if we know the values of the circuit components, the source voltage phasor, and the frequency of the source.

In-Class Exercises

Using the expression for the circuit resonance frequency ω_0 previously introduced in **Pb. 6.32**, for the *RLC* circuit:

Pb. 6.51 Show that the system's total impedance can be written as:

$$Z = R + j\omega_0 L\left(\nu - \frac{1}{\nu}\right), \quad \text{where} \quad \nu = \frac{\omega}{\omega_0} = \omega\sqrt{LC}$$

Pb. 6.52 Show that $Z(\nu) = \overline{Z}(1/\nu)$; and from this result, deduce the value of ν at which the impedance is entirely real.

Pb. 6.53 Find the magnitude and the phase of the total impedance.

Pb. 6.54 Selecting for the values of the circuit elements $LC = 1$, $RC = 3$, and $\omega = 1$, compare the results that you obtain through the phasor analytical method with the numerical results for the voltage across the capacitor in an *RLC* circuit that you found while solving Eq. (4.36).

The Transfer Function

As you would have discovered solving **Pb. 6.54**, the ratio of the phasor of the potential difference across the capacitor with that of the ac source can be directly calculated once the value of the current phasor is known. This ratio is called the Transfer Function for this circuit if the voltage across the capacitor is taken as the output of this circuit. It is obtained by combining Eqs. (6.85) and (6.86) and is given by:

$$\frac{\tilde{V}_c}{\tilde{V}_s} = \frac{1}{(j\omega RC - \omega^2 LC + 1)} = H(\omega) \tag{6.87}$$

The Transfer Function concept can be generalized to any ac circuit. It refers to the ratio of the output voltage phasor to the input voltage phasor. It incorporates all the relevant information on the details of the circuit. It is the standard form for representing the response of a circuit to a single sinusoidal function input.

Homework Problem

Pb. 6.55 Plot the magnitude and the phase of theTransfer Function given in Eq. (6.87) as a function of ω, for $LC = 1$, $RC = 3$.

6.8.2 The Infinite *LC* Ladder

The *LC* ladder consists of an infinite repetition of the basic elements shown in Figure 6.2.

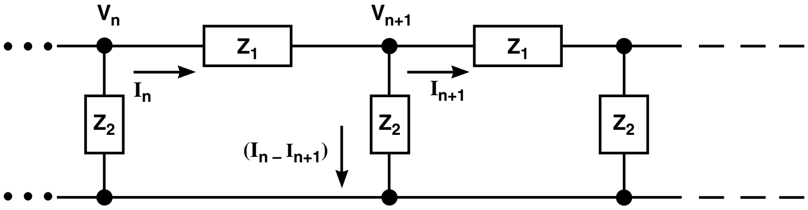

FIGURE 6.2
The circuit of an infinite LC ladder.

Using the definition of impedances, the phasors of the n and $(n + 1)$ voltages and currents are related through:

$$\tilde{V}_n - \tilde{V}_{n+1} = Z_1 \tilde{I}_n \tag{6.88}$$

$$\tilde{V}_{n+1} = (\tilde{I}_n - \tilde{I}_{n+1}) Z_2 \tag{6.89}$$

From Eq. (6.88), we deduce the following expressions for $\tilde{I}_n$ and $\tilde{I}_{n+1}$:

$$\tilde{I}_n = \frac{\tilde{V}_n - \tilde{V}_{n+1}}{Z_1} \tag{6.90}$$

$$\tilde{I}_{n+1} = \frac{\tilde{V}_{n+1} - \tilde{V}_{n+2}}{Z_1} \tag{6.91}$$

Substituting these values for the currents in Eq. (6.89), we deduce a second-order difference equation for the voltage phasor:

$$\tilde{V}_{n+2} - \left(\frac{Z_1}{Z_2} + 2\right)\tilde{V}_{n+1} + \tilde{V}_n = 0 \tag{6.92}$$

The solution of this difference equation can be directly obtained by the techniques discussed in Chapter 2 for obtaining solutions of homogeneous difference equations. The physically meaningful solution is given by:

$$\lambda = 1 + \frac{1}{Z_2}\left\{\frac{Z_1}{2} - \sqrt{\frac{Z_1^2}{4} + Z_2 Z_1}\right\} \tag{6.93}$$

and the voltage phasor at node n is then given by:

$$\tilde{V}_n = \tilde{V}_s \lambda^n \tag{6.94}$$

We consider the model where $Z_1 = j\omega L$ and $Z_2 = 1/(j\omega C)$, respectively, for an inductor and a capacitor. The expression for λ then takes the following form:

$$\lambda = \left(1 - \frac{\upsilon^2}{2}\right) - j\left(\upsilon^2 - \frac{\upsilon^4}{4}\right)^{1/2} \tag{6.95}$$

where the normalized frequency is defined by $\upsilon = \omega / \omega_0 = \omega\sqrt{LC}$. We plot in Figure 6.3 the magnitude and the phase of the root λ as function of the normalized frequency.

As can be directly observed from an examination of Figure 6.3, the magnitude of λ is equal to 1 (i.e., the magnitude of $\tilde{V}_n$ is also 1) for $\upsilon < \upsilon_{cutoff} = 2$, while it drops precipitously after that, with the dropoff in the potential much steeper with increasing node number. Physically, this represents extremely short penetration through the ladder for signals with frequencies larger than the cutoff frequency. Furthermore, note that for $\upsilon < \upsilon_{cutoff} = 2$, the phase of $\tilde{V}_n$ increases linearly with the index n; and because it is negative, it corresponds to a delay in the signal as it propagates down the ladder, which corresponds to a finite velocity of propagation for the signal.

Before we leave this ladder circuit, it is worth addressing a practical concern. While it is impossible to realize an infinite-dimensional ladder, the above conclusions do not change by much if we replace the infinite ladder by a finite ladder and we terminate it after awhile by a resistor with resistance equal to $\sqrt{L/C}$.

In-Class Exercise

Pb. 6.56 Repeat the analysis given above for the *LC* ladder circuit, if instead we were to:

a. Interchange the positions of the inductors and the capacitors in the ladder circuit. Based on this result and the above *LC* result, can you design a bandpass filter with a flat response?

b. Interchange the inductor elements by resistors. In particular, compute the input impedance of this circuit.

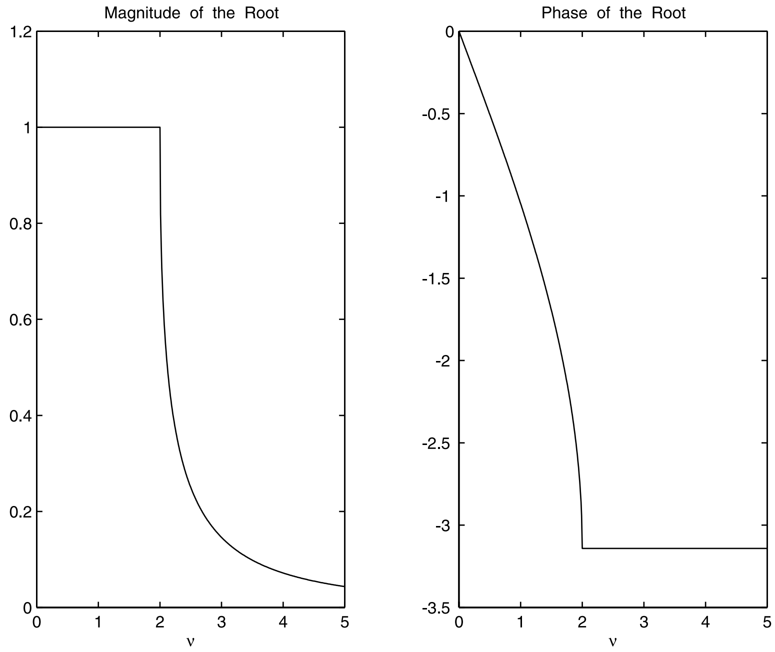

FIGURE 6.3
The magnitude (left panel) and the phase (right panel) of the characteristic root of the infinite LC ladder.

6.9 Transfer Function for a Difference Equation with Constant Coefficients*

In Section 6.8.1, we found the Transfer Function for what essentially was a simple ODE. In this section, we generalize the technique to find the Transfer Function of a difference equation with constant coefficients. The form of the difference equation is given by:

$$\begin{aligned} y(k) = b_0 u(k) + b_1 u(k-1) + \ldots + b_m u(k-m) \\ - a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n) \end{aligned} \tag{6.96}$$

Along the same route that we followed in the phasor treatment of ODE, assume that both the input and output are of the form:

$$u(k) = Ue^{j\Omega k} \text{ and } y(k) = Ye^{j\Omega k} \tag{6.97}$$

where Ω is a normalized frequency; typically, in electrical engineering applications, the real frequency multiplied by the sampling time. Replacing these expressions in the difference equation, we obtain:

$$\frac{Y}{U} = \frac{\sum_{l=0}^{m} b_l e^{-j\Omega l}}{1 + \sum_{l=1}^{n} a_l e^{-j\Omega l}} = \frac{\sum_{l=0}^{m} b_l z^{-l}}{1 + \sum_{l=1}^{n} a_l z^{-l}} \equiv H(z) \tag{6.98}$$

where, by convention, $z = e^{j\Omega}$.

Example 6.10

Find the Transfer Function of the following difference equation:

$$y(k) = u(k) + \frac{2}{3} y(k-1) - \frac{1}{3} y(k-2) \tag{6.99}$$

Solution: By direct substitution into Eq. (6.98), we find:

$$H(z) = \frac{1}{1 - \frac{2}{3} z^{-1} + \frac{1}{3} z^{-2}} = \frac{z^2}{z^2 - \frac{2}{3} z + \frac{1}{3}} \tag{6.100}$$

It is to be noted that the Transfer Function is a ratio of two polynomials. The zeros of the numerator are called the zeros of the Transfer Function, while the zeros of the denominator are called its poles. If the coefficients of the difference equations are real, then by the Fundamental Theorem of Algebra, the zeros and the poles are either real or are pairs of complex conjugate numbers.

The Transfer Function fully describes any linear system. As will be shown in linear systems courses, the z-transform of the Transfer Function gives the weights for the solution of the difference equation, while the values of the poles of the Transfer Function determine what are called the system modes of the solution. These are the modes intrinsic to the circuit, and they do not depend on the specific form of the input function.

Furthermore, it is worth noting that the study of recursive filters, the backbone of digital signal processing, can be simply reduced to a study of the Transfer Function under different configurations. In Applications 2 and 3 that follow, we briefly illustrate two particular digital filters in wide use.

Application 1

Using the Transfer Function formalism, we want to estimate the accuracy of the three integrating schemes discussed in Chapter 4. We want to compare the Transfer Function of each of those algorithms to that of the exact result, obtained upon integrating exactly the function $e^{j\omega t}$.

The exact result for integrating the function $e^{j\omega t}$ is, of course, $\frac{e^{j\omega t}}{j\omega}$, thus giving for the exact Transfer Function for integration the expression:

$$H_{exact} = \frac{1}{j\omega} \tag{6.101}$$

Before proceeding with the computation of the transfer function for the different numerical schemes, let us pause for a moment and consider what we are actually doing when we numerically integrate a function. We go through the following steps:

1. We discretize the time interval over which we integrate; that is, we define the sampling time Δt, such that the discrete points abscissa are given by $k(\Delta t)$, where k is an integer.
2. We write a difference equation for the integral relating its values at the discrete points with its values and that of the integrand at discrete points with equal or smaller indices.
3. We obtain the value of the integral by iterating the defining difference equation.

The test function used for the estimation of the integration methods accuracy is written at the discrete points as:

$$y(k) = e^{jk\omega(\Delta t)} \tag{6.102}$$

The difference equations associated with each of the numerical integration schemes are:

$$I_T(k+1) = I_T(k) + \frac{\Delta t}{2}(y(k+1) + y(k)) \tag{6.103}$$

$$I_{MP}(k+1) = I_{MP}(k) + \Delta t y(k+1/2) \tag{6.104}$$

$$I_S(k+1) = I_S(k-1) + \frac{\Delta t}{3}(y(k+1) + 4y(k) + y(k-1)) \tag{6.105}$$

leading to the following expressions for the respective Transfer Functions:

$$H_T = \frac{\Delta t}{2}\frac{e^{j\omega(\Delta t)}+1}{e^{j\omega(\Delta t)}-1} \tag{6.106}$$

$$H_{MP} = \Delta t\frac{e^{j\omega(\Delta t)/2}}{e^{j\omega(\Delta t)}-1} \tag{6.107}$$

$$H_S = \frac{\Delta t}{3}\frac{(e^{j\omega(\Delta t)}+4+e^{-j\omega(\Delta t)})}{e^{j\omega(\Delta t)}-e^{-j\omega(\Delta t)}} \tag{6.108}$$

The measures of accuracy of the integration scheme are the ratios of these Transfer Functions to that of the exact expression. These are given, respectively, by:

$$R_T = \frac{(\omega\Delta t/2)}{\sin(\omega\Delta t/2)}\cos(\omega\Delta t/2) \tag{6.109}$$

$$R_{MP} = \frac{(\omega\Delta t/2)}{\sin(\omega\Delta t/2)} \tag{6.110}$$

$$R_S = \left(\frac{\omega\Delta t}{3}\right)\frac{\cos(\omega\Delta t)+2}{\sin(\omega\Delta t)} \tag{6.111}$$

Table 6.1 gives the value of this ratio as a function of the number of sampling points, per oscillation period, selected in implementing the different integration subroutines:

TABLE 6.1

Accuracy of the Different Elementary Numerical Integrating Methods

Number of Sampling Points in a Period	R_T	R_{MP}	R_S
100	0.9997	1.0002	1.0000
50	0.9986	1.0007	1.0000
40	0.9978	1.0011	1.0000
30	0.9961	1.0020	1.0000
20	0.9909	1.0046	1.0001
10	0.9591	1.0206	1.0014
5	0.7854	1.1107	1.0472

As can be noted, the error is less than 1% for any of the discussed methods as long as the number of points in one oscillation period is larger than 20, although the degree of accuracy is best, as we expected based on geometrical arguments, for Simpson's rule.

In a particular application, where a finite number of frequencies are simultaneously present, the choice of (Δt) for achieving a specified level of accuracy in the integration subroutine should ideally be determined using the shortest of the periods present in the integrand.

Application 2

As mentioned earlier, the Transfer Function technique is the prime tool for the analysis and design of digital filters. In this and the following application, we illustrate its use in the design of a low-pass digital filter and a digital prototype bandpass filter.

The low-pass filter, as its name indicates, filters out the high-frequency components from a signal.

Its defining difference equation is given by:

$$y(k) = (1-a)y(k-1) + au(k) \tag{6.112}$$

giving for its Transfer Function the expression:

$$H(z) = \frac{a}{1-(1-a)z^{-1}} \tag{6.113}$$

Written as a function of the normalized frequency, it is given by:

$$H(e^{j\Omega}) = \frac{ae^{j\Omega}}{e^{j\Omega}-(1-a)} \tag{6.114}$$

We plot, in Figure 6.4, the magnitude and the phase of the transfer function as a function of the normalized frequency for the value of $a = 0.1$. Note that the gain is equal to 1 for $\Omega = 0$, and decreases monotonically thereafter.

To appreciate the operation of this filter, consider a sinusoidal signal that has been contaminated by the addition of noise. We can simulate the noise by adding to the original signal an array consisting of random numbers with maximum amplitude equal to 20% of the original signal. The top panel of Figure 6.5 represents the contaminated signal. If we pass this signal through a low-pass filter, the lower panel of Figure 6.5 shows the outputted filtered signal.

As can be observed, the noise, which is a high-frequency signal, has been filtered out and the signal shape has been almost restored to its original shape before that noise was added.

The following *script M-file* simulates the above operations:

```
t=linspace(0,4*pi,300);
N=length(t);
s=sin(t);
n=0.3*rand(1,N);
u=s+n;
```

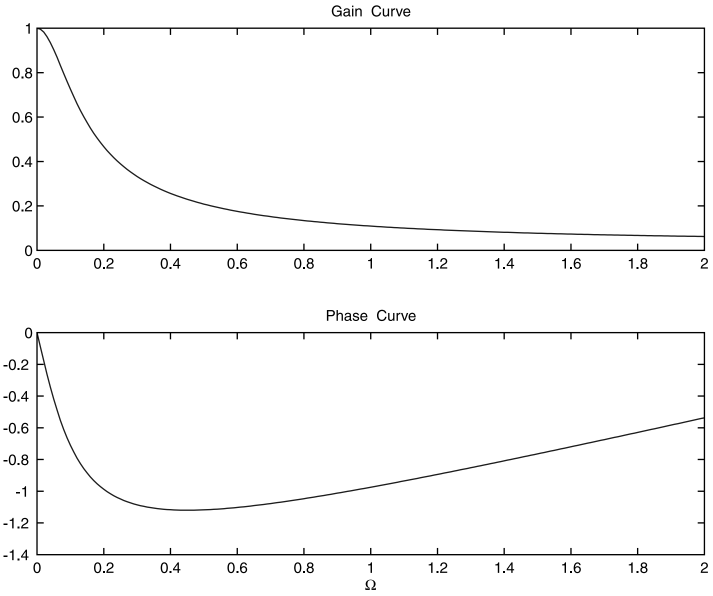

FIGURE 6.4
The gain (top panel) and phase (bottom panel) responses of a low-pass filter as a function of the frequency.

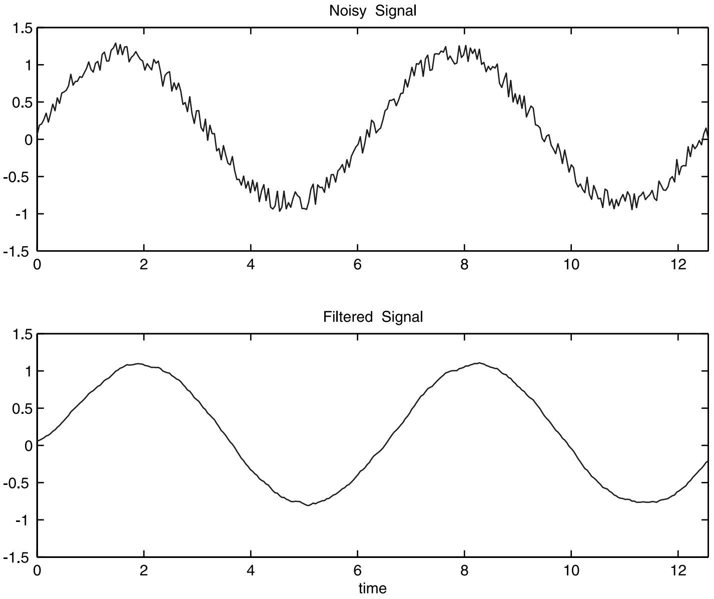

FIGURE 6.5
The action of a low-pass filter. Top panel: Profile of the signal contaminated by noise. Bottom panel: Profile of the filtered signal.

```
y(1)=u(1);
  for k=2:N
    y(k)=+0.9*y(k-1)+0.1*u(k);
  end
subplot(2,1,1)
plot(t,u)
axis([0 4*pi -1.5 1.5]);
title('Noisy Signal')
subplot(2,1,2)
plot(t,y)
title('Filtered Signal')
axis([0 4*pi -1.5 1.5]);
```

Application 3

The digital prototype bandpass filter ideally filters out from a signal all frequencies lower than a given frequency and higher than another frequency. In practice, the cutoffs are not so sharp and the lower and higher cut-off frequencies of the bandpass are defined as those at which the gain curve (i.e., the magnitude of the Transfer Function as function of the frequency) is at $(1/\sqrt{2})$ its maximum value.

The difference equation that describes this prototype filter is

$$\begin{aligned} y(k) &= \{(1-r)\sqrt{1-2r\cos(2\Omega_0)+r^2}\}u(k) \\ &+ 2r\cos(\Omega_0)y(k-1) - r^2y(k-2) \end{aligned} \tag{6.115}$$

where Ω_0 is the normalized frequency with maximum gain and r is a number close to 1.

The purpose of the following analysis is, given the lower and higher cutoff normalized frequencies, to find the quantities Ω_0 and r in the above difference equation.

The Transfer Function for the above difference equation is given by:

$$H(z) = \frac{g_0 z^2}{z^2 - 2r\cos(\Omega_0)z + r^2} \tag{6.116}$$

where

$$g_0 = (1-r)\sqrt{1-2r\cos(2\Omega_0)+r^2} \tag{6.117}$$

and

$$z = e^{j\Omega}$$

The gain of this filter, or equivalently the magnitude of the Transfer Function, is

$$\left|H(e^{j\Omega})\right| = \frac{(1-r)\sqrt{1-2r\cos(2\Omega_0)+r^2}}{(1+Ar+Br^2+Ar^3+r^4)} \tag{6.118}$$

where

$$A = -4\cos(\Omega)\cos(\Omega_0) \tag{6.119}$$

$$B = 4\cos^2(\Omega) + 4\cos^2(\Omega_0) - 2 \tag{6.120}$$

The lower and upper cutoff frequencies are defined, as previously noted, by the condition:

$$\left|H(e^{j\Omega_{(1,2)}})\right| = \frac{1}{\sqrt{2}} \tag{6.121}$$

Substituting condition (6.121) in the gain expression (6.118) leads to the conclusion that the cutoff frequencies are obtained from the solutions of the following quadratic equation:

$$\begin{aligned} &\cos^2(\Omega) - \left[\frac{(1+r^2)\cos(\Omega_0)}{r}\right]\cos(\Omega) \\ &+ \frac{(1-r)^2}{4r^2}[4r\cos(2\Omega_0) - (1-r)^2] + \cos^2(\Omega_0) = 0 \end{aligned} \tag{6.122}$$

Adding and subtracting the roots of this equation, we deduce after some straightforward algebra, the following determining equations for Ω_0 and r:

1. r is the root in the interval [0, 1] of the following eighth-degree polynomial:

$$r^8 + (a-b)r^6 - 8ar^5 + (14a-2b-2)r^4 - 8ar^3 + (a-b)r^2 + 1 = 0 \tag{6.123}$$

where

$$a = (\cos(\Omega_1) + \cos(\Omega_2))^2 \tag{6.124}$$

$$b = (\cos(\Omega_1) - \cos(\Omega_2))^2 \tag{6.125}$$

2. Ω_0 is given by:

$$\Omega_0 = \cos^{-1}\left[\frac{ra^{1/2}}{1+r^2}\right] \tag{6.126}$$

Example 6.12

Write a program to determine the parameters r and Ω_0 of a prototype bandpass filter if the cutoff frequencies and the sampling time are given.

Solution: The following *script M-file* implements the above target:

```
f1= ;             %enter the lower cutoff
f2= ;             %enter the upper cutoff
tau= ;            %enter the sampling time
w1=2*pi*f1*tau;
w2=2*pi*f2*tau;
a=(cos(w1)+cos(w2))^2;
b=(cos(w1)-cos(w2))^2;
p=[1 0 a-b -8*a 14*a-2*b-2 -8*a a-b 0 1];
rr=roots(p);
r=rr(find(rr>0 & rr<1 & imag(rr)==0))
w0=acos((r*a^(1/2))/(1+r^2));
f0=(1/(2*pi*tau))*w0
```

In Figure 6.6, we show the gain and phase response for this filter, for the case that the cutoff frequencies are chosen to be 1000 Hz and 1200 Hz, and the sampling rate is 10 μs.

To test the action of this filter, we input into it a signal that consists of a mixture of a sinusoid having a frequency at the frequency of the maximum gain of this filter and a number of its harmonics; for example,

$$u(t) = \sin(2\pi f_0 t) + 0.5\sin(4\pi f_0 t) + 0.6\sin(6\pi f_0 t) \tag{6.127}$$

We show in Figure 6.7 the input and the filtered signals. As expected from an analysis of the gain curve, only the fundamental frequency signal has survived. The amplitude of the filtered signal settles to that of the fundamental frequency signal following a short transient period.

NOTE Before leaving this topic, it is worth noting that the above prototype bandpass filter can have sharper cutoff features (i.e., decreasing the value of

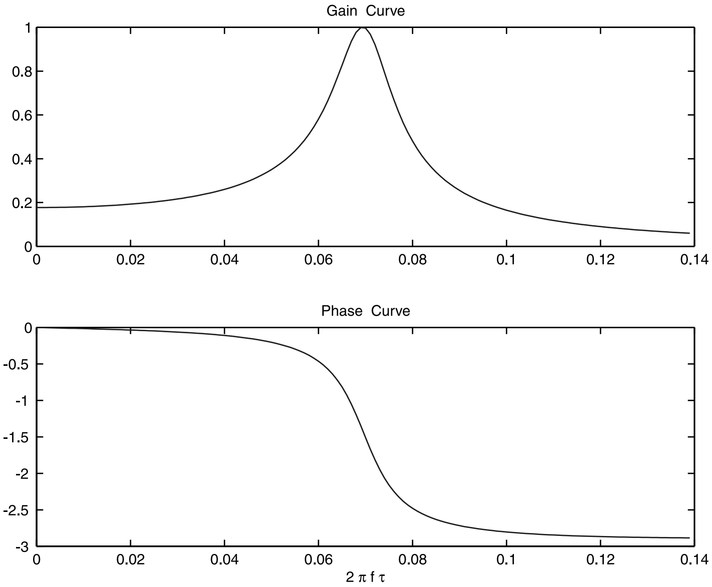

FIGURE 6.6
The transfer function of a prototype bandpass filter. Top panel: Plot of the gain curve as function of the normalized frequency. Bottom panel: Plot of the phase curve as function of the normalized frequency.

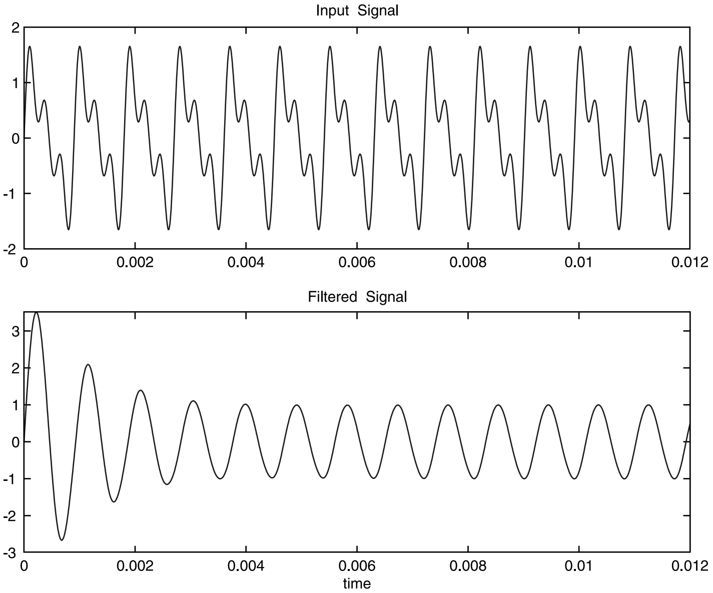

FIGURE 6.7
The filtering action of a prototype bandpass filter. Top panel: Input signal consists of a combination of a fundamental frequency signal (equal to the frequency corresponding to the filter maximum gain) and two of its harmonics. Bottom panel: Filtered signal.

the gain curve for frequencies below the lower cutoff and higher than the upper cutoff) through having many of these prototype filters in cascade. This will be a topic of study in future linear system or filter design courses.

In-Class Exercises

Pb. 6.59 Work out the missing algebraic steps in the derivation leading to Eqs. (6.123) through (6.126).

Pb. 6.60 Given the following values for the lower and upper cutoff frequencies and the sampling time:

$$f_1 = 200 \text{ Hz}; \quad f_2 = 400 \text{ Hz}; \quad \tau = 10^{-5} \text{ s}$$

find f_0 and plot the gain curve as function of the normalized frequency for the bandpass prototype filter.

6.10 MATLAB Commands Review

`abs`	Computes the modulus of a complex number.
`angle`	Computes the argument of a complex number.
`conj`	Computes the complex conjugate of a complex number.
`find`	Finds the locations of elements in an array that satifies certain specified conditions.
`imag`	Computes the imaginary part of a complex number.
`real`	Computes the real part of a complex number.

7

Vectors

7.1 Vectors in Two Dimensions (2-D)

A vector in 2-D is defined by its length and the angle it makes with a reference axis (usually the x-axis). This vector is represented graphically by an arrow. The tail of the arrow is called the initial point of the vector and the tip of the arrow is the terminal point. Two vectors are equal when both their length and angle with a reference axis are equal.

7.1.1 Addition

The sum of two vectors $\vec{u} + \vec{v} = \vec{w}$ is a vector constructed graphically as follows. At the tip of the first vector, draw a vector equal to the second vector, such that its tail coincides with the tip of the first vector. The resultant vector has as its tail that of the first vector, and as its tip, the tip of the just-drawn second vector (the Parallelogram Rule) (see Figure 7.1).

The negative of a vector is that vector whose tip and tail have been exchanged from those of the vector. This leads to the conclusion that the difference of two vectors is the other diagonal in the parallelogram (Figure 7.2).

7.1.2 Multiplication of a Vector by a Real Number

If we multiply a vector $\vec{v}$ by a real number k, the result is a vector whose length is k times the length of $\vec{v}$, and whose direction is that of $\vec{v}$ if k is positive, and opposite if k is negative.

7.1.3 Cartesian Representation

It is most convenient for a vector to be described by its projections on the x-axis and on the y-axis, respectively; these are denoted by (v_1, v_2) or (v_x, v_y). In this representation:

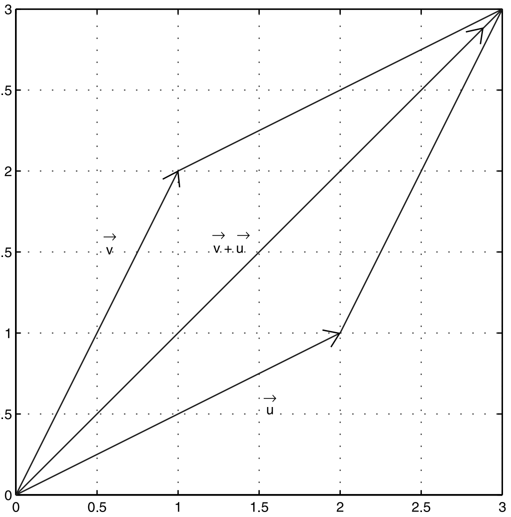

FIGURE 7.1
Sum of two vectors.

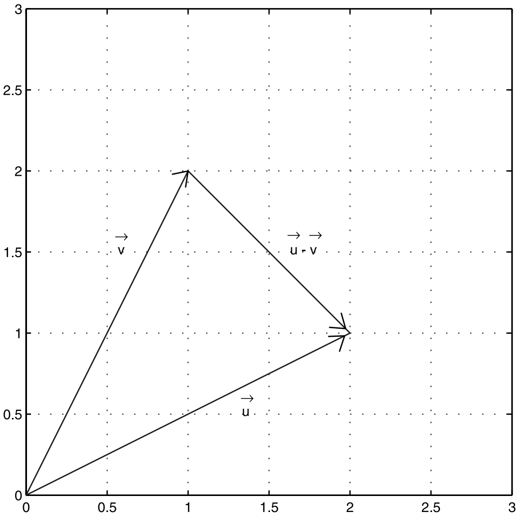

FIGURE 7.2
Difference of two vectors.

$$\vec{u} = (u_1, u_2) = (u_1)\hat{e}_1 + (u_2)\hat{e}_2 \tag{7.1}$$

where $\hat{e}_1$ and $\hat{e}_2$ are the unit vectors (length is 1) parallel to the x-axis and y-axis, respectively. In terms of this representation, we can write the zero vector, the sum of two vectors, and the multiplication of a vector by a real number as follows:

$$\vec{0} = (0, 0) = 0\hat{e}_1 + 0\hat{e}_2 \tag{7.2}$$

$$\vec{u} + \vec{v} = \vec{w} = (u_1 + v_1, u_2 + v_2) = (u_1 + v_1)\hat{e}_1 + (u_2 + v_2)\hat{e}_2 \tag{7.3}$$

$$k\vec{u} = (ku_1, ku_2) = (ku_1)\hat{e}_1 + (ku_2)\hat{e}_2 \tag{7.4}$$

Preparatory Exercise

Pb. 7.1 Using the above definitions and properties, prove the following identities:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$$

$$\vec{u} + (-\vec{u}) = \vec{0}$$

$$k(l\vec{u}) = (kl)\vec{u}$$

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$$

$$(k + l)\vec{u} = k\vec{u} + l\vec{u}$$

The norm of a vector is the length of this vector. Using the Pythagorean theorem, its square is:

$$\|\vec{u}\|^2 = u_1^2 + u_2^2 \tag{7.5}$$

and therefore the unit vector in the $\vec{u}$ direction, denoted by $\hat{e}_u$, is given by:

$$\hat{e}_u = \frac{1}{\sqrt{u_1^2 + u_2^2}}(u_1, u_2) \tag{7.6}$$

All of the above can be generalized to 3-D, or for that matter to n-dimensions. For example:

$$\hat{e}_u = \frac{1}{\sqrt{u_1^2 + u_2^2 + \dots u_n^2}}(u_1, u_2, \dots, u_n) \tag{7.7}$$

7.1.4 MATLAB Representation of the Above Results

MATLAB distinguishes between two kinds of vectors: the column vector and the row vector. As long as the components of the vectors are all real, the difference between the two is in the structure of the array. In the column vector case, the array representation is vertical and in the row vector case, the array representation is horizontal. This distinction is made for the purpose of including in a consistent structure the formulation of the dot product and the definition of matrix multiplication.

Example 7.1

Type and execute the following commands, while interpreting the output at each step:

```
V=[1 3 5 7]
W=[1;3;5;7]
V'
U=3*V
Z=U+V
Y=V+W       %you cannot add a row vector and a column
            %vector
```

You would have observed that:

1. The difference in the representation of the column and row vectors is in the manner they are separated inside the square brackets.
2. The single quotation mark following a vector with real components changes that vector from being a column vector to a row vector, and vice versa.
3. Multiplying a vector by a scalar simply multiplies each component of this vector by this scalar.
4. You can add two vectors of the same kind and the components would be adding by pairs.
5. You cannot add two vectors of different kinds; the computer will give you an error message alerting you that you are adding two quantities of different dimensions.

The MATLAB command for obtaining the norm of a vector is **norm**. Using this notation, it is a simple matter to define the unit vector in the same direction as a given vector.

Example 7.2

Find the length of the vector and the unit vector $u = [1 \quad 5 \quad 3 \quad 2]$ and the unit vector parallel to it.

```
u=[1 5 3 2]
lengthu=norm(u)          %length of vector u
unitu=u/(norm(u))        %unit vector parallel to u
lengthunitu=norm(unitu) %verify length of unit vector
```

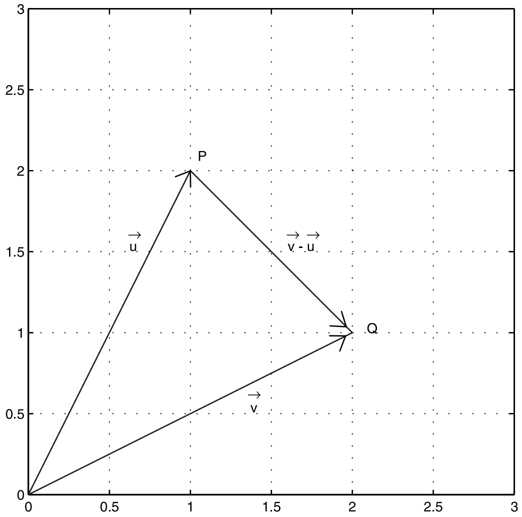

FIGURE 7.3
The geometry of the generalized Pythagorean theorem.

7.2 Dot (or Scalar) Product

If the angle between the vectors $\vec{u}$ and $\vec{v}$ is θ, then the dot product of the two vectors is:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta) \tag{7.8}$$

The dot product can also be expressed as a function of the vectors components. Referring to Figure 7.3, we know from trigonometry the relation relating the length of one side of a triangle with the length of the other two sides and the cosine of the angle between the other two sides. This relation is the generalized Pythagorean theorem. Referring to Figure 7.3, this gives:

$$\|PQ\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta) \tag{7.9}$$

but since:

$$\overline{P}\vec{Q} = \vec{v} - \vec{u} \tag{7.10}$$

$$\Rightarrow \|\vec{u}\|\|\vec{v}\|\cos(\theta) = \frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2) \tag{7.11}$$

and the dot product can be written as:

$$\vec{u} \cdot \vec{v} = \frac{1}{2}(u_1^2 + u_2^2 + v_1^2 + v_2^2 - (v_1 - u_1)^2 - (v_2 - u_2)^2 = u_1 v_1 + u_2 v_2 \tag{7.12}$$

In an *n*-dimensional space, the above expression is generalized to:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \tag{7.13}$$

and the norm square of the vector can be written as the dot product of the vector with itself; that is,

$$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \ldots + u_n^2 \tag{7.14}$$

Example 7.3

Parallelism and orthogonality of two vectors in a plane. Let the vectors $\vec{u}$ and $\vec{v}$ be given by: $\vec{u} = 3\hat{e}_1 + 4\hat{e}_2$ and $\vec{v} = a\hat{e}_1 + 7\hat{e}_2$. What is the value of *a* if the vectors are parallel, and if the vectors are orthogonal?

Solution:

Case 1: If the vectors are parallel, this means that they make the same angle with the *x*-axis. The tangent of this angle is equal to the ratio of the vector *x*-component to its *y*-component. This means that:

$$\frac{a}{7} = \frac{3}{4} \Rightarrow a = 21/4$$

Case 2: If the vectors are orthogonal, this means that the angle between them is 90°, and their dot product will be zero because the cosine for that angle is zero. This implies that:

$$3a + 28 = 0 \Rightarrow a = -28/3$$

Example 7.4

Find the unit vector in 2-D that is perpendicular to the line $ax + by + c = 0$.

Solution: Choose two arbitrary points on this line. Denote their coordinates by (x_1, y_1) and (x_2, y_2); being on the line, they satisfy the equation of the line:

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$

Substracting the first equation from the second equation, we obtain:

$$a(x_2 - x_1) + b(y_2 - y_1) = 0$$

which means that $(a, b) \perp (x_2 - x_1, y_2 - y_1)$, and the unit vector perpendicular to the line is:

$$\hat{e}_{\perp} = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$$

Example 7.5

Find the angle that the lines $3x + 2y + 2 = 0$ and $2x - y + 1 = 0$ and make together.

Solution: The angle between two lines is equal to the angle between their normal unit vectors. The unit vectors normal to each of the lines are, respectively:

$$\hat{n}_1 = \left(\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right) \quad \text{and} \quad \hat{n}_2 = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right)$$

Having the two orthogonal unit vectors, it is a simple matter to compute the angle between them:

$$\cos(\theta) = \hat{n}_1 \cdot \hat{n}_2 = \frac{4}{\sqrt{65}} \Rightarrow \theta = 1.0517 \text{ radians}$$

7.2.1 MATLAB Representation of the Dot Product

The dot product is written as the product of a row vector by a column vector of the same length.

Example 7.6

Find the dot product of the vectors:

$$u = [1 \quad 5 \quad 3 \quad 7] \quad \text{and} \quad v = [2 \quad 4 \quad 6 \quad 8]$$

Solution: Type and execute each of the following commands, while interpreting each output:

```
u=[1  5  3  7]
v=[2  4  6  8]
u*v'
v'*u
u*v              %you  cannot  multiply  two  rows
u'*v
u*u'
(norm(u))^2
```

As observed from the above results, in MATLAB, the dot product can be obtained only by the multiplication of a row on the left and a column of the same length on the right. If the order of a row and column are exchanged, we obtain a two-dimensional array structure (i.e., a matrix, the subject of Chapter 8). On the other hand, if we multiply two rows, MATLAB gives an error message about the non-matching of dimensions.

Observe further, as pointed out previously, the relation between the length of a vector and its dot product with itself.

In-Class Exercises

Pb. 7.2 Generalize the analytical technique, as previously used in Example 7.4 for finding the normal to a line in 2-D, to find the unit vector in 3-D that is perpendicular to the plane:

$$ax + by + cz + d = 0$$

(*Hint:* A vector is perpendicular to a plane if it is perpendicular to two non-collinear vectors in that plane.)

Pb. 7.3 Find, in 2-D, the distance of the point $P(x_0, y_0)$ from the line $ax + by + c = 0$. (*Hint:* Remember the geometric definition of the dot product.)

Pb. 7.4 Prove the following identities:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}, \quad \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}, \quad k \cdot (\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v}$$

7.3 Components, Direction Cosines, and Projections

7.3.1 Components

The *components* of a vector are the values of each element in the defining n-tuplet representation. For example, consider the vector $\vec{u} = [1 \quad 5 \quad 3 \quad 7]$ in real 4-D. We say that its first, second, third, and fourth components are 1, 5, 3, and 7, respectively. (We are maintaining, in this section, the arrow notation for the vectors, irrespective of the dimension of the space.)

The simplest basis of a n-dimensional vector space is the collection of n unit vectors, each having only one of their components that is non-zero and such that the location of this non-zero element is different for each of these basis vectors. This basis is not unique.

For example, in 4-D space, the canonical four-unit orthonormal basis vectors are given, respectively, by:

$$\hat{e}_1 = [1 \quad 0 \quad 0 \quad 0] \tag{7.15}$$

$$\hat{e}_2 = [0 \quad 1 \quad 0 \quad 0] \tag{7.16}$$

$$\hat{e}_3 = [0 \quad 0 \quad 1 \quad 0] \tag{7.17}$$

$$\hat{e}_4 = [0 \quad 0 \quad 0 \quad 1] \tag{7.18}$$

and the vector $\vec{u}$ can be written as a linear combination of the basis vectors:

$$\vec{u} = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3 + u_4\hat{e}_4 \tag{7.19}$$

The basis vectors are chosen to be orthonormal, which means that in addition to requiring each one of them to have unit length, they are also orthogonal two by two to each other. These properties of the basis vectors leads us to the following important result: the m^{th} component of a vector is obtained by taking the dot product of the vector with the corresponding unit vector, that is,

$$u_m = \hat{e}_m \cdot \vec{u} \tag{7.20}$$

7.3.2 Direction Cosines

The *direction cosines* are defined by:

$$\cos(\gamma_m) = \frac{u_m}{\|\vec{u}\|} = \frac{\hat{e}_m \cdot \vec{u}}{\|\vec{u}\|} \tag{7.21}$$

In 2-D or 3-D, these quantities have the geometrical interpretation of being the cosine of the angles that the vector $\vec{u}$ makes with the x, y, and z axes.

7.3.3 Projections

The *projection* of a vector $\vec{u}$ over a vector $\vec{a}$ is a vector whose magnitude is the dot product of the vector $\vec{u}$ with the unit vector in the direction of $\vec{a}$, denoted by $\hat{e}_a$, and whose orientation is in the direction of $\hat{e}_a$:

$$proj_{\vec{a}}(\vec{u}) = (\vec{u} \cdot \hat{e}_a)\hat{e}_a = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|} \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} \tag{7.22}$$

The component of $\vec{u}$ that is perpendicular to $\vec{a}$ is obtained by subtracting from $\vec{u}$ the projection vector of $\vec{u}$ over $\vec{a}$.

MATLAB Example

Assume that we have the vector $\vec{u} = \hat{e}_1 + 5\hat{e}_2 + 3\hat{e}_3 + 7\hat{e}_4$ and the vector $\vec{a} = 2\hat{e}_1 + 3\hat{e}_2 + \hat{e}_3 + 4\hat{e}_4$. We desire to obtain the components of each vector, the projection of $\vec{u}$ over $\vec{a}$, and the component of $\vec{u}$ orthogonal to $\vec{a}$.

Type, execute, and interpret at each step, each of the following commands using the above definitions:

```
u=[1 5 3 7]
a=[2 3 1 4]
u(1)
a(2)
prjuovera=((u*a')/(norm(a)^2))*a
orthoutoa=u-prjuovera
prjuovera*orthoutoa'
```

The last command should give you an answer that is zero, up to machine round-up errors because the projection of $\vec{u}$ over $\vec{a}$ and the component of $\vec{u}$ orthogonal to $\vec{a}$ are perpendicular.

7.4 The Dirac Notation and Some General Theorems*

Thus far, we have established some key practical results in real finite dimensional vector spaces; namely:

1. A vector can be decomposed into a linear combination of the basis vectors.
2. The dot product of two vectors can be written as the multiplication of a row vector by a column vector, each of whose elements are the components of the respective vectors.
3. The norm of a vector, a non-negative quantity, is the square root of the dot product of the vector with itself.
4. The unit vector parallel to a specific vector is that vector divided by its norm.
5. The projection of a vector on another can be deduced from the dot product of the two vectors.

To facilitate the statement of these results in a notation that will be suitable for infinite-dimensional vector spaces (which is very briefly introduced in Section 7.7), Dirac in his elegant formulation of quantum mechanics introduced a simple notation that we now present.

The Dirac notation represents the row vector by what he called the "bra-vector" and the column vector by what he called the "ket-vector," such that when a dot product is obtained by joining the two vectors, the result will be the scalar "bra-ket" quantity. Specifically:

$$\text{Column vector } \vec{u} \Rightarrow |u\rangle \tag{7.23}$$

$$\text{Row vector } \vec{v} \Rightarrow \langle v| \tag{7.24}$$

$$\text{Dot product } \vec{v} \cdot \vec{u} \Rightarrow \langle v|u\rangle \tag{7.25}$$

The *orthonormality* of the basis vectors is written as:

$$\langle m|n\rangle = \delta_{m,n} \tag{7.26}$$

where the basis vectors are referred to by their indices, and where $\delta_{m,n}$ is the Kroenecker delta, equal to 1 when its indices are equal, and zero otherwise.

The *norm of a vector,* a non-negative quantity, is given by:

$$(\mathit{norm\ \ of}\ \ |u\rangle)^2 = \|u\|^2 = \langle u|u\rangle \tag{7.27}$$

The *Decomposition rule* is written as:

$$|u\rangle = \sum_n c_n |n\rangle \tag{7.28}$$

where the components are obtained by multiplying Eq. (7.28) on the left by $\langle m|$. Using Eq. (7.26), we deduce:

$$\langle m|u\rangle = \sum_n c_n \langle m|n\rangle = \sum_n c_n \delta_{m,n} = c_m \tag{7.29}$$

Next, using the Dirac notation, we present the proofs of two key theorems of vector algebra: the Cauchy-Schwartz inequality and the triangle inequality.

7.4.1 Cauchy-Schwartz Inequality

Let $|u\rangle$ and $|v\rangle$ be any non-zero vectors; then:

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle \tag{7.30}$$

PROOF Let $\varepsilon = \pm 1$, ($\varepsilon^2 = 1$); then

$$\langle u|v\rangle = \varepsilon|\langle u|v\rangle| \quad \text{such that} \quad \begin{cases} \varepsilon = 1 & \text{if} \quad \langle u|v\rangle \geq 0 \\ \varepsilon = -1 & \text{if} \quad \langle u|v\rangle \leq 0 \end{cases} \tag{7.31}$$

Now, consider the ket $|\varepsilon u + tv\rangle$; its norm is always non-negative. Computing this norm square, we obtain:

$$\begin{aligned} \langle \varepsilon u + tv|\varepsilon u + tv\rangle &= \varepsilon^2\langle u|u\rangle + \varepsilon t\langle u|v\rangle + t\varepsilon\langle v|u\rangle + t^2\langle v|v\rangle \\ &= \langle u|u\rangle + 2\varepsilon t\langle u|v\rangle + t^2\langle v|v\rangle \\ &= \langle u|u\rangle + 2t|\langle u|v\rangle| + t^2\langle v|v\rangle \end{aligned} \tag{7.32}$$

The RHS of this quantity is a positive quadratic polynomial in t, and can be written in the standard form:

$$at^2 + bt + c \geq 0 \tag{7.33}$$

The non-negativity of this quadratic polynomial means that it can have at most one real root. This means that the descriminant must satisfy the inequality:

$$b^2 - 4ac \leq 0 \tag{7.34}$$

Replacing a, b, c by their values from Eq. (7.32), we obtain:

$$4|\langle u|v\rangle|^2 - 4\langle u|u\rangle\langle v|v\rangle \leq 0 \tag{7.35}$$

$$\Rightarrow |\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle \tag{7.36}$$

which is the desired result. Note that the equality holds if and only if the two vectors are linearly dependent (i.e., one vector is equal to a scalar multiplied by the other vector).

Example 7.7

Show that for any three non-zero numbers, u_1, u_2, and u_3, the following inequality always holds:

$$9 \leq (u_1 + u_2 + u_3)\left(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3}\right) \tag{7.37}$$

PROOF Choose the vectors $|v\rangle$ and $|w\rangle$ such that:

$$|v\rangle = |u_1^{1/2}, u_2^{1/2}, u_3^{1/2}\rangle \tag{7.38}$$

$$|w\rangle = \left|\left(\frac{1}{u_1}\right)^{1/2}, \left(\frac{1}{u_2}\right)^{1/2}, \left(\frac{1}{u_3}\right)^{1/2}\right\rangle \tag{7.39}$$

then:

$$\langle v|w\rangle = 3 \tag{7.40}$$

$$\langle v|v\rangle = (u_1 + u_2 + u_3) \tag{7.41}$$

$$\langle w|w\rangle = \left(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3}\right) \tag{7.42}$$

Applying the Cauchy-Schwartz inequality in Eq. (7.36) establishes the desired result. The above inequality can be trivially generalized to n-elements, which leads to the following important result for the equivalent resistance for resistors all in series or all in parallel.

Application

The equivalent resistance of n-resistors all in series and the equivalent resistance of the same n-resistors all in parallel obey the relation:

$$n^2 \leq \frac{R_{series}}{R_{parallel}} \tag{7.43}$$

PROOF The proof is straightforward. Using Eq. (7.37) and recalling Ohm's law for n resistors $\{R_1, R_2, \ldots, R_n\}$, the equivalent resistances for this combination, when all resistors are in series or are all in parallel, are given respectively by:

$$R_{series} = R_1 + R_2 + \ldots + R_n \tag{7.44}$$

and

$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \tag{7.45}$$

Question: Can you derive a similar theorem for capacitors all in series and all in parallel? (Remember that the equivalent capacitance law is different for capacitors than for resistors.)

7.4.2 Triangle Inequality

This is, as the name implies, a generalization of a theorem from Euclidean geometry in 2-D that states that the length of one side of a triangle is smaller or equal to the sum of the the other two sides. Its generalization is

$$\|u + v\| \leq \|u\| + \|v\| \tag{7.46}$$

PROOF Using the relation between the norm and the dot product, we have:

$$\begin{aligned} \|u + v\|^2 &= \langle u + v \mid u + v \rangle = \langle u \mid v \rangle + 2\langle u \mid v \rangle + \langle v \mid v \rangle \\ &= \|u\|^2 + 2\langle u \mid v \rangle + \|v\|^2 \leq \|u\|^2 + 2\left|\langle u \mid v \rangle\right| + \|v\|^2 \end{aligned} \tag{7.47}$$

Using the Cauchy-Schwartz inequality for the dot product appearing in the previous inequality, we deduce that:

$$\|u + v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = \left(\|u\| + \|v\|\right)^2 \tag{7.48}$$

which establishes the theorem.

Homework Problems

Pb. 7.5 Using the Dirac notation, generalize to n-dimensions the 2-D geometry Parallelogram theorem, which states that: *The sum of the squares of the diagonals of a parallelogram is equal to twice the sum of the squares of the side; or that:*

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

Pb. 7.6 Referring to the inequality of Eq. (7.43), which relates the equivalent resistances of n-resistors in series and in parallel, under what conditions does the equality hold?

7.5 Cross Product and Scalar Triple Product*

In this section and in Sections 7.6 and 7.7, we restrict our discussions to vectors in a 3-D space, and use the more familiar conventional vector notation.

7.5.1 Cross Product

DEFINITION If two vectors are given by $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ then their cross product, denoted by $\vec{u}\times\vec{v}$, is a vector given by:

$$\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \tag{7.49}$$

By simple substitution, we can infer the following properties for the cross product as summarized in the preparatory exercises below.

Preparatory Exercises

Pb. 7.7 Show, using the above definition for the cross product, that:

a. $\vec{u}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{u}\times\vec{v}) = 0 \Rightarrow \vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

b. $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\,\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2$ Called the Lagrange Identity

c. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$ Noncommutativity

d. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$ Distributive property

e. $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$

f. $\vec{u}\times\vec{0} = \vec{0}$

g. $\vec{u}\times\vec{u} = \vec{0}$

Pb. 7.8 Verify the following relations for the basis unit vectors:

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3; \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_1; \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2$$

Pb. 7.9 Ask your instructor to show you how the Right Hand rule is used to determine the direction of a vector equal to the cross product of two other vectors.

7.5.2 Geometric Interpretation of the Cross Product

As noted in **Pb. 7.7a**, the cross product is a vector that is perpendicular to its two constituents. This determines the resultant vector's direction. To determine its magnitude, consider the Lagrange Identity. If the angle between $\vec{u}$ and $\vec{v}$ is θ, then:

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2(\theta) \tag{7.50}$$

and

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin(\theta) \tag{7.51}$$

that is, the magnitude of the cross product of two vectors is the area of the parallelogram formed by these vectors.

7.5.3 Scalar Triple Product

DEFINITION If $\vec{u}, \vec{v}, \text{and } \vec{w}$ are vectors in 3-D, then $\vec{u} \cdot (\vec{v} \times \vec{w})$ is called the scalar triple product of $\vec{u}, \vec{v}, \text{and } \vec{w}$.

PROPERTY

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v}) \tag{7.52}$$

This property can be trivially proven by writing out the components expansions of the three quantities.

7.5.3.1 Geometric Interpretation of the Scalar Triple Product

If the vectors' $\vec{u}, \vec{v}, \text{and } \vec{w}$ original points are brought to the same origin, these three vectors define a parallelepiped. The absolute value of the scalar triple product can then be interpreted as the volume of this parallelepiped. We have shown earlier that $\vec{v} \times \vec{w}$ is a vector that is perpendicular to both $\vec{v}$ and $\vec{w}$, and whose magnitude is the area of the base parallelogram. From the definition of the scalar product, dotting this vector with $\vec{u}$ will give a scalar that is the product of the area of the parallelepiped base multiplied by the parallelepiped height, whose magnitude is exactly the volume of the parallelepiped.

The circular permutation property of Eq. (7.52) then has a very simple geometric interpretation: in computing the volume of a parallelepiped, it does not matter which surface we call base.

MATLAB Representation

The cross product of the vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is found using the **`cross(u,v)`** command.

The triple scalar product of the vectors $\vec{u}, \vec{v},$ and $\vec{w}$ is found through the **`det([u;v;w])`** command. Make sure that the vectors defined as arguments of these functions are defined as 3-D vectors, so that the commands work and the results make sense.

Example 7.8

Given the vectors $\vec{u}$ = (2, 1, 0), $\vec{v}$ = (0, 3, 0), $\vec{w}$ = (1, 2, 3), find the cross product of the separate pairs of these vectors, and the volume of the parallelepiped formed by the three vectors.

Solution: Type, execute, and interpret at each step, each of the following commands, using the above definitions:

```
u=[2 1 0]
v=[0 3 0]
w=[1 2 3]
ucrossv=cross(u,v)
ucrossw=cross(u,w)
vcrossw=cross(v,w)
paralvol=abs(det([u;v;w]))
paralvol2=abs(cross(u,v)*w')
```

Question: Verify that the last command is an alternate way of writing the volume of the parallelepiped expression.

In-Class Exercises

Pb. 7.10 Compute the shortest distance from New York to London. (*Hint:* (1) A great circle is the shortest path between two points on a sphere; (2) the angle between the radial unit vectors passing through each of the cities can be obtained from their respective latitude and longitude.)

Pb. 7.11 Find two unit vectors that are orthogonal to both vectors given by:

$$\vec{a} = (2,-1,2) \quad \text{and} \quad \vec{b} = (1,2,-3)$$

Pb. 7.12 Find the area of the triangle with vertices at the points:

$$A(0,-1,1), \quad B(3,1,0), \quad \text{and} \quad C(-2,0,2)$$

Pb. 7.13 Find the volume of the parallelepiped formed by the three vectors:

$$\vec{u} = (1,2,0), \quad \vec{v} = (0,3,0), \quad \vec{w} = (1,2,3)$$

Pb. 7.14 Determine the equation of a plane that passes through the point (1, 1, 1) and is normal to the vector (2, 1, 2).

Pb. 7.15 Find the angle of intersection of the planes:

$$x + y - z = 0 \quad \text{and} \quad x - 3y + z - 1 = 0$$

Pb. 7.16 Find the distance between the point (3, 1, –2) and the plane $z = 2x - 3y$.

Pb. 7.17 Find the equation of the line that contains the point (3, 2, 1) and is perpendicular to the plane $x + 2y - 2z = 2$. Write the parametric equation for this line.

Pb. 7.18 Find the point of intersection of the plane $2x - 3y + z = 6$ and the line

$$\frac{x-1}{3} = \frac{y+1}{1} = \frac{z-2}{2}$$

Pb. 7.19 Show that the points (1, 5), (3, 11), and (5, 17) are collinear.

Pb. 7.20 Show that the three vectors $\vec{u}, \vec{v}, \text{and } \vec{w}$ are coplanar:

$$\vec{u} = (2,3,5); \quad \vec{v} = (2,8,1); \quad \vec{w} = (8,22,12)$$

Pb. 7.21 Find the unit vector normal to the plane determined by the points (0, 0, 1), (0, 1, 0), and (1, 0, 0).

Homework Problem

Pb. 7.22 Determine the tetrahedron with the largest surface area whose vertices P_0, P_1, P_2, and P_3 are on the unit sphere $x^2 + y^2 + z^2 = 1$.

(*Hints:* (1) Designate the point P_0 as north pole and confine P_1 to the zero meridian. With this choice, the coordinates of the vertices are given by:

$$P_0 = (\theta_0 = \pi / 2, \phi_0 = 0)$$
$$P_1 = (\theta_1, \phi_1 = 0)$$
$$P_2 = (\theta_2, \phi_2)$$
$$P_3 = (\theta_3, \phi_3)$$

(2) From symmetry, the optimal tetrahedron will have a base (P_1, P_2, P_3) that is an equilateral triangle in a plane parallel to the equatorial plane. The latitude of (P_1, P_2, P_3) is θ, while their longitudes are (0, 2π/3, –2π/3), respectively. (3) The area of the tetrahedron is the sum of the areas of the four triangles (012), (023), (031), (123), where we are indicating each point by its subscript. (4) Express the area as function of θ. Find the value of θ that maximizes this quantity.)

7.6 Vector Valued Functions

As you may recall, in Chapter 1 we described curves in 2-D and 3-D by parametric equations. Essentially, we gave each of the coordinates as a function of a parameter. In effect, we generated a vector valued function because the position of the point describing the curve can be written as:

$$\vec{R}(t) = x(t)\hat{e}_1 + y(t)\hat{e}_2 + z(t)\hat{e}_3 \tag{7.53}$$

If the parameter t was chosen to be time, then the tip of the vector $\vec{R}(t)$ would be the position of a point on that curve as a function of time. In mechanics, finding $\vec{R}(t)$ is ultimately the goal of any problem in the dynamics of a point particle. In many problems of electrical engineering design of tubes and other microwave engineering devices, we need to determine the position of electrons whose motion we control by a variety of electrical and magnetic fields geometries. The following are the kinematics variables of the problem. The dynamics form the subject of mechanics courses.

To help visualize the shape of a curve generated by the tip of the position vector $\vec{R}(t)$, we introduce the tangent vector and the normal vector to the curve and the curvature of the curve.

The velocity vector field associated with the above position vector is defined through:

$$\frac{d\vec{R}(t)}{dt} = \frac{dx(t)}{dt}\hat{e}_1 + \frac{dy(t)}{dt}\hat{e}_2 + \frac{dz(t)}{dt}\hat{e}_3 \tag{7.54}$$

and the unit vector tangent to the curve is given by:

$$\hat{T}(t) = \frac{\dfrac{d\vec{R}(t)}{dt}}{\left\|\dfrac{d\vec{R}(t)}{dt}\right\|} \tag{7.55}$$

This is, of course, the unit vector that is always in the direction of the velocity of the particle.

LEMMA

If a vector valued function $\vec{V}(t)$ *has a constant value, then its derivative* $\dfrac{d\vec{V}(t)}{dt}$ *is orthogonal to it.*

PROOF The proof of this lemma is straightforward. If the length of the vector is constant, then its dot product with itself is a constant; that is, $\vec{V}(t)\cdot\vec{V}(t) = C.$ Differentiating both sides of this equation gives $\dfrac{d\vec{V}(t)}{dt}\cdot\vec{V}(t) = 0,$ and the orthogonality between the two vectors is thus established.

The tangential unit vector $\hat{T}(t)$ is, by definition, constructed to have unit length. We construct the norm to the curve by taking the unit vector in the direction of the time-derivative of the tangential vector; that is,

$$\hat{N}(t) = \frac{\dfrac{d\hat{T}(t)}{dt}}{\left\|\dfrac{d\hat{T}(t)}{dt}\right\|} \tag{7.56}$$

The curvature of the curve is

$$\kappa = \frac{\left\|\dfrac{d\hat{T}(t)}{dt}\right\|}{\left\|\dfrac{d\vec{R}(t)}{dt}\right\|} \tag{7.57}$$

Example 7.9

Find the tangent, normal, and curvature of the trajectory of a particle moving in uniform circular motion of radius a and with angular frequency ω.

Solution: The parametric equation of motion is

$$\vec{R}(t) = a\cos(\omega t)\hat{e}_1 + a\sin(\omega t)\hat{e}_2$$

The velocity vector is

$$\frac{d\vec{R}(t)}{dt} = -a\omega\sin(\omega t)\hat{e}_1 + a\omega\cos(\omega t)\hat{e}_2$$

and its magnitude is $a\omega$.

The tangent vector is therefore:

$$\hat{T}(t) = -\sin(\omega t)\hat{e}_1 + \cos(\omega t)\hat{e}_2$$

The normal vector is

$$\hat{N}(t) = -\cos(\omega t)\hat{e}_1 - \sin(\omega t)\hat{e}_2$$

The radius of curvature is

$$\kappa(t) = \frac{\left\|\frac{d\hat{T}(t)}{dt}\right\|}{\left\|\frac{d\vec{R}(t)}{dt}\right\|} = \frac{\|-\omega\cos(\omega t)\hat{e}_1 - \omega\sin(\omega t)\hat{e}_2\|}{\|-a\omega\sin(\omega t)\hat{e}_1 + a\omega\cos(\omega t)\hat{e}_2\|} = \frac{1}{a} = \text{constant}$$

Homework Problems

Pb. 7.23 Show that in 2-D the radius of curvature can be written as:

$$\kappa = \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{3/2}}$$

where the prime refers to the first derivative with respect to time, and the double-prime refers to the second derivative with respect to time.

Pb. 7.24 Using the parametric equations for an ellipse given in Example 1.13, find the curvature of the ellipse as function of *t*.

a. At what points is the curvature a minimum, and at what points is it a maximum?

b. What does the velocity do at the points of minimum and maximum curvature?

c. On what dates of the year does the planet Earth pass through these points on its trajectory around the sun?

7.7 Line Integral

As you may have already learned in an elementary physics course: if a force $\vec{F}$ is applied to a particle that moves by an infinitesimal distance $\Delta\vec{l}$, then the infinitesimal work done by the force on the particle is the scalar product of the force by the displacement; that is,

$$\Delta W = \vec{F} \cdot \Delta\vec{l} \tag{7.58}$$

Now, to calculate the work done when the particle moves along a curve *C*, located in a plane, we need to define the concept of a line integral.

Suppose that the curve is described parametrically [i.e., $x(t)$ and $y(t)$ are given]. Furthermore, suppose that the vector field representing the force is given by:

$$\vec{F} = P(x, y)\hat{e}_x + Q(x, y)\hat{e}_y \tag{7.59}$$

The displacement element is given by:

$$\Delta l = \Delta x\hat{e}_x + \Delta y\hat{e}_y \tag{7.60}$$

The infinitesimal element of work, which is the dot product of the above two quantities, can then be written as:

$$\Delta W = P\Delta x + Q\Delta y \tag{7.61}$$

This expression can be simplified if the curve is written in parametric form. Assuming the parameter is *t*, then ΔW can be written as a function of the single parameter *t*:

$$\Delta W = P(t)\frac{dx}{dt}\Delta t + Q(t)\frac{dy}{dt}\Delta t = \left(P(t)\frac{dx}{dt} + Q(t)\frac{dy}{dt}\right)\Delta t \tag{7.62}$$

and the total work can be written as an integral over the single variable t:

$$W = \int_{t_0}^{t_1}\left(P(t)\frac{dx}{dt} + Q(t)\frac{dy}{dt}\right)dt \tag{7.63}$$

Homework Problems

Pb. 7.25 How much work is done in moving the particle from the point (0, 0) to the point (3, 9) in the presence of the force $\vec{F}$ along the following two different paths?

a. The parabola $y = x^2$.

b. The line $y = 3x$.

The force is given by:

$$\vec{F} = xy\hat{e}_x + (x^2 + y^2)\hat{e}_y$$

Pb. 7.26 Let $\vec{F} = y\hat{e}_x + x\hat{e}_y$. Calculate the work moving from (0, 0) to (1, 1) along each of the following curves:

a. The straight line $y = x$.

b. The parabola $y = x^2$.

c. The curve C described by the parametric equations:

$$x(t) = t^{3/2} \quad \text{and} \quad y(t) = t^5$$

A vector field such as the present one, whose line integral is independent of the path chosen between fixed initial and final points, is said to be conservative. In your vector calculus course, you will establish the necessary and sufficient conditions for a vector field to be conservative. The importance of conservative fields lies in the ability of their derivation from a scalar potential. More about this topic will be discussed in electromagnetic courses.

7.8 Infinite Dimensional Vector Spaces*

This chapter section introduces some preliminary ideas on infinite-dimensional vector spaces. We assume that the components of this vector space are complex numbers rather than real numbers, as we have restricted ourselves thus far. Using these ideas, we discuss, in a very preliminary fashion, Fourier series and Legendre polynomials.

We use the Dirac notation to stress the commonalties that unite the finite- and infinite-dimensional vector spaces. We, at this level, sacrifice the mathematical rigor for the sake of simplicity, and even commit a few sins in our treatment of limits. A more formal and rigorous treatment of this subject can be found in many books on functional analysis, to which we refer the interested reader for further details.

A Hilbert space is much the same type of mathematical object as the vector spaces that you have been introduced to in the preceding sections of this chapter. Its elements are functions, instead of *n*-dimensional vectors. It is infinite-dimensional because the function has a value, say a component, at each point in space, and space is continuous with an infinite number of points.

The Hilbert space has the following properties:

1. The space is linear under the two conditions that:
   a. If a is a constant and $|\varphi\rangle$ is any element in the space, then $a|\psi\rangle$ is also an element of the space; and
   b. If a and b are constants, and $|\varphi\rangle$ and $|\psi\rangle$ are elements belonging to the space, then $a|\varphi\rangle + b|\psi\rangle$ is also an element of the space.
2. There is an inner (dot) product for any two elements in the space. The definition adopted here for this inner product for functions defined in the interval $t_{\min} \leq t \leq t_{\max}$ is:

$$\langle \psi | \varphi \rangle = \int_{t_{\min}}^{t_{\max}} \overline{\psi}(t)\varphi(t)dt \tag{7.64}$$

3. Any element of the space has a norm ("length") that is positive and related to the inner product as follows:

$$\|\varphi\|^2 = \langle \varphi | \varphi \rangle = \int_{t_{\min}}^{t_{\max}} \overline{\varphi}(t)\varphi(t)dt \tag{7.65}$$

   Note that the requirement for the positivity of a norm is that which necessitated the complex conjugation in the definition of the bra-vector.
4. The Hilbert space is complete; or loosely speaking, the Hilbert space contains all its limit points. This condition is too technical and will not be further discussed here.

In this Hilbert space, we define similar concepts to those in finite-dimensional vector spaces:

- *Orthogonality*. Two vectors are orthogonal if:

$$\langle \psi | \varphi \rangle = \int_{t_{\min}}^{t_{\max}} \overline{\psi}(t)\varphi(t)dt = 0 \tag{7.66}$$

- *Basis vectors*. Any function in Hilbert space can be expanded in a linear combination of the basis vectors $\{u_n\}$, such that:

$$|\varphi\rangle = \sum_n c_n |u_n\rangle \tag{7.67}$$

  and such that the elements of the basis vectors obey the orthonormality relations:

$$\langle u_m | u_n \rangle = \delta_{m,n} \tag{7.68}$$

- *Decomposition rule*. To find the c_n's, we follow the same procedure adopted for finite-dimensional vector spaces; that is, take the inner product of the expansion in Eq. (7.67) with the bra $\langle u_m|$. We obtain, using the orthonormality relations [Eq. (7.68)], the following:

$$\langle u_m | \varphi \rangle = \sum_n c_n \langle u_m | u_n \rangle = \sum_n c_n \delta_{m,n} = c_m \tag{7.69}$$

  Said differently, c_m is the projection of the ket $|\varphi\rangle$ onto the bra $\langle u_m|$.
- *The norm as a function of the components*. The norm of a vector can be expressed as a function of its components. Using Eqs. (7.67) and (7.68), we obtain:

$$\|\varphi\|^2 = \langle \varphi | \varphi \rangle = \sum_n \sum_m \bar{c}_n c_m \langle u_n | u_m \rangle = \sum_n \sum_m \bar{c}_n c_m \delta_{n,m} = \sum_n |c_n|^2 \tag{7.70}$$

  Said differently, the norm square of a vector is equal to the sum of the magnitude square of the components.

Application 1: The Fourier Series

The theory of Fourier series, as covered in your calculus course, states that a function that is periodic, with period equal to 1, in some normalized units can be expanded as a linear combination of the sequence $\{\exp(j2\pi nt)\}$, where n is an integer that goes from minus infinity to plus infinity. The purpose here is to recast the familiar Fourier series results within the language and notations of the above formalism.

Basis:

$$\left|u_n\right\rangle = \exp(j2\pi nt) \quad \text{and} \quad \left\langle u_n\right| = \exp(-j2\pi nt) \tag{7.71}$$

Orthonormality of the basis vectors:

$$\left\langle u_m \middle| u_n\right\rangle = \int_{-1/2}^{1/2} \exp(-j2\pi mt)\exp(j2\pi nt)dt = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases} \tag{7.72}$$

Decomposition rule:

$$\left|\varphi\right\rangle = \sum_{n=-\infty}^{\infty} c_n \left|u_n\right\rangle = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi nt) \tag{7.73}$$

where

$$c_n = \left\langle u_n \middle| \varphi\right\rangle = \int_{-1/2}^{1/2} \exp(-j2\pi nt)\varphi(t)dt \tag{7.74}$$

Parseval's identity:

$$\|\varphi\|^2 = \left\langle\varphi \middle| \varphi\right\rangle = \int_{-1/2}^{1/2} \overline{\varphi}(t)\varphi(t)dt = \int_{-1/2}^{1/2} |\varphi(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{7.75}$$

Example 7.9

Derive the analytic expression for the potential difference across the capacitor in the *RLC* circuit of Figure 4.5 if the temporal profile of the source potential is a periodic function, of period 1, in some normalized units.

Solution:

1. Because the potential is periodic with period 1, it can be expanded using Eq. (7.73) in a Fourier series with basis functions $\{e^{j2\pi nt}\}$:

$$V_s(t) = \text{Re}\left\{\sum_n \tilde{V}_s^n e^{j2\pi nt}\right\} \tag{7.76}$$

where $\tilde{V}_s^n$ is the phasor associated with the frequency mode ($2\pi n$). (Note that n in the expressions for the phasors is a superscript and not a power.)

2. We find $\tilde{V}_c^n$, the capacitor response phasor associated with the $\tilde{V}_s^n$ excitation. This can be found by noting that the voltage across the capacitor is equal to the capacitor impedance multiplied by the current phasor, giving:

$$\tilde{V}_c^n = Z_c^n \tilde{I}^n = \frac{Z_c^n \tilde{V}_s^n}{Z_c^n + Z_R^n + Z_L^n} \tag{7.77}$$

where from the results of Section 6.8, particularly Eqs. (6.83) through (6.85), we have:

$$Z_c^n = \frac{1}{j2\pi nC} \tag{7.78}$$

$$Z_L^n = j2\pi nL \tag{7.79}$$

$$Z_R^n = R \tag{7.80}$$

3. Finally, we use the linearity of the ODE system and write the solution as the linear superposition of the solutions corresponding to the response to each of the basis functions; that is,

$$V_c(t) = \mathrm{Re}\left\{\sum_n \frac{Z_c^n \tilde{V}_s^n}{Z_c^n + Z_R^n + Z_L^n} e^{j2\pi nt}\right\} \tag{7.81}$$

leading to the expression:

$$V_c(t) = \mathrm{Re}\left\{\sum_n \frac{\tilde{V}_s^n}{1-(2\pi n)^2 LC + j(2\pi n)RC} e^{j2\pi nt}\right\} \tag{7.82}$$

Homework Problem

Pb. 7.27 Consider the *RLC* circuit. Assuming the same notation as in Section 6.5.3, but now assume that the source potential is given by:

$$V_s = V_0 \cos^6(\omega t)$$

a. Find analytically the potential difference across the capacitance. (*Hint:* Write the power of the trigonometric function as function of the different multiples of the angle.)

b. Find numerically the steady-state solution to this problem using the techniques of Chapter 4, and assume for some normalized units the following values for the parameters:

$$LC = 1, \quad RC = 1, \quad \omega = 2\pi$$

c. Compare your numerical results with the analytical results.

Application 2: The Legendre Polynomials

We propose to show that the Legendre polynomials are an orthonormal basis for all functions of compact support over the interval $-1 \leq x \leq 1$. Thus far, we have encountered the Legendre polynomials twice before. They were defined through their recursion relations in **Pb. 2.25,** and in Section 4.7.1 through their defining ODE. In this application, we define the Legendre polynomials through their generating function; show how their definitions through their recursion relation, or through their ODE, can be deduced from their definition through their generating function; and show that they constitute an orthonormal basis for functions defined on the interval $-1 \leq x \leq 1$.

1. The generating function for the Legendre polynomials is given by the simple form:

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l \tag{7.83}$$

2. The lowest orders of $P_l(x)$ can be obtained from the small t-expansion of $G(x, t)$; therefore, expanding Eq. (7.83) to first order in t gives:

$$1 + xt + \mathrm{O}(t^2) = P_0(x) + tP_1(x) + \mathrm{O}(t^2) \tag{7.84}$$

from which, we can deduce that:

$$P_0(x) = 1 \tag{7.85}$$

$$P_1(x) = x \tag{7.86}$$

3. By inspection, it is straightforward to verify by substitution that the generating function satisfies the equation:

$$(1-2xt+t^2)\frac{\partial G}{\partial t} + (t-x)G = 0 \tag{7.87}$$

Because power series can be differentiated term by term, Eq. (7.87) gives:

$$(1-2xt+t^2)\sum_{l=0}^{\infty} lP_l(x)t^{l-1} + (t-x)\sum_{l=0}^{\infty} P_l(x)t^l = 0 \tag{7.88}$$

Since this equation should hold true for all values of *t*, this means that all coefficients of any power of *t* should be zero; therefore:

$$(l+1)P_l(x) - 2lxP_l(x) + (l-1)P_{l-1}(x) + P_{l-1}(x) - xP_l(x) = 0 \tag{7.89}$$

or collecting terms, this can be written as:

$$(l+1)P_l(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0 \tag{7.90}$$

This is the recursion relation of **Pb. 2.25**.

4. By substitution in the explicit expression of the generating function, we can also verify that:

$$(1-2xt+t^2)\frac{\partial G}{\partial x} - tG = 0 \tag{7.91}$$

which leads to:

$$(1-2xt+t^2)\sum_{l=0}^{\infty}\frac{dP_l(x)}{dx} - \sum_{l=0}^{\infty} P_l(x)t^{l+1} = 0 \tag{7.92}$$

Again, looking at the coefficients of the same power of *t* permits us to obtain another recursion relation:

$$\frac{dP_{l+1}(x)}{dx} - 2x\frac{dP_l(x)}{dx} + \frac{dP_{l-1}(x)}{dx} - P_l(x) = 0 \tag{7.93}$$

Differentiating Eq. (7.90), we first eliminate $\frac{dP_{l-1}(x)}{dx}$ and then $\frac{dP_l(x)}{dx}$ from the resulting equation, and use Eq. (7.93) to obtain two new recursion relations:

$$\frac{dP_{l+1}(x)}{dx} - x\frac{dP_l(x)}{dx} = (l+1)P_l(x) \tag{7.94}$$

and

$$x\frac{dP_l(x)}{dx} - \frac{dP_{l-1}(x)}{dx} = lP_l(x) \tag{7.95}$$

Adding Eqs. (7.94) and (7.95), we obtain the more symmetric formula:

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} = (2l+1)P_l(x) \tag{7.96}$$

Replacing l by $l - 1$ in Eq. (7.94) and eliminating $P'_{l-1}(x)$ from Eq. (7.95), we find that:

$$(1-x^2)\frac{dP_l(x)}{dx} = lP_{l-1}(x) - lxP_l(x) \tag{7.97}$$

Differentiating Eq. (7.97) and using Eq. (7.95), we obtain:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] + l(l+1)P_l(x) = 0 \tag{7.98a}$$

which can be written in the equivalent form:

$$(1-x^2)\frac{d^2P_l(x)}{dx^2} - 2x\frac{dP_l(x)}{dx} + l(l+1)P_l(x) = 0 \tag{7.98b}$$

which is the ODE for the Legendre polynomial, as previously pointed out in Section 4.7.1.

5. Next, we want to show that if $l \neq m$, we have the orthogonality between any two elements (with different indices) of the basis; that is

$$\int_{-1}^{1} P_l(x)P_m(x)dx = 0 \tag{7.99}$$

To show this relation, we multiply Eq. (7.98) on the left by $P_m(x)$ and integrate to obtain:

$$\int_{-1}^{1} P_m(x)\left\{\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right]+l(l+1)P_l(x)\right\}dx = 0 \tag{7.100}$$

Integrating the first term by parts, we obtain:

$$\int_{-1}^{1}\left\{(x^2-1)\frac{dP_m(x)}{dx}\frac{dP_l(x)}{dx}+l(l+1)P_m(x)P_l(x)\right\}dx = 0 \tag{7.101}$$

Similarly, we can write the ODE for $P_m(x)$, and multiply on the left by $P_l(x)$; this results in the equation:

$$\int_{-1}^{1}\left\{(x^2-1)\frac{dP_l(x)}{dx}\frac{dP_m(x)}{dx}+m(m+1)P_l(x)P_m(x)\right\}dx = 0 \tag{7.102}$$

Now, subtracting Eq. (7.102) from Eq. (7.101), we obtain:

$$[m(m+1)-l(l+1)]\int_{-1}^{1} P_l(x)P_m(x)dx = 0 \tag{7.103}$$

But because $l \neq m$, this can only be satisfied if the integral is zero, which is the result that we are after.

6. Finally, we compute the normalization of the basis functions; that is, compute:

$$\int_{-1}^{1} P_l(x)P_l(x)dx = N_l^2 \tag{7.104}$$

From Eq. (7.90), we can write:

$$P_l(x)-(2l-1)xP_{l-1}(x)+(l-1)P_{l-2}(x) = 0 \tag{7.105}$$

If we multiply this equation by $(2l + 1)P_l(x)$ and subtract from it Eq. (7.90), which we multiplied by $(2l + 1)P_{l-1}(x)$, we obtain:

$$\begin{aligned} &l(2l+1)P_l^2(x)+(2l-1)(l-1)P_{l-1}(x)P_{l-2}(x) \\ &\quad -(l+1)(2l-1)P_{l-1}(x)P_{l+1}(x)-l(2l-1)P_{l-1}^2(x) = 0 \end{aligned} \tag{7.106}$$

Now integrate over the interval [–1, 1] and using Eq. (7.103), we obtain, for $l = 2, 3, \ldots$:

$$\int_{-1}^{1} P_l^2(x)dx = \frac{(2l-1)}{(2l+1)}\int_{-1}^{1} P_{l-1}^2(x)dx \tag{7.107}$$

Repeated applications of this formula and the use of Eq. (7.86) yields:

$$\int_{-1}^{1} P_l^2(x)dx = \frac{3}{(2l+1)}\int_{-1}^{1} P_1^2(x)dx = \frac{2}{(2l+1)} \tag{7.108}$$

Direct calculations show that this is also valid for $l = 0$ and $l = 1$. Therefore, the orthonormal basis functions are given by:

$$\left| u_l \right\rangle = \sqrt{l+\frac{1}{2}}\, P_l(x) \tag{7.109}$$

The general theorem that summarizes the decomposition of a function into the Legendre polynomials basis states:

THEOREM

If the real function $f(x)$ *defined over the interval* $[-1, 1]$ *is piecewise smooth and if the integral* $\int_{-1}^{1} f^2(x)dx < \infty,$ *then the series:*

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x) \tag{7.110}$$

where

$$c_l = \left(l+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_l(x)dx \tag{7.111}$$

converges to $f(x)$ at every continuity point of the function.

The proof of this theorem is not given here.

Example 7.10

Find the decomposition into Legendre polynomials of the following function:

$$f(x) = \begin{cases} 0 & \text{for} \quad -1 \le x \le a \\ 1 & \text{for} \quad a < x \le 1 \end{cases} \tag{7.112}$$

Solution: The conditions for the above theorem are satisfied, and

$$c_l = \left(l + \frac{1}{2}\right)\int_a^1 P_l(x)dx \tag{7.113}$$

From Eq. (7.96), and noting that $P_l(1) = 1$, we find that:

$$c_0 = \frac{1}{2}(1 - a) \tag{7.114}$$

and

$$c_l = -\frac{1}{2}[P_{l+1}(a) - P_{l-1}(a)] \tag{7.115}$$

We show in Figure 7.4 the sum of the truncated decomposition for Example 7.10 for different values of $l_{\max}$.

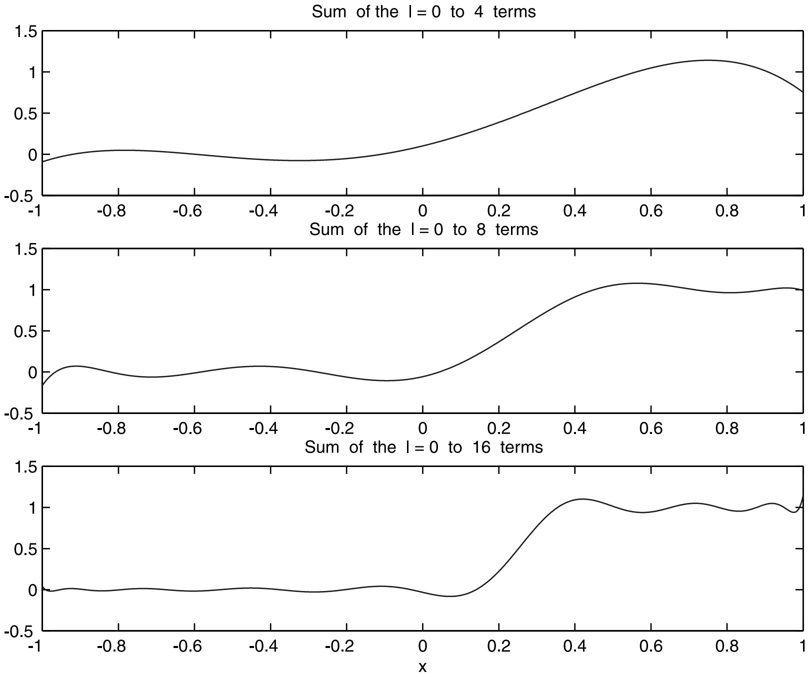

FIGURE 7.4
The plot of the truncated Legendre polynomials expansion of the discontinuous function given by Eq. (7.112), for a = 0.25. Top panel: l_{max} = 4. Middle panel: l_{max} = 8. Bottom panel: l_{max} = 16.

7.9 MATLAB Commands Review

' Transposition (i.e., for vectors with real components, this changes a row into a column).

norm Computes the Euclidean length of a vector.

cross Calculates the cross product of two 3-D vectors.

det Determinant; used here to compute the triple scalar product.

8

Matrices

8.1 Setting up Matrices

DEFINITION A matrix is a collection of numbers arranged in a two-dimensional (2-D) array structure. Each element of the matrix, call it $M_{i,j}$, occupies the i^{th} row and j^{th} column.

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \cdots & M_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & M_{m3} & \cdots & M_{mn} \end{bmatrix} \tag{8.1}$$

We say that **M** is an $(m \otimes n)$ matrix, which means that it has m rows and n columns. If $m = n$, we call the matrix square. If $m = 1$, the matrix is a row vector; and if $n = 1$, the matrix is a column vector.

8.1.1 Creating Matrices in MATLAB

8.1.1.1 Entering the Elements

In this method, the different elements of the matrix are keyed in; for example:

```
M=[1 3 5 7 11; 13 17 19 23 29; 31 37 41 47 53]
```

gives

```
M =
     1    3    5    7   11
    13   17   19   23   29
    31   37   41   47   53
```

To find the size of the matrix (i.e., the number of rows and columns), enter:

```
size(M)
```

gives

```
ans =
     3  5
```

To view a particular element, for example, the (2, 4) element, enter:

```
M(2,4)
```

gives

```
ans =
     23
```

To view a particular row such as the 3rd row, enter:

```
M(3,:)
```

gives

```
ans =
     31  37  41  47  53
```

To view a particular column such as the 4th column, enter:

```
M(:,4)
```

gives

```
ans =
      7
     23
     47
```

If we wanted to construct a submatrix of the original matrix, for example, one that includes the block from the 2nd to 3rd row (included) and from the 2nd column to the 4th column (included), enter:

```
M(2:3,2:4)
```

gives

```
ans =
     17  19  23
     37  41  47
```

8.1.1.2 Retrieving Special Matrices from the MATLAB Library

MATLAB has some commonly used specialized matrices in its library that can be called as needed. For example:

- The matrix of size ($m \otimes n$) with all elements being zero is **`M=zeros(m,n);`**

For example:

```
M=zeros(3,4)
```

gives

```
M =
     0  0  0  0
     0  0  0  0
     0  0  0  0
```

- The matrix of size ($m \otimes n$) with all elements equal to 1 is **`N=ones(m,n)`**:

For example:

```
N=ones(4,3)
```

produces

```
N =
     1  1  1
     1  1  1
     1  1  1
     1  1  1
```

- The matrix of size ($n \otimes n$) with only the diagonal elements equal to one, otherwise zero, is **`P=eye(n,n)`**:

For example:

```
P=eye(4,4)
```

gives

```
P =
    1  0  0  0
    0  1  0  0
    0  0  1  0
    0  0  0  1
```

- The matrix of size ($n \otimes n$) with elements randomly chosen from the interval [0, 1], such as:

```
Q=rand(4,4)
```

gives, in one instance:

```
Q =
    0.9708  0.4983  0.9601  0.2679
    0.9901  0.2140  0.7266  0.4399
    0.7889  0.6435  0.4120  0.9334
    0.4387  0.3200  0.7446  0.6833
```

- We can select to extract the upper triangular part of the **Q** matrix, but assign to all the lower triangle elements the value zero:

```
upQ=triu(Q)
```

produces

```
upQ =
    0.9708  0.4983  0.9601  0.2679
    0       0.2140  0.7266  0.4399
    0       0       0.4120  0.9334
    0       0       0       0.6833
```

or extract the lower triangular part of the **Q** matrix, but assign to all the upper triangle elements the value zero:

```
loQ=tril(Q)
```

produces

```
loQ =
0.9708  0       0        0
0.9901  0.2140  0        0
0.7889  0.6435  0.4120   0
0.4387  0.3200  0.7446   0.6833
```

- The single quotation mark (′) after the name of a matrix changes the matrix rows into becoming its columns, and vice versa, if the elements are all real. If the matrix has complex numbers as elements, it also takes their complex conjugate in addition to the transposition.
- Other specialized matrices, including the whole family of sparse matrices, are also included in the MATLAB library. You can find more information about them in the **`help`** documentation.

8.1.1.3 Functional Construction of Matrices

The third method for generating matrices is to give, if it exists, an algorithm that generates each element of the matrix. For example, suppose we want to generate the Hilbert matrix of size $(n \otimes n)$, where $n = 4$ and the functional form of the elements are: $M_{mn} = \frac{1}{m+n}$. The routine for generating this matrix will be as follows:

```
M=zeros(4,4);
for m=1:4
  for n=1:4
  M(m,n)=1/(m+n);
  end
end
M
```

- We can also create new matrices by appending known matrices. For example:

Let the matrices **A** and **B** be given by:

```
A=[1 2 3 4];
B=[5 6 7 8];
```

We want to expand the matrix **A** by the matrix **B** along the horizontal (this is allowed only if both matrices have the same number of rows). Enter:

```
C=[A B]
```

gives

```
C =
     1  2  3  4  5  6  7  8
```

Or, we may want to expand **A** by stacking it on top of **B** (this is allowed only if both matrices have the same number of columns). Enter:

```
D=[A;B]
```

produces

```
D =
     1  2  3  4
     5  6  7  8
```

We illustrate the appending operations for larger matrices: define **E** as the (2 ⊗ 3) matrix with one for all its elements, and we desire to append it horizontally to **D**. This is allowed because both have the same number of rows (= 2). Enter:

```
E=ones(2,3)
```

produces

```
E =
     1  1  1
     1  1  1
```

Enter:

```
F=[D E]
```

produces

```
F =
     1  2  3  4  1  1  1
     5  6  7  8  1  1  1
```

Or, we may want to stack two matrices in a vertical configuration. This requires that the two matrices have the same number of columns. Enter:

```
G=ones(2,4)
```

gives

```
G =
     1  1  1  1
     1  1  1  1
```

Enter

```
H=[D;G]
```

produces

```
H =
     1  2  3  4
     5  6  7  8
     1  1  1  1
     1  1  1  1
```

Finally, the command sum applied to a matrix gives a row in which *m*-element is the sum of all the elements of the m^{th} column in the original matrix. For example, entering:

```
sum(H)
```

produces

```
ans =
     8  10  12  14
```

8.2 Adding Matrices

Adding two matrices is only possible if they have equal numbers of rows and equal numbers of columns; or, said differently, they both have the same size.

The addition operation is the obvious one. That is, the (*m*, *n*) element of the sum (**A+B**) is the sum of the (*m*, *n*) elements of respectively **A** and **B**:

$$(\mathbf{A}+\mathbf{B})_{mn} = A_{mn} + B_{mn} \tag{8.2}$$

Entering

```
A=[1 2 3 4];
B=[5 6 7 8];
A+B
```

produces

```
ans =
    6  8  10  12
```

If we had subtraction of two matrices, it would be the same syntax as above but using the minus sign between the matrices.

8.3 Multiplying a Matrix by a Scalar

If we multiply a matrix by a number, each element of the matrix is multiplied by that number.

Entering:

```
3*A
```

produces

```
ans =
    3  6  9  12
```

Entering:

```
3*(A+B)
```

produces

```
ans =
    18  24  30  36
```

8.4 Multiplying Matrices

Two matrices **A**$(m \otimes n)$ and **B**$(r \otimes s)$ can be multiplied only if $n = r$. The size of the product matrix is $(m \otimes s)$. An element of the product matrix is obtained from those of the constitutent matrices through the following rule:

$$(\mathbf{AB})_{kl} = \sum_{h} A_{kh} B_{hl} \tag{8.3}$$

This result can be also interpreted by observing that the (*k, l*) element of the product is the dot product of the *k*-row of **A** and the *l*-column of **B**.

In MATLAB, we denote the product of the matrices **A** and **B** by `A*B`.

Example 8.1

Write the different routines for performing the matrix multiplication from the different definitions of the matrix product.

Solution: Edit and execute the following *script M-file:*

```
D=[1 2 3; 4 5 6];
E=[3 6 9 12; 4 8 12 16; 5 10 15 20];

F=D*E

F1=zeros(2,4);
for i=1:2
  for j=1:4
    for k=1:3
    F1(i,j)=F1(i,j)+D(i,k)*E(k,j);
    end
  end
end
F1

F2=zeros(2,4);
for i=1:2
  for j=1:4
    F2(i,j)=D(i,:)*E(:,j);
  end
end
F2
```

The result `F` is the one obtained using the MATLAB built-in matrix multiplication; the result `F1` is that obtained from Eq. (8.3) and `F2` is the answer obtained by performing, for each element of the matrix product, the dot product of the appropriate row from the first matrix with the appropriate column from the second matrix. Of course, all three results should give the same answer, which they do.

8.5 Inverse of a Matrix

In this section, we assume that we are dealing with square matrices ($n \otimes n$) because these are the only class of matrices for which we can define an inverse.

DEFINITION A matrix $\mathbf{M}^{-1}$ is called the inverse of matrix **M** if the following conditions are satisfied:

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I} \tag{8.4}$$

(The identity matrix is the ($n \otimes n$) matrix with ones on the diagonal and zero everywhere else; the matrix **`eye(n,n)`** in MATLAB.)

EXISTENCE The existence of an inverse of a matrix hinges on the condition that the determinant of this matrix is non-zero [**`det(M)`** in MATLAB]. We leave the proof of this theorem to future courses in linear algebra. For now, the formula for generating the value of the determinant is given here.

- The determinant of a square matrix **M**, of size ($n \otimes n$), is a number equal to:

$$\det(\mathbf{M}) = \sum_{P} (-1)^{P} M_{1k_1} M_{2k_2} M_{3k_3} \ldots M_{nk_n} \tag{8.5}$$

where P is the n! permutation of the first n-integers. The sign in front of each term is positive if the number of transpositions relating

$$(1, 2, 3, \ldots, n) \quad \text{and} \quad (k_1, k_2, k_3, \ldots, k_n)$$

is even, while the sign is negative otherwise.

Example 8.2

Using the definition for a determinant, as given in Eq. (8.5), find the expression for the determinant of a ($2 \otimes 2$) and a ($3 \otimes 3$) matrix.

Solution:

a. If n = 2, there are only two possibilities for permuting these two numbers, giving the following: (1, 2) and (2, 1). In the first permutation, no transposition was necessary; that is, the multiplying factor in Eq. (8.5) is 1. In the second term, one transposition is needed; that is, the multiplying factor in Eq. (8.5) is –1, giving for the determinant the value:

$$\Delta = M_{11}M_{22} - M_{12}M_{21} \tag{8.6}$$

b. If n = 3, there are only six permutations for the sequence (1, 2, 3): namely, (1, 2, 3), (2, 3, 1), and (3, 1, 2), each of which is an even permutation and (3, 2, 1), (2, 1, 3), and (1, 3, 2), which are odd permutations, thereby giving for the determinant the value:

$$\begin{aligned}\Delta = {} & M_{11}M_{22}M_{33} + M_{12}M_{23}M_{31} + M_{13}M_{21}M_{32} \\ & - (M_{13}M_{22}M_{31} + M_{12}M_{21}M_{33} + M_{11}M_{23}M_{32})\end{aligned} \tag{8.7}$$

MATLAB Representation

Compute the determinant and the inverse of the matrices **M** and **N**, as keyed below:

```
M=[1 3 5; 7 11 13; 17 19 23];
detM=det(M)
invM=inv(M)
```

gives

```
detM=
    -84
invM=
    -0.0714  -0.3095   0.1905
    -0.7143   0.7381  -0.2619
     0.6429  -0.3810   0.1190
```

On the other hand, entering:

```
N=[2 4 6; 3 5 7; 5 9 13];
detN=det(N)
invN=inv(N)
```

produces

```
detN =
    0
invN
    Warning: Matrix is close to singular or badly
    scaled.
```

Homework Problems

Pb. 8.1 As earlier defined, a square matrix in which all elements above (below) the diagonal are zeros is called a lower (upper) triangular matrix. Show that the determinant of a triangular $n \otimes n$ matrix is

$$\det(\mathbf{T}) = T_{11}T_{22}T_{33} \ldots T_{nn}$$

Pb. 8.2 If **M** is an $n \otimes n$ matrix and k is a constant, show that:

$$\det(k\mathbf{M}) = k^n \det(\mathbf{M})$$

Pb. 8.3 Assuming the following result, which will be proven to you in linear algebra courses:

$$\det(\mathbf{MN}) = \det(\mathbf{M}) \times \det(\mathbf{N})$$

Prove that if the inverse of the matrix M exists, then:

$$\det(\mathbf{M}^{-1}) = \frac{1}{\det(\mathbf{M})}$$

8.6 Solving a System of Linear Equations

Let us assume that we have a system of n linear equations in n unknowns that we want to solve:

$$\begin{aligned}
&M_{11}\,x_1 + M_{12}\,x_2 + M_{13}\,x_3 + \ldots + M_{1n}\,x_n = b_1 \\
&M_{21}\,x_1 + M_{22}\,x_2 + M_{23}\,x_3 + \ldots + M_{2n}\,x_n = b_2 \\
&\vdots \\
&M_{n1}\,x_1 + M_{n2}\,x_2 + M_{n3}\,x_3 + \ldots + M_{nn}\,x_n = b_n
\end{aligned} \tag{8.8}$$

The above equations can be readily written in matrix notation:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \cdots & M_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ M_{n1} & M_{n2} & M_{n3} & \cdots & M_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} \tag{8.9}$$

or

$$\mathbf{MX} = \mathbf{B} \tag{8.10}$$

where the column of b' s and x' s are denoted by **B** and **X**. Multiplying, on the left, both sides of this matrix equation by $\mathbf{M}^{-1}$, we find that:

$$\mathbf{X} = \mathbf{M}^{-1}\mathbf{B} \tag{8.11}$$

As pointed out previously, remember that the condition for the existence of solutions is a non-zero value for the determinant of **M**.

Example 8.3

Use MATLAB to solve the system of equations given by:

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 22 \\ 7x_1 + 11x_2 - 13x_3 &= -10 \\ 17x_1 + 19x_2 - 23x_3 &= -14 \end{aligned}$$

Solution: Edit and execute the following *script M-file:*

```
M=[1 3 5; 7 11 -13; 17 19 -23];
B=[22;-10;-14];
detM=det(M);
invM=inv(M);
X=inv(M)*B.
```

Verify that the vector **X** could also have been obtained using the left slash notation: `X=M\B`.

NOTE In this and the immediately preceding chapter sections, we said very little about the algorithm used for computing essentially the inverse of a matrix. This is a subject that will be amply covered in your linear algebra courses. What the interested reader needs to know at this stage is that the

Gaussian elimination technique (and its different refinements) is essentially the numerical method of choice for the built-in algorithms of numerical softwares, including MATLAB. The following two examples are essential building blocks in such constructions.

Example 8.4

Without using the MATLAB inverse command, solve the system of equations:

$$\mathbf{LX} = \mathbf{B} \tag{8.12}$$

where **L** is a lower triangular matrix.

Solution: In matrix form, the system of equations to be solved is

$$\begin{bmatrix} L_{11} & 0 & 0 & \cdots & 0 \\ L_{21} & L_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} \tag{8.13}$$

The solution of this system can be directly obtained if we proceed iteratively. That is, we find in the following order: $x_1, x_2, \ldots, x_n$, obtaining:

$$\begin{aligned} x_1 &= \frac{b_1}{L_{11}} \\ x_2 &= \frac{(b_2 - L_{21}x_1)}{L_{22}} \\ &\vdots \\ x_k &= \frac{\left(b_k - \sum_{j=1}^{k-1} L_{kj}x_j \right)}{L_{kk}} \end{aligned} \tag{8.14}$$

The above solution can be implemented by executing the following *script M-file*:

```
L=[ ];                  % enter the L matrix
b=[ ];                  % enter the B column
n=length(b);
x=zeros(n,1);
```

```
x(1)=b(1)/L(1,1);
  for k=2:n
  x(k)=(b(k)-L(k,1:k-1)*x(1:k-1))/L(k,k);
  end
x
```

Example 8.5

Solve the system of equations: **UX** = **B**, where **U** is an upper triangular matrix.

Solution: The matrix form of the problem becomes:

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & U_{n-1\,n-1} & U_{n-1\,n} \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \tag{8.15}$$

In this case, the solution of this system can also be directly obtained if we proceed iteratively, but this time in the backward order $x_n, x_{n-1}, \ldots, x_1$, obtaining:

$$\begin{aligned} x_n &= \frac{b_n}{U_{nn}} \\ x_{n-1} &= \frac{(b_{n-1} - U_{n-1\,n}\, x_n)}{U_{n-1\,n-1}} \\ &\vdots \\ x_k &= \frac{\left(b_k - \sum_{j=k+1}^{n} U_{kj} x_j \right)}{U_{kk}} \end{aligned} \tag{8.16}$$

The corresponding *script M-file* is

```
U=[ ];                   % enter the U matrix
b=[ ];                   % enter the B column
n=length(b);
x=zeros(n,1);
x(n)=b(n)/U(n,n);
  for k=n-1:-1:1
```

```
    x(k)=(b(k)-U(k,k+1:n)*x(k+1:n))/U(k,k);
    end
x
```

8.7 Application of Matrix Methods

This section provides seven representative applications that illustrate the immense power that matrix formulation and tools can provide to diverse problems of common interest in electrical engineering.

8.7.1 dc Circuit Analysis

Example 8.6

Find the voltages and currents for the circuit given in Figure 8.1.

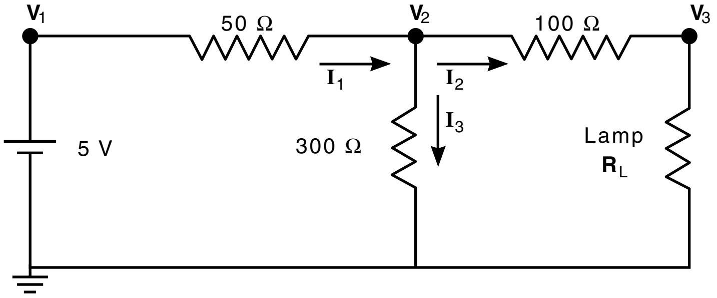

FIGURE 8.1
Circuit of Example 8.6.

Solution: Using Kirchoff's current and voltage laws and Ohm's law, we can write the following equations for the voltages and currents in the circuit, assuming that $R_L = 2\Omega$:

$$V_1 = 5$$

$$V_1 - V_2 = 50I_1$$

$$V_2 - V_3 = 100I_2$$

$$V_2 = 300I_3$$

$$V_3 = 2I_2$$

$$I_1 = I_2 + I_3$$

NOTE These equations can be greatly simplified if we use the method of elimination of variables. This is essentially the method of nodes analysis covered in circuit theory courses. At this time, our purpose is to show a direct numerical method for obtaining the solutions.

If we form column vector **VI**, the top three components referring to the voltages V_1, V_2, V_3, and the bottom three components referring to the currents I_1, I_2, I_3, then the following *script M-file* provides the solution to the above circuit:

```
M=[1 0 0 0 0 0;1 -1 0 -50 0 0;0 1 -1 0 -100 0;...
0 1 0 0 0 -300;0 0 1 0 -2 0;0 0 0 1 -1 -1];
Vs=[5;0;0;0;0;0];
VI=M\Vs
```

In-Class Exercise

Pb. 8.4 Use the same technique as shown in Example 8.6 to solve for the potentials and currents in the circuit given in Figure 8.2.

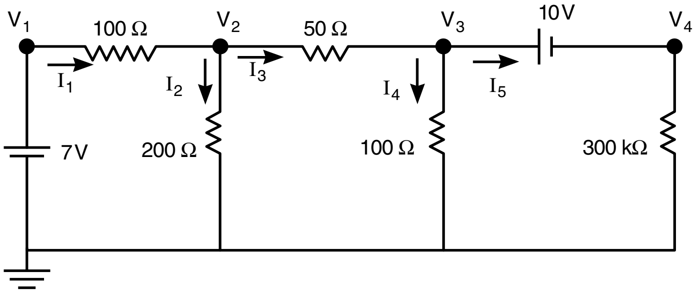

FIGURE 8.2
Circuit of Pb. 8.4.

8.7.2 dc Circuit Design

In design problems, we are usually faced with the reverse problem of the direct analysis problem, such as the one solved in Section 8.7.1.

Example 8.7

Find the value of the lamp resistor in Figure 8.1, so that the current flowing through it is given, *a priori.*

Solution: We approach this problem by defining a function file for the relevant current. In this case, it is

```
function  ilamp=circuit872(RL)
M=[1  0  0  0  0  0;1  -1  0  -50  0  0;0  1  -1  0  -100  0;...
0  1  0  0  0  -300;0  0  1  0  -RL  0;0  0  0  1  -1  -1];
Vs=[5;0;0;0;0;0];
VI=M\Vs;
ilamp=VI(5);
```

Then, from the command window, we proceed by calling this function and plotting the current in the lamp as a function of the resistance. Then we graphically read for the value of R_L, which gives the desired current value.

In-Class Exercise

Pb. 8.5 For the circuit of Figure 8.1, find R_L that gives a 22-mA current in the lamp. (*Hint:* Plot the current as function of the load resistor.)

8.7.3 ac Circuit Analysis

Conceptually, there is no difference between performing an ac steady-state analysis of a circuit with purely resistive elements, as was done in Subsection 8.7.1, and performing the analysis for a circuit that includes capacitors and inductors, if we adopt the tool of impedance introduced in Section 6.8, and we write the circuit equations instead with phasors. The only modification from an all-resistors circuit is that matrices now have complex numbers as elements, and the impedances have frequency dependence. For convenience, we illustrate again the relationships of the voltage-current phasors across resistors, inductors, and capacitors:

$$\tilde{V}_R = \tilde{I}R \tag{8.17}$$

$$\tilde{V}_L = \tilde{I}(j\omega L) \tag{8.18}$$

$$\tilde{V}_C = \frac{\tilde{I}}{(j\omega C)} \tag{8.19}$$

and restate Kirchoff's laws again:

- Kirchoff's voltage law: The sum of all voltage drops around a closed loop is balanced by the sum of all voltage sources around the same loop.

- Kirchoff's current law: The algebraic sum of all currents entering (exiting) a circuit node must be zero.

In-Class Exercise

Pb. 8.6 In a bridged-T filter, the voltage $V_s(t)$ is the input voltage, and the output voltage is that across the load resistor R_L. The circuit is given in Figure 8.3.

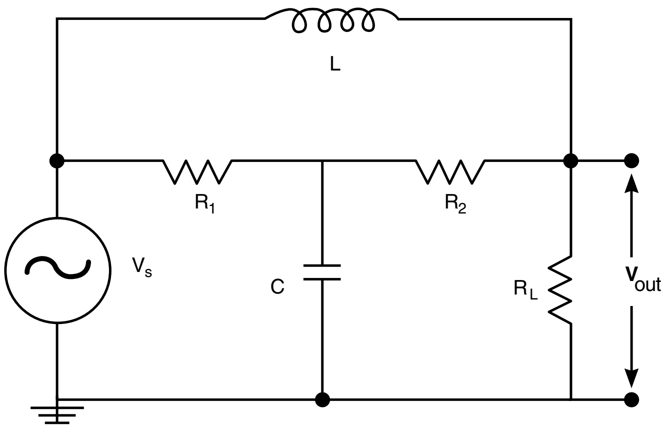

FIGURE 8.3
Bridged-T filter. Circuit of Pb. 8.6.

Assuming that $R_1 = R_2 = 3\ \Omega$, $R_L = 2\ \Omega$, $C = 0.25$ F, and $L = 1$ H:

a. Write the equations for the phasors of the voltages and currents.

b. Form the matrix representation for the equations found in part (**a**).

c. Plot the magnitude and phase of $\dfrac{\tilde{V}_{\text{out}}}{\tilde{V}_S}$ as a function of the frequency.

d. Compare the results obtained in part (**c**) with the analytical results of the problem, given by:

$$\frac{\tilde{V}_{\text{out}}}{\tilde{V}_S} = \frac{N(\omega)}{D(\omega)}$$

$$N(\omega) = R_2R_L(R_1 + R_2) + j\omega R_2^2(L + CR_1R_L)$$

$$D(\omega) = R_2[R_1R_L + R_2R_L - \omega^2 LCR_1(R_2 + R_L)]$$
$$+ j\omega[L(R_1R_2 + R_1R_L + R_2R_L) + CR_1R_2^2R_L]$$

8.7.4 Accuracy of a Truncated Taylor Series

In this subsection and subection 8.7.5, we illustrate the use of matrices as a convenient constructional tool to state and manipulate problems with two indices. In this application, we desire to verify the accuracy of the truncated Taylor series $S = \sum_{n=0}^{N} \frac{x^n}{n!}$ as an approximation to the function $y = \exp(x)$, over the interval $0 \leq x < 1$.

Because this application's purpose is to illustrate a constructional scheme, we write the code lines as we are proceeding with the different computational steps:

1. We start by dividing the (0, 1) interval into equally spaced segments. This array is given by:

```
x=[0:0.01:1];
M=length(x);
```

2. Assume that we are truncating the series at the value $N = 10$:

```
N=10;
```

3. Construct the matrix **W** having the following form:

$$\mathbf{W} = \begin{bmatrix} 1 & x_1 & \frac{x_1^2}{2!} & \frac{x_1^3}{3!} & \cdots & \frac{x_1^N}{N!} \\ 1 & x_2 & \frac{x_2^2}{2!} & \frac{x_2^3}{3!} & \cdots & \frac{x_2^N}{N!} \\ 1 & x_3 & \frac{x_3^2}{2!} & \frac{x_3^3}{3!} & \cdots & \frac{x_3^N}{N!} \\ \vdots & & \vdots & & \cdots & \\ \vdots & & \vdots & & \ddots & \vdots \\ 1 & x_M & \frac{x_M^2}{2!} & \frac{x_M^3}{3!} & \cdots & \frac{x_M^N}{N!} \end{bmatrix} \tag{8.20}$$

Specify the size of **W**, and then give the induction rule to go from one column to the next:

$$W(i, j) = x(i) * \frac{W(i, j-1)}{j-1} \tag{8.21}$$

This is implemented in the code as follows:

```
W=ones(M,N);
for i=1:M
  for j=2:N
  W(i,j)=x(i)*W(i,j-1)/(j-1);
  end
end
```

4. The value of the truncated series at a specific point is the sum of the row elements corresponding to its index; however since MATLAB command **sum** acting on a matrix adds the column elements, we take the sum of the adjoint (the matrix obtained, for real elements, by changing the rows to columns and vice versa) of **W** to obtain our result. Consequently, add to the code:

   ```
   serexp=sum(W');
   ```

5. Finally, compare the values of the truncated series with that of the exponential function

   ```
   y=exp(x);
   plot(x,serexp,x,y,'--")
   ```

In examining the plot resulting from executing the above instructions, we observe that the truncated series give a very good approximation to the exponential over the whole interval.

If you would also like to check the error of the approximation as a function of x, enter:

```
dy=abs(y-serexp);
semilogy(x,dy)
```

Examining the output graph, you will find, as expected, that the error increases with an increase in the value of x. However, the approximation of the exponential by the partial sum of the first ten elements of the truncated Taylor series is accurate over the whole domain considered, to an accuracy of better than one part per million.

Question: Could you have estimated the maximum error in the above computed value of **dy** by evaluating the first neglected term in the Taylor's series at $x = 1$?

In-Class Exercise

Pb. 8.7 Verify the accuracy of truncating at the fifth element the following Taylor series, in a domain that you need to specify, so the error is everywhere less than one part in 10,000:

a. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

b. $\sin(x) = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n+1}}{(2n+1)!}$

c. $\cos(x) = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!}$

8.7.5 Reconstructing a Function from Its Fourier Components

From the results of Section 7.9, where we discussed the Fourier series, it is a simple matter to show that any even periodic function with period 2π can be written in the form of a cosine series, and that an odd periodic function can be written in the form of a sine series of the fundamental frequency and its higher harmonics.

Knowing the coefficients of its Fourier series, we would like to plot the function over a period. The purpose of the following example is two-fold:

1. On the mechanistic side, to illustrate again the setting up of a two indices problem in a matrix form.
2. On the mathematical contents side, examining the effects of truncating a Fourier series on the resulting curve.

Example 8.8

Plot $y(x) = \sum_{k=1}^{M} C_k \cos(kx)$, if $C_k = \frac{(-1)^k}{k^2+1}$. Choose successively for M the values 5, 20, and 40.

Solution: Edit and execute the following *script M-file:*

```
M= ;
p=500;
k=1:M;
```

```
n=0:p;
x=(2*pi/p)*n;
a=cos((2*pi/p)*n'*k);
c=((-1).^k)./(k.^2+1);
y=a*c';
plot(x,y)
axis([0 2*pi -1 1.2])
```

Draw in your notebook the approximate shape of the resulting curve for different values of **M**.

In-Class Exercises

Pb. 8.8 For different values of the cutoff, plot the resulting curves for the functions given by the following Fourier series:

$$y_1(x) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{1}{(2k-1)^2}\right)\cos((2k-1)x)$$

$$y_2(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{(2k-1)}\right)\cos((2k-1)x)$$

$$y_3(x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin((2k-1)x)$$

Pb. 8.9 The purpose of this problem is to explore the Gibbs phenomenon. This phenomenon occurs as a result of truncating the Fourier series of a discontinuous function. Examine, for example, this phenomenon in detail for the function $y_3(x)$ given in **Pb. 8.8**.

The function under consideration is given analytically by:

$$y_3(x) = \begin{cases} 0.5 & \text{for} \quad 0 < x < \pi \\ -0.5 & \text{for} \quad \pi < x < 2\pi \end{cases}$$

a. Find the value where the truncated Fourier series overshoots the value of 0.5. (Answer: The limiting value of this first maximum is 0.58949).

b. Find the limiting value of the first local minimum. (Answer: The limiting value of this first minimum is 0.45142).

c. Derive, from first principles, the answers to parts (**a**) and (**b**). (*Hint:* Look up in a standard integral table the sine integral function.)

NOTE An important goal of filter theory is to find methods to smooth these kinds of oscillations.

8.7.6 Interpolating the Coefficients of an (n – 1)-degree Polynomial from n Points

The problem at hand can be posed as follows:

Given the coordinates of n points: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, we want to find the polynomial of degree $(n - 1)$, denoted by $p_{n-1}(x)$, whose curve passes through these points.

Let us assume that the polynomial has the following form:

$$p_{n-1}(x) = a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1} \tag{8.22}$$

From a knowledge of the column vectors **X** and **Y**, we can formulate this problem in the standard linear system form. In particular, in matrix form, we can write:

$$\mathbf{V} * \mathbf{A} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \mathbf{Y} \tag{8.23}$$

Knowing the matrix **V** and the column **Y**, it is then a trivial matter to deduce the column **A**:

$$\mathbf{A} = \mathbf{V}^{-1} * \mathbf{Y} \tag{8.24}$$

What remains to be done is to generate in an efficient manner the matrix **V** using the column vector **X** as input. We note the following recursion relation for the elements of **V**:

$$V(k, j) = x(k) * V(k, j - 1) \tag{8.25}$$

Furthermore, the first column of **V** has all its elements equal to 1.

The following routine computes **A**:

```
X=[x1;x2;x3;.......;xn];
Y=[y1;y2;y3;.......;yn];
n=length(X);
V=ones(n,n);
  for j=2:n
  V(:,j)=X.*V(:,j-1);
  end

A=V\Y
```

In-Class Exercises

Find the polynomials that are defined through:

Pb. 8.10 The points (1, 5), (2, 11), and (3, 19).

Pb. 8.11 The points (1, 8), (2, 39), (3, 130), (4, 341), and (5, 756).

8.7.7 Least Square Fit of Data

In Section 8.7.6, we found the polynomial of degree (n – 1) that was uniquely determined by the coordinates of n points on its curve. However, when data fitting is the tool used by experimentalists to verify a theoretical prediction, many more points than the minimum are measured in order to minimize the effects of random errors generated in the acquisition of the data. But this over-determination in the system parameters faces us with the dilemma of what confidence level one gives to the accuracy of specific data points, and which data points to accept or reject. *A priori,* one takes all data points, and resorts to a determination of the vector **A** whose corresponding polynomial comes closest to all the experimental points. Closeness is defined through the Euclidean distance between the experimental points and the predicted curve. This method for minimizing the sum of the square of the Euclidean distance between the optimal curve and the experimental points is referred to as the least-square fit of the data.

To have a geometrical understanding of what we are attempting to do, consider the conceptually analogous problem in 3-D of having to find the plane with the least total square distance from five given data points. So what do we do? Using the projection procedure derived in Chapter 7, we deduce each point's distance from the plane; then we go ahead and adjust the parameters of the plane equation to obtain the smallest total square distance between the points and the plane. In linear algebra courses, using generalized optimization techniques, you will be shown that the best fit to **A** (i.e., the one called least-square fit) is given (using the rotation of the previous subsection) by:

$$\mathbf{A}_N = (\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{Y} \tag{8.26}$$

A MATLAB routine to fit a number of (n) points to a polynomial of order ($m - 1$) now reads:

```
X=[x1;x2;x3;.......;xn];
Y=[y1;y2;y3;.......;yn];
n=length(X);
m=         %(m-1) is the degree of the polynomial
V=ones(n,m);
  for j=2:m
  V(:,j)=X.*V(:,j-1);
  end

AN=inv(V'*V)*(V'*Y)
```

MATLAB also has a built-in command to achieve the least-square fit of data. Look up the **`polyfit`** function in your help documentation, and learn its use and point out what difference exists between its notation and that of the above routine.

In-Class Exercise

Pb. 8.12 Find the second-degree polynomials that best fit the data points: (1, 8.1), (2, 24.8), (3, 52.5), (4, 88.5), (5, 135.8), and (6, 193.4).

8.8 Eigenvalues and Eigenvectors*

DEFINITION If **M** is a square $n \otimes n$ matrix, then a vector $|v\rangle$ is called an eigenvector and λ, a scalar, is called an eigenvalue, if they satisfy the relation:

$$\mathbf{M}|v\rangle = \lambda|v\rangle \tag{8.27}$$

that is, the vector $\mathbf{M}|v\rangle$ is a scalar multiplied by the vector $|v\rangle$.

8.8.1 Finding the Eigenvalues of a Matrix

To find the eigenvalues, note that the above definition of eigenvectors and eigenvalues can be rewritten in the following form:

$$(\mathbf{M} - \lambda\mathbf{I})|v\rangle = 0 \tag{8.28}$$

where **I** is the identity $n \otimes n$ matrix. The above set of homogeneous equations admits a solution only if the determinant of the matrix multiplying the vector $|v\rangle$ is zero. Therefore, the eigenvalues are the roots of the polynomial $p(\lambda)$, defined as follows:

$$p(\lambda) = \det(\mathbf{M} - \lambda\mathbf{I}) \tag{8.29}$$

This equation is called the characteristic equation of the matrix **M**. It is of degree n in λ. (This last assertion can be proven by noting that the contribution to the determinant of ($\mathbf{M} - \lambda\mathbf{I}$), coming from the product of the diagonal elements of this matrix, contributes a factor of λ^n to the expression of the determinant.)

Example 8.9

Find the eigenvalues and the eigenvectors of the matrix **M**, defined as follows:

$$\mathbf{M} = \begin{pmatrix} 2 & 4 \\ 1/2 & 3 \end{pmatrix}$$

Solution: The characteristic polynomial for this matrix is given by:

$$p(\lambda) = (2 - \lambda)(3 - \lambda) - (4)(1/2) = \lambda^2 - 5\lambda + 4$$

The roots of this polynomial (i.e., the eigenvalues of the matrix) are, respectively,

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 4$$

To find the eigenvectors corresponding to the above eigenvalues, which we shall denote respectively by $|v_1\rangle$ and $|v_2\rangle$, we must satisfy the following two equations separately:

$$\begin{pmatrix} 2 & 4 \\ 1/2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1\begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 4 \\ 1/2 & 3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 4\begin{pmatrix} c \\ d \end{pmatrix}$$

From the first set of equations, we deduce that: $b = -a/4$; and from the second set of equations that $d = c/2$, thus giving for the eigenvectors $|v_1\rangle$ and $|v_2\rangle$, the following expressions:

$$|v_1\rangle = a\begin{pmatrix} -1 \\ 1/4 \end{pmatrix}$$

$$|v_2\rangle = c\begin{pmatrix} -1 \\ -1/2 \end{pmatrix}$$

It is common to give the eigenvectors in the normalized form (that is, fix a and c to make $\langle v_1|v_1\rangle = \langle v_2|v_2\rangle = 1$, thus giving for $|v_1\rangle$ and $|v_2\rangle$, the normalized values:

$$|v_1\rangle = \sqrt{\frac{16}{17}}\begin{pmatrix} -1 \\ 1/4 \end{pmatrix} = \begin{pmatrix} -0.9701 \\ 0.2425 \end{pmatrix}$$

$$|v_2\rangle = \sqrt{\frac{4}{5}}\begin{pmatrix} -1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -0.8944 \\ -0.4472 \end{pmatrix}$$

8.8.2 Finding the Eigenvalues and Eigenvectors Using MATLAB

Given a matrix **M**, the MATLAB command to find the eigenvectors and eigenvalues is given by **`[V,D]=eig(M)`**; the columns of **`V`** are the eigenvectors and **`D`** is a diagonal matrix whose elements are the eigenvalues. Entering the matrix **M** and the eigensystem commands gives:

```
V =
    -0.9701  -0.8944
     0.2425  -0.4472
D =
    1  0
    0  4
```

Finding the matrices **V** and **D** is referred to as diagonalizing the matrix **M**. It should be noted that this is not always possible. For example, the matrix is not diagonalizable when one or more of the roots of the characteristic polynomial is zero. In courses of linear algebra, you will study the necessary and sufficient conditions for **M** to be diagonalizable.

In-Class Exercises

Pb. 8.13 Show that if $\mathbf{M}|v\rangle = \lambda|v\rangle$, then $\mathbf{M}^n|v\rangle = \lambda^n|v\rangle$. That is, the eigenvalues of $\mathbf{M}^n$ are λ^n; however, the eigenvectors $|v\rangle$'s remain the same as those of **M**.

Verify this theorem using the choice in Example 8.9 for the matrix **M**.

Pb. 8.14 Find the eigenvalues of the upper triangular matrix:

$$\mathbf{T} = \begin{pmatrix} 1/4 & 0 & 0 \\ -1 & 1/2 & 0 \\ 2 & -3 & 1 \end{pmatrix}$$

Generalize your result to prove analytically that the eigenvalues of any triangular matrix are its diagonal elements. (*Hint:* Use the previously derived result in **Pb. 8.1** for the expression of the determinant of a triangular matrix.)

Pb. 8.15 A general theorem, which will be proven to you in linear algebra courses, states that if a matrix is diagonalizable, then, using the above notation:

$$\mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \mathbf{M}$$

Verify this theorem for the matrix **M** of Example 8.9.

a. Using this theorem, show that:

$$\det(\mathbf{M}) = \det(\mathbf{D}) = \prod_i^n \lambda_i$$

b. Also show that:

$$\mathbf{V}\mathbf{D}^n\mathbf{V}^{-1} = \mathbf{M}^n$$

c. Apply this theorem to compute the matrix $\mathbf{M}^5$, for the matrix **M** of Example 8.9.

Pb. 8.16 Find the non-zero eigenvalues of the 2 ⊗ 2 matrix **A** that satisfies the equation:

$$\mathbf{A} = \mathbf{A}^3$$

Homework Problems

The function of a matrix can formally be defined through a Taylor series expansion. For example, the exponential of a matrix **M** can be defined through:

$$\exp(\mathbf{M}) = \sum_{n=0}^{\infty} \frac{\mathbf{M}^n}{n!}$$

Pb. 8.17 Use the results from **Pb. 8.15** to show that:

$$\exp(\mathbf{M}) = \mathbf{V}\exp(\mathbf{D})\mathbf{V}^{-1}$$

where, for any diagonal matrix:

$$\exp\begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} = \begin{pmatrix} \exp(\lambda_1) & 0 & \cdots & \cdots & 0 \\ 0 & \exp(\lambda_2) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \exp(\lambda_{n-1}) & 0 \\ 0 & 0 & \cdots & 0 & \exp(\lambda_n) \end{pmatrix}$$

Pb. 8.18 Using the results from **Pb. 8.17**, we deduce a direct technique for solving the initial value problem for any system of coupled linear ODEs with constant coefficients.

Find and plot the solutions in the interval $0 \le t \le 1$ for the following set of ODEs:

$$\frac{dx_1}{dt} = x_1 + 2x_2$$

$$\frac{dx_2}{dt} = 2x_1 - 2x_2$$

with the initial conditions: $x_1(0) = 1$ and $x_2(0) = 3$. (*Hint:* The solution of $\frac{dX}{dt} = \mathbf{A}\mathbf{X}$ is $\mathbf{X}(t) = \exp(\mathbf{A}t)\mathbf{X}(0)$, where **X** is a time-dependent vector and **A** is a time-independent matrix.)

Pb. 8.19 MATLAB has a shortcut for computing the exponential of a matrix. While the command **`exp(M)`** takes the exponential of each element of the matrix, the command **`expm(M)`** computes the matrix exponential. Verify your results for **Pb. 8.18** using this built-in function.

8.9 The Cayley-Hamilton and Other Analytical Techniques*

In Section 8.8, we presented the general techniques for computing the eigenvalues and eigenvectors of square matrices, and showed their power in solving systems of coupled linear differential equations. In this section, we add to our analytical tools arsenal some techniques that are particularly powerful when elegant solutions are desired in low-dimensional problems. We start with the Cayley-Hamilton theorem.

8.9.1 Cayley-Hamilton Theorem

The matrix **M** satisfies its own characteristic equation.

PROOF As per Eq. (8.29), the characteristic equation for a matrix is given by:

$$p(\lambda) = \det(\mathbf{M} - \lambda\mathbf{I}) = 0 \tag{8.30}$$

Let us now form the polynomial of the matrix **M** having the same coefficients as that of the characteristic equation, $p(\mathbf{M})$. Using the result from **Pb. 8.15**, and assuming that the matix is diagonalizable, we can write for this polynomial:

$$p(\mathbf{M}) = \mathbf{V}p(\mathbf{D})\mathbf{V}^{-1} \tag{8.31}$$

where

$$p(\mathbf{D}) = \begin{pmatrix} p(\lambda_1) & 0 & \cdots & \cdots & 0 \\ 0 & p(\lambda_2) & & & 0 \\ \vdots & & \ddots & & \\ \vdots & & & p(\lambda_{n-1}) & 0 \\ 0 & 0 & \cdots & 0 & p(\lambda_n) \end{pmatrix} \tag{8.32}$$

However, we know that $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$ are all roots of the characteristic equation. Therefore,

$$p(\lambda_1) = p(\lambda_2) = \ldots = p(\lambda_{n-1}) = p(\lambda_n) = 0 \tag{8.33}$$

thus giving:

$$p(\mathbf{D}) = 0 \tag{8.34}$$

$$\Rightarrow p(\mathbf{M}) = 0 \tag{8.35}$$

Example 8.10

Using the Cayley-Hamilton theorem, find the inverse of the matrix **M** given in Example 8.9.

Solution: The characteristic equation for this matrix is given by:

$$p(\mathbf{M}) = \mathbf{M}^2 - 5\mathbf{M} + 4\mathbf{I} = 0$$

Now multiply this equation by $\mathbf{M}^{-1}$ to obtain:

$$\mathbf{M} - 5\mathbf{I} + 4\mathbf{M}^{-1} = 0$$

and

$$\Rightarrow \mathbf{M}^{-1} = 0.25(5\mathbf{I} - \mathbf{M}) = \begin{pmatrix} \frac{3}{4} & -1 \\ -\frac{1}{8} & \frac{1}{2} \end{pmatrix}$$

Example 8.11

Reduce the following fourth-order polynomial in **M**, where **M** is given in Example 8.9, to a first-order polynomial in **M**:

$$P(\mathbf{M}) = \mathbf{M}^4 + \mathbf{M}^3 + \mathbf{M}^2 + \mathbf{M} + \mathbf{I}$$

Solution: From the results of Example 8.10 , we have:

$$\mathbf{M}^2 = 5\mathbf{M} - 4\mathbf{I}$$

$$\mathbf{M}^3 = 5\mathbf{M}^2 - 4\mathbf{M} = 5(5\mathbf{M} - 4\mathbf{I}) - 4\mathbf{M} = 21\mathbf{M} - 20\mathbf{I}$$

$$\mathbf{M}^4 = 21\mathbf{M}^2 - 20\mathbf{M} = 21(5\mathbf{M} - 4I) - 20\mathbf{M} = 85\mathbf{M} - 84\mathbf{I}$$

$$\Rightarrow P(\mathbf{M}) = 112\mathbf{M} - 107\mathbf{I}$$

Verify the answer numerically using MATLAB.

8.9.2 Solution of Equations of the Form $\frac{dX}{dt} = AX$

We sketched a technique in **Pb. 8.17** that uses the eigenvectors matrix and solves this equation. In Example 8.12, we solve the same problem using the Cayley-Hamilton technique.

Example 8.12

Using the Cayley-Hamilton technique, solve the system of equations:

$$\frac{dx_1}{dt} = x_1 + 2x_2$$

$$\frac{dx_2}{dt} = 2x_1 - 2x_2$$

with the initial conditions: $x_1(0) = 1$ and $x_2(0) = 3$

Solution: The matrix **A** for this system is given by:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

and the solution of this system is given by:

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(0)$$

Given that **A** is a 2 ⊗ 2 matrix, we know from the Cayley-Hamilton result that the exponential function of **A** can be written as a first-order polynomial in **A**; thus:

$$P(\mathbf{A}) = e^{\mathbf{A}t} = a\mathbf{I} + b\mathbf{A}$$

To determine a and b, we note that the polynomial equation holds as well for the eigenvalues of **A**, which are equal to –3 and 2; therefore:

$$e^{-3t} = a - 3b$$

$$e^{2t} = a + 2b$$

giving:

$$a = \frac{2}{5}e^{-3t} + \frac{3}{5}e^{2t}$$

$$b = \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{1}{5}e^{-3t} + \frac{4}{5}e^{2t} & \frac{2}{5}e^{2t} - \frac{2}{5}e^{-3t} \\ \frac{2}{5}e^{2t} - \frac{2}{5}e^{-3t} & \frac{4}{5}e^{-3t} + \frac{1}{5}e^{2t} \end{pmatrix}$$

Therefore, the solution of the system of equations is

$$X(t) = \begin{pmatrix} 2e^{2t} - e^{-3t} \\ e^{2t} + 2e^{-3t} \end{pmatrix}$$

8.9.3 Solution of Equations of the Form $\frac{dX}{dt} = AX + B(t)$

Multiplying this equation on the left by e^{-At}, we obtain:

$$e^{-\mathbf{A}t}\frac{d\mathbf{X}}{dt} = e^{-\mathbf{A}t}\mathbf{AX} + e^{-\mathbf{A}t}\mathbf{B}(t) \tag{8.36}$$

Rearranging terms, we write this equation as:

$$e^{-\mathbf{A}t}\frac{d\mathbf{X}}{dt} - e^{-\mathbf{A}t}\mathbf{AX} = e^{-\mathbf{A}t}\mathbf{B}(t) \tag{8.37}$$

We note that the LHS of this equation is the derivative of $e^{-\mathbf{A}t}\mathbf{X}$. Therefore, we can now write Eq. (8.37) as:

$$\frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{X}(t)] = e^{-\mathbf{A}t}\mathbf{B}(t) \tag{8.38}$$

This can be directly integrated to give:

$$[e^{-\mathbf{A}t}\mathbf{X}(t)]\Big|_0^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}(\tau)d\tau \tag{8.39}$$

or, written differently as:

$$e^{-\mathbf{A}t}\mathbf{X}(t) - \mathbf{X}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}(\tau)d\tau \tag{8.40a}$$

which leads to the standard form of the solution:

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}(\tau)d\tau \tag{8.40b}$$

We illustrate the use of this solution in finding the classical motion of an electron in the presence of both an electric field and a magnetic flux density.

Example 8.13

Find the motion of an electron in the presence of a constant electric field and a constant magnetic flux density that are parallel.

Solution: Let the electric field and the magnetic flux density be given by:

$$\vec{E} = E_0\hat{e}_3$$

$$\vec{B} = B_0\hat{e}_3$$

Newton's equation of motion in the presence of both an electric field and a magnetic flux density is written as:

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

where $\vec{v}$ is the velocity of the electron, and m and q are its mass and charge, respectively. Writing this equation in component form, it reduces to the following matrix equation:

$$\frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \alpha\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $\alpha = \frac{qB_0}{m}$ and $\beta = \frac{qE_0}{m}$.

This equation can be put in the above standard form for an inhomogeneous first-order equation if we make the following identifications:

$$\mathbf{A} = \alpha\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \beta\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

First, we note that the matrix **A** is block diagonalizable; that is, all off-diagonal elements with 3 as either the row or column index are zero, and therefore we can separately do the exponentiation of the third component giving $e^0 = 1$; the exponentiation of the top block can be performed along the same steps, using the Cayley-Hamilton techniques from Example 8.12 , giving finally:

$$e^{\mathbf{A}t} = \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) & 0 \\ -\sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, we can write the solutions for the electron's velocity components as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) & 0 \\ -\sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1(0) \\ v_2(0) \\ v_3(0) \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$$

or equivalently:

$$v_1(t) = v_1(0)\cos(\alpha t) + v_2(0)\sin(\alpha t)$$
$$v_2(t) = -v_1(0)\sin(\alpha t) + v_2(0)\cos(\alpha t)$$
$$v_3(t) = v_3(0) + \beta t$$

In-Class Exercises

Pb. 8.20 Plot the 3-D curve, with time as parameter, for the tip of the velocity vector of an electron with an initial velocity $v = v_0\hat{e}_1$, where $v_0 = 10^5$ m/s, entering a region of space where a constant electric field and a constant magnetic flux density are present and are described by: $\vec{E} = E_0\hat{e}_3$, where $E_0 = -10^4$ V/m, and $\vec{B} = B_0\hat{e}_3$, where $B_0 = 10^{-2}$ Wb/m². The mass of the electron is $m_e = 9.1094 \times 10^{-31}$ kg, and the magnitude of the electron charge is $e = 1.6022 \times 10^{-19}$ *C*.

Pb. 8.21 Integrate the expression of the velocity vector in **Pb. 8.20** to find the parametric equations of the electron position vector for the preceding problem configuration, and plot its 3-D curve. Let the origin of the axis be fixed to where the electron enters the region of the electric and magnetic fields.

Pb. 8.22 Find the parametric equations for the electron velocity if the electric field and the magnetic flux density are still parallel, the magnetic flux density is still constant, but the electric field is now described by $\vec{E} = E_0\cos(\omega t)\hat{e}_3$.

Example 8.14

Find the motion of an electron in the presence of a constant electric field and a constant magnetic flux density perpendicular to it.

Solution: Let the electric field and the magnetic flux density be given by:

$$\vec{E} = E_0\hat{e}_3$$

$$\vec{B} = B_0\hat{e}_1$$

The matrix **A** is given in this instance by:

$$\mathbf{A} = \alpha\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

while the vector **B** is still given by:

$$\mathbf{B} = \beta\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The matrix $e^{\mathbf{A}t}$ is now given by:

$$e^{\mathbf{A}t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha t) & \sin(\alpha t) \\ 0 & -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix}$$

and the solution for the velocity vector is for this configuration given, using Eq. (8.40), by:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha t) & \sin(\alpha t) \\ 0 & -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix}\begin{pmatrix} v_1(0) \\ v_2(0) \\ v_3(0) \end{pmatrix} +$$

$$+\int_0^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\alpha(t-\tau)] & \sin[\alpha(t-\tau)] \\ 0 & -\sin[\alpha(t-\tau)] & \cos[\alpha(t-\tau)] \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} d\tau$$

leading to the following parametric representation for the velocity vector:

$$v_1(t) = v_1(0)$$

$$v_2(t) = v_2(0)\cos(\alpha t) + v_3(0)\sin(\alpha t) + \frac{\beta}{\alpha}[1 - \cos(\alpha t)]$$

$$v_3(t) = -v_2(0)\sin(\alpha t) + v_3(0)\cos(\alpha t) + \frac{\beta}{\alpha}\sin(\alpha t)$$

Homework Problems

Pb. 8.23 Plot the 3-D curve, with time as parameter, for the tip of the velocity vector of an electron with an initial velocity $\vec{v}(0) = \frac{v_0}{\sqrt{3}}(\hat{e}_1 + \hat{e}_2 + \hat{e}_3)$, where $v_0 = 10^5$ m/s, entering a region of space where the electric field and the magnetic flux density are constant and described by $\vec{E} = E_0\hat{e}_3$, where $E_0 = -10^4$ V/m; and $\vec{B} = B_0\hat{e}_1$, where $B_0 = 10^{-2}$ Wb/m^2.

Pb. 8.24 Find the parametric equations for the position vector for **Pb. 8.23**, assuming that the origin of the axis is where the electron enters the region of the force fields. Plot the 3-D curve that describes the position of the electron.

8.9.4 Pauli Spinors

We have shown thus far in this section the power of the Cayley-Hamilton theorem in helping us avoid the explicit computation of the eigenvectors while still analytically solving a number of problems of linear algebra where the dimension of the matrices was essentially 2 ⊗ 2, or in some special cases 3 ⊗ 3. In this subsection, we discuss another analytical technique for matrix manipulation, one that is based on a generalized underlying abstract algebraic structure: the Pauli spin matrices. This is the prototype and precursor to more advanced computational techniques from a field of mathematics called Group Theory. The Pauli matrices are 2 ⊗ 2 matrices given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.41a}$$

$$\sigma_2 = j\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{8.41b}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.41c}$$

These matrices have the following properties, which can be easily verified by inspection:

Property 1: $$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbf{I} \tag{8.42}$$

where I is the $2 \otimes 2$ identity matrix.

Property 2: $$\sigma_1\sigma_2 + \sigma_2\sigma_1 = \sigma_1\sigma_3 + \sigma_3\sigma_1 = \sigma_2\sigma_3 + \sigma_3\sigma_2 = 0 \tag{8.43}$$

Property 3: $$\sigma_1\sigma_2 = j\sigma_3; \quad \sigma_2\sigma_3 = j\sigma_1; \quad \sigma_3\sigma_1 = j\sigma_2 \tag{8.44}$$

If we define the quantity $\vec{\sigma} \cdot \vec{v}$ to mean:

$$\vec{\sigma} \cdot \vec{v} = \sigma_1 v_1 + \sigma_2 v_2 + \sigma_3 v_3 \tag{8.45}$$

that is, $\vec{v} = (v_1, v_2, v_3)$, where the parameters v_1, v_2, v_3 are represented as the components of a vector, the following theorem is valid.

THEOREM

$$(\vec{\sigma} \cdot \vec{v})(\vec{\sigma} \cdot \vec{w}) = (\vec{v} \cdot \vec{w})\mathbf{I} + j\vec{\sigma} \cdot (\vec{v} \times \vec{w}) \tag{8.46}$$

where the vectors' dot and cross products have the standard definition.

PROOF The left side of this equation can be expanded as follows:

$$\begin{aligned}(\vec{\sigma} \cdot \vec{v})(\vec{\sigma} \cdot \vec{w}) &= (\sigma_1 v_1 + \sigma_2 v_2 + \sigma_3 v_3)(\sigma_1 w_1 + \sigma_2 w_2 + \sigma_3 w_3) \\ &= (\sigma_1^2 v_1 w_1 + \sigma_2^2 v_2 w_2 + \sigma_3^2 v_3 w_3) + (\sigma_1\sigma_2 v_1 w_2 + \sigma_2\sigma_1 v_2 w_1) + \\ &\quad + (\sigma_1\sigma_3 v_1 w_3 + \sigma_3\sigma_1 v_3 w_1) + (\sigma_2\sigma_3 v_2 w_3 + \sigma_3\sigma_2 v_3 w_2)\end{aligned} \tag{8.47}$$

Using property 1 of the Pauli's matrices, the first parenthesis on the RHS of Eq. (8.47) can be written as:

$$(\sigma_1^2 v_1 w_1 + \sigma_2^2 v_2 w_2 + \sigma_3^2 v_3 w_3) = (v_1 w_1 + v_2 w_2 + v_3 w_3)\mathbf{I} = (\vec{v} \cdot \vec{w})\mathbf{I} \tag{8.48}$$

Using properties 2 and 3 of the Pauli's matrices, the second, third, and fourth parentheses on the RHS of Eq. (8.47) can respectively be written as:

$$(\sigma_1\sigma_2 v_1 w_2 + \sigma_2\sigma_1 v_2 w_1) = j\sigma_3(v_1 w_2 - v_2 w_1) \tag{8.49}$$

$$(\sigma_1\sigma_3 v_1 w_3 + \sigma_3\sigma_1 v_3 w_1) = j\sigma_2(-v_1 w_3 + v_3 w_1) \tag{8.50}$$

$$(\sigma_2\sigma_3 v_2 w_3 + \sigma_3\sigma_2 v_3 w_2) = j\sigma_1(v_2 w_3 - v_3 w_2) \tag{8.51}$$

Recalling that the cross product of two vectors $(\vec{v} \times \vec{w})$ can be written from Eq. (7.49) in components form as:

$$(\vec{v} \times \vec{w}) = (v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1)$$

the second, third, and fourth parentheses on the RHS of Eq. (8.47) can be combined to give $j\vec{\sigma} \cdot (\vec{v} \times \vec{w})$, thus completing the proof of the theorem.

COROLLARY
If $\hat{e}$ is a unit vector, then:

$$(\vec{\sigma} \cdot \hat{e})^2 = \mathbf{I} \tag{8.52}$$

PROOF Using Eq. (8.46), we have:

$$(\vec{\sigma} \cdot \hat{e})^2 = (\hat{e} \cdot \hat{e})\mathbf{I} + j\vec{\sigma} \cdot (\hat{e} \times \hat{e}) = \mathbf{I}$$

where, in the last step, we used the fact that the norm of a unit vector is one and that the cross product of any vector with itself is zero.

A direct result of this corollary is that:

$$(\vec{\sigma} \cdot \hat{e})^{2m} = \mathbf{I} \tag{8.53}$$

and

$$(\vec{\sigma} \cdot \hat{e})^{2m+1} = (\vec{\sigma} \cdot \hat{e}) \tag{8.54}$$

From the above results, we are led to the theorem:

THEOREM

$$\exp(j\vec{\sigma} \cdot \hat{e}\phi) = \cos(\phi) + j\vec{\sigma} \cdot \hat{e}\sin(\phi) \tag{8.55}$$

PROOF If we Taylor expand the exponential function, we obtain:

$$\exp(j\vec{\sigma}\cdot\hat{e}\phi) = \sum_m \frac{[j\phi(\vec{\sigma}\cdot\hat{e})]^m}{m!} \tag{8.56}$$

Now separating the even power and odd power terms, using the just derived result for the odd and even powers of $(\vec{\sigma}\cdot\hat{e})$, and Taylor expansions of the cosine and sine functions, we obtain the desired result.

Example 8.15

Find the time development of the spin state of an electron in a constant magnetic flux density.

Solution: [For readers not interested in the physical background of this problem, they can immediately jump to the paragraph following Eq. (8.59).]

Physical Background: In addition to the spatio-temporal dynamics, the electron and all other elementary particles of nature also have internal degrees of freedom; which means that even if the particle has no translational motion, its state may still evolve in time. The spin of a particle is such an internal degree of freedom. The electron spin internal degree of freedom requires for its representation a two-dimensional vector, that is, two fundamental states are possible. As may be familiar to you from your elementary chemistry courses, the up and down states of the electron are required to satisfactorily describe the number of electrons in the different orbitals of the atoms. For the up state, the eigenvalue of the spin matrix is positive; while for the down state, the eigenvalue is negative (respectively $\hbar/2$ and $-\hbar/2$, where $\hbar = 1.0546 \times 10^{-34}$ J.s $= h/(2\pi)$, and h is Planck's constant).

Due to spin, the quantum mechanical dynamics of an electron in a magnetic flux density does not only include quantum mechanically the time development equivalent to the classical motion that we described in Examples 8.13 and 8.14; it also includes precession of the spin around the external magnetic flux density, similar to that experienced by a small magnet dipole in the presence of a magnetic flux density.

The magnetic dipole moment due to the spin internal degree of freedom of an electron is proportional to the Pauli's spin matrix; specifically:

$$\vec{\mu} = -\mu_B\vec{\sigma} \tag{8.57}$$

where $\mu_B = 0.927 \times 10^{-23}$ J/Tesla.

In the same notation, the electron spin angular momentum is given by:

$$\vec{\mathbf{S}} = \frac{\hbar}{2}\vec{\sigma} \tag{8.58}$$

The electron magnetic dipole, due to spin, interaction with the magnetic flux density is described by the potential:

$$\mathbf{V} = \mu_B \vec{\sigma} \cdot \vec{B} \tag{8.59}$$

and the dynamics of the electron spin state in the magnetic flux density is described by Schrodinger's equation:

$$j\hbar \frac{d}{dt}|\psi\rangle = \mu_B \vec{\sigma} \cdot \vec{B} |\psi\rangle \tag{8.60}$$

where, as previously mentioned, the Dirac ket-vector is two-dimensional.

Mathematical Problem: To put the problem in purely mathematical form, we are asked to find the time development of the two-dimensional vector $|\psi\rangle$ if this vector obeys the system of equations:

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = -j\frac{\Omega}{2}(\vec{\sigma} \cdot \hat{e})\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \tag{8.61}$$

where $\frac{\Omega}{2} = \frac{\mu_B B_0}{\hbar}$, and is called the Larmor frequency, and the magnetic flux density is given by $\vec{B} = B_0 \hat{e}$. The solution of Eq. (8.61) can be immediately written because the magnetic flux density is constant. The solution at an arbitrary time is related to the state at the origin of time through:

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \exp\left[-j\frac{\Omega}{2}(\vec{\sigma} \cdot \hat{e})t\right]\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} \tag{8.62}$$

which from Eq. (8.55) can be simplified to read:

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \left[\cos\left(\frac{\Omega}{2}t\right)I - j(\vec{\sigma} \cdot \hat{e})\sin\left(\frac{\Omega}{2}t\right)\right]\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} \tag{8.63}$$

If we choose the magnetic flux density to point in the z-direction, then the solution takes the very simple form:

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} e^{-j\frac{\Omega}{2}t} a(0) \\ e^{j\frac{\Omega}{2}t} b(0) \end{pmatrix} \tag{8.64}$$

Physically, the above result can be interpreted as the precession of the electron around the direction of the magnetic flux density. To understand this statement, let us find the eigenvectors of the $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_y$ matrices. These are given by:

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{8.65a}$$

$$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\j\end{pmatrix} \quad \text{and} \quad \beta_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-j\end{pmatrix} \tag{8.65b}$$

The eigenvalues of $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_y$ corresponding to the eigenvectors $\boldsymbol{\alpha}$ are equal to 1, while those corresponding to the eigenvectors $\boldsymbol{\beta}$ are equal to –1.

Now, assume that the electron was initially in the state $\boldsymbol{\alpha}_x$:

$$\begin{pmatrix}a(0)\\b(0)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \alpha_x \tag{8.66}$$

By substitution in Eq. (8.64), we can compute the electron spin state at different times. Thus, for the time indicated, the electron spin state is given by the second column in the list below:

$$t = \frac{\pi}{2\Omega} \Rightarrow |\psi\rangle = e^{-j\pi/4}\alpha_y \tag{8.67}$$

$$t = \frac{\pi}{\Omega} \Rightarrow |\psi\rangle = e^{-j\pi/2}\beta_x \tag{8.68}$$

$$t = \frac{3\pi}{2\Omega} \Rightarrow |\psi\rangle = e^{-j3\pi/4}\beta_y \tag{8.69}$$

$$t = \frac{2\pi}{\Omega} \Rightarrow |\psi\rangle = e^{-j\pi}\alpha_x \tag{8.70}$$

In examining the above results, we note that, up to an overall phase, the electron spin state returns to its original state following a cycle. During this cycle, the electron "pointed" successively in the positive *x*-axis, the positive *y*-axis, the negative *x*-axis, and the negative *y*-axis before returning again to the positive *x*-axis, thus mimicking the hand of a clock moving in the counterclockwise direction. It is this "motion" that is referred to as the electron spin precession around the direction of the magnetic flux density.

In-Class Exercises

Pb. 8.25 Find the Larmor frequency for an electron in a magnetic flux density of 100 Gauss (10^{-2} Tesla).

Pb. 8.26 Similar to the electron, the proton and the neutron also have spin as one of their internal degrees of freedom, and similarly attached to this spin, both the proton and the neutron each have a magnetic moment. The magnetic moment attached to the proton and neutron have, respectively, the values $\mu_n = -1.91\ \mu_N$ and $\mu_p = 2.79\ \mu_N$, where μ_N is called the nuclear magneton and is equal to $\mu_N = 0.505 \times 10^{-26}$ Joule/Tesla.

Find the precession frequency of the proton spin if the proton is in the presence of a magnetic flux density of strength 1 Tesla.

Homework Problem

Pb. 8.27 Magnetic resonance imaging (MRI) is one of the most accurate techniques in biomedical imaging. Its principle of operation is as follows. A strong dc magnetic flux density aligns in one of two possible orientations the spins of the protons of the hydrogen nuclei in the water of the tissues (we say that it polarizes them). The other molecules in the system have zero magnetic moments and are therefore not affected. In thermal equilibrium and at room temperature, there are slightly more protons aligned parallel to the magnetic flux density because this is the lowest energy level in this case. A weaker rotating ac transverse flux density attempts to flip these aligned spins. The energy of the transverse field absorbed by the biological system, which is proportional to the number of spin flips, is the quantity measured in an MRI scan. It is a function of the density of the polarized particles present in that specific region of the image, and of the frequency of the ac transverse flux density.

In this problem, we want to find the frequency of the transverse field that will induce the maximum number of spin flips.

The ODE describing the spin system dynamics in this case is given by:

$$\frac{d}{dt}|\psi\rangle = j[\Omega_\perp \cos(\omega t)\sigma_1 + \Omega_\perp \sin(\omega t)\sigma_2 + \Omega\sigma_3]|\psi\rangle$$

where $\Omega = \frac{\mu_p B_0}{\hbar}, \Omega_\perp = \frac{\mu_p B_\perp}{\hbar}$, μ_p is given in **Pb. 8.26**, and the magnetic flux density is given by

$$\vec{B} = B_\perp \cos(\omega t)\hat{e}_1 + B_\perp \sin(\omega t)\hat{e}_2 + B_0\hat{e}_3$$

Assume for simplicity the initial state $|\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and denote the state of the system at time t by $|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$:

a. Find numerically at which frequency ω the magnitude of $b(t)$ is maximum.

b. Once you have determined the optimal ω, go back and examine what strategy you should adopt in the choice of $\Omega_\perp$ to ensure maximum resolution.

c. Verify your numerical answers with the analytical solution of this problem, which is given by:

$$|b(t)|^2 = \frac{\Omega_\perp^2}{\tilde{\omega}^2} \sin^2(\tilde{\omega} t)$$

where $\tilde{\omega}^2 = (\Omega - \omega / 2)^2 + \Omega_\perp^2$.

8.10 Special Classes of Matrices*

8.10.1 Hermitian Matrices

Hermitian matrices of finite or infinite dimensions (operators) play a key role in quantum mechanics, the primary tool for understanding and solving physical problems at the atomic and subatomic scales. In this section, we define these matrices and find key properties of their eigenvalues and eigenvectors.

DEFINITION The Hermitian adjoint of a matrix **M**, denoted by $\mathbf{M}^\dagger$ is equal to the complex conjugate of its transpose:

$$\mathbf{M}^\dagger = \overline{\mathbf{M}}^T \tag{8.71}$$

For example, in complex vector spaces, the bra-vector will be the Hermitian adjoint of the corresponding ket-vector:

$$\langle v| = (|v\rangle)^\dagger \tag{8.72}$$

LEMMA

$$(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger \tag{8.73}$$

PROOF From the definition of matrix multiplication and Hermitian adjoint, we have:

$$\begin{aligned}[(\mathbf{AB})^\dagger]_{ij} &= (\overline{\mathbf{A}}\,\overline{\mathbf{B}})_{ji} \\ &= \sum_k \overline{\mathbf{A}}_{jk}\overline{\mathbf{B}}_{ki} = \sum_k (\mathbf{A}^\dagger)_{kj}(\mathbf{B}^\dagger)_{ik} \\ &= \sum_k (\mathbf{B}^\dagger)_{ik}(\mathbf{A}^\dagger)_{kj} = (\mathbf{B}^\dagger\mathbf{A}^\dagger)_{ij}\end{aligned}$$

DEFINITION A matrix is Hermitian if it is equal to its Hermitian adjoint; that is

$$\mathbf{H}^\dagger = \mathbf{H} \tag{8.74}$$

THEOREM 1
The eigenvalues of a Hermitian matrix are real.

PROOF Let λ_m be an eigenvalue of **H** and let $|v_m\rangle$ be the corresponding eigenvector; then:

$$\mathbf{H}|v_m\rangle = \lambda_m |v_m\rangle \tag{8.75}$$

Taking the Hermitian adjoints of both sides, using the above lemma, and remembering that **H** is Hermitian, we successively obtain:

$$(\mathbf{H}|v_m\rangle)^\dagger = \langle v_m|\mathbf{H}^\dagger = \langle v_m|\mathbf{H} = \langle v_m|\overline{\lambda}_m \tag{8.76}$$

Now multiply (in an inner-product sense) Eq. (8.75) on the left with the bra $\langle v_m|$ and Eq. (8.76) on the right by the ket-vector $|v_m\rangle$, we obtain:

$$\langle v_m|\mathbf{H}|v_m\rangle = \lambda_m\langle v_m|v_m\rangle = \overline{\lambda}_m\langle v_m|v_m\rangle \Rightarrow \lambda_m = \overline{\lambda}_m \tag{8.77}$$

THEOREM 2
The eigenvectors of a Hermitian matrix corresponding to different eigenvalues are orthogonal; that is, given that:

$$\mathbf{H}\left|v_m\right\rangle = \lambda_m\left|v_m\right\rangle \tag{8.78}$$

$$\mathbf{H}\left|v_n\right\rangle = \lambda_n\left|v_n\right\rangle \tag{8.79}$$

and

$$\lambda_m \neq \lambda_n \tag{8.80}$$

then:

$$\left\langle v_n \middle| v_m\right\rangle = \left\langle v_m \middle| v_n\right\rangle = 0 \tag{8.81}$$

PROOF Because the eigenvalues are real, we can write:

$$\left\langle v_n\right|\mathbf{H} = \left\langle v_n\right|\lambda_n \tag{8.82}$$

Dot this quantity on the right by the ket $\left|v_m\right\rangle$ to obtain:

$$\left\langle v_n\right|\mathbf{H}\left|v_m\right\rangle = \left\langle v_n\right|\lambda_n\left|v_m\right\rangle = \lambda_n\left\langle v_n \middle| v_m\right\rangle \tag{8.83}$$

On the other hand, if we dotted Eq. (8.78) on the left with the bra-vector $\left\langle v_n\right|$, we obtain:

$$\left\langle v_n\right|\mathbf{H}\left|v_m\right\rangle = \left\langle v_n\right|\lambda_m\left|v_m\right\rangle = \lambda_m\left\langle v_n \middle| v_m\right\rangle \tag{8.84}$$

Now compare Eqs. (8.83) and (8.84). They are equal, or that:

$$\lambda_m\left\langle v_n \middle| v_m\right\rangle = \lambda_n\left\langle v_n \middle| v_m\right\rangle \tag{8.85}$$

However, because $\lambda_m \neq \lambda_n$, this equality can only be satisfied if $\left\langle v_n \middle| v_m\right\rangle = 0$, which is the desired result.

In-Class Exercises

Pb. 8.28 Show that any Hermitian $2 \otimes 2$ matrix has a unique decomposition into the Pauli spin matrices and the identity matrix.

Pb. 8.29 Find the multiplication rule for two 2 ⊗ 2 Hermitian matrices that have been decomposed into the Pauli spin matrices and the identity matrix; that is

If: $\mathbf{M} = a_0\mathbf{I} + a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2 + a_3\boldsymbol{\sigma}_3$

and $\mathbf{N} = b_0\mathbf{I} + b_1\boldsymbol{\sigma}_1 + b_2\boldsymbol{\sigma}_2 + b_3\boldsymbol{\sigma}_3$

Find: the p-components in: $\mathbf{P} = \mathbf{MN} = p_0\mathbf{I} + p_1\boldsymbol{\sigma}_1 + p_2\boldsymbol{\sigma}_2 + p_3\boldsymbol{\sigma}_3$

Homework Problem

Pb. 8.30 The Calogero and Perelomov matrices of dimensions $n \otimes n$ are given by:

$$M_{lk} = (1-\delta_{lk})\left\{1 + j\cot\left[\frac{(l-k)\pi}{n}\right]\right\}$$

a. Verify that their eigenvalues are given by:

$$\lambda_s = 2s - n - 1$$

where $s = 1, 2, 3, \ldots, n$.

b. Verify that their eigenvectors matrices are given by:

$$V_{ls} = \exp\left(-j\frac{2\pi}{n}ls\right)$$

c. Use the above results to derive the Diophantine summation rule:

$$\sum_{l=1}^{n-1}\cot\left(\frac{l\pi}{n}\right)\sin\left(\frac{2sl\pi}{n}\right) = n - 2s$$

where $s = 1, 2, 3, \ldots, n - 1$.

8.10.2 Unitary Matrices

DEFINITION A unitary matrix has the property that its Hermitian adjoint is equal to its inverse:

$$\mathbf{U}^{\dagger} = \mathbf{U}^{-1} \tag{8.86}$$

An example of a unitary matrix would be the matrix $e^{j\mathbf{H}t}$, if **H** was Hermitian.

THEOREM 1
The eigenvalues of a unitary matrix all have magnitude one.

PROOF The eigenvalues and eigenvectors of the unitary matrix satisfy the usual equations for these quantities; that is:

$$\mathbf{U}\left|v_n\right\rangle = \lambda_n\left|v_n\right\rangle \tag{8.87}$$

Taking the Hermitian conjugate of this equation, we obtain:

$$\left\langle v_n\right|\mathbf{U}^{\dagger} = \left\langle v_n\right|\mathbf{U}^{-1} = \left\langle v_n\right|\bar{\lambda}_n \tag{8.88}$$

Multiplying Eq. (8.87) on the left by Eq. (8.88), we obtain:

$$\left\langle v_n\right|\mathbf{U}^{-1}\mathbf{U}\left|v_n\right\rangle = \left\langle v_n \middle| v_n\right\rangle = \left|\lambda_n\right|^2\left\langle v_n \middle| v_n\right\rangle \tag{8.89}$$

from which we deduce the desired result that: $\left|\lambda_n\right|^2 = 1$.

A direct corollary of the above theorem is that $|\det(\mathbf{U})| = 1$. This can be proven directly if we remember the result of **Pb. 8.15**, which states that the determinant of any diagonalizable matrix is the product of its eigenvalues, and the above theorem that proved that each of these eigenvalues has unit magnitude.

THEOREM 2
A transformation represented by a unitary matrix keeps invariant the scalar (dot, or inner) product of two vectors.

PROOF The matrix **U** acting on the vectors $|\varphi\rangle$ and $|\psi\rangle$ results in two new vectors, denoted by $|\varphi'\rangle$ and $|\psi'\rangle$ and such that:

$$\left|\varphi'\right\rangle = \mathbf{U}\left|\varphi\right\rangle \tag{8.90}$$

$$\left|\psi'\right\rangle = \mathbf{U}\left|\psi\right\rangle \tag{8.91}$$

Taking the Hermitian adjoint of Eq. (8.90), we obtain:

$$\left\langle\varphi'\right| = \left\langle\varphi\right|\mathbf{U}^{\dagger} = \left\langle\varphi\right|\mathbf{U}^{-1} \tag{8.92}$$

Multiplying Eq. (8.91) on the left by Eq. (8.92), we obtain:

$$\langle \varphi' | \psi' \rangle = \langle \varphi | \mathbf{U}^{-1} \mathbf{U} | \psi \rangle = \langle \varphi | \psi \rangle \tag{8.93}$$

which is the result that we are after. In particular, note that the norm of the vector under this matrix multiplication remains invariant. We will have the opportunity to study a number of examples of such transformations in Chapter 9.

8.10.3 Unimodular Matrices

DEFINITION A unimodular matrix has the defining property that its determinant is equal to one. In the remainder of this section, we restrict our discussion to 2 ⊗ 2 unimodular matrices, as these form the tools for the matrix formulation of ray optics and Gaussian optics, which are two of the major sub-fields of photonics engineering.

Example 8.16

Find the eigenvalues and eigenvectors of the 2 ⊗ 2 unimodular matrix.

Solution: Let the matrix **M** be given by the following expression:

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8.94}$$

The unimodularity condition is then written as:

$$\det(\mathbf{M}) = ad - bc = 1 \tag{8.95}$$

Using Eq. (8.95), the eigenvalues of this matrix are given by:

$$\lambda_{\pm} = \frac{1}{2}[(a+d) \pm \sqrt{(a+d)^2 - 4}] \tag{8.96}$$

Depending on the value of $(a + d)$, these eigenvalues can be parameterized in a simple expression. We choose, here, the range $-2 \leq (a + d) \leq 2$ for illustrative purposes. Under this constraint, the following parameterization is convenient:

$$\cos(\theta) = \frac{1}{2}(a+d) \tag{8.97}$$

(For the ranges below –2 and above 2, the hyperbolic cosine function will be more appropriate and similar steps to the ones that we will follow can be repeated.)

Having found the eigenvalues, which can now be expressed in the simple form:

$$\lambda_{\pm} = e^{\pm j\theta} \tag{8.98}$$

let us proceed to find the matrix **V**, defined as:

$$\mathbf{M} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} \quad \text{or} \quad \mathbf{M}\mathbf{V} = \mathbf{V}\mathbf{D} \tag{8.99}$$

and where **D** is the diagonal matrix of the eigenvalues. By direct substitution, in the matrix equation defining **V**, Eq. (8.99), the following relations can be directly obtained:

$$\frac{V_{11}}{V_{21}} = \frac{\lambda_{+} - d}{c} \tag{8.100}$$

and

$$\frac{V_{12}}{V_{22}} = \frac{\lambda_{-} - d}{c} \tag{8.101}$$

If we choose for convenience $V_{11} = V_{22} = c$ (which is always possible because each eigenvector can have the value of one of its components arbitrary chosen with the other components expressed as functions of it), the matrix **V** can be written as:

$$\mathbf{V} = \begin{pmatrix} e^{j\theta} - d & e^{-j\theta} - d \\ c & c \end{pmatrix} \tag{8.102}$$

and the matrix **M** can be then written as:

$$\mathbf{M} = \frac{\begin{pmatrix} e^{j\theta} - d & e^{-j\theta} - d \\ c & c \end{pmatrix}\begin{pmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{pmatrix}\begin{pmatrix} c & d - e^{-j\theta} \\ -c & e^{j\theta} - d \end{pmatrix}}{(2j\sin(\theta))} \tag{8.103}$$

Homework Problem

Pb. 8.31 Use the decomposition given by Eq. (8.103) and the results of **Pb. 8.15** to prove the Sylvester theorem for the unimodular matrix, which states that:

$$\mathbf{M}^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} \dfrac{\sin[(n+1)\theta] - D\sin(n\theta)}{\sin(\theta)} & \dfrac{B\sin(n\theta)}{\sin(\theta)} \\ \dfrac{C\sin(n\theta)}{\sin(\theta)} & \dfrac{D\sin(n\theta) - \sin[(n-1)\theta]}{\sin(\theta)} \end{pmatrix}$$

where θ is defined in Equation 8.97.

Application: Dynamics of the Trapping of an Optical Ray in an Optical Fiber

Optical fibers, the main waveguides of land-based optical broadband networks are hair-thin glass fibers that transmit light pulses over very long distances with very small losses. Their waveguiding property is due to a quadratic index of refraction radial profile built into the fiber. This profile is implemented in the fiber manufacturing process, through doping the glass with different concentrations of impurities at different radial distances.

The purpose of this application is to explain how waveguiding can be achieved if the index of refraction inside the fiber has the following profile:

$$n = n_0\left(1 - \frac{n_2^2}{2}r^2\right) \tag{8.104}$$

where r is the radial distance from the fiber axis and $n_2^2 r^2$ is a number smaller than 0.01 everywhere inside the fiber.

This problem can, of course, be solved by finding the solution of Maxwell equations, or the differential equation of geometrical optics for ray propagation in a non-uniform medium. However, we will not do this in this application. Here, we use only Snell's law of refraction (see Figure 8.4), which states that at the boundary between two transparent materials with two different indices of refraction, light refracts such that the product of the index of refraction of each medium multiplied by the sine of the angle that the ray makes with the normal to the interface in each medium is constant, and Sylvester's theorem derived in **Pb. 8.31**.

Let us describe a light ray going through the fiber at any point z along its length, by the distance r that the ray is displaced from the fiber axis, and by the small angle α that the ray's direction makes with the fiber axis. Now consider two points on the fiber axis separated by the small distance δz. We want

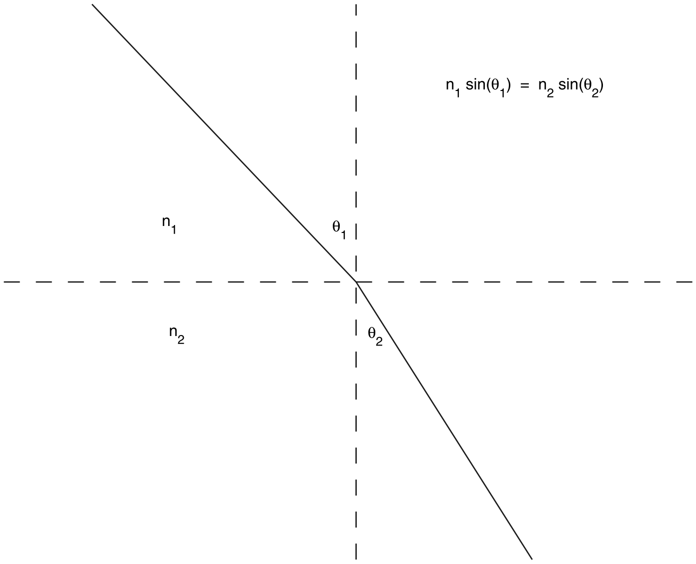

FIGURE 8.4
Parameters of Snell's law of refraction.

to find $r(z + \delta z)$ and $\alpha(z + \delta z)$, knowing $r(z)$ and $\alpha(z)$. We are looking for the iteration relation that successive applications will permit us to find the ray displacement r and α slope at any point inside the fiber if we knew their values at the fiber entrance plane.

We solve the problem in two steps. We first assume that there was no bending in the ray, and then find the ray transverse displacement following a small displacement. This is straightforward from the definition of the slope of the ray:

$$\delta r = \alpha(z)\delta z \tag{8.105}$$

Because the angle α is small, we approximated the tangent of the angle by the value of the angle in radians.

Therefore, if we represent the position and slope of the ray as a column matrix, Eq. (8.105) can be represented by the following matrix representation:

$$\begin{pmatrix} r(z+\delta z) \\ \alpha(z+\delta z) \end{pmatrix} = \begin{pmatrix} 1 & \delta z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r(z) \\ \alpha(z) \end{pmatrix} \tag{8.106}$$

Next, we want to find the bending experienced by the ray in advancing through the distance δz. Because the angles that should be used in Snell's law are the complementary angles to those that the ray forms with the axis of the fiber, and recalling that the glass index of refraction is changing only in the radial direction, we deduce from Snell's law that:

$$n(r+\delta r)\cos(\alpha+\delta\alpha) = n(r)\cos(\alpha) \tag{8.107}$$

Now, taking the leading terms of a Taylor expansion of the LHS of this equation leads us to:

$$\left[n(r)+\frac{dn(r)}{dr}\delta r\right]\left[1-\frac{(\alpha+\delta\alpha)^2}{2}\right] \approx n(r)\left(1-\frac{\alpha^2}{2}\right) \tag{8.108}$$

Further simplification of this equation gives to first order in the variations:

$$\delta\alpha \approx \frac{1}{\alpha n(r)}\frac{dn(r)}{dr}\delta r \approx \frac{1}{n_0}(-n_0 n_2^2 r)\delta z = -(n_2^2\delta z)r \tag{8.109}$$

which can be expressed in matrix form as:

$$\begin{pmatrix} r(z+\delta z) \\ \alpha(z+\delta z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -n_2^2\delta z & 1 \end{pmatrix}\begin{pmatrix} r(z) \\ \alpha(z) \end{pmatrix} \tag{8.110}$$

The total variation in the values of the position and slope of the ray can be obtained by taking the product of the two matrices in Eqs. (8.106) and (8.110), giving:

$$\begin{pmatrix} r(z+\delta z) \\ \alpha(z+\delta z) \end{pmatrix} = \begin{pmatrix} 1-(n_2\delta z)^2 & \delta z \\ -n_2^2\delta z & 1 \end{pmatrix}\begin{pmatrix} r(z) \\ \alpha(z) \end{pmatrix} \tag{8.111}$$

Equation (8.111) provides us with the required recursion relation to numerically iterate the progress of the ray inside the fiber. Thus, the ray distance from the fiber axis and the angle that it makes with this axis can be computed at any z in the fiber if we know the values of the ray transverse coordinate and its slope at the entrance plane.

The problem can also be solved analytically if we note that the determinant of this matrix is 1 (the matrix is unimodular). Sylvester's theorem provides the means to obtain the following result:

$$\begin{pmatrix} r(z) \\ \alpha(z) \end{pmatrix} = \begin{pmatrix} \cos(n_2 z) & \dfrac{\sin(n_2 z)}{n_2} \\ -n_2\sin(n_2 z) & \cos(n_2 z) \end{pmatrix}\begin{pmatrix} r(0) \\ \alpha(0) \end{pmatrix} \tag{8.112}$$

Homework Problems

Pb. 8.32 Consider an optical fiber of radius $a = 30\mu$, $n_0 = 4/3$, and $n_2 = 10^3\ m^{-1}$. Three ray enters this fiber parallel to the fiber axis at distances of 5μ, 10μ, and 15μ from the fiber's axis.

a. Write a MATLAB program to follow the progress of the rays through the fiber, properly choosing the δz increment.

b. Trace these rays going through the fiber.

Figure 8.5 shows the answer that you should obtain for a fiber length of 3 cm.

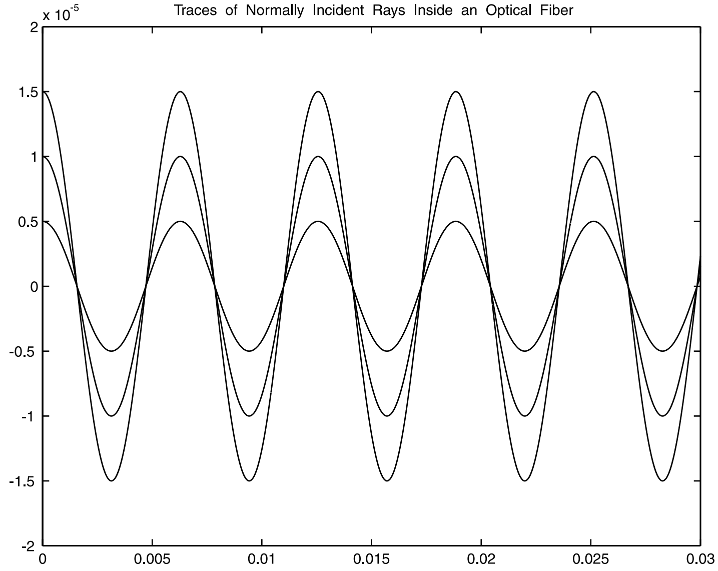

FIGURE 8.5
Traces of rays, originally parallel to the fiber's axis, when propagating inside an optical fiber.

Pb. 8.33 Using Sylvester's theorem, derive Eq. (8.112). (*Hint:* Define the angle θ, such that $\sin\left(\frac{\theta}{2}\right)=\frac{\alpha\delta z}{2}$, and recall that while δz goes to zero, its product with the number of iterations is finite and is equal to the distance of propagation inside the fiber.)

Pb. 8.34 Find the maximum angle that an incoming ray can have so that it does not escape from the fiber. (Remember to include the refraction at the entrance of the fiber.)

8.11 MATLAB Commands Review

det	Compute the determinant of a matrix.
expm	Computes the matrix exponential.

`eye`	Identity matrix.
`inv`	Find the inverse of a matrix.
`ones`	Matrix with all elements equal to 1.
`polyfit`	Fit polynomial to data.
`triu`	Extract upper triangle of a matrix.
`tril`	Extract lower triangle of a matrix.
`zeros`	Matrix with all elements equal to zero.
`[V,D]=eig(M)`	Finds the eigenvalues and eigenvectors of a matrix.

9

Transformations

The theory of transformations concerns itself with changes in the coordinates and shapes of objects upon the action of geometrical operations, dynamical boosts, or other operators. In this chapter, we deal only with linear transformations, using examples from both plane geometry and relativistic dynamics (space-time geometry). We also show how transformation techniques play an important role in image processing. We formulate both the problems and their solutions in the language of matrices. Matrices are still denoted by boldface type and matrix multiplication by an asterisk.

9.1 Two-Dimensional (2-D) Geometric Transformations

We first concern ourselves with the operations of inversion about the origin of axes, reflection about the coordinate axes, rotation around the origin, scaling, and translation. But prior to going into the details of these transformations, we need to learn how to draw closed polygonal figures in MATLAB so that we can implement and graph the different cases.

9.1.1 Polygonal Figures Construction

Consider a polygonal figure whose vertices are located at the points:

$$(x_1, y_1),\ (x_2, y_2),\ \dots,\ (x_n, y_n)$$

The polygonal figure can then be thought off as line segments (edges) connecting the vertices in a given order, including the edge connecting the last point to the initial point to ensure that we obtain a closed figure. The implementation of the steps leading to the drawing of the figure follows:

1. Label all vertex points.
2. Label the path you follow.

3. Construct a (2 ⊗ (n + 1) matrix, the **G** matrix, where the elements of the first row consist of the ordered (n + 1)-tuplet, (x_1, x_2, x_3, …, x_n, x_1), and those of the second row consists of the corresponding y coordinates (n + 1)-tuplet.
4. Plot the second row of **G** as function of its first row.

Example 9.1

Plot the trapezoid whose vertices are located at the points (2, 1), (6, 1), (5, 3), and (3, 3).

Solution: Enter and execute the following commands:

```
G=[2 6 5 3 2; 1 1 3 3 1];
plot(G(1,:),G(2,:))
```

To ensure that the exact geometrical shape is properly reproduced, remember to instruct your computer to choose the axes such that you have equal x-range and y-range and an aspect ratio of 1. If you would like to add any text anywhere in the figure, use the command **`gtext`**.

9.1.2 Inversion about the Origin and Reflection about the Coordinate Axes

We concern ourselves here with inversion with respect to the origin and with reflection about the x- or y-axis. Inversion about other points or reflection about other than the coordinate axes can be deduced from a composition of the present transformations and those discussed later.

- The inversion about the origin changes the coordinates as follows:

$$\begin{aligned} x' &= -x \\ y' &= -y \end{aligned} \tag{9.1}$$

In matrix form, this transformation can be represented by:

$$\mathbf{P} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9.2}$$

- For the reflection about the x-axis, denoted by $\mathbf{P}_x$, and the reflection about the y-axis, denoted by $\mathbf{P}_y$, the transformation matrices are given by:

$$\mathbf{P}_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9.3}$$

$$\mathbf{P}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9.4}$$

In-Class Exercise

Pb. 9.1 Using the trapezoid of Example 9.1, obtain all the transformed **G**′s as a result of the action of each of the three transformations defined in Eqs. (9.2) through (9.4), and plot the transformed figures on the same graph.

Pb. 9.2 In drawing the original trapezoid, we followed the counterclockwise direction in the sequencing of the different vertices. What is the sequencing of the respective points in each of the transformed **G**′s?

Pb. 9.3 Show that the quantity $(x^2 + y^2)$ is invariant under separately the action of $\mathbf{P}_x$, $\mathbf{P}_y$, or **P**.

9.1.3 Rotation around the Origin

The new coordinates of a point in the *x*-*y* plane rotated by an angle θ around the *z*-axis can be directly derived through some elementary trigonometry. Here, instead, we derive the new coordinates using results from the complex numbers chapter (Chapter 6). Recall that every point in a 2-D plane represents a complex number, and multiplication by a complex number of modulus 1 and argument θ results in a rotation of angle θ of the original point. Therefore:

$$\begin{aligned} z' &= ze^{j\theta} \\ x' + jy' &= (x + jy)(\cos(\theta) + j\sin(\theta)) \\ &= (x\cos(\theta) - y\sin(\theta)) + j(x\sin(\theta) + y\cos(\theta)) \end{aligned} \tag{9.5}$$

Equating separately the real parts and the imaginary parts, we deduce the action of rotation on the coordinates of a point:

$$\begin{aligned} x' &= x\cos(\theta) - y\sin(\theta) \\ y' &= x\sin(\theta) + y\cos(\theta) \end{aligned} \tag{9.6}$$

The above transformation can also be written in matrix form. That is, if the point is represented by a size 2 column vector, then the new vector is related to the old one through the following transformation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{R}(\theta) \begin{bmatrix} x \\ y \end{bmatrix} \tag{9.7}$$

The convention for the sign of the angle is the same as that used in Chapter 6, namely that it is measured positive when in the counterclockwise direction.

Preparatory Exercises

Using the above form for the rotation matrix, verify the following properties:

Pb. 9.4 Its determinant is equal to 1.

Pb. 9.5 $\mathbf{R}(-\theta) = [\mathbf{R}(\theta)]^{-1} = [\mathbf{R}(\theta)]^T$

Pb. 9.6 $\mathbf{R}(\theta_1) * \mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2) = \mathbf{R}(\theta_2) * \mathbf{R}(\theta_1)$

Pb. 9.7 $(x')^2 + (y')^2 = x^2 + y^2$

Pb. 9.8 Show that $\mathbf{P} = \mathbf{R}(\theta = \pi)$. Also show that there is no rotation that can reproduce $\mathbf{P}_x$ or $\mathbf{P}_y$.

In-Class Exercises

Pb. 9.9 Find the coordinates of the image of the point (x, y) obtained by reflection about the line $y = x$. Test your results using MATLAB.

Pb. 9.10 Find the transformation matrix corresponding to a rotation of $-\pi/3$, followed by an inversion around the origin. Solve the problem in two different ways.

Pb. 9.11 By what angle should you rotate the trapezoid so that point (6, 1) of the trapezoid of Example 9.1 is now on the y-axis?

9.1.4 Scaling

If the x-coordinate of each point in the plane is multiplied by a positive constant s_x, then the effect of this transformation is to expand or compress each plane figure in the x-direction. If $0 < s_x < 1$, the result is a compression; and if $s_x > 1$, the result is an expansion. The same can also be done along the y-axis. This class of transformations is called scaling.

The matrices corresponding to these transformations, in 2-D, are respectively:

$$\mathbf{S_x} = \begin{bmatrix} s_x & 0 \\ 0 & 1 \end{bmatrix} \tag{9.8}$$

$$\mathbf{S_y} = \begin{bmatrix} 1 & 0 \\ 0 & s_y \end{bmatrix} \tag{9.9}$$

In-Class Exercises

Pb. 9.12 Find the transformation matrix for simultaneously compressing the x-coordinate by a factor of 2, while expanding the y-coordinate by a factor of 2. Apply this transformation to the trapezoid of Example 9.1 and plot the result.

Pb. 9.13 Find the inverse matrices for $\mathbf{S_x}$ and $\mathbf{S_y}$.

9.1.5 Translation

A translation is defined by a vector $\vec{T} = (t_x, t_y)$, and the transformation of the coordinates is given simply by:

$$\begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned} \tag{9.10}$$

or, written in matrix form as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{9.11}$$

The effect of translation over the matrix **G** is described by the relation:

$$\mathbf{G_T} = \mathbf{G} + \mathbf{T} * \mathbf{ones}(1, n+1) \tag{9.12}$$

where n is the number of points being translated.

In-Class Exercise

Pb. 9.14 Translate the trapezoid of Example 9.1 by a vector of length 5 that is making an angle of 30° with the x-axis.

9.2 Homogeneous Coordinates

As we have seen in Section 9.1, inversion about the origin, reflection about the coordinate axes, rotation, and scaling are operations that can be represented by a multiplicative matrix, and therefore the composite operation of acting successively on a figure by one or more of these operations can be described by a product of matrices. The translation operation, on the other hand, is represented by an addition, and thus cannot be incorporated, as yet, into the matrix multiplication scheme; and consequently, the expression for composite operations becomes less tractable. We illustrate this situation with the following example:

Example 9.2

Find the new G that results from rotating the trapezoid of Example 9.1 by a $\pi/4$ angle around the point Q (–5, 5).

Solution: Because we have thus far defined the rotation matrix only around the origin, our task here is to generalize this result. We solve the problem by reducing it to a combination of elementary operations thus far defined. The strategy for solving the problem goes as follows:

1. Perform a translation to place Q at the origin of a new coordinate system.
2. Perform a $\pi/4$ rotation around the new origin, using the above form for rotation.
3. Translate back the origin to its initial location.

Written in matrix form, the above operations can be written sequentially as follows:

1.
$$\mathbf{G_1} = \mathbf{G} + \mathbf{T} * \mathbf{ones}(1, n+1) \qquad (9.13)$$

where
$$\mathbf{T} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \qquad (9.14)$$

and $n = 4$.

2. $$\mathbf{G}_2 = \mathbf{R}(\pi / 4) * \mathbf{G}_1 \tag{9.15}$$

3. $$\mathbf{G}_3 = \mathbf{G}_2 - \mathbf{T} * \mathbf{ones}(1, n+1) \tag{9.16}$$

and the final result can be written as:

$$\mathbf{G}_3 = \mathbf{R}(\pi / 4) * \mathbf{G} + [(\mathbf{R}(\pi / 4) - 1) * \mathbf{T}] * \mathbf{ones}(1, n+1) \tag{9.17}$$

We can implement the above sequence of transformations through the following *script M-file*:

```
plot(-5,5,'*')
hold on
G=[2 6 5 3 2; 1 1 3 3 1];
plot(G(1,:),G(2,:),'b')
T=[5;-5];
G1=G+T*ones(1,5);
plot(G1(1,:),G1(2,:), 'r')
R=[cos(pi/4) -sin(pi/4);sin(pi/4) cos(pi/4)];
G2=R*G1;
plot(G2(1,:),G2(2,:),'g')
G3=G2-T*ones(1,5);
plot(G3(1,:),G3(2,:),'k')
axis([-12 12 -12 12])
axis square
```

Although the above formulation of the problem is absolutely correct, the number of terms in the final expression for the image can wind up, in more involved problems, being large and cumbersome because of the existence of sums and products in the intermediate steps. Thus, the question becomes: can we incorporate all the transformations discussed thus far into only multiplicative matrices?

The answer comes from an old trick that mapmakers have used successfully; namely, the technique of homogeneous coordinates. In this technique, as applied to the present case, we append to any column vector the row with value 1, that is, the point (x_m, y_m) is now represented by the column vector:

$$\begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} \tag{9.18}$$

Similarly in the definition of **G**, we should append to the old definition, a row with all elements being 1.

In this coordinate representation, the different transformations thus far discussed are now multiplicative and take the following forms:

$$\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.19}$$

$$\mathbf{P_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.20}$$

$$\mathbf{P_y} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.21}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.22}$$

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.23}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \tag{9.24}$$

The composite matrix of any two transformations can now be written as the product of the matrices representing the constituent transformations. Of course, this economizes on the writing of expressions and makes the calculations less prone to trivial errors originating in the expansion of products of sums.

Example 9.3

Repeat Example 9.2, but now use the homogeneous coordinates.

Solution: The following *script M-file* implements the required task:

```
plot(-5,5,'*')
hold on
G=[2 6 5 3 2; 1 1 3 3 1;1 1 1 1 1];
plot(G(1,:),G(2,:),'b')
T=[1 0 5;0 1 -5;0 0 1];
G1=T*G;
plot(G1(1,:),G1(2,:), 'r')
R=[cos(pi/4) -sin(pi/4) 0;sin(pi/4) cos(pi/4) 0;...
  0 0 1];
G2=R*G1;
plot(G2(1,:),G2(2,:),'g')
G3=inv(T)*G2;
plot(G3(1,:),G3(2,:),'k')
axis([-12 12 -12 12])
axis square
hold off
```

9.3 Manipulation of 2-D Images

Currently more and more images are being stored or transmitted in digital form. What does this mean?

To simplify the discussion, consider a black and white image and assume that it has a square boundary. The digital image is constructed by the optics of the detecting system (i.e., the camera) to form on a plane containing a 2-D array of detectors, instead of the traditional photographic film. Each of these detectors, called a pixel (picture element), measures the intensity of light falling on it. The image is then represented by a matrix having the same size as the detectors' 2-D array structure, and such that the value of each of the matrix elements is proportional to the intensity of the light falling on the associated detector element. Of course, the resolution of the picture increases as the number of arrays increases.

9.3.1 Geometrical Manipulation of Images

Having the image represented by a matrix, it is now possible to perform all kinds of manipulations on it in MATLAB. For example, we could flip it in the left/right directions (**fliplr**), or in the up/down direction (**flipud**), or rotate it by 90° (**rot90**), or for that matter transform it by any matrix transformation. In the remainder of this section, we explore some of the techniques commonly employed in the handling and manipulation of digital images.

Let us explore and observe the structure of a matrix subjected to the above elementary trasformations. For this purpose, execute and observe the outputs from each of the following commands:

```
M=(1/25)*[1 2 3 4 5;6 7 8 9 10;11 12 13 14 15;16
  17 18 19 20;21 22 23 24 25]
lrM=fliplr(M)
udM=flipud(M)
Mr90=rot90(M)
```

A careful examination of the resulting matrix elements will indicate the general features of each of these transformations. You can also see in a visually more suggestive form how each of the transformations changed the image of the original matrix, if we render the image of **M** and its transform in false colors, that is, we assign a color to each number.

To perform this task, choose the **colormap(hot)** command to obtain the images. In this mapping, the program assigns a color to each pixel, varying from black-red-yellow-white, depending on the magnitude of the intensity at the corresponding detector.

Enter, in the following sequence, each of the following commands and at each step note the color distributions of the image:

```
colormap(hot)
imagesc(M,[0 1])
imagesc(lrM,[0 1])
imagesc(udM,[0 1])
imagesc(Mr90,[0 1])
```

The command **imagesc** produces an intensity image of a data matrix that spans a given range of values.

9.3.2 Digital Image Processing

A typical problem in digital image processing involves the analysis of the raw data of an image that was subject, during acquisition, to a blur due to the movement of the camera or to other sources of noise. An example of this situation occurs in the analysis of aerial images; the images are blurred due, *inter alia,* to the motion of the plane while the camera shutter is open. The question is, can we do anything to obtain a crisper image from the raw data if we know the speed and altitude of the plane when it took the photograph?

The answer is affirmative. We consider for our example the photograph of a rectangular board. Construct this image by entering:

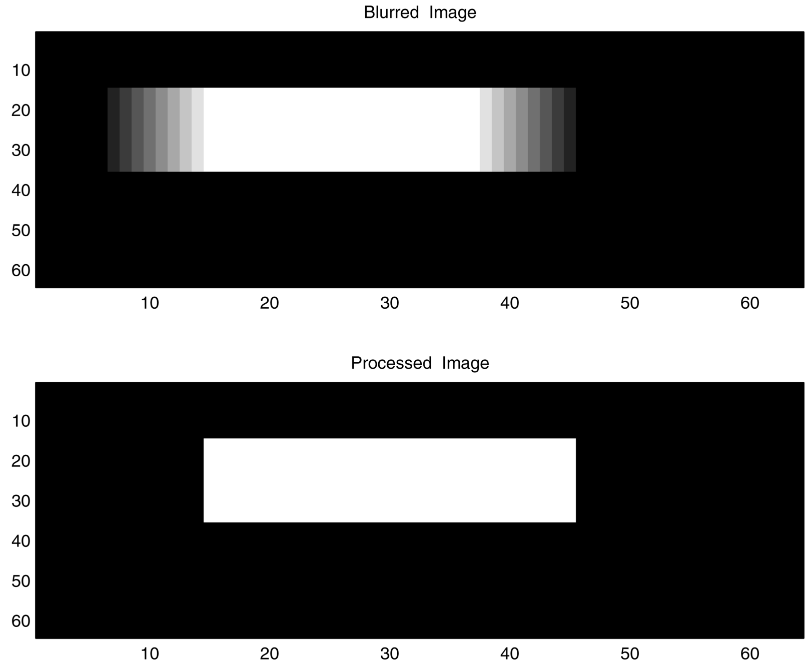

FIGURE 9.1
The raw and processed images of a rectangular board photographed from a moving plane. Top panel: Raw (blurred) image. Bottom panel: Processed image.

```
N=64;
A=zeros(N,N);
A(15:35,15:45)=1;
colormap(gray);
imagesc(A,[0 1])
```

where (N N) is the size of the image (here, N = 64).

Now assume that the camera that took the image had moved while the shutter was open by a distance that would correspond in the image plane to L pixels. What will the image look like now? (See Figure 9.1.)

The blurring operation was modeled here by the matrix **B**. The blurred image is simulated through the matrix product:

$$\mathbf{A1} = \mathbf{A} * \mathbf{B} \tag{9.25}$$

where **B**, the blurring matrix, is given by the following Toeplitz matrix:

```
L=9;
B=toeplitz([ones(L,1);zeros(N-L,1)],[1;zeros(N-
  1,1)])/L;
```

Here, the blur length was L = 9, and the blurred image **A1** was obtained by executing the following commands:

```
A1=A*B;
imagesc(A1,[0 1])
```

To bring back the unblurred picture, simply multiply the matrix `A1` on the right by `inv(B)` and obtain the original image.

In practice, one is given the blurred image and asked to reconstruct it while correcting for the blur. What to do?

1. Compute the blur length from the plane speed and height.
2. Construct the Toeplitz matrix, and take its inverse.
3. Apply the inverse of the Toeplitz matrix to the blurred image matrix, obtaining the processed image.

9.3.3 Encrypting an Image

If for any reason, two individuals desire to exchange an image but want to keep its contents only to themselves, they may agree beforehand on a scrambling matrix that the first individual applies to scramble the sent image, while the second individual applies the inverse of the scramble matrix to unscramble the received image.

Given that an average quality image currently has a minimum size of about (1000×1000) pixels, reconstructing the scrambling matrix, if chosen cleverly, would be inaccessible except to the most powerful and specialized computers.

The purpose of the following problems is to illustrate an efficient method for building a scrambling matrix.

In-Class Exercises

Assume for simplicity that the 2-D array size is (10×10), and that the scrambling matrix is chosen such that each row has one element equal to 1, while the others are 0, and no two rows are equal.

Pb. 9.15 For the (10×10) matrix dimension, how many possible scrambling matrices **S**, constructed as per the above prescription, are there? If the matrix size is (1000×1000), how many such scrambling matrices will there be?

Pb. 9.16 An original figure was scrambled by the scrambling matrix **S** to obtain the image shown in Figure 9.2. The matrix **S** is (10×10) and has all its elements equal to zero, except $S(1, 6) = S(2, 3) = S(3, 2) = S(4, 1) = S(5, 9) = S(6, 4) = S(7, 10) = S(8, 7) = S(9, 8) = S(10, 5) = 1$. Find the original image.

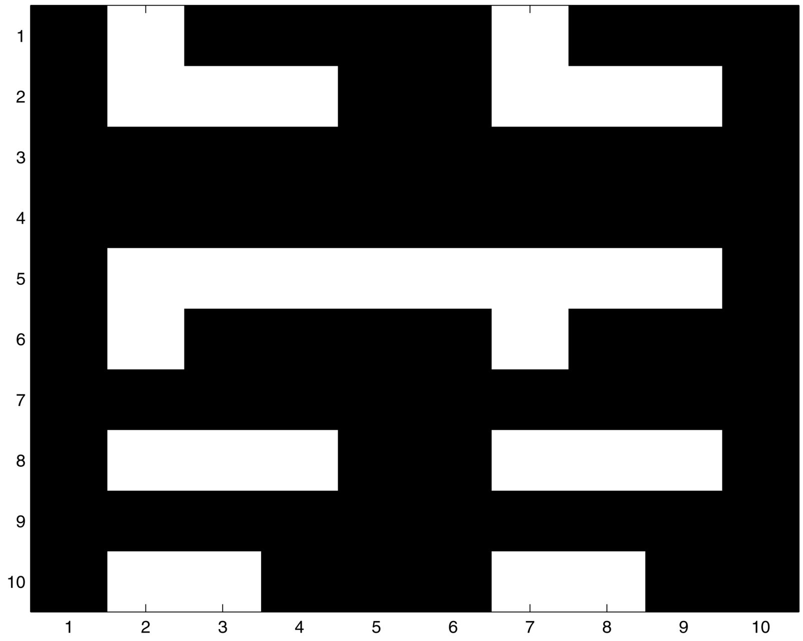

FIGURE 9.2
Scrambled image of Pb. 9.16.

9.4 Lorentz Transformation*

9.4.1 Space-Time Coordinates

Einstein's theory of special relativity studies the relationship of the dynamics of a system, if described in two coordinate systems moving with constant speed one from the other. The theory of special relativity does not assume, as classical mechanics does, that there exists an absolute time common to all coordinate systems. It associates with each coordinate system a four-dimensional space (three space coordinates and one time coordinate). The theory of special relativity associates a space-time transformation to go between two coordinate systems moving uniformily with respect to each other. Each real point event (e.g., the arrival of a light flash on a screen) will be measured in both systems. If we distinguish by primes the data of the second observer from those of the first, then the first observer will ascribe to the event the coordinates (x, y, z, t), while the second observer will ascribe to it the coordinates (x', y', z', t'); that is, there is no absolute time. The Lorentz transformation gives the rules for going from one coordinate system to the other.

Assuming that the velocity v between the two systems has the same direction as the positive x-axis and where the x-axis direction continuously coincides with that of the x'-axis; and furthermore, that the origin of the spatial coordinates of one system at time $t = 0$ coincides with the origin of the other system at time $t' = 0$, Einstein, on the basis of two postulates, derived the following transformation relating the coordinates of the two systems:

$$x' = \frac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{9.26}$$

where c is the velocity of light in vacuum. The derivation of these formulae are detailed for you in electromagnetic theory or modern physics courses and are not the subject of discussions here. Our purpose here is to show that knowing the above transformations, we can deduce many interesting physical observations as a result thereof.

Preparatory Exercise

Pb. 9.17 Show that, upon a Lorentz transformation, we have the equality:

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

This is referred to as the Lorentz invariance of the norm of the space-time four-vectors. What is the equivalent invariant in 3-D Euclidean geometry?

If we rename our coordinates such that:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = jct \tag{9.27}$$

the Lorentz transformation takes the following matricial form:

$$\mathbf{L}_\beta = \begin{bmatrix} \dfrac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \dfrac{j\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{j\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-\beta^2}} \end{bmatrix} \tag{9.28}$$

where $\beta = \dfrac{v}{c}$, and the relations that were given earlier relating the primed and unprimed coordinates can be summarized by:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{j\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{j\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{9.29}$$

In-Class Exercises

Pb. 9.18 Write the above transformation for the case that the two coordinate systems are moving from each other at half the speed of light, and find (x', y', z', t') if

$$x = 2, \quad y = 3, \quad z = 4, \quad ct = 3$$

Pb. 9.19 Find the determinant of $\mathbf{L}_\beta$.

Pb. 9.20 Find the multiplicative inverse of $\mathbf{L}_\beta$, and compare it to the transpose.

Pb. 9.21 Find the approximate expression of $\mathbf{L}_\beta$ for $\beta << 1$. Give a physical interpretation to your result using Newtonian mechanics.

9.4.2 Addition Theorem for Velocities

The physical problem of interest here is: assuming that a point mass is moving in the primed system in the x'-y' plane with uniform speed u' and its trajectory is making an angle θ' with the x'-axis, what is the speed of this particle, as viewed in the unprimed system, and what is the angle that its trajectory makes with the x-axis, as observed in the unprimed system?

In the unprimed and primed systems, respectively, the parametric equations for the point particle motion are given by:

$$x = ut\cos(\theta), \quad y = ut\sin(\theta) \tag{9.30}$$

$$x' = u't'\cos(\theta), \quad y' = u't'\sin(\theta') \tag{9.31}$$

where u and u' are the speeds of the particle in the unprimed and primed systems, respectively. Note that if the prime system moves with velocity v with respect to the unprimed system, then the unprimed system moves with a velocity $-v$ with respect to the primed system, and using the Lorentz transformation, we can write the following equalities:

$$ut\cos(\theta) = \frac{(u'\cos(\theta') + v)}{\sqrt{1-\beta^2}}t' \tag{9.32}$$

$$ut\sin(\theta) = u't'\sin(\theta') \tag{9.33}$$

$$t = \frac{[1 + (u'v / c^2)\cos(\theta')]}{\sqrt{1-\beta^2}}t' \tag{9.34}$$

Dividing Eqs. (9.32) and (9.33) by Eq. (9.34), we obtain:

$$u\cos(\theta) = \frac{(u'\cos(\theta') + v)}{[1 + (u'v / c^2)\cos(\theta')]} \tag{9.35}$$

$$u\sin(\theta) = \frac{u'\sin(\theta')\sqrt{1-\beta^2}}{[1 + (u'v / c^2)\cos(\theta')]} \tag{9.36}$$

From this we can deduce the magnitude and direction of the velocity of the particle, as measured in the unprimed system:

$$u^2 = \frac{u'^2 + v^2 + 2u'v\cos(\theta') - (u'^2v^2 / c^2)\sin^2(\theta')}{[1 + (u'v / c^2)\cos(\theta')]^2} \tag{9.37}$$

$$\tan(\theta) = \frac{u'\sin(\theta')\sqrt{1-\beta^2}}{u'\cos(\theta') + v} \tag{9.38}$$

Preparatory Exercises

Pb. 9.22 Find the velocity of a photon (the quantum of light) in the unprimed system if its velocity in the primed system is $u' = c$.

(Note the constancy of the velocity of light, if measured from either the primed or the unprimed system. As previously mentioned, this constituted one of only two postulates in Einstein's formulation of the theory of special relativity, which determined uniquely the form of the dynamical boost transformation.)

Pb. 9.23 Show that if u' is parallel to the x'-axis, then the velocity addition formula takes the following simple form:

$$u = \frac{u' + v}{1 + \dfrac{u'v}{c^2}}$$

Pb. 9.24 Find the approximate form of the above expression for u when $\beta << 1$, and show that it reduces to the expression of velocity addition in Newtonian mechanics.

In-Class Exercises

Pb. 9.25 Find the angle θ, if $\theta' = \frac{\pi}{2}$ and $u' = v = \frac{c}{2}$.

Pb. 9.26 Plot the angle θ as a function of θ' when $v/c = 0.99$ and $u'/c = 1$.

Pb. 9.27 Let the variable ϕ be defined such that $\tanh(\phi) = \beta$. Write the Lorentz transformation matrix as function of ϕ. Can you give the Lorentz transformation a geometric interpretation in non-Euclidean geometry?

Pb. 9.28 Using the result of **Pb. 9.27**, write the resultant transformation from a boost with parameter ϕ_1, followed by another boost with parameter ϕ_2. Does this rule for composition of Lorentz transformations remind you of a similar transformation that you studied previously in this chapter?

9.5 MATLAB Commands Review

`colormap`	Control the color mix of an image.
`fliplr`	Flip a matrix left to right.
`flipud`	Flip a matrix in the up-to-down direction.
`imagesc`	Create a pixel intensity map from data stored in a matrix.
`load`	Import data files from outside MATLAB.
`rot90`	Rotate a matrix by 90°.
`toeplitz`	Specialized matrix constructor that describes, *inter alia,* the operation of a blur in an image.

10

*A Taste of Probability Theory**

10.1 Introduction

In addition to its everyday use in all aspects of our public, personal, and leisure lives, probability plays an important role in electrical engineering practice in at least three important aspects. It is the mathematical tool to deal with three broad areas:

1. *The problems associated with the inherent uncertainty in the input of certain systems.* The random arrival time of certain inputs to a system cannot be predetermined; for example, the log-on and the log-off times of terminals and workstations connected to a computer network, or the data packets' arrival time to a computer network node.
2. *The problems associated with the distortion of a signal due to noise.* The effects of noise have to be dealt with satisfactorily at each stage of a communication system from the generation, to the transmission, to the detection phases. The source of this noise may be due to either fluctuations inherent in the physics of the problem (e.g., quantum effects and thermal effects) or due to random distortions due to externally generated uncontrollable parameters (e.g., weather, geography, etc.).
3. *The problems associated with inherent human and computing machine limitations while solving very complex systems.* Individual treatment of the dynamics of very large number of molecules in a material, in which more than 10^{22} molecules may exist in a quart-size container, is not possible at this time, and we have to rely on statistical averages when describing the behavior of such systems. This is the field of statistical physics and thermodynamics.

Furthermore, probability theory provides the necessary mathematical tools for error analysis in all experimental sciences. It permits estimation of the error bars and the confidence level for any experimentally obtained result, through a methodical analysis and reduction of the raw data.

In future courses in probability, random variables, stochastic processes (which is random variables theory with time as a parameter), information theory, and statistical physics, you will study techniques and solutions to the different types of problems from the above list. In this very brief introduction to the subject, we introduce only the very fundamental ideas and results — where more advanced courses seem to almost always start.

10.2 Basics

Probability theory is best developed mathematically based on a set of axioms from which a well-defined deductive theory can be constructed. This is referred to as the axiomatic approach. We concentrate, in this section, on developing the basics of probability theory, using a physical description of the underlying concepts of probability and related simple examples, to lead us intuitively to what is usually the starting point of the set theoretic axiomatic approach.

Assume that we conduct n independent trials under identical conditions, in each of which, depending on chance, a particular event A of particular interest either occurs or does not occur. Let $n(A)$ be the number of experiments in which A occurs. Then, the ratio $n(A)/n$, called the relative frequency of the event A to occur in a series of experiments, clusters for $n \to \infty$ about some constant. This constant is called the probability of the event A, and is denoted by:

$$P(A) = \lim_{n\to\infty} \frac{n(A)}{n} \tag{10.1}$$

From this definition, we know specifically what is meant by the statement that the probability for obtaining a head in the flip of a fair coin is 1/2.

Let us consider the rolling of a single die as our prototype experiment :

1. The possible outcomes of this experiment are elements belonging to the set:

$$S = \{1, 2, 3, 4, 5, 6\} \tag{10.2}$$

If the die is fair, the probability for each of the elementary elements of this set to occur in the roll of a die is equal to:

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6} \tag{10.3}$$

2. The observer may be interested not only in the elementary elements occurrence, but in finding the probability of a certain event which may consist of a set of elementary outcomes; for example:
   a. An event may consist of "obtaining an even number of spots on the upward face of a randomly rolled die." This event then consists of all successful trials having as experimental outcomes any member of the set:

$$E = \{2, 4, 6\} \tag{10.4}$$

   b. Another event may consist of "obtaining three or more spots" (hence, we will use this form of abbreviated statement, and not keep repeating: on the upward face of a randomly rolled die). Then, this event consists of all successful trials having experimental outcomes any member of the set:

$$B = \{3, 4, 5, 6\} \tag{10.5}$$

   Note that, in general, events may have overlapping elementary elements.

For a fair die, using the definition of the probability as the limit of a relative frequency, it is possible to conclude, based on experimental trials, that:

$$P(E) = P(2) + P(4) + P(6) = \frac{1}{2} \tag{10.6}$$

while

$$P(B) = P(3) + P(4) + P(5) + P(6) = \frac{2}{3} \tag{10.7}$$

and

$$P(S) = 1 \tag{10.8}$$

The last equation [Eq. (10.8)] is the mathematical expression for the statement that the probability of the event that includes all possible elementary outcomes is 1 (i.e., certainty).

It should be noted that if we define the events O and C to mean the events of "obtaining an odd number" and "obtaining a number smaller than 3," respectively, we can obtain these events' probabilities by enumerating the elements of the subsets of S that represent these events; namely:

$$P(O) = P(1) + P(3) + P(5) = \frac{1}{2} \tag{10.9}$$

$$P(C) = P(1) + P(2) = \frac{1}{3} \tag{10.10}$$

However, we also could have obtained these same results by noting that the events E and O (B and C) are disjoint and that their union spanned the set S. Therefore, the probabilities for events O and C could have been deduced, as well, through the relations:

$$P(O) = 1 - P(E) \tag{10.11}$$

$$P(C) = 1 - P(B) \tag{10.12}$$

From the above and similar observations, it would be a satisfactory representation of the physical world if the above results were codified and elevated to the status of axioms for a formal theory of probability. However, the question becomes how many of these basic results (the axioms) one really needs to assume, such that it will be possible to derive all other results of the theory from this seed. This is the starting point for the formal approach to the probability theory.

The following axioms were proven to be a satisfactory starting point. Assign to each event A, consisting of elementary occurrences from the set S, a number $P(A)$, which is designated as the probability of the event A, and such that:

1. $$0 \le P(A) \tag{10.13}$$

2. $$\mathrm{P(S)} = 1 \tag{10.14}$$

3. If: $A \cap B = \varnothing$, where $\varnothing$ is the empty set (10.15)
   Then: $P(A \cup B) = P(A) + P(B)$

In the following examples, we illustrate some common techniques for finding the probabilities for certain events. Look around, and you will find plenty more.

Example 10.1

Find the probability for getting three sixes in a roll of three dice.

Solution: First, compute the number of elements in the total sample space. We can describe each roll of the dice by a 3-tuplet (a, b, c), where $a, b,$ and c can take the values 1, 2, 3, 4, 5, 6. There are $6^3 = 216$ possible 3-tuplets. The event that we are seeking is realized only in the single elementary occurrence when the 3-tuplet (6, 6, 6) is obtained; therefore, the probability for this event, for fair dice, is

$$P(A) = \frac{1}{216}$$

Example 10.2

Find the probability of getting only two sixes in a roll of three dice.

Solution: The event in this case consists of all elementary occurrences having the following forms:

$$(a, 6, 6), \quad (6, b, 6), \quad (6, 6, c)$$

where $a = 1, \ldots, 5$; $b = 1, \ldots, 5$; and $c = 1, \ldots, 5$. Therefore, the event A consists of elements corresponding to 15 elementary occurrences, and its probability is

$$P(A) = \frac{15}{216}$$

Example 10.3

Find the probability that, if three individuals are asked to guess a number from 1 to 10, their guesses will be different numbers.

Solution: There are 1000 distinct equiprobable 3-tuplets (a, b, c), where each component of the 3-tuplet can have any value from 1 to 10. The event A occurs when all components have unequal values. Therefore, while a can have any of 10 possible values, b can have only 9, and c can have only 8. Therefore, $n(A) = 8 \times 9 \times 10$, and the probability for the event A is

$$P(A) = \frac{8 \times 9 \times 10}{1000} = 0.72$$

Example 10.4

An inspector checks a batch of 100 microprocessors, 5 of which are defective. He examines ten items selected at random. If none of the ten items is defective, he accepts the batch. What is the probability that he will accept the batch?

Solution: The number of ways of selecting 10 items from a batch of 100 items is:

$$N = \frac{100!}{10!(100-10)!} = \frac{100!}{10!90!} = C_{10}^{100}$$

where C_k^n is the binomial coefficient and represents the number of combinations of n objects taken k at a time without regard to order. It is equal to $\frac{n!}{k!(n-k)!}$. All these combinations are equally probable.

If the event A is that where the batch is accepted by the inspector, then A occurs when all ten items selected belong to the set of acceptable quality units. The number of elements in A is

$$N(A) = \frac{95!}{10!\,85!} = C_{10}^{95}$$

and the probability for the event A is

$$P(A) = \frac{C_{10}^{95}}{C_{10}^{100}} = \frac{86 \times 87 \times 88 \times 89 \times 90}{96 \times 97 \times 98 \times 99 \times 100} = 0.5837$$

In-Class Exercises

Pb. 10.1 A cube whose faces are colored is split into 125 smaller cubes of equal size.

a. Find the probability that a cube drawn at random from the batch of randomly mixed smaller cubes will have three colored faces.

b. Find the probability that a cube drawn from this batch will have two colored faces.

Pb. 10.2 An urn has three blue balls and six red balls. One ball was randomly drawn from the urn and then a second ball, which was blue. What is the probability that the first ball drawn was blue?

Pb. 10.3 Find the probability that the last two digits of the cube of a random integer are 1. Solve the problem analytically, and then compare your result to a numerical experiment that you will conduct and where you compute the cubes of all numbers from 1 to 1000.

Pb. 10.4 From a lot of n resistors, p are defective. Find the probability that k resistors out of a sample of m selected at random are found defective.

Pb. 10.5 Three cards are drawn from a deck of cards.

a. Find the probability that these cards *are* the Ace, the King, and the Queen of Hearts.

b. Would the answer change if the statement of the problem was "an Ace, a King, and a Queen"?

Pb. 10.6 Show that:

$$P(\overline{A}) = 1 - P(A)$$

where $\overline{A}$, the complement of A, are all events in S having no element in common with A.

NOTE In solving certain category of probability problems, it is often convenient to solve for $P(A)$ by computing the probability of its complement and then applying the above relation.

Pb. 10.7 Show that if $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

(*Hint:* Use mathematical induction and Eq. (10.15).)

10.3 Addition Laws for Probabilities

We start by reminding the reader of the key results of elementary set theory:

- The Commutative law states that:

$$A \cap B = B \cap A \tag{10.16}$$

$$A \cup B = B \cup A \tag{10.17}$$

- The Distributive laws are written as:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{10.18}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{10.19}$$

- The Associative laws are written as:

$$(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C \tag{10.20}$$

$$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C \tag{10.21}$$

- De Morgan's laws are

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \tag{10.22}$$

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B} \tag{10.23}$$

- The Duality principle states that: If in an identity, we replace unions by intersections, intersections by unions, S by $\varnothing$, and $\varnothing$ by S, then the identity is preserved.

THEOREM 1

If we define the difference of two events $A_1 - A_2$ to mean the events in which A_1 occurs but not A_2, the following equalities are valid:

$$P(A_1 - A_2) = P(A_1) - P(A_1 \cap A_2) \tag{10.24}$$

$$P(A_2 - A_1) = P(A_2) - P(A_1 \cap A_2) \tag{10.25}$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \tag{10.26}$$

PROOF From the basic set theory algebra results, we can deduce the following equalities:

$$A_1 = (A_1 - A_2) \cup (A_1 \cap A_2) \tag{10.27}$$

$$A_2 = (A_2 - A_1) \cup (A_1 \cap A_2) \tag{10.28}$$

$$A_1 \cup A_2 = (A_1 - A_2) \cup (A_2 - A_1) \cup (A_1 \cap A_2) \tag{10.29}$$

Further note that the events $(A_1 - A_2)$, $(A_2 - A_1)$, and $(A_1 \cap A_2)$ are mutually exclusive. Using the results from **Pb. 10.7**, Eqs. (10.27) and (10.28), and the preceding comment, we can write:

$$P(A_1) = P(A_1 - A_2) + P(A_1 \cap A_2) \tag{10.30}$$

$$P(A_2) = P(A_2 - A_1) + P(A_1 \cap A_2) \tag{10.31}$$

which establish Eqs. (10.24) and (10.25). Next, consider Eq. (10.29); because of the mutual exclusivity of each event represented by each of the parenthesis on its LHS, we can use the results of **Pb. 10.7**, to write:

$$P(A_1 \cup A_2) = P(A_1 - A_2) + P(A_2 - A_1) + P(A_1 \cap A_2) \tag{10.32}$$

using Eqs. (10.30) and (10.31), this can be reduced to Eq. (10.26).

THEOREM 2

Given any n events $A_1, A_2, \ldots, A_n$ and defining $P_1, P_2, P_3, \ldots, P_n$ to mean:

$$P_1 = \sum_{i=1}^{n} P(A_i) \tag{10.33}$$

$$P_2 = \sum_{1 \le i < j \le n} P(A_i \cap A_j) \tag{10.34}$$

$$P_3 = \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) \tag{10.35}$$

etc. …, then:

$$P\left(\bigcup_{k=1}^{n} A_k\right) = P_1 - P_2 + P_3 - P_4 + \ldots + (-1)^{n-1} P_n \tag{10.36}$$

This theorem can be proven by mathematical induction (we do not give the details of this proof here).

Example 10.5

Using the events *E*, *O*, *B*, *C* as defined in Section 10.1, use Eq. (10.36) to show that: $P(E \cup O \cup B \cup C) = 1$.

Solution: Using Eq. (10.36), we can write:

$$\begin{aligned}
P(E \cup O \cup B \cup C) &= P(E) + P(O) + P(B) + P(C) \\
&\quad - [P(E \cap O) + P(E \cap B) + P(E \cap C) + P(O \cap B) + P(O \cap C) + P(B \cap C)] \\
&\quad + [P(E \cap O \cap B) + P(E \cap O \cap C) + P(E \cap B \cap C) + P(O \cap B \cap C)] \\
&\quad - P(E \cap O \cap B \cap C) \\
&= \left[\frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{1}{3}\right] - \left[0 + \frac{2}{6} + \frac{1}{6} + \frac{2}{6} + \frac{1}{6} + 0\right] + [0 + 0 + 0 + 0] - [0] = 1
\end{aligned}$$

Example 10.6

Show that for any *n* events $A_1, A_2, \ldots, A_n$, the following inequality holds:

$$P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$$

Solution: We prove this result by mathematical induction:

- For $n = 2$, the result holds because by Eq. (10.26) we have:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

and since any probability is a non-negative number, this leads to the inequality:

$$P(A_1 \cup A_2) \le P(A_1) + P(A_2)$$

- Assume that the theorem is true for $(n - 1)$ events, then we can write:

$$P\left(\bigcup_{k=2}^{n} A_k\right) \le \sum_{k=2}^{n} P(A_k)$$

- Using associativity, Eq. (10.26), the result for $(n - 1)$ events, and the non-negativity of the probability, we can write:

$$\begin{aligned} P\left(\bigcup_{k=1}^{n} A_k\right) &= P\left(A_1 \cup \left(\bigcup_{k=1}^{n} A_k\right)\right) = P(A_1) + P\left(\bigcup_{k=2}^{n} A_k\right) - P\left(A_1 \cap \left(\bigcup_{k=2}^{n} A_k\right)\right) \\ &\le P(A_1) + \sum_{k=2}^{n} P(A_k) - P\left(A_1 \cap \left(\bigcup_{k=2}^{n} A_k\right)\right) \le \sum_{k=1}^{n} P(A_k) \end{aligned}$$

which is the desired result.

In-Class Exercises

Pb. 10.8 Show that if the events $A_1, A_2, \ldots, A_n$ are such that:

$$A_1 \subset A_2 \subset \ldots \subset A_n$$

then:

$$P\left(\bigcup_{k=1}^{n} A_k\right) = P(A_n)$$

Pb. 10.9 Show that if the events $A_1, A_2, \ldots, A_n$ are such that:

$$A_1 \supset A_2 \supset \ldots \supset A_n$$

then:

$$P\left(\bigcap_{k=1}^{n} A_k\right) = P(A_n)$$

Pb. 10.10 Find the probability that a positive integer randomly selected will be non-divisible by:

a. 2 and 3.

b. 2 or 3.

Pb. 10.11 Show that the expression for Eq. (10.36) simplifies to:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = C_1^n P(A_1) - C_2^n P(A_1 \cap A_2) + C_3^n P(A_1 \cap A_2 \cap A_3) -$$
$$\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

when the probability for the intersection of any number of events is independent of the indices.

Pb. 10.12 A filing stack has n drawers, and a secretary randomly files m-letters in these drawers.

a. Assuming that $m > n$, find the probability that there will be at least one letter in each drawer.

b. Plot this probability for $n = 12$, and $15 \leq m \leq 50$.

(*Hint:* Take the event A_j to mean that no letter is filed in the j^{th} drawer and use the result of **Pb. 10.11**.)

10.4 Conditional Probability

The conditional probability of an event A assuming C and denoted by $P(A|C)$ is, by definition, the ratio:

$$P(A|C) = \frac{P(A \cap C)}{P(C)} \tag{10.37}$$

Example 10.7

Considering the events E, O, B, C as defined in Section 10.2 and the above definition for conditional probability, find the probability that the number of spots showing on the die is even, assuming that it is equal to or greater than 3.

Solution: In the above notation, we are asked to find the quantity $P(E|B)$. Using Eq. (10.37), this is equal to:

$$P(E|B) = \frac{P(E \cap B)}{P(B)} = \frac{P(\{4,6\})}{P(\{3,4,5,6\})} = \frac{\left(\frac{2}{6}\right)}{\left(\frac{4}{6}\right)} = \frac{1}{2}$$

In this case, $P(E|B) = P(E)$. When this happens, we say that the two events E and B are independent.

Example 10.8

Find the probability that the number of spots showing on the die is even, assuming that it is larger than 3.

Solution: Call D the event of having the number of spots larger than 3. Using Eq. (10.37), $P(E|D)$ is equal to:

$$P(E|D) = \frac{P(E \cap D)}{P(D)} = \frac{P(\{4,6\})}{P(\{4,5,6\})} = \frac{\left(\frac{2}{6}\right)}{\left(\frac{3}{6}\right)} = \frac{2}{3}$$

In this case, $P(E|D) \neq P(E)$; and thus the two events E and D are not independent.

Example 10.9

Find the probability of picking a blue ball first, then a red ball from an urn that contains five red balls and four blue balls.

Solution: From the definition of conditional probability [Eq. (10.37)], we can write:

$$P(\text{Blue ball first and Red ball second}) =$$

$$P(\text{Red ball second}|\text{Blue ball first}) \times P(\text{Blue ball first})$$

The probability of picking a blue ball first is

$$P\left(\text{Blue ball first}\right) = \frac{\text{Original number of Blue balls}}{\text{Total number of balls}} = \frac{4}{9}$$

The conditional probability is given by:

$$P(\text{Red ball second}|\text{Blue ball first}) =$$

$$\frac{\text{Number of Red balls}}{\text{Number of balls remaining after first pick}} = \frac{5}{8}$$

giving:

$$P(\text{Blue ball first and Red ball second}) = \frac{4}{9} \times \frac{5}{8} = \frac{5}{18}$$

10.4.1 Total Probability and Bayes Theorems

TOTAL PROBABILITY THEOREM

If $[A_1, A_2, \ldots, A_n]$ is a partition of the total elementary occurrences set S, that is,

$$\bigcup_{i=1}^{n} A_i = S \quad \text{and} \quad A_i \cap A_j = \varnothing \quad \text{for} \quad i \neq j$$

and B is an arbitrary event, then:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n) \tag{10.38}$$

PROOF From the algebra of sets, and the definition of a partition, we can write the following equalities:

$$\begin{aligned} B = B \cap S &= B \cap (A_1 \cup A_2 \cup \ldots \cup A_n) \\ &= (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n) \end{aligned} \tag{10.39}$$

Since the events $(B \cap A_i)$ and $(B \cap A_j)$ and are mutually exclusive for $i \neq j$, then using the results of **Pb. 10.7**, we can deduce that:

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n) \tag{10.40}$$

Now, using the conditional probability definition [Eq. (10.38)], Eq. (10.40) can be written as:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n) \tag{10.41}$$

This result is known as the Total Probability theorem.

BAYES THEOREM

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n)} \tag{10.42}$$

PROOF From the definition of the conditional probability [Eq. (10.37)], we can write:

$$P(B \cap A_i) = P(A_i|B)P(B) \tag{10.43}$$

Again, using Eqs. (10.37) and (10.43), we have:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} \tag{10.44}$$

Now, substituting Eq. (10.41) in the denominator of Eq. (10.44), we obtain Eq. (10.42).

Example 10.10

A digital communication channel transmits the signal as a collection of ones (1s) and zeros (0s). Assume (statistically) that 40% of the 1s and 33% of the 0s are changed upon transmission. Suppose that, in a message, the ratio between the transmitted 1 and the transmitted 0 was 5/3. What is the probability that the received signal is the same as the transmitted signal if:

a. The received signal was a 1?

b. The received signal was a 0?

Solution: Let O be the event that 1 was received, and Z be the event that 0 was received. If H_1 is the hypothesis that 1 was received and H_0 is the hypothesis that 0 was received, then from the statement of the problem, we know that:

$$\frac{P(H_1)}{P(H_0)} = \frac{5}{3} \quad \text{and} \quad P(H_1) + P(H_0) = 1$$

giving:

$$P(H_1) = \frac{5}{8} \quad \text{and} \quad P(H_0) = \frac{3}{8}$$

Furthermore, from the text of the problem, we know that:

$$P(O|H_1) = \frac{3}{5} \quad \text{and} \quad P(Z|H_1) = \frac{2}{5}$$

$$P(O|H_0) = \frac{1}{3} \quad \text{and} \quad P(Z|H_0) = \frac{2}{3}$$

From the total probability result [Eq. (10.41)], we obtain:

$$\begin{aligned} P(O) &= P(O|H_1)P(H_1) + P(O|H_0)P(H_0) \\ &= \frac{3}{5} \times \frac{5}{8} + \frac{1}{3} \times \frac{3}{8} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} P(Z) &= P(Z|H_1)P(H_1) + P(Z|H_0)P(H_0) \\ &= \frac{2}{5} \times \frac{5}{8} + \frac{2}{3} \times \frac{3}{8} = \frac{1}{2} \end{aligned}$$

The probability that the received signal is 1 if the transmitted signal was 1 from Bayes theorem:

$$P(H_1|O) = \frac{P(H_1)P(O|H_1)}{P(O)} = \frac{\frac{5}{3}\frac{3}{5}}{\frac{1}{2}} = \frac{3}{4}$$

Similarly, we can obtain the probability that the received signal is 0 if the transmitted signal is 0:

$$P(H_0|Z) = \frac{P(H_0)P(Z|H_0)}{P(Z)} = \frac{\frac{3}{8}\frac{2}{3}}{\frac{1}{2}} = \frac{1}{2}$$

In-Class Exercises

Pb. 10.13 Show that when two events A and B are independent, the addition law for probability becomes:

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

Pb. 10.14 Consider four boxes, each containing 1000 resistors. Box 1 contains 100 defective items; Box 2 contains 400 defective items; Box 3 contains 50 defective items; and Box 4 contains 80 defective items.

a. What is the probability that a resistor chosen at random from any of the boxes is defective?

b. What is the probability that if the resistor is found defective, it came from Box 2?

(*Hint:* The randomness in the selection of the box means that: $P(B_1) = P(B_2) = P(B_3) = P(B_4) = 0.25$.)

10.5 Repeated Trials

Bernoulli trials refer to identical, successive, and independent trials, in which an elementary event A can occur with probability:

$$p = P(A) \tag{10.45}$$

or fail to occur with probability:

$$q = 1 - p \tag{10.46}$$

In the case of n consecutive Bernoulli trials, each elementary event can be described by a sequence of 0s and 1s, such as in the following:

$$\omega = \underbrace{1\,0\,0\,0\,1 \ldots 0\,1}_{n \text{ digits} - k \text{ ones}} \tag{10.47}$$

where n is the number of trials, k is the number of successes, and $(n - k)$ is the number of failures. Because the trials are independent, the probability for the above single occurrence is:

$$P(\omega) = p^k q^{n-k} \tag{10.48}$$

The total probability for the event with k successes in n trials is going to be the probability of the single event multiplied by the number of configurations with a given number of digits and a given number of 1s. The number of such configurations is given by the binomial coefficient C_k^n. Therefore:

$$P(k \text{ successes in } n \text{ trials}) = C_k^n p^k q^{n-k} \tag{10.49}$$

Example 10.11

Find the probability that the number 3 will appear twice in five independent rolls of a die.

Solution: In a single trial, the probability of success (i.e., 3 showing up) is

$$p = \frac{1}{6}$$

Therefore, the probability that it appears twice in five independent rolls will be

$$P(2 \text{ successes in } 5 \text{ trials}) = C_2^5 p^2 q^5 = \frac{5!}{2!\,3!}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 0.16075$$

Example 10.12

Find the probability that in a roll of two dice, three occurrences of snake-eyes (one spot on each die) are obtained in ten rolls of the two dice.

Solution: The space S of the roll of two dice consists of 36 elementary elements (6×6), only one of which results in a snake-eyes configuration; therefore:

$$p = 1/36; \quad k = 3; \quad n = 10$$

and

$$P(3 \text{ successes in } 10 \text{ trials}) = C_3^{10} p^3 q^7 = \frac{10!}{3!7!}\left(\frac{1}{36}\right)^3\left(\frac{35}{36}\right)^7 = 0.00211$$

In-Class Exercises

Pb. 10.15 Assuming that a batch of manufactured components has an 80% chance of passing an inspection, what is the chance that at least 16 batches in a lot of 20 would pass the inspection?

Pb. 10.16 In an experiment, we keep rolling a fair die until it comes up showing three spots. What are the probabilities that this will take:

a. Exactly four rolls?

b. At least four rolls?

c. At most four rolls?

Pb. 10.17 Let X be the number of successes in a Bernoulli trials experiment with n trials and the probability of success p in each trial. If the mean number of successes m, also called average value $\overline{X}$ and expectation value $E(X)$, is defined as:

$$m \equiv \overline{X} \equiv E(X) \equiv \sum XP(X)$$

and the variance is defined as:

$$V(X) \equiv E((X - \overline{X})^2)$$

show that:

$$\overline{X} = np \quad \text{and} \quad V(X) = np(1-p)$$

10.5.1 Generalization of Bernoulli Trials

In the above Bernoulli trials, we considered the case of whether or not a single event A was successful (i.e., two choices). This was the simplest partition of the set S.

In cases where we partition the set S in r subsets: $S = \{A_1, A_2, \ldots, A_r\}$, and the probabilities for these single events are, respectively: $\{p_1, p_2, \ldots, p_r\}$, where $p_1 + p_2 + \ldots + p_r = 1$, it can be easily proven that the probability in n independent trials for the event A_1 to occur k_1 times, the event A_2 to occur k_1 times, etc., is given by:

$$P(k_1, k_2, \ldots, k_r; n) = \frac{n!}{k_1! k_2! \ldots k_r!} p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \tag{10.50}$$

where $k_1 + k_2 + \ldots + k_r = n$

Example 10.13

Consider the sum of the spots in a roll of two dice. We partition the set of outcomes $\{2, 3, \ldots, 11, 12\}$ into the three events $A_1 = \{2, 3, 4, 5\}$, $A_2 = \{6, 7\}$, $A_3 = \{8, 9, 10, 11, 12\}$. Find $P(1, 7, 2; 10)$.

Solution: The probabilities for each of the events are, respectively:

$$p_1 = \frac{10}{36}, \quad p_2 = \frac{11}{36}, \quad p_3 = \frac{15}{36}$$

and

$$P(1,7,2;10) = \frac{10!}{1!7!2!}\left(\frac{10}{36}\right)^1\left(\frac{11}{36}\right)^7\left(\frac{15}{36}\right)^2 = 0.00431$$

10.6 The Poisson and the Normal Distributions

In this section, we obtain approximate expressions for the binomial distribution in different limits. We start by considering the expression for the probability of k successes in n Bernoulli trials with two choices for outputs; that is, Eq. (10.49).

10.6.1 The Poisson Distribution

Consider the limit when $p << 1$, but $np \equiv a \approx O(1)$. Then:

$$P(k=0) = \frac{n!}{0!\,n!}p^0(1-p)^n = \left(1-\frac{a}{n}\right)^n \tag{10.51}$$

But in the limit $n \to \infty$,

$$\left(1-\frac{a}{n}\right)^n = e^{-a} \tag{10.52}$$

giving:

$$P(k=0) = e^{-a} \tag{10.53}$$

Now consider $P(k = 1)$; it is equal to:

$$\lim_{n\to\infty} P(k=1) = \frac{n!}{1!\,(n-1)!}p^1(1-p)^{n-1} \approx a\left(1-\frac{a}{n}\right)^n \approx ae^{-a} \tag{10.54}$$

For $P(k = 2)$, we obtain:

$$\lim_{n\to\infty} P(k=2) = \frac{n!}{2!\,(n-2)!}p^2(1-p)^{n-2} \approx \frac{a^2}{2!}\left(1-\frac{a}{n}\right)^n \approx \frac{a^2}{2!}e^{-a} \tag{10.55}$$

Similarly,

$$\lim_{n \to \infty} P(k) \approx \frac{a^k}{k!} e^{-a} \tag{10.56}$$

We compare in Figure 10.1 the exact with the approximate expression for the probability distribution, in the region of validity of the Poisson approximation.

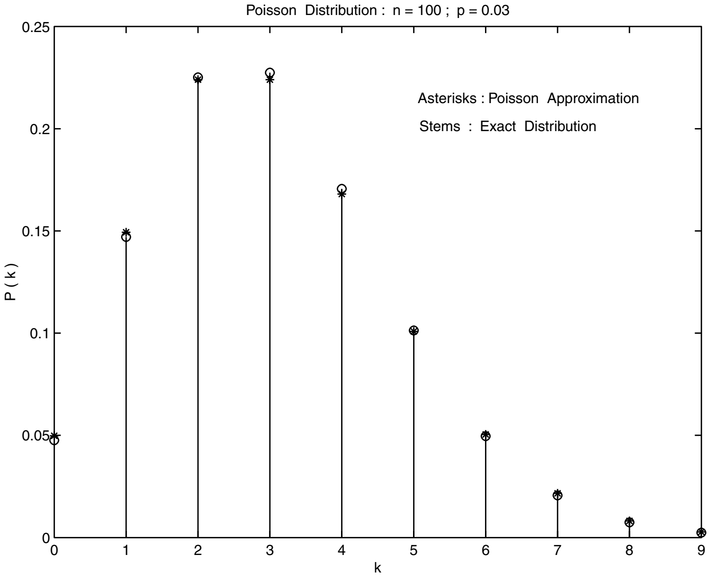

FIGURE 10.1
The Poisson distribution.

Example 10.14

A massive parallel computer system contains 1000 processors. Each processor fails independently of all others and the probability of its failure is 0.002 over a year. Find the probability that the system has no failures during one year of operation.

Solution: This is a problem of Bernoulli trials with $n = 1000$ and $p = 0.002$:

$$P(k = 0) = C_0^{1000} p^0 (1 - p)^{1000} = (0.998)^{1000} = 0.13506$$

or, using the Poisson approximate formula, with $a = np = 2$:

$$P(k = 0) \approx e^{-a} = e^{-2} \approx 0.13533$$

Example 10.15

Due to the random vibrations affecting its supporting platform, a recording head introduces glitches on the recording medium at the rate of $n = 100$ glitches per minute. What is the probability that $k = 3$ glitches are introduced in the recording over any interval of time $\Delta t = 1\text{s}$?

Solution: If we choose an interval of time equal to 1 minute, the probability for an elementary event to occur in the subinterval Δt in this 1 minute interval is

$$p = \frac{1}{60}$$

The problem reduces to finding the probability of $k = 3$ in $n = 100$ trials.

The Poisson formula gives this probability as:

$$P(3) = \frac{1}{3!}\left(\frac{100}{60}\right)^3 \exp\left(-\frac{100}{60}\right) = 0.14573$$

where $a = 100/60$. (For comparison purposes, the exact value for this probability, obtained using the binomial distribution expression, is 0.1466.)

Homework Problem

Pb. 10.18 Let $A_1, A_2, \ldots, A_{m+1}$ be a partition of the set S, and let $p_1, p_2, \ldots, p_{m+1}$ be the probabilities associated with each of these events. Assuming that n Bernoulli trials are repeated, show, using Eq. (10.50), that the probability that the event A_1 occurs k_1 times, the event A_2 occurs k_2 times, etc., is given in the limit $n \to \infty$ by:

$$\lim_{n\to\infty} P(k_1, k_2, \ldots, k_{m+1}; n) = \frac{(a_1)^{k_1} e^{-a_1}}{k_1!} \frac{(a_2)^{k_2} e^{-a_2}}{k_2!} \cdots \frac{(a_m)^{k_m} e^{-a_m}}{k_m!}$$

where $a_i = np_i$.

10.6.2 The Normal Distribution

Prior to considering the derivation of the normal distribution, let us recall Sterling's formula, which is the approximation of $n!$ when $n \to \infty$:

$$\lim_{n\to\infty} n! \approx \sqrt{2\pi n}\; n^n e^{-n} \tag{10.57}$$

We seek the approximate form of the binomial distribution in the limit of very large n and $npq >> 1$. Using Eq. (10.57), the expression for the probability given in Eq. (10.49), reduces to:

$$P(k \text{ successes in } n \text{ trials}) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{k(n-k)}}\left(\frac{np}{k}\right)^{k}\left(\frac{nq}{(n-k)}\right)^{n-k} \qquad (10.58)$$

Now examine this expression in the neighborhood of the mean (see **Pb. 10.17**). We define the distance from this mean, normalized to the square root of the variance, as:

$$x = \frac{k-np}{\sqrt{npq}} \qquad (10.59)$$

Using the leading two terms of the power expansion of $(\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + \ldots)$, the natural logarithm of the two parentheses on the RHS of Eq. (10.58) can be approximated by:

$$\ln\left(\frac{k}{np}\right)^{-k} \approx -(np + \sqrt{npq}\ x)\left(\sqrt{\frac{q}{np}}x - \frac{1}{2}\frac{q}{np}x^2\right) \qquad (10.60)$$

$$\ln\left(\frac{n-k}{nq}\right)^{-(n-k)} \approx -(nq - \sqrt{npq}\ x)\left(-\sqrt{\frac{p}{nq}}x - \frac{1}{2}\frac{p}{nq}x^2\right) \qquad (10.61)$$

Adding Eqs. (10.61) and (10.62), we deduce that:

$$\lim_{n\to\infty}\left(\frac{np}{k}\right)^{k}\left(\frac{nq}{(n-k)}\right)^{n-k} = e^{-x^2} \qquad (10.62)$$

Furthermore, we can approximate the square root term on the RHS of Eq. (10.58) by its value at the mean; that is

$$\sqrt{\frac{n}{n(n-k)}} \approx \frac{1}{\sqrt{npq}} \qquad (10.63)$$

Combining Eqs. (10.62) and (10.63), we can approximate Eq. (10.58), in this limit, by the Gaussian distribution:

$$P(k \text{ successes in } n \text{ trials}) = \frac{1}{\sqrt{2\pi npq}} \exp\left[-\frac{(k-np)^2}{2npq}\right] \qquad (10.64)$$

This result is known as the De Moivre-Laplace theorem. We compare in Figure 10.2 the binomial distribution and its Gaussian approximation in the region of the validity of the approximation.

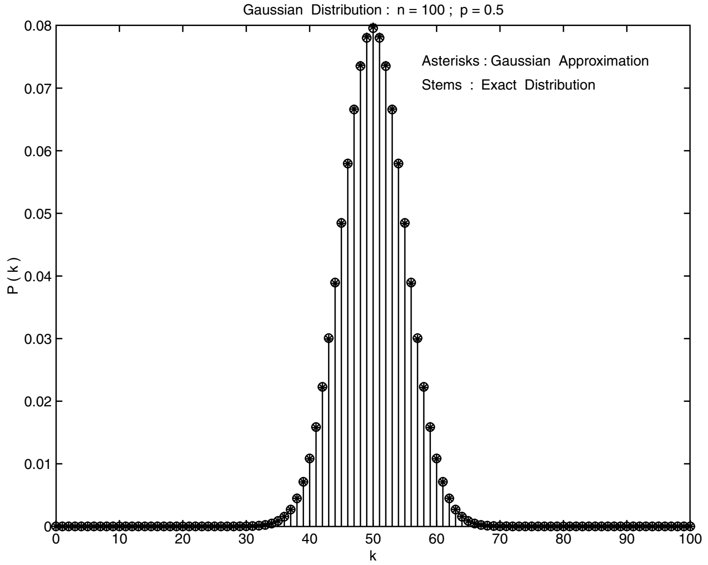

FIGURE 10.2
The normal (Gaussian) distribution.

Example 10.16

A fair die is rolled 400 times. Find the probability that an even number of spots show up 200 times, 210 times, 220 times, and 230 times.

Solution: In this case, n = 400; p = 0.5; np = 200; and $\sqrt{npq} = 10.$

Using Eq. (10.65), we get: $\begin{cases} P(200\text{ even}) = 0.03989; \quad P(210\text{ even}) = 0.02419 \\ P(220\text{ even}) = 0.00540; \quad P(230\text{ even}) = 4.43\times 10^{-4} \end{cases}$

Homework Problems

Pb. 10.19 Using the results of **Pb. 4.34**, relate in the region of validity of the Gaussian approximation the quantity:

$$\sum_{k=k_1}^{k_2} P(k \text{ successes in } n \text{ trials})$$

to the Gaussian integral, specifying each of the parameters appearing in your expression. (*Hint:* First show that in this limit, the summation can be approximated by an integration.)

Pb. 10.20 Let A_1, A_2, …, A_r be a partition of the set S, and let p_1, p_2, …, p_r be the probabilities associated with each of these events. Assuming n Bernoulli trials are repeated, show that, in the limit $n \to \infty$ and where k_i are in the vicinity of $np_i >> 1$, the following approximation is valid:

$$P(k_1, k_2, \ldots, k_r; n) = \frac{\exp\left\{-\frac{1}{2}\left[\frac{(k_1 - np_1)^2}{np_1} + \ldots + \frac{(k_r - np_r)^2}{np_r}\right]\right\}}{\sqrt{(2\pi n)^{r-1} p_1 \ldots p_r}}$$

Supplement: Review of Elementary Functions

In this supplement, we review the basic features and characteristics of the simple elementary functions.

S.1 Affine Functions

By an affine function, we mean an expression of the form

$$y(x) = ax + b \tag{S.1}$$

In the special case where $b = 0$, we say that y is a linear function of x.

We can interpret the parameters in the above function as representing the slope-intercept form of a straight line. Here, a is the slope, which is a measure of the steepness of a line; and b is the y-intercept (i.e., the line intersects the y-axis at the point $(0, b)$).

The following cases illustrate the different possibilities:

1. $a = 0$: this specifies a horizontal line at a height b above the x-axis and that has zero slope.
2. $a > 0$: the height of a point on the line (i.e., the y-value) increases as the value of x increases.
3. $a < 0$: the height of the line decreases as the value of x increases.
4. $b > 0$: the line y-intercept is positive.
5. $b < 0$: the line y-intercept is negative.
6. $x = k$: this function represents a vertical line passing through the point $(k, 0)$.

It should be noted that:

- If two lines have the same slope, they are parallel.
- Two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of each other. (It is easy to deduce this

property if you remember the relationship that you learned in trigonometry relating the sine and cosine of two angles that differ by π/2.) See Section S.4 for more details.

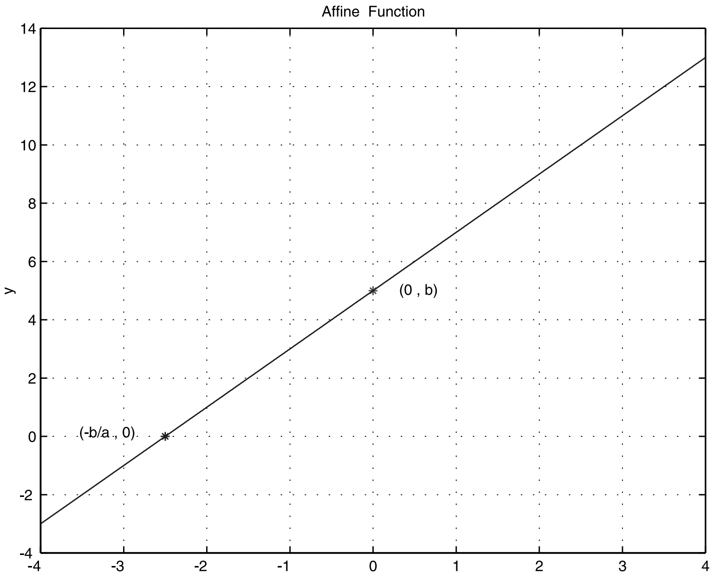

FIGURE S.1
Graph of the line y = ax + b (a = 2, b = 5).

S.2 Quadratic Functions

Parabola

A quadratic parabolic function is an expression of the form:

$$y(x) = ax^2 + bx + c \quad \text{where} \quad a \neq 0 \tag{S.2}$$

Any x for which $ax^2 + bx + c = 0$ is called a root or a zero of the quadratic function. The graphs of quadratic functions are called parabolas.

If we plot these parabolas, we note the following characteristics:

1. For $a > 0$, the parabola opens up (convex curve) as shown in Figure S.2.
2. For $a < 0$, the parabola opens down (concave curve) as shown in Figure S.2.

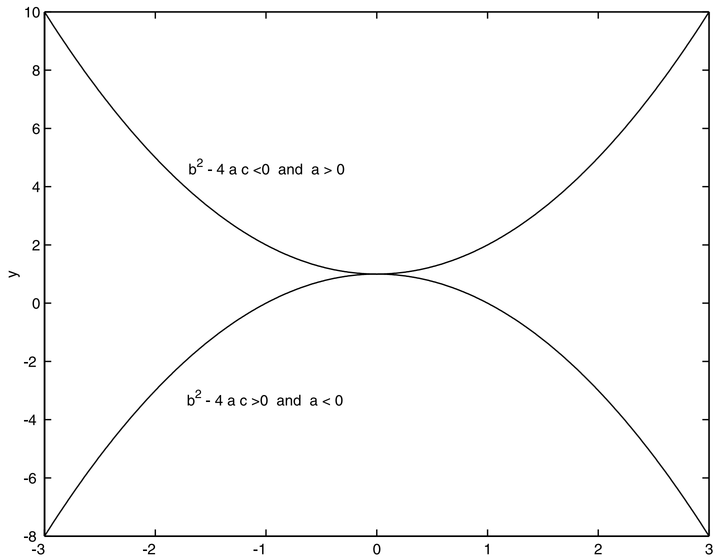

FIGURE S.2
Graph of a quadratic parabolic (second-order polynomial) function with 0 or 2 roots.

3. The parabola does not always intersect the *x*-axis; but where it does, this point's abscissa is a real root of the quadratic equation.

A parabola can cross the *x*-axis in either 0 or 2 points, or the *x*-axis can be tangent to it at one point. If the vertex of the parabola is above the *x*-axis and the parabola opens up, there is no intersection, and hence, no real roots. If, on the other hand, the parabola opens down, the curve will intersect at two values of *x* equidistant from the vertex position. If the vertex is below the *x*-axis, we reverse the convexity conditions for the existence of two real roots. We recall that the roots of a quadratic equation are given by:

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{S.3}$$

When $b^2 - 4ac < 0$, the parabola does not intersect the *x*-axis. There are no real roots; the roots are said to be complex conjugates. When $b^2 - 4ac = 0$, the *x*-axis is tangent to the parabola and we have one double root.

Geometrical Description of a Parabola

The parabola can also be described through the following geometric construction: a parabola is the locus of all points *P* in a plane that are equidistant from a fixed line (called the directrix) and a fixed point (called the focus) not situated on the line.

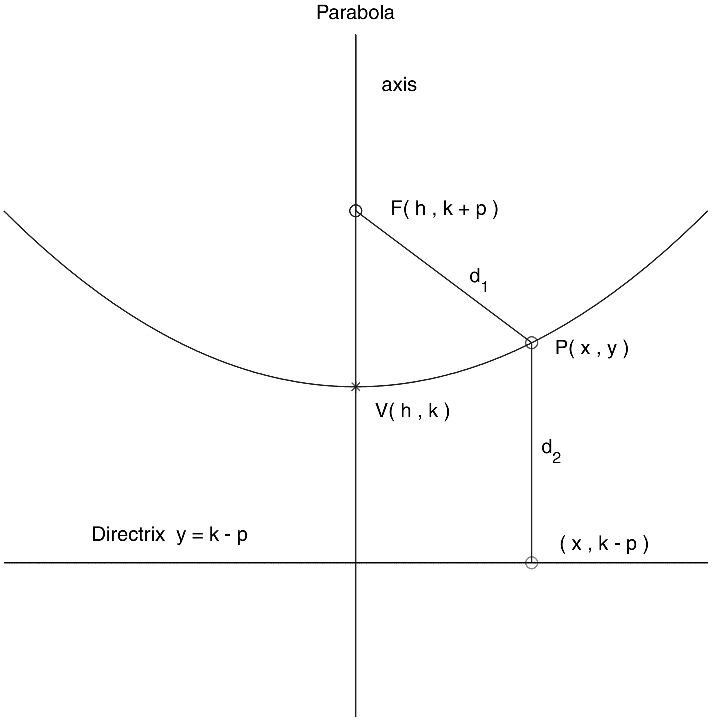

FIGURE S.3
Graph of a parabola defined through geometric parameters. (Parameter values: h = 2, k = 2, p = 1.)

$$d_1 = d_2 \tag{S.4}$$

The algebraic expression for the parabola, using the above geometric parameters, can be obtained by specifically writing and equating the expressions for the distances of a point on the parabola from the focus and from the directrix:

$$\sqrt{(x-h)^2 + (y-(k+p))^2} = |y-(k-p)| \tag{S.5}$$

Squaring both sides of this equation, this equality reduces to:

$$(x-h)^2 = 4p(y-k) \tag{S.6}$$

or in standard form, it can be written:

$$y = \frac{x^2}{4p} - \frac{h}{2p}x + \left(\frac{h^2+4pk}{4p}\right) \tag{S.7}$$

Ellipse

The standard form of the equation describing an ellipse is given by:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \tag{S.8}$$

The ellipse's center is located at (h, k), and assuming $a > b$, the major axis length is equal to $2a$, the minor axis length is equal to $2b$, the foci are located at $(h - c, k)$ and $(h + c, k)$, and those of the vertices at $(h - a, k)$ and $(h + a, k)$; where

$$c^2 = a^2 - b^2 \tag{S.9}$$

Geometric Definition of an Ellipse

An ellipse is the locus of all points P such that the sum of the distance between P and two distinct points (called the foci) is constant and greater than the distance between the two foci.

$$d_1 + d_2 = 2a \tag{S.10}$$

The center of the ellipse is the midpoint between foci, and the two points of intersection of the line through the foci and the ellipse are called the vertices.

The eccentricity of an ellipse is the ratio of the distance between the center and a focus over the distance between the center and a vertex; that is

$$\varepsilon = c/a \tag{S.11}$$

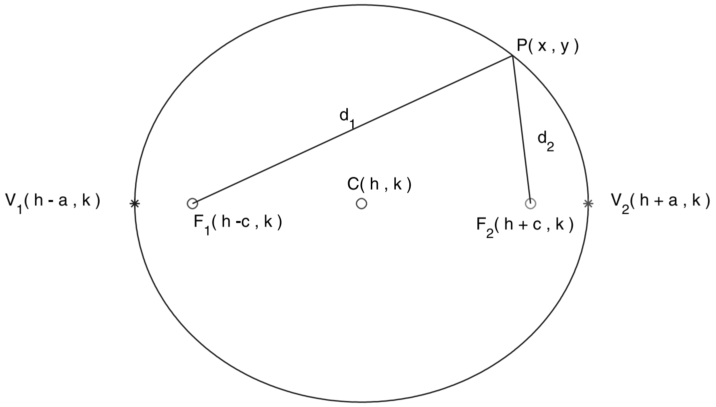

FIGURE S.4
Graph of an ellipse defined through geometric parameters. (Parameter values: h = 2, k = 2, a = 3, b = 2.)

Hyperbola

The standard form of the equation describing a hyperbola is given by:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{S.12}$$

The center of the hyperbola is located at (h, k), and assuming $a > b$, the major axis length is equal to $2a$, the minor axis length is equal to $2b$, the foci are located at $(h - c, k)$ and $(h + c, k)$, and those of the vertices at $(h - a, k)$ and $(h + a, k)$. In this case, $c > a > 0$ and $c > b > 0$ and

$$c^2 = a^2 + b^2 \tag{S.13}$$

Geometric Definition of a Hyperbola

A hyperbola is the locus of all points P in a plane such that the absolute value of the difference of the distances between P and the two foci is constant and is less than the distance between the two foci; that is

$$|d_1 - d_2| = 2a \tag{S.14}$$

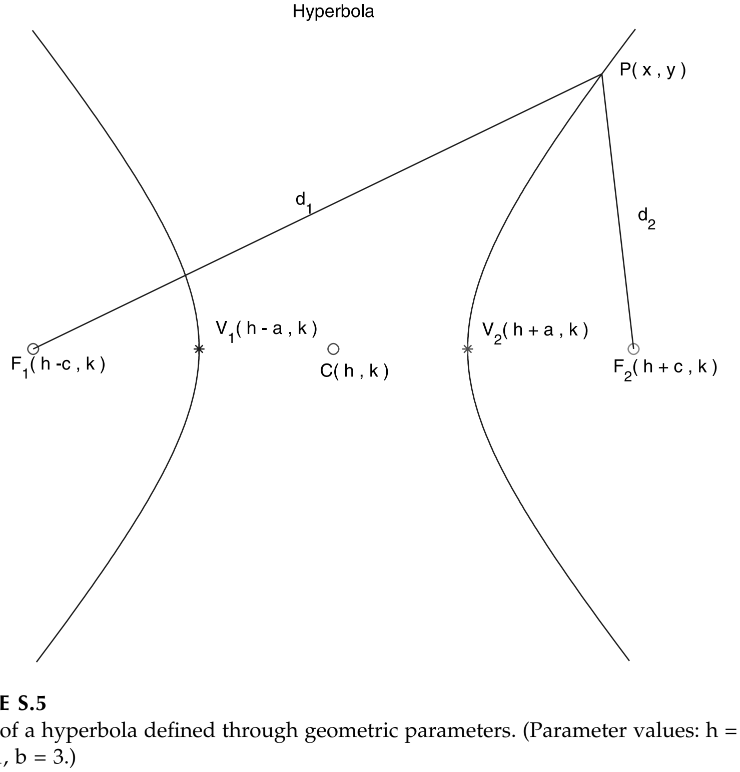

FIGURE S.5
Graph of a hyperbola defined through geometric parameters. (Parameter values: h = 2, k = 2, a = 1, b = 3.)

The center of the hyperbola is the midpoint between foci, and the two points of intersection of the line through the foci and the hyperbola are called the vertices.

S.3 Polynomial Functions

A polynomial function is an expression of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{S.15}$$

where $a_n \neq 0$ for an n^{th} degree polynomial.

The Fundamental Theorem of Algebra states that, for the above polynomial, there are exactly n complex roots; furthermore, if all the polynomial coefficients are real, then the complex roots always come in pairs consisting of a complex number and its complex conjugate.

S.4 Trigonometric Functions

The trigonometric circle is defined as the circle with center at the origin of the coordinates axes and having radius 1.

The trigonometric functions are defined as functions of the components of a point P on the trigonometric circle. Specifically, if we define the angle θ as the angle between the x-axis and the line OP, then:

- cos(θ) is is the x-component of the point P.
- sin(θ) is the y-component of the point P.

Using the Pythagorean theorem in the right angle triangle OQP, one deduces that:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{S.16}$$

Using the above definitions for the sine and cosine functions and elementary geometry, it is easy to note the following properties for the trigonometric functions:

$$\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \cos(-\theta) = \cos(\theta) \tag{S.17}$$

$$\sin(\theta + \pi) = -\sin(\theta) \quad \text{and} \quad \cos(\theta + \pi) = -\cos(\theta) \tag{S.18}$$

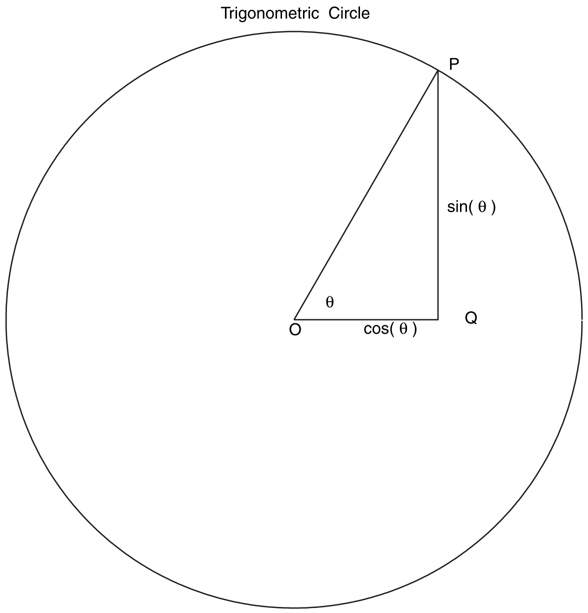

FIGURE S.6
The trigonometric circle.

$$\sin(\theta + \pi / 2) = \cos(\theta) \quad \text{and} \quad \cos(\theta + \pi / 2) = -\sin(\theta) \tag{S.19}$$

$$\sin(\pi / 2 - \theta) = \cos(\theta) \quad \text{and} \quad \cos(\pi / 2 - \theta) = \sin(\theta) \tag{S.20}$$

The tangent and cotangent functions are defined as:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \text{and} \quad \cot(\theta) = \frac{1}{\tan(\theta)} \tag{S.21}$$

Other important trigonometric relations relate the angles and sides of a triangle. These are the so-called Law of Cosines and Law of Sines in a triangle:

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \tag{S.22}$$

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \tag{S.23}$$

where the sides of the triangle are a, b, c, and the angles opposite, respectively, of each of these sides are denoted by α, β, γ.

S.5 Inverse Trigonometric Functions

The inverse of a function $y = f(x)$ is a function, denoted by $x = f^{-1}(y)$, having the property that $y = f(f^{-1}(y))$. It is important to note that a function $f(x)$ that is single-valued (i.e., to each element x in its domain, there corresponds one, and only one, element y in its range) may have an inverse that is multi-valued (i.e., many x values may correspond to the same y). Typical examples of multi-valued inverse functions are the inverse trigonometric functions. In such instances, a single-valued inverse function can be defined if the range of the inverse function is defined on a more limited region of space. For example, the $\cos^{-1}$ function (called arc cosine) is single-valued if $0 \leq x \leq \pi$.

Note that the above notation for the inverse of a function should not be confused with the negative-one power of the function $f(x)$, which should be written as:

$$(f(x))^{-1} \quad \text{or} \quad 1/f(x)$$

Also note that because the inverse function reverses the role of the x- and y-coordinates, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$ (i.e., the first bisector of the coordinate axes).

S.6 The Natural Logarithmic Function

The natural logarithmic function is defined by the following integral:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \tag{S.24}$$

The following properties of the logarithm can be directly deduced from the above definition:

$$\ln(ab) = \ln(a) + \ln(b) \tag{S.25}$$

$$\ln(a^r) = r\ln(a) \tag{S.26}$$

$$\ln\left(\frac{1}{a}\right) = -\ln(a) \tag{S.27}$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b) \tag{S.28}$$

To illustrate the technique for deriving any of the above relations, let us consider the first of them:

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \tag{S.29}$$

The first term on the RHS is ln(a), while the second term through the substitution $u = t/a$ reduces to the definition of ln(b).

Note that:

$$\ln(1) = 0 \tag{S.30}$$

$$\ln(e) = 1 \tag{S.31}$$

where $e = 2.71828$.

S.7 The Exponential Function

The exponential function is defined as the inverse function of the natural logarithmic function; that is

$$\exp(\ln(x)) = x \quad \text{for all } x > 0 \tag{S.32}$$

$$\ln(\exp(y)) = y \quad \text{for all } y \tag{S.33}$$

The following properties of the exponential function hold for all real numbers:

$$\exp(a)\exp(b) = \exp(a+b) \tag{S.34}$$

$$(\exp(a))^b = \exp(ab) \tag{S.35}$$

$$\exp(-a) = \frac{1}{\exp(a)} \tag{S.36}$$

$$\frac{\exp(a)}{\exp(b)} = \exp(a-b) \tag{S.37}$$

It should be pointed out that any of the above properties can be directly obtained from the definition of the exponential function and the properties of the logarithmic function. For example, the first of these relations can be derived as follows:

$$\ln(\exp(a)\exp(b)) = \ln(\exp(a)) + \ln(\exp(b)) = a + b \tag{S.38}$$

Taking the exponential of both sides of this equation, we obtain:

$$\exp(\ln(\exp(a)\exp(b))) = \exp(a)\exp(b) = \exp(a + b) \tag{S.39}$$

which is the desired result.

Useful Features of the Exponential Function

If the exponential function is written in the form:

$$y(x) = \exp(-bx) \tag{S.40}$$

the following features are apparent:

1. If $b > 0$, then the function is convergent at (+ infinity) and goes to zero there.
2. If $b < 0$, then the function blows up at (+ infinity).
3. If $b = 0$, then the function is everywhere equal to a constant $y = 1$.
4. The exponential functions are monotonically increasing for $b < 0$, and monotonically decreasing for $b > 0$.
5. If $b_1 > b_2 > 0$, then everywhere on the positive x-axis, $y_1(x) < y_2(x)$.
6. The exponential function has no roots.
7. For $b > 0$, the product of the exponential function with any polynomial goes to zero at (+ infinity).

We plot in Figures S.7 and S.8 examples of the exponential function for different values of the parameters. The first six properties above are clearly exhibited in these figures.

S.8 The Hyperbolic Functions

The hyperbolic cosine function is defined by:

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2} \tag{S.41}$$

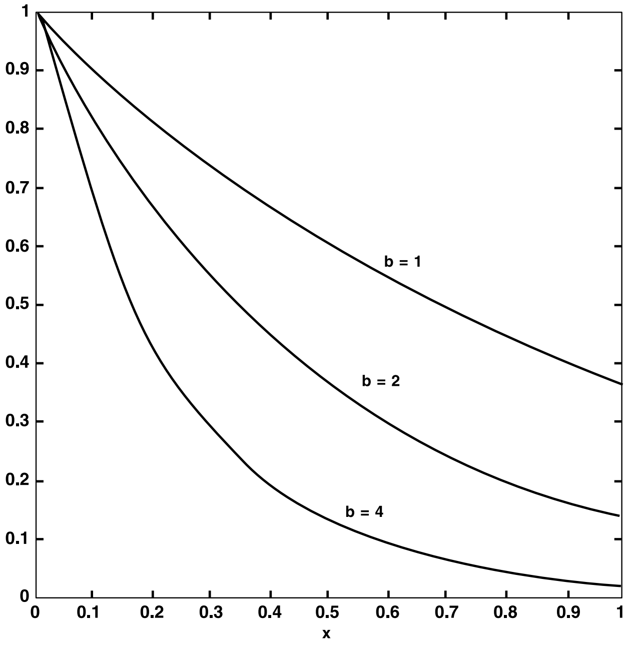

FIGURE S.7
The graph of the function y = exp(–bx), for different positive values of b.

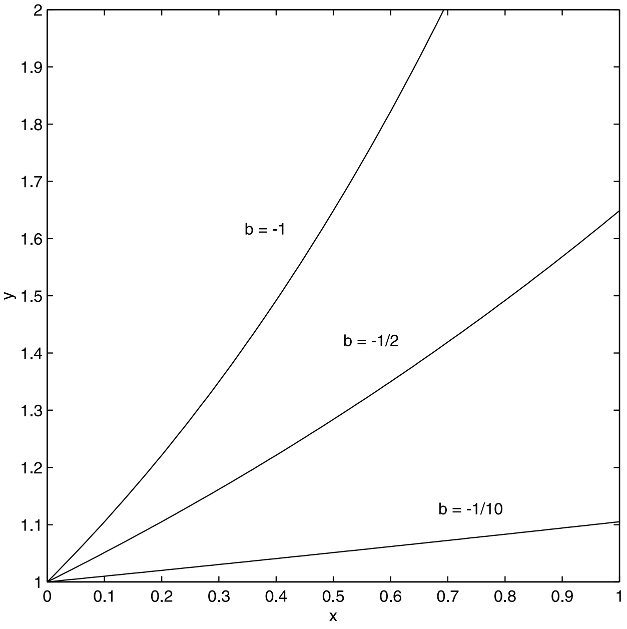

FIGURE S.8
The graph of the function y = exp(–bx), for different negative values of b.

and the hyperbolic sine function is defined by:

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2} \tag{S.42}$$

Using the above definitions, it is straightforward to derive the following relations:

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{S.43}$$

$$1 - \tan^2(x) = \text{sech}^2(x) \tag{S.44}$$

S.9 The Inverse Hyperbolic Functions

$$y = \sinh^{-1}(x) \quad \text{if} \quad x = \sinh(y) \tag{S.45}$$

Using the definition of the hyperbolic functions, we can write the inverse hyperbolic functions in terms of logarithmic functions. For example, considering the inverse hyperbolic sine function from above, we obtain:

$$e^y - 2x - e^{-y} = 0 \tag{S.46}$$

multiplying by e^y everywhere, we obtain a second-degree equation in e^y:

$$e^{2y} - 2xe^y - 1 = 0 \tag{S.47}$$

Solving this quadratic equation, and choosing the plus term in front of the discriminant, since e^y is everywhere positive, we obtain:

$$e^y = x + \sqrt{x^2 + 1} \tag{S.48}$$

giving, for the inverse hyperbolic sine function, the expression:

$$y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \tag{S.49}$$

In a similar manner, one can show the following other identities:

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) \tag{S.50}$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{S.51}$$

$$\mathrm{sech}^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \tag{S.52}$$

Appendix: Some Useful Formulae

Sum of Integers and Their Powers

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^3$$

$$\sum_{k=1}^{n} k^4 = \frac{n(n+2)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{k=1}^{n} (2k-1) = n^2$$

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3}$$

$$\sum_{k=1}^{n} (2k-1)^3 = n^2(2n^2-1)$$

$$\sum_{k=1}^{n} k(k+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$$

Arithmetic Series

$$\sum_{k=0}^{n-1}(a+kr)=\frac{n}{2}[2a+(n-1)r]$$

Geometric Series

$$\sum_{k=1}^{n} aq^{k-1}=\frac{a(q^n-1)}{q-1} \qquad q\neq 1$$

Arithmo-Geometric Series

$$\sum_{k=0}^{n-1}(a+kr)q^k=\frac{a-[a+(n-1)r]q^n}{(1-q)}+\frac{rq(1-q^{n-1})}{(1-q)^2} \qquad q\neq 1$$

Taylor's Series

$$f(x+a)=\sum_{k=0}^{\infty} f^{(k)}(x)\frac{a^k}{k!}$$

$$f(x+a,y+b)=f(x,y)+a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}+\frac{1}{2!}\left[a^2\frac{\partial^2 f}{\partial x^2}+b^2\frac{\partial^2 f}{\partial y^2}+2ab\frac{\partial^2 f}{\partial x\partial y}\right]+\dots$$

Trigonometric Functional Relations

$$\sin(x) \pm \sin(y) = 2\sin\left[\frac{1}{2}(x \pm y)\right]\cos\left[\frac{1}{2}(x \mp y)\right]$$

$$\cos(x) + \cos(y) = 2\cos\left[\frac{1}{2}(x + y)\right]\cos\left[\frac{1}{2}(x - y)\right]$$

$$\cos(x) - \cos(y) = 2\sin\left[\frac{1}{2}(x + y)\right]\sin\left[\frac{1}{2}(y - x)\right]$$

$$\sin\left(\frac{1}{2}x\right) = \pm\sqrt{\frac{1}{2}(1 - \cos(x))}$$

$$\cos\left(\frac{1}{2}x\right) = \pm\sqrt{\frac{1}{2}(1 + \cos(x))}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

$$\sin(4x) = \cos(x)[4\sin(x) - 8\sin^3(x)]$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$$

Relation of Trigonometric and Hyperbolic Functions

$$\sin(x) = -j\sinh(jx)$$

$$\cos(x) = \cosh(jx)$$

$$\tan(x) = \frac{1}{j}\tanh(jx)$$

Expansion of Elementary Functions in Power Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$