# Residue Calculus Evaluation of Infinite Integrals

**Integrals of Certain Rational Functions** We consider rational functions of a complex variable $z$ taking the form

$$f(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials in $z$ with real coefficients:

$$p(z) = p_0 z^m + p_1 z^{m-1} + \cdots + p_{m-1} z + p_m, \;\; m \geq 0,$$

$$q(z) = q_0 z^n + q_1 z^{n-1} + \cdots + q_{n-1} z + q_n, \;\; n \geq m + 2.$$

In addition, we suppose that $q(z)$ has no zeros on the real axis. We want to evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx.$$

Since $q(z)$ has real coefficients and has no zeros on the real axis, all of its zeros occur in complex conjugate pairs and we conclude $n = 2\nu$. We may suppose that $q(z)$ has zeros

$$z_1,\, \overline{z_1},\, z_2,\, \overline{z_2}, \cdots, z_\nu,\, \overline{z_\nu},$$

and we will suppose that $z_1,\, z_2, \cdots z_\nu$ are the zeros in the upper half plane.

We construct a positively oriented "path, or "contour", $\mathcal{C}_r$ consisting of the interval $[-r,\, r]$ on the real axis together with the semicircle

$$\mathcal{S}_r = \{ z \mid |z| = r,\; \mathrm{Re}\, z > 0 \}.$$

Further, we restrict consideration to values of $r$ such that $r > |z_k|, \; k = 1, 2, ..., \nu$. Applying the Residue Theorem, we have

$$\int_{\mathcal{C}_r} \frac{p(z)}{q(z)} dz = 2\pi i \left( \sum_{k=1}^{\nu} \text{Res} \left. \frac{p(z)}{q(z)} \right|_{z=z_k} \right) \equiv 2\pi i \sum_{k=1}^{\nu} \rho_k.$$

Then

$$\int_{-r}^{r} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{k=1}^{\nu} \rho_k - \int_{\mathcal{S}_r} \frac{p(z)}{q(z)} dz.$$

The idea now is to show that $\lim_{r \to \infty} \int_{\mathcal{S}_r} \frac{p(z)}{q(z)} dz = 0$. To this end we observe that

$$\left| \frac{p(z)}{q(z)} \right| = \frac{1}{|z|^{n-m}} \left| \frac{p_0 + p_1 z^{-1} + \cdots + p_{m-1} z^{-(m-1)} + p_m z^{-m}}{q_0 + q_1 z^{-1} + \cdots + q_{n-1} z^{-(n-1)} + q_n z^{-n}} \right|.$$

from which it is clear that there are positive numbers $p$ and $r_0$ such that

$$\max_{z \in \mathcal{S}_r} \left| \frac{p(z)}{q(z)} \right| \le \frac{p}{r^2}, \;\; r \ge r_0.$$

Then we can use the estimate, valid for $r \ge r_0$,

$$\left| \int_{\mathcal{S}_r} \frac{p(z)}{q(z)} dz \right| \le \max_{z \in \mathcal{S}_r} \left| \frac{p(z)}{q(z)} \right| \text{length } (\mathcal{S}_r) \le \frac{p}{r^2} \pi r = \frac{p \pi}{r}$$

to see that it is, indeed, true that $\lim_{r \to \infty} \int_{\mathcal{S}_r} \frac{p(z)}{q(z)} dz = 0$. That being the case, we now have

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = \lim_{r \to \infty} \int_{\mathcal{C}_r} \frac{p(z)}{q(z)} dz = 2\pi i \sum_{k=1}^{\nu} \rho_k,$$

the $\rho_k$, $k = 1, 2, \cdots, \nu$ being the residues of $\frac{p(z)}{q(z)}$ lying in the upper half plane.

We choose as our first example an integral which can be evaluated using the standard methods of calculus.

**Example 1** We evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$. From the standard calculus we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{r \to \infty} \tan^{-1}(x)\Big|_{-r}^{r} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Using the residue calculus we note that $\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$ has a single isolated singularity in the upper half plane at $z = i$. That singularity is a pole of order 1, so the residue there is

$$\lim_{z \to i} (z-i)\frac{1}{(z-i)(z+i)} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

Then

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i\,\frac{1}{2i} = \pi.$$

The next example is one for which the use of the standard calculus would be substantially more difficult.

**Example 2** We compute $\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^4}$.

The zeros of $q(z) = 1 + z^4$ are the fourth roots of $-1$. Of these only

$$z_1 = \frac{1+i}{\sqrt{2}}, \quad z_2 = \frac{-1+i}{\sqrt{2}},$$

lie in the upper half plane. Writing

$$\frac{z^2}{1+z^4} = \frac{z^2}{(z-z_1)\,(z-z_2)\left(z^2+\sqrt{2}i\,z-1\right)},$$

the residue at $z_1 = \frac{1+i}{\sqrt{2}}$ is

$$\lim_{z \to z_1} (z-z_1)\,\frac{z^2}{(z-z_1)\,(z-z_2)\left(z^2+\sqrt{2}i\,z-1\right)} = \lim_{z \to z_1} \frac{z^2}{(z-z_2)\left(z^2+\sqrt{2}i\,z-1\right)}$$

$$= \frac{i}{\left(\frac{1+i}{\sqrt{2}} - \frac{-1+i}{\sqrt{2}}\right)\left(i + \sqrt{2}i\frac{1+i}{\sqrt{2}} - 1\right)} = \frac{i}{2\sqrt{2}(i-1)} = \frac{1-i}{4\sqrt{2}}.$$

A similar computation gives the residue at $z_2 = \frac{-1+i}{\sqrt{2}}$ as $\frac{i}{2\sqrt{2}(-i-1)} = \frac{-1-i}{4\sqrt{2}}$. The value of the integral is thus

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^4} = 2\pi i \left(\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

**Evaluation of Integrals of the Form** $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \left\{ \begin{matrix} \cos \alpha x \\ \sin \alpha x \end{matrix} \right\} dx$

Here we make the same assumptions on $p(x)$ and $q(x)$ as in the foregoing discussion, but now it is enough to assume $n \geq m+1$; the alternating character of the trigonometric component of the integrand assures the existence of the integral as long as $\frac{p(x)}{q(x)}$ is ultimately monotone decreasing, or monotone increasing, to 0– and that is the case under these circumstances. Assuming $\alpha > 0$, we use the same contour $\mathcal{C}_r$ as previously (if $\alpha < 0$ we use a comparable contour in the lower half plane; if $\alpha = 0$ we return to the earlier discussion with $n \geq m+2$). We remind ourselves that

$$\int_{\mathcal{C}_r} \frac{p(z)\,e^{i\alpha z}\,dz}{q(z)} = \int_{\mathcal{C}_r} \frac{p(z)\,\cos \alpha z\,dz}{q(z)} + i \int_{\mathcal{C}_r} \frac{p(z)\,\sin \alpha z\,dz}{q(z)}$$

and thus

$$\int_{\mathcal{C}_r} \frac{p(z)\,\cos \alpha z\,dz}{q(z)} = \mathrm{Re}\left(\int_{\mathcal{C}_r} \frac{p(z)\,e^{i\alpha z}\,dz}{q(z)}\right),$$

$$\int_{\mathcal{C}_r} \frac{p(z)\,\sin \alpha z\,dz}{q(z)} = \mathrm{Im}\left(\int_{\mathcal{C}_r} \frac{p(z)\,e^{i\alpha z}\,dz}{q(z)}\right).$$

In this situation, rather than considering just the semicircle $\mathcal{S}_r$ as earlier, we decompose the contour $\mathcal{C}_r$ into the interval $[-r\,r]$ along the real axis, the set $\mathcal{T}_r$, consisting of two sub-arcs of $\mathcal{S}_r$ defined by

$$\mathcal{T}_r = \left\{ z = x + iy \in \mathcal{S}_r \mid 0 < y \le \sqrt{|x|} \right\}$$

and

$$\mathcal{U}_r = \left\{ z = x + iy \in \mathcal{S}_r \mid 0 < \sqrt{|x|} < y \right\}.$$

In the upper half plane $y > 0$ we have

$$\left| e^{i\alpha z} \right| = \left| e^{-\alpha y} e^{i\alpha x} \right| = e^{-\alpha y},$$

which, in particular, is less than 1. Using much the same estimate as in the earlier discussion we can then see that for $r \ge r_0$ and $\delta$ chosen so that the length of one of the arcs of $\mathcal{T}_r$ is less than or equal to $(1+\delta)\sqrt{r}$ for $r \ge r_0$,

$$\left| \int_{\mathcal{T}_r} \frac{p(z)\, e^{i\alpha z}}{q(z)} dz \right| \le \max_{z \in \mathcal{T}_r} \left| \frac{p(z)}{q(z)} \right| \text{ length } (\mathcal{T}_r) \le 2(1+\delta) \frac{p}{r} \sqrt{r} = 2(1+\delta) \frac{p}{\sqrt{r}}.$$

It is clear then that

$$\lim_{r \to \infty} \left| \int_{\mathcal{T}_r} \frac{p(z)\, e^{i\alpha z}}{q(z)} dz \right| = 0.$$

On $\mathcal{U}_r$, on the other hand, we have

$$\left| e^{i\alpha z} \right| = \left| e^{-\alpha y} e^{i\alpha x} \right| = e^{-\alpha y} \le e^{-\alpha \sqrt{r}},$$

and we see that

$$\left| \int_{\mathcal{U}_r} \frac{p(z)\, e^{i\alpha z}}{q(z)} dz \right| \le \max_{z \in \mathcal{U}_r} \left| \frac{p(z)\, e^{i\alpha z}}{q(z)} \right| \text{ length } (\mathcal{U}_r) \le \frac{p\, e^{-\alpha\sqrt{r}}}{r} \to 0, \ r \to \infty.$$

Combining the two results we see that

$$\lim_{r \to \infty} \int_{\mathcal{S}_r} \frac{p(z)\, e^{i\alpha z}}{q(z)} dz = 0$$

so that, taking the limit as $r \to \infty$ just as before and defining the residues $\rho_k$ as before,

$$\int_{-\infty}^{\infty} \frac{p(x)\, e^{i\alpha x}}{q(x)} dx \;=\; 2\pi i \sum_{k=1}^{\nu} \rho_k.$$

That being the case, we then have

$$\int_{\mathcal{C}_r} \frac{p(z)\, \cos \alpha z\, dz}{q(z)} \;=\; \mathrm{Re}\left(2\pi i \sum_{k=1}^{\nu} \rho_k\right),$$

$$\int_{\mathcal{C}_r} \frac{p(z)\, \cos \alpha z\, dz}{q(z)} \;=\; \mathrm{Im}\left(2\pi i \sum_{k=1}^{\nu} \rho_k\right).$$

**Example 3** We compute $\int_{-\infty}^{\infty} \frac{x\, \sin 2x\, dx}{1+x^2}$. The corresponding contour integral is $\int_{\mathcal{C}_r} \frac{z\, e^{i\,2z}\, dz}{1+z^2}$. The only singularity of the integrand in the interior of $\mathcal{C}_r$, for $r > 1$, occurs at $z = i$ where the residue is

$$\lim_{z \to i} (z-i) \frac{z\, e^{i\,2z}}{(z-i)(z+i)} \;=\; \lim_{z \to i} \frac{z\, e^{i\,2z}}{z+i} \;=\; \frac{i\, e^{-2}}{2i} \;=\; \frac{e^{-2}}{2}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{x\, \sin 2x\, dx}{1+x^2} \;=\; \mathrm{Im}\left(2\pi i\, \frac{e^{-2}}{2}\right) \;=\; \pi\, e^{-2}.$$

Correspondingly

$$\int_{-\infty}^{\infty} \frac{x\, \cos 2x\, dx}{1+x^2} \;=\; \mathrm{Re}\left(2\pi i\, \frac{e^{-2}}{2}\right) \;=\; 0,$$

which is clear in any case because the integrand is an odd function of $x$.