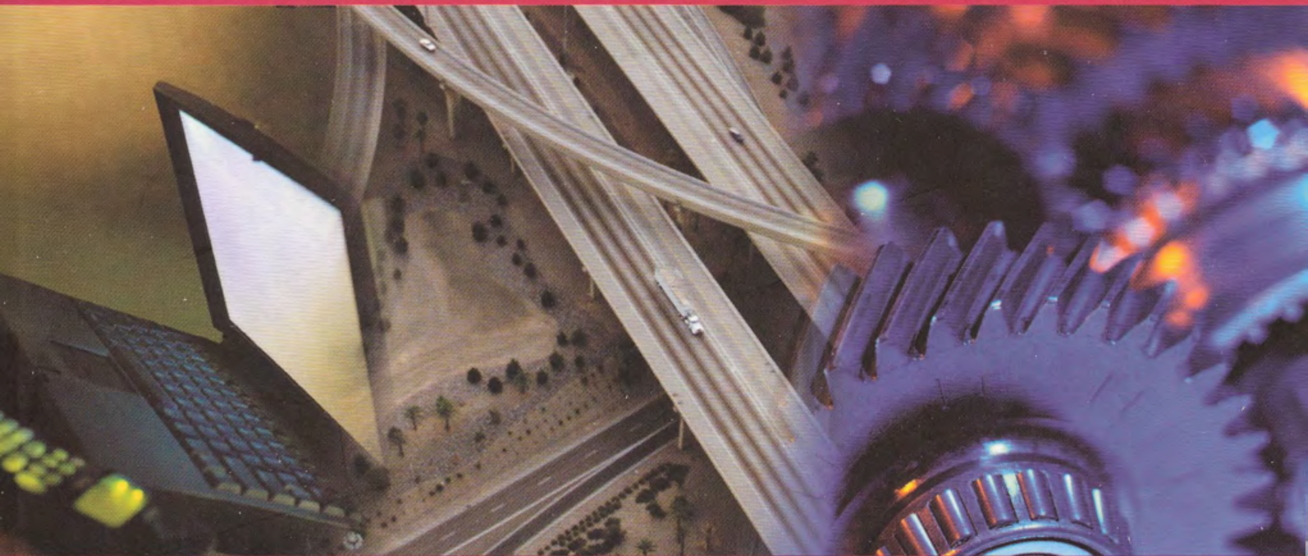
ADVANCED ENGINEERING MATHEMATICS



K.A. STROUD

WITH ADDITIONS BY DEXTER J. BOOTH

FOURTH EDITION

Advanced Engineering Mathematics

A new edition of Further Engineering Mathematics

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with additions by

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Contents

Preface to the First Edition

The purpose of this book is essentially to provide a sound second year course in Mathematics appropriate to studies leading to B.Sc. Engineering Degrees and other qualifications of a comparable level. The emphasis throughout is on techniques and applications, supported by sufficient formal proofs to warrant the methods being employed.

The structure of the text and the techniques used follow closely those of the author's first year book, *Engineering Mathematics – Programmes and Problems*, to which this further book is a companion volume and a continuation of the highly successful learning strategies devised. As with the previous work, the text is based on a series of self-instructional programmes arising from extensive research and rigid evaluation in a variety of relevant courses and, once again, the individualised nature of the development makes the book eminently suitable both for general class use and for personal study.

Each of the course programmes guides the student through the development of a particular topic, with numerous worked examples to demonstrate the techniques and with increased responsibility passing to the student as mastery is achieved. Revision exercises are provided where appropriate and each programme terminates with a *Revision Summary* of the main points covered, a *Test Exercise* based directly on the work of the programme and a set of *Further Problems* which provides opportunity for the additional practice that is essential for ensured success. The ability to work at one's own pace throughout is of utmost importance in maintaining motivation and in achieving mastery.

In several instances, the topic of a programme is a direct extension of basic work covered in *Engineering Mathematics* and where this is so, the title page of the programme carries a brief reference to the relevant programme in the first year treatment. This clearly directs the student to worthwhile revision of the prerequisites assumed in the further development of the subject matter.

A complete set of Answers to all problems and a detailed Index are provided at the end of the book.

Grateful acknowledgement is made of the constructive suggestions and cooperation received from many quarters both in the development of the original programmes and in the final preparation of the text. Recognition must also be made of the many sources from which

examples have been gleaned over the years and which contribute in no small measure to the success of the work.

Finally my sincere appreciation is due to the publishers for their patience, advice and ready cooperation in the preparation of the text for publication.

K.A. Stroud

Preface to the Second Edition

Since the first publication of *Further Engineering Mathematics* as core material for a typical second year engineering degree course, requests have been received from time to time for the inclusion of further topics to cover the particular requirements of individual syllabuses.

Some limit, inevitably, has to be placed on the physical size of the text, but it has been possible at least to include a programme on *Linear Optimisation* (*Linear Programming*) which was one of the subjects most frequently required.

The treatment of the additional material follows the structure of the rest of the book and the emphasis is largely on the practical use of the *simplex method* for the solution of both maximisation and minimisation problems.

The opportunity has also been taken to amend and clarify a number of minor points in the existing text and my thanks are due to those correspondents who have undertaken to write with constructive comment. Such feedback is always welcome.

K.A.S.

Preface to the Third Edition

With the new edition of *Further Engineering Mathematics*, the opportunity has been taken to incorporate a number of minor revisions and amendments to the previous text.

The format of the pages has been changed and the publishers have undertaken the complete resetting of the text to result in a more open presentation of the material and to facilitate the learning process still further.

Once again, my sincere thanks are due to all those correspondents who have kindly written with constructive comment concerning the book and to the publishers for their continued support, advice and cooperation throughout the preparation, production and marketing of the work.

K.A.S.

Preface to the Fourth Edition

It is now nearly 20 years since *Advanced Engineering Mathematics* (in earlier editions called *Further Engineering Mathematics*) by Ken Stroud was published and from the start it has been one of the most widely used and successful textbooks for science and engineering students at this level. I am delighted to have been asked to contribute to a new edition. As with the fifth edition of *Engineering Mathematics* I have endeavoured to retain the very essence of the book that has contributed to so many students' mathematical abilities over the years, particularly the time-tested Stroud format with its close attention to technique development throughout. In my task I have been greatly assisted by a first-rate team of academics who have worked alongside me in the development of this edition. To them I should like to express my sincere gratitude for all the detailed care and consideration they have given to all my contributions.

Immediately noticeable is the title change from *Further Engineering Mathematics* to *Advanced Engineering Mathematics* which, it is felt, more clearly describes the contents to a world-wide audience. Because a substantial amount of material in the first two Programmes of the earlier editions is no longer taught in the detail given, the first significant change to the contents has been their consolidation into a single Programme called *Numerical solutions of equations and interpolation*. To cater for continual changes in engineering mathematics four new Programmes have been added: *Z transforms, Introduction to the Fourier transform, Numerical solutions of partial differential equations* and *Complex analysis 3*, the last dealing with complex integration. The two original Programmes dealing with the *Laplace transform* have been separated into three Programmes with the addition of new material on harmonic oscillators. Sturm–Liouville systems have been introduced into the Programme *Power series solutions of ordinary differential equations* and predictor–corrector methods have been added to the Programme *Numerical solutions of ordinary differential equations*.

To follow the format of the fifth edition of *Engineering Mathematics* and to give as much assistance as possible in organising the student's study I have introduced specific **Learning outcomes** at the beginning and **Can You?** checklists at the end of each Programme. In this way the learning experience is made more explicit and the student is given greater confidence in what has been learnt.

It is only in working on this new edition, just as with the earlier book *Engineering Mathematics*, that the enormity of Ken Stroud's achievement can be really understood. The vast amount of work involved, the care and attention to detail and above all the complete understanding of his students and their learning processes are apparent in every page. It has been both a challenge and an honour to be able to work on such a book. I should like to thank the Stroud family again for their support in my work for this new edition. I should also like to thank my Editor, Helen Bugler, and her erstwhile assistant, Esther Thackeray, for their continued good humour, care and professionalism that have been invaluable in the creation of this new edition.

Huddersfield
February 2003

Dexter J. Booth

Hints on using the Book

This book contains twenty-three Programmes, each of which has been written in such a way as to make learning more effective and more interesting. It is almost like having a personal tutor, for you proceed at your own rate of learning and any difficulties you may have are cleared before you have the chance to practise incorrect ideas or techniques.

You will find that each Programme is divided into sections called frames. When you start a Programme, begin at Frame 1. Read each frame carefully and carry out any instructions or exercise which you are asked to do. In almost every frame, you are required to make a response of some kind, testing your understanding of the information in the frame, and you can immediately compare your answer with the correct answer given in the next frame. To obtain the greatest benefit, you are strongly advised to cover up the following frame, where necessary, until you have made your response. When a series of dots occurs, you are expected to supply the missing word, phrase, or number. At every stage, you will be guided along the right path. There is no need to hurry: read the frames carefully and follow the directions exactly. In this way, you must learn.

At the end of each Programme, you will find a **Revision summary** and a **Can You?** checklist that matches the **Learning outcomes** given at the beginning of the Programme. Read these carefully to make sure you have not missed anything. Next you will find a short **Test exercise**. This is set directly on what you have learned in the Programme: the questions are straightforward and contain no tricks. When you have completed these, return to the **Can You?** checklist as a final reminder of the contents of the Programme. To provide you with the necessary practice, a set of **Further problems** is also included. Remember that in mathematics, as in many other situations, practice makes perfect – or more nearly so.

Even if you feel you have done some of the topics before, work steadily through each Programme: it will serve as useful revision and fill in any gaps in your knowledge that you may have.

Useful background information

1 Algebraic identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

2 Trigonometrical identities

(1) $\sin^2\theta + \cos^2\theta = 1$; $\quad \sec^2\theta = 1 + \tan^2\theta$;

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(2) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

(3) Let $A = B = \theta$ $\quad\therefore\quad \sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

(4) Let $\theta = \dfrac{\phi}{2}$ $\quad\therefore\quad \sin\phi = 2\sin\dfrac{\phi}{2}\cos\dfrac{\phi}{2}$

$$\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}$$
$$= 1 - 2\sin^2\frac{\phi}{2} = 2\cos^2\frac{\phi}{2} - 1$$

$$\tan\phi = \frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}$$

(5) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(6) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(7) Negative angles: $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

(8) Angles having the same trigonometrical ratios:

(a) Same sine: θ and $(180^\circ - \theta)$

(b) Same cosine: θ and $(360^\circ - \theta)$, i.e. $(-\theta)$

(c) Same tangent: θ and $(180^\circ + \theta)$

(9) $a\sin\theta + b\cos\theta = A\sin(\theta+\alpha)$

$a\sin\theta - b\cos\theta = A\sin(\theta-\alpha)$

$a\cos\theta + b\sin\theta = A\cos(\theta-\alpha)$

$a\cos\theta - b\sin\theta = A\cos(\theta+\alpha)$

where $\begin{cases} A = \sqrt{a^2+b^2} \\ \alpha = \tan^{-1}\frac{b}{a} \quad (0^\circ < \alpha < 90^\circ) \end{cases}$

3 Standard curves

(a) *Straight line*

Slope, $m = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$

Angle between two lines, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

For parallel lines, $m_2 = m_1$

For perpendicular lines, $m_1 m_2 = -1$

Equation of a straight line (slope $= m$)

(1) Intercept c on real y-axis: $y = mx + c$

(2) Passing through (x_1, y_1): $y - y_1 = m(x - x_1)$

(3) Joining (x_1, y_1) and (x_2, y_2): $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

(b) *Circle*

Centre at origin, radius r: $x^2 + y^2 = r^2$

Centre (h, k), radius r: $(x - h)^2 + (y - k)^2 = r^2$

General equation: $x^2 + y^2 + 2gx + 2fy + c = 0$

with centre $(-g, -f)$: radius $= \sqrt{g^2 + f^2 - c}$

Parametric equations: $x = r\cos\theta$, $y = r\sin\theta$

(c) *Parabola*

Vertex at origin, focus $(a, 0)$: $y^2 = 4ax$

Parametric equations: $x = at^2$, $y = 2at$

(d) *Ellipse*

Centre at origin, foci $\left(\pm\sqrt{a^2 + b^2}, 0\right)$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where $a =$ semi-major axis, $b =$ semi-minor axis

Parametric equations: $x = a\cos\theta$, $y = b\sin\theta$

(e) *Hyperbola*

Centre at origin, foci $\left(\pm\sqrt{a^2 + b^2}, 0\right)$: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Parametric equations: $x = a\sec\theta$, $y = b\tan\theta$

Rectangular hyperbola:

Centre at origin, vertex $\pm\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$: $xy = \dfrac{a^2}{2} = c^2$

where $c = \dfrac{a}{\sqrt{2}}$ i.e. $xy = c^2$

Parametric equations: $x = ct$, $y = c/t$

4 Laws of mathematics

(a) *Associative laws* – for addition and multiplication

$a + (b + c) = (a + b) + c$

$a(bc) = (ab)c$

(b) *Commutative laws* – for addition and multiplication

$a + b = b + a$

$ab = ba$

(c) *Distributive laws* – for multiplication and division

$a(b + c) = ab + ac$

$\dfrac{b + c}{a} = \dfrac{b}{a} + \dfrac{c}{a}$ (provided $a \neq 0$)

Numerical solutions of equations and interpolation

Frames 1 to 87

Learning outcomes

When you have completed this Programme you will be able to:

- Appreciate the Fundamental Theorem of Algebra
- Find the two roots of a quadratic equation and recognise that for polynomial equations with real coefficients complex roots exist in complex conjugate pairs
- Use the relationships between the coefficients and the roots of a polynomial equation to find the roots of the polynomial
- Transform a cubic equation to its reduced form
- Use Tartaglia's solution to find the real root of a cubic equation
- Find the solution of the equation $f(x) = 0$ by the method of bisection
- Solve equations involving a single real variable by iteration and use a spreadsheet for efficiency
- Solve equations using the Newton–Raphson iterative method
- Use the modified Newton–Raphson method to find the first approximation when the derivative is small
- Understand the meaning of *interpolation* and use simple linear and graphical interpolation
- Use the Gregory–Newton interpolation formula with forward and backward differences for equally spaced domain points
- Use the Gauss interpolation formulas using central differences for equally spaced domain points
- Use Lagrange interpolation when the domain points are not equally spaced

Introduction

1

In this Programme we shall be looking at analytic and numerical methods of solving the general equation in a single variable, $f(x) = 0$. In addition, a functional relationship can be exhibited in the form of a collection of ordered pairs rather than in the form of an algebraic expression. We shall be looking at interpolation methods of estimating values of $f(x)$ for intermediate values of x between those listed among the ordered pairs.

First we shall look at the **Fundamental Theorem of Algebra**, which deals with the factorisation of polynomials.

The Fundamental Theorem of Algebra

2

The *Fundamental Theorem of Algebra* can be stated as follows

Every polynomial expression $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can be written as a product of n linear factors in the form

$$f(x) = a_n(x - r_1)(x - r_2)(\cdots)(x - r_n)$$

As an immediate consequence of this we can see that there are n values of x that satisfy the polynomial equation $f(x) = 0$, namely $x = r_1$, $x = r_2, \ldots, x = r_n$. We call these values the *roots* of the polynomial, but be aware that they may not all be distinct. Furthermore, the polynomial coefficients a_i and the polynomial roots r_i may be real, imaginary or complex.

For example the quadratic equation

$x^2 + 5x + 6 = 0$ can be written $(x + 2)(x + 3) = 0$ so it has the two *distinct* roots $x = -2$ and $x = -3$

$x^2 - 4x + 4 = 0$ can be written as $(x - 2)(x - 2) = 0$ so it has the two *coincident* roots $x = 2$ and $x = 2$

$x^2 + x + 1 = 0$ can be written as $(x + a)(x + b) = 0$ so it has the two roots $x = -a$ and $x = -b$

To find the numerical values of a and b we need to use the formula for finding the roots of a general quadratic equation. Can you recall what it is? If not, then refer to Frame 14 of Programme F.6 in *Engineering Mathematics, Fifth Edition.*

The solution to the quadratic equation $ax^2 + bx + c = 0$ is ....

The answer is in the next frame

3

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

So the roots of $x^2 + x + 1 = 0$ are ....

Next frame

4

$$\boxed{x = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}}$$

Because

$$a = b = c = 1 \text{ and so } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2}$$
$$= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

This quadratic equation has two distinct *complex* roots. Notice that the two roots form a *complex conjugate pair* – each is the complex conjugate of the other. **Whenever a polynomial with *real coefficients* a_i has a complex root it also has the complex conjugate as another root.**

So given that $x = -2 + j\sqrt{5}$ is one root of a quadratic equation with real coefficients then

the other root is ....

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$$\boxed{x = -2 - j\sqrt{5}}$$

Because

The complex conjugate of $x = -2 + j\sqrt{5}$ is $x = -2 - j\sqrt{5}$ and complex roots of a polynomial equation with real coefficients always appear as conjugate pairs.

The quadratic equation with these two roots is ....

6

$$\boxed{x^2 + 4x + 9 = 0}$$

Because

If $x = a$ and $x = b$ are the two roots of a quadratic equation then $(x - a)(x - b) = 0$ gives the quadratic equation. That is $(x - a)(x - b) = x^2 - (a + b)x + ab = 0$.

Here, the two roots are $x = -2 + j\sqrt{5}$ and $x = -2 - j\sqrt{5}$ so that

$$\left(x - \left[-2 + j\sqrt{5}\right]\right)\left(x - \left[-2 - j\sqrt{5}\right]\right) = 0$$

That is $x^2 - x\left[-2 + j\sqrt{5} - 2 - j\sqrt{5}\right] + \left[-2 + j\sqrt{5}\right]\left[-2 - j\sqrt{5}\right] = 0$.

So $x^2 + 4x + 9 = 0$.

Notice that the coefficients are …………

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$$\boxed{\text{Real}}$$

Relations between the coefficients and the roots of a polynomial equation

Let α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$. Then, writing the expression $x^3 + px^2 + qx + r$ in terms of α, β, γ gives

$$x^3 + px^2 + qx + r = \ldots\ldots\ldots\ldots$$

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$$\boxed{(x - \alpha)(x - \beta)(x - \gamma)}$$

Therefore

$$\begin{aligned} x^3 + px^2 + qx + r &= (x - \alpha)(x - \beta)(x - \gamma) \\ &= \left(x^2 - [\alpha + \beta]x + \alpha\beta\right)(x - \gamma) \\ &= x^3 - (\alpha + \beta)x^2 + \alpha\beta x - \gamma x^2 + (\alpha + \beta)\gamma x - \alpha\beta\gamma \\ &= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma \end{aligned}$$

$\therefore$ equating coefficients

(a) $\alpha + \beta + \gamma = \ldots\ldots\ldots\ldots$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \ldots\ldots\ldots\ldots$

(c) $\alpha\beta\gamma = \ldots\ldots\ldots\ldots$

9

$$\boxed{\text{(a)}\ -p;\quad \text{(b)}\ q;\quad \text{(c)}\ -r}$$

This, of course, applies to a cubic equation. Let us extend this to a more general equation.

So on to the next frame

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In general, if $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ are roots of the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0 \quad (p_0 \neq 0)$$

then sum of the roots $= -\dfrac{p_1}{p_0}$

sum of products of the roots, two at a time $= \dfrac{p_2}{p_0}$

sum of products of the roots, three at a time $= -\dfrac{p_3}{p_0}$

sum of products of the roots, n at a time $= (-1)^n.\dfrac{p_n}{p_0}$

So for the equation $3x^4 + 2x^3 + 5x^2 + 7x - 4 = 0$, if $\alpha, \beta, \gamma, \delta$ are the four roots, then

(a) $\alpha + \beta + \gamma + \delta = \dots\dots\dots\dots$

(b) $\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \delta\beta + \gamma\alpha = \dots\dots\dots\dots$

(c) $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = \dots\dots\dots\dots$

(d) $\alpha\beta\gamma\delta = \dots\dots\dots\dots$

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(a) $-\dfrac{2}{3}$; (b) $\dfrac{5}{3}$; (c) $-\dfrac{7}{3}$; (d) $-\dfrac{4}{3}$

Now for a problem or two on the same topic.

Example 1

Solve the equation $x^3 - 8x^2 + 9x + 18 = 0$ given that the sum of two of the roots is 5.

Using the same approach as before, if α, β, γ are the roots, then

(a) $\alpha + \beta + \gamma = \dots\dots\dots\dots$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \dots\dots\dots\dots$

(c) $\alpha\beta\gamma = \dots\dots\dots\dots$

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(a) 8; (b) 9; (c) -18

So we have $\alpha + \beta + \gamma = 8 \qquad$ Let $\alpha + \beta = 5$

$$\therefore\ 5 + \gamma = 8 \quad \therefore\ \gamma = 3$$

$$\text{Also} \quad \alpha\beta\gamma = -18 \qquad \alpha\beta(3) = -18 \qquad \therefore\ \alpha\beta = -6$$

$$\alpha + \beta = 5 \quad \therefore\ \beta = 5 - \alpha \quad \therefore\ \alpha(5 - \alpha) = -6$$

$$\alpha^2 - 5\alpha - 6 = 0 \quad \therefore\ (\alpha - 6)(\alpha + 1) = 0 \quad \therefore\ \alpha = -1 \text{ or } 6$$

$$\therefore\ \beta = 6 \text{ or } -1$$

Roots are $x = -1, 3, 6$

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Example 2

Solve the equation $2x^3 + 3x^2 - 11x - 6 = 0$ given that the three roots form an arithmetic sequence.

Let us represent the roots by $(a - k)$, a, $(a + k)$

Then the sum of the roots $= 3a = \dots\dots\dots\dots$

and the product of the roots $= a(a - k)(a + k) = \dots\dots\dots\dots$

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$$\boxed{3a = -\frac{3}{2}; \quad a(a+k)(a-k) = \frac{6}{2} = 3}$$

$$\therefore\ a = -\frac{1}{2} \qquad -\frac{1}{2}\left(\frac{1}{4} - k^2\right) = 3 \qquad \therefore\ k = \pm\frac{5}{2}$$

If $k = \frac{5}{2}$ $\quad a = -\frac{1}{2}$; $\quad a - k = -3$; $\quad a + k = 2$

If $k = -\frac{5}{2}$ $\quad a = -\frac{1}{2}$; $\quad a - k = 2$; $\quad a + k = -3$

$\therefore$ required roots are $-3, -\frac{1}{2}, 2$

Here is a similar one.

Example 3

Solve the equation $x^3 + 3x^2 - 6x - 8 = 0$ given that the three roots are in geometric sequence.

This time, let the roots be $\frac{a}{k}$, a, ak

Then $\frac{a}{k} = a + ak = \dots\dots\dots\dots$ and $\left(\frac{a}{k}\right)(a)(ak) = \dots\dots\dots\dots$

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$$\boxed{\text{sum of roots} = -3; \quad \text{product of roots} = 8}$$

It then follows that the roots are $\dots\dots\dots\dots$, $\dots\dots\dots\dots$, $\dots\dots\dots\dots$

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$$\boxed{-4, \quad 2, \quad -1}$$

The working rests on the relationships between the roots and the coefficients, i.e. if α, β, γ are the roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

then (a) $\alpha + \beta + \gamma = \dots\dots\dots\dots$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \dots\dots\dots\dots$

(c) $\alpha\beta\gamma = \dots\dots\dots\dots$

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$$\boxed{\text{(a) } -\frac{b}{a}; \quad \text{(b) } \frac{c}{a}; \quad \text{(c) } -\frac{d}{a}}$$

In each of the three examples reconstruct the cubic to confirm that they are correct.

Now on to the next stage

Cubic equations

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The Fundamental Theorem of Algebra tells us that every cubic expression

$$f(x) = ax^3 + bx^2 + cx + d$$

can be written as a product of three linear factors

$$f(x) = a(x - r_1)(x - r_2)(x - r_3)$$

Consequently, every cubic equation

$$f(x) = a(x - r_1)(x - r_2)(x - r_3) = 0$$

has three roots which may be distinct or coincident and which may be real or complex. However, because complex roots of a polynomial with real coefficients always appear in complex conjugate pairs we can say that every such cubic equation has

at least one . . . .

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$$\boxed{\text{at least one real root}}$$

To find the value of this real root we can employ a formula equivalent to the formula used to find the two roots of the general quadratic. This is called Tartaglia's method but before we can proceed to look at that we must first consider how to transform the general cubic to its **reduced form**.

Next frame

Transforming a cubic to reduced form

20

In every case, an equation of the form

$$x^3 + ax^2 + bx + c = 0$$

can be converted into the reduced form $y^3 + py + q = 0$ by the substitution $x = y - \frac{a}{3}$.

The example overleaf will demonstrate the method. ▶

Example 4

Express $f(x) = x^3 + 6x^2 - 4x + 5 = 0$ in reduced form.

Substitute $x = y - \frac{a}{3}$, i.e. $x = y - \frac{6}{3} = y - 2$. Put $x = y - 2$.

The equation then becomes

$$(y-2)^3 + 6(y-2)^2 - 4(y-2) + 5 = 0$$

$$(y^3 - 3y^2 2 + 3y4 - 8) + 6(y^2 - 4y + 4) - 4(y-2) + 5 = 0$$

which simplifies to ....

21

$$\boxed{y^3 - 16y + 29 = 0}$$

Tartaglia's solution for a real root

In the sixteenth century, Tartaglia discovered that a root of the cubic equation $x^3 + ax + b = 0$, where $a > 0$, is given by

$$x = \left\{-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3} + \left\{-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3}$$

That looks pretty formidable, but it is a good deal easier than it appears. Notice that $\frac{b}{2}$ and $\sqrt{\frac{a^3}{27} + \frac{b^2}{4}}$ occur twice and it is convenient to evaluate these first and then substitute the results in the main expression for x.

Example 5

Find a real root of $x^3 + 2x + 5 = 0$.

Here, $a = 2$, $b = 5$ $\therefore$ $\frac{b}{2} = 2{\cdot}5$

$\sqrt{\frac{a^3}{27} + \frac{b^2}{4}} = \sqrt{\frac{8}{27} + \frac{25}{4}} = \sqrt{6{\cdot}5463} = 2{\cdot}5586$

Then $x = (-2{\cdot}5 + 2{\cdot}5586)^{1/3} + (-2{\cdot}5 - 2{\cdot}5586)^{1/3}$

$\quad = 0{\cdot}3884 - 1{\cdot}7166 = -1{\cdot}3282 \qquad x = -1{\cdot}328$

Once we have a real root, the equation can be reduced to a quadratic and the remaining two roots determined. They are $x = 0{\cdot}664 + j\,1{\cdot}823$ and $x = 0{\cdot}664 - j\,1{\cdot}823$ (see *Engineering Mathematics, Fifth Edition*, Programme F.6).

Example 6

Determine a real root of $2x^3 + 3x - 4 = 0$.

This is first written $x^3 + 1{\cdot}5x - 2 = 0$ $\therefore$ $a = 1{\cdot}5$, $b = -2$

Now you can evaluate $\frac{b}{2}$ and $\sqrt{\frac{a^3}{27} + \frac{b^2}{4}}$ and so determine

$$x = \ldots\ldots\ldots\ldots$$

22

$\boxed{0{\cdot}8796}$

Because

$$\left\{-\frac{b}{2}+\sqrt{\frac{a^3}{27}+\frac{b^2}{4}}\right\}^{1/3}=\{2{\cdot}06066\}^{1/3}=1{\cdot}2725 \text{ and}$$

$$\left\{-\frac{b}{2}-\sqrt{\frac{a^3}{27}+\frac{b^2}{4}}\right\}^{1/3}=\{-0{\cdot}6066\}^{1/3}=-0{\cdot}3929,$$

therefore $x = 1{\cdot}2725 - 0{\cdot}3929 = 0{\cdot}8796$

Note: If you wish to find the real root of a cubic using Tartaglia's method and $a < 0$ then just multiply the entire equation by -1.

Next frame

Numerical methods

23

The methods that we have used so far to solve quadratic equations and to find the real root of a cubic equation are called *analytic methods*. These analytic methods used straightforward algebraic techniques to develop a formula for the answer. The numerical value of the answer can then be found by simple substitution of numbers for the variables in the formula. Unfortunately, general polynomial equations of order five or higher cannot by solved by analytic methods. Instead, we must resort to what are termed *numerical methods*. The simplest method of finding the solution to the equation $f(x) = 0$ is the *bisection* method.

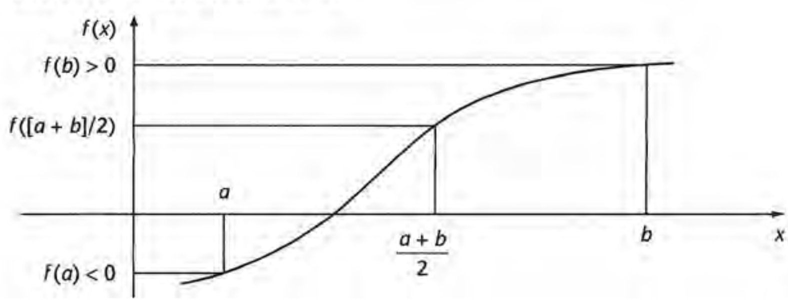
Bisection

The bisection method of finding a solution to the equation $f(x) = 0$ consists of

Finding a value of x, say $x = a$, such that $f(a) < 0$

Finding a value of x, say $x = b$, such that $f(b) > 0$

The solution to the equation $f(x) = 0$ must then lie between a and b. Furthermore, it must lie either in the first half of the interval between a and b or in the second half.



Find the value of $f([a+b]/2)$ – that is halfway between a and b.

If $f([a+b]/2) > 0$ then the solution lies in the first half and if $f([a+b]/2) < 0$ then it lies in the second half. This procedure is repeated, narrowing down the width of the interval by a half each time. An example should clarify all this.

Example 7

Find the positive value of x that satisfies the equation $x^2 - 2 = 0$.

Firstly we note that if $x = 1$ then $x^2 - 2 < 0$, and that if $x = 2$ then $x^2 - 2 > 0$, so the solution that we seek must lie between 1 and 2.

We look for the ....

24

The mid-point between 1 and 2 which is 1·5

Now, when $x = 1{\cdot}5$, $x^2 - 2 = 0{\cdot}25 > 0$

so the solution must lie between ....

25

1 and 1·5

The mid-point between 1 and 1·5 is 1·25. When $x = 1{\cdot}25$, $x^2 - 2 = -0{\cdot}4375 < 0$

so the solution must lie between ....

26

1·25 and 1·5

The mid-point between 1·25 and 1·5 is 1·375. We now evaluate $x^2 - 2$ at this point and determine in which half interval the solution lies. This process is repeated and the following table displays the results. In each block of six numbers the first column lists the end points of the interval and the mid-point. The second column contains the respective values $f(x) = x^2 - 2$. Construct the table as follows.

(a) For each block of six numbers copy the last number in the first column into the second place of the first column of the following block. This represents the centre point of the previous interval.

(b) For each block of six numbers copy the number that represents the other end point of the new interval from the first column into the first place of the first column of the following block. Look at the signs in the second column of the first block to decide which is the appropriate number.

▶

a	1·0000	−1·0000	1·0000	−1·0000	1·5000	0·2500	1·5000	0·2500
b	2·0000	2·0000	1·5000	0·2500	1·2500	−0·4375	1·3750	−0·1094
$(a+b)/2$	1·5000	0·2500	1·2500	−0·4375	1·3750	−0·1094	1·4375	0·0664
a	1·3750	−0·1094	1·4375	0·0664	1·4063	−0·0225	1·4219	0·0217
b	1·4375	0·0664	1·4063	−0·0225	1·4219	0·0217	1·4141	−0·0004
$(a+b)/2$	1·4063	−0·0225	1·4219	0·0217	1·4141	−0·0004	1·4180	0·0106
a	1·4141	−0·0004	1·4141	−0·0004	1·4141	−0·0004	1·4141	−0·0004
b	1·4180	0·0106	1·4160	0·0051	1·4150	0·0023	1·4146	0·0010
$(a+b)/2$	1·4160	0·0051	1·4150	0·0023	1·4146	0·0010	1·4143	0·0003
a	1·4141	−0·0004	1.4143	0·0003	1·4142	−0·0001		
b	1·4143	0·0003	1·4142	−0·0001	1·4142	0·0001		
$(a+b)/2$	1·4142	−0·0001	1·4142	0·0001	1·4142	0·0000		

The final result to four decimal places is $x = 1{\cdot}4142$ which is the correct answer to that level of accuracy – but it has taken a lot of activity to produce it. A much faster way of solving this equation is to use an iteration formula that was first devised by Newton.

Next frame

Numerical solution of equations by iteration

27

The process of finding the numerical solution to the equation

$$f(x) = 0$$

by iteration is performed by first finding an approximate solution and then using this approximate solution to find a more accurate solution. This process is repeated until a solution is found to the required level of accuracy. For example, Newton showed that the square root of a number a can be found from the iteration equation

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{a}{x_i}\right), \quad i = 0, 1, 2, \ldots$$

where x_0 is the approximation that starts the iteration off. So, to find a succession of approximate values of $\sqrt{2}$, each of increasing accuracy, we proceed as follows. Let $x_0 = 1{\cdot}5$ – found by the first stage of the bisection method. Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) = 0{\cdot}5(1{\cdot}5 + 2/1{\cdot}5) = 1{\cdot}4166\ldots$$

This value is then used to find x_2.

By rounding x_1 to 1·4167, the value of x_2 is found to be ....

28

$$\boxed{x_2 = 1{\cdot}4142}$$

Because

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = 0{\cdot}5(1{\cdot}4167 + 2/1{\cdot}4167) = 1{\cdot}4142\ldots$$

This has achieved the same level of accuracy as the bisection method in just two steps.

Using a spreadsheet

This simple iteration procedure is more efficiently performed using a spreadsheet. If the use of a spreadsheet is a totally new experience for you then you are referred to Programme 4 of *Engineering Mathematics, Fifth Edition* where the spreadsheet is introduced as a tool for constructing graphs of functions. If you have a limited knowledge then you will be able to follow the text from here. The spreadsheet we shall be using here is Microsoft Excel, though all commercial spreadsheets possess the equivalent functionality.

Open your spreadsheet and in cell A1 enter n and press **Enter**. In this first column we are going to enter the iteration numbers. In cell A2 enter the number 0 and press **Enter**. Place the cell highlight in cell A2 and highlight the block of cells A2 to A7 by holding down the mouse button and wiping the highlight down to cell A7. Click the **Edit** command on the Command bar and point at **Fill** from the drop-down menu. Select **Series** from the next drop-down menu and accept the default **Step value** of 1 by clicking OK in the Series window.

The cells A3 to A7 fill with ....

29

$$\boxed{\text{the numbers 1 to 5}}$$

In cell B1 enter the letter x – this column is going to contain the successive x-values obtained by iteration. In cell B2 enter the value of x_0, namely 1·5.

In cell B3 enter the formula

= 0·5*(B2+2/B2)

The number that appears in cell B3 is then ....

30

$$\boxed{1{\cdot}416667}$$

Place the cell highlight in cell B3, click the command **Edit** on the Command bar and select **Copy** from the drop-down menu. You have now copied the formula in cell B3 onto the Clipboard. Highlight the cells B4 to B7 and then click the **Edit** command again but this time select **Paste** from the drop-down menu.

The cells B4 to B7 fill with numbers to provide the display

....

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n	x
0	1·5
1	1·416667
2	1·414216
3	1·414214
4	1·414214
5	1·414214

By using the various formatting facilities provided by the spreadsheet the display can be amended to provide the following

n	x
0	1·500000000000000
1	1·416666666666670
2	1·414215686274510
3	1·414213562374690
4	1·414213562373090
5	1·414213562373090

The number of decimal places here is 15, which is far greater than is normally required but it does demonstrate how effective a spreadsheet can be. In future we shall restrict the displays to 6 decimal places.

Notice that to find a value accurate to a given number of decimal places or significant figures it is sufficient to repeat the iterations until there is no change in the result from one iteration to the next.

Save your spreadsheet under some suitable name such as *Newton* because you may wish to use it again.

Now we shall look at this spreadsheet a little more closely

Relative addresses

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Place the cell highlight in cell B3 and the formula that it contains is =0·5*(B2+2/B2). Now place the cell highlight in cell B4 and the formula there is =0·5*(B3+2/B3). Why the difference?

When you enter the cell address B2 in the formula in B3 the spreadsheet understands that to mean *the contents of the cell immediately above*. It is this meaning that is copied into cell B4 where the *cell immediately above* is B3. If you wish to refer to a specific cell in a formula then you must use an **absolute address**.

Place the cell highlight in cell C1 and enter the number 2. Now place the cell highlight in cell B3 and re-enter the formula

=0·5*(B2+$C$1/B2)

and copy this into cells B4 to B7. The numbers in the second column have not changed but the formulas have because in cells B3 to B7 the same reference is made to cell C1. *The use of the dollar signs has indicated an absolute address.* So why would we do this?

Change the number in cell C1 to 3 to obtain the display ....

33

n	x
0	1·500000000000000
1	1·750000000000000
2	1·732142857142860
3	1·732050810014730
4	1·732050807568880
5	1·732050807568880

These are the iterated values of $\sqrt{3}$ – the square root of the contents of cell C1. We can now use the same spreadsheet to find the square root of any positive number.

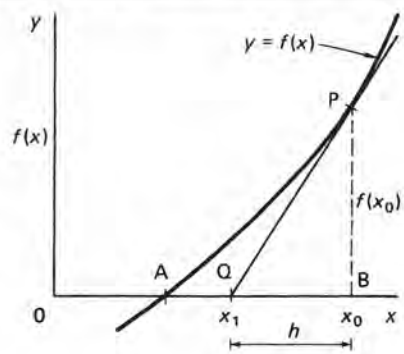
Newton's iterative procedure to find the square root of a positive number is a special case of the **Newton–Raphson** procedure to find the solution of the general equation $f(x) = 0$, and we shall look at this in the next frame.

Newton–Raphson iterative method

34

Consider the graph of $y = f(x)$ as shown. Then the x-value at the point A, where the graph crosses the x-axis, gives a solution of the equation $f(x) = 0$.

If P is a point on the curve near to A, then $x = x_0$ is an approximate value of the root of $f(x) = 0$, the error of the approximation being given by AB.



Let PQ be the tangent to the curve as P, crossing the x-axis at Q $(x_1, 0)$. Then $x = x_1$ is a better approximation to the required root.

From the diagram, $\dfrac{\text{PB}}{\text{QB}} = \left[\dfrac{dy}{dx}\right]_{\text{P}}$ i.e. the value of the derivative of y at the point P, $x = x_0$.

$$\therefore \quad \frac{\text{PB}}{\text{QB}} = f'(x_0) \quad \text{and} \quad \text{PB} = f(x_0)$$

$$\therefore \quad \text{QB} = \frac{\text{PB}}{f'(x_0)} = \frac{f(x_0)}{f'(x_0)} = h \text{ (say)}$$

$$x_1 = x_0 - h \qquad \therefore \; x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

▶

If we begin, therefore, with an approximate value (x_0) of the root, we can determine a better approximation (x_1). Naturally, the process can be repeated to improve the result still further. Let us see this in operation.

On to the next frame

35

Example 1

The equation $x^3 - 3x - 4 = 0$ is of the form $f(x) = 0$ where $f(1) < 0$ and $f(3) > 0$ so there is a solution to the equation between 1 and 3. We shall take this to be 2, by bisection. Find a better approximation to the root.

We have $f(x) = x^3 - 3x - 4 \quad \therefore \; f'(x) = 3x^2 - 3$

If the first approximation is $x_0 = 2$, then

$$f(x_0) = f(2) = -2 \quad \text{and} \quad f'(x_0) = f'(2) = 9$$

A better approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0{}^3 - 3x_0 - 4}{3x_0{}^2 - 3}$$

$$x_1 = 2 - \frac{(-2)}{9} = 2{\cdot}22$$

$$\therefore \; x_0 = 2; \quad x_1 = 2{\cdot}22$$

If we now start from x_1 we can get a better approximation still by repeating the process.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1{}^3 - 3x_1 - 4}{3x_1{}^2 - 3}$$

Here $x_1 = 2{\cdot}22 \qquad f(x_1) = \ldots\ldots\ldots\ldots; \quad f'(x_1) = \ldots\ldots\ldots\ldots$

36

$$\boxed{f(x_1) = 0{\cdot}281; \quad f'(x_1) = 11{\cdot}785}$$

Then $x_2 = \ldots\ldots\ldots\ldots$

37

$$\boxed{x_2 = 2{\cdot}196}$$

Because

$$x_2 = 2{\cdot}22 - \frac{0{\cdot}281}{11{\cdot}79} = 2{\cdot}196$$

Using $x_2 = 2{\cdot}196$ as a starter value, we can continue the process until successive results agree to the desired degree of accuracy.

$x_3 = \ldots\ldots\ldots\ldots$

38

$$\boxed{x_3 = 2{\cdot}196}$$

Because

$$f(x_2) = f(2{\cdot}196) = 0{\cdot}002026; \qquad f'(x_2) = f'(2{\cdot}196) = 11{\cdot}467$$

$$\therefore\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2{\cdot}196 - \frac{0{\cdot}00203}{11{\cdot}467} = 2{\cdot}196 \text{ (to 4 sig. fig.)}$$

The process is simple but effective and can be repeated again and again. Each repetition, or *iteration*, usually gives a result nearer to the required root $x = x_A$.

In general $x_{n+1} =$ ....

39

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Tabular display of results

Open your spreadsheet and in cells A1 to D1 enter the headings n, x, $f(x)$ and $f'(x)$

Fill cells A2 to A6 with the numbers 0 to 4

In cell B2 enter the value for x_0, namely 2

In cell C2 enter the formula for $f(x_0)$, namely =B2^3 – 3*B2 – 4 and copy into cells C3 to C6

In cell D2 enter the formula for $f'(x_0)$, namely =3*B2^2 – 3 and copy into cells D3 to D6

In cell B3 enter the formula for x_1, namely =B2 – C2/D2 and copy into cells B4 to B6.

The final display is ....

40

n	x	$f(x)$	$f'(x)$
0	2	−2	9
1	2·222222	0·30727	11·81481
2	2·196215	0·004492	11·47008
3	2·195823	1·01E-06	11·46492
4	2·195823	5·15E-14	11·46492

As soon as the number in the second column is repeated then we know that we have arrived at that particular level of accuracy. The required root is therefore $x = 2{\cdot}195823$ to 6 dp. Save the spreadsheet so that it can be used as a template for other such problems.

Now let us have another example.

Next frame

41

Example 2

The equation $x^3 + 2x^2 - 5x - 1 = 0$ is of the form $f(x) = 0$ where $f(1) < 0$ and $f(2) > 0$ so there is a solution to the equation between 1 and 2. We shall take this to be $x = 1{\cdot}5$. Use the Newton–Raphson method to find the root to six decimal places.

Use the previous spreadsheet as a template and make the following amendments

In cell B2 enter the number ....

42

1·5

Because

That is the value of x_0 that is used to start the iteration

In cell C2 enter the formula ....

43

= B2^3 + 2*B2^2 – 5*B2 – 1

Because

That is the value of $f(x_0) = {x_0}^3 + 2{x_0}^2 - 5x_0 - 1$. Copy the contents of cell C2 into cells C3 to C5.

In cell D2 enter the formula ....

44

= 3*B2^2 + 4*B2 – 5

Because

That is the value of $f'(x_0) = 3{x_0}^2 + 4x_0 - 5$. Copy the contents of cell D2 into cells D3 to D5.

In cell B2 the formula remains the same as ....

45

= B2 – C2/D2

The final display is then ....

46

n	x	$f(x)$	$f'(x)$
0	1·5	−0·625	7·75
1	1·580645	0·042798	8·817898
2	1·575792	0·000159	8·752524
3	1·575773	2·21E-09	8·75228

We cannot be sure that the value 1·575773 is accurate to the sixth decimal place so we must extend the table.

Highlight cells A5 to D5, click **Edit** on the Command bar and select **Copy** from the drop-down menu.

Place the cell highlight in cell A6, click **Edit** and then **Paste**.

The seventh row of the spreadsheet then fills to produce the display

n	x	$f(x)$	$f'(x)$
0	1·5	−0·625	7·75
1	1·580645	0·042798	8·817898
2	1·575792	0·000159	8·752524
3	1·575773	2·21E-09	8·75228
4	1·575773	−8·9E-16	8·75228

And the repetition of the x-value ensures that the solution $x = 1{\cdot}575773$ is indeed accurate to 6 dp.

Now do one completely on your own.

Next frame

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Example 3

The equation $2x^3 - 7x^2 - x + 12 = 0$ has a root near to $x = 1{\cdot}5$. Use the Newton–Raphson method to find the root to six decimal places.

The spreadsheet solution produces . . . .

48

$$\boxed{x = 1{\cdot}686141 \text{ to 6 dp}}$$

Because

Fill cells A2 to A6 with the numbers 0 to 4

In cell B2 enter the value for x_0, namely 1·5

In cell C2 enter the formula for $f(x_0)$, namely =2*B2^3 – 7*B2^2 – B2 + 12 and copy into cells C3 to C6

In cell D2 enter the formula for $f'(x_0)$, namely =6*B2^2 – 14*B2 – 1 and copy into cells D3 to D6

In cell B3 enter the formula for x_1, namely =B2 – C2/D2 and copy into cells B4 to B6.

The final display is . . . .

49

n	x	$f(x)$	$f'(x)$
0	1·5	1·5	−8·5
1	1·676471	0·073275	−7·60727
2	1·686103	0·000286	−7·54778
3	1·686141	4·46E-09	−7·54755
4	1·686141	0	−7·54755

As soon as the number in the second column is repeated then we know that we have arrived at that particular level of accuracy. The required root is therefore $x = 1{\cdot}686141$ to 6 dp.

First approximations

The whole process hinges on knowing a 'starter' value as first approximation. If we are not given a hint, this information can be found by either

(a) applying the remainder theorem if the function is a polynomial
(b) drawing a sketch graph of the function.

Example 4

Find the real root of the equation $x^3 + 5x^2 - 3x - 4 = 0$ correct to six significant figures.

Application of the remainder theorem involves substituting $x = 0$, $x = \pm 1$, $x = \pm 2$, etc. until two adjacent values give a change in sign.

$f(x) = x^3 + 5x^2 - 3x - 4$

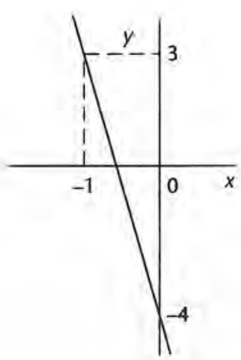
$f(0) = -4;\quad f(1) = -1;\quad f(-1) = 3$

The sign changes from $f(0)$ to $f(-1)$. There is thus a root between $x = 0$ and $x = -1$.

Therefore choose $x = -0{\cdot}5$ as the first approximation and then proceed as before.

Complete the table and obtain the root

$x = \ldots\ldots\ldots\ldots$



50

$$\boxed{x = -0{\cdot}675527}$$

The final spreadsheet display is

n	x	$f(x)$	$f'(x)$
0	−0·5	−1·375	−7·25
1	−0·689655	0·11907	−8·469679
2	−0·675597	0·000582	−8·386675
3	−0·675527	1·43E-08	−8·386262
4	−0·675527	0	−8·386262

51

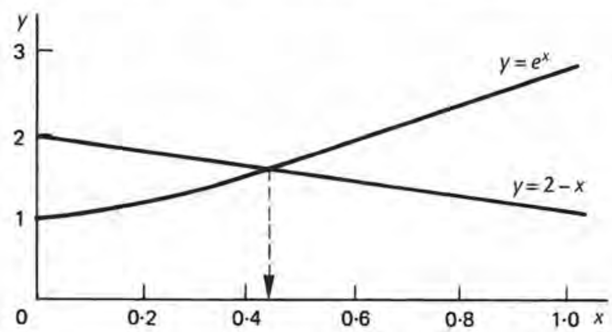
Example 5

Solve the equation $e^x + x - 2 = 0$ giving the root to 6 significant figures.

It is sometimes more convenient to obtain a first approximation to the required root from a sketch graph of the function, or by some other graphical means.

In this case, the equation can be rewritten as $e^x = 2 - x$ and we therefore sketch graphs of $y = e^x$ and $y = 2 - x$.

x	0·2	0·4	0·6	0·8	1
e^x	1·22	1·49	1·82	2·23	2·72
$2 - x$	1·8	1·6	1·4	1·2	1



It can be seen that the two curves cross over between $x = 0{\cdot}4$ and $x = 0{\cdot}6$.

Approximate root $x = 0{\cdot}4$

$$f(x) = e^x + x - 2 \qquad f'(x) = e^x + 1$$

$$x = \ldots\ldots\ldots\ldots$$

Finish it off

52

$$\boxed{x = 0{\cdot}442854}$$

The final spreadsheet display is

n	x	$f(x)$	$f'(x)$
0	0·4	−0·10818	2·491825
1	0·443412	0·001426	2·558014
2	0·442854	2·42E-07	2·557146
3	0·442854	7·11E-15	2·557146

Note: There are times when the normal application of the Newton–Raphson method fails to converge to the required root. This is particularly so when $f'(x_0)$ is very small, so before we leave this section let us consider this difficulty.

Modified Newton–Raphson method

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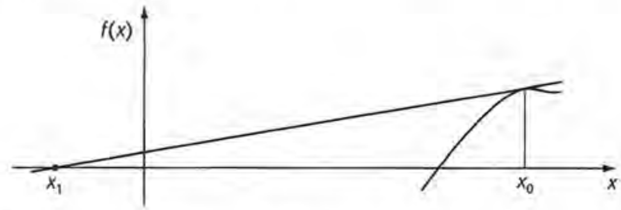
If the slope of the curve at $x = x_0$ is small, the value of the second approximation $x = x_1$ may be further from the exact root at A than the first approximation.



If $x = x_0$ is an approximate solution of $f(x) = 0$ and $x = x_0 - h$ is the exact solution then $f(x_0 - h) = 0$. By Taylor's series

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \ldots = 0$$

(a) If we assume that h is small enough to neglect terms of the order h^2 and higher then this equation can be written as

$f(x_0 - h) \approx f(x_0) - hf'(x_0)$, that is $f(x_0) - hf'(x_0) \approx 0$ and so

$h \approx \dfrac{f(x_0)}{f'(x_0)}$ giving $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ as a better approximation to the solution of $f(x) = 0$.

This is, of course, the relationship we have been using and which may fail when $f'(x)$ is small.

Notice: h is positive unless the sign of $f(x_0)$ is the opposite of the sign of $f'(x_0)$.

(b) If we consider the first three terms then

$f(x_0 - h) \approx f(x_0) - hf'(x_0) + \dfrac{h^2}{2!}f''(x_0) \approx 0$, that is

$$2f(x_0) - 2hf'(x_0) + h^2 f''(x_0) \approx 0$$

Since $f'(x_0)$ is small we shall assume that we can neglect it so

$h = \pm\sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$

That is $h = \sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$ unless the signs of $f(x_0)$ and $f'(x_0)$ are different when it is $h = -\sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$. We use this result only when $f'(x_0)$ is found to be very small. Having found x_1 from x_0 we then revert to the normal relationship $x_{n+1} = x_n - \dfrac{f(x_0)}{f'(x_0)}$ for subsequent iterations.

Note this

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Example 6

The equation $x^3 - 1{\cdot}3x^2 + 0{\cdot}4x - 0{\cdot}03 = 0$ is known to have a root near $x = 0{\cdot}7$. Determine the root to 6 significant figures.

We start off in the usual way.

$$f(x) = x^3 - 1{\cdot}3x^2 + 0{\cdot}4x - 0{\cdot}03$$
$$f'(x) = 3x^2 - 2{\cdot}6x + 0{\cdot}4$$

and complete the first line of the normal table.

n	x_n	$f(x_n)$	$f'(x_n)$	$h = \dfrac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n - h$
0	0·7				

Complete just the first line of values.

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We have

n	x_n	$f(x_n)$	$f'(x_n)$	$h = \dfrac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n - h$
0	0·7	−0·044	0·05	−0·88	1·58

We notice at once that

(a) The value of x_1 is well away from the approximate value (0·7) of the root.

(b) The value of $f'(x_0)$ is small, i.e. 0·05.

To obtain x_1 we therefore make a fresh start, using the modified relationship $x_1 = \dots\dots\dots\dots$

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$$\boxed{x_1 = x_0 \pm \sqrt{\frac{-2f(x_0)}{f''(x_0)}}}$$

$$\begin{aligned} f(x) &= x^3 - 1{\cdot}3x^2 + 0{\cdot}4x - 0{\cdot}03 = [(x - 1{\cdot}3)x + 0{\cdot}4]x - 0{\cdot}03 \\ f'(x) &= 3x^2 - 2{\cdot}6x + 0{\cdot}4 \qquad = (3x - 2{\cdot}6)x + 0{\cdot}4 \\ f''(x) &= 6x - 2{\cdot}6 \end{aligned}$$

n	x_0	$f(x_0)$	$f''(x_0)$	$h = \sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$	$x_1 = x_0 \pm h$
0	0·7	−0·044			

Complete the line

57

n	x_0	$f(x_0)$	$f''(x_0)$	$h = \sqrt{\dfrac{-2f(x_0)}{f''(x_0)}}$	$x_1 = x_0 \pm h$
0	0·7	−0·044	1·6	0·2345	0·9345

Note that in the expression $x_1 = x_0 \pm h$, we chose the positive sign since at $x_0 = 0{\cdot}7$, $f(x_0)$ is negative and the slope $f'(x_0)$ is positive.

Having established that $x_1 = 0{\cdot}9345$, we now revert to the usual $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ for the rest of the calculation. Complete the table therefore and obtain the required root.

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The final spreadsheet display is

n	x	$f(x)$	$f'(x)$	$f''(x)$
0	0·7	−0·044	0·05	1·6
1	0·934521	0·024625	0·590233	
2	0·892801	0·002544	0·469997	
3	0·887387	4·02E-05	0·45516	
4	0·887298	1·06E-08	0·454919	
5	0·887298	9·16E-16	0·454919	

Therefore to six decimal places the required root is $x = 0{\cdot}887298$.

Note that we only used the modified method to find x_1. After that the normal relationship is used.

And now . . .

To date our task has been to find a value of x that satisfies an explicit equation $f(x) = 0$. This is quite general because *any* equation in x can be written in this form. For example, the equation

$$\sin x = x - e^{3x}$$

can always be written as

$$\sin x - x + e^{3x} = 0$$

and then approached by one of the methods that we have discussed so far.

What we want to do now is to work the other way – given a value of x, to find the corresponding value of $f(x)$. If $f(x)$ is given explicitly then this is no problem, it is just a matter of substituting the value of x in the formula and working it out. However, many times a function exists but it is not given explicitly, as in the case of a set of readings compiled as a result of an experiment or practical test. We shall consider this problem in the following frames.

Next frame

Interpolation

59

When a function is defined by a well-understood expression such as

$$f(x) = 4x^3 - 3x^2 + 7$$

or

$$f(x) = 5\sin(\exp[x])$$

the values of the dependent variable $f(x)$ corresponding to given values of the independent variable x can be found by direct substitution. Sometimes, however, a function is not defined in this way but by a collection of ordered pairs of numbers.

Example 1

A function can be defined by the following set of data:

x	$f(x)$
1	4
2	14
3	40
4	88
5	164
6	274

Intermediate values, for example, $x = 2{\cdot}5$, can be estimated by a process called **interpolation**.

The value of $f(2{\cdot}5)$ will clearly lie between 14 and 40, the function values for $x = 2$ and $x = 3$.

Purely as an estimate, $f(2{\cdot}5) = \ldots\ldots\ldots\ldots$

What do you suggest?

60

27

1 Linear interpolation

If you gave the result as 27, you no doubt agreed that $x = 2{\cdot}5$ is midway between $x = 2$ and $x = 3$, and that therefore $f(2{\cdot}5)$ would be midway between 14 and 40, i.e. 27. This is the simplest form of interpolation, but there is no evidence that there is a linear relationship between x and $f(x)$, and the result is therefore suspect.

Of course, we could have estimated the function value at $x = 2{\cdot}5$ by other means, such as

....

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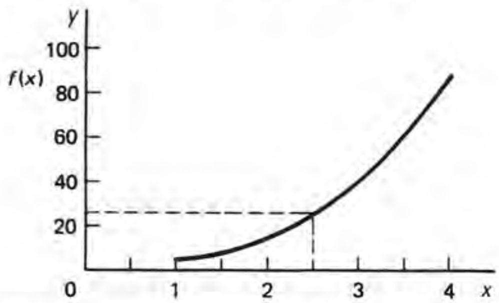
by drawing the graph of $f(x)$ against x

2 Graphical interpolation

We could, indeed, plot the graph of $f(x)$ against x and, from it, estimate the value of $f(x)$ at $x = 2{\cdot}5$.



This method is also approximate and time consuming.

$f(2{\cdot}5) \approx 26$

In what follows we shall look at interpolation using *finite differences*, which work well and quickly when the values of x are equally spaced. When the values of x are not equally spaced we need to resort to the more involved algebraic method called *Lagrangian interpolation* (which could also be used for equally spaced points).

Next frame

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3 Gregory–Newton interpolation formula using forward finite differences

x	$f(x)$
$\vdots$	$\vdots$
x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	$\vdots$

$\Delta f_0 = f(x_1) - f(x_0)$

We assume that $x_0, x_1, \ldots$ are distinct, equally spaced apart, and $x_0 < x_1 < \ldots$

For each pair of consecutive function values, $f(x_0)$ and $f(x_1)$, in the table, the *forward difference* Δf_0 is calculated by subtracting $f(x_0)$ from $f(x_1)$. This difference is written in a third column of the table, midway between the lines carrying $f(x_0)$ and $f(x_1)$.

x	$f(x)$	Δf
1	4	
		10
2	14	
		26
3	40	
$\vdots$	$\vdots$	

Complete the table for the data given in Frame 59 which then becomes …………

63

x	$f(x)$	Δf
1	4	
		10
2	14	
		26
3	40	
		48
4	88	
		76
5	164	
		110
6	274	

We now form a fourth column, the forward differences of the values of Δf, denoted by $\Delta^2 f$, and again written midway between the lines of Δf. These are the second forward differences of $f(x)$.

So the table then becomes …………

64

x	$f(x)$	Δf	$\Delta^2 f$
1	4		
		10	
2	14		16
		26	
3	40		22
		48	
4	88		28
		76	
5	164		34
		110	
6	274		

A further column can now be added in like manner, giving the third differences, denoted by $\Delta^3 f$, so that we then have …………

65

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1	4			
		10		
2	14		16	
		26		6
3	40		22	
		48		6
4	88		28	
		76		6
5	164		34	
		110		
6	274			

Notice that the table has now been completed, for the third differences are constant and all subsequent differences would be zero.

Now we shall see how to use the table. So move on

66

To find $f(2{\cdot}5)$

	x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
	1	4			
			10		
$x_0 \rightarrow$	2	14		16	
h x_p			26		6
$x_1 \rightarrow$	3	40		22	
			48		6
	4	88		28	
			76		6
	5	164		34	
			110		
	6	274			

We have to find $f(2{\cdot}5)$. Therefore denote $\left.\begin{array}{l} x = 2 \text{ as } x_0 \\ x = 3 \text{ as } x_1 \end{array}\right\} x = 2{\cdot}5 \text{ as } x_p$

Let $h =$ the constant range between successive values of x,
i.e. $h = x_1 - x_0$

Express $(x_p - x_0)$ as a fraction of h, i.e. $p = \dfrac{x_p - x_0}{h}$, $\quad 0 < p < 1$

Therefore, in the case above, $h = 1$ and $p = \dfrac{2{\cdot}5 - 2{\cdot}0}{1} = 0{\cdot}5$.

All we now use from the table is the set of values underlined by the broken line drawn diagonally from $f(x_0)$.

So we have

$$p = \ldots\ldots\ldots\ldots; \quad f_0 = \ldots\ldots\ldots\ldots; \quad \Delta f_0 = \ldots\ldots\ldots\ldots;$$
$$\Delta^2 f_0 = \ldots\ldots\ldots\ldots; \quad \Delta^3 f_0 = \ldots\ldots\ldots\ldots$$

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$$\boxed{p = 0{\cdot}5 \quad f_0 = 14; \quad \Delta f_0 = 26; \quad \Delta^2 f_0 = 22; \quad \Delta^3 f_0 = 6}$$

Now we are ready to deal with the *Gregory–Newton forward difference interpolation formula*

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{1 \times 2}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{1 \times 2 \times 3}\Delta^3 f_0 + \ldots$$

This is sometimes written in operator form

$$f_p = \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} f_0$$

which you no doubt recognise as the binomial expansion of

$$f_p = (1 + \Delta)^p \times f_0$$

Substituting the values in the above example gives

$$f(2{\cdot}5) = f_p = \ldots\ldots\ldots\ldots$$

68

$$\boxed{24{\cdot}625}$$

Because

$$f_p = 14 + 0{\cdot}5(26) + \frac{0{\cdot}5(-0{\cdot}5)}{1 \times 2}(22) + \frac{0{\cdot}5(-0{\cdot}5)(-1{\cdot}5)}{1 \times 2 \times 3}(6)$$
$$= 14 + 13 - 2{\cdot}75 + 0{\cdot}375$$
$$= 27{\cdot}375 - 2{\cdot}75 = 24{\cdot}625$$

Comparing the results of the three methods we have discussed

(a) Linear interpolation $f(2{\cdot}5) = 27$

(b) Graphical interpolation $f(2{\cdot}5) = 26$

(c) Gregory–Newton formula $f(2{\cdot}5) = 24{\cdot}625$ – the true value

Example 2

x	$f(x)$
2	14
4	88
6	274
8	620
10	1174

It is required to determine the value of $f(x)$ at $x = 5{\cdot}5$.

In this case

$x_0 = \ldots\ldots\ldots\ldots$ $x_1 = \ldots\ldots\ldots\ldots$

$h = \ldots\ldots\ldots\ldots$ $p = \ldots\ldots\ldots\ldots$

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$$\boxed{x_0 = 4; \quad x_1 = 6; \quad h = 2; \quad p = 0{\cdot}75}$$

Because

$$h = x_1 - x_0 = 6 - 4 = 2$$
$$p = \frac{x_p - x_0}{h} = \frac{5{\cdot}5 - 4}{2} = \frac{1{\cdot}5}{2} = 0{\cdot}75$$

First compile the table of forward differences …………

70

	x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
	2	14			
			74		
$x_0 \longrightarrow$	4	88		112	
			186		48
$x_1 \longrightarrow$	6	274		160	
			346		48
	8	620		208	
			554		
	10	1174			

The Gregory–Newton forward difference interpolation formula is

$$f_p = (1 + \Delta)^p \times f_0$$

i.e. $f_p = \ldots\ldots\ldots\ldots$

71

$$
\boxed{\begin{aligned} f_p &= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} f_0 \\ &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \ldots \end{aligned}}
$$

So, substituting the relevant values from the table, gives

$$f(5{\cdot}5) = f_p = \ldots\ldots\ldots\ldots$$

72

$$\boxed{214{\cdot}4}$$

Because

	x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
	2	14			
			74		
$x_0 \longrightarrow$	4	88		112	
$x_p \longrightarrow$			186		48
$x_1 \longrightarrow$	6	274		160	
			346		48
	8	620		208	
			554		
	10	1174			

$$
\begin{aligned} f(5{\cdot}5) = f_p &= 88 + 0{\cdot}75(186) + \frac{0{\cdot}75(-0{\cdot}25)}{1 \times 2}(160) \\ &\quad + \frac{0{\cdot}75(-0{\cdot}25)(-1{\cdot}25)}{1 \times 2 \times 3}(48) \\ &= 88 + 139{\cdot}5 - 15 + 1{\cdot}875 = 214{\cdot}375 \\ \therefore\ f(5{\cdot}5) &= 214{\cdot}4 \end{aligned}
$$

Finally, one more.

Example 3

Determine the value of $f(-1)$ from the set of function values.

x	-4	-2	0	2	4	6	8
$f(x)$	541	55	1	-53	-155	31	1225

Complete the working and then check with the next frame.

73

$$\boxed{f(-1) = 10}$$

Here is the working; method as before.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
-4	541				
		-486			
$x_0 \rightarrow$ -2	55		432		
$x_p \rightarrow$		-54		-432	
$x_1 \rightarrow$ 0	1		0		384
		-54		-48	
2	-53		-48		384
		-102		336	
4	-155		288		384
		186		720	
6	31		1008		
		1194			
8	1225				

$x_0 = -2$; $x_1 = 0$; $x_p = -1$; $\quad \therefore\ h = 2$; $p = \frac{1}{2}$

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{1 \times 2}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{1 \times 2 \times 3}\Delta^3 f_0 + \frac{p(p-1)(p-2)(p-3)}{1 \times 2 \times 3 \times 4}\Delta^4 f_0$$

$$= 55 + \frac{1}{2}(-54) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \times 2}(0) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \times 2 \times 3}(-48) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \times 2 \times 3 \times 4}(384)$$

$$= 55 - 27 + 0 - 3 - 15 = 10$$

$\therefore\ f_p = f(-1) = 10$

This table of data does have its restrictions. For example, if we had wanted to find $f(2{\cdot}5)$ from the table we would have run out of data because there is no $\Delta^4 f$ entry available. In such a case we can resort to a zig-zag path through the table using **central differences**.

Next frame

Central differences

74

The central difference operator δ is defined by its action on the expression $f(x)$ as

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

and using this operator the interpolated value of $f(x)$ near to the given value of f_0 is defined by the **Gauss forward** formula as

$$f_p = f_0 + p\delta f_{0+\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0+\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \dots$$

or by the **Gauss backward** formula as

$$f_p = f_0 + p\delta f_{0-\frac{1}{2}} + \frac{(p+1)p}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0-\frac{1}{2}} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 f_0 + \dots$$

There are no tabulated values at the half-interval values $x_0 + h/2$ and $x_0 - h/2$ and so these are taken to be the differences evaluated at mid-interval as given in the forward difference table. This means that the tables for the Gregory–Newton forward differences and the central differences are identical (apart, that is, from the column headings); the method of tracing through the table, however, is different. For example, to find $f(2{\cdot}5)$ for the example given in Frame 59

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$
1	4			
		10		
2	14		16	
		26		6
3	40		22	
		48		6
4	88		28	
		76		6
5	164		34	
		110		
6	274			

Here $x_0 = 2$, $f_0 = 14$, $\delta f_{0+\frac{1}{2}} = 26$, $\delta^2 f_0 = 16$, $\delta^3 f_{0+\frac{1}{2}} = 6$, $\delta^4 f_0 = 0$ and $p = 0{\cdot}5$. Thus

$$f_p = 14 + (0{\cdot}5)26 + \frac{(0{\cdot}5)(-0{\cdot}5)}{2}16 + \frac{(0{\cdot}5)(-0{\cdot}5)(-1{\cdot}5)}{6}6$$
$$= 14 + 13 - 2 - 0{\cdot}375 = 24{\cdot}625$$

which agrees with the value found using the Gregory–Newton forward difference formula.

Try one for yourself. The given tabulated values are

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$
0	−5			
		3		
1	−2		6	
		9		12
2	7		18	
		27		12
3	34		30	
		57		
4	91			

Using the Gauss forward difference formula, the interpolated value of

$f(2{\cdot}2) = \ldots\ldots\ldots\ldots$

Next frame

75

$\boxed{10{\cdot}576}$

Because

Using $f_p = f_0 + p\delta f_{0+\frac{1}{2}} + \dfrac{p(p-1)}{2!}\delta^2 f_0 + \dfrac{p(p-1)(p+1)}{3!}\delta^3 f_{0+\frac{1}{2}} + \ldots$ and following the solid line through the table where

$x_0 = 2,\quad f_0 = 7,\quad \delta f_{0+\frac{1}{2}} = 27,\quad \delta^2 f_0 = 18,\quad \delta^3 f_{0+\frac{1}{2}} = 12$ and $p = 0{\cdot}2$,

$$\text{then } f_p = 7 + (0{\cdot}2)27 + \frac{(0{\cdot}2)(-0{\cdot}8)}{2}18 + \frac{(0{\cdot}2)(-0{\cdot}8)(1{\cdot}2)}{6}12$$

$$= 7 + 5{\cdot}4 - 1{\cdot}44 - 0{\cdot}384$$

$$= 10{\cdot}576$$

Using the Gauss backward difference formula (following the broken line)

$$f_p = f_0 + p\delta f_{0-\frac{1}{2}} + \frac{p(p+1)}{2!}\delta^2 f_0 + \frac{p(p-1)(p+1)}{3!}\delta^3 f_{0-\frac{1}{2}} + \ldots$$

where here $\delta f_{0-\frac{1}{2}} = 9$ and $\delta^3 f_{0-\frac{1}{2}} = 12$ and so

$$f_p = 7 + (0{\cdot}2)9 + \frac{(0{\cdot}2)(1{\cdot}2)}{2}18 + \frac{(0{\cdot}2)(1{\cdot}2)(-0{\cdot}8)}{6}12$$
$$= 7 + 1{\cdot}8 + 2{\cdot}16 - 0{\cdot}384 = 10{\cdot}576$$

as found with the Gauss forward difference formula.

Next frame

Gregory–Newton backward differences

76

We have seen that the Gregory–Newton forward difference procedure loses terms if the interpolation is for points sufficiently forward in the table. We have also seen how this difficulty can be avoided by using central differences. However, even with central differences we can run out of data before completing a full traverse of the table. In such a situation we resort to the Gregory–Newton backward difference formula

$$f_p = f_0 + p\Delta f_{-1} + \frac{p(p+1)}{2!}\Delta^2 f_{-2} + \frac{p(p+1)(p+2)}{3!}\Delta^3 f_{-3} + \ldots$$

As an example, consider the table of Frame 74.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1	4			
		10		
2	14		16	
		26		6
3	40		22	
		48		6
4	88		28	
		76		6
5	164		34	
		110		
6	274			

Using this table we can calculate $f(5{\cdot}5)$ by tracing back through the table (see broken line) as

$$\begin{aligned} f(5{\cdot}5) &= f_0 + (0{\cdot}5)\Delta f_{-1} + \frac{(0{\cdot}5)(1{\cdot}5)}{2}\Delta^2 f_{-2} + \frac{(0{\cdot}5)(1{\cdot}5)(2{\cdot}5)}{6}\Delta^3 f_{-3} \\ &= 164 + (0{\cdot}5)76 + \frac{(0{\cdot}5)(1{\cdot}5)28}{2} + \frac{(0{\cdot}5)(1{\cdot}5)(2{\cdot}5)6}{6} \\ &= 214{\cdot}375 \end{aligned}$$

77

In each of the examples that we have looked at so far the tabular display of differences eventually results in a column of zeros and this determines the number of terms in an interpolation calculation. The zeros have arisen because all the examples have been derived from polynomials. The following example deals with a tabular display of differences which does not result in a column of zeros. In this case the number of terms used in the interpolation calculation determines confidence in the accuracy of the result.

Example

Use the Gregory–Newton forward difference method to find $f(0{\cdot}15)$ to 4 decimal places from the following finite difference table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	0·000000			
		0·099833		
0·1	0·099833		−0·000998	
		0·098836		−0·000988
0·2	0·198669		−0·001985	
		0·096851		−0·000968
0·3	0·295520		−0·002953	
		0·093898		−0·000938
0·4	0·389418		−0·003891	
		0·090007		
0·5	0·479426			

Here $x_0 = 0{\cdot}1$, $x_1 = 0{\cdot}2$, $x_p = 0{\cdot}15$ and therefore $p = 0{\cdot}5$, and

$$\begin{aligned}
f_p &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \dots \\
&= 0{\cdot}099833 + \frac{1}{2}(0{\cdot}098836) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-0{\cdot}001985)/2 \\
&\quad + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-0{\cdot}000969)/6 + \dots \\
&= 0{\cdot}099833 + 0{\cdot}049418 + 0{\cdot}000248 - 0{\cdot}000061 + \dots \\
&= 0{\cdot}1494 \text{ to 4 dp}
\end{aligned}$$

As you can see, the calculation can continue indefinitely and termination is dictated by the number of decimal places required in the final answer.

Lagrange interpolation

78

If the straight line $p(x) = a_0 + a_1 x$ passes through the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, where a_0 and a_1 are constants, then the equation for this line can also be written as

$$p(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

For example, the straight line $p(x) = 3 + 2x$ passes through the two points (1, 5) and (2, 7). Substituting the values for the variables in the above equation demonstrates this alternative form for the equation

$$p(x) = \frac{x - 2}{1 - 2} 5 + \frac{x - 1}{2 - 1} 7 = 10 - 5x + 7x - 7 = 3 + 2x$$

So, given the two data points from Frame 59, (2, 14) and (3, 40), using linear interpolation

$$f(2{\cdot}5) \approx p(2{\cdot}5) = \ldots\ldots\ldots\ldots$$

79

$$\boxed{27}$$

Because

$$\begin{aligned} p(x) &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \\ &= \frac{x - 3}{2 - 3} 14 + \frac{x - 2}{3 - 2} 40 = 26x - 38 \end{aligned}$$

and so

$$f(2{\cdot}5) \approx p(x) = 26(2{\cdot}5) - 38 = 27$$

80

The principle of Lagrange interpolation is that a function $f(x)$ whose values are given at a collection of points is assumed to be approximately represented by a polynomial $p(x)$ that passes through each and every point. The polynomial is called the **interpolation polynomial** and it is of degree one less than the number of points given. For two data points the interpolating polynomial is taken to be a linear polynomial, as you have just seen in the last example. For three data points the interpolating polynomial is taken to be a quadratic, for four data points the interpolation polynomial is taken to be a cubic, and so on.

▶

In the same manner as before it can be shown that the quadratic

$$p(x) = a_0 + a_1x + a_2x^2$$

that passes through the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written as

$$p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}f(x_1)$$
$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}f(x_2)$$

So let's try one. Given the collection of values

x	$f(x)$
1·5	0·405
2·1	0·742
3	1·099

by Lagrangian interpolation, $f(1{\cdot}8) \approx$ .... to 2 decimal places

81

0·58

Because

$$p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}f(x_1)$$
$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}f(x_2)$$
$$= \frac{(x - 2{\cdot}1)(x - 3)}{(1{\cdot}5 - 2{\cdot}1)(1{\cdot}5 - 3)}0{\cdot}405 + \frac{(x - 1{\cdot}5)(x - 3)}{(2{\cdot}1 - 1{\cdot}5)(2{\cdot}1 - 3)}0{\cdot}742$$
$$+ \frac{(x - 1{\cdot}5)(x - 2{\cdot}1)}{(3 - 1{\cdot}5)(3 - 2{\cdot}1)}1{\cdot}099$$
$$= \frac{(x^2 - 5{\cdot}1x + 6{\cdot}3)}{0{\cdot}9}0{\cdot}405 + \frac{(x^2 - 4{\cdot}5x + 4{\cdot}5)}{(-0{\cdot}54)}0{\cdot}742$$
$$+ \frac{(x^2 - 3{\cdot}6x + 3{\cdot}15)}{1{\cdot}35}1{\cdot}099$$
$$= -0{\cdot}11x^2 + 0{\cdot}958x - 0{\cdot}784$$

So that

$$f(1{\cdot}8) \approx p(1{\cdot}8) = 0{\cdot}58 \text{ to 2 decimal places.}$$

By carefully considering the interpolating polynomials for two and three data points you should be able to see a pattern. Write down what you think the interpolating polynomial should be for four data points:

....

82

$$p(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

Use this interpolating polynomial for the data points

x	$f(x)$
1	0·368
1·2	0·301
1·3	0·273
1·5	0·223

To 2 decimal places, $f(1{\cdot}4) \approx$ ....

83

0·25

Because $p(x)$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$= \frac{(x-1{\cdot}2)(x-1{\cdot}3)(x-1{\cdot}5)}{(1-1{\cdot}2)(1-1{\cdot}3)(1-1{\cdot}5)}0{\cdot}368 + \frac{(x-1)(x-1{\cdot}3)(x-1{\cdot}5)}{(1{\cdot}2-1)(1{\cdot}2-1{\cdot}3)(1{\cdot}2-1{\cdot}5)}0{\cdot}301 + \frac{(x-1)(x-1{\cdot}2)(x-1{\cdot}5)}{(1{\cdot}3-1)(1{\cdot}3-1{\cdot}2)(1{\cdot}3-1{\cdot}5)}0{\cdot}273 + \frac{(x-1)(x-1{\cdot}2)(x-1{\cdot}3)}{(1{\cdot}5-1)(1{\cdot}5-1{\cdot}2)(1{\cdot}5-1{\cdot}3)}0{\cdot}223$$

$$= \frac{(x^3-4x^2+5{\cdot}31x-2{\cdot}34)}{(-0{\cdot}03)}0{\cdot}368 + \frac{(x^3-3{\cdot}8x^2+4{\cdot}75x-1{\cdot}95)}{0{\cdot}006}0{\cdot}301 + \frac{(x^3-3{\cdot}7x^2+4{\cdot}5x-1{\cdot}8)}{(-0{\cdot}006)}0{\cdot}273 + \frac{(x^3-3{\cdot}5x^2+4{\cdot}06x-1{\cdot}56)}{0{\cdot}03}0{\cdot}223$$

$$= -0{\cdot}167x^3 + 0{\cdot}767x^2 - 1{\cdot}415x + 1{\cdot}183$$

So that

$f(1{\cdot}4) \approx p(1{\cdot}4) = 0{\cdot}25$ to 2 decimal places

The general Lagrange interpolation polynomial for $n+1$ data points at $x_0, x_1, \dots, x_n$ is

$$p(x) = \frac{(x-x_1)(x-x_2)(\cdots)(x-x_n)}{(x_0-x_1)(x_0-x_2)(\cdots)(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)(\cdots)(x-x_n)}{(x_1-x_0)(x_1-x_2)(\cdots)(x_1-x_n)}f(x_1) + \cdots \cdots + \frac{(x-x_0)(x-x_1)(\cdots)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(\cdots)(x_n-x_{n-1})}f(x_n)$$

▶

This now completes the work of this Programme. What follows is a **Revision summary** and a **Can You?** checklist. Read the summary carefully and respond to the questions in the checklist. When you feel sure that you are happy with the content of this Programme, try the **Test exercise**. Take your time, there is no need to hurry. Finally, a collection of **Further problems** provides valuable additional practice.

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Revision summary 1

1 The *Fundamental Theorem of Algebra* can be stated as follows:
Every polynomial expression $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can be written as a product of n linear factors in the form
$f(x) = a_n(x - r_1)(x - r_2)(\cdots)(x - r_n)$

2 *Relations between the coefficients and the roots of a polynomial equation*
Whenever a polynomial with *real coefficients* a_i has a complex root it also has the complex conjugate as another root.
If α, β, γ, ... are the roots of the equation

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

then, provided $p_0 \neq 0$

sum of roots $= -\dfrac{p_1}{p_0}$

sum of the product of the roots, taken two at a time $= \dfrac{p_2}{p_0}$

sum of the product of the roots, taken three at a time $= -\dfrac{p_3}{p_0}$

sum of the product of the roots, taken n at a time $= (-1)^n \dfrac{p_n}{p_0}$.

3 *Cubic equations*
Every cubic equation with real coefficients has at least one real root that can be found by **Tartaglia's solution**. The real root of $x^3 + ax + b = 0$, $a > 0$ is

$$x = \left\{-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3} + \left\{-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3}$$

Reduced form
Every cubic equation of the form $x^3 + ax^2 + bx + c = 0$ can be written in reduced form $y^3 + py + q = 0$ by using the transformation $x = y - \dfrac{a}{3}$.

4 *Numerical methods*
Bisection
The bisection method of finding a solution to the equation $f(x) = 0$ consists of

Finding a value of x such that $f(x) < 0$, say $x = a$
Finding a value of x such that $f(x) > 0$, say $x = b$.

The solution to the equation $f(x) = 0$ must then lie between a and b. Furthermore, it must lie either in the first half of the interval between a and b or in the second half.

5 *Numerical solution of equations by iteration*
The process of finding the numerical solution to the equation

$$f(x) = 0$$

by iteration is performed by first finding an approximate solution and then using this approximate solution to find a more accurate solution. This process is repeated until a solution is found to the required level of accuracy.

6 *Using a spreadsheet*
Iteration procedures are more efficiently performed using a spreadsheet.

7 *Newton–Raphson iteration method*
If $x = x_0$ is an approximate solution to the equation $f(x) = 0$, a better approximation $x = x_1$ is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \text{ and in general } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

8 *Modified Newton–Raphson iteration method*
If, in the Newton–Raphson procedure $f'(x_0)$ is sufficiently small enough to cause the value of x_1 to be a worse approximation to the solution than x_0, then x_1 is obtained from the relationship

$$x_1 = x_0 \pm \sqrt{\frac{-2f(x_0)}{f''(x_0)}}$$

Subsequent iterations then use $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

9 *Interpolation*
Linear
Graphical

10 *Gregory–Newton interpolation formulas using central finite differences*

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_o + \cdots$$

11 *Gauss interpolation formulas using central finite differences*
Gauss forward formula

$$f_p = f_0 + p\delta f_{0+\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0+\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \cdots$$

Gauss backward formula

$$f_p = f_0 + p\delta f_{0-\frac{1}{2}} + \frac{(p+1)p}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{0-\frac{1}{2}} + \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 f_0 + \cdots$$

▶

12 *Gregory–Newton interpolation formula using backward finite differences*

$$f_p = f_0 + p\Delta f_{-1} + \frac{p(p+1)}{2!}\Delta^2 f_{-2} + \frac{p(p+1)(p+2)}{3!}\Delta^3 f_{-3} + \cdots$$

13 *Lagrange interpolation*

If the straight line $p(x) = a_0 + a_1 x$ passes through the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, where a_0 and a_1 are constants, then the interpolation polynomial (straight line) for this line can be written as

$$p(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

The quadratic interpolating polynomial that passes through the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written as

$$p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

The cubic interpolating polynomial that passes through the four data points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $(x_3, f(x_3))$ can be written as

$$\begin{aligned} p(x) = {} & \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) \\ & + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) \\ & + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) \\ & + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3) \end{aligned}$$

The interpolating polynomial that passes through $n+1$ data points is

$$\begin{aligned} p(x) = {} & \frac{(x - x_1)(x - x_2)(\cdots)(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(\cdots)(x_0 - x_n)} f(x_0) \\ & + \frac{(x - x_0)(x - x_2)(\cdots)(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(\cdots)(x_1 - x_n)} f(x_1) + \cdots \\ & \cdots + \frac{(x - x_0)(x - x_1)(\cdots)(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(\cdots)(x_n - x_{n-1})} f(x_n) \end{aligned}$$

Can You?

Checklist 1

85

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Appreciate the Fundamental Theorem of Algebra?	1 to 3
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Find the two roots of a quadratic equation and recognise that for polynomial equations with real coefficients complex roots exist in complex conjugate pairs?	4 to 6
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use the relationships between the coefficients and the roots of a polynomial equation to find the roots of the polynomial?	7 to 17
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Transform a cubic equation to reduced form?	18 to 20
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use Tartaglia's solution to find the real root of a cubic equation?	21 and 22
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Find the solution of the equation $f(x) = 0$ by the method of bisection?	23 to 26
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Solve equations involving a single real variable by iteration and use a spreadsheet for efficiency?	27 to 33
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Solve equations using the Newton–Raphson iterative method?	34 to 52
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use the modified Newton–Raphson method to find the first approximation when the derivative is small?	53 to 58
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Understand the meaning of interpolation and use simple linear and graphical interpolation?	59 to 61
Yes ☐ ☐ ☐ ☐ ☐ *No*	

▶

- Use the Gregory–Newton interpolation formula using forward and backward differences for equally spaced domain points? 62 to 73

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the Gauss interpolation formulas using central differences for equally spaced domain points? 74 to 77

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use Lagrange interpolation when the domain points are not equally spaced? 78 to 83

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 1

86

1 Given that $x = -1 + j\sqrt{3}$ is one root of a quadratic equation with real coefficients, find the other root and hence the quadratic equation.

2 Solve the cubic equation $2x^3 - 7x^2 - 42x + 72 = 0$.

3 Write the cubic $3x^3 + 5x^2 + 3x + 5$ in reduced form and use Tartaglia's method to find the real root.

4 Use the method of bisection to find a solution to $x^3 - 5 = 0$ correct to 4 significant figures.

5 Use the Newton–Raphson method to find a positive solution of the following equation, correct to 6 decimal places:

$$\cos 3x = x^2$$

6 Use the modified Newton–Raphson method to find the solution correct to 6 decimal places near to $x = 2$ of the equation

$$x^3 - 6x^2 + 13x - 9 = 0$$

7 Given the table of values

x	$f(x)$
1	0
2	19
3	70
4	171
5	340
6	595

estimate

(a) $f(2{\cdot}5)$ using the Gregory–Newton forward difference formula
(b) $f(3{\cdot}4)$ using the Gauss central difference formula
(c) $f(5{\cdot}6)$ using the Gregory–Newton backward difference formula. ▶

8 Given the table of values

x	$f(x)$
1	4
2	−9
5	−108

use Lagrangian interpolation to estimate the value of $f(2{\cdot}2)$.

Further problems 1

87

1 Given that $x = \dfrac{-1 - j\sqrt{3}}{2}$ and $x = \dfrac{-1 + j}{\sqrt{2}}$ are two roots of a quartic equation with real coefficients, find the other two roots and hence the quartic equation.

2 Solve the equation $x^3 - 5x^2 - 8x + 12 = 0$, given that the sum of two of the roots is 7.

3 Find the values of the constants p and q such that the function $f(x) = 2x^3 + px^2 + qx + 6$ may be exactly divisible by $(x - 2)(x + 1)$.

4 If $f(x) = 4x^4 + px^3 - 23x^2 + qx + 11$ and when $f(x)$ is divided by $2x^2 + 7x + 3$ the remainder is $3x + 2$, determine the values of p and q.

5 If one root of the equation $x^3 - 2x^2 - 9x + 18 = 0$ is the negative of another, determine the three roots.

6 Solve the equation $x^3 - 7x^2 - 21x + 27 = 0$, given that the roots form a geometric sequence.

7 Form the equation whose roots are those of the equation $x^3 + x^2 + 9x + 9 = 0$ each increased by 2.

8 Form the equation whose roots exceed by 3 the roots of the equation $x^3 - 4x^2 + x + 6 = 0$.

9 If the equation $4x^3 - 4x^2 - 5x + 3 = 0$ is known to have two roots whose sum is 2, solve the equation.

10 Solve the equation $x^3 - 10x^2 + 8x + 64 = 0$, given that the product of two of the roots is the negative of the third.

11 Form the equation whose roots exceed those of the equation $2x^3 - 3x^2 - 11x + 6 = 0$ by 2.

12 If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, prove that $\alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q$.

13 Using Tartaglia's solution, find the real root of the equation $2x^3+4x-5=0$ giving the result to 4 significant figures.

14 Solve the equation $x^3-6x-4=0$.

15 Rewrite the equation $x^3+6x^2+9x+4=0$ in reduced form and hence determine the three roots.

16 Show that the equation $x^3+3x^2-4x-6=0$ has a root between $x=1$ and $x=2$, and use the Newton–Raphson iterative method to evaluate this root to 4 significant figures.

17 Find the real root of the equations:

(a) $x^3+4x+3=0$ (b) $5x^3+2x-1=0$.

18 Solve the following equations:

(a) $x^3-5x+1=0$ (b) $x^3+2x-3=0$
(c) $x^3-4x+1=0$.

19 Express the following in reduced form and determine the roots:

(a) $x^3+6x^2+9x+5=0$
(b) $8x^3+20x^2+6x-9=0$
(c) $4x^3-9x^2+42x-10=0$.

20 Use the Newton–Raphson iterative method to solve the following.

(a) Show that a root of the equation $x^3+3x^2+5x+9=0$ occurs between $x=-2$ and $x=-3$. Evaluate the root to four significant figures.

(b) Show graphically that the equation $e^{2x}=25x-10$ has two real roots and find the larger root correct to four significant figures.

(c) Verify that the equation $x-\cos x=0$ has a root near to $x=0{\cdot}8$ and determine the root correct to three significant figures.

(d) Obtain graphically an approximate root of the equation $2\ln x=3-x$. Evaluate the root correct to four significant figures.

(e) Verify that the equation $x^4+5x-20=0$ has a root at approximately $x=1{\cdot}8$. Determine the root correct to five significant figures.

(f) Show that the equation $x+3\sin x=2$ has a root between $x=0{\cdot}4$ and $x=0{\cdot}6$. Evaluate the root correct to five significant figures.

(g) The equation $2\cos x=e^x-1$ has a real root between $x=0{\cdot}8$ and $x=0{\cdot}9$. Evaluate the root correct to four significant figures.

(h) The equation $20x^3-22x^2+5x-1=0$ has a root at approximately $x=0{\cdot}6$. Determine the value of the root correct to four significant figures.

▶

21 A polynomial function is defined by the following set of function values

x	2	4	6	8	10
$y = f(x)$	$-7{\cdot}00$	$9{\cdot}00$	$97{\cdot}0$	305	681

Find

(a) $f(4{\cdot}8)$ using the Gregory–Newton forward difference formula

(b) $f(7{\cdot}2)$ using the Gauss central difference formula

(c) $f(8{\cdot}5)$ using the Gregory–Newton backward difference formula.

22 For the function $f(x)$

x	4	5	6	7	8	9	10
$f(x)$	-10	12	56	128	234	380	572

Find

(a) $f(4{\cdot}5)$ and $f(6{\cdot}4)$ using the Gregory–Newton forward difference formula

(b) $f(7{\cdot}1)$ and $f(8{\cdot}9)$ using the Gregory–Newton backward difference formula.

23

x	2	4	6	8	10	12
$f(x)$	-9	35	231	675	1463	2691

For the function defined in the table above, evaluate (a) $f(2{\cdot}6)$ and (b) $f(7{\cdot}2)$.

24 A function $f(x)$ is defined by the following table

x	-4	-2	0	2	4	6	8
$f(x)$	277	51	1	-17	-147	-533	-1319

Find

(a) $f(-3)$ and $f(1{\cdot}6)$ using the Gregory–Newton forward difference formula

(b) $f(0{\cdot}2)$ and $f(3{\cdot}1)$ using the Gauss central difference formula

(c) $f(4{\cdot}4)$ and $f(7)$ using the Gregory–Newton backward difference formula.

▶

25 Given the table of values

x	$f(x)$
-1	$-2{\cdot}71828$
3	$-0{\cdot}04979$
5	$-0{\cdot}00674$

use Lagrangian interpolation to find the value of $f(3{\cdot}4)$.

26 Given the table of values

x	$f(x)$
6	$0{\cdot}801153$
$7{\cdot}2$	$-0{\cdot}82236$
9	$-0{\cdot}73922$
13	$0{\cdot}994808$

use Lagrangian interpolation to find the value of $f(8)$.

27 Given the table of values

x	$f(x)$
-2	$-2{\cdot}63906$
0	$-2{\cdot}48491$
5	$-1{\cdot}94591$
6	$-1{\cdot}79176$

use Lagrangian interpolation to find the values of

(a) $f(-0{\cdot}8)$

(b) $f(0{\cdot}8)$

(c) $f(5{\cdot}5)$.

Laplace transforms 1

Frames 1 to 93

Learning outcomes

When you have completed this Programme you will be able to:

- Obtain the Laplace transforms of simple standard expressions
- Use the first shift theorem to find the Laplace transform of a simple expression multiplied by an exponential
- Find the Laplace transform of a simple expression multiplied or divided by a variable
- Use partial fractions to find the inverse Laplace transform
- Use the 'cover up' rule
- Use the Laplace transforms of derivatives to solve differential equations
- Use the Laplace transform to solve simultaneous differential equations

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme 26 Introduction to Laplace transforms

Introduction

1

The solution of a linear, ordinary differential equation with constant coefficients such as the second-order equation

$$af''(t) + bf'(t) + cf(t) = g(t)$$

can be solved by first obtaining the general form for the expression $f(t)$. This general form will contain a number of integration constants whose values can be found by applying the appropriate boundary conditions (see *Engineering Mathematics, Fifth Edition*, Programme 25). A more systematic way of solving such equations is to use the Laplace transform which converts the differential equation into an algebraic equation and has the added advantage of incorporating the boundary conditions from the beginning. Furthermore, in situations where $f(t)$ represents a function with discontinuities, the Laplace transform method can succeed where other methods fail.

Laplace transform techniques also provide powerful tools in numerous fields of technology such as Control Theory where a knowledge of the system transfer function is essential and where the Laplace transform comes into its own. Let us see what it is all about. (For a more detailed introduction see *Engineering Mathematics, Fifth Edition*, Programme 26.)

Laplace transforms

The Laplace transform of an expression $f(t)$ is denoted by $L\{f(t)\}$ and is defined as the semi-infinite integral

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt \qquad (1)$$

The parameter s is assumed to be positive and large enough to ensure that the integral converges. In more advanced applications s may be complex and in such cases the real part of s must be positive and large enough to ensure convergence.

In determining the transform of an expression, you will appreciate that the limits of the integral are substituted for t, so that the result will be an expression in s. Therefore

$$L\{f(t)\} = \int_{t=0}^{\infty} f(t)e^{-st}\,dt = F(s)$$

Make a note of this general definition: then we can apply it

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So we have $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t = F(s)$

Example 1

To find the Laplace transform of $f(t) = a$ (constant).

$$L\{a\} = \int_0^\infty ae^{-st}\mathrm{d}t = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}\left[e^{-st}\right]_0^\infty$$
$$= -\frac{a}{s}\{0-1\} = \frac{a}{s}$$
$$\therefore\ L\{a\} = \frac{a}{s} \qquad (s > 0) \qquad (2)$$

Example 2

To find the Laplace transform of $f(t) = e^{at}$ (a constant). As with all cases, we multiply $f(t)$ by e^{-st} and integrate between $t = 0$ and $t = \infty$.

$$\therefore\ L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t$$
$$= \ldots\ldots\ldots\ldots$$

Finish it off.

3

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

Because

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$
$$= -\frac{1}{s-a}\{0-1\} = \frac{1}{s-a}$$
$$\therefore\ L\{e^{at}\} = \frac{1}{s-a} \qquad (s > a) \qquad (3)$$

So we already have two standard transforms

$$L\{a\} = \frac{a}{s} \quad \text{and} \quad L\{e^{at}\} = \frac{1}{s-a}$$

$$\therefore\quad L\{4\} = \ldots\ldots\ldots\ldots; \qquad L\{e^{4t}\} = \ldots\ldots\ldots\ldots$$
$$L\{-5\} = \ldots\ldots\ldots\ldots; \qquad L\{e^{-2t}\} = \ldots\ldots\ldots\ldots$$

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$$L\{4\} = \frac{4}{s}; \qquad L\{e^{4t}\} = \frac{1}{s-4}$$
$$L\{-5\} = -\frac{5}{s}; \qquad L\{e^{-2t}\} = \frac{1}{s+2}$$

Note that, as we said earlier, the Laplace transform is always an expression in s.

Now for some more examples

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Example 3

To find the Laplace transform of $f(t) = \sin at$. We could, of course, apply the definition and evaluate

$$L\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\,dt$$

using integration by parts.

However, it is much shorter if we use the fact that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

so that $\sin\theta$ is the imaginary part of $e^{j\theta}$, written $\mathscr{I}(e^{j\theta})$.

The function $\sin at$ can therefore be written $\mathscr{I}(e^{jat})$ so that

$$L\{\sin at\} = L\{\mathscr{I}(e^{jat})\} = \mathscr{I}\int_0^\infty e^{jat}e^{-st}\,dt = \mathscr{I}\int_0^\infty e^{-(s-ja)t}\,dt$$
$$= \mathscr{I}\left\{\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right]_0^\infty\right\} = \mathscr{I}\left\{-\frac{1}{(s-ja)}[0-1]\right\}$$
$$= \mathscr{I}\left\{\frac{1}{s-ja}\right\}$$

We can rationalise the denominator by multiplying top and bottom by ....

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$$\boxed{s + ja}$$

$$\therefore\ L\{\sin at\} = \mathscr{I}\left\{\frac{s+ja}{s^2+a^2}\right\} = \frac{a}{s^2+a^2}$$
$$\therefore\ L\{\sin at\} = \frac{a}{s^2+a^2} \qquad (4)$$

We can use the same method to determine $L\{\cos at\}$ since $\cos at$ is the real part of e^{jat}, written $\Re(e^{jat})$.

Then $L\{\cos at\} = $ ....

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$$\boxed{L\{\cos at\} = \frac{s}{s^2 + a^2}} \qquad (5)$$

Because

$$L\{\cos at\} = \Re\left\{\frac{s + ja}{s^2 + a^2}\right\} = \frac{s}{s^2 + a^2}$$

Recapping then: $L\{1\} = \ldots\ldots\ldots\ldots;$ $\quad L\{e^{3t}\} = \ldots\ldots\ldots\ldots$

$L\{\sin 2t\} = \ldots\ldots\ldots\ldots;$ $\quad L\{\cos 4t\} = \ldots\ldots\ldots\ldots$

8

$$\boxed{\begin{aligned} L\{1\} &= \frac{1}{s}; & L\{e^{3t}\} &= \frac{1}{s-3} \\ L\{\sin 2t\} &= \frac{2}{s^2+4}; & L\{\cos 4t\} &= \frac{s}{s^2+16} \end{aligned}}$$

Example 4

To find the transform of $f(t) = t^n$ where n is a positive integer.

By the definition $\quad L\{t^n\} = \int_0^\infty t^n e^{-st}\,\mathrm{d}t.$

Integrating by parts

$$\begin{aligned} L\{t^n\} &= \left[t^n\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,\mathrm{d}t \\ &= -\frac{1}{s}\left[t^n e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t \end{aligned}$$

We said earlier that in a product such as $t^n e^{-st}$ the numerical value of s is large enough to make the product converge to zero as $t \to \infty$

$$\therefore \quad \left[t^n e^{-st}\right]_0^\infty = 0 - 0 = 0$$

$$\therefore \; L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t \qquad (6)$$

You will notice that $\int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t$ is identical to $\int_0^\infty t^n e^{-st}\,\mathrm{d}t$ except that n is replaced by $(n-1)$.

$\therefore$ If $I_n = \int_0^\infty t^n e^{-st}\,\mathrm{d}t$, then $I_{n-1} = \int_0^\infty t^{n-1}e^{-st}\,\mathrm{d}t$

and the result (6) becomes $I_n = \frac{n}{s}.I_{n-1} \qquad (7)$

This is a reduction formula, and if we now replace n by $(n-1)$ we get

$$I_{n-1} = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_{n-1} = \frac{n-1}{s}.I_{n-2}}$$

If we replace n by $(n-1)$ again in this last result, we have

$$I_{n-2} = \frac{n-2}{s}.I_{n-3}$$

$$\text{So } I_n = \int_0^\infty t^n e^{-st}\,\mathrm{d}t = \frac{n}{s}.I_{n-1}$$

$$= \frac{n}{s}.\frac{n-1}{s}.I_{n-2}$$

$$= \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.I_{n-3} \quad \text{etc.}$$

$$= \ldots\ldots\ldots\ldots \text{ (next line)}$$

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$$\boxed{I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}.I_{n-4}}$$

So finally, we have

$$I_n = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\frac{n-3}{s}\ldots\frac{n-(n-1)}{s}.I_0$$

But $\quad I_0 = L\{t^0\} = L\{1\} = \dfrac{1}{s}$

$$\therefore\ I_n = \frac{n(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$\therefore\ L\{t^n\} = \frac{n!}{s^{n+1}} \tag{8}$$

$$\therefore\ L\{t\} = \frac{1}{s^2};\quad L\{t^2\} = \frac{2}{s^3};\quad L\{t^3\} = \frac{6}{s^4}$$

and with $n = 0$, since $0! = 1$, the general result includes $L\{1\} = \dfrac{1}{s}$ which we have already established.

Example 5

Laplace transforms of $f(t) = \sinh at$ and $f(t) = \cosh at$.

Starting from the exponential definitions of $\sinh at$ and $\cosh at$, i.e.

$$\sinh at = \tfrac{1}{2}(e^{at} - e^{-at}) \quad \text{and} \quad \cosh at = \tfrac{1}{2}(e^{at} + e^{-at})$$

we proceed as follows.

(a) $f(t) = \sinh at.\quad L\{\sinh at\} = \displaystyle\int_0^\infty \sinh at\, e^{-st}\,\mathrm{d}t$

$$= \frac{1}{2}\int_0^\infty (e^{at} - e^{-at})e^{-st}\,\mathrm{d}t$$

$$= \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\}\mathrm{d}t$$

$$= \ldots\ldots\ldots\ldots$$

Complete it

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$$\boxed{L\{\sinh at\} = \frac{a}{s^2 - a^2}}$$

Because

$$\frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} \mathrm{d}t = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2}$$

$$\therefore \; L\{\sinh at\} = \frac{a}{s^2 - a^2} \qquad (9)$$

(b) $f(t) = \cosh at$. Proceeding in the same way

$$L\{\cosh at\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{\cosh at\} = \frac{s}{s^2 - a^2}}$$

$$L\{\cosh at\} = \frac{1}{2}\int_0^\infty (e^{at} + e^{-at})e^{-st}\,\mathrm{d}t = \frac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\}\mathrm{d}t$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{s^2 - a^2}\right\} = \frac{s}{s^2 - a^2}$$

$$\therefore \; L\{\cosh at\} = \frac{s}{s^2 - a^2} \qquad (10)$$

So we have accumulated several standard results:

$$L\{a\} = \frac{a}{s}; \qquad L\{e^{at}\} = \frac{1}{s-a}; \qquad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}; \qquad L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}; \qquad L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Make a note of this list if you have not already done so: it forms the basis of much that is to follow.

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The Laplace transform is a linear transform, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\} = kL\{f(t)\}$$

Note: Two transforms must **not** be multiplied together to form the transform of a product of expressions – we shall see later that the product of two transforms is the transform of the *convolution* of two expressions.

Example 6

(a) $$\begin{aligned} L\{2e^{-t} + t\} &= L\{2e^{-t}\} + L\{t\} \\ &= 2L\{e^{-t}\} + L\{t\} \\ &= \frac{2}{s+1} + \frac{1}{s^2} = \frac{2s^2+s+1}{s^2(s+1)} \end{aligned}$$

(b) $$\begin{aligned} L\{2\sin 3t + \cos 3t\} &= 2L\{\sin 3t\} + L\{\cos 3t\} \\ &= 2.\frac{3}{s^2+9} + \frac{s}{s^2+9} = \frac{s+6}{s^2+9} \end{aligned}$$

(c) $$\begin{aligned} L\{4e^{2t} + 3\cosh 4t\} &= 4L\{e^{2t}\} + 3L\{\cosh 4t\} \\ &= 4.\frac{1}{s-2} + 3.\frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16} \\ &= \frac{7s^2-6s-64}{(s-2)(s^2-16)} \end{aligned}$$

So 1. $L\{2\sin 3t + 4\sinh 3t\} = \ldots\ldots\ldots\ldots$

2. $L\{5e^{4t} + \cosh 2t\} = \ldots\ldots\ldots\ldots$

3. $L\{t^3 + 2t^2 - 4t + 1\} = \ldots\ldots\ldots\ldots$

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1. $\dfrac{18(s^2+3)}{s^4-81}$; 2. $\dfrac{6s^2-4s-20}{(s-4)(s^2-4)}$; 3. $\dfrac{1}{s^4}\{s^3-4s^2+4s+6\}$

The working is straightforward.

1. $$\begin{aligned} L\{2\sin 3t + 4\sinh 3t\} &= 2.\frac{3}{s^2+9} + 4.\frac{3}{s^2-9} \\ &= \frac{6}{s^2+9} + \frac{12}{s^2-9} = \frac{18(s^2+3)}{s^4-81} \end{aligned}$$

2. $$L\{5e^{4t} + \cosh 2t\} = \frac{5}{s-4} + \frac{s}{s^2-4} = \frac{6s^2-4s-20}{(s-4)(s^2-4)}$$

▶

3. $L\{t^3 + 2t^2 - 4t + 1\} = \frac{3!}{s^4} + 2.\frac{2!}{s^3} - 4.\frac{1!}{s^2} + \frac{1}{s}$
$$= \frac{1}{s^4}\{s^3 - 4s^2 + 4s + 6\}$$

We have been building up a list of standard transforms of simple expressions. Before we leave this part of the work, there are three important and useful theorems which enable us to deal with rather more complicated expressions.

Theorem 1 The first shift theorem

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The first shift theorem states that if $L\{f(t)\} = F(s)$ then

$L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,\mathrm{d}t = F(s+a)$

That is

$$L\{e^{-at}f(t)\} = F(s+a)$$

The transform $L\{e^{-at}f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

For example $L\{\sin 2t\} = \frac{2}{s^2+4}$

then $L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$

Similarly, $L\{t^2\} = \frac{2}{s^3}$ $\therefore$ $L\{t^2e^{4t}\} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{2}{(s-4)^3}}$$

Because $L\{t^2\} = \frac{2}{s^3}$. $\therefore$ $L\{t^2e^{4t}\}$ is the same with s replaced by $(s-4)$.

$\therefore$ $L\{t^2e^{4t}\} = \frac{2}{(s-4)^3}$

Here is a short exercise by way of practice.

Exercise

Determine the following.

1. $L\{e^{-2t}\cosh 3t\}$
2. $L\{2e^{3t}\sin 3t\}$
3. $L\{4te^{-t}\}$
4. $L\{e^{2t}\cos t\}$
5. $L\{e^{3t}\sinh 2t\}$
6. $L\{t^3e^{-4t}\}$

Complete all six and then check with the results in the next frame

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Here they are.

1. $L\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$ $\quad \therefore L\{e^{-2t}\cosh 3t\} = \dfrac{s+2}{(s+2)^2 - 9} = \dfrac{s+2}{s^2 + 4s - 5}$

2. $L\{\sin 3t\} = \dfrac{3}{s^2 + 9}$ $\quad \therefore L\{2e^{3t}\sin 3t\} = \dfrac{6}{(s-3)^2 + 9} = \dfrac{6}{s^2 - 6s + 18}$

3. $L\{4t\} = 4.\dfrac{1}{s^2}$ $\quad \therefore L\{4te^{-t}\} = \dfrac{4}{(s+1)^2}$

4. $L\{\cos t\} = \dfrac{s}{s^2 + 1}$ $\quad \therefore L\{e^{2t}\cos t\} = \dfrac{s-2}{(s-2)^2 + 1} = \dfrac{s-2}{s^2 - 4s + 5}$

5. $L\{\sinh 2t\} = \dfrac{2}{s^2 - 4}$ $\quad \therefore L\{e^{3t}\sinh 2t\} = \dfrac{2}{(s-3)^2 - 4} = \dfrac{2}{s^2 - 6s + 5}$

6. $L\{t^3\} = \dfrac{3!}{s^4}$ $\quad \therefore L\{t^3 e^{-4t}\} = \dfrac{6}{(s+4)^4}$

Now let us deal with the next theorem

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Theorem 2 Multiplying by *t* and *t*^{n}

If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -F'(s)$

Because $L\{tf(t)\} = \int_{t=0}^{\infty} tf(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)\left(-\dfrac{\mathrm{d}e^{-st}}{\mathrm{d}s}\right)\mathrm{d}t$

$$= -\frac{\mathrm{d}}{\mathrm{d}s}\int_{t=0}^{\infty} f(t)e^{-st}\,\mathrm{d}t = -F'(s)$$

That is

$$L\{tf(t)\} = -F'(s)$$

For example, $\quad L\{\sin 2t\} = \dfrac{2}{s^2 + 4}$

$$\therefore L\{t\sin 2t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

and similarly, $\quad L\{t\cosh 3t\} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{s^2+9}{(s^2-9)^2}}$$

Because $L\{t\cosh 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = -\frac{(s^2-9)-s(2s)}{(s^2-9)^2} = \frac{s^2+9}{(s^2-9)^2}$

We could, if necessary, take this a stage further and find $L\{t^2\cosh 3t\}$

$$L\{t^2\cosh 3t\} = L\{t(t\cosh 3t)\} = -\frac{d}{ds}\left\{\frac{s^2+9}{(s^2-9)^2}\right\}$$
$$= \frac{2s(s^2+27)}{(s^2-9)^3}$$

Likewise, starting with $L\{\sin 4t\} = \frac{4}{s^2+16}$

$L\{t\sin 4t\} = \ldots\ldots\ldots\ldots$ and $L\{t^2\sin 4t\} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{8s}{(s^2+16)^2};\quad \frac{8(3s^2-16)}{(s^2+16)^3}}$$

applying $L\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$ in each case.

Theorem 2 obviously extends the range of functions that we can deal with.

So, in general, if $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Make a note of this in your record book

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Theorem 3 Dividing by *t*

If $L\{f(t)\} = F(s)$ then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma$

provided $\displaystyle\underset{t\to 0}{Lim}\left(\frac{f(t)}{t}\right)$ exists. To demonstrate this we start from the right-hand side of the result

$$\begin{aligned}\int_{\sigma=s}^{\infty} F(\sigma)\,d\sigma &= \int_{\sigma=s}^{\infty}\left\{\int_{t=0}^{\infty} f(t)e^{-\sigma t}\,dt\right\}d\sigma\\ &= \int_{t=0}^{\infty}\int_{\sigma=s}^{\infty} f(t)e^{-\sigma t}\,d\sigma\,dt\\ &= \int_{t=0}^{\infty} f(t)\left\{\int_{\sigma=s}^{\infty} e^{-\sigma t}\,d\sigma\right\}dt\\ &= \int_{t=0}^{\infty} f(t)\frac{e^{-st}}{t}\,dt\\ &= L\left\{\frac{f(t)}{t}\right\}\end{aligned}$$

Notice the dummy variable σ. The end result is an expression in s which comes from the lower limit of the integral so the variable of integration, which is absorbed during the process of integration, is changed to σ. Notice also that we interchange the order of integration.

This rule is somewhat restricted in use, since it is applicable only if $\displaystyle\underset{t\to 0}{Lim}\left(\frac{f(t)}{t}\right)$ exists. In indeterminate cases, we use L'Hôpital's rule to find out. Let's try a couple of examples.

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Example 1

Determine $L\left\{\dfrac{\sin at}{t}\right\}$

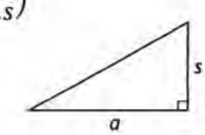
First we test $\displaystyle\underset{t\to 0}{Lim}\left\{\frac{\sin at}{t}\right\} = \left\{\frac{0}{0}\right\} = ?$ By L'Hôpital's rule, we differentiate top and bottom separately and substitute $t = 0$ in the result to ascertain the limit of the new expression.

$\displaystyle\underset{t\to 0}{Lim}\left\{\frac{\sin at}{t}\right\} = \underset{t\to 0}{Lim}\left\{\frac{a\cos at}{1}\right\} = a$, that is, the limit exists and the theorem can therefore be applied.

So $L\{\sin at\} = \dfrac{a}{s^2+a^2}$, therefore

$$\begin{aligned}L\left\{\frac{\sin at}{t}\right\} &= \int_s^{\infty}\frac{a}{\sigma^2+a^2}\,d\sigma\\ &= \left[\arctan\left(\frac{\sigma}{a}\right)\right]_s^{\infty}\\ &= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)\\ &= \arctan\left(\frac{a}{s}\right)\end{aligned}$$

Notice that $\arctan\left(\dfrac{a}{s}\right) + \arctan\left(\dfrac{s}{a}\right) = \dfrac{\pi}{2}$, as can be seen from the figure



▶

Example 2

Determine $L\left\{\dfrac{1-\cos 2t}{t}\right\}$

First we test whether $\lim\limits_{t\to 0}\left\{\dfrac{1-\cos 2t}{t}\right\}$ exists. Result? . . . .

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$$\boxed{\text{the limit exists}}$$

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\}=\frac{1-1}{0}=\frac{0}{0}=?\quad\therefore\text{ Apply l'Hôpital's rule.}$$

$$\lim_{t\to 0}\left\{\frac{1-\cos 2t}{t}\right\}=\lim_{t\to 0}\left\{\frac{2\sin 2t}{1}\right\}=\frac{0}{1}=0\quad\therefore\text{ limit exists.}$$

$$L\{1-\cos 2t\}=\frac{1}{s}-\frac{s}{s^2+4}$$

Then, by Theorem 3

$$L\left\{\frac{1-\cos 2t}{t}\right\}=\int_{\sigma=s}^{\infty}\left\{\frac{1}{\sigma}-\frac{\sigma}{\sigma^2+4}\right\}d\sigma$$

$$=\left[\ln\sigma-\frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^{\infty}=\frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^{\infty}$$

When $\sigma\to\infty$, $\ln\left(\dfrac{\sigma^2}{\sigma^2+4}\right)\to\ln 1=0$

$$\text{Therefore, } L\left\{\frac{1-\cos 2t}{t}\right\}=\ldots\ldots\ldots\ldots$$

Complete it

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$$\boxed{\ln\sqrt{\frac{s^2+4}{s^2}}}$$

Because

$$L\left\{\frac{1-\cos 2t}{t}\right\}=-\frac{1}{2}\ln\left(\frac{s^2}{s^2+4}\right)=\ln\left(\frac{s^2}{s^2+4}\right)^{-1/2}$$

$$=\ln\sqrt{\frac{s^2+4}{s^2}}$$

Let us pause here for a while and take stock, for we have met a number of results important in the future work.

▶

1 *Standard transforms*

$f(t)$	$L\{f(t)\} = F(s)$
a	$\dfrac{a}{s}$
e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$
t^n	$\dfrac{n!}{s^{n+1}}$

(n a positive integer)

2 *Theorem 1* The first shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$

3 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\dfrac{\mathrm{d}}{\mathrm{d}s}\{F(s)\}$

4 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_{\sigma=s}^{\infty} F(\sigma)\,\mathrm{d}\sigma$

provided $\displaystyle\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists.

Now let us work through a short revision exercise, so move on

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Exercise

Determine the Laplace transforms of the following expressions.

1 $\sin 3t$

2 $\cos 2t$

3 e^{4t}

4 $6t^2$

5 $\sinh 3t$

6 $t\cosh 4t$

7 $t^2 - 3t + 4$

8 $\dfrac{e^{3t}-1}{t}$

9 $e^{3t}\cos 4t$

10 $t^2 \sin t$

Complete the whole set and then check results with the next frame

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Here are the results.

1 $\dfrac{3}{s^2+9}$ **6** $\dfrac{s^2+16}{(s^2-16)^2}$

2 $\dfrac{s}{s^2+4}$ **7** $\dfrac{1}{s^3}(4s^2-3s+2)$

3 $\dfrac{1}{s-4}$ **8** $\ln\left(\dfrac{s}{s-3}\right)$

4 $\dfrac{12}{s^3}$ **9** $\dfrac{s-3}{s^2-6s+25}$

5 $\dfrac{3}{s^2-9}$ **10** $\dfrac{6s^2-2}{(s^2+1)^3}$

It is just a case of applying the standard tranforms and the three theorems.

Now on to the next piece of work

Inverse transforms

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Here we have the reverse process, i.e. given a Laplace transform, we have to find the function of t to which it belongs.

For example, we know that $\dfrac{a}{s^2+a^2}$ is the Laplace transform of $\sin at$, so we can now write $L^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$, the symbol L^{-1} indicating the inverse transform and **not** a reciprocal.

$\therefore$ (a) $L^{-1}\left\{\dfrac{1}{s-2}\right\} = \dots\dots\dots\dots$; (c) $L^{-1}\left\{\dfrac{4}{s}\right\} = \dots\dots\dots\dots$

(b) $L^{-1}\left\{\dfrac{s}{s^2+25}\right\} \dots\dots\dots\dots$; (d) $L^{-1}\left\{\dfrac{12}{s^2-9}\right\} = \dots\dots\dots\dots$

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(a) $L^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$; (c) $L^{-1}\left\{\dfrac{4}{s}\right\} = 4$

(b) $L^{-1}\left\{\dfrac{s}{s^2+25}\right\} = \cos 5t$; (d) $L^{-1}\left\{\dfrac{12}{s^2-9}\right\} = 4\sinh 3t$

Therefore, given a transform, we can write down the corresponding expression in t, provided we can recognise it from our table of transforms.

▶

But what about $L^{-1}\left\{\dfrac{3s+1}{s^2-s-6}\right\}$? This certainly did not appear in our list of standard transforms.

In considering $L^{-1}\left\{\dfrac{3s+1}{s^2-s-6}\right\}$, it happens that we can write $\dfrac{3s+1}{s^2-s-6}$ as the sum of two simpler functions $\dfrac{1}{s+2}+\dfrac{2}{s-3}$ which, of course, makes all the difference, since we can now proceed

$$L^{-1}\left\{\frac{3s+1}{s^2-s-6}\right\}=L^{-1}\left\{\frac{1}{s+2}+\frac{2}{s-3}\right\}$$

which we immediately recognise as ....

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$$\boxed{e^{-2t}+2e^{3t}}$$

The two simpler expressions $\dfrac{1}{s+2}$ and $\dfrac{2}{s-3}$ are called the *partial fractions* of $\dfrac{3s+1}{s^2-s-6}$, and the ability to represent a complicated algebraic fraction in terms of its partial fractions is the key to much of this work. Let us take a closer look at the rules.

Rules of partial fractions

1. The numerator must be of lower degree than the denominator. This is usually the case in Laplace transforms. If it is not, then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fractions.
3. A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where A is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\dfrac{A}{(s+a)}+\dfrac{B}{(s+a)^2}$.
5. Similarly $(s+a)^3$ gives $\dfrac{A}{(s+a)}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$.
6. A quadratic factor (s^2+ps+q) gives $\dfrac{Ps+Q}{s^2+ps+q}$.
7. Repeated quadratic factors $(s^2+ps+q)^2$ give

   $$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}.$$

So $\dfrac{s-19}{(s+2)(s-5)}$ has partial fractions of the form ....

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$$\boxed{\frac{A}{s+2}+\frac{B}{s-5}}$$

and $\dfrac{3s^2-4s+11}{(s+3)(s-2)^2}$ has partial fractions of the form ....

Be careful of the repeated factor.

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$$\boxed{\frac{A}{s+3}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}}$$

Let us work through the various steps with an example.

Example 1

To determine $L^{-1}\left\{\dfrac{5s+1}{s^2-s-12}\right\}$.

(a) First we check that the numerator is of lower degree than the denominator. In fact, this is so.

(b) Factorise the denominator $\dfrac{5s+1}{s^2-s-12}=\dfrac{5s+1}{(s-4)(s+3)}$.

(c) Then the partial fractions are of the form ....

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$$\boxed{\frac{A}{s-4}+\frac{B}{s+3}}$$

We therefore have the identity

$$\frac{5s+1}{s^2-s-12}\equiv\frac{A}{s-4}+\frac{B}{s+3}$$

If we multiply through both sides by the denominator $s^2-s-12\equiv(s-4)(s+3)$ we have

$$5s+1\equiv A(s+3)+B(s-4)$$

This is also an identity and true for any value of s we care to substitute – our job is now to find the values of A and B.

We now substitute convenient values for s

(a) Let $(s-4)=0$, i.e. $s=4$ $\quad\therefore\ 21=A(7)+B(0)$ $\quad\therefore\ A=3$

(b) Let $(s+3)=0$, i.e. $s=-3$ and we get ....

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$$\boxed{B = 2}$$

$$\therefore \quad \frac{5s+1}{s^2-s-12} \equiv \frac{3}{s-4} + \frac{2}{s+3}$$

$$\therefore \quad L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\} = \dots\dots\dots\dots$$

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$$\boxed{3e^{4t} + 2e^{-3t}}$$

Example 2

Determine $L^{-1}\left\{\dfrac{9s-8}{s^2-2s}\right\}$.

Working as before, $f(t) = \dots\dots\dots\dots$

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$$\boxed{4 + 5e^{2t}}$$

Because

$$L\{f(t)\} = \frac{9s-8}{s^2-2s}.$$

(a) Numerator of first degree; denonominator of second degree. Therefore rule satisfied.

(b) $\dfrac{9s-8}{s(s-2)} \equiv \dfrac{A}{s} + \dfrac{B}{s-2}$.

(c) Multiply by $s(s-2)$. $\therefore\ 9s - 8 = A(s-2) + Bs$.

(d) Put $s = 0$. $\quad -8 = A(-2) + B(0) \ \therefore\ A = 4$.

(e) Put $s - 2 = 0$, i.e. $s = 2$. $\quad 10 = A(0) + B(2) \ \therefore\ B = 5$.

$$\therefore\ f(t) = L^{-1}\left\{\frac{4}{s} + \frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Example 3

Express $F(s) = \dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ in partial fractions and hence determine its inverse transform.

$\dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ has partial fractions of the form $\dots\dots\dots\dots$

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$$\boxed{\frac{A}{s+2}+\frac{B}{s-3}+\frac{C}{(s-3)^2}}$$

Now we multiply throughout by $(s+2)(s-3)^2$ and get

$$s^2-15s+41 \equiv A(s-3)^2+B(s+2)(s-3)+C(s+2)$$

Putting $(s-3)=0$ and then $(s+2)=0$ we obtain ....

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$$\boxed{A=3 \text{ and } C=1}$$

Now that we have run out of 'crafty' substitutions, we equate coefficients of the highest power of s on each side, i.e. the coefficients of s^2. This gives ....

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$$\boxed{1=A+B \quad \therefore\ 1=3+B \quad \therefore\ B=-2}$$

So $\dfrac{s^2-15s+41}{(s+2)(s-3)^2}=\dfrac{3}{s+2}-\dfrac{2}{s-3}+\dfrac{1}{(s-3)^2}$

Now $L^{-1}\left\{\dfrac{3}{s+2}\right\}=$ .... and $L^{-1}\left\{\dfrac{2}{s-3}\right\}=$ ....

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$$\boxed{3e^{-2t} \text{ and } 2e^{3t}}$$

But what about $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$?

We remember that $L^{-1}\left\{\dfrac{1}{s^2}\right\}=$ ....

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$$\boxed{t}$$

and that by Theorem 1, if $L\{f(t)\}=F(s)$ then $L\{e^{-at}f(t)\}=F(s+a)$.

$\therefore\ \dfrac{1}{(s-3)^2}$ is like $\dfrac{1}{s^2}$ with s replaced by $(s-3)$ i.e. $a=-3$.

$$\therefore\ L^{-1}\left\{\frac{1}{(s-3)^2}\right\}=te^{3t}$$

$$\therefore\ L^{-1}\left\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\right\}=3e^{-2t}+2e^{3t}+te^{3t}$$

▶

Example 4

Determine $L^{-1}\left\{\dfrac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$.

Notice that this time we have a quadratic factor in the denominator

$$\frac{4s^2-5s+6}{(s+1)(s^2+4)} \equiv \frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$
$$\therefore\ 4s^2-5s+6 \equiv A(s^2+4)+(Bs+C)(s+1).$$

(a) Putting $(s+1)=0$, i.e. $s=-1$, $15=5A$ $\therefore\ A=3$

(b) Equate coefficients of highest power, i.e. s^2

$4=A+B$ $\therefore\ 4=3+B$ $\therefore\ B=1$

(c) We now equate the lowest power on each side, i.e. the constant term

$6=4A+C$ $\therefore\ 6=12+C$ $\therefore\ C=-6$

Now you can finish it off. $f(t)=$ ....

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$$\boxed{f(t)=3e^{-t}+\cos 2t-3\sin 2t}$$

Because

$$L\{f(t)\}=\frac{3}{s+1}+\frac{s}{s^2+4}-\frac{6}{s^2+4}$$
$$\therefore\ f(t)=3e^{-t}+\cos 2t-3\sin 2t$$

The 'cover up' rule

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While we can always find A, B, C, etc., there are many cases where we can use the 'cover up' methods and write down the values of the constant coefficients almost on sight. However, this method only works when the denominator of the original fraction has non-repeated, linear factors. The following examples illustrate the method.

▶

Example 1

We know that $F(s) = \dfrac{9s-8}{s(s-2)}$ has partial fractions of the form $\dfrac{A}{s} + \dfrac{B}{s-2}$.

By the 'cover up' rule, the constant A, that is the coefficient of $\dfrac{1}{s}$, is found by temporarily covering up the factor s in the denominator of $F(s)$ and finding the limiting value of what remains when s (the factor covered up) tends to zero.

Therefore $A =$ coefficient of $\dfrac{1}{s} = \underset{s\to 0}{Lim}\left\{\dfrac{9s-8}{s-2}\right\} = 4$. That is $A = 4$.

Similarly, B, the coefficient of $\dfrac{1}{s-2}$, is obtained by covering up the factor $(s-2)$ in the denominator of $F(s)$ and finding the limiting value of what remains when $(s-2) \to 0$, that is $s \to 2$.
Therefore $B =$ coefficient of $\dfrac{1}{s-2} = \underset{s\to 2}{Lim}\left\{\dfrac{9s-8}{s}\right\} = 5$. That is $B = 5$.

So that

$$\frac{9s-8}{s(s-2)} = \frac{4}{s} + \frac{5}{s-2}$$

Another example

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Example 2

$$F(s) = \frac{s+17}{(s-1)(s+2)(s-3)} \equiv \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}.$$

A: cover up $(s-1)$ in $F(s)$ and find

$$\underset{s\to 1}{Lim}\left\{\frac{s+17}{(s+2)(s-3)}\right\} = \frac{18}{-6} \quad \therefore\ A = -3$$

Similarly

B: .... $\quad \therefore\ B =$ ....

C: .... $\quad \therefore\ C =$ ....

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$$B = \underset{s\to -2}{Lim}\left\{\frac{s+17}{(s-1)(s-3)}\right\} = \frac{15}{(-3)(-5)} = 1 \qquad \therefore\ B = 1$$

$$C = \underset{s\to 3}{Lim}\left\{\frac{s+17}{(s-1)(s+2)}\right\} = \frac{20}{(2)(5)} = 2 \qquad \therefore\ C = 2$$

$$\therefore\ F(s) = \frac{1}{s+2} + \frac{2}{s-3} - \frac{3}{s-1}$$

So $f(t) = e^{-2t} + 2e^{3t} - 3e^{t}$

Every entry in our table of standard transforms gives rise to a corresponding entry in a similar table of inverse transforms. Let us tabulate such a list.

45 Table of inverse transforms

$F(s)$	$f(t)$	
$\dfrac{a}{s}$	a	
$\dfrac{1}{s+a}$	e^{-at}	
$\dfrac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$	(n a positive integer)
$\dfrac{a}{s^2+a^2}$	$\sin at$	
$\dfrac{s}{s^2+a^2}$	$\cos at$	
$\dfrac{a}{s^2-a^2}$	$\sinh at$	
$\dfrac{s}{s^2-a^2}$	$\cosh at$	

Theorem 1

The first shift theorem can be stated as follows.

If $F(s)$ is the Laplace transform of $f(t)$ then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$.

Here is a short revision exercise.

Exercise

1 Find the inverse transforms of

(a) $\dfrac{1}{2s-3}$; (b) $\dfrac{5}{(s-4)^3}$; (c) $\dfrac{3s+4}{s^2+9}$.

2 Express in partial fractions

(a) $\dfrac{22s+16}{(s+1)(s-2)(s+3)}$; (b) $\dfrac{s^2-11s+6}{(s+1)(s-2)^2}$.

3 Determine

(a) $L^{-1}\left\{\dfrac{4s^2-17s-24}{s(s+3)(s-4)}\right\}$; (b) $L^{-1}\left\{\dfrac{5s^2-4s-7}{(s-3)(s^2+4)}\right\}$.

▶

1 (a) $\frac{1}{2}e^{3t/2}$;	(b) $5t^2e^{4t}$;	(c) $3\cos 3t + \frac{4}{3}\sin 3t$
2 (b) $\frac{1}{s+1} + \frac{4}{s-2} - \frac{5}{s+3}$;	(b) $\frac{2}{s+1} - \frac{1}{s-2} - \frac{4}{(s-2)^2}$	
3 (a) $2 + 3e^{-3t} - e^{4t}$;	(b) $2e^{3t} + 3\cos 2t + \frac{5}{2}\sin 2t$	

Solution of differential equations by Laplace transforms

To solve a differential equation by Laplace transforms, we go through four distinct stages **46**

(a) Rewrite the equation in terms of Laplace transforms.

(b) Insert the given initial conditions.

(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

We have spent some time finding the transforms of a variety of functions of t and the inverse transforms of functions of s, i.e. we have largely covered steps (a) and (d) of the above list. However, to write a differential equation in Laplace transforms, we must obtain the transforms of the derivatives $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$.

Transforms of derivatives

Let $f'(t)$ denote the first derivative of $f(t)$ with respect to t,

$f''(t)$ denote the second derivative of $f(t)$ with respect to t, etc.

Then $L\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt$ by definition.

Integrating by parts

$$L\{f'(t)\} = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty f(t)\{-se^{-st}\}\,dt$$

When $t \to \infty$, $e^{-st}f(t) \to$ ....

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$$\boxed{0}$$

Because s is positive and large enough to ensure that e^{-st} decays faster than any possible growth of $f(t)$.

$$\therefore\ L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

Replacing $f(t)$ by $f'(t)$ gives

$$L\{f''(t)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$

Because

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

so

$$\begin{aligned} L\{f''(t)\} &= -f'(0) + sL\{f'(t)\} \\ &= -f'(0) + s(-f(0) + sL\{f(t)\}) \end{aligned}$$

Writing $L\{f(t)\} = F(s)$ as usual, we have

$$\begin{aligned} L\{f(t)\} &= F(s) \\ L\{f'(t)\} &= sF(s) - f(0) \\ L\{f''(t)\} &= s^2F(s) - sf(0) - f'(0) \end{aligned}$$

We can see a pattern emerging

$$L\{f'''(t)\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'0) - f''(0)}$$

Alternative notation

We make the working neater by adopting the following notation.
Let $x = f(t)$ and at $t = 0$, we write

$$\begin{aligned} x &= x_0 & &\text{i.e. } f(0) = x_0 \\ \frac{dx}{dt} &= x_1 & &\text{i.e. } f'(0) = x_1 \\ \frac{d^2x}{dt^2} &= x_2 & &\text{i.e. } f''(0) = x_2 \text{ etc.} \\ \therefore\ \frac{d^nx}{dt^n} &= x_n & &\text{i.e. } f^n(0) = x_n \end{aligned}$$

Also we denote the Laplace transform of x by $\bar{x}$,

i.e. $\bar{x} = L\{x\} = L\{f(t)\} = F(s)$.

So, using the 'dot' notation for derivatives, the previous results can be written ....

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$$\begin{aligned} L\{x\} &= \bar{x} \\ L\{\dot{x}\} &= s\bar{x} - x_0 \\ L\{\ddot{x}\} &= s^2\bar{x} - sx_0 - x_1 \\ L\{\dddot{x}\} &= s^3\bar{x} - s^2x_0 - sx_1 - x_2 \end{aligned}$$

In each case, the subscript indicates the order of the derivative, i.e. x_n = the value of $\dfrac{d^n x}{dt^n}$ at $t = 0$.
Notice the pattern of the results.

$$L\{\ddddot{x}\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{\ddddot{x}\} = s^4\bar{x} - s^3x_0 - s^2x_1 - sx_2 - x_3}$$

Now, at long last, we can start solving differential equations.

Solution of first-order differential equations

Example 1

Solve the equation $\dfrac{dx}{dt} - 2x = 4$ given that at $t = 0$, $x = 1$.
We go through the four stages.
(a) Rewrite the equation in Laplace transforms, using the last notation

$$L\{x\} = \bar{x}; \quad L\{\dot{x}\} = \ldots\ldots\ldots\ldots$$
$$L\{4\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{L\{\dot{x}\} = s\bar{x} - x_0; \quad L\{4\} = \frac{4}{s}}$$

Then the equation becomes $\quad (s\bar{x} - x_0) - 2\bar{x} = \dfrac{4}{s}$

(b) Insert the initial condition that at $t = 0$, $x = 1$, i.e. $x_0 = 1$

$$\therefore\ s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}$$

(c) Now we rearrange this to give an expression for $\bar{x}$

$$\bar{x} = \ldots\ldots\ldots\ldots$$

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$$\bar{x} = \frac{s+4}{s(s-2)}$$

(d) Finally, we take inverse transforms to obtain x.

$\dfrac{s+4}{s(s-2)}$ in partial fractions gives ....

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$$\frac{3}{s-2} - \frac{2}{s}$$

Because

$$\frac{s+4}{s(s-2)} \equiv \frac{A}{s} + \frac{B}{s-2} \qquad \therefore\ s+4 = A(s-2) + Bs$$

(1) Put $(s-2) = 0$, i.e. $s = 2$ $\qquad \therefore\ 6 = B(2) \qquad \therefore\ B = 3$

(2) Put $s = 0$ $\qquad \therefore\ 4 = A(-2) \qquad \therefore\ A = -2$

$$\therefore\ \bar{x} = \frac{s+4}{s(s-2)} = \frac{3}{s-2} - \frac{2}{s}$$

Therefore, taking inverse transforms

$$x = L^{-1}\left\{\frac{s+4}{s(s-2)}\right\} = L^{-1}\left\{\frac{3}{s-2} - \frac{2}{s}\right\} = \ldots\ldots\ldots\ldots$$

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$$x = 3e^{2t} - 2$$

This solution should now be substituted back into the differential equation to verify that it is, indeed, correct.

Example 2

Solve the equation $\dfrac{dx}{dt} + 2x = 10e^{3t}$ given that at $t = 0$, $x = 6$.

(a) Convert the equations to Laplace transforms, i.e.

....

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$$(s\bar{x} - x_0) + 2\bar{x} = \frac{10}{s-3}$$

(b) Insert the initial condition, $x_0 = 6$

$$s\bar{x} - 6 + 2\bar{x} = \frac{10}{s-3}$$

(c) Rearrange to obtain $\bar{x} = \ldots\ldots\ldots\ldots$

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$$\bar{x} = \frac{6s-8}{(s+2)(s-3)}$$

(d) Taking inverse transforms to obtain x

$$x = L^{-1}\left\{\frac{6s-8}{(s+2)(s-3)}\right\} = \ldots\ldots\ldots\ldots$$

Complete the solution

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$$x = 4e^{-2t} + 2e^{3t}$$

Because

$$\frac{6s-8}{(s+2)(s-3)} \equiv \frac{A}{s+2} + \frac{B}{s-3}$$

$$\therefore\ 6s-8 = A(s-3) + B(s+2)$$

(1) Put $(s-3)=0$, i.e. $s=3$ $\quad\therefore\quad 10 = B(5) \quad\therefore\ B=2$

(2) Put $(s+2)=0$, i.e. $s=-2$. $\quad\therefore\quad -20 = A(-5) \quad\therefore\ A=4$

$$\therefore\ \bar{x} = \frac{6s-8}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{2}{s-3}$$

$$\therefore\ x = L^{-1}\left\{\frac{4}{s+2} + \frac{2}{s-3}\right\} = 4e^{-2t} + 2e^{3t}$$

Example 3

Solve the equation $\dfrac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$, given that at $t=0$, $x=0$.

Work this through the four steps in the same way as before and complete it on your own.

$$x = \ldots\ldots\ldots\ldots$$

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$$\boxed{x = e^{4t} - e^{2t} + te^{4t}}$$

The working is quite standard.

$$\frac{dx}{dt} - 4x = 2e^{2t} + e^{4t}$$

(a) $(s\bar{x} - x_0) - 4\bar{x} = \dfrac{2}{s-2} + \dfrac{1}{s-4}$

(b) $x_0 = 0 \quad \therefore \; s\bar{x} - 4\bar{x} = \dfrac{2}{s-2} + \dfrac{1}{s-4}$

(c) $\therefore \; \bar{x} = \dfrac{2}{(s-2)(s-4)} + \dfrac{1}{(s-4)^2}$

(d) $\dfrac{2}{(s-2)(s-4)} \equiv \dfrac{A}{s-2} + \dfrac{B}{s-4} \quad \therefore \; 2 = A(s-4) + B(s-2)$

Putting $(s-2) = 0$, i.e. $s = 2 \quad \therefore \; 2 = A(-2) \quad \therefore \; A = -1$

Putting $(s-4) = 0$, i.e. $s = 4 \quad \therefore \; 2 = B(2) \quad \therefore \; B = 1$

$\therefore \; \bar{x} = \dfrac{1}{s-4} - \dfrac{1}{s-2} + \dfrac{1}{(s-4)^2}$

$\therefore \; x = e^{4t} - e^{2t} + te^{4t}$

Now on to the next frame

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Solution of second-order differential equations

The method is, in effect, the same as before, going through the same four distinct stages.

Example 1

Solve the equation $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 2e^{3t}$, given that at $t = 0$, $x = 5$ and $\dfrac{dx}{dt} = 7$.

(a) We rewrite the equation in terms of its transforms, remembering that

$$L\{x\} = \bar{x}$$

$$L\{\dot{x}\} = s\bar{x} - x_0$$

$$L\{\ddot{x}\} = s^2\bar{x} - sx_0 - x_1$$

The equation becomes ....

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$$\boxed{(s^2\bar{x} - sx_0 - x_1) - 3(s\bar{x} - x_0) + 2\bar{x} = \frac{2}{s-3}}$$

(b) Insert the initial conditions. In this case $x_0 = 5$ and $x_1 = 7$

$$\therefore\ (s^2\bar{x} - 5s - 7) - 3(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}$$

(c) Rearrange to obtain $\bar{x} = \dots\dots\dots\dots$

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$$\boxed{\bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}}$$

Because

$$s^2\bar{x} - 5s - 7 - 3s\bar{x} + 15 + 2\bar{x} = \frac{2}{s-3}$$

$$(s^2 - 3s + 2)\bar{x} - 5s + 8 = \frac{2}{s-3}$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s-3} + 5s - 8 = \frac{2 + 5s^2 - 23s + 24}{s-3}$$

$$\therefore\ \bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

(d) Now for partial fractions

$$\frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\therefore\ 5s^2 - 23s + 26 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

So that $A = \dots\dots\dots\dots$; $B = \dots\dots\dots\dots$; $C = \dots\dots\dots\dots$

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$$\boxed{A = 4;\quad B = 0;\quad C = 1}$$

$$\therefore\ \bar{x} = \frac{4}{s-1} + \frac{1}{s-3}$$

$$\therefore\ x = \dots\dots\dots\dots$$

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$$\boxed{x = 4e^t + e^{3t}}$$

As you see, the Laplace transform method can be considerably shorter than the classical method which requires

(a) determination of the complementary function

(b) determination of a particular integral

(c) obtaining the general solution, before

(d) arriving at the particular solution by substitution of the initial conditions in the general solution.

▶

Here is another example.

Example 2

Solve $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4x = 24\cos 2t$ given that at $t = 0$, $x = 3$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 4$.

(a) In Laplace transforms ....

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$$\boxed{\left(s^2\bar{x} - sx_0 - x_1\right) - 4\bar{x} = \frac{24s}{s^2+4}}$$

(b) Insert initial condition, i.e. $x_0 = 3$; $x_1 = 4$

$$s^2\bar{x} - 3s - 4 - 4\bar{x} = \frac{24s}{s^2+4}$$

$$\therefore\ (s^2 - 4)\bar{x} = 3s + 4 + \frac{24s}{s^2+4}$$

$$= \frac{3s^3 + 4s^2 + 36s + 16}{s^2+4}$$

(c) $\bar{x} = \dfrac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)}$

Expressed in partial fractions, this becomes

....

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$$\boxed{\frac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)} \equiv \frac{As+B}{s^2+4} + \frac{C}{s-2} + \frac{D}{s+2}}$$

$$\therefore\ 3s^3 + 4s^2 + 36s + 16 \equiv (As+B)(s-2)(s+2) + C(s^2+4)(s+2) + D(s^2+4)(s-2)$$

Putting $(s-2) = 0$, i.e. $s = 2$, gives $C = 4$

Putting $(s+2) = 0$, i.e. $s = -2$, gives $D = 2$

Equating coefficients of s^3 and also the constant terms gives $A = -3$ and $B = 0$.

$$\therefore\ \bar{x} = \frac{3s^3 + 4s^2 + 36s + 16}{(s^2+4)(s-2)(s+2)} = \frac{4}{s-2} + \frac{2}{s+2} - \frac{3s}{s^2+4}$$

$$\therefore\ x = \text{............}$$

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$$\boxed{x = 4e^{2t} + 2e^{-2t} - 3\cos 2t}$$

Now let us solve another equation, this time using the 'cover up' rule.

Example 3

Solve $\ddot{x} + 5\dot{x} + 6x = 4t$, given that at $t = 0$, $x = 0$ and $\dot{x} = 0$.

As usual we begin $(s^2\bar{x} - sx_0 - x_1) + 5(s\bar{x} - x_0) + 6\bar{x} = \dfrac{4}{s^2}$

$$x_0 = 0;\ x_1 = 0 \quad \therefore\ (s^2 + 5s + 6)\bar{x} = \frac{4}{s^2}$$

$$\therefore\ \bar{x} = \frac{4}{s^2(s+2)(s+3)}$$

The s^2 in the denominator can be awkward, so we introduce a useful trick and detach one factor s outside the main expression, thus

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{s(s+2)(s+3)}\right\} = \frac{1}{s}\left\{\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}\right\}$$

Applying the 'cover up' rule to the expressions within the brackets

$$\bar{x} = \frac{1}{s}\left\{\frac{4}{6}.\frac{1}{s} - \frac{2}{(s+2)} + \frac{4}{3}.\frac{1}{s+3}\right\}$$

Now we bring the external $\dfrac{1}{s}$ back into the fold

$$\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{2}{s(s+2)} + \frac{4}{3}.\frac{1}{s(s+3)}$$

and the second and third terms can be expressed in simple partial fractions so that

$$\bar{x} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+2} + \frac{4}{9}.\frac{1}{s} - \frac{4}{9}.\frac{1}{s+3}}$$

which can now be simplified into

$$\bar{x} = \frac{2}{3}.\frac{1}{s^2} - \frac{5}{9}.\frac{1}{s} + \frac{1}{s+2} - \frac{4}{9}.\frac{1}{s+3}$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

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$$x = \frac{2}{3}t - \frac{5}{9} + e^{-2t} - \frac{4}{9}e^{-3t}$$

There are times when a quadratic coefficient of $\bar{x}$ cannot be expressed in simple linear factors. In that case, we merely complete the square converting the expression into $(s \pm k)^2 \pm a^2$. Let us see such an example.

Example 4

Solve $\ddot{x} - 2\dot{x} + 10x = e^{2t}$, given that at $t = 0$, $x = 0$ and $\dot{x} = 1$.

We find the expression for $\bar{x}$ as before.

$\bar{x} = \ldots\ldots\ldots\ldots$

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$$\bar{x} = \frac{s-1}{(s-2)(s^2 - 2s + 10)}$$

Because

$$(s^2\bar{x} - sx_0 - x_1) - 2(s\bar{x} - x_0) + 10\bar{x} = \frac{1}{s-2}$$

$$x_0 = 0;\ x_1 = 1 \qquad \therefore\ s^2\bar{x} - 1 - 2s\bar{x} + 10\bar{x} = \frac{1}{s-2}$$

$$\therefore\ (s^2 - 2s + 10)\bar{x} = 1 + \frac{1}{s-2} = \frac{s-1}{s-2}$$

$$\therefore\ \bar{x} = \frac{s-1}{(s-2)(s^2 - 2s + 10)}$$

Expressing this in partial fractions

$\bar{x} = \ldots\ldots\ldots\ldots$ Evaluate the coefficients.

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$$\bar{x} = \frac{1}{10}\left\{\frac{1}{s-2} - \frac{s-10}{s^2 - 2s + 10}\right\}$$

Because

$$\frac{s-1}{(s-2)(s^2 - 2s + 10)} \equiv \frac{A}{(s-2)} + \frac{Bs + C}{s^2 - 2s + 10}$$

$$\therefore\ s - 1 = A(s^2 - 2s + 10) + (s-2)(Bs + C)$$

Put $(s-2) = 0$, i.e. $s = 2$ $\qquad 1 = A(4 - 4 + 10) \qquad \therefore\ A = \frac{1}{10}$

$[s^2]$ $\qquad 0 = A + B \qquad \therefore\ B = -\frac{1}{10}$

[CT] $\qquad -1 = 10A - 2C \quad \therefore\ 2C = 2 \qquad \therefore\ C = 1$

$$\therefore\ \bar{x} = \frac{1}{10}\left\{\frac{1}{s-2} - \frac{s-10}{s^2 - 2s + 10}\right\}$$

Now we have to find the inverse transforms to obtain x. The first term $\dfrac{1}{s-2}$ is easy enough, but what of $\dfrac{s-10}{s^2-2s+10}$? The denominator will not factorise into simple linear factors; therefore we complete the square in the denominator and write it as

$$\frac{s-10}{s^2-2s+10}=\frac{s-10}{(s-1)^2+9}$$

and then we improve this still further and write it in the form $\dfrac{(s-1)-9}{(s-1)^2+9}$. We are quite happy with this, for $\dfrac{s-1}{(s-1)^2+9}$ is merely $\dfrac{s}{s^2+9}$ with s replaced by $(s-1)$, which indicates an extra factor e^t in the final function of t (Theorem 1).

So $\quad \bar{x}=\dfrac{1}{10}\left\{\dfrac{1}{s-2}-\dfrac{s-1}{(s-1)^2+9}+\dfrac{9}{(s-1)^2+9}\right\}$

$\therefore\ x=\ldots\ldots\ldots\ldots$

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$$\boxed{x=\frac{1}{10}\left\{e^{2t}-e^t\cos 3t+3e^t\sin 3t\right\}}$$

Just try one more like this one

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Example 5

Solve $\ddot{x}+\dot{x}+x=e^{-t}$ given that at $t=0$, $x=0$ and $\dot{x}=1$. We find the expression for $\bar{x}$ as before.

$\bar{x}=\ldots\ldots\ldots\ldots$

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$$\boxed{\bar{x}=\frac{s+2}{(s+1)(s^2+s+1)}}$$

Because $(s^2\bar{x}-sx_0-x_1)+(s\bar{x}-x_0)+\bar{x}=\dfrac{1}{s+1}$ where $x_0=0$ and $x_1=1$ so that

$$s^2\bar{x}-1+s\bar{x}+\bar{x}=\frac{1}{s+1}$$

therefore

$$\bar{x}(s^2+s+1)=1+\frac{1}{s+1}=\frac{s+2}{s+1}$$

giving

$$\bar{x}=\frac{s+2}{(s+1)(s^2+s+1)}$$

Expressing this in partial fractions

$\bar{x}=\ldots\ldots\ldots\ldots$

Evaluate the coefficients

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$$\boxed{\bar{x} = \frac{1}{s+1} + \frac{s-1}{s^2+s+1}}$$

Because

$$\bar{x} = \frac{s+2}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$$

so that

$$s+2 = A\left(s^2+s+1\right) + (Bs+C)(s+1)$$

Put $s+1=0$, that is $s=-1$ then

$$\begin{array}{lll} & 1 = A(1-1+1) & \text{so that } A=1 \\ \left[s^2\right] & 0 = A+B & \text{so that } B=-1 \\ [\text{CT}] & 2 = A+C & \text{so that } C=1 \end{array}$$

Therefore

$$\bar{x} = \frac{1}{s+1} - \frac{s-1}{s^2+s+1}$$

Completing the squares in the second term gives

$$\frac{s-1}{s^2+s+1} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{s-1}{s^2+s+1} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

Because

$$\begin{aligned} \frac{s-1}{s^2+s+1} &= \frac{s-1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} \\ &= \frac{s+\frac{1}{2}-\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \end{aligned}$$

so that

$$\bar{x} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\bar{x} = \frac{1}{s+1} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\sqrt{3}\times\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

and so

$$x = \ldots\ldots\ldots\ldots$$

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$$x = e^{-t} - e^{-t/2}\cos\frac{\sqrt{3}t}{2} + \sqrt{3}e^{-t/2}\sin\frac{\sqrt{3}t}{2}$$

Before we leave this topic, the same general approach can be employed for solving simultaneous differential equations.

Let us see an example in the next frame

Simultaneous differential equations

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Example 1

Solve the pair of simultaneous equations

$$\dot{y} - x = e^t$$
$$\dot{x} + y = e^{-t}$$

given that at $t = 0$, $x = 0$ and $y = 0$.

(a) We first express both equations in Laplace transforms.

$$(s\bar{y} - y_0) - \bar{x} = \frac{1}{s-1}$$
$$(s\bar{x} - x_0) + \bar{y} = \frac{1}{s+1}$$

(b) Then we insert the initial conditions, $x_0 = 0$ and $y_0 = 0$.

$$\therefore \left.\begin{aligned} s\bar{y} - \bar{x} &= \frac{1}{s-1} \\ s\bar{x} + \bar{y} &= \frac{1}{s+1} \end{aligned}\right\} \qquad (1)$$

(c) We now solve these for $\bar{x}$ and $\bar{y}$ by the normal algebraic method. Eliminating $\bar{y}$ we have

$$s\bar{y} - \bar{x} = \frac{1}{s-1}$$
$$s\bar{y} + s^2\bar{x} = \frac{s}{s+1}$$
$$\therefore\ (s^2+1)\bar{x} = \frac{2}{s+1} - \frac{1}{s-1} = \frac{s^2-2s-1}{(s+1)(s-1)}$$
$$\therefore\ \bar{x} = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)}$$

Representing this in partial fractions gives ....

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$$\boxed{\bar{x} = -\frac{1}{2}.\frac{1}{s-1} - \frac{1}{2}.\frac{1}{s+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}}$$

Because

$$\bar{x} = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)} \equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore\ s^2-2s-1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (s-1)(s+1)(Cs+D)$$

Putting $s=1$ and $s=-1$ gives $A=-\frac{1}{2}$ and $B=-\frac{1}{2}$.
Comparing coefficients of s^3 and the constant terms gives $C=1$ and $D=1$.

$$\therefore\ \bar{x} = \frac{1}{2}.\frac{1}{s-1} - \frac{1}{2}.\frac{1}{s+1} + \frac{s+1}{s^2+1}$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

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$$\boxed{x = -\tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t} + \cos t + \sin t}$$

We now revert to equations (1) and eliminate $\bar{x}$ to obtain $\bar{y}$ and hence y, in the same way. Do this on your own.

$$y = \ldots\ldots\ldots\ldots$$

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$$\boxed{y = \tfrac{1}{2}e^t + \tfrac{1}{2}e^{-t} - \cos t + \sin t}$$

Here is the working.

$$\left.\begin{aligned} s^2\bar{y} - s\bar{x} &= \frac{s}{s-1} \\ \bar{y} + s\bar{x} &= \frac{1}{s+1} \end{aligned}\right\}$$

$$\therefore\ (s^2+1)\bar{y} = \frac{s}{s-1} + \frac{1}{s+1} = \frac{s^2+2s-1}{(s-1)(s+1)}$$

$$\therefore\ \bar{y} = \frac{s^2+2s-1}{(s-1)(s+1)(s^2+1)} \equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

$$\therefore\ s^2+2s-1 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (s-1)(s+1)(Cs+D)$$

Putting $s=1$ and $s=-1$ gives $A=\frac{1}{2}$ and $B=\frac{1}{2}$.
Equating coefficients of s^3 and the constant terms gives $C=-1$ and $D=1$.

$$\therefore\ \bar{y} = \frac{1}{2}.\frac{1}{s-1} + \frac{1}{2}.\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore\ y = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - \cos t + \sin t$$

▶

So the results are

$$x = -\frac{1}{2}(e^t + e^{-t}) + \sin t + \cos t = \sin t + \cos t - \cosh t$$

$$y = \frac{1}{2}(e^t + e^{-t}) + \sin t - \cos t = \sin t - \cos t + \cosh t$$

$$\therefore \; x = \sin t + \cos t - \cosh t; \quad y = \sin t - \cos t + \cosh t$$

Simultaneous equations are all solved in much the same way. Here is another.

Example 2

Solve the equations

$$2\dot{y} - 6y + 3x = 0$$
$$3\dot{x} - 3x - 2y = 0$$

given that at $t = 0$, $x = 1$ and $y = 3$.

Expressing these in Laplace transforms, we have

....

....

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$$\boxed{\begin{aligned} &2(s\bar{y} - y_0) - 6\bar{y} + 3\bar{x} = 0 \\ &3(s\bar{x} - x_0) - 3\bar{x} - 2\bar{y} = 0 \end{aligned}}$$

Then we insert the initial conditions and simplify, obtaining

....

....

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$$\boxed{\begin{aligned} &3\bar{x} + (2s - 6)\bar{y} = 6 \qquad (1) \\ &(3s - 3)\bar{x} - 2\bar{y} = 3 \qquad (2) \end{aligned}}$$

(a) *To find* $\bar{x}$

$$\begin{array}{ll} (1) & 3\bar{x} + (2s-6)\bar{y} = 6 \\ (2) \times (s-3) & (s-3)(3s-3)\bar{x} - (2s-6)\bar{y} = 3(s-3) \\ \text{Adding,} & [(s-3)(3s-3) + 3]\bar{x} = 3s - 9 + 6 \\ & \therefore \; (3s^2 - 12s + 12)\bar{x} = 3s - 3 \\ & (s^2 - 4s + 4)\bar{x} = s - 1 \end{array}$$

$$\therefore \; \bar{x} = \frac{s-1}{(s-2)^2} \equiv \frac{A}{s-2} + \frac{B}{(s-2)^2} = \frac{A(s-2)+B}{(s-2)^2}$$

$$\therefore \; s - 1 = A(s-2) + B \quad \text{giving} \quad A = 1 \quad \text{and} \quad B = 1$$

$$\therefore \; \bar{x} = \frac{1}{s-2} + \frac{1}{(s-2)^2} \qquad \therefore \; x = e^{2t} + te^{2t}$$

(b) Going back to equations (1) and (2), we can find y.

$y =$ ....

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$$\boxed{y = \tfrac{1}{2}\{6e^{2t} + 3te^{2t}\}}$$

Because, eliminating $\bar{x}$ we get

$$\bar{y} = \frac{6s-9}{2(s-2)^2} \equiv \frac{1}{2}\left\{\frac{A}{s-2} + \frac{B}{(s-2)^2}\right\} = \frac{1}{2}\left\{\frac{A(s-2)+B}{(s-2)^2}\right\}$$

$\therefore\ 6s - 9 = A(s-2) + B \qquad \therefore\ A = 6;\quad B = 3$

$$\therefore\ \bar{y} = \frac{1}{2}\left\{\frac{6}{s-2} + \frac{3}{(s-2)^2}\right\} \qquad \therefore\ y = \tfrac{1}{2}\{6e^{2t} + 3te^{2t}\}$$

Simultaneous second-order equations are solved in like manner. Again, with all these solutions it is a worthwhile exercise to substitute the solution back into the differential equation to verify that the solution is correct.

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Example 3

If x and y are functions of t, solve the equations

$$\ddot{x} + 2x - y = 0$$
$$\ddot{y} + 2y - x = 0$$

given that at $t = 0$, $x_0 = 4$; $y_0 = 2$; $x_1 = 0$; $y_1 = 0$.

We start off as usual with $(s^2\bar{x} - sx_0 - x_1) + 2\bar{x} - \bar{y} = 0$

and $(s^2\bar{y} - sy_0 - y_1) + 2\bar{y} - \bar{x} = 0$

Inserting the initial conditions, we have

$$s^2\bar{x} - 4s + 2\bar{x} - \bar{y} = 0$$
$$s^2\bar{y} - 2s + 2\bar{y} - \bar{x} = 0$$

Simplifying these we can eliminate $\bar{y}$ to obtain $\bar{x}$ and hence x.

$x = \dots\dots\dots\dots$

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$$\boxed{x = 3\cos t + \cos\left(\sqrt{3}t\right)}$$

Because

$$(s^2+2)\bar{x} - \bar{y} = 4s \quad (1)$$

$$-\bar{x} + (s^2+2)\bar{y} = 2s \quad (2)$$

Eliminating $\bar{y}$ and simplifying gives

$$\bar{x} = \frac{4s^3+10s}{(s^2+1)(s^2+3)}$$

$$\therefore\ \bar{x} = \frac{4s^3+10s}{(s^2+1)(s^2+3)} \equiv \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+3}$$

$$\therefore\ 4s^3+10s = \left(s^2+3\right)(As+B) + \left(s^2+1\right)(Cs+D)$$

Equating coefficients of like powers of s

$[s^3]$ $\quad 4 = A + C \quad \therefore\ A + C = 4$

[CT] $\quad 0 = 3B + D \quad \therefore\ 3B + D = 0$

Putting $s = 1$, $\quad 14 = 4A + 4B + 2C = 2D \quad \therefore\ 2A + 2B + C + D = 7$

Putting $s = -1$ $\quad -14 = -4A + 4B - 2C + 2D \quad \therefore\ 2A - 2B + C - D = 7$

Putting $C = 4 - A$ and $D = -3B$ in the last two leads to

$A = \ldots\ldots\ldots\ldots;\quad B = \ldots\ldots\ldots\ldots;$

$C = \ldots\ldots\ldots\ldots;\quad D = \ldots\ldots\ldots\ldots$

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$$\boxed{A = 3;\quad B = 0;\quad C = 1;\quad D = 0}$$

$$\therefore\ \bar{x} = \frac{3s}{s^2+1} + \frac{s}{s^2+3}$$

$$\therefore\ x = \ldots\ldots\ldots\ldots$$

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$$x = 3\cos t + \cos\left(\sqrt{3}t\right)$$

To find y we could return to equations (1) and (2) and repeat the process, eliminating $\bar{x}$ so as to obtain $\bar{y}$ and hence y.

But always keep an eye on the original equations, the first of which is

$$\ddot{x} + 2x - y = 0$$

Therefore, in this particular case, $y = \ddot{x} + 2x$.

So all we have to do is to differentiate x twice and substitute

$$x = 3\cos t + \cos\left(\sqrt{3}t\right)$$
$$\dot{x} = -3\sin t - \sqrt{3}\sin\left(\sqrt{3}t\right)$$
$$\ddot{x} = -3\cos t - 3\cos\left(\sqrt{3}t\right)$$
$$\therefore\ y = -3\cos t - 3\cos\left(\sqrt{3}t\right) + 6\cos t + 2\cos\left(\sqrt{3}t\right)$$
$$\therefore\ y = 3\cos t - \cos\left(\sqrt{3}t\right)$$

which is a good deal quicker.

So, as we have seen, the method of solving differential equations by Laplace transforms follows a general routine.

(a) Express the equation in Laplace transforms
(b) Insert the initial conditions
(c) Simplify to obtain the transform of the solution
(d) Rewrite the final transform in partial fractions
(e) Determine the inverse transforms

and, by now, you are fully aware of the importance of *partial fractions*!

That brings us to the end of this particular Programme. We shall continue our study of Laplace transforms in the next Programme. Meanwhile, be sure you are familiar with the items listed in the **Revision summary** that follows, and respond to the questions in the **Can You?** checklist. You will then have no difficulty with the **Test Exercise** and the **Further problems** provide additional practice.

Revision summary 2

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1 *Laplace transform* $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$.

2 *Table of transforms*

$f(t)$	$L\{f(t)\} = F(s)$	
a	$\dfrac{a}{s}$	
e^{at}	$\dfrac{1}{s-a}$	
$\sin at$	$\dfrac{a}{s^2+a^2}$	
$\cos at$	$\dfrac{s}{s^2+a^2}$	
$\sinh at$	$\dfrac{a}{s^2-a^2}$	
$\cosh at$	$\dfrac{s}{s^2-a^2}$	
t^n	$\dfrac{n!}{s^{n+1}}$	(n a positive integer)

3 *Linearity of the Laplace transform*

(a) The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(t)\}$$

(b) The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is

$$L\{kf(t)\} = kL\{f(t)\}$$

4 *Theorem 1* First shift theorem

If $L\{f(t)\} = F(s)$, then $L\{e^{-at}f(t)\} = F(s+a)$.

5 *Theorem 2* Multiplying by t

If $L\{f(t)\} = F(s)$, then $L\{tf(t)\} = -\dfrac{d}{ds}\{F(s)\}$.

6 *Theorem 3* Dividing by t

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

provided that $\lim\limits_{t\to 0}\left\{\dfrac{f(t)}{t}\right\}$ exists.

7 *Inverse transform*

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\} = f(t)$.

▶

8 *Rules of partial fractions*

(a) The numerator must be of lower degree than the denominator. If not, divide out.

(b) Factorise the denominator into its prime factors.

(c) A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where A is a constant to be determined.

(d) A repeated factor $(s+a)^2$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}$.

(e) Similarly $(s+a)^3$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$.

(f) A quadratic factor (s^2+ps+q) gives $\dfrac{Ps+Q}{s^2+ps+q}$.

(g) A repeated quadratic factor $(s^2+ps+q)^2$ gives

$$\frac{Ps+Q}{s^2+ps+q}+\frac{Rs+T}{(s^2+ps+q)^2}.$$

9 *The 'cover up' rule*

The 'cover up' rule often enables the values of the constant coefficients to be written down almost on sight. However, this method only works when the denominator of the original fraction has non-repeated, linear factors.

10 *Table of inverse transforms*

$F(s)$	$f(t)$	
$\dfrac{a}{s}$	a	
$\dfrac{1}{s+a}$	e^{-at}	
$\dfrac{n!}{s^{n+1}}$	t^n	(n a positive integer)
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$	
$\dfrac{a}{s^2+a^2}$	$\sin at$	
$\dfrac{s}{s^2+a^2}$	$\cos at$	
$\dfrac{a}{s^2-a^2}$	$\sinh at$	
$\dfrac{s}{s^2-a^2}$	$\cosh at$	

By the first shift theorem

If $F(s)$ is the Laplace transform of $f(t)$

then $F(s+a)$ is the Laplace transform of $e^{-at}f(t)$.

▶

11 *Laplace transforms of derivatives*

$$L\{x\} = \bar{x}$$

$$L\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = L\{\dot{x}\} = s\bar{x} - x_0$$

$$L\left\{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right\} = L\{\ddot{x}\} = s\bar{x} - sx_0 - x_1 \text{ etc.}$$

where $x_0 =$ value of x at $t = 0$

$x_1 =$ value of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ at $t = 0$, etc.

12 *Solution of differential equations*

(a) Rewrite the equation in terms of Laplace transforms.
(b) Insert the given initial conditions.
(c) Rearrange the equation algebraically to give the transform of the solution.
(d) Express the transform in standard forms by partial fractions.
(e) Determine the inverse transforms to obtain the particular solution.

13 *Simultaneous differential equations*

Convert the simultaneous differential equations into simultaneous algebraic equations by taking the Laplace transform of each equation in turn. Insert the initial values. Solve the simultaneous algebraic equations in the usual manner and take the inverse Laplace transform of the algebraic solutions to find the solutions to the simultaneous differential equations.

☑ Can You?

Checklist 2

91

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Obtain the Laplace transforms of simple standard expressions? 1 to 14

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the first shift theorem to find the Laplace transform of a simple expression multiplied by an exponential? 15 to 17

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the Laplace transform of a simple expression multiplied or divided by a variable? 18 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Use partial fractions to find the inverse Laplace transform? 27 to 41

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the 'cover up' rule? 42 to 45

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the Laplace transforms of derivatives to solve differential equations? 46 to 78

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the Laplace transform to solve simultaneous differential equations? 79 to 89

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 2

92

1 Determine the Laplace transforms of the following functions.

(a) $3e^{-4t} - 5e^{4t}$ (b) $\sin 4t + \cos 4t$ (c) $t^3 + 2t^2 - t + 4$

(d) $e^{-2t}\cos 5t$ (e) $t\sin 3t$ (f) $\dfrac{e^{-t} - e^{-2t}}{t}$.

2 Determine the inverse transforms of the following.

(a) $\dfrac{s-5}{(s-3)(s-4)}$ (b) $\dfrac{s^2+3s-7}{(s-1)(s^2+2)}$

(c) $\dfrac{s^2-3s-4}{(s-3)(s-1)^2}$ (d) $\dfrac{2s^2-6s-1}{(s-3)(s^2-2s+5)}$.

3 Solve the following equations by Laplace transforms.

(a) $\dfrac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{-2t}$ given that $x = 2$ when $t = 0$

(b) $3\dot{x} - 6x = \sin 2t$ given that $x = 1$ when $t = 0$

(c) $\ddot{x} - 7\dot{x} + 12x = 2$ given that at $t = 0$, $x = 1$ and $\dot{x} = 5$

(d) $\ddot{x} - 2\dot{x} + x = te^t$ given that at $t = 0$, $x = 1$ and $\dot{x} = 0$.

4 Solve the following pair of simultaneous equations where x and y are functions of t and given that at $t = 0$, $x = 4$ and $y = -1$.

$$\dot{x} + \dot{y} + x + 2y = e^{-3t}$$
$$\dot{x} + 3x + 5y = 5e^{-2t}$$

Further problems 2

93

1 Determine the Laplace transforms of the following functions.

(a) $e^{4t}\cos 2t$ (b) $t\sin 2t$ (c) $t^3 + 4t^2 + 5$

(d) $e^{3t}(t^2 + 4)$ (e) $t^2\cos t$ (f) $\dfrac{\sinh 2t}{t}$.

▶

2 Determine the inverse transforms of the following.

(a) $\dfrac{2s-6}{(s-2)(s-4)}$ (b) $\dfrac{5s-8}{s(s-4)}$ (c) $\dfrac{s^2-2s+3}{(s-2)^3}$

(d) $\dfrac{2-11s}{(s-2)(s^2+2s+2)}$ (e) $\dfrac{s}{(s^2+1)(s^2+4)}$ (f) $\dfrac{s-5}{s^2+4s+20}$.

In Questions 3 to 11, solve the equations by Laplace transforms.

3 $\dot{x}-4x=8$ at $t=0$, $x=2$.

4 $3\dot{x}-4x=\sin 2t$ at $t=0$, $x=\frac{1}{3}$.

5 $\ddot{x}-2\dot{x}+x=2(t+\sin t)$ at $t=0$, $x=6$, $\dot{x}=5$.

6 $\ddot{x}-6\dot{x}+8x=e^{3t}$ at $t=0$, $x=0$, $\dot{x}=2$.

7 $\ddot{x}+9x=\cos 2t$ at $t=0$, $x=1$, $\dot{x}=3$.

8 $\ddot{x}-2\dot{x}+5x=e^{2t}$ at $t=0$, $x=0$, $\dot{x}=1$.

9 $\ddot{x}+4\dot{x}+4x=t^2+e^{-2t}$ at $t=0$, $x=\frac{1}{2}$, $\dot{x}=0$.

10 $\ddot{x}+8\dot{x}+32x=32\sin 4t$ at $t=0$, $x=\dot{x}=0$.

11 $\ddot{x}+25x=10(\cos 5t-2\sin 5t)$ at $t=0$, $x=1$, $\dot{x}=2$.

In Questions 12 to 17, solve the pairs of simultaneous equations by Laplace transforms.

12 $\left.\begin{aligned}\dot{y}+3x&=e^{-2t}\\ \dot{x}-3y&=e^{2t}\end{aligned}\right\}$ at $t=0$, $x=y=0$.

13 $\left.\begin{aligned}4\dot{x}-2\dot{y}+10x-5y&=0\\ \dot{y}-18x+15y&=10\end{aligned}\right\}$ at $t=0$, $y=4$, $x=2$.

14 $\left.\begin{aligned}\dot{x}-2\dot{y}-3x+6y&=12\\ 3\dot{y}+5x+2y&=16\end{aligned}\right\}$ at $t=0$, $x=12$, $y=8$.

15 $\left.\begin{aligned}2\dot{x}+3\dot{y}+7x&=14t+7\\ 5\dot{x}-3\dot{y}+4x+6y&=14t-14\end{aligned}\right\}$ at $t=0$, $x=y=0$.

16 $\left.\begin{aligned}2\dot{x}+2x+3\dot{y}+6y&=56e^{t}-3e^{-t}\\ \dot{x}-2x-\dot{y}-3y&=-21e^{t}-7e^{-t}\end{aligned}\right\}$ at $t=0$, $x=8$, $y=3$.

17 $\left.\begin{aligned}\ddot{x}-\ddot{y}+x-y&=5e^{2t}\\ 2\dot{x}-\dot{y}+y&=0\end{aligned}\right\}$ at $t=0$, $x=1$, $y=2$, $\dot{x}=0$.

18 Find an expression for x in terms of t, given that

$\ddot{y}-\dot{x}+2x=10\sin 2t$

$\dot{y}+2y+x=0$ and when $t=0$, $x=y=0$.

19 If $\ddot{x}+8x+2y=24\cos 4t$
and $\ddot{y}+2x+5y=0$
and at $t=0$, $x=y=0$, $\dot{x}=1$, $\dot{y}=2$, determine an expression for y in terms of t.

20 Solve completely, the pair of simultaneous equations

$5\ddot{x}+12\ddot{y}+6x=0$

$5\ddot{x}+16\ddot{y}+6y=0$

given that, at $t=0$, $x=\frac{7}{4}$, $y=1$, $\dot{x}=0$, $\dot{y}=0$.

Laplace transforms 2

Frames 1 to 42

Learning outcomes

When you have completed this Programme you will be able to:

- Use the Heaviside unit step function to 'switch' expressions on and off
- Obtain the Laplace transform of expressions involving the Heaviside unit step function

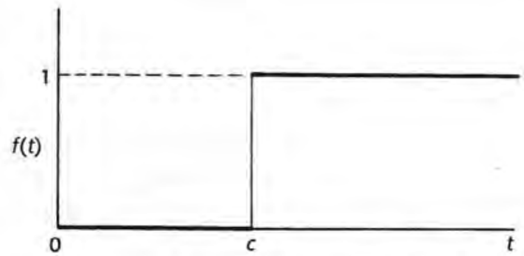
Introduction

1

In the previous Programme, we dealt with the Laplace transforms of continuous functions of t. In practical applications, it is convenient to have a function which, in effect, 'switches on' or 'switches off' a given term at pre-described values of t. This we can do with the *Heaviside unit step function.*

Heaviside unit step function

Consider a function that maintains a zero value for all values of t up to $t = c$ and a unit value for $t = c$ and all values of $t \geq c$.



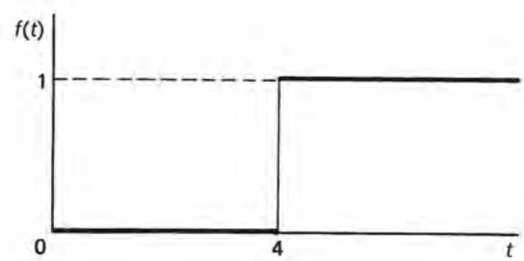
$$f(t) = 0 \quad \text{for } t < c$$
$$f(t) = 1 \quad \text{for } t \geq c$$

This function is the *Heaviside unit step function* and is denoted by

$f(t) = u(t - c)$

where the c indicates the value of t at which the function changes from a value of 0 to a value of 1.

Thus, the function



is denoted by $f(t) = \ldots\ldots\ldots\ldots$

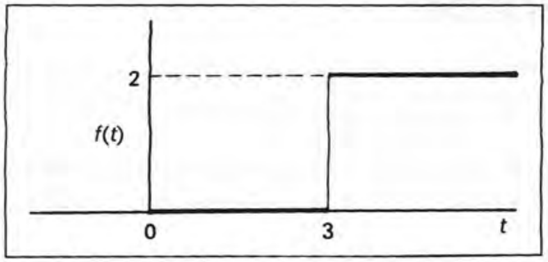
2

$$\boxed{f(t) = u(t - 4)}$$

Similarly, the graph of $f(t) = 2u(t - 3)$ is

$\ldots\ldots\ldots\ldots$

3



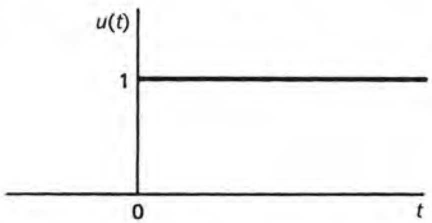
So $u(t-c)$ has just two values

for $t < c$, $u(t-c) = \ldots\ldots\ldots\ldots$
for $t \geq c$, $u(t-c) = \ldots\ldots\ldots\ldots$

4

$$\boxed{t < c,\ u(t-c) = 0; \quad t \geq c,\ u(t-c) = 1}$$

Unit step at the origin

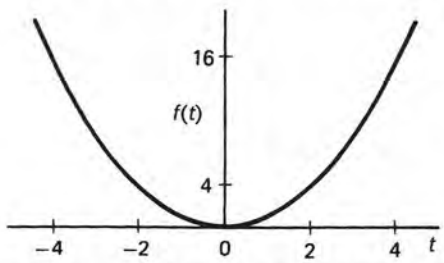


If the unit step occurs at the origin, then $c = 0$ and $f(t) = u(t-c)$ becomes

$$f(t) = u(t)$$

i.e. $u(t) = 0$ for $t < 0$
$u(t) = 1$ for $t \geq 0$.

Effect of the unit step function



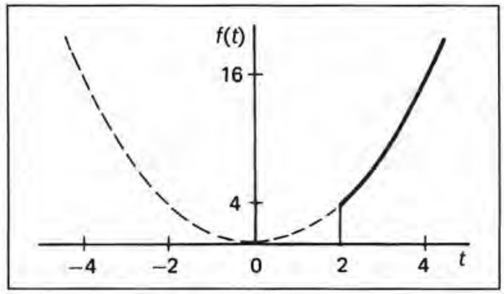
The graph of $f(t) = t^2$ is, of course, as shown.

Remembering the definition of $u(t-c)$, the graph of

$f(t) = u(t-2) \cdot t^2$ is

$\ldots\ldots\ldots\ldots$

5



For $t < 2$, $u(t-2) = 0 \quad \therefore \; u(t-2) \cdot t^2 = 0 \cdot t^2 = 0$

$t \geq 2$, $u(t-2) = 1 \quad \therefore \; u(t-2) \cdot t^2 = 1 \cdot t^2 = t^2$

So the function $u(t-2)$ suppresses the function t^2 for all values of t up to $t = 2$ and 'switches on' the function t^2 at $t = 2$.

Now we can sketch the graphs of the following functions.

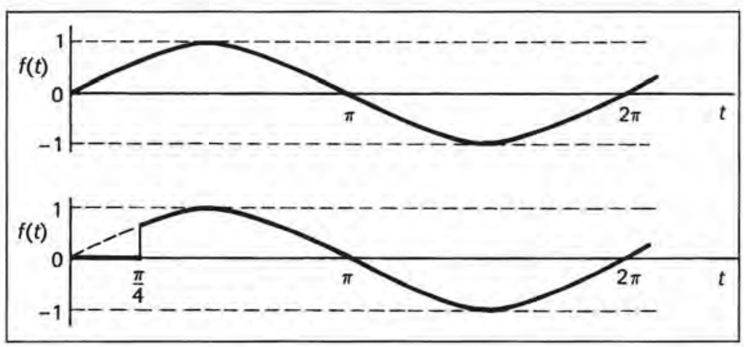
(a) $f(t) = \sin t \quad$ for $0 < t < 2\pi$

(b) $f(t) = u(t - \pi/4) \cdot \sin t \quad$ for $0 < t < 2\pi$.

These give ....

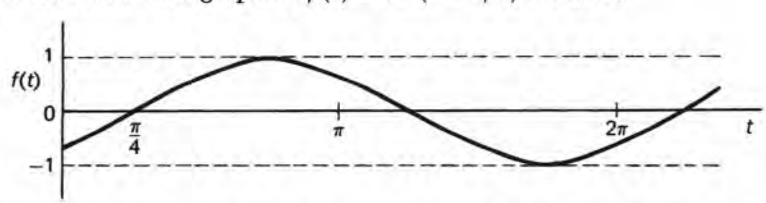
and ....

6



That is, the graph of $f(t) = u(t - \pi/4) \cdot \sin t$ is the graph of $f(t) = \sin t$ but suppressed for all values prior to $t = \pi/4$.

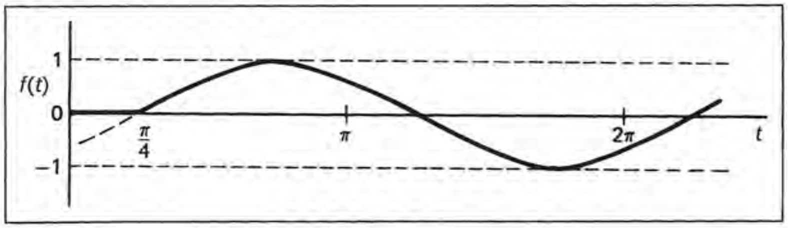
If we sketch the graph of $f(t) = \sin(t - \pi/4)$ we have



Since $u(t - c)$ has the effect of suppressing a function for $t < c$, then the graph of $f(t) = u(t - \pi/4) \cdot \sin(t - \pi/4)$ is

....

7



That is, the graph of $f(t) = u(t - \pi/4) \cdot \sin(t - \pi/4)$ is the graph of $f(t) = \sin t$ $(t > 0)$, shifted $\pi/4$ units along the t-axis.

In general, the graph of $f(t) = u(t - c) \cdot \sin(t - c)$ is the graph of $f(t) = \sin t$ $(t > 0)$, shifted along the t-axis through an interval of c units.

Similarly, for $t > 0$, sketch the graphs of

(a) $f(t) = e^{-t}$

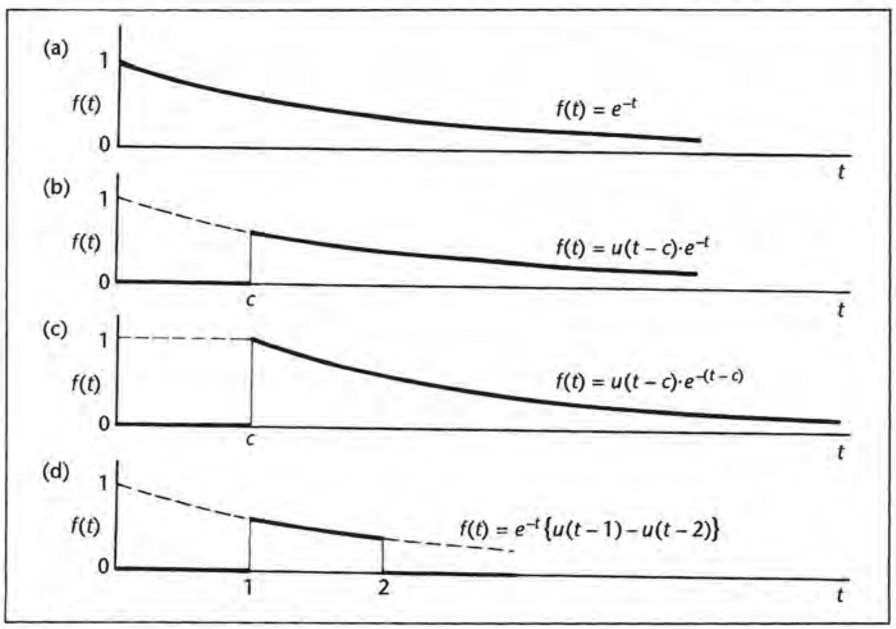
(b) $f(t) = u(t - c) \cdot e^{-t}$

(c) $f(t) = u(t - c) \cdot e^{-(t-c)}$

(d) $f(t) = e^{-t}\{u(t - 1) - u(t - 2)\}$.

Arrange the graphs under each other to show the important differences.

8



In (a), we have the graph of $f(t) = e^{-t}$
In (b), the same graph is suppressed prior to $t = c$
In (c), the graph of $f(t) = e^{-t}$ is shifted c units along the t-axis
In (d), the graph of $f(t) = e^{-t}$ is turned on at $t = 1$ and off at $t = 2$.

Laplace transform of $u(t - c)$

$$L\{u(t-c)\} = \frac{e^{-cs}}{s}$$

Because

$$L\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)\,\mathrm{d}t$$

but

$$e^{-st}u(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$L\{u(t-c)\} = \int_0^\infty e^{-st}u(t-c)\,\mathrm{d}t = \int_c^\infty e^{-st}\,\mathrm{d}t$$
$$= \left[\frac{e^{-st}}{-s}\right]_c^\infty = \frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of the unit step at the origin is

$$L\{u(t)\} = \ldots\ldots\ldots\ldots$$

9

$$\boxed{\frac{1}{s}}$$

Because $c = 0$.

So $$L\{u(t-c)\} = \frac{e^{-cs}}{s}$$

and $$L\{u(t)\} = \frac{1}{s}.$$

Also from the definition of $u(t)$:

$$L(1) = L\{1 \cdot u(t)\}$$
$$L(t) = L\{t \cdot u(t)\}$$
$$L\{f(t)\} = L\{f(t) \cdot u(t)\}$$

Make a note of these results: we shall be using them

10

As we have seen, the unit step function $u(t-c)$ is often combined with other functions of t, so we now consider the Laplace transform of $u(t-c)\cdot f(t-c)$.

Laplace transform of $u(t-c)\cdot f(t-c)$ (the second shift theorem)

$$L\{u(t-c)\cdot(f(t-c)\}=e^{-cs}L\{f(t)\}=e^{-cs}F(s)$$

Because

$$L\{u(t-c)\cdot f(t-c)\}=\int_0^\infty e^{-st}u(t-c)\cdot f(t-c)\,dt$$

$$\text{but } e^{-st}u(t-c)=\begin{cases}0 & \text{for } 0<t<c\\ e^{-st} & \text{for } t\geq c\end{cases}$$

so that

$$L\{u(t-c)\cdot f(t-c)\}=\int_c^\infty e^{-st}f(t-c)\,dt$$

We now make the substitution $t-c=v$ so that $t=c+v$ and $dt=dv$. Also for the limits, when $t=c$, $v=0$ and when $t\to\infty$, $v\to\infty$. Therefore

$$\begin{aligned}L\{u(t-c)\cdot f(t-c)\}&=\int_0^\infty e^{-s(c+v)}f(v)\,dv\\&=e^{-cs}\int_0^\infty e^{-sv}f(v)\,dv\end{aligned}$$

Now $\int_0^\infty e^{-sv}f(v)\,dv$ has exactly the same value as $\int_0^\infty e^{-st}f(t)\,dt$ which is, of course, the Laplace transform of $f(t)$. Therefore

$$L\{u(t-c)\cdot f(t-c)\}=e^{-cs}L\{f(t)\}=e^{-cs}F(s)$$

11

$$\boxed{L\{u(t-c)\cdot f(t-c)\}=e^{-cs}\cdot F(s)\quad\text{where } F(s)=L\{f(t)\}}$$

$$\begin{aligned}\text{So } L\left\{u(t-4)\cdot(t-4)^2\right\}&=e^{-4s}\cdot F(s)\quad\text{where } F(s)=L\{t^2\}\\&=e^{-4s}\left(\frac{2!}{s^3}\right)=\frac{2e^{-4s}}{s^3}\end{aligned}$$

Note that $F(s)$ is the transform of t^2 and *not* of $(t-4)^2$.

In the same way:

$$L\{u(t-3)\cdot\sin(t-3)\}=\ldots\ldots\ldots\ldots$$

12

$$\boxed{\dfrac{e^{-3s}}{s^2+1}}$$

Because $L\{u(t-3)\cdot\sin(t-3)\} = e^{-3s}\cdot F(s)$ where $F(s) = L\{\sin t\}$

$$= \frac{1}{s^2+1}$$

$$\therefore\ L\{u(t-3)\cdot\sin(t-3)\} = e^{-3s}\left(\frac{1}{s^2+1}\right)$$

So now do these in the same way.

(a) $L\left\{u(t-2)\cdot(t-2)^3\right\} = \ldots\ldots\ldots\ldots$

(b) $L\{u(t-1)\cdot\sin 3(t-1)\} = \ldots\ldots\ldots\ldots$

(c) $L\{u(t-5)\cdot e^{(t-5)}\} = \ldots\ldots\ldots\ldots$

(d) $L\{u(t-\pi/2)\cdot\cos 2(t-\pi/2)\} = \ldots\ldots\ldots\ldots$

13

Here they are

(a) $L\left\{u(t-2)\cdot(t-2)^3\right\} = e^{-2s}\cdot F(s)$ where $F(s) = L\{t^3\}$

$$= e^{-2s}\left(\frac{3!}{s^4}\right) = \frac{6e^{-2s}}{s^4}$$

(b) $L\{u(t-1)\cdot\sin 3(t-1)\} = e^{-s}\cdot F(s)$ where $F(s) = L\{\sin 3t\}$

$$= e^{-s}\left(\frac{3}{s^2+9}\right) = \frac{3e^{-s}}{s^2+9}$$

(c) $L\left\{u(t-5)\cdot e^{(t-5)}\right\} = e^{-5s}\cdot F(s)$ where $F(s) = L\{e^t\}$

$$= e^{-5s}\left(\frac{1}{s-1}\right) = \frac{e^{-5s}}{s-1}$$

(d) $L\{u(t-\pi/2)\cdot\cos 2(t-\pi/2)\} = e^{-\pi s/2}\cdot F(s)$ where $F(s) = L\{\cos 2t\}$

$$= e^{-\pi s/2}\left(\frac{s}{s^2+4}\right) = \frac{s\cdot e^{-\pi s/2}}{s^2+4}$$

So $L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s)$ where $F(s) = L\{f(t)\}$.

Written in reverse, this becomes

If $F(s) = L\{f(t)\}$, then $e^{-cs}\cdot F(s) = L\{u(t-c)\cdot f(t-c)\}$

where c is real and positive.

This is known as the *second shift theorem.*

Make a note of it: then we will use it

14

$$\boxed{\text{If } F(s) = L\{f(t)\}, \text{ then } e^{-cs} \cdot F(s) = L\{u(t-c) \cdot f(t-c)\}}$$

This is useful in finding inverse transforms, as we shall now see.

Example 1

Find the function whose transform is $\dfrac{e^{-4s}}{s^2}$.

The numerator corresponds to e^{-cs} where $c = 4$ and therefore indicates $u(t-4)$.

Then $\dfrac{1}{s^2} = F(s) = L\{t\} \quad \therefore\ f(t) = t$.

$$\therefore\ L^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} = u(t-4)\cdot(t-4)$$

Remember that in writing the final result, $f(t)$ is replaced by

....

15

$$\boxed{f(t-c)}$$

Example 2

Determine $L^{-1}\left\{\dfrac{6e^{-2s}}{s^2+4}\right\}$.

The numerator contains e^{-2s} and therefore indicates ....

16

$$\boxed{u(t-2)}$$

The remainder of the transform, i.e. $\dfrac{6}{s^2+4}$, can be written as $3\left(\dfrac{2}{s^2+4}\right)$

$$\therefore\ \frac{6}{s^2+4} = F(s) = L\{\ldots\ldots\ldots\}$$

17

$$\boxed{L\{3\sin 2t\}}$$

$$\therefore\ L^{-1}\left\{\frac{6e^{-2s}}{s^2+4}\right\} = \ldots\ldots\ldots$$

18

$$\boxed{3u(t-2)\cdot\sin 2(t-2)}$$

Because

$$L^{-1}\left\{\frac{6e^{-2s}}{s^2+4}\right\} = u(t-2)\cdot f(t-2) \quad \text{where } f(t) = L^{-1}\left\{\frac{6}{s^2+4}\right\}$$
$$= u(t-2)\cdot 3\sin 2(t-2)$$

Example 3

Determine $L^{-1}\left\{\dfrac{s\cdot e^{-s}}{s^2+9}\right\}$.

This, in similar manner, is ....

19

$$\boxed{u(t-1)\cdot\cos 3(t-1)}$$

Because the numerator contains e^{-s} which indicates $u(t-1)$.
Also $\dfrac{s}{s^2+9} = F(s) = L\{\cos 3t\}$

$\therefore\ f(t) = \cos 3t \quad \therefore\ f(t-1) = \cos 3(t-1)$.

$$\therefore\ L^{-1}\left\{\frac{s\cdot e^{-s}}{s^2+9}\right\} = u(t-1)\cdot\cos 3(t-1)$$

Remember that, having obtained $f(t)$, the result contains $f(t-c)$.
Here is a short exercise by way of practice.

Exercise

Determine the inverse transforms of the following.

(a) $\dfrac{2e^{-5s}}{s^3}$ (d) $\dfrac{2s\cdot e^{-3s}}{s^2-16}$

(b) $\dfrac{3e^{-2s}}{s^2-1}$ (e) $\dfrac{5e^{-s}}{s}$

(c) $\dfrac{8e^{-4s}}{s^2+4}$ (f) $\dfrac{s\cdot e^{-s/2}}{s^2+2}$

20

Results – all very straightforward.

(a) $u(t-5)\cdot(t-5)^2$

(b) $3u(t-2)\cdot\sinh(t-2)$

(c) $4u(t-4)\cdot\sin 2(t-4)$

(d) $2u(t-3)\cdot\cosh 4(t-3)$

(e) $5u(t-1)$

(f) $u(t-1/2)\cdot\cos\sqrt{2}(t-1/2)$.

Before looking at a more interesting example, let us collect our results together as far as we have gone.

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The main points are

$$\text{(a)}\ u(t-c) = 0 \quad 0 < t < c \qquad (1)$$
$$= 1 \quad t \geq c$$

$$\text{(b)}\ L\{u(t-c)\} = \frac{e^{-cs}}{s} \qquad (2)$$
$$L\{u(t)\} = \frac{1}{s}$$

$$\text{(c)}\ L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s) \quad \text{where } F(s) = L\{f(t)\} \qquad (3)$$

$$\text{(d) If } F(s) = L\{f(t)\}, \text{ then } e^{-cs}\cdot F(s) = L\{u(t-c)\}\cdot f(t-c)\} \qquad (4)$$

Now let us apply these to some further examples.

Example 1

Determine the expression $f(t)$ for which

$$L\{f(t)\} = \frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{5e^{-2s}}{s^2}$$

We take each term in turn and find its inverse transform.

(a) $L^{-1}\left\{\frac{3}{s}\right\} = 3L^{-1}\left\{\frac{1}{s}\right\} = 3$ i.e. $3u(t)$

(b) $L^{-1}\left\{\frac{4e^{-s}}{s^2}\right\} = u(t-1)\cdot 4(t-1)$

(c) $L^{-1}\left\{\frac{5e^{-2s}}{s^2}\right\} = \ldots\ldots\ldots\ldots$

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$$\boxed{u(t-2)\cdot 5(t-2)}$$

So we have $L^{-1}\left\{\frac{3}{s}\right\} = 3u(t)$

$$L^{-1}\left\{\frac{4e^{-s}}{s^2}\right\} = u(t-1)\cdot 4(t-1)$$

$$L^{-1}\left\{\frac{5e^{-2s}}{s^2}\right\} = u(t-2)\cdot 5(t-2)$$

$$\therefore\ F(t) = 3u(t) - u(t-1)\cdot 4(t-1) + u(t-2)\cdot 5(t-2)$$

To sketch the graph of $f(t)$ we consider the values of the function within the three sections $0 < t < 1$, $1 < t < 2$, and $2 < t$.

Between $t = 0$ and $t = 1$, $f(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{f(t) = 3}$$

Because in this interval, $u(t) = 1$, but $u(t-1) = 0$ and $u(t-2) = 0$. In the same way, between $t = 1$ and $t = 2$, $f(t) = \ldots\ldots\ldots\ldots$

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$$\boxed{f(t) = 7 - 4t}$$

Because between $t = 1$ and $t = 2$, $u(t) = 1$, $u(t-1) = 1$, but $u(t-2) = 0$.

$$\therefore\ f(t) = 3 - 4(t-1) + 0 = 3 - 4t + 4 = 7 - 4t$$

Similarly, for $t > 2$, $f(t) = \dots\dots\dots\dots$

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$$\boxed{f(t) = t - 3}$$

Because for $t > 2$, $u(t) = 1$, $u(t-1) = 1$ and $u(t-2) = 1$

$$\begin{aligned}\therefore\ f(t) &= 3 - 4(t-1) + 5(t-2)\\ &= 3 - 4t + 4 + 5t - 10 = t - 3\end{aligned}$$

So, collecting the results together, we have

$$\begin{array}{llll}\text{for} & 0 < t < 1, & f(t) = 3 & \\ & 1 < t < 2, & f(t) = 7 - 4t & (t = 1, f(t) = 3;\ t = 2, f(t) = -1)\\ & 2 < t, & f(t) = t - 3 & (t = 2, f(t) = -1;\ t = 3, f(t) = 0)\end{array}$$

Using these facts we can sketch the graph of $f(t)$, which is

$\dots\dots\dots\dots$

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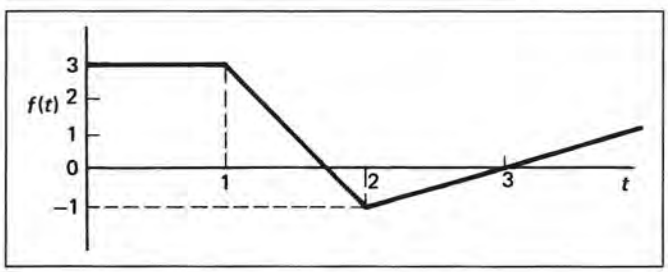

Here is another.

Example 2

Determine the expression $f(t) = L^{-1}\left\{\dfrac{2}{s} + \dfrac{3e^{-s}}{s^2} - \dfrac{3e^{-3s}}{s^2}\right\}$ and sketch the graph of $f(t)$.

First we express the inverse transform of each term in terms of the unit step function.

This gives $\dots\dots\dots\dots$

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$$L^{-1}\left\{\frac{2}{s}\right\} = 2u(t); \quad L^{-1}\left\{\frac{3e^{-s}}{s^2}\right\} = u(t-1)\cdot 3(t-1)$$
$$L^{-1}\left\{\frac{3e^{-3s}}{s^2}\right\} = u(t-3)\cdot 3(t-3)$$

$$\therefore \; f(t) = 2u(t) + u(t-1)\cdot 3(t-1) - u(t-3)\cdot 3(t-3)$$

So there are 'break points', i.e. changes of function, at $t = 1$ and $t = 3$, and we investigate $f(t)$ within the three intervals.

$0 < t < 1 \quad f(t) = \ldots\ldots\ldots\ldots$
$1 < t < 3 \quad f(t) = \ldots\ldots\ldots\ldots$
$3 < t \quad f(t) = \ldots\ldots\ldots\ldots$

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$$0 < t < 1,\ f(t) = 2; \quad 1 < t < 3,\ f(t) = 3t - 1; \quad 3 < t,\ f(t) = 8$$

Because with

$0 < t < 1,\quad u(t) = 1$, but $u(t-1) = u(t-3) = 0 \qquad \therefore\ f(t) = 2$
$1 < t < 3,\quad u(t) = 1,\ u(t-1) = 1$, but $u(t-3) = 0$
$\qquad \therefore\ f(t) = 2 + 3(t-1) = 3t - 1 \qquad \therefore\ f(t) = 3t - 1$
$3 < t,\quad u(t) = 1,\ u(t-1) = 1,\ u(t-3) = 1$
$\qquad \therefore\ f(t) = 2 + 3t - 3 - 3t + 9 \qquad \therefore\ f(t) = 8$

Therefore, the graph of $f(t)$ is ....

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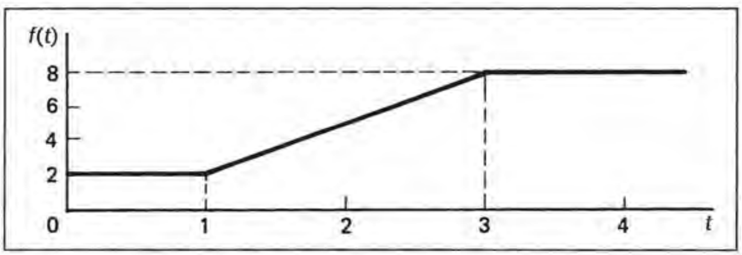

Between the break points, $f(t) = 3t - 1 \qquad \begin{cases} t = 1,\ f(t) = 2 \\ t = 3,\ f(t) = 8 \end{cases}$

Now move on for the next example

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Example 3

If $f(t) = L^{-1}\left\{\dfrac{(1 - e^{-2s})(1 + e^{-4s})}{s^2}\right\}$, determine $f(t)$ and sketch the graph of the function.

Although at first sight this looks more complicated, we simply multiply out the numerator and proceed as before.

▶

$$f(t) = L^{-1}\left\{\frac{1 - e^{-2s} + e^{-4s} - e^{-6s}}{s^2}\right\}$$
$$= L^{-1}\left\{\frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2} - \frac{e^{-6s}}{s^2}\right\}$$

We now write down the inverse transform of each term in terms of the unit function, so that

$$f(t) = \ldots\ldots\ldots\ldots$$

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$$\boxed{f(t) = u(t)\cdot t - u(t-2)\cdot(t-2) + u(t-4)\cdot(t-4) - u(t-6)\cdot(t-6)}$$

and we can see there are break points at $t = 2$, $t = 4$, $t = 6$.

For	$0 < t < 2$,	$f(t) = t - 0 + 0 - 0$	$f(t) = t$
	$2 < t < 4$,	$f(t) = t - (t-2) + 0 - 0$	$f(t) = 2$
	$4 < t < 6$,	$f(t) = t - (t-2) + (t-4) - 0$	$f(t) = t - 2$
	$6 < t$,	$f(t) = t - (t-2) + (t-4) - (t-6)$	$f(t) = 4$

The second and fourth components are constant, but before sketching the graph of the function, we check the values of $f(t) = t$ and $f(t) = t - 2$ at the relevant break points.

$f(t) = t$. At $t = 0$, $f(t) = 0$; at $t = 2$, $f(t) = 2$

$f(t) = t - 2$. At $t = 4$, $f(t) = 2$; at $t = 6$, $f(t) = 4$.

So the graph of the function is $\ldots\ldots\ldots\ldots$

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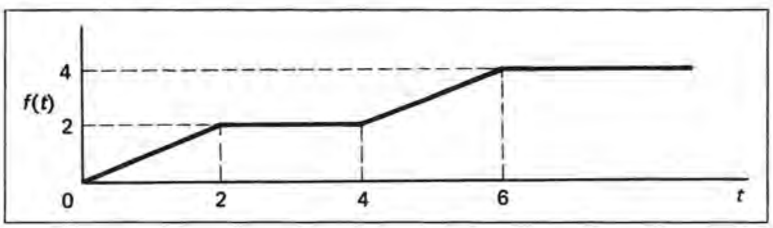

It is always wise to calculate the function values at break points, since discontinuities, or jumps, sometimes occur.

On to the next frame

33

Now for one in reverse.

Example 4

A function $f(t)$ is defined by

$$f(t) = 4 \quad \text{for } 0 < t < 2$$
$$= 2t - 3 \quad \text{for } 2 < t.$$

Sketch the graph of the function and determine its Laplace transform.

We see that for $t = 0$ to $t = 2$, $f(t) = 4$.

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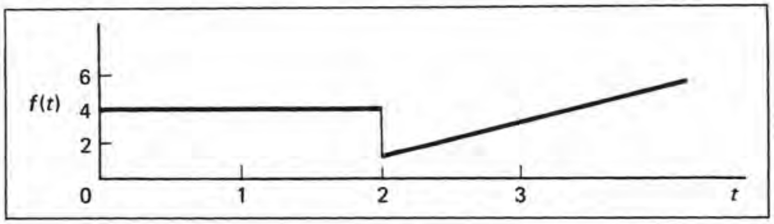

Notice the discontinuity at $t = 2$.

Expressing the function in unit step form:

$$f(t) = 4u(t) - 4u(t-2) + u(t-2) \cdot (2t-3)$$

Note that the second term cancels $f(t) = 4$ at $t = 2$ and that the third switches on $f(t) = 2t - 3$ at $t = 2$.

Before we can express this in Laplace transforms, $(2t - 3)$ in the third term must be written as a function of $(t - 2)$ to correspond to $u(t - 2)$. Therefore, we write $2t - 3$ as $2(t - 2) + 1$.

$$\begin{aligned}\text{Then } f(t) &= 4u(t) - 4u(t-2) + u(t-2) \cdot \{2(t-2) + 1\} \\ &= 4u(t) - 4u(t-2) + u(t-2) \cdot 2(t-2) + u(t-2) \\ &= 4u(t) - 3u(t-2) + u(t-2) \cdot 2(t-2)\end{aligned}$$

$$\therefore \; L\{f(t)\} = \dots\dots\dots\dots$$

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$$\boxed{L\{f(t)\} = \frac{4}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-2s}}{s^2}}$$

Here is one for you to work through in much the same way.

Example 5

A function is defined by
$$\begin{aligned} f(t) &= 6 && 0 < t < 1 \\ &= 8 - 2t && 1 < t < 3 \\ &= 4 && 3 < t. \end{aligned}$$

Sketch the graph and find the Laplace transform of the function.

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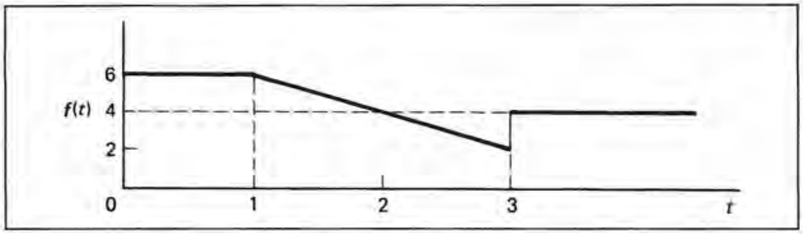

Expressing this in unit step form we have

$$f(t) = 6u(t) - 6u(t-1) + u(t-1)\cdot(8-2t)$$
$$\quad - u(t-3)\cdot(8-2t) + u(t-3)\cdot 4$$

where the second term switches off the first function $f(t) = 6$ at $t = 1$ and the third term switches on the second function $f(t) = 8 - 2t$, which in turn is switched off by the fourth term at $t = 3$ and replaced by $f(t) = 4$ in the fifth term.

Before we can write down the transforms of the third and fourth terms, we must express $f(t) = 8 - 2t$ in terms of $(t - 1)$ and $(t - 3)$ respectively.

$$8 - 2t = 6 + 2 - 2t = 6 - 2(t-1)$$
$$8 - 2t = 2 + 6 - 2t = 2 - 2(t-3)$$
$$\begin{aligned}\therefore\ f(t) &= 6u(t) - 6u(t-1) + u(t-1)\cdot\{6 - 2(t-1)\}\\ &\quad - u(t-3)\cdot\{2 - 2(t-3)\} + 4u(t-3)\\ &= 6u(t) - 6u(t-1) + 6u(t-1)\\ &\quad - u(t-1)\cdot 2(t-1) - 2u(t-3)\\ &\quad + u(t-3)\cdot 2(t-3) + 4u(t-3)\end{aligned}$$

which simplifies finally to $f(t) = \ldots\ldots\ldots\ldots$

$$\boxed{f(t) = 6u(t) - u(t-1)\cdot 2(t-1) + u(t-3)\cdot 2(t-3) + 2u(t-3)}$$ 37

from which $L\{f(t)\} = \ldots\ldots\ldots\ldots$

$$\boxed{L\{f(t)\} = \frac{6}{s} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2} + \frac{2e^{-3s}}{s}}$$ 38

Note that, in building up the function in unit step form

(a) to 'switch on' a function $f(t)$ at $t = c$, we add the term $u(t-c)\cdot f(t-c)$

(b) to 'switch off' a function $f(t)$ at $t = c$, we subtract $u(t-c)\cdot f(t-c)$.

You have now reached the end of this Programme and this brings you to the **Revision summary** and the **Can You?** checklist. Following that is the **Test exercise**. Work through this *at your own pace*. A set of **Further problems** provides additional valuable practice.

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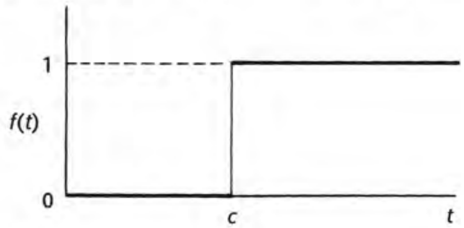
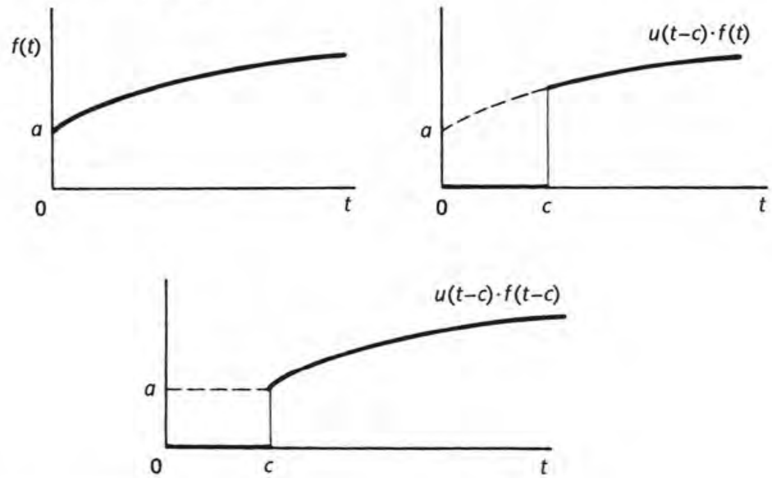
Revision summary 3

1 *Heaviside unit step function: $u(t-c)$*



$$f(t) = 0 \quad 0 < t < c$$
$$ = 1 \quad c < t$$

2 *Suppression and shift*



3 *Laplace transform of $u(t-c)$*

$$L\{u(t-c)\} = \frac{e^{-cs}}{s}; \quad L\{u(t)\} = \frac{1}{s}.$$

4 *Laplace transform of $u(t-c) \cdot f(t-c)$*

$$L\{u(t-c) \cdot f(t-c)\} = e^{-cs} \cdot F(s) \quad \text{where } F(s) = L\{f(t)\}.$$

5 *Second shift theorem*

If $F(s) = L\{f(t)\}$, then $e^{-cs} \cdot F(s) = L\{u(t-c) \cdot f(t-c)\}$ where c is real and positive.

Can You?

Checklist 3

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Use the Heaviside unit step function to 'switch' expressions on and off? 1 to 8

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the Laplace transform of expressions involving the Heaviside unit step function? 8 to 38

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 3

41

1 In each of the following cases, sketch the graph of the function and find its Laplace transform.

(a) $f(t) = 3t \quad 0 < t < 2$
$\quad\;\; = 6 \quad 2 < t$

(b) $f(t) = e^{-2t} \quad 0 < t < 3$
$\quad\;\; = 0 \quad 3 < t$

(c) $f(t) = t^2 \quad 0 < t < 2$
$\quad\;\; = 2 \quad 2 < t < 3$
$\quad\;\; = 4 \quad 3 < t$

(d) $f(t) = \sin 2t \quad 0 < t < \pi$
$\quad\;\; = 0 \quad \pi < t.$

2 Determine the function $f(t)$ whose transform $F(s)$ is

$$F(s) = \frac{1}{s}\left\{2 - 5e^{-s} + 8e^{-3s}\right\}.$$

Sketch the graph of the function between $t = 0$ and $t = 4$.

3 If $f(t) = L^{-1}\left\{\dfrac{(1+3e^{-2s})(1-e^{-3s})}{s^2}\right\}$, determine $f(t)$ and sketch the graph of the function.

4 Determine the function $f(t)$ for which

$$f(t) = L^{-1}\left\{\frac{2(1-e^{-s})}{s(1-e^{-3s})}\right\}.$$

Sketch the waveform and express the function in analytical form.

Further problems 3

42

1 If $L\{f(t)\} = \dfrac{1}{s^2}\left\{3s + 2e^{-2s} - 2e^{-5s}\right\}$, determine $f(t)$.

2 If $f(t) = L^{-1}\left\{\dfrac{(1-e^{-s})(1+e^{-2s})}{s^2}\right\}$, find $f(t)$ in terms of the unit step function.

3 A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= 4 && 0 < t < 3 \\ &= 2t+1 && 3 < t. \end{aligned}$$

Sketch the graph of the function and determine its Laplace transform.

4 Express in terms of the Heaviside unit step function

(a) $$\begin{aligned} f(t) &= t^2 && 0 < t < 3 \\ &= 5t && 3 < t. \end{aligned}$$

(b) $$\begin{aligned} f(t) &= \cos t && 0 < t < \pi \\ &= \cos 2t && \pi < t < 2\pi \\ &= \cos 3t && 2\pi < t. \end{aligned}$$

5 A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= 0 && 0 < t < 2 \\ &= t+1 && 2 < t < 3 \\ &= 0 && 3 < t. \end{aligned}$$

Determine $L\{f(t)\}$.

6 A function $f(t)$ is defined by

$$\begin{aligned} f(t) &= t^2 && 0 < t < 2 \\ &= 4 && 2 < t < 5 \\ &= 0 && 5 < t. \end{aligned}$$

Determine (a) the function in terms of the unit step function
(b) the Laplace transform of $f(t)$.

Laplace transforms 3

Frames 1 to 70

Learning outcomes

When you have completed this Programme you will be able to:

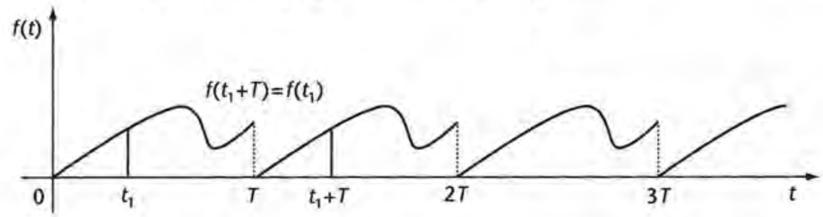
- Find the Laplace transforms of periodic functions
- Obtain the inverse Laplace transforms of transforms of periodic functions
- Describe and use the unit impulse to evaluate integrals
- Obtain the Laplace transform of the unit impulse
- Use the Laplace transform to solve differential equations involving the unit impulse
- Solve the equation and describe the behaviour of an harmonic oscillator

Laplace transforms of periodic functions

1

Periodic functions

Let $f(t)$ represent a periodic function with period T so that $f(t+nT) = f(t)$ with a graph of the following form



If we describe the first cycle by $\bar{f}(t)$ then

$$\bar{f}(t) = \begin{cases} f(t) & \text{for } 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

The second cycle is identical to the first cycle except that it is shifted by T units of time along the t-axis. Therefore the second cycle can be described in terms of the Heaviside unit step function as $\bar{f}(t-T)u(t-T)$. That is

$$\bar{f}(t-T)u(t-T) = \begin{cases} f(t) & \text{for } T \le t < 2T \\ 0 & \text{otherwise} \end{cases}$$

By this reasoning the periodic function $f(t)$ is represented by

$$f(t) = \bar{f}(t)u(t) + \dots\dots\dots\dots$$

2

$$\boxed{f(t) = \bar{f}(t)u(t) + \bar{f}(t-T)u(t-T) + \bar{f}(t-2T)u(t-2T) + \cdots}$$

Because

$u(t)$ switches on $\bar{f}(t)$ at time $t = 0$, $u(t-T)$ switches on $\bar{f}(t-T)$ at time $t = T$ and $u(t-2T)$ switches on $\bar{f}(t-2T)$ at time $t = 2T$, etc.

Consider now the Laplace transform of $\bar{f}(t)$. By definition

$$L\{\bar{f}(t)\} = \int_0^\infty e^{-st}\bar{f}(t)\,\mathrm{d}t = \int_0^T e^{-st}f(t)\,\mathrm{d}t = \bar{F}(s)$$

because for $t > T$, $\bar{f}(t) = 0$ and so the semi-infinite integral becomes an integral just over the period of $f(t)$. Using the second shift theorem (see Frame 10 of Programme 3), the Laplace transform of $f(t)$ is

$$L\{f(t)\} = L\{\bar{f}(t)u(t)\} + L\{\bar{f}(t-T)u(t-T)\} + L\{\bar{f}(t-2T)u(t-2T)\} + \cdots$$

That is

$$L\{f(t)\} = \dots\dots\dots\dots$$

3

$$\boxed{L\{f(t)\} = \bar{F}(s) + e^{-sT}\bar{F}(s) + e^{-2sT}\bar{F}(s) + \cdots}$$

Because

$L\{\bar{f}(t)u(t-c)\} = e^{-sc}L\{\bar{f}(t)\}$ by the second shift theorem.

We can factor out $\bar{F}(s)$ and write $L\{f(t)\}$ as

$$L\{f(t)\} = (1 + e^{-sT} + e^{-2sT} + \ldots)\bar{F}(s)$$

Now, do you remember the series $1 + x + x^2 + x^3 + \ldots$? This can be written in closed form as

$$1 + x + x^2 + x^3 + \ldots = \ldots\ldots\ldots\ldots$$

4

$$\boxed{1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}}$$

Because

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

either by the binomial theorem or by performing the long division.

So, if we let $x = e^{-sT}$ then

$$1 + e^{-sT} + e^{-2sT} + \ldots = \ldots\ldots\ldots\ldots$$

5

$$\boxed{1 + e^{-sT} + e^{-2sT} + \ldots = \frac{1}{1-e^{-sT}}}$$

And so the Laplace transform of $f(t)$ is given as

$L\{f(t)\} = (1 + e^{-sT} + e^{-2sT} + \ldots)\bar{F}(s) = \ldots\ldots\ldots\ldots$ where $\bar{F}(s) = \ldots\ldots\ldots\ldots$

6

$$\boxed{L\{f(t)\} = \frac{1}{(1-e^{-sT})}\bar{F}(s) \text{ where } \bar{F}(s) = \int_0^T e^{-st}f(t)\,dt}$$

Note that we integrate $e^{-st}f(t)$ over one cycle, that is from $t = 0$ to $t = T$, and not from $t = 0$ to $t = \infty$ as we did previously.

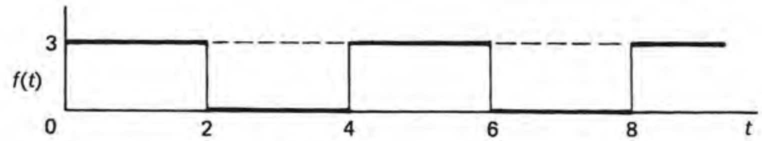
This is an important result. Make a note of it – then we shall apply it

▶

Example 1

Find the Laplace transform of the function $f(t)$ defined by

$$\left.\begin{aligned} f(t) &= 3 \quad 0 < t < 2 \\ &= 0 \quad 2 < t < 4 \end{aligned}\right\} \quad f(t+4) = f(t)$$



The expression for $L\{f(t)\}$ is

.... (do not evaluate it yet)

7

$$\boxed{L\{f(t)\} = \frac{1}{1 - e^{-4s}} \int_0^4 e^{-st} \cdot f(t)\,dt}$$

Because the period = 4, i.e. $T = 4$.

The function $f(t) = 3$ for $0 < t < 2$ and $f(t) = 0$ for $2 < t < 4$.

$$\therefore\ L\{f(t)\} = \frac{1}{1 - e^{-4s}} \int_0^2 e^{-st} \cdot 3\,dt = \ldots\ldots\ldots\ldots$$

8

$$\boxed{L\{f(t)\} = \frac{3}{s(1 + e^{-2s})}}$$

Because

$$L\{f(t)\} = \frac{3}{1 - e^{-4s}} \left[\frac{e^{-st}}{-s}\right]_0^2 = \frac{3}{1 - e^{-4s}} \left\{ \left(\frac{e^{-2s}}{-s}\right) - \left(\frac{1}{-s}\right) \right\}$$

$$= \frac{3}{1 - e^{-4s}} \left\{ \frac{1 - e^{-2s}}{s} \right\} = \frac{3}{s(1 + e^{-2s})}$$

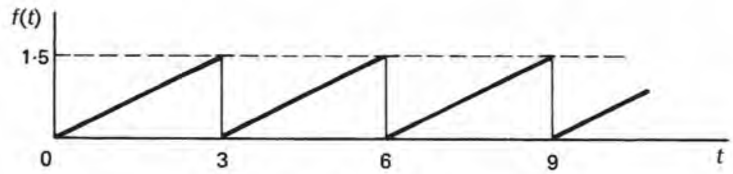
That is all there is to it. Now for another, so move on

9

Example 2

Find the Laplace transform of the periodic function defined by

$$f(t) = t/2 \qquad 0 < t < 3$$
$$f(t+3) = f(t)$$



Because in this case, period $= 3$, i.e. $T = 3$.

$$\therefore\ L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\cdot f(t)\,dt$$
$$= \frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\cdot\left(\frac{t}{2}\right)dt$$
$$\therefore\ 2(1-e^{-3s})L\{f(t)\} = \int_0^3 t\cdot e^{-st}\,dt$$

Integrating by parts and simplifying the result gives

$L\{f(t)\} = \ldots\ldots\ldots\ldots$

10

$$\boxed{L\{f(t)\} = \frac{1}{2s^2}\left\{1 - \frac{3s}{e^{3s}-1}\right\}}$$

Because

$$2(1-e^{-3s})L\{f(t)\} = \int_0^3 te^{-st}\,dt$$
$$= \left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^3 + \frac{1}{s}\int_0^3 e^{-st}\,dt$$
$$= -\frac{3e^{-3s}}{s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^3$$
$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}$$
$$\therefore\ L\{f(t)\} = \frac{1}{2s^2}\left\{1 - \frac{3se^{-3s}}{1-e^{-3s}}\right\}$$
$$= \frac{1}{2s^2}\left\{1 - \frac{3s}{e^{3s}-1}\right\}$$

▶

Example 3

Sketch the graph of the function

$$f(t) = e^t \qquad 0 < t < 5$$
$$f(t+5) = f(t)$$

and determine its Laplace transform.

First we sketch the graph of $f(t)$, which is ....

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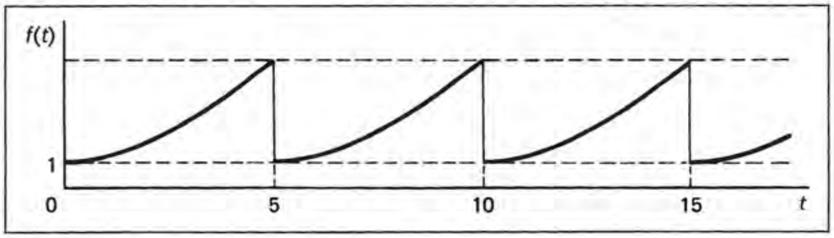

Clearly, period $= 5 \quad \therefore\ T = 5$

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}\cdot f(t)\,dt \quad \text{gives}$$

$$L\{f(t)\} = \ldots\ldots\ldots\ldots$$

Complete the working

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$$\boxed{L\{f(t)\} = \frac{1-e^{-5(s-1)}}{(s-1)(1-e^{-5s})}}$$

Because

$$L\{f(t)\} = \frac{1}{1-e^{-5s}}\int_0^5 e^{-st}\cdot e^t\,dt$$

$$\therefore\ (1-e^{-5s})L\{f(t)\} = \int_0^5 e^{-(s-1)t}\,dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^5 = \frac{1}{s-1}\left\{1-e^{-5(s-1)}\right\}$$

$$\therefore\ L\{f(t)\} = \frac{1-e^{-5(s-1)}}{(s-1)(1-e^{-5s})}$$

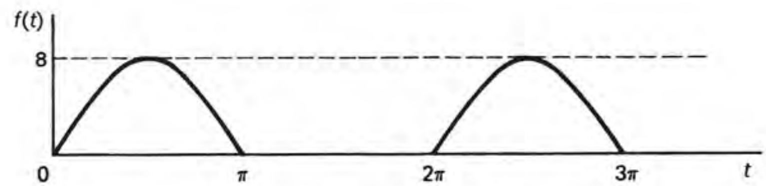
All very straightforward.

▶

Example 4

Determine the Laplace transform of the half-wave rectifier output waveform defined by

$$\left.\begin{aligned} f(t) &= 8\sin t \quad 0 < t < \pi \\ &= 0 \qquad\quad \pi < t < 2\pi \end{aligned}\right\} \quad f(t + 2\pi) = f(t)$$



Here the period is 2π i.e. $T = 2\pi$.

In general, for a periodic function of period T

$$L\{f(t)\} = \dots\dots\dots\dots$$

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$$\boxed{L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\, \mathrm{d}t}$$

So, for this example

$$L\{f(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} \cdot f(t)\, \mathrm{d}t$$

$$\therefore \ (1 - e^{-2\pi s})L\{f(t)\} = \int_0^{\pi} e^{-st} \cdot 8\sin t\, \mathrm{d}t$$

Writing $\sin t$ as the imaginary part of e^{jt}, i.e. $\sin t \equiv \mathscr{I} e^{jt}$,

$$\begin{aligned} (1 - e^{-2\pi s})L\{f(t)\} &= 8\mathscr{I} \int_0^{\pi} e^{-st} \cdot e^{jt}\, \mathrm{d}t \\ &= 8\mathscr{I} \int_0^{\pi} e^{-(s-j)t}\, \mathrm{d}t \end{aligned}$$

and this you can finish off in the usual manner, giving

$$L\{f(t)\} = \dots\dots\dots\dots$$

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$$\boxed{L\{f(t)\} = \frac{8}{(s^2+1)(1-e^{-\pi s})}}$$

Because

$$(1-e^{-2\pi s})L\{f(t)\} = 8 \cdot \mathscr{I}\int_0^\pi e^{-(s-j)t}\,dt$$

$$= 8 \cdot \mathscr{I}\left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^\pi$$

$$= \mathscr{I}\left\{\frac{-8}{s-j}\left[e^{-(s-j)\pi} - 1\right]\right\}$$

$$= 8 \cdot \mathscr{I}\left\{\frac{1}{s-j}\left[1 - e^{-s\pi}e^{j\pi}\right]\right\}$$

But $e^{j\pi} = \cos\pi + j\sin\pi = -1$.

$$\therefore\ (1-e^{-2\pi s})L\{f(t)\} = 8 \cdot \mathscr{I}\left\{\frac{1}{s-j}(1+e^{-s\pi})\right\}$$

$$= 8 \cdot \mathscr{I}\left\{\frac{s+j}{s^2+1}(1+e^{-\pi s})\right\} = 8\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$$

$$\therefore\ L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \times 8\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$$

$$= \frac{8}{(1-e^{-\pi s})(s^2+1)}$$

Now let us consider the corresponding inverse transforms when periodic functions are involved.

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Inverse transforms

Finding inverse transforms of functions of s which are transforms of periodic functions is not as straightforward as in earlier examples, for the transforms result from integration over one cycle and not from $t = 0$ to $t = \infty$. Hence we have no simple table of inverse transforms upon which to draw.

However, all difficulties can be surmounted and an example will show how we deal with this particular problem.

Example 1

Determine the inverse transform

$$L^{-1}\left\{\frac{2+e^{-2s}-3e^{-s}}{s(1-e^{-2s})}\right\}$$

The first thing we see is the factor $(1-e^{-2s})$ in the denominator, which suggests a periodic function of period 2 units, i.e. $\dfrac{1}{1-e^{-Ts}}$ where $T = 2$.

The key to the solution is to write $(1-e^{-2s})$ in the denominator as $(1-e^{-2s})^{-1}$ in the numerator and to expand this as a binomial series.

We remember that $(1-x)^{-1} = \ldots\ldots\ldots\ldots$

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$$\boxed{(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots}$$

$$\therefore\ (1-e^{-2s})^{-1} = 1 + (e^{-2s}) + (e^{-2s})^2 + (e^{-2s})^3 + \dots$$
$$= 1 + e^{-2s} + e^{-4s} + e^{-6s} + \dots$$
$$\therefore\ L\{f(t)\} = \frac{2 + e^{-2s} - 3e^{-s}}{s(1-e^{-2s})} = \frac{1}{s}(2 + e^{-2s} - 3e^{-s})(1-e^{-2s})^{-1}$$
$$= \frac{1}{s}(2 + e^{-2s} - 3e^{-s})(1 + e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + \dots)$$

We now multiply the second series by each term of the first in turn and collect up like terms, giving

$$L\{f(t)\} = \frac{1}{s}\left\{\begin{array}{lllllll} 2 & & +2e^{-2s} & & +2e^{-4s} & & +2e^{-6s}\ \dots \\ & & +\ e^{-2s} & & +\ e^{-4s} & & +\ e^{-6s}\ \dots \\ & -3e^{-s} & & -3e^{-3s} & & -3e^{-5s} & \quad\ \ \dots \end{array}\right\}$$
$$= \dots\dots\dots\dots$$

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$$\boxed{L\{f(t)\} = \frac{1}{s}\{2 - 3e^{-s} + 3e^{-2s} - 3e^{-3s} + 3e^{-4s} - 3e^{-5s} + \dots\}}$$

Each term is of the form $\dfrac{e^{-cs}}{s}$, so, expressing $f(t)$ in unit step form, we have

$$f(t) = \dots\dots\dots\dots$$

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$$\boxed{f(t) = 2u(t) - 3u(t-1) + 3u(t-2) - 3u(t-3) + 3u(t-4)\dots}$$

and from this we can sketch the waveform, which is therefore

$\dots\dots\dots\dots$

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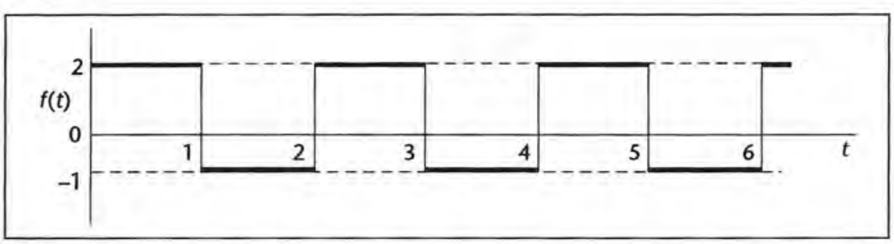

We can finally define this periodic function in analytical terms.

$$f(t) = \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} f(t) &= 2 \quad 0 < t < 1 \\ &= -1 \quad 1 < t < 2 \end{aligned} \Bigg\} \; f(t+2) = f(t)}$$

The key to the whole process is thus to ....

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express $(1 - e^{-Ts})$ in the denominator as $(1 - e^{-Ts})^{-1}$ in the numerator and to expand this as a binomial series.

We do this by making use of the basic series

$$(1 - x)^{-1} = \dots\dots\dots\dots$$

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$$\boxed{(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots}$$

Example 2

Determine $L^{-1}\left\{\dfrac{3(1 - e^{-s})}{s(1 - e^{-3s})}\right\}$ and sketch the resulting waveform of $f(t)$.

$$L\{f(t)\} = \frac{3}{s}(1 - e^{-s})(1 - e^{-3s})^{-1}$$
$$= \dots\dots\dots\dots \qquad \text{(next step)}$$

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$$\boxed{L\{f(t)\} = \frac{3}{s}(1 - e^{-s})(1 + e^{-3s} + e^{-6s} + e^{-9s} + \dots)}$$

which multiplied out gives

$$\begin{aligned} L\{f(t)\} &= \frac{3}{s}(1 - e^{-s} + e^{-3s} - e^{-4s} + e^{-6s} - e^{-7s} + \dots) \\ &= \frac{3}{s} - \frac{3e^{-s}}{s} + \frac{3e^{-3s}}{s} - \frac{3e^{-4s}}{s} + \frac{3e^{-6s}}{s} - \dots \end{aligned}$$

And in unit step form, this gives

$$f(t) = \dots\dots\dots\dots$$

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$$\boxed{f(t) = 3u(t) - 3u(t-1) + 3u(t-3) - 3u(t-4) + \dots}$$

The waveform is thus ....

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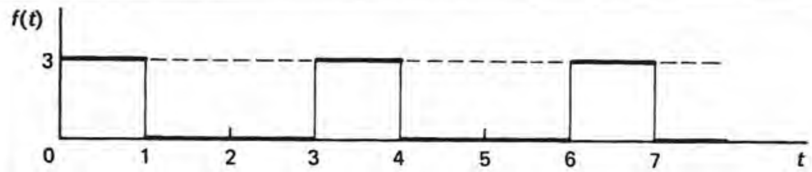

$$\left.\begin{array}{ll} f(t) = 3 & 0 < t < 1 \\ f(t) = 0 & 1 < t < 3 \end{array}\right\} f(t+3) = f(t)$$

And now, one more. They are all done in the same way

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Example 3

If $L\{f(t)\} = \dfrac{1}{2s^2} - \dfrac{2e^{-4s}}{s(1-e^{-4s})}$, determine $f(t)$ and sketch the waveform.

The first term is easy enough. In unit step form $L^{-1}\left\{\dfrac{1}{2s^2}\right\} = \dfrac{t}{2} \cdot u(t)$

From the second term

$$\begin{aligned} \frac{2e^{-4s}}{s(1-e^{-4s})} &= \frac{2}{s}\left\{e^{-4s}\left(1-e^{-4s}\right)^{-1}\right\} \\ &= \frac{2}{s}\left\{e^{-4s}\left(1+e^{-4s}+e^{-8s}+e^{-12s}+\ldots\right)\right\} \\ &= \frac{2e^{-4s}}{s} + \frac{2e^{-8s}}{s} + \frac{2e^{-12s}}{s} + \frac{2e^{-16s}}{s} + \ldots \end{aligned}$$

$\therefore\ f(t) = \ldots\ldots\ldots\ldots$ (in unit step form)

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$$\boxed{f(t) = \frac{t}{2} \cdot u(t) - 2u(t-4) - 2u(t-8) - 2u(t-12) - \ldots}$$

Now we have to draw the waveform. Consider the function terms up to each break point in turn.

$$\begin{array}{lll} 0 < t < 4 & f(t) = \dfrac{t}{2} & f(0) = 0;\ \ f(4) = 2 \\ 4 < t < 8 & f(t) = \dfrac{t}{2} - 2 & f(4) = 0;\ \ f(8) = 2 \\ 8 < t < 12 & f(t) = \dfrac{t}{2} - 2 - 2 & f(8) = 0;\ \ f(12) = 2 \text{ etc.} \end{array}$$

So the waveform is $\ldots\ldots\ldots\ldots$

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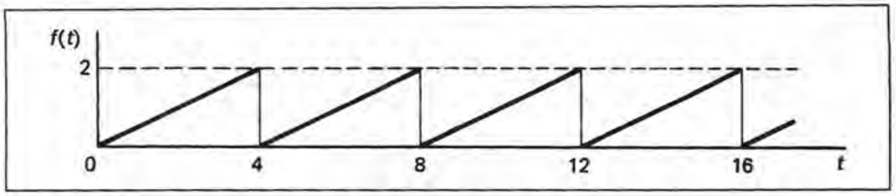

Expressed analytically, we finally have

$$f(t) = \frac{t}{2} \quad 0 < t < 4, \quad f(t+4) = f(t)$$

The Dirac delta – the unit impulse

29

So far we have dealt with a number of standard Laplace transforms and then the Heaviside unit step function with some of its applications. We now come to consider an entity that is different from any of the functions we have used before because it is not a proper function. Rather than being defined by its inputs and corresponding outputs it is defined by its effect on other functions. If $f(t)$ represents a function then the Dirac delta $\delta(t)$ is defined by the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$$

$\delta(t)$ is often referred to as the **Dirac delta function** even though it is not a function in the conventional sense of being completely defined in terms of its outputs for the corresponding inputs. The nearest that can be achieved in defining it in function terms is

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

From the definition, if $f(t) = 1$ then

$$\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = 1}$$

Because

$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$ and $f(t) = 1$ so $f(a) = 1$, therefore

$\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = 1$ hence the name *unit impulse*.

Also, if $p < a < q$ then

$$\int_{p}^{q} \delta(t-a)\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_p^q \delta(t-a)\,\mathrm{d}t = 1}$$

Because

$$\int_{-\infty}^{\infty} \delta(t-a)\,\mathrm{d}t = \int_{-\infty}^{p} \delta(t-a)\,\mathrm{d}t + \int_p^q \delta(t-a)\,\mathrm{d}t + \int_q^{\infty} \delta(t-a)\,\mathrm{d}t$$

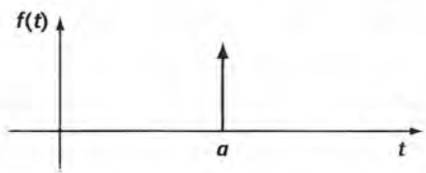
$$= 0 + \int_p^q \delta(t-a)\,\mathrm{d}t + 0 \quad \text{since } \delta(t-a) = 0 \text{ for } -\infty < t \le p \text{ and } q \le t < \infty$$

$$= 1$$

So that $\displaystyle\int_p^q \delta(t-a)\,\mathrm{d}t = 1$

Graphical representation

Graphically the Dirac delta or unit impulse $\delta(t-a)$ is represented by the horizontal axis with a vertical line of infinite length at $t = a$.



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So far, then, we have

(a) $\displaystyle\int_p^q \delta(t-a)\,\mathrm{d}t = 1$

(b) $\displaystyle\int_p^q f(t)\cdot\delta(t-a)\,\mathrm{d}t = f(a)$

provided, in each case, that $p < a < q$.

Example 1

To evaluate $\displaystyle\int_1^3 (t^2+4)\cdot\delta(t-2)\,\mathrm{d}t$.

The factor $\delta(t-2)$ shows that the impulse occurs at $t = 2$, i.e. $a = 2$.

$f(t) = t^2 + 4 \qquad \therefore\ f(a) = f(2) = 4 + 4 = 8$

$\therefore\ \displaystyle\int_1^3 (t^2+4)\cdot\delta(t-2)\,\mathrm{d}t = f(2) = 8$

Example 2

To evaluate $\int_0^{\pi} \cos 6t \cdot \delta(t - \pi/2)\,dt$.

$$\int_0^{\pi} \cos 6t \cdot \delta(t - \pi/2)\,dt = f(\pi/2) = \cos 3\pi = -1$$

and in the same way

(a) $\int_0^6 5 \cdot \delta(t-3)\,dt = \dots\dots\dots\dots$

(b) $\int_2^5 e^{-2t} \cdot \delta(t-4)\,dt = \dots\dots\dots\dots$

(c) $\int_0^{\infty} (3t^2 - 4t + 5) \cdot \delta(t-2)\,dt = \dots\dots\dots\dots$

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(a) $\int_0^6 5 \cdot \delta(t-3)\,dt = 5 \times 1 = 5$

(b) $\int_2^5 e^{-2t} \cdot \delta(t-4)\,dt = f(4) = \left[e^{-2t}\right]_{t=4} = e^{-8}$

(c) $\int_0^{\infty} (3t^2 - 4t + 5) \cdot \delta(t-2)\,dt = 12 - 8 + 5 = 9$

Nothing could be easier. It all rests on the fact that, provided $p < a < q$

$$\int_p^q f(t) \cdot \delta(t-a)\,dt = \dots\dots\dots\dots$$

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$$\boxed{f(a)}$$

Now let us consider the Laplace transform of $\delta(t-a)$.

On then to the next frame

35 Laplace transform of $\delta(t-a)$

We have already shown that

$$\int_p^q f(t) \cdot \delta(t-a)\,dt = f(a) \qquad p < a < q$$

Therefore, if $p = 0$ and $q = \infty$

$$\int_0^{\infty} f(t) \cdot \delta(t-a)\,dt = f(a)$$

Hence, if $f(t) = e^{-st}$, this becomes

$$\int_0^{\infty} e^{-st} \cdot \delta(t-a)\,dt = L\{\delta(t-a)\}$$

$$= \dots\dots\dots\dots$$

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$$\boxed{e^{-as}}$$

i.e. the value of $f(t)$, i.e. e^{-st}, at $t = a$.

$L\{\delta(t-a)\} = e^{-as}$

It follows from this that the Laplace transform of the impulse function at the origin is ....

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$$\boxed{1}$$

Because, for $a = 0$, $L\{\delta(t-a)\} = L\{\delta(t)\} = e^0 = 1$

$$\therefore\ L\{\delta(t)\} = 1$$

Finally, let us deal with the more general case of $L\{f(t)\cdot\delta(t-a)\}$. We have $L\{f(t)\cdot\delta(t-a)\} = \int_0^\infty e^{-st}\cdot f(t)\cdot\delta(t-a)\,dt$. Now the integrand $e^{-st}\cdot f(t)\cdot\delta(t-a) = 0$ for all values of t except at $t = a$ at which point $e^{-st} = e^{-as}$, and $f(t) = f(a)$.

$$\therefore\ L\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}\int_0^\infty \delta(t-a)\,dt$$
$$= f(a)\cdot e^{-as}(1)$$
$$\therefore\ L\{f(t)\cdot\delta(t-a)\} = f(a)e^{-as}$$

Another important result to note. Then let us deal with some examples

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We have $L\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}$

Therefore

(a) $L\{6\cdot\delta(t-4)\}$ $\quad a = 4$, $\quad\therefore\ L\{6\cdot\delta(t-4)\} = 6e^{-4s}$
(b) $L\{t^3\cdot\delta(t-2)\}$ $\quad a = 2$, $\quad\therefore\ L\{t^3\cdot\delta(t-2)\} = 8e^{-2s}$

Similarly

(c) $L\{\sin 3t\cdot\delta(t-\pi/2)\} =$ ....

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$$\boxed{-e^{-\pi s/2}}$$

Because

$$L\{\sin 3t\cdot\delta(t-\pi/2)\} = [\sin 3t]_{t=\pi/2}\cdot e^{-\pi s/2} = -e^{-\pi s/2}$$

and

(d) $L\{\cosh 2t\cdot\delta(t)\} =$ ....

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$\boxed{1}$

Because

$$L\{\cosh 2t \cdot \delta(t)\} = [\cosh 2t]_{t=0} \cdot e^0 = \cosh 0 \cdot (1) = 1$$

So our main conclusions so far are as follows.

(1) $\int_p^q \delta(t-a)\,dt = \ldots\ldots\ldots\ldots$ provided $\ldots\ldots\ldots\ldots$

(2) $\int_p^q f(t)\cdot\delta(t-a)\,dt = \ldots\ldots\ldots\ldots$ provided $\ldots\ldots\ldots\ldots$

(3) $L\{\delta(t-a)\} = \ldots\ldots\ldots\ldots$

(4) $L\{\delta(t)\} = \ldots\ldots\ldots\ldots$

(5) $L\{f(t)\cdot\delta(t-a)\} = \ldots\ldots\ldots\ldots$

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(1) $\int_p^q \delta(t-a)\,dt = 1$ provided $p < a < q$

(2) $\int_p^q f(t)\cdot\delta(t-a)\,dt = f(a)$ provided $p < a < q$

(3) $L\{\delta(t-a)\} = e^{-as}$

(4) $L\{\delta(t)\} = 1$

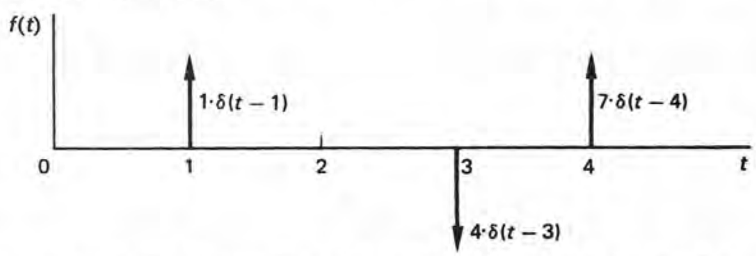
(5) $L\{f(t)\cdot\delta(t-a)\} = f(a)\cdot e^{-as}$

Just check that you have noted this important list – the basis of all work on the Dirac delta function.

Now for one further example on this section

Example

Impulses of 1, 4, 7 units occur at $t = 1$, $t = 3$ and $t = 4$ respectively, in the directions shown.



Write down an expression for $f(t)$ and determine its Laplace transform.

We have $f(t) = 1\cdot\delta(t-1) - 4\cdot\delta(t-3) + 7\cdot\delta(t-4)$.

Then $L\{f(t)\} = \ldots\ldots\ldots\ldots$

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$$\boxed{L\{f(t)\} = e^{-s} - 4e^{-3s} + 7e^{-4s}}$$

and that is all there is to that.

The derivative of the unit step function

One further consideration is interesting.

Consider some function $f(t)$ that is zero outside some finite interval $[a, b]$ of the real line. That is, $f(t) = 0$ for $t < a$ and $t > b$, then

$$\int_{-\infty}^{\infty} [u(t)f(t)]' \, dt = [u(t)f(t)]_{-\infty}^{\infty} = 0$$

where $u(t)$ is the unit step function and $f(t)$ is zero at the limits. Now

$$\int_{-\infty}^{\infty} [u(t)f(t)]' \, dt = \int_{-\infty}^{\infty} u'(t)f(t) \, dt + \int_{-\infty}^{\infty} u(t)f'(t) \, dt$$

and so

$$\int_{-\infty}^{\infty} u'(t)f(t) \, dt = -\int_{-\infty}^{\infty} u(t)f'(t) \, dt$$

This means that

$$\begin{aligned}
\int_{-\infty}^{\infty} u'(t)f(t) \, dt &= -\int_{-\infty}^{\infty} u(t)f'(t) \, dt \\
&= -\int_{0}^{\infty} f'(t) \, dt && \text{Because the unit step is zero for negative } t \\
&= -\Big[f(t)\Big]_{0}^{\infty} \\
&= -f(\infty) + f(0) \\
&= f(0) && \text{Because } f(\infty) = 0 \text{ by definition} \\
&= \int_{-\infty}^{\infty} \delta(t)f(t) \, dt && \text{By the definition of the Dirac delta}
\end{aligned}$$

and so $u'(t) = \delta(t)$ – *the unit impulse is equal to the derivative of the unit step function.*

Differential equations involving the unit impulse

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Example 1

A system has the equation of motion

$$\ddot{x} + 6\dot{x} + 8x = g(t)$$

where $g(t)$ is an impulse of 4 units applied at $t = 5$. At $t = 0$, $x = 0$ and $\dot{x} = 3$. Determine an expression for the displacement x in terms of t.

The impulse of 4 units is applied at $t = 5$. $\therefore\ g(t) = 4 \cdot \delta(t - 5)$.

$$\therefore\ \ddot{x} + 6\dot{x} + 8x = 4 \cdot \delta(t - 5) \quad \text{At } t = 0,\ x = 0,\ \dot{x} = 3.$$

Taking Laplace transforms this differential equation becomes

....

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$$\boxed{(s^2\bar{x} - sx_0 - x_1) + 6(s\bar{x} - x_0) + 8\bar{x} = 4e^{-5s}}$$

Now $x_0 = 0$; $x_1 = 3$

$$\therefore\ s^2\bar{x} - 3 + 6s\bar{x} + 8\bar{x} = 4e^{-5s}$$

$$\therefore\ (s^2 + 6s + 8)\bar{x} = 3 + 4e^{-5s}$$

$$\therefore\ \bar{x} = (3 + 4e^{-5s})\frac{1}{(s+2)(s+4)}$$

Writing $\dfrac{1}{(s+2)(s+4)}$ in partial fractions, we get

$\bar{x} =$ ....

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$$\boxed{\bar{x} = (3 + 4e^{-5s})\left\{\frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{2} \cdot \frac{1}{s+4}\right\}}$$

$$\therefore\ \bar{x} = \frac{3}{2}\left\{\frac{1}{s+2} - \frac{1}{s+4}\right\} + 2\left\{\frac{e^{-5s}}{s+2} - \frac{e^{-5s}}{s+4}\right\}$$

Taking inverse transforms

$$x = \frac{3}{2}\{e^{-2t} - e^{-4t}\} + 2\left\{e^{-2(t-5)} \cdot u(t-5) - e^{-4(t-5)} \cdot u(t-5)\right\}$$

$$= \frac{3}{2}\{e^{-2t} - e^{-4t}\} + 2\{e^{-2t} \cdot e^{10} \cdot u(t-5) - e^{-4t} \cdot e^{20} \cdot u(t-5)\}$$

which simplifies to $x =$ ....

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$$\boxed{x = e^{-2t}\left\{\frac{3}{2} + 2e^{10} \cdot u(t-5)\right\} - e^{-4t}\left\{\frac{3}{2} + 2e^{20} \cdot u(t-5)\right\}}$$

Example 2

Solve the equation $\ddot{x} + 4\dot{x} + 13x = 2 \cdot \delta(t)$ where, at $t = 0$, $x = 2$ and $\dot{x} = 0$.

$$\ddot{x} + 4\dot{x} + 13x = 2 \cdot \delta(t) \qquad x_0 = 2;\ x_1 = 0$$

Expressing in Laplace transforms, we have

....

47

$$\boxed{(s^2\bar{x} - sx_0 - x_1) + 4(s\bar{x} - x_0) + 13\bar{x} = 2 \cdot (1)}$$

Inserting the initial conditions and simplifying,

$$\bar{x} = \text{............}$$

48

$$\boxed{\bar{x} = (2s + 10)\frac{1}{s^2 + 4s + 13}}$$

Rearranging the denominator by completing the square, this can be written

$$\bar{x} = (2s + 10)\frac{1}{(s+2)^2 + 9}$$

$$\therefore\ x = \text{...........}$$

49

$$\boxed{x = 2e^{-2t}\{\cos 3t + \sin 3t\}}$$

Because

$$\bar{x} = \frac{2(s+2)}{(s+2)^2 + 9} + \frac{6}{(s+2)^2 + 9}$$

$$\therefore\ x = 2e^{-2t}\cos 3t + 2e^{-2t}\sin 3t$$

$$\therefore\ x = 2e^{-2t}\{\cos 3t + \sin 3t\}$$

Now for one further example for you to work through on your own.

So move on

50

Example 3

The equation of motion of a system is

$$\ddot{x} + 5\dot{x} + 4x = g(t) \text{ where } g(t) = 3 \cdot \delta(t-2).$$

At $t = 0$, $x = 2$ and $\dot{x} = -2$. Determine an expression for the displacement x in terms of t.

We have $\ddot{x} + 5\dot{x} + 4x = 3 \cdot \delta(t-2)$ with $x_0 = 2$ and $x_1 = -2$.

As before, you can express this in Laplace transforms, substitute the initial conditions, simplify to obtain an expression for x and finally take inverse transforms to determine the required expression for x.

Work right through it carefully. It is good revision and there are no snags.

$$x = \ldots\ldots\ldots\ldots$$

51

$$\boxed{x = e^{-t}\{2 + e^2 \cdot u(t-2)\} - e^8 \cdot e^{-4t} \cdot u(t-2)}$$

Here is the working for you to check.

$$\ddot{x} + 5\dot{x} + 4x = 3 \cdot \delta(t-2) \text{ with } x_0 = 2 \text{ and } x_1 = -2$$

$$(s^2\bar{x} - sx_0 - x_1) + 5(s\bar{x} - x_0) + 4\bar{x} = 3e^{-2s}$$

$$s^2\bar{x} - 2s + 2 + 5s\bar{x} - 10 + 4\bar{x} = 3e^{-2s}$$

$$(s^2 + 5s + 4)\bar{x} - 2s - 8 = 3e^{-2s}$$

$$\therefore \quad (s+1)(s+4)\bar{x} = 2s + 8 + 3e^{-2s}$$

$$\therefore \quad \bar{x} = \frac{2(s+4)}{(s+1)(s+4)} + e^{-2s} \cdot \frac{3}{(s+1)(s+4)}$$

$$= \frac{2}{s+1} + e^{-2s}\left\{\frac{1}{s+1} - \frac{1}{s+4}\right\}$$

$$\therefore \quad \bar{x} = \frac{2}{s+1} + \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+4}$$

$$\therefore \quad x = 2e^{-t} + u(t-2) \cdot e^{-(t-2)} - u(t-2) \cdot e^{-4(t-2)}$$

$$= 2e^{-t} + u(t-2) \cdot e^2 \cdot e^{-t} - u(t-2) \cdot e^8 \cdot e^{-4t}$$

$$x = e^{-t}\{2 + e^2 \cdot u(t-2)\} - e^8 \cdot e^{-4t} \cdot u(t-2)$$

Harmonic oscillators

52

If the position of a system at time t is described by the expression $f(t)$ where $f(t)$ satisfies the differential equation

$af''(t) + bf(t) = 0$, $f(0) = \alpha$ and $f'(0) = \beta$

(and where a and b have the same sign)

then, taking Laplace transforms of both sides gives

$$L\{af''(t) + bf(t)\} = L\{0\}$$

That is

$$a[s^2F(s) - s\alpha - \beta] + b[F(s)] = 0$$

Collecting like terms gives

$$(as^2 + b)F(s) = s\alpha + \beta$$

giving

$$F(s) = \frac{s\alpha + \beta}{as^2 + b}$$

Therefore $F(s) = \dfrac{s(\alpha/a)}{s^2 + (b/a)} + \dfrac{\beta/a}{s^2 + (b/a)}$ and so

$$f(t) = \frac{\alpha}{a}\cos\sqrt{\frac{b}{a}}t + \frac{\beta}{a}\sin\sqrt{\frac{b}{a}}t$$

The system executes *simple harmonic, oscillatory motion with frequency* $\sqrt{\dfrac{b}{a}}$ radians per unit of time and with period $\dfrac{2\pi}{\sqrt{b/a}} = 2\pi\sqrt{\dfrac{a}{b}}$. It is called an **harmonic oscillator**. Let's try some examples.

Example 1

Find the solution to the harmonic oscillator

$f''(t) + 16f(t) = 0$ where $f(0) = 1$ and $f'(0) = 0$

Taking Laplace transforms gives

$$F(s) = \ldots\ldots\ldots\ldots$$

53

$$\boxed{F(s) = \frac{s}{s^2 + 16}}$$

Because

Taking Laplace transforms $L\{f''(t) + 16f(t)\} = L\{0\}$.

That is $s^2F(s) - s + 16F(s) = 0$ and so

$$F(s) = \frac{s}{s^2 + 16}$$

This means that

$$f(t) = \ldots\ldots\ldots\ldots$$

54

$$\boxed{f(t) = \cos 4t}$$

Because

$F(s) = \dfrac{s}{s^2+16} = \dfrac{s}{s^2+4^2}$ so $f(t) = \cos 4t$ from the Table of Laplace transforms on page 68.

The motion of this system is then periodic with frequency 4 radians per unit of time and with period $2\pi/4 = \pi/2$ units of time.

Example 2

The frequency and period of the harmonic oscillator whose position $f(t)$ satisfies the differential equation

$$5f''(t) + 10f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 4$$

is given as

frequency .... radians per unit of time
and period .... units of time

55

$$\boxed{\text{frequency } \sqrt{2} \text{ and period } \sqrt{2}\pi}$$

Because

Taking Laplace transforms gives

$L\{5f''(t) + 10f(t)\} = L\{0\}$ that is $5s^2F(s) - 4 + 10F(s) = 0$ so that

$$F(s) = \frac{4}{5s^2+10} = \frac{4/5}{s^2+2}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{2\sqrt{2}}{5}\sin\sqrt{2}t$$

This is periodic with frequency $\sqrt{2}$ radians per unit of time and period $2\pi/\sqrt{2} = \sqrt{2}\pi$ units of time.

Notice that the amplitude of the motion is $\dfrac{2\sqrt{2}}{5}$.

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Damped motion

Consider the equation

$$5f''(t) + 5f'(t) + 10f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 4$$

This is the same as the last equation in Frame 54 with an extra term added, namely $5f'(t)$. This term describes a particular effect on the system as you will see from the solution.

Solving the differential equation gives

$$f(t) = \ldots\ldots\ldots\ldots$$

57

$$\boxed{f(t) = \frac{8}{5\sqrt{7}}e^{-t/2}\sin\left(\sqrt{7}t/2\right)}$$

Because

Taking Laplace transforms gives

$L\{5f''(t) + 5f'(t) + 10f(t)\} = L\{0\}$ that is

$5(s^2F(s) - 4) + 5sF(s) + 10F(s) = 0$

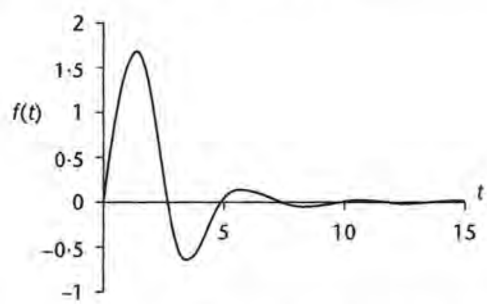
so that

$$F(s) = \frac{20}{5s^2 + 5s + 10} = \frac{4}{s^2 + s + 2} = \frac{4}{(s + 1/2)^2 + (\sqrt{7}/2)^2}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{8}{\sqrt{7}}e^{-t/2}\sin\left(\sqrt{7}t/2\right)$$

This is periodic with frequency 1 radian per unit of time and period 2π units of time but with an amplitude that is decreasing with time. The graph of this function is as follows



The effect of the $5f'(t)$ in the differential equation is to introduce **damping** into the oscillatory motion so causing the oscillations to decay. Let's try another example.

Example 3

Consider the equation

$5f''(t) + f'(t) + 10f(t) = 0$ where $f(0) = 0$ and $f'(0) = 4$

This equation is again similar to the previous equation but with a smaller damping term of $f'(t)$ instead of $5f'(t)$. Then here

$$f(t) = \ldots\ldots\ldots\ldots$$

58

$$\boxed{f(t) = \frac{4}{\sqrt{1{\cdot}99}} e^{-0.1t} \sin \sqrt{1{\cdot}99}t}$$

Because

Taking Laplace transforms gives

$L\{5f''(t) + f'(t) + 10f(t)\} = L\{0\}$ that is

$5(s^2F(s) - 4) + sF(s) + 10F(s) = 0$

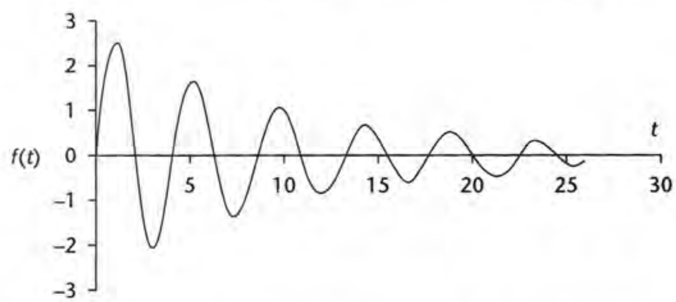
so that

$$F(s) = \frac{20}{5s^2 + 1s + 10} = \frac{4}{s^2 + 0{\cdot}2s + 2} = \frac{4}{(s + 0{\cdot}1)^2 + 1{\cdot}99}$$

and from the Table of Laplace transforms on page 68

$$f(t) = \frac{4}{\sqrt{1{\cdot}99}} e^{-0.1t} \sin \sqrt{1{\cdot}99}t$$

This is periodic with frequency $\sqrt{1{\cdot}99}$ radians per unit of time and period $2\pi/\sqrt{1{\cdot}99}$ units of time and with an amplitude that is decreasing with time. The graph of this function is as follows



Again, the effect of the $f'(t)$ in the differential equation is to introduce damping into the oscillatory motion so causing it to decay. Also because the coefficient of $f'(t)$ is smaller in this example, the damping is less severe.

Forced harmonic motion with damping

59

The equation

$$f''(t)+f'(t)+f(t)=e^t \text{ where } f(0)=0 \text{ and } f'(0)=0$$

we know would represent damped harmonic motion were it not for the exponential on the right-hand side. To see the effect of the exponential we solve the equation.

Taking Laplace transforms we see that

$$F(s)= \ldots\ldots\ldots\ldots$$

60

$$\boxed{F(s)=\frac{1}{(s-1)(s^2+s+1)}}$$

Because

$$L\{f''(t)+f'(t)+f(t)\}=L\{e^t\} \text{ that is } (s^2+s+1)F(s)=\frac{1}{s-1} \text{ so}$$

$$F(s)=\frac{1}{(s-1)(s^2+s+1)}$$

Separating into partial fractions gives

$$F(s)= \ldots\ldots\ldots\ldots$$

61

$$\boxed{F(s)=\frac{1}{3(s-1)}-\frac{s+2}{3(s^2+s+1)}}$$

Because

$$\frac{1}{(s-1)(s^2+s+1)}=\frac{A}{(s-1)}+\frac{Bs+C}{(s^2+s+1)}$$
$$=\frac{A(s^2+s+1)+(Bs+C)(s-1)}{(s-1)(s^2+s+1)}$$

Equating numerators and then comparing coefficients of powers of s gives

$$1=A(s^2+s+1)+(Bs+C)(s-1)$$

$[s^2]$:	$0=A+B$	(1)	So (2) + (3):	$1=2A-B$
$[s]$:	$0=A-B+C$	(2)	$2\times(1)$:	$0=2A+2B$
[CT]:	$1=A-C$	(3)	Therefore:	$-1=3B$

so $B=-1/3=-A$ and $C=-2/3$

$$\text{Thus } F(s)=\frac{1}{(s-1)(s^2+s+1)}=\frac{1}{3(s-1)}-\frac{s+2}{3(s^2+s+1)}$$

Consequently

$$f(t)= \ldots\ldots\ldots\ldots$$

62

$$\boxed{f(t)=\frac{e^t}{3}-\frac{1}{3}e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t+\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)}$$

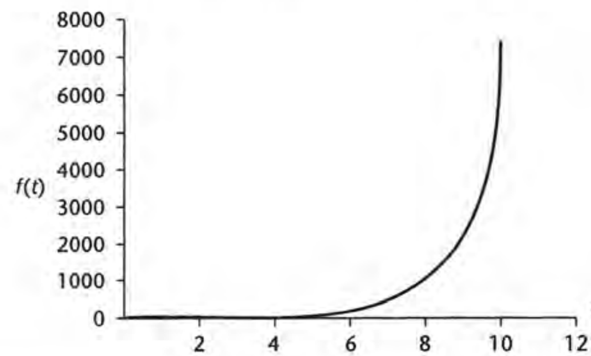
Because

$$\begin{aligned}F(s)&=\frac{1}{3(s-1)}-\frac{s+2}{3(s^2+s+1)}\\&=\frac{1}{3(s-1)}-\frac{s+\frac{1}{2}}{3\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)}-\frac{\frac{3}{2}}{3\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)}\end{aligned}$$

So

$$f(t)=\frac{e^t}{3}-\frac{1}{3}e^{-t/2}\left(\cos\tfrac{\sqrt{3}}{2}t+\sqrt{3}\sin\tfrac{\sqrt{3}}{2}t\right)$$

from the Table of Laplace transforms on page 68.



Notice that the term $\frac{1}{3}e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t+\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)$ represents damped harmonic motion and is called the **transient** term whereas the term $\frac{e^t}{3}$ represents a **steady-state** term, so called because as the transient term decays the steady-state term remains the dominant part of the solution. The steady-state solution is a direct consequence of the term on the right-hand side of the differential equation.

Try another one for yourself. The transient and steady-state terms of the system described by the differential equation

$$f''(t)+2f'(t)+5f(t)=e^{2t} \text{ where } f(0)=0 \text{ and } f'(0)=1$$

are Transient term .... Steady-state term ....

63

$$\boxed{-\frac{1}{13}e^{-t}\cos 2t+\frac{5}{13}e^{-t}\sin 2t,\ \frac{1}{13}e^{2t}}$$

Because

Taking Laplace transforms, $L\{f''(t)+2f'(t)+5f(t)\}=L\{e^{2t}\}$. That is

$[s^2F(s)-1]+2sF(s)+5F(s)=\dfrac{1}{s-2}$, that is

$(s^2+2s+5)F(s)=1+\dfrac{1}{s-2}=\dfrac{s-1}{s-2}$

So that $F(s)=\dfrac{s-1}{(s-2)(s^2+2s+5)}=\dfrac{A}{s-2}+\dfrac{Bs+C}{s^2+2s+5}$. Hence

$s-1=A(s^2+2s+5)+(Bs+C)(s-2)$. Equating powers of s gives

$[s^2]$: $\quad 0=A+B$

$[s]$: $\quad 1=2A-2B+C$

$[CT]$: $\quad -1=5A-2C$

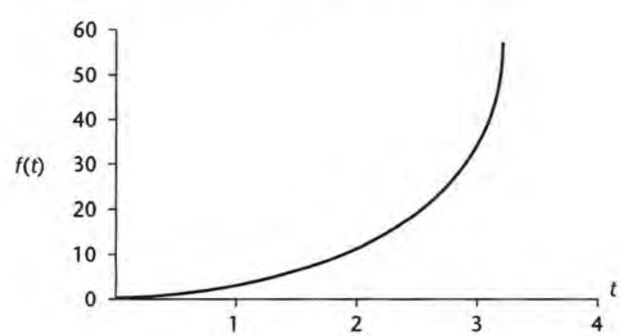
Solving these three equations gives $A=1/13$, $B=-1/13$ and $C=9/13$ so that

$$\begin{aligned}F(s)&=\frac{1}{13(s-2)}-\frac{s-9}{13(s^2+2s+5)}\\&=\frac{1}{13(s-2)}-\frac{s-9}{13\left((s+1)^2+2^2\right)}.\text{ That is}\end{aligned}$$

$$F(s)=\frac{1}{13(s-2)}-\frac{s+1}{13\left((s+1)^2+2^2\right)}+\frac{10}{13\left((s+1)^2+2^2\right)}$$

Therefore

$$f(t)=\frac{1}{13}e^{2t}-\frac{1}{13}e^{-t}\cos 2t+\frac{5}{13}e^{-t}\sin 2t$$



Next frame

64

Resonance

These differential equations with a function on the right-hand side are called **inhomogeneous differential equations**. They represent systems whose behaviour $f(t)$ is dictated by the structure of the left-hand side and the **forcing function** on the right-hand side. If an undamped and unforced system which exhibits periodic behaviour has a periodic forcing function applied that has the same period then **resonance** will occur and the system will undergo periodic behaviour with an increasing amplitude. An example will illustrate this.

The differential equation

$$f''(t) + f(t) = 0 \text{ where } f(0) = 0 \text{ and } f'(0) = 1$$

represents an undamped, unforced system with behaviour

$$f(t) = \ldots\ldots\ldots\ldots$$

65

$$\boxed{f(t) = \sin t}$$

Because

Taking the Laplace transform of both sides of the equation gives

$L\{f''(t) + f(t)\} = L\{0\}$ that is $s^2F(s) - 1 + F(s) = 0$ so that

$F(s) = \dfrac{1}{s^2+1}$ giving $f(t) = \sin t$

If the forcing term $-2\sin t$ is applied to the right-hand side of the equation it has the same period as the natural frequency of the system being forced and so resonance will set in. The differential equation to solve is then

$$f''(t) + f(t) = -2\sin t \text{ where } f(0) = 0 \text{ and } f'(0) = 1$$

This has the solution $f(t) = \ldots\ldots\ldots\ldots$

66

$$\boxed{f(t) = t\cos t}$$

Because

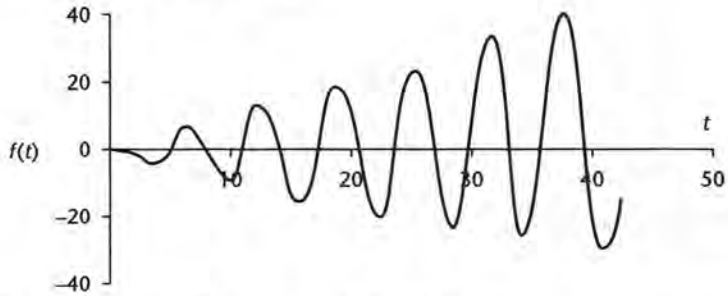
Taking the Laplace transform of both sides of the equation gives

$L\{f''(t) + f(t)\} = L\{-2\sin t\}$ that is $s^2F(s) - 1 + F(s) = -\dfrac{2}{s^2+1}$

so that $F(s) = \dfrac{1}{s^2+1} - \dfrac{2}{(s^2+1)^2}$ giving $F(s) = \dfrac{s^2-1}{(s^2+1)^2}$. Now, the

Laplace transform of $\cos t$ is $\dfrac{s}{s^2+1}$ and $\left(\dfrac{s}{s^2+1}\right)' = -\dfrac{s^2-1}{(s^2+1)^2}$.

Therefore $f(t) = t\cos t$



The system undergoes periodic behaviour with an increasing amplitude.

You have now reached the end of this Programme and this brings you to the **Revision summary** and the **Can You?** checklist. Following that is the **Test exercise**. Work through this *at your own pace*. A set of **Further problems** provides additional valuable practice.

Revision summary 4

67

1 *Periodic functions*

$f(t) = f(t + nT) \qquad n = 1, 2, 3, \ldots \qquad$ Period $= T$.

2 *Laplace transform of a periodic function with period T*

$$L\{f(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} \cdot f(t)\,dt.$$

3 *Inverse transforms involving periodic functions*

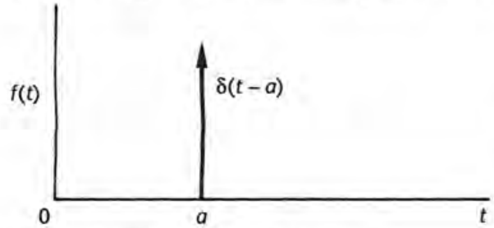
e.g. $L^{-1}\left\{\dfrac{1 + 2e^{-3s} - 3e^{-2s}}{s(1 - e^{-3s})}\right\}$

Expand $(1 - e^{-3s})^{-1}$ as a binomial series, like

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

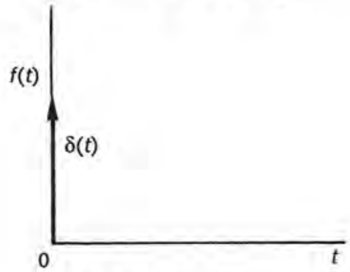
Multiply out and take inverse transforms of each term in turn.

4 *Dirac delta function* or unit impulse function



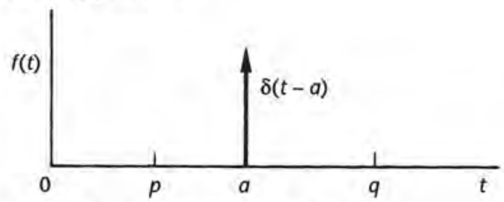
$$\delta(t-a) = 0 \qquad t \neq a$$
$$= \infty \qquad t = a.$$

5 *Delta function at the origin*



$$a = 0 \quad \therefore \ \delta(t) = 0 \qquad t \neq 0$$
$$= \infty \qquad t = 0.$$

6 *Area of pulse = 1*



$$\therefore \ \int_p^q \delta(t-a)\,\mathrm{d}t = 1$$
$$p < a < q$$

7 *Integration of the impulse function*

$$\int_p^q f(t) \cdot \delta(t-a)\,\mathrm{d}t = f(a) \qquad p < a < q$$

8 *Laplace transform of $\delta(t-a)$*

$L\{\delta(t-a)\} = e^{-as}$

$L\{\delta(t)\} = 1$ because $a = 0$

$L\{f(t) \cdot \delta(t-a)\} = f(a) \cdot e^{-as}$.

9 *Harmonic oscillators*

The equation of $af''(t) + bf(t) = 0$, $f(0) = \alpha$ and $f'(0) = \beta$, where a and b are of the same sign, represents a system undergoing simple harmonic motion and is referred to as an harmonic oscillator. The system oscillates with a frequency of $\sqrt{\dfrac{b}{a}}$ radians per unit of time and with period $\dfrac{2\pi}{\sqrt{b/a}} = 2\pi\sqrt{\dfrac{a}{b}}$ units of time. If a first derivative term is added to the left-hand side of the equation then, provided all three coefficients have the same sign, the system will undergo damped harmonic motion.

▶

10 *Forced harmonic motion*
Forced harmonic motion is achieved by the existence of a term on the right-hand side of the equation giving rise to transient and steady-state parts of the solution.

11 *Resonance*
Resonance is exhibited by a system undergoing periodic behaviour with a growing amplitude of vibration. Resonance occurs when a system, whose unforced behaviour is periodic, is forced with the same period.

☑ Can You?

Checklist 4 68

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5, how confident are you that you can: **Frames**

- Find the Laplace transforms of periodic functions? 1 to 14
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Obtain the inverse Laplace transforms of transforms of periodic functions? 15 to 28
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Describe and use the unit impulse to evaluate integrals? 29 to 34
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Obtain the Laplace transform of the unit impulse? 35 to 42
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the Laplace transform to solve differential equations involving the unit impulse? 43 to 51
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Solve the equation and describe the behaviour of an harmonic oscillator? 52 to 66
  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 4

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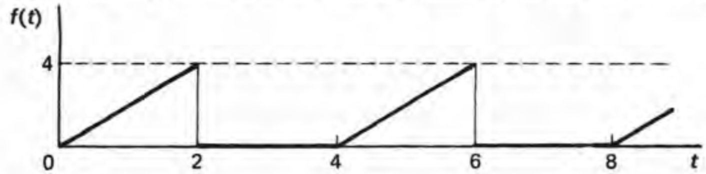
1 Determine the Laplace transform of the periodic function shown.



2 Evaluate

(a) $\displaystyle\int_0^4 e^{-3t} \cdot \delta(t-2)\,dt$

(b) $\displaystyle\int_0^\infty \sin 3t \cdot \delta(t-\pi)\,dt$

(c) $\displaystyle\int_1^3 (2t^2+3) \cdot \delta(t-2)\,dt.$

3 Determine (a) $L\{4 \cdot \delta(t-3)\}$, (b) $L\{e^{-3t} \cdot \delta(t-2)\}$.

4 Sketch the graph of $f(t) = 3 \cdot \delta(t) + 4 \cdot \delta(t-2) - 3 \cdot \delta(t-4)$ and determine its Laplace transform.

5 Solve the equation $\ddot{x} + 6\dot{x} + 10x = 7 \cdot \delta(t)$ given that, at $t = 0$, $x = -1$ and $\dot{x} = 0$.

6 The equation of motion of a system is
$\ddot{x} + 3\dot{x} + 2x = 3 \cdot \delta(t-4)$.
At $t = 0$, $x = 2$ and $\dot{x} = -4$. Determine an expression for the displacement x in terms of t.

7 Find the frequency, periodic time and solution for each of the following harmonic oscillators.
(a) $f''(t) + f(t) = 0$ given that $f(0) = 0$ and $f'(0) = 1$
(b) $6f''(t) + 2f'(t) + 9f(t) = 0$ given that $f(0) = 0$ and $f'(0) = 3$.

8 Find the transient and steady-state solutions of the forced harmonic oscillator
$f''(t) + 2f'(t) + 3f(t) = 4e^{5t}$ given that $f(0) = -2$ and $f'(0) = 6$.

Further problems 4

70

1 If $f(t) = a\sin t \quad 0 < t < \pi$; $= 0 \quad \pi < t < 2\pi$; $f(t + 2\pi) = f(t)$,

$$f(t) = \left.\begin{array}{ll} a\sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{array}\right\} \; f(t+2\pi) = f(t),$$

prove that $L\{f(t)\} = \dfrac{a}{(s^2+1)(1-e^{-\pi s})}$.

2 If $f(t) = a\sin t \quad 0 < t < \pi \quad f(t+\pi) = f(t)$, determine $L\{f(t)\}$.

3 Find the Laplace transforms of the following periodic functions.

(a) $f(t) = t \quad 0 < t < T \quad f(t+T) = f(t)$

(b) $f(t) = e^t \quad 0 < t < 2\pi \quad f(t+2\pi) = f(t)$

(c) $$f(t) = \left.\begin{array}{ll} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{array}\right\} \quad f(t+2) = f(t)$$

(d) $$f(t) = \left.\begin{array}{ll} t^2 & 0 < t < 2 \\ 4 & 2 < t < 3 \end{array}\right\} \quad f(t+3) = f(t)$$

4 A mass M is attached to a spring of stiffness $\omega^2 M$ and is set in motion at $t = 0$ by an impulsive force P. The equation of motion is

$$M\ddot{x} + M\omega^2 x = P\cdot\delta(t).$$

Obtain an expression for x in terms of t.

5 An impulsive voltage E is applied at $t = 0$ to a series circuit containing inductance L and capacitance C. Initially, the current and charge are zero. The current i at time t is given by

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{q}{C} = E\cdot\delta(t)$$

where q is the instantaneous value of the charge on the capacitor. Since $i = \dfrac{\mathrm{d}q}{\mathrm{d}t}$, determine an expression for the current i in the circuit at time t.

6 A system has the equation of motion

$$\ddot{x} + 5\dot{x} + 6x = F(t)$$

where, at $t = 0$, $x = 0$ and $\dot{x} = 2$. If $F(t)$ is an impulse of 20 units applied at $t = 4$, determine an expression for x in terms of t.

7 Find the frequency, periodic time and solution for each of the following harmonic oscillators.

(a) $12f''(t) + f(t) = 0$ given that $f(0) = -1$ and $f'(0) = 2$

(b) $f''(t) + 12f(t) = 0$ given that $f(0) = 2$ and $f'(0) = -1$.

8 Solve for each of the following harmonic oscillators.

(a) $4{\cdot}6f''(t) + 2{\cdot}2f(t) = 0$ given that $f(0) = 1{\cdot}6$ and $f'(0) = -3{\cdot}1$

(b) $\sqrt{2}f''(t) + \sqrt{3}f(t) = 0$ given that $f(0) = 0$ and $f'(0) = \pi$.

9 Find the transient and steady-state solutions of the forced harmonic oscillator

$$4f''(t) + 3f'(t) + 2f(t) = e^t$$

given that $f(0) = 0$ and $f'(0) = 6$.

Z transforms

Frames 1 to 43

Learning outcomes

When you have completed this Programme you will be able to:

- Define the Z transform of a sequence and derive transforms of specified sequences
- Make reference to a table of standard Z transforms
- Recognise the Z transform as being a linear transform and so obtain the transform of linear combinations of standard sequences
- Apply the first and second shift theorems, the translation theorem, the initial and final value theorems and the derivative theorem
- Use partial fractions to derive the inverse transforms
- Solve linear, first-order, constant coefficient recurrence relations
- Demonstrate the relationship between the Laplace transform and the Z transform

Introduction

1

The Laplace transform deals with continuous functions and can be used to solve many differential equations that arise in science and engineering. There are occasions, however, when we have to deal with discrete functions – *sequences* – and their associated **difference equations**. For example, the central processing unit of your computer can only handle information in the form of pulses of electricity. This information transmission is called **digital** transmission. There are, however, times when information is fed into the computer in the form of a continuously varying signal called an **analogue** signal. For instance, a mouse can be moved about the flat surface of your desk in a continuous manner but the central processing unit will only recognise position on the screen to the nearest pixel. The analogue signal coming from the mouse needs to be converted into a digital signal for recognition by the computer's central processing unit. This conversion of a signal from analogue to digital is achieved by a device called a **demodulator** that *samples* the analogue signal at regular intervals of time and outputs the sampled values as the digital signal – as a sequence of values. The Z transform, which is allied to the Laplace transform, deals with such sequences and the recurrence relations – or difference equations – that arise.

Sequences

2

The sequence $\dots, 3^{-2}, 3^{-1}, 3^0, 3, 3^2, 3^3, \dots$ has a general term of the form 3^k and as a shorthand notation we use $\{3^k\}_{-\infty}^{\infty}$ to represent this sequence and to indicate that the powers range from $-\infty$ to ∞. The sum

$$\sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k = \dots + \left(\frac{3}{z}\right)^{-1} + \left(\frac{3}{z}\right)^0 + \left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \dots$$

is called the **Z transform** of the sequence, $Z\{3^k\}_{-\infty}^{\infty}$, and is denoted by $F(z)$, where the complex number z is chosen to ensure that the sum is finite. We say that

$$\{3^k\}_{-\infty}^{\infty} \text{ and } Z\{3^k\}_{-\infty}^{\infty} = F(z) = \sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k \text{ form a } Z \text{ transform pair.}$$

For our purposes we shall consider only *causal sequences* of the form $\{x_k\}_0^\infty$ where $x_k = 0$ for $k < 0$ which for brevity we shall denote by $\{x_k\}$ with corresponding Z transform

$$Z\{x_k\} = F(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}.$$

Notice that this is the *definition* of the Z transform of the sequence $\{x_k\}$. For example, the *unit impulse* sequence $\{\delta_k\} = \{1, 0, 0, 0, \ldots\}$ has the Z transform

$Z\{\delta_k\} = \ldots\ldots\ldots\ldots$ valid for $\ldots\ldots\ldots\ldots$ values of z

3

$$\boxed{Z\{\delta_k\} = 1 \text{ valid for all values of } z}$$

Because

$$Z\{\delta_k\} = \sum_{k=0}^{\infty} \frac{\delta_k}{z_k}$$
$$= 1 + \frac{0}{z} + \frac{0}{z^2} + \ldots = 1$$

Try another.

The sequence $\{u_k\} = \{1, 1, 1, \ldots\} = \{1\}$ is called the *unit step* sequence and has the Z transform

$\ldots\ldots\ldots\ldots$ provided $|z| \ldots\ldots\ldots\ldots$

Next frame

4

$$\boxed{\frac{z}{z-1} \text{ provided } |z| > 1}$$

Because

$$Z\{u_k\} = F(z)$$
$$= \sum_{k=0}^{\infty} \frac{u_k}{z^k} = \sum_{k=0}^{\infty} \frac{1}{z^k}$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots$$

Comparing this to the series expansion of $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ then

$$F(z) = \frac{1}{1 - \frac{1}{z}} \text{ provided } \left|\frac{1}{z}\right| < 1$$
$$= \frac{z}{z-1} \text{ provided } |z| > 1$$

And another.

Given the causal sequence $\{x_k\} = \{1, a, a^2, a^3, a^4, \ldots\} = \{a^k\}$

the Z transform is $\ldots\ldots\ldots\ldots$

Next frame

5

$$\boxed{\frac{z}{z-a} \text{ provided } |z| > a}$$

Because

$$\begin{aligned} Z\{a^k\} &= \sum_{k=0}^{\infty} \frac{a^k}{z^k} \\ &= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k \\ &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \end{aligned}$$

Comparing this to the series expansion of $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ which is valid for $|x| < 1$ then

$$\begin{aligned} F(z) &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \\ &= \frac{1}{1 - \dfrac{a}{z}} \text{ provided } \left|\frac{a}{z}\right| < 1. \end{aligned}$$

That is, multiplying numerator and denominator by z

$$F(z) = \frac{z}{z-a} \text{ provided } |z| > |a|$$

Therefore $\{a^k\}$ and $F(z) = \dfrac{z}{z-a}$, ($|z| > |a|$) form a Z transform pair.

Let's try another. The sequence $\{x_k\} = \{0, 1, 2, 3, 4, \dots\} = \{k\}$ has the Z transform

$$Z\{k\} = F(z) = \dots\dots\dots\dots$$

Answer in the next frame

6

$$\boxed{F(z) = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \dots}$$

Because

$$\begin{aligned} Z\{k\} &= F(z) \\ &= \sum_{k=0}^{\infty} \frac{x_k}{z^k} \\ &= \sum_{k=0}^{\infty} \frac{k}{z^k} \\ &= 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \dots \end{aligned}$$

By comparing this sequence with the derivative of $(1-x)^{-1}$ and its series representation, this sequence can be written as a rational expression in z as $F(z) = \dots\dots\dots\dots$

7

$$\boxed{F(z) = \frac{z}{(z-1)^2}}$$

Because

$$F(z) = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \dots$$

Comparing this with the series expansion

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \dots &= \frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + \dots) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1} = \frac{1}{(1-x)^2} \end{aligned}$$

then we can see that by multiplying $F(z)$ by z

$$zF(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \dots = \frac{1}{(1-1/z)^2}$$

so, dividing both sides by z gives

$$F(z) = \frac{1}{z(1-1/z)^2} = \frac{z}{(z-1)^2}$$

Next frame

Table of Z transforms

8

We list the results that we have obtained so far as well as some additional ones for future reference.

Sequence	**Transform $F(z)$**	**Permitted values of z**
$\{\delta_k\} = \{1, 0, 0, \dots\}$	1	All values of z
$\{u_k\} = \{1, 1, 1, \dots\}$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \dots\}$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \dots\}$	$\dfrac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \dots\}$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \dots\}$	$\dfrac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \dots\}$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

Next frame

Properties of *Z* transforms

1 *Linearity*

9

The Z transform is a linear transform. That is, if a and b are constants then

$$Z(a\{x_k\}+b\{y_k\})=aZ\{x_k\}+bZ\{y_k\}$$

For example, the Z transform of the sequence $\{k\}$ is $Z\{k\}=$ .... and the Z transform of the sequence $\{e^{-2k}\}$ is $Z\{e^{-2k}\}=$ ....

10

$$\boxed{Z\{k\}=\frac{z}{(z-1)^2}\text{ and }Z\{e^{-2k}\}=\frac{z}{z-e^{-2}}}$$

Because

$Z\{k\}=\dfrac{z}{(z-1)^2}$ from the table and, also from the table,

$Z\{a^k\}=\dfrac{z}{z-a}$ so when $a=e^{-2}$,

$$Z\{e^{-2k}\}=\frac{z}{z-e^{-2}}$$

Consequently, the Z transform of $3\{k\}-5\{e^{-2k}\}$ is ....

11

$$\boxed{\frac{-5z^3+13z^2-z(3e^{-2}+5)}{(z-1)^2(z-e^{-2})}}$$

Because

$$\begin{aligned}Z(3\{k\}-5\{e^{-2k}\})&=3Z\{k\}-5Z\{e^{-2k}\}\\&=\frac{3z}{(z-1)^2}-\frac{5z}{(z-e^{-2})}\\&=\frac{3z(z-e^{-2})-5z(z-1)^2}{(z-1)^2(z-e^{-2})}\\&\frac{3z^2-3ze^{-2}-5z^3+10z^2-5z}{(z-1)^2(z-e^{-2})}\\&=\frac{-5z^3+13z^2-z(3e^{-2}+5)}{(z-1)^2(z-e^{-2})}\end{aligned}$$

▶

2 ***First shift theorem*** (shifting to the left)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

is the Z transform of the sequence that has been shifted by m places to the left. For example

$$Z\{x_{k+1}\} = zF(z) - zx_0$$
$$Z\{x_{k+2}\} = z^2F(z) - z^2x_0 - zx_1$$

These will be used later when solving difference equations. Note the similarity between these results and the Laplace transforms for the first and second derivatives for continuous functions.

For example, given that $Z\{4^k\} = \dfrac{z}{z-4}$ then

$$Z\{4^{k+3}\} = \ldots\ldots\ldots\ldots$$

12

$$\boxed{\dfrac{64z}{z-4}}$$

Because

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

so

$$\begin{aligned}
Z\{4^{k+3}\} &= z^3 Z\{4^k\} - \left[z^3 4^0 + z^2 4^1 + z4^2\right] \text{ where } Z\{4^k\} = \frac{z}{z-4}\\
&= z^3\frac{z}{z-4} - \left[z^3 + 4z^2 + 16z\right]\\
&= \frac{z^4}{z-4} - \left[z^3 + 4z^2 + 16z\right]\\
&= \frac{z^4 - (z^3 + 4z^2 + 16z)(z-4)}{z-4}\\
&= \frac{z^4 - (z^4 - 64z)}{z-4}\\
&= \frac{64z}{z-4}
\end{aligned}$$

In this way we have derived the Z transform of the sequence $\{64, 256, 1024, \ldots\}$ by shifting the sequence $\{1, 4, 16, 64, 256, \ldots\}$ three places to the left and losing the first three terms.

Try another. Given that $Z\{k\} = \dfrac{z}{(z-1)^2}$ then

$$Z\{(k+1)\} = \ldots\ldots\ldots\ldots$$

13

$$\boxed{\frac{z^2}{(z-1)^2}}$$

Because

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

so

$$Z\{k+1\} = z\frac{z}{(z-1)^2} - [z \times 0]$$
$$= \frac{z^2}{(z-1)^2}$$

3 *Second shift theorem* (shifting to the right)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

For example, given that $Z\{x_k\} = \dfrac{z}{z-1}$ then

$$Z\{x_{k-3}\} = \ldots\ldots\ldots\ldots$$

14

$$\boxed{\frac{1}{z^2(z-1)}}$$

Because

$$Z\{x_{k-m}\} = z^{-m}F(z)$$

so

$$Z\{x_{k-3}\} = z^{-3}\frac{z}{z-1}$$
$$= \frac{1}{z^2(z-1)}$$

In this way we have derived the Z transform of the sequence $\{0, 0, 0, 1, 1, 1, \ldots\}$ by shifting the sequence $\{1, 1, 1, 1, \ldots\}$ three places to the right and defining the first three terms as zeros.

Try this one. The sequence $\{x_k\}$ with Z transform

$$Z\{x_k\} = \frac{1}{(z-a)}, \text{ where } a \text{ is a constant, is } \{\ldots\ldots\ldots\ldots\}$$

15

$$\boxed{\{a^{k-1}\}}$$

Because

From the table of transforms the nearest transform to the one in question is $\dfrac{z}{(z-a)}$ which is the Z transform of$\{a^k\}$. Now

$$\begin{aligned}\frac{1}{(z-a)} &= \frac{1}{z} \times \frac{z}{(z-a)} \\ &= z^{-1}F(z) \quad \text{where } F(z) = Z\{a^k\}\end{aligned}$$

and so

$$\frac{1}{(z-a)} = Z\{a^{k-1}\}$$

which is the Z transform of $\{a^k\}$, shifted one place to the right.

4 *Translation*

If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

For example, $Z\{k\} = \dfrac{z}{(z-1)^2}$ so that $Z\{2^k k\} = \ldots\ldots\ldots\ldots$

16

$$\boxed{\frac{2z}{(z-2)^2}}$$

Because

Since $Z\{k\} = \dfrac{z}{(z-1)^2} = F(z)$ then by the translation property

$$\begin{aligned}Z\{2^k k\} &= F(2^{-1}z) \\ &= \frac{2^{-1}z}{(2^{-1}z-1)^2} \\ &= \frac{2z}{(z-2)^2}\end{aligned}$$

▶

5 *Final value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\underset{k\to\infty}{Lim}\, x_k = \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \text{ provided that } \underset{k\to\infty}{Lim}\, x_k \text{ exists.}$$

For example, the sequence $\left\{\left(\frac{1}{2}\right)^k\right\}$ has the Z transform

$$F(z) = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}.$$

Now

$$\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} = \underset{z\to 1}{Lim}\left\{\frac{2(z-1)}{2z-1}\right\} = 0$$

and

$$\underset{k\to\infty}{Lim}\left\{\left(\frac{1}{2}\right)^k\right\} = 0 \text{ which confirms the final value theorem.}$$

Using the final value theorem the final value of the sequence with the Z transform

$$F(z) = \frac{10z^2+2z}{(z-1)(5z-1)^2} \text{ is } \ldots\ldots\ldots\ldots$$

17

$$\boxed{0{\cdot}75}$$

Because

$$\begin{aligned}\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} &= \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\}\\ &= \underset{z\to 1}{Lim}\left\{\frac{10z+2}{(5z-1)^2}\right\}\\ &= \frac{12}{16}\\ &= 0{\cdot}75\end{aligned}$$

6 *The initial value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim}\,\{F(z)\}$$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and $\underset{z\to\infty}{Lim}\, F(z) = \underset{z\to\infty}{Lim}\dfrac{z}{z-a} = \underset{z\to\infty}{Lim}\dfrac{1}{1} = 1$ by L'Hôpital's rule. Furthermore $x_0 = a^0 = 1$ so demonstrating the validity of the theorem.

▶

7 The derivative of the transform

If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$

This is easily proved.

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k} \text{ and so } F'(z) = \sum_{k=0}^{\infty} x_k(-k)z^{-k-1} = -\frac{1}{z}\sum_{k=0}^{\infty} x_k k z^{-k}$$

$$= -\frac{1}{z}Z\{kx_k\}$$

and so $-zF'(z) = Z\{kx_k\}$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and so the sequence $\{ka^k\}$ has Z transform

$$Z\{kx_k\} = -zF'(z) = \ldots\ldots\ldots\ldots$$

18

$$\boxed{Z\{kx_k\} = \frac{az}{(z-a)^2}}$$

Because

$$-zF'(z) = -z\left(\frac{z}{z-a}\right)' = -z\left(\frac{z-a-z}{(z-a)^2}\right) = \frac{az}{(z-a)^2}$$

Notice that this is in agreement with the Table of transforms in Frame 8.

Next frame

Inverse transforms

19

If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}$$

There are many times when, given the Z transform of a sequence, it is not possible to immediately read off the sequence from the Table of transforms. Instead some manipulation may be required and, as with Laplace transforms, very often this involves using partial fractions.

Example

The sequence $\{x_k\}$ has Z transform $F(z) = \dfrac{z}{z^2 - 5z + 6}$. To find the inverse transform, and hence the sequence, we recognise that the denominator can be factorised and separated into partial fractions as

$$F(z) = \ldots\ldots\ldots\ldots$$

20

$$\boxed{F(z) = \frac{3}{z-3} - \frac{2}{z-2}}$$

Because

$$\begin{aligned} F(z) &= \frac{z}{z^2 - 5z + 6} \\ &= \frac{z}{(z-2)(z-3)} \\ &= \frac{A}{z-2} + \frac{B}{z-3} \\ &= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)} \end{aligned}$$

Equating numerators gives $z = A(z-3) + B(z-2)$, giving $A + B = 1$ and $-3A - 2B = 0$. From these two equations we find that $A = -2$ and $B = 3$. So

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

The nearest Z transform in the table to either of these two partial fractions is $Z\{a^k\} = \dfrac{z}{z-a}$. Therefore if we write

$$\begin{aligned} F(z) &= \frac{3}{z-3} - \frac{2}{z-2} \\ &= \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2} \end{aligned}$$

so

$$Z^{-1}F(z) = \ldots\ldots\ldots\ldots$$

21

$$\boxed{Z^{-1}F(z) = \{3^k - 2^k\}}$$

Because

$$\begin{aligned} F(z) &= \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2} \\ &= 3 \times z^{-1}Z\{3^k\} - 2 \times z^{-1}Z\{2^k\} \end{aligned}$$

and so

$$\begin{aligned} Z^{-1}F(z) &= 3 \times \{3^{k-1}\} - 2 \times \{2^{k-1}\} \text{ by the second shift theorem} \\ &= \{3^k\} - \{2^k\} \\ &= \{3^k - 2^k\} \text{ giving } x_k = 3^k - 2^k \end{aligned}$$

There is a simpler way of doing this without employing the second shift theorem. Recognising that z appears in the numerator of $F(z)$, we consider instead the partial fraction breakdown of $\dfrac{F(z)}{z}$

$$\frac{F(z)}{z} = \ldots\ldots\ldots\ldots$$

22

$$\boxed{\frac{1}{z-3} - \frac{1}{z-2}}$$

Because

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z} \times \frac{z}{z^2 - 5z + 6} \\
&= \frac{1}{z^2 - 5z + 6} \\
&= \frac{1}{(z-2)(z-3)} \\
&= \frac{A}{z-2} + \frac{B}{z-3} \\
&= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}
\end{aligned}$$

Equating numerators gives $1 = A(z-3) + B(z-2)$, giving

[z]: $A + B = 0$

[CT]: $-3A - 2B = 1$ with solution $A = -1$ and $B = 1$. So that

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z-3} - \frac{1}{z-2} \text{ that is} \\
F(z) &= \frac{z}{z-3} - \frac{z}{z-2} \\
&= Z\{3^k\} - Z\{2^k\} \text{ and so} \\
Z^{-1}F(z) &= \{3^k\} - \{2^k\} \\
&= \{3^k - 2^k\}
\end{aligned}$$

Thus the use of the second shift theorem is avoided.

So try one yourself. The sequence $\{x_k\}$ has Z transform

$$F(z) = \frac{5z}{(z^2 - 4z + 4)(z + 2)}$$

therefore $\{x_k\} = \dots\dots\dots\dots$

23

$$\boxed{\{x_k\} = \left\{\frac{5k}{4} - \frac{5}{16} \times \left(2^k + (-2)^k\right)\right\}}$$

Because

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{z} \times \frac{5z}{(z^2-4z+4)(z+2)}\\ &= \frac{5}{(z-2)^2(z+2)}\\ &= \frac{A}{(z-2)^2} + \frac{B}{z-2} + \frac{C}{z+2}\\ &= \frac{A(z+2) + B(z-2)(z+2) + C(z-2)^2}{(z-2)^2(z+2)}\end{aligned}$$

Equating numerators gives $5 = A(z+2) + B(z^2-4) + C(z^2-4z+4)$, giving

$[z^2]$: $\quad B + C = 0$

$[z]$: $\quad A - 4C = 0$

$[\mathrm{CT}]$: $\quad 2A - 4B + 4C = 5$

with solution $A = 5/4$, $B = -5/16$ and $C = 5/16$, so

$$\frac{F(z)}{z} = \frac{5/4}{(z-2)^2} - \frac{5/16}{z-2} + \frac{5/16}{z+2} \text{ giving}$$

$$F(z) = \frac{5}{8} \times \frac{2z}{(z-2)^2} - \frac{5}{16} \times \frac{z}{z-2} + \frac{5}{16} \times \frac{z}{z+2} \text{ and so}$$

$$\begin{aligned}Z^{-1}F(z) &= \frac{5}{8} \times \{k2^k\} - \frac{5}{16} \times \{2^k\} + \frac{5}{16} \times \left\{(-2)^k\right\}\\ &= \left\{\frac{5}{16}\left[(2k-1)2^k + (-2)^k\right]\right\}\end{aligned}$$

Next frame

Recurrence relations

24

Sometimes adjacent terms of a sequence are related to each other. For example the terms of the sequence

$$\{x_k\} = \{2^k\}$$

are such that $x_{k+1} = 2^{k+1} = 2 \times 2^k = 2x_k$. That is

$$x_{k+1} = 2x_k$$

This equation holds true for all adjacent terms of the sequence – it *recurs* for all values of k. The equation is called a **linear, first-order, constant coefficient recurrence relation**. The order of the equation is given by the maximum shift between related terms – here it is 1. Clearly, the recurrence relation

$x_{k+2} - x_{k+1} - x_k = 1$ is of order ....

25

$$\boxed{2}$$

Because

The maximum shift between terms in the relation is 2 – that is from k to $k+2$.

Initial terms

A recurrence relation can be used to generate the terms of a sequence provided initial terms are given – equal in number to the order of the equation. For example, given the sequence $\{x_k\}$ where $x_{k+1} = 3x_k$ with the initial term $x_0 = 2$ generates the sequence of terms

$$\{x_k\} = \{2, \ \ldots, \ \ldots, \ \ldots, \ \ldots\}$$

26

$$\boxed{\{x_k\} = \{2, 6, 18, 54, \ldots\}}$$

Because

Since $x_{k+1} = 3x_k$ where $x_0 = 2$ then

$$x_1 = 3x_0 = 3 \times 2 = 6$$
$$x_2 = 3x_1 = 3 \times 6 = 18$$
$$x_3 = 3x_2 = 3 \times 18 = 54$$

Similarly, if another sequence has terms that satisfy the second-order recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = 1$$

then the first five terms of the sequence are

$$\{x_k\} = \{0, 1, \ \ldots, \ \ldots, \ \ldots, \ \ldots\}$$

27

$$\boxed{\{x_k\} = \{0, 1, 4, 11, 26, \ldots\}}$$

Because

Since $x_{k+2} - 3x_{k+1} + 2x_k = 1$ where $x_0 = 0$ and $x_1 = 1$ then

$$x_2 - 3x_1 + 2x_0 = 1 \text{ that is } x_2 - 3 \times 1 + 2 \times 0 = 1 \text{ and so } x_2 = 4$$
$$x_3 - 3x_2 + 2x_1 = 1 \text{ that is } x_3 - 3 \times 4 + 2 \times 1 = 1 \text{ and so } x_3 = 11$$
$$x_4 - 3x_3 + 2x_2 = 1 \text{ that is } x_4 - 3 \times 11 + 2 \times 4 = 1 \text{ and so } x_4 = 26$$

Try another yourself.

The sequence $\{x_k\}$ has terms that satisfy the second-order recurrence relation

$$x_{k+2} - x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = -1$$

The first six terms of this sequence are

$$\{x_k\} = \{0, \ -1, \ \ldots, \ \ldots, \ \ldots, \ \ldots, \ \ldots\}$$

28

$$\boxed{\{x_k\} = \{0, \ -1, 1, 0, 2, 1, \ldots\}}$$

Because

Since $x_{k+2} - x_k = 1$ where $x_0 = 0$ and $x_1 = -1$ then

$x_2 - x_0 = 1$ that is $x_2 - 0 = 1$ and so $x_2 = 1$

$x_3 - x_1 = 1$ that is $x_3 + 1 = 1$ and so $x_3 = 0$

$x_4 - x_2 = 1$ that is $x_4 - 1 = 1$ and so $x_4 = 2$

$x_5 - x_3 = 1$ that is $x_5 - 0 = 1$ and so $x_5 = 1$

Therefore $\{x_k\} = \{0, \ -1, 1, 0, 2, 1, \ldots\}$

Next frame

Solving the recurrence relation

29

If a sequence $\{x_k\}$ satisfies a recurrence relation with given initial conditions then the general term of the sequence can be found by using the Z transform where $Z\{x_k\} = F(z)$. This is referred to as *solving the recurrence relation*. For example, solve the recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = 1$$

Because this recurrence relation is true for all values of k it can itself be used to form a sequence $\{y_k\}$, namely

$$\{y_k\} = \{x_{k+2} - 3x_{k+1} + 2x_k\} = \{1\}$$

Now, taking the Z transform of both sides of this equation gives

$$Z\{y_k\} = Z\{x_{k+2} - 3x_{k+1} + 2x_k\} = Z\{1\} \text{ that is}$$

$$Z\{x_{k+2}\} - 3Z\{x_{k+1}\} + 2\{x_k\} = Z\{1\}$$

Using the first shift theorem and $Z\{x_k\} = F(z)$ this then becomes

$$(z^2F(z) - z^2x_0 - zx_1) - 3(zF(z) - zx_0) + 2F(z) = \frac{z}{z-1}$$

Collecting like terms and substituting for the initial terms $x_0 = 0$ and $x_1 = 1$ gives

$$(z^2 - 3z + 2)F(z) - z = \frac{z}{z-1} \text{ so } (z^2 - 3z + 2)F(z) = z + \frac{z}{z-1} = \frac{z^2}{z-1}$$

That is $F(z) = \dfrac{z^2}{(z-1)(z^2 - 3z + 2)} = \dfrac{z^2}{(z-1)^2(z-2)}$

and so $\dfrac{F(z)}{z} = \dfrac{z}{(z-1)^2(z-2)}$

This has the partial fraction breakdown

$$\frac{F(z)}{z} = \frac{\cdots\cdots}{(z-1)^2} \cdots \frac{\cdots\cdots}{z-1} \cdots \frac{\cdots\cdots}{z-2}$$

30

$$\boxed{\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}}$$

Because

$$\begin{aligned}\frac{F(z)}{z} &= \frac{z}{(z-1)^2(z-2)}\\ &= \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2}\\ &= \frac{A(z-2) + B(z-1)(z-2) + C(z-1)^2}{(z-1)^2(z-2)}\end{aligned}$$

and so

$z = A(z-2) + B(z-1)(z-2) + C(z-1)^2$ giving

$[z^2]$: $B + C = 0$

$[z^1]$: $A - 3B - 2C = 1$

[CT]: $-2A + 2B + C = 0$

with solution $A = -1$, $B = -2$ and $C = 2$

Therefore

$$\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}$$

Taking the inverse Z transform of $F(z)$ yields the sequence

$$Z^{-1}F(z) = \ldots\ldots\ldots\ldots$$

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$$\boxed{Z^{-1}F(z) = \{-k - 2 + 2^{k+1}\}}$$

Because

$$\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2} \text{ and so}$$

$$F(z) = -\frac{z}{(z-1)^2} - \frac{2z}{z-1} + \frac{2z}{z-2}$$

Therefore

$$\begin{aligned} Z^{-1}F(z) &= -Z^{-1}\left(\frac{z}{(z-1)^2}\right) - 2Z^{-1}\left(\frac{z}{z-1}\right) + 2Z^{-1}\left(\frac{z}{z-2}\right) \\ &= \{-k - 2x_k + 2(2^k)\} \\ &= \{-k - 2 + 2^{k+1}\} \text{ since } x_k = 1 \end{aligned}$$

Indeed, $\{x_k\} = \{-k - 2 + 2^{k+1}\}$ is the solution to the recurrence relation as can be seen by substituting back

$$\begin{aligned} &x_{k+2} - 3x_{k+1} + 2x_k \\ &= \left(-[k+2] - 2 + 2^{[k+2]+1}\right) - 3\left(-[k+1] - 2 + 2^{[k+1]+1}\right) \\ &\quad + 2(-k - 2 + 2^{k+1}) \\ &= (-k - 4 + 8 \times 2^k) - 3(-k - 3 + 4 \times 2^k) + 2(-k - 2 + 2 \times 2^k) \\ &= -k - 4 + 8 \times 2^k + 3k + 9 - 12 \times 2^k - 2k - 4 + 4 \times 2^k \\ &= 1 \end{aligned}$$

Try one yourself.

The solution of the second-order recurrence relation

$x_{k+2} - x_k = 1$ where $x_0 = 0$ and $x_1 = -1$ is $x_k = \ldots\ldots\ldots\ldots$

32

$$\boxed{x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}}$$

Because

Taking the Z transform of the recurrence relation gives

$Z\{x_{k+2} - x_k\} = Z\{1\}$. That is, $Z\{x_{k+2}\} - Z\{x_k\} = Z\{1\}$ so that

$$(z^2F(z) - z^2x_0 - zx_1) - F(z) = \frac{z}{z-1}.$$

Substituting for $x_0 = 0$ and $x_1 = -1$ gives

$$F(z) = \ldots\ldots\ldots\ldots$$

33

$$\boxed{F(z) = \frac{-z+2}{(z+1)(z-1)^2}}$$

Because

$$\left(z^2F(z) - z^2x_0 - zx_1\right) - F(z) = \frac{z}{z-1} \text{ where } x_0 = 0 \text{ and } x_1 = -1 \text{ giving}$$

$$(z^2-1)F(z) + z = \frac{z}{z-1} \text{ so}$$

$$F(z) = \frac{z}{(z^2-1)(z-1)} - \frac{z}{(z^2-1)} \text{ so}$$

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{(z+1)(z-1)^2} - \frac{1}{(z+1)(z-1)} \\ &= \frac{1-(z-1)}{(z+1)(z-1)^2} \\ &= \frac{-z+2}{(z+1)(z-1)^2}\end{aligned}$$

Separating into partial fractions gives

$$\frac{F(z)}{z} = \dots\dots\dots\dots$$

34

$$\boxed{\frac{F(z)}{z} = \frac{3}{4}\frac{z}{z+1} - \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}}$$

Because

$$\begin{aligned}\frac{F(z)}{z} &= \frac{-z+2}{(z+1)(z-1)^2} \\ &= \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \\ &= \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}\end{aligned}$$

Equating numerators and comparing coefficients of powers of z gives

$[z^2]$: $A + B = 0$

$[z]$: $-2A + C = -1$

[CT]: $A - B + C = 2$ with solution $A = 3/4$, $B = -3/4$ and $C = 1/2$

so that $F(z) = \dfrac{3}{4}\dfrac{z}{z+1} - \dfrac{3}{4}\dfrac{z}{z-1} + \dfrac{1}{2}\dfrac{z}{(z-1)^2}$

By inverting the transform we find that

$$x_k = \dots\dots\dots\dots$$

35

$$x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}$$

Because

$$F(z) = \frac{3}{4}\frac{z}{z+1} - \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

and

$$Z^{-1}\left\{\frac{z}{z+1}\right\} = \left\{(-1)^k\right\} \text{ so } Z^{-1}\left\{(3/4)\frac{z}{z+1}\right\} = (3/4)\left\{(-1)^k\right\}$$

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = \{1^k\} \text{ so } Z^{-1}\left\{(-3/4)\frac{z}{z-1}\right\} = (-3/4)\{1^k\}$$

$$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = \{k\} \text{ so } Z^{-1}\left\{(1/2)\frac{z}{(z-1)^2}\right\} = (1/2)\{k\}$$

Therefore $\{x_k\} = \left\{(3/4)(-1)^k - (3/4) + (k/2)\right\}$

so that $x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}$

Next frame

Sampling

36

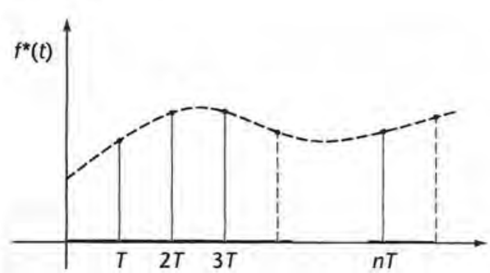
If a continuous function $f(t)$ of time t progresses from $t = 0$ onwards and is measured at every time interval T then what will result is the sequence of values

$$\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\}$$

A new, piecewise continuous function $f^*(t)$ can then be created from the sequence of sampled values such that

$$f^*(t) = \begin{cases} f(kT) & \text{if } t = kT \\ 0 & \text{otherwise} \end{cases}$$

The graph of this new function consists of a series of spikes at the regular intervals $t = kT$



▶

This function can alternatively be described in terms of the delta function $\delta(t)$ as

$$f^*(t) = f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + f(3T)\delta(t-3T) + \dots$$
$$= \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

The Laplace transform of $f^*(t)$ is then given as

$$F^*(s) = L\{f^*(t)\}$$
$$= \int_0^{\infty} \{f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \dots\}e^{-st}\,dt$$
$$= f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + f(3T)e^{-3sT} + \dots$$
$$= \sum_{k=0}^{\infty} f(kT)e^{-ksT}$$

Define a new variable $z = e^{sT}$ and we see that

$$L\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$$

which is the Z transform of the sequence$\{f(kT)\}$.

Example 1

The function $f(t) = e^{-at}$ is sampled every interval of T.

The Z transform of the sampled function is then ....

37

$$\boxed{F(z) = \frac{z}{z - e^{-aT}}}$$

Because

Defining $f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT) = \sum_{k=0}^{\infty} e^{-akT}\delta(t-kT)$ then the Laplace transform of $f^*(t)$ is given as

$$F^*(s) = \sum_{k=0}^{\infty} e^{-kaT}e^{-ksT}$$

This means that the Z transform of $\{f(kT)\}$ is

$$F(z) = \sum_{k=0}^{\infty} \frac{e^{-kaT}}{z^k} = \frac{1}{1 - \dfrac{e^{-aT}}{z}} = \frac{z}{z - e^{-aT}}$$

Notice that this agrees with the Z transform of the sequence $\{b^k\}$ $\left(\text{which is } \dfrac{z}{z-b}\right)$ when b is replaced by e^{-aT}.

Try another.

Example 2

The function $f(t) = t$ is sampled every interval of T.

The Z transform of the sampled function is then ....

38

$$\boxed{F(z) = \frac{Tz}{(z-1)^2}}$$

Because

The Z transform of $\{f(kT)\}$ is $F(z) = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$. Here $f(kT) = kT$ and so

$$\begin{aligned}
F(z) &= \sum_{k=0}^{\infty} \frac{kT}{z^k} \\
&= T\left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots\right) \\
&= \frac{T}{z}(1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \dots) \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}(1 + z^{-1} + z^{-2} + z^{-3} + \dots\} \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{Tz}{(z-1)^2}
\end{aligned}$$

Example 3

The function $f(t) = \cos t$ is sampled every interval of T.

The Z transform of the sampled function is then ....

39

$$\boxed{F(z) = \frac{z(z - \cos T)}{z^2 - 2\cos T + 1}}$$

Because

$f(t) = \cos t = \dfrac{e^{jT} + e^{-jT}}{2}$ and the Z transform of $\{e^{-kaT}\}$ is $F(z) = \dfrac{z}{z - e^{-aT}}$.

Therefore the Z transform of $\dfrac{e^{jT} + e^{-jT}}{2}$ is

$$\begin{aligned}
\frac{1}{2}\left(\frac{z}{z - e^{-jT}} + \frac{z}{z - e^{jT}}\right) &= \frac{1}{2}\left(\frac{z(z - e^{jT}) + z(z - e^{-jT})}{(z - e^{-jT})(z - e^{jT})}\right) \\
&= \frac{1}{2}\left(\frac{2z^2 - z(e^{jT} + e^{-jT})}{z^2 - [e^{jT} + e^{-jT}]z + 1}\right) \\
&= \frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}
\end{aligned}$$

▶

And that is the end of the Programme on Z transforms. All that remain are the **Revision summary** and the **Can You?** checklist. Read through these closely and make sure that you understand all the workings of this Programme. Then try the **Test exercise**; there is no need to hurry, take your time and work through the questions carefully. The **Further problems** then provide a valuable collection of additional exercises for you to try.

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Revision summary 5

1 *Sequences*

The sequence $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ is represented by the notation $\{x_k\}_{-\infty}^{\infty}$. The sequence $\{x_k\}_0^{\infty}$ is called a causal sequence and is denoted simply by $\{x_k\}$.

2 *Z transform*

The Z transform of the causal sequence $\{x_k\}$ is

$$Z\{x_k\} = \sum_{k=0}^{\infty}\left(\frac{x_k}{z^k}\right) = F(z)$$ where the value of z is chosen to ensure that the sum converges.

$\{x_k\}$ and $Z\{x_k\}$ form a Z transform pair.

3 *Table of Z transforms*

Sequence	**Transform** $F(z)$	**Permitted values of z**
$\{\delta_k\} = \{1, 0, 0, \ldots\}$	1	All values of z
$\{x_k\} = \{1, 1, 1, \ldots\}$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\dfrac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\dfrac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

4 *Linearity*

The Z transform is a linear transform. That is, if a and b are constants then

$$Z(a\{x_k\} + b\{y_k\}) = aZ\{x_k\} + bZ\{y_k\}.$$

▶

5 *First shift theorem* (shifting to the left)
If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

the Z transform of the sequence that has been shifted by m places to the left.

6 *Second shift theorem* (shifting to the right)
If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m} F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

7 *Translation*
If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

8 *Final value theorem*
For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\underset{k\to\infty}{Lim}\, x_k = \underset{z\to 1}{Lim} \left\{ \left(\frac{z-1}{z} \right) F(z) \right\} \text{ provided that } \underset{k\to\infty}{Lim}\, x_k \text{ exists.}$$

9 *The initial value theorem*
For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim} \{F(z)\}.$$

10 *The derivative of the transform*
If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$.

11 *Inverse transformations*
If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}.$$

12 *Recurrence relations*
A recurrence relation expresses the relationship that adjacent terms of a series hold to each other. The order of the equation is given by the maximum shift between related terms.

Initial terms
A recurrence relation can be used to generate the terms of a sequence provided initial terms are given – equal in number to the order of the equation.

Solving the recurrence relation
If a sequence $\{x_k\}$ satisfies a recurrence relation with given initial conditions then the general term of the sequence can be found by using the Z transform where $Z\{x_k\} = F(z)$. This is referred to as *solving the recurrence relation*.

▶

13 *Sampling*

If a continuous function $f(t)$ is sampled at equal intervals, the resulting sequence has a Z transform that is related to the Laplace transform of the piecewise function created from the sequence of sample values.

$$L\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k} = Z\{f(kT)\}$$

where

$$\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\},$$

$$f^*(t) = \begin{cases} f(kT) & \text{if } t = k \\ 0 & \text{otherwise} \end{cases}$$

and

$$z = e^{sT}.$$

☑ Can You?

41 Checklist 5

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Define the Z transform of a sequence and derive transforms of specified sequences?	1 to 7
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Make reference to a table of standard Z transforms?	8
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Recognise the Z transform as being a linear transform and so obtain the transform of linear combinations of standard sequences?	9 to 11
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply the first and second shift theorems, the translation theorem, the initial and final value theorems and the derivative theorem?	11 to 18
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use partial fractions to derive the inverse transforms?	19 to 23
Yes ☐ ☐ ☐ ☐ ☐ *No*	

▶

- Solve linear, constant coefficient recurrence relations? 24 to 35

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Demonstrate the relationship between the Laplace transform and the Z transform? 36 to 39

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 5

42

1. Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = (-1)^k$.
2. Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = 4k - 2a^k$.
3. Find the Z transform of the causal sequences:
   (a) $\{k - 3\}$
   (b) $\{5^{k+2}\}$
4. Find the inverse Z transformation of
   $F(z) = \dfrac{z^2(z-3)}{(z^2 - 2z + 1)(z-2)}$.
5. Solve the recurrence relation
   $$x_{k+2} - 4x_{k+1} + 4x_k = 3 \text{ where } x_0 = 1 \text{ and } x_1 = 0.$$
6. The function $f(t) = \sin t$ is sampled at equal intervals of $t = T$. Find the Z transform of the resulting sequence of values.

Further problems 5

43

1. Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = (-a)^k$ where $a > 0$.
2. Solve each of the following recurrence relations.
   (a) $x_{k+2} + 5x_{k+1} + 6x_k = 1$ where $x_0 = 0$ and $x_1 = 1$
   (b) $3x_{k+2} - 7x_{k+1} + 2x_k = k$ where $x_0 = 1$ and $x_1 = 0$
   (c) $x_{k+2} - 9x_k = 2k$ where $x_0 = 1$ and $x_1 = 1$.
3. Given that $y_{k+1} = v_k$ and $v_{k+1} = w_k$ where $w_k = x_k - y_k$, show that $y_{k+2} + y_k = x_k$ and solve for y_k when $\{x_k\} = \{\delta_k\}$, the unit impulse sequence where $y_0 = 0$, $y_1 = 1$.

▶

4 If

$$p_{k+1} = q_k$$
$$q_{k+1} = r_k$$
$$r_k = x_k - \alpha q_k - \beta p_k$$ where α and β are constants, show that

$$p_{k+2} + \alpha p_{k+1} + \beta p_k = x_k$$

Solve this recurrence relation when $p_0 = 1$, $p_1 = 0$ for

(a) $\alpha = 4$, $\beta = 4$ and $\{x_k\} = \{\delta_k\}$, the unit impulse sequence

(b) $\alpha = 4$, $\beta = 4$ and $\{x_k\} = \{u_k\}$ the unit step sequence.

5 Find the Z transform of each of the following sequences.

(a) $\{1, 0, 1, 0, 1, 0, \ldots\}$

(b) $\{0, 1, 0, 1, 0, 1, \ldots\}$

(c) $\{1, 0, 1, 1, 0, 0, 0, 1\}$

(d) $\{1, 1, 1, 0, 0, 0, 1, 1\}$

(e) $\{0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1\}$

(f) $\{1, 1, 0, 0, 0, 1, 1\}$

Note that the last four of these are finite sequences.

6 Find the inverse transform of

(a) $F(z) = \dfrac{z}{(z+1)(z+2)(z+3)}$

(b) $F(z) = \dfrac{z^2}{(z+1)(z+2)(z+3)}$

(c) $F(z) = \dfrac{z(3z+1)}{(z-2)(z-3)}$

(d) $F(z) = \dfrac{z^2}{2 - 3z + z^2}$.

7 Given

$$F(z) = \frac{3z^2}{z^2 - z + 1}$$

show that

$$Z^{-1}F(z) = \{3, 3, -3, -3, \ldots\}.$$

Hint: Use long division on $F(z)$.

8 Given

$$F(z) = \left(1 + \frac{2}{z}\right)^{-3}$$

show that

$$Z^{-1}F(z) = \{1, -6, 24, -48, \ldots\}.$$

Hint: Use the binomial theorem on $F(z)$.

▶

9 Find the final value of the sequence $\{x_k\}$ with Z transform

$$F(z) = \frac{4z^2 - z}{2z^2 - 3z + 1}.$$

10 What is the initial value of the sequence whose Z transform is given by

$$F(z) = \frac{2z^2 - z + 1}{5 - 3z - 7z^2}?$$

11 Given the sequence of n terms $\{x_k\}$ for $0 \leq k \leq n - 1$ with Z transform $F_n(z)$, show that the Z transform of the sequence formed by continually repeating the terms $\{x_k\}$ is given as

$$F(z) = \frac{F_n(z)}{1 - z^{-n}}.$$

12 Using the result of Question 11, show that the Z transform of the sequence obtained by continually repeating the three term sequence $\{1, 0, -1\}$ is

$$F(z) = \frac{z^2}{z^2 + 1}.$$

13 Find the Z transforms of the sequence of values obtained when $f(t)$ is sampled at regular intervals of $t = T$ where

(a) $f(t) = \sinh t$

(b) $f(t) = \cosh at$

(c) $f(t) = e^{-at} \cosh bt$.

Programme 6

Fourier series

Frames 1 to 89

Learning outcomes

When you have completed this Programme you will be able to:

- Determine the period and amplitude of a periodic function
- Write down the harmonics of a periodic trigonometric function
- Give an analytic description of a non-sinusoidal periodic function
- Evaluate integrals with periodic integrands
- Demonstrate the orthogonality of the trigonometric functions $\sin nx$ and $\cos nx$ for $n = 0, 1, 2, \ldots$
- Describe a periodic function as a Fourier series subject to Dirichlet conditions
- Obtain the Fourier coefficients and hence the Fourier series of a periodic function
- Describe the effects of the harmonics in the construction of the Fourier series
- Find the value of the Fourier series at a point of discontinuity of the periodic function
- Derive the Fourier series of non-sinusoidal periodic functions
- Recognise even and odd functions and their products
- Derive the Fourier sine and cosine series for odd and even functions respectively
- Derive half-range Fourier series
- Recognise the condition for the Fourier series to contain only odd or only even harmonics
- Explain the significance of the term $a_0/2$

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 15 Integration 1 and 17 Reduction formulas

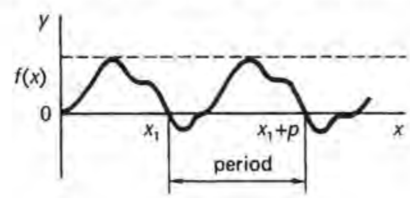
Introduction

1

We have seen earlier that many functions can be expressed in the form of infinite series. Problems involving various forms of oscillations are common in fields of modern technology and *Fourier series*, with which we shall now be concerned, enable us to represent a periodic function as an infinite trigonometrical series in sine and cosine terms. One important advantage of a Fourier series is that it can represent a function containing discontinuities, whereas Maclaurin's and Taylor's series require the function to be continuous throughout.

Periodic functions

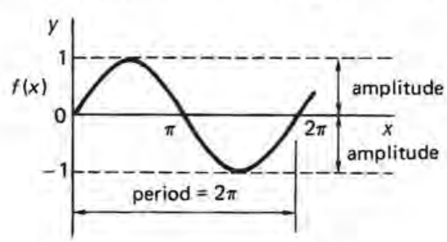
A function $f(x)$ is said to be *periodic* if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the *period* of the oscillations.



Graphs of $y = A \sin nx$

(a) $y = \sin x$

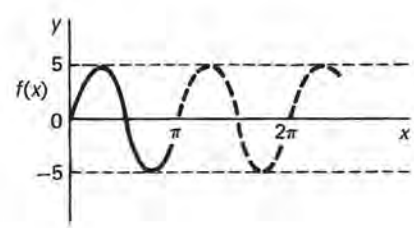
The obvious example of a periodic function is $y = \sin x$, which goes through its complete range of values while x increases from 0° to 360°. The period is therefore 360° or 2π radians and the amplitude, the maximum displacement from the position of rest, is 1.



(b) $y = 5 \sin 2x$

The amplitude is 5.

The period is 180° and there are thus 2 complete cycles in 360°.



▶

(c) $y = A \sin nx$
Thinking along the same lines, the function $y = A \sin nx$ has amplitude ....; period ....;
and will have .... complete cycles in 360°.

2

$$\boxed{\text{amplitude} = A;\ \text{period} = \frac{360^\circ}{n} = \frac{2\pi}{n};\ n \text{ cycles in } 360^\circ}$$

Graphs of $y = A \cos nx$ have the same characteristics.

By way of revising earlier work, then, complete the following short exercise.

Exercise

In each of the following, state (a) the amplitude and (b) the period.

1 $y = 3 \sin 5x$ **5** $y = 5 \cos 4x$
2 $y = 2 \cos 3x$ **6** $y = 2 \sin x$
3 $y = \sin \dfrac{x}{2}$ **7** $y = 3 \cos 6x$
4 $y = 4 \sin 2x$ **8** $y = 6 \sin \dfrac{2x}{3}$

Deal with all eight. They will not take much time.

3

No.	Amplitude	Period	No.	Amplitude	Period
1	3	$2\pi/5$	**5**	5	$\pi/2$
2	2	$2\pi/3$	**6**	2	π
3	1	4π	**7**	3	$\pi/3$
4	4	π	**8**	6	3π

Harmonics

A function $f(x)$ is sometimes expressed as a series of a number of different sine components. The component with the largest period is the *first harmonic*, or *fundamental* of $f(x)$.

$y = A_1 \sin x$ is the first harmonic or fundamental
$y = A_2 \sin 2x$ is the second harmonic
$y = A_3 \sin 3x$ is the third harmonic, etc.

and in general
$y = A_n \sin nx$ is the .... harmonic, with amplitude .... and period ....

4

$$\boxed{n\text{th harmonic; amplitude } A_n\text{; period} = \frac{2\pi}{n}}$$

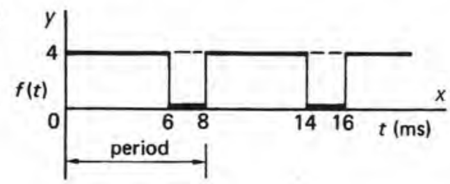
Non-sinusoidal periodic functions

Although we introduced the concept of a periodic function via a sine curve, a function can be periodic without being obviously sinusoidal in appearance.

Example

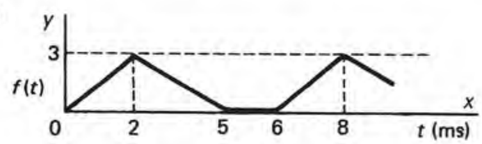
In the following cases, the x-axis carries a scale of t in milliseconds.

(a)



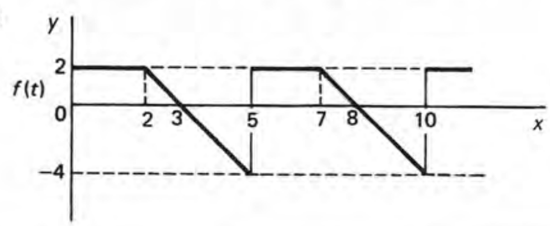
period = 8 ms

(b)



period = ...

(c)



period = ...

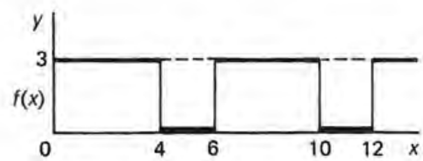
5

(b) period = 6 ms; (c) period = 5 ms

Analytic description of a periodic function

A periodic function can be defined analytically in many cases.

Example 1



(a) Between $x = 0$ and $x = 4$, $y = 3$, i.e. $f(x) = 3 \quad 0 < x < 4$

(b) Between $x = 4$ and $x = 6$, $y = 0$, i.e. $f(x) = 0 \quad 4 < x < 6$

So we could define the function by

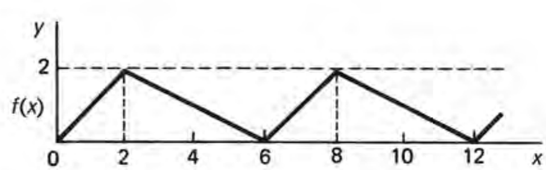
$$f(x) = \begin{cases} 3 & 0 < x < 4 \\ 0 & 4 < x < 6 \end{cases}$$
$$f(x+6) = f(x)$$

the last line indicating that ....

6

the function is periodic with period 6 units

Example 2



In this case

(a) Between $x = 0$ and $x = 2$, $y = x$ i.e. $f(x) = x \quad 0 < x < 2$

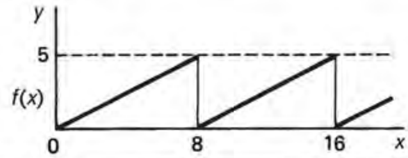
(b) Between $x = 2$ and $x = 6$, $y = -\dfrac{x}{2} + 3$, i.e. $f(x) = 3 - \dfrac{x}{2} \quad 2 < x < 6$

(c) The period is 6 units i.e. $f(x+6) = f(x)$.

So we have

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 3 - \dfrac{x}{2} & 2 < x < 6 \end{cases}$$
$$f(x+6) = f(x).$$

Example 3



In this case ....
....

7

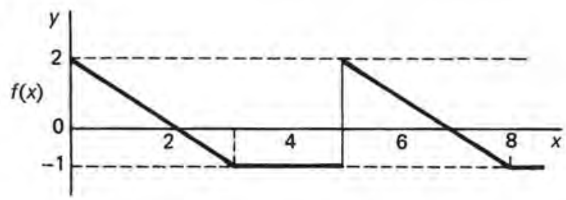
$$f(x) = \frac{5x}{8} \qquad 0 < x < 8$$
$$f(x+8) = f(x)$$

Here is a short exercise.

Exercise

Define analytically the periodic functions shown.

1



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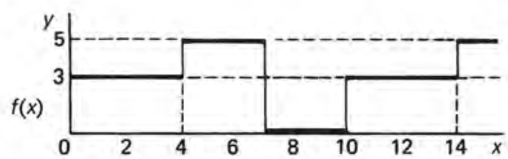

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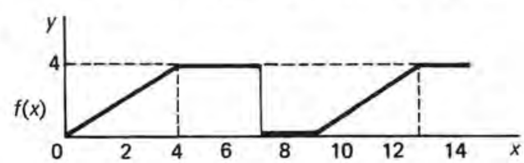

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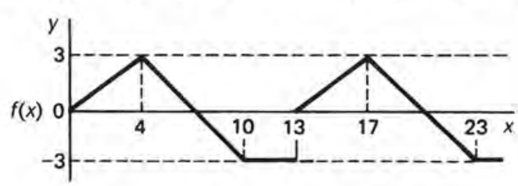

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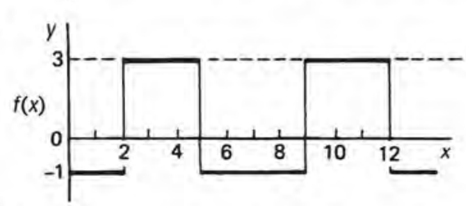


Finish all five and then check the results.

8

Here are the details.

1
$$f(x) = \begin{cases} 2 - x & 0 < x < 3 \\ -1 & 3 < x < 5 \end{cases}$$
$f(x+5) = f(x)$.

2
$$f(x) = \begin{cases} 3 & 0 < x < 4 \\ 5 & 4 < x < 7 \\ 0 & 7 < x < 10 \end{cases}$$
$f(x+10) = f(x)$.

3
$$f(x) = \begin{cases} x & 0 < x < 4 \\ 4 & 4 < x < 7 \\ 0 & 7 < x < 9 \end{cases}$$
$f(x+9) = f(x)$.

4
$$f(x) = \begin{cases} \dfrac{3x}{4} & 0 < x < 4 \\ 7 - x & 4 < x < 10 \\ -3 & 10 < x < 13 \end{cases}$$
$f(x+13) = f(x)$.

5
$$f(x) = \begin{cases} -1 & 0 < x < 2 \\ 3 & 2 < x < 5 \\ -1 & 5 < x < 7 \end{cases}$$
$f(x+7) = f(x)$.

Now we have the same thing in reverse.

Exercise

Sketch the graphs of the following, inserting relevant values.

1
$$f(x) = \begin{cases} 4 & 0 < x < 5 \\ 0 & 5 < x < 8 \end{cases}$$
$f(x+8) = f(x)$.

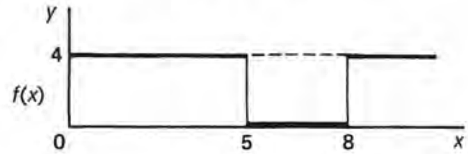
2
$$f(x) = 3x - x^2 \qquad 0 < x < 3$$
$f(x+3) = f(x)$.

3
$$f(x) = \begin{cases} 2\sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$
$f(x+2\pi) = f(x)$.

4
$$f(x) = \begin{cases} \dfrac{x}{2} & 0 < x < \pi \\ \pi - \dfrac{x}{2} & \pi < x < 2\pi \end{cases}$$
$f(x+2\pi) = f(x)$.

5
$$f(x) = \begin{cases} \dfrac{x^2}{4} & 0 < x < 4 \\ 4 & 4 < x < 6 \\ 0 & 6 < x < 8 \end{cases}$$
$f(x+8) = f(x)$.

Here they are: check carefully. 9

1


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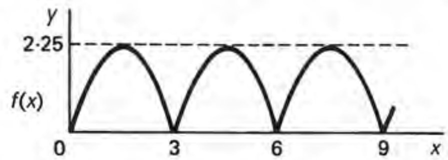

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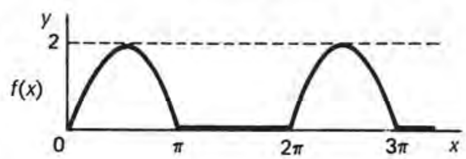

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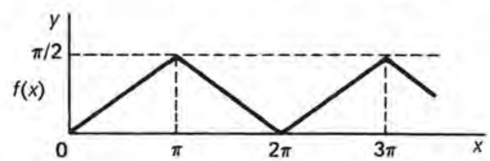

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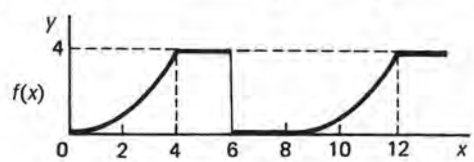

All this is in preparation for what is to come, so let us now consider Fourier series.

Move on then to the next frame

Integrals of periodic functions

Before we proceed we need to consider some specific integrals involving integers m and n. These are integrals over a single period of periodic integrands. You will already know some of these and the others you will easily be able to work out. The integrals that we are concerned with are those of sines, cosines and their combinations where the integration is over a single period from $-\pi$ to π. First, though, we list the integral of the unit constant over the period. 10

▶

1 $\displaystyle\int_{-\pi}^{\pi} dx = \Big[x\Big]_{-\pi}^{\pi} = 2\pi$

2 $\displaystyle\int_{-\pi}^{\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$

$$= \frac{\sin n\pi}{n} - \frac{\sin(-n\pi)}{n}$$

$$= 0 \quad \text{because } \sin n\pi = 0$$

3 $\displaystyle\int_{-\pi}^{\pi} \sin nx\, dx = \ldots\ldots\ldots\ldots \qquad (n \neq 0)$

11

$$\boxed{\int_{-\pi}^{\pi} \sin nx\, dx = 0}$$

Because

$$\int_{-\pi}^{\pi} \sin nx\, dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$

$$= -\frac{\cos n\pi}{n} + \frac{\cos(-n\pi)}{n}$$

$$= 0 \quad \text{because } \cos(-x) = \cos x$$

4 $\displaystyle\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \frac{\cos 2nx + 1}{2}\, dx \quad \text{because } \cos 2A = 2\cos^2 A - 1$

$$= \left[\frac{\sin 2nx}{4n} + \frac{x}{2}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$

$$= \frac{\sin 2n\pi}{4n} + \frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n} - \frac{(-\pi)}{2}$$

$$= \pi$$

5 $\displaystyle\int_{-\pi}^{\pi} \sin^2 nx\, dx = \ldots\ldots\ldots\ldots \qquad (n \neq 0)$

12

$$\boxed{\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi}$$

Because

$$\int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2}\, dx \quad \text{because } \cos 2A = 1 - 2\sin^2 A$$

$$= \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} \qquad (n \neq 0)$$

$$= \frac{\pi}{2} - \frac{\sin 2n\pi}{4n} - \frac{(-\pi)}{2} + \frac{\sin(-2n\pi)}{4n}$$

$$= \pi$$

▶

6 $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$

because $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$$= \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} \qquad (m \neq n)$$

$$= \frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n}$$

$$= 0$$

7 $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \ldots\ldots\ldots\ldots \qquad (m \neq n)$

13

$$\boxed{\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n}$$

Because

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \, dx$$

because $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} \qquad (m \neq n)$$

$$= \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} + \frac{\sin(m+n)(-\pi)}{m+n}$$

$$= 0$$

8 $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx \, dx \qquad (m \neq n)$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m+n)x - \sin(m-n)x] \, dx$$

because $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi} \qquad (m \neq n)$$

$$= \frac{1}{2}\left(-\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} + \frac{\cos(m+n)(-\pi)}{m+n} - \frac{\cos(m-n)(-\pi)}{m-n}\right)$$

$$= 0 \quad \text{because } \cos(-x) = \cos x$$

And finally, when $m = n$

9 $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin mx \, dx = \ldots\ldots\ldots\ldots$

14

$$\boxed{\int_{-\pi}^{\pi} \cos mx \sin mx \, dx = 0}$$

Because

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos mx \sin mx \, dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx \, dx \quad \text{because } \sin 2A = 2\sin A \cos A \\
&= \frac{1}{2}\left[-\frac{\cos 2mx}{2m}\right]_{-\pi}^{\pi} \qquad (m \neq 0) \\
&= \frac{1}{2}\left(-\frac{\cos 2m\pi}{2m} + \frac{\cos 2m(-\pi)}{2m}\right) \\
&= 0 \quad \text{because } \cos(-x) = \cos x
\end{aligned}$$

15

Summary

1 $\displaystyle\int_{-\pi}^{\pi} dx = \left[x\right]_{-\pi}^{\pi} = 2\pi$

2 $\displaystyle\int_{-\pi}^{\pi} \cos nx \, dx = 0$

3 $\displaystyle\int_{-\pi}^{\pi} \sin nx \, dx = 0$

4 $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi\delta_{mn} \quad \text{where } \delta_{mn} = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases}$

(δ_{mn} is called the Kronecker delta)

5 $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi\delta_{mn}$

6 $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$

Note that the same results are obtained no matter what the end points of the integrals are, *provided that the interval between them is one period.* So, for example

$$\begin{aligned}
\int_{k}^{k+2\pi} \cos nx \, dx &= \left[\frac{\sin nx}{n}\right]_{k}^{k+2\pi} \qquad (n \neq 0) \\
&= \frac{\sin(nk + 2n\pi)}{n} - \frac{\sin nk}{n} \\
&= 0 \quad \text{because } \sin(x + 2n\pi) = \sin x
\end{aligned}$$

Now to put all these integrals to practical use

Orthogonal functions

16

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \leq x \leq b$ and

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0$$

then we say that the two functions are **orthogonal** to each other on the interval $a \leq x \leq b$. In the previous frames we have seen that the trigonometric functions $\sin nx$ and $\cos nx$ where $n = 0, 1, 2, \ldots$ form an infinite collection of periodic functions that are mutually orthogonal on the interval $-\pi \leq x \leq \pi$, indeed on any interval of width 2π. That is

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,\mathrm{d}x = 0 \quad \text{for } m \neq n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\,\mathrm{d}x = 0 \quad \text{for } m \neq n$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,\mathrm{d}x = 0$$

Fourier series

17

Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic functions just discussed. That is, if $f(x)$ is defined on the interval $-\pi \leq x \leq \pi$ where $f(x + 2n\pi) = f(x)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

This is the **Fourier series** expansion of $f(x)$ where the a_n and b_n are constants called the *Fourier coefficients*. But how do we find the values of these constants? Quite easily. We make use of the mutual orthogonality of the trigonometric functions in the expansion.

▶

For example, to find a_{10} we multiply $f(x)$ by $\cos 10x$ and integrate over a period. That is

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos 10x\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)\cos 10x\,\mathrm{d}x \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos 10x\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos 10x\,\mathrm{d}x \\
&\qquad + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\cos 10x\,\mathrm{d}x \\
&= \frac{a_0}{2}\times 0 + \sum_{n=1}^{\infty} a_n\pi\delta_{n,10} + \sum_{n=1}^{\infty} b_n\times 0 \\
&= a_0\pi\times 0 + a_1\pi\times 0 + \ldots + a_9\pi\times 0 + a_{10}\pi\times 1 + a_{11}\pi\times 0 + \ldots \\
&= a_{10}\pi
\end{aligned}$$

So that

$$a_{10} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 10x\,\mathrm{d}x$$

In just the same way $\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x = \ldots\ldots\ldots\ldots$

18

$$\boxed{\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x = a_m\pi}$$

Because

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)\cos mx\,\mathrm{d}x \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,\mathrm{d}x \\
&\qquad + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,\mathrm{d}x \\
&= \frac{a_0}{2}\times 0 + \sum_{n=1}^{\infty} a_n\pi\delta_{n,m} + \sum_{n=1}^{\infty} b_n\times 0 \\
&= a_m\pi
\end{aligned}$$

and so

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x$$

Finally

$$\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x = b_m\pi}$$

Because

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)\sin mx\,\mathrm{d}x \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\sin mx\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\sin mx\,\mathrm{d}x \\
&\qquad + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\sin mx\,\mathrm{d}x \\
&= \frac{a_0}{2}\times 0 + \sum_{n=1}^{\infty} a_n\times 0 + \sum_{n=1}^{\infty} b_n\pi\delta_{n,m} \\
&= b_m\pi
\end{aligned}$$

and so

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x$$

Summary

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Given that certain conditions are satisfied, if $f(x)$ is defined on the interval $-\pi \le x \le \pi$ and where $f(x+2n\pi) = f(x)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

This is the **Fourier series** expansion of $f(x)$ where the a_n and b_n are constants called the *Fourier coefficients* and where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x,\ n = 0,\ 1,\ 2, \dots$$

Look in particular at the constant function $f(x) = c$ which can be considered as a periodic function with any period we wish to choose. Choosing the period to be 2π then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c\cos nx\,\mathrm{d}x = \frac{c}{\pi}\int_{-\pi}^{\pi}\cos nx\,\mathrm{d}x = 2c\delta_{n,0}.$$

That is $a_0 = 2c$ so $c = \dfrac{a_0}{2}$ as expected.

Also

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c\sin nx\,\mathrm{d}x = \frac{c}{\pi}\int_{-\pi}^{\pi}\sin nx\,\mathrm{d}x = 0$$

▶

From this we see that we have two choices to represent the Fourier series. We can either write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

or we can write

$$f(x) = \sum_{n=0}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n \neq 0)$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

We choose the former and so avoid having a separate integral for a_0.

Dirichlet conditions

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If a function $f(x)$ is such that

(a) $f(x)$ is defined, single-valued and periodic with period 2π

(b) $f(x)$ and $f'(x)$ have at most a finite number of finite discontinuities over a single period – that is they are *piecewise continuous*

then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ converges to $f(x)$ when $(x, f(x))$ is a point of continuity.

The Dirichlet conditions are sufficient for the Fourier series to represent $f(x)$ not only at a point of continuity but, with a slight modification, also at a point of discontinuity, as we shall see later in Frame 36. Also the periodicity of the function need not be restricted to 2π, as we shall see from Frame 38 onwards.

Note that these conditions, while being sufficient, are not necessary because there are functions that do not satisfy these conditions which still possess a convergent Fourier series. However, the cases met in science and engineering do generally meet these conditions.

Exercise

If the following functions are defined over the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$, state whether or not each function can be represented by a Fourier series.

1. $f(x) = x^3$
2. $f(x) = 4x - 5$
3. $f(x) = \dfrac{2}{x}$
4. $f(x) = \dfrac{1}{x-5}$
5. $f(x) = \tan x$
6. $f(x) = y \quad$ where $x^2 + y^2 = 9$

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1. Yes
2. Yes
3. No: infinite discontinuity at $x = 0$
4. Yes
5. No: infinite discontinuity at $x = \pi/2$
6. No: two valued

On then

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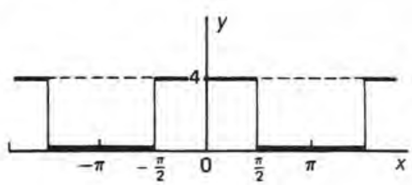
Example 1



Find the Fourier series for the function shown.

Consider one cycle between $x = -\pi$ and $x = \pi$.

The function can be defined by $f(x) = \begin{cases} 0 & -\pi < x < -\dfrac{\pi}{2} \\ 4 & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$

$$f(x+2\pi) = f(x).$$

(a) As before $f(x) = \dfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$

The expression for a_0 is ....

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$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

This gives

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} 0\,dx + \int_{-\pi/2}^{\pi/2} 4\,dx + \int_{\pi/2}^{\pi} 0\,dx\right\}$$

$$= \frac{1}{\pi}\Big[4x\Big]_{-\pi/2}^{\pi/2} \qquad \therefore\ a_0 = 4$$

(b) *To find* a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$\therefore\ a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (0)\cos nx\,dx + \int_{-\pi/2}^{\pi/2} 4\cos nx\,dx + \int_{\pi/2}^{\pi} (0)\cos nx\,dx\right\}$$

$$\therefore\ a_n = \dots\dots\dots\dots$$

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$$\boxed{a_n = \frac{8}{\pi n}\sin\frac{n\pi}{2}}$$

Then considering different integer values of n, we have

If n is even	$a_n = 0$
If $n = 1, 5, 9, \dots$	$a_n = \dfrac{8}{n\pi}$
If $n = 3, 7, 11, \dots$	$a_n = -\dfrac{8}{n\pi}$

We keep these in mind while we find b_n.

(c) *To find* b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \dots\dots\dots\dots$$

$$\boxed{b_n = 0}$$

Because we have

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(0)\sin nx\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2} 4\sin nx\,\mathrm{d}x + \int_{\pi/2}^{\pi}(0)\sin nx\,\mathrm{d}x\right\}$$

$$= \frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\sin nx\,\mathrm{d}x = \frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi/2}^{\pi/2}$$

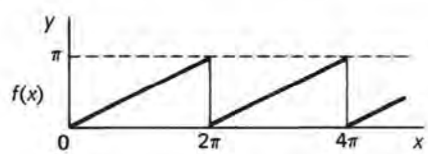
$$= -\frac{4}{n\pi}\left\{\cos\frac{n\pi}{2} - \cos\left(\frac{-n\pi}{2}\right)\right\} = 0 \qquad \therefore\ b_n = 0$$

So with $a_0 = 4$; a_n as stated above; $b_n = 0$; the Fourier series is

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \dots\right\}$$

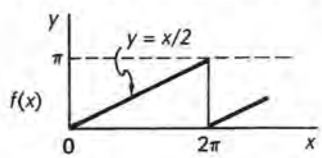
In this particular example, there are, in fact, no sine terms.

Example 2



Determine the Fourier series to represent the periodic function shown.

It is more convenient here to take the limits as 0 to 2π.



The function can be defined as

$$f(x) = \frac{x}{2} \qquad 0 < x < 2\pi$$

$$f(x + 2\pi) = f(x) \quad \text{i.e. period} = 2\pi.$$

Now to find the coefficients.

(a) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\mathrm{d}x = \frac{1}{4\pi}\left[x^2\right]_0^{2\pi}$$

$$= \pi \qquad \therefore\ a_0 = \pi$$

(b) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,\mathrm{d}x$$

$= \dots\dots\dots\dots$ (integrating by parts)

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$$\boxed{a_n = 0}$$

Because

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{2\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}(0)\right\} = 0 \qquad \therefore\ a_n = 0$$

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin nx\,dx = \ldots\ldots\ldots\ldots$

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$$\boxed{b_n = -\frac{1}{n}}$$

Straightforward integration by parts, as for a_n, gives the result stated. So we now have

$$a_0 = \ldots\ldots\ldots\ldots;\quad a_n = \ldots\ldots\ldots\ldots;\quad b_n = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_0 = \pi;\quad a_n = 0;\quad b_n = -\frac{1}{n}}$$

Now the general expression for a Fourier series is

$\ldots\ldots\ldots\ldots$

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$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}}$$

Therefore in this case

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\{b_n\sin nx\} \qquad \text{because } a_n = 0$$

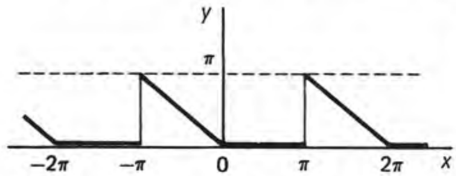
$$= \frac{\pi}{2} + \left\{-\frac{1}{1}\sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x - \ldots\right\}$$

$$\therefore\ f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right\}$$

Note that in this example, the series contains a constant term and sine terms only.

Example 3

Find the Fourier series for the function defined by



$$f(x) = -x \qquad -\pi < x < 0$$
$$f(x) = 0 \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

The general expressions for a_0, a_n, b_n are

$$a_0 = \dots\dots\dots\dots$$
$$a_n = \dots\dots\dots\dots$$
$$b_n = \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \end{aligned}}$$

With that reminder, in this example $a_0 = \dots\dots\dots\dots$

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$$\boxed{a_0 = \frac{\pi}{2}}$$

Because

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \dfrac{1}{\pi}\int_{-\pi}^{0} (-x)\,\mathrm{d}x = \dfrac{1}{\pi}\left[-\dfrac{x^2}{2}\right]_{-\pi}^{0} = \dfrac{\pi}{2}$

(b) To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \dots\dots\dots\dots$$

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$$\boxed{a_n = -\frac{2}{\pi n^2} \quad (n \text{ odd}); \quad 0 \quad (n \text{ even})}$$

Because

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx\,\mathrm{d}x \\
&= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,\mathrm{d}x \\
&= -\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x\right\} \\
&= -\frac{1}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{1}{\pi n^2}\{1-\cos n\pi\}
\end{aligned}$$

But $\cos n\pi = 1$ (n even) or -1 (n odd)

$$\therefore\ a_n = -\frac{2}{\pi n^2} \quad (n \text{ odd}) \quad \text{or } 0 \quad (n \text{ even})$$

(c) *Now to find* b_n. Working as for a_n, we obtain

$$b_n = \ldots\ldots\ldots\ldots$$

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$$\boxed{b_n = -\frac{1}{n} \quad (n \text{ even}) \quad \text{or} \quad \frac{1}{n} \quad (n \text{ odd})}$$

Because

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,\mathrm{d}x \\
&= -\frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,\mathrm{d}x \\
&= -\frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,\mathrm{d}x\right\} \\
&= -\frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}\right\} = -\frac{\cos n\pi}{n}
\end{aligned}$$

$$\therefore\ b_n = -\frac{1}{n} \quad (n \text{ even}); \quad \frac{1}{n} \quad (n \text{ odd})$$

So we have $a_0 = \frac{\pi}{2}$; $a_n = 0$ (n even) or $-\frac{2}{\pi n^2}$ (n odd)

$$b_n = -\frac{1}{n} \quad (n \text{ even}) \quad \text{or} \quad \frac{1}{n} \quad (n \text{ odd})$$

$$\therefore\ f(x) = \ldots\ldots\ldots\ldots$$

Complete the series

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$$\boxed{\begin{aligned} f(x) &= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right) \\ &\quad + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right) \end{aligned}}$$

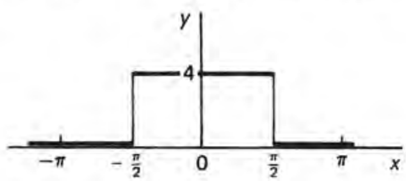
It is just a case of substituting $n = 1, 2, 3$, etc.

In this particular example, we have a constant term and both sine and cosine terms.

Effect of harmonics

It is interesting to see just how accurately the Fourier series represents the function with which it is associated. The complete representation requires an infinite number of terms, but we can, at least, see the effect of including the first few terms of the series.

Let us consider the waveform shown. We established earlier in Frames 23–26 that the function



$$f(x) = \begin{cases} 0 & -\pi < x < -\dfrac{\pi}{2} \\ 4 & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$$

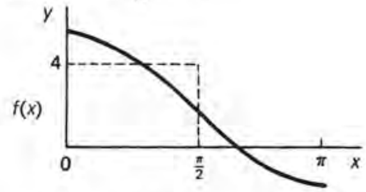
$$f(x + 2\pi) = f(x)$$

gives the Fourier series

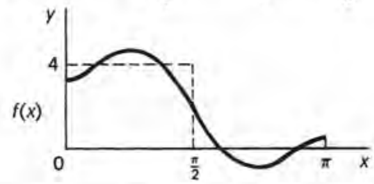
$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots\right\}$$

If we start with just one cosine term, we can then see the effect of including subsequent harmonics. Let us restrict our attention to just the right-hand half of the symmetrical waveform. Detailed plotting of points gives the development on the next page.

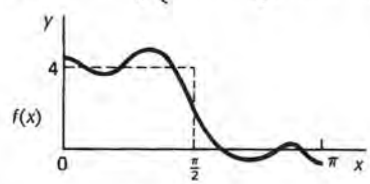
1 $f(x) = 2 + \frac{8}{\pi}\cos x$



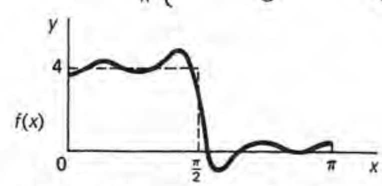
2 $f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x\right\}$



3 $f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x\right\}$



4 $f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x\right\}$



As the number of terms is increased, the graph gradually approaches the shape of the original square waveform. The ripples increase in number and, apart from the one nearest to the step, decrease in amplitude. A perfectly square waveform is unattainable in practice. For practical purposes, the first few terms normally suffice to give an accuracy of acceptable level.

Gibbs' phenomenon

You will notice from the previous two diagrams that near the discontinuity, as more terms are taken into acount, the series tends to overshoot on one side and undershoot on the other. This over and undershooting on either side of the discontinuity does not go away as the number of terms in the Fourier series that are taken into account is increased, rather it tends to two spikes on either side of the discontinuity. This effect is called the *Gibbs' phenomenon*.

Sum of a Fourier series at a point of discontinuity

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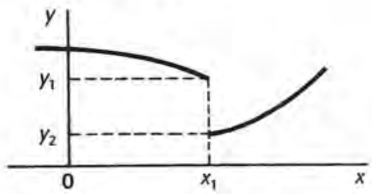
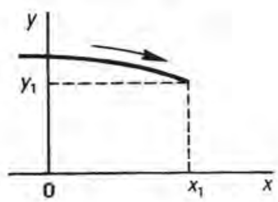
$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

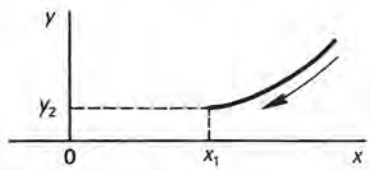
If $f(x)$ is continuous at $x = x_1$, the series converges to the value $f(x_1)$ as the number of terms included increases to infinity. A particular point of interest occurs at a point of finite discontinuity or 'jump' of the function $y = f(x)$.



As $x \to x_1$, the expression $f(x)$ approaches y_1 or y_2 depending on the direction of approach.



If we approach $x = x_1$ from below that value, the limiting value of $f(x)$ is y_1.



If we approach $x = x_1$ from above that value, the limiting value of $f(x)$ is y_2.

To distinguish between these two values we write

$y_1 = f(x_1 - 0)$ denoting immediately before $x = x_1$

$y_2 = f(x_1 + 0)$ denoting immediately after $x = x_1$.

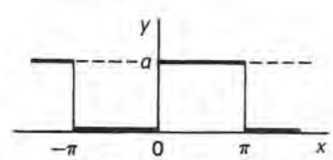
In fact, if we substitute $x = x_1$ in the Fourier series for $f(x)$, it can be shown that the series converges to the value $\frac{1}{2}\{f(x_1 - 0) + f(x_1 + 0)\}$ i.e. $\frac{1}{2}(y_1 + y_2)$, the average of y_1 and y_2.

Example

Consider the function

$$f(x) = \begin{cases} 0 & -\pi < x < \pi \\ a & \pi < x < 2\pi \end{cases}$$

$$f(x + 2\pi) = f(x).$$



First of all, determine the Fourier series to represent the function. There are no snags.

$$f(x) = \ldots\ldots\ldots\ldots$$

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$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\{\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \dots\}$$

Check the working

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} a\,dx = \frac{1}{\pi}\Big[ax\Big]_0^{\pi} = a \qquad \therefore\ a_0 = a$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} a\cos nx\,dx$

$= \frac{a}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0 \qquad \therefore\ a_n = 0$

(c) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} a\sin nx\,dx$

$= \frac{a}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{a}{n\pi}(1 - \cos n\pi) = \frac{a}{n\pi}(1-(-1)^n)$

and because

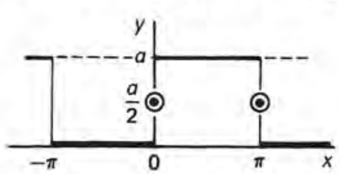
$\cos n\pi = 1 \quad (n \text{ even}) \text{ and } -1 \quad (n \text{ odd})$

$b_n = 0 \quad (n \text{ even}); \quad \frac{2a}{n\pi} \quad (n \text{ odd})$

$$\therefore\ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore\ f(x) = \frac{a}{2} + \frac{2a}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}$$

A finite discontinuity, or 'jump', occurs at $x = 0$. If we substitute $x = 0$ in the series obtained, all the sine terms vanish and we get $f(x) = a/2$, which is, in fact, the average of the two function values at $x = 0$.



Note also that at $x = \pi$, another finite discontinuity occurs and substituting $x = \pi$ in the series gives the same result.

Now on to something new

Functions with periods other than 2π

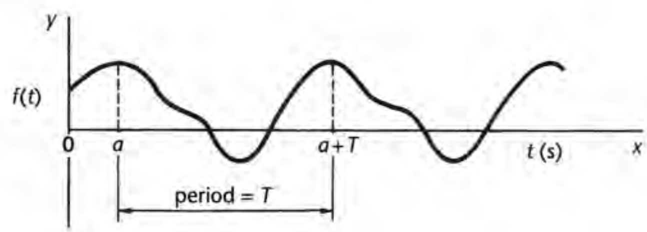
So far, we have considered functions $f(x)$ with period 2π. In practice, we often encounter functions defined over periodic intervals other than 2π, e.g. from 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, etc.

Functions with period T

If $y = f(x)$ is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$, i.e. has a period T, we can convert this to an interval of 2π by changing the units of the independent variable.

In many practical cases involving physical oscillations, the independent variable is time (t) and the periodic interval is normally denoted by T, i.e.

$$f(t+T) = f(t)$$



Each cycle is therefore completed in T seconds and the frequency f **hertz** (oscillations per second) of the periodic function is therefore given by $f = \frac{1}{T}$. If the angular velocity, ω radians per second, is defined by $\omega = 2\pi f$, then

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad T = \frac{2\pi}{\omega}.$$

The angle, x radians, at any time t is therefore $x = \omega t$ and the Fourier series to represent the function can be expressed as

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right\} \end{aligned}$$

39 Fourier coefficients

With the new variable, the Fourier coefficients become

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t \qquad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,\mathrm{d}t$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,\mathrm{d}t = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,\mathrm{d}t = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,\mathrm{d}t.$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T, $-\dfrac{T}{2}$ to $\dfrac{T}{2}$, $-\dfrac{\pi}{\omega}$ to $\dfrac{\pi}{\omega}$, 0 to $\dfrac{2\pi}{\omega}$ etc. as is convenient, so long as they cover a complete period.

Example

Determine the Fourier series for a periodic function defined by

$$f(t) = \begin{cases} 2(1+t) & -1 < t < 0 \\ 0 & 0 < t < 1 \end{cases}$$

$$f(t+2) = f(t)$$

The first step is to sketch the waveform which is ....

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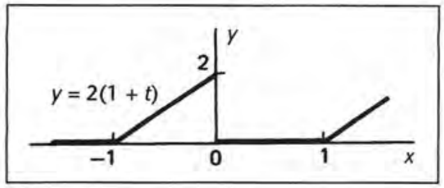

We have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos n\pi t + b_n \sin n\pi t\} \quad \text{because } T = 2$$

Therefore

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \int_{-1}^{1} f(t)\,dt = \int_{-1}^{0} 2(1+t)\,dt + \int_{0}^{1}(0)\,dt$$

$$= \left[2t + t^2\right]_{-1}^{0} = 1$$

and

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\pi t\,dt = \int_{-1}^{1} f(t)\cos n\pi t\,dt$$

$$= \int_{-1}^{0} 2(1+t)\ \cos n\pi t\,dt = \dots\dots\dots\dots$$

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$$\boxed{a_n = 0\ (n \text{ even});\quad a_n = \frac{4}{n^2\pi^2}\ (n \text{ odd})}$$

Because

$$a_n = \int_{-1}^{0} 2(1+t)\cos n\pi t\,dt$$

$$= 2\left\{\left[(1+t)\frac{\sin\ n\pi t}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0}\sin\ n\pi t\,dt\right\}$$

$$= 2\left\{(0-0) - \frac{1}{n\pi}\left[-\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0}\right\} = \frac{2}{n^2\pi^2}(1-\cos n\pi)$$

$$= \frac{2}{n^2\pi^2}(1-(-1)^n)$$

so that

$$a_n = 0 \quad (n \text{ even}),\quad a_n = \frac{4}{n^2\pi^2} \quad (n \text{ odd})$$

Now for b_n

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt = \dots\dots\dots\dots$$

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$$\boxed{b_n = -\frac{2}{n\pi}}$$

Because

$$\begin{aligned}
b_n &= \int_{-1}^{0} 2(1+t)\sin n\pi t \, dt \\
&= 2\left\{\left[(1+t)\frac{-\cos n\pi t}{n\pi}\right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0} \cos n\pi t \, dt\right\} \\
&= 2\left\{-\frac{1}{n\pi} + \left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0}\right\} = -\frac{2}{n\pi} + \frac{2}{n^2\pi^2}(\sin n\pi) = -\frac{2}{n\pi}
\end{aligned}$$

So the first few terms of the series give

$$f(t) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} f(t) = \frac{1}{2} + \frac{4}{\pi^2}&\left\{\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \ldots\right\} \\ &- \frac{2}{\pi}\left\{\sin \pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{4}\sin 4\pi t + \ldots\right\}\end{aligned}}$$

The Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\}$$

can also be written in the form

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} B_n \sin(n\omega t + \phi_n)$$

Comparing these two expressions we see that $A_0 = a_0$, $B_n \sin \phi_n = a_n$ and $B_n \cos \phi_n = b_n$. From this it follows that

$$B_n = \ldots\ldots\ldots\ldots \text{ and } \phi_n = \ldots\ldots\ldots\ldots$$

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$$\boxed{B_n = \sqrt{a_n^2 + b_n^2}; \quad \phi_n = \arctan\left(\frac{a_n}{b_n}\right)}$$

So

$B_1 \sin(\omega t + \phi_1)$	is the first harmonic or fundamental (lowest frequency)
$B_2 \sin(2\omega t + \phi_2)$	is the second harmonic (frequency twice that of the fundamental)
$B_n \sin(n\omega t + \phi_n)$	is the nth harmonic (frequency n times that of the fundamental).

And for the series to converge, the values of B_n must eventually decrease with higher-order harmonics, i.e. $B_n \to 0$ as $n \to \infty$.

Odd and even functions

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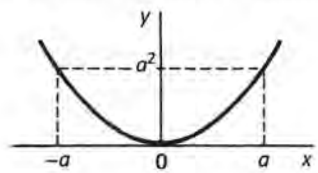
(a) *Even functions*

A function $f(x)$ is said to be *even* if

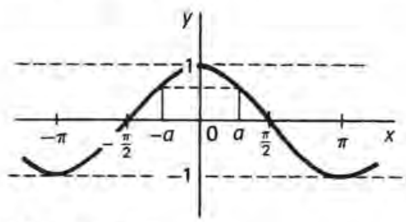
$$f(-x) = f(x)$$

i.e. the function value for a particular negative value of x is the same as that for the corresponding positive value of x. The graph of an even function is therefore *symmetrical about the y-axis*.



$y = f(x) = x^2$ is an even function because

$$f(-2) = 4 = f(2)$$
$$f(-3) = 9 = f(3) \quad \text{etc.}$$



$y = f(x) = \cos x$ is an even function because

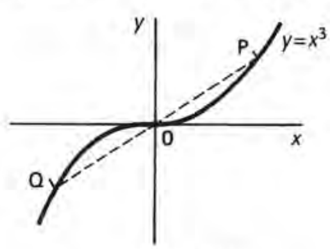
$$\cos(-x) = \cos x$$
$$f(-a) = \cos a = f(a).$$

(b) *Odd functions*

A function $f(x)$ is said to be *odd* if

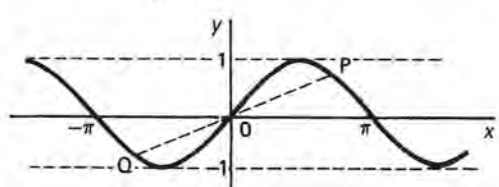
$$f(-x) = -f(x)$$

i.e. the function value for a particular negative value of x is numerically equal to that for the corresponding positive value of x but opposite in sign. The graph of an odd function is thus *symmetrical about the origin*.



$y = f(x) = x^3$ is an odd function because

$$f(-2) = -8 = -f(2)$$
$$f(-5) = -125 = -f(5) \quad \text{etc.}$$

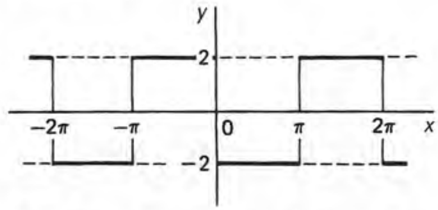


$y = f(x) = \sin x$ is an odd function because

$$\sin(-x) = -\sin x$$
$$f(-a) = -f(a).$$

So, for an even function $f(-x) = f(x)$, symmetrical about the y-axis
for an odd function $f(-x) = -f(x)$, symmetrical about the origin.

Example 1



$f(x)$ shown by the waveform is therefore an .... function because it is ....

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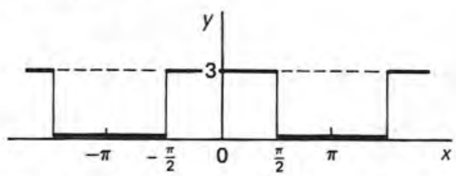
odd; symmetrical about the origin, i.e. $f(-x) = -f(x)$

Example 2



Hence the waveform of $y = f(x)$ depicts an .... function, because it is ....

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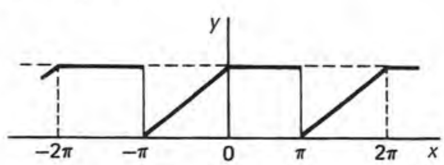
even; symmetrical about the y-axis, i.e. $f(-x) = f(x)$

Example 3



In this case, the waveform shows a function that is .... because ....

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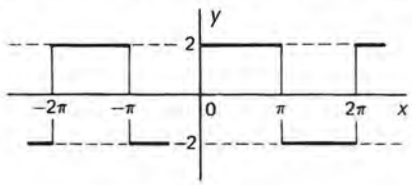
neither even nor odd; not symmetrical about either the y-axis or the origin

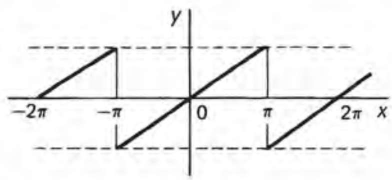
Exercise

State whether each of the following functions is odd, even, or neither.

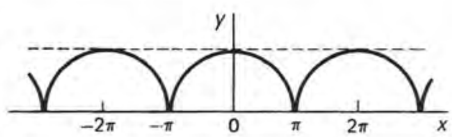
1



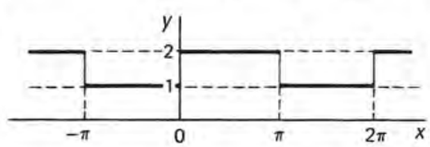
2



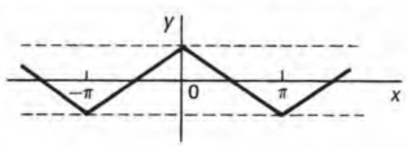
3



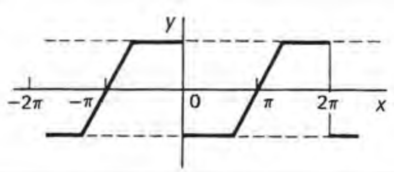
4



5



6



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1 odd	**2** odd	**3** even
4 neither	**5** even	**6** odd

We shall shortly see that a knowledge of odd and even functions can save a lot of unnecessary calculation.

First, however, let us consider products of odd and even functions in the next frame

50 Products of odd and even functions

The rules closely resemble the elementary rules of signs.

$$\begin{aligned}
&(\text{even}) \times (\text{even}) = (\text{even}) \quad \text{like} \quad && (+) \times (+) = (+)\\
&(\text{odd}) \times (\text{odd}) = (\text{even}) && (-) \times (-) = (+)\\
&(\text{odd}) \times (\text{even}) = (\text{odd}) && (-) \times (+) = (-).
\end{aligned}$$

The results can easily be proved.

(a) *Two even functions*

Let $F(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are even functions.

Then $F(-x) = f(-x)g(-x) = f(x)g(x)$ since $f(x)$ and $g(x)$ are even

$\therefore\ F(-x) = F(x)$ i.e. $F(x)$ is even

(b) *Two odd functions*

Let $F(x) = f(x)g(x)$ where $f(x)$ and $g(x)$ are odd functions.

$$\begin{aligned}
\text{Then } F(-x) &= f(-x)g(-x)\\
&= \{-f(x)\}\{-g(x)\} \text{ since } f(x) \text{ and } g(x) \text{ are odd}\\
&= f(x)g(x) = F(x)
\end{aligned}$$

$\therefore\ F(-x) = F(x)$ i.e. $F(x)$ is even

Finally

(c) *One odd and one even function*

Let $F(x) = f(x)g(x)$ where $f(x)$ is odd and $g(x)$ even.

Then $F(-x) = f(-x)g(-x) = -f(x)g(x) = -F(x)$

$\therefore\ F(-x) = -F(x)$ i.e. $F(x)$ is odd

So if $f(x)$ and $g(x)$ are both even, then $f(x)g(x)$ is even
and if $f(x)$ and $g(x)$ are both odd, then $f(x)g(x)$ is even
but if either $f(x)$ or $g(x)$ is even and the other odd, then $f(x)g(x)$ is odd.

Now for a short exercise, so move on

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Exercise

State whether each of the following products is odd, even, or neither.

1 $x^2 \sin 2x$

2 $x^3 \cos x$

3 $\cos 2x \cos 3x$

4 $x \sin nx$

5 $3 \sin x \cos 4x$

6 $(2x+3) \sin 4x$

7 $\sin^2 x \cos 3x$

8 $x^3 e^x$

9 $(x^4 + 4) \sin 2x$

10 $\dfrac{1}{x+2} \cosh x$

Finish all ten and then check with the next frame

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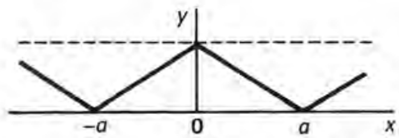
1	odd (E)(O) = (O)	**6**	neither (N)(O) = (N)
2	odd (O)(E) = (O)	**7**	even (E)(E) = (E)
3	even (E)(E) = (E)	**8**	neither (O)(N) = (N)
4	even (O)(O) = (E)	**9**	odd (E)(O) = (O)
5	odd (O)(E) = (O)	**10**	neither (N)(E) = (N)

Two useful facts emerge from odd and even functions. Thinking in terms of areas under the graphs

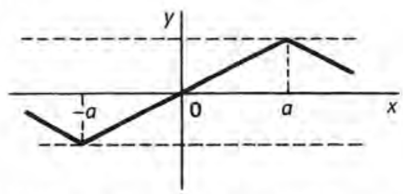
(a)



For an *even* function

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

(b)



For an *odd* function

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0$$

We can now look at two important theorems concerning odd and even functions.

Theorem 1

If $f(x)$ is defined over the interval $-\pi < x < \pi$ and $f(x)$ is *even*, then the Fourier series for $f(x)$ contains *cosine terms* only. Included in this is a_0 which may be regarded as $a_n \cos nx$ with $n = 0$.

Proof: Since $f(x)$ is even, $\int_{-\pi}^{0} f(x)\,\mathrm{d}x = \int_{0}^{\pi} f(x)\,\mathrm{d}x$

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,\mathrm{d}x \qquad \therefore\ a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,\mathrm{d}x$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x.$

But $f(x)\cos nx$ is the product of two even functions and therefore itself even.

$$\therefore\ a_n = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx}$$

Because as the integrand is even,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Arguing along similar lines, this gives $b_n = \dots\dots\dots\dots$

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$$\boxed{b_n = 0}$$

Because, since $f(x)\sin nx$ is the product of an even function and an odd function, it is itself odd.

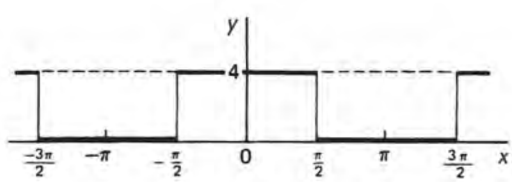
$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0. \quad \therefore\ b_n = 0$$

Therefore, there are no sine terms in the Fourier series for $f(x)$.

Now for an example.

Example

The waveform shown is symmetrical about the y-axis. The function is therefore even and there will be no sine terms in the series.



$$\therefore\ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi/2} 4\,dx = \dfrac{2}{\pi}\Big[4x\Big]_0^{\pi/2} = 4$

(b) $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \dfrac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \dots\dots\dots\dots$ Finish the integration.

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$$a_n = 0 \quad (n \text{ even}); \quad a_n = \frac{8}{\pi n} \quad (n = 1, 5, 9, \ldots);$$
$$a_n = -\frac{8}{\pi n} \quad (n = 3, 7, 11, \ldots)$$

Because

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \,dx = \frac{2}{\pi}\int_0^{\pi/2} 4\cos nx \,dx$$
$$= \frac{8}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

But $\sin\frac{n\pi}{2} = 0$ for n even

$= 1$ for $n = 1, 5, 9, \ldots$

$= -1$ for $n = 3, 7, 11, \ldots$ Hence the result stated.

(c) We know that $b_n = 0$, because $f(x)$ is an even function. Therefore, the required series is

$$f(x) = \ldots\ldots\ldots\ldots$$

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$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \ldots\right\}$$

If you care to look back to Example 1 in Frame 23, you will see how much time and effort we have saved by not having to evaluate b_n.

A similar theorem applies to odd functions.

Theorem 2

If $f(x)$ is an *odd* function defined over the interval $-\pi < x < \pi$, then the Fourier series for $f(x)$ contains *sine terms* only.

Proof: Since $f(x)$ is an odd function, $\int_{-\pi}^{0} f(x)\,dx = -\int_0^{\pi} f(x)\,dx$.

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$. But $f(x)$ is odd $\quad \therefore\ a_0 = 0$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

Remembering that $f(x)$ is odd and $\cos nx$ is even, the product $f(x)\cos nx$ is …………

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$$\boxed{\text{odd}}$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\text{odd function})\,dx = 0$$

$\therefore\ a_n = 0 \quad$ (including $a_0 = 0$)

Now for b_n we have

(c) $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ and because $f(x)$ and $\sin nx$ are each odd, the product $f(x)\sin nx$ is ....

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$$\boxed{\text{even}}$$

$$\text{Then } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\text{even function})\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

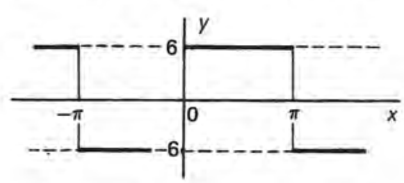
$$\therefore\ b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

So, if $f(x)$ is odd, $a_0 = 0$; $\quad a_n = 0$; $\quad b_n = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\sin nx\,dx$

i.e. the Fourier series contains sine terms only.

Example

Consider the function shown.



$$f(x) = -6 \qquad -\pi < x < 0$$
$$f(x) = 6 \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

Before we do any evaluation, we can see that this is .... and therefore ....

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$$\boxed{\text{an odd function; sine terms only, i.e. } a_0 = 0 \text{ and } a_n = 0}$$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$ $\qquad f(x)\sin nx$ is a product of two odd functions and is therefore even.

$$\therefore\ b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = \text{...........}$$

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$$\boxed{b_n = 0 \quad (n \text{ even}) \quad \text{or} \quad \frac{24}{\pi n} \quad (n \text{ odd})}$$

Because

$$b_n = \frac{2}{\pi}\int_0^{\pi} 6 \sin nx \, dx = \frac{12}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{12}{\pi n}(1 - \cos n\pi).$$

Hence the result stated above.

So the series is $f(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{f(x) = \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\}}$$

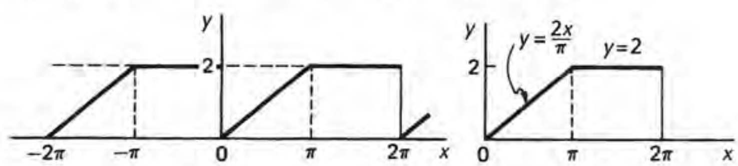
Because $\cos n\pi = (-1)^n$.

Of course, if $f(x)$ is neither an odd nor an even function, then we must obtain expressions for a_0, a_n and b_n in full.

One more example

Example

Determine the Fourier series for the function shown.



This is neither odd nor even. Therefore we must find a_0, a_n and b_n.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

(a) $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\, dx = \dfrac{1}{\pi}\left\{\int_0^{\pi}\frac{2}{\pi}x\,dx + \int_{\pi}^{2\pi} 2\,dx\right\}$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{\pi}\right]_0^{\pi} + \left[2x\right]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\{\pi + 4\pi - 2\pi\} = 3 \qquad \therefore\ a_0 = 3$$

(b) $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos nx\, dx = \dfrac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\right)\cos nx\, dx + \int_{\pi}^{2\pi} 2\cos nx\, dx\right\}$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{x \sin nx}{n}\right]_0^{\pi} - \frac{1}{\pi n}\int_0^{\pi}\sin nx\, dx + \int_{\pi}^{2\pi}\cos nx\, dx\right\}$$

$= \ldots\ldots\ldots\ldots$

Finish it off

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$$\boxed{a_n = 0 \quad (n \text{ even}); \quad a_n = \frac{-4}{\pi^2 n^2} \quad (n \text{ odd})}$$

Because

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left\{\frac{1}{\pi}(0-0) - \frac{1}{\pi n}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{\pi n^2}(-(-1)^n + 1) + (0-0)\right\} \\
&= -\frac{2}{\pi^2 n^2}(1-(-1)^n)
\end{aligned}$$

and so

$$a_n = 0 \quad (n \text{ even}) \text{ and } a_n = -\frac{4}{\pi^2 n^2} \quad (n \text{ odd})$$

(c) To find b_n, we proceed in the same general manner

$b_n = \ldots\ldots\ldots\ldots$

Complete it on your own

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$$\boxed{b_n = -\frac{2}{\pi n}}$$

Here is the working.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\right)\sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right\} \\
&= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos nx\,dx + \int_{\pi}^{2\pi}\sin nx\,dx\right\} \\
&= \frac{2}{\pi}\left\{\frac{1}{\pi n}(-\pi\cos n\pi) + \frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi + (0-0) - \frac{1}{n}(\cos 2\pi n - \cos n\pi)\right\} \\
&= \frac{2}{\pi}\left\{-\frac{1}{n}\cos 2n\pi\right\} = -\frac{2}{\pi n}\cos 2n\pi
\end{aligned}$$

But $\cos 2n\pi = 1$. $\quad \therefore\ b_n = -\dfrac{2}{\pi n}$

So the required series is $f(x) = \ldots\ldots\ldots\ldots$

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$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right\}$$
$$- \frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \dots\right\}$$

At this stage, let us take stock of our findings so far.
If a function $f(x)$ is defined over the range $-\pi$ to π, or any other periodic interval of 2π, then the Fourier series for $f(x)$ is of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

We also know that

(a) if $f(x)$ is an *even* function, the series will contain *no sine terms*
(b) if $f(x)$ is an *odd* function, the series will contain *only sine terms*
(c) if $f(x)$ is *neither odd nor even*, the series will, in general, contain a constant term, cosine terms and sine terms.

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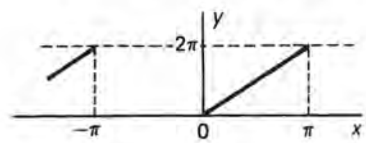
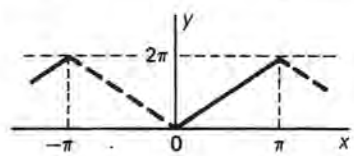
Half-range series

Sometimes a function of period 2π is defined over the range 0 to π, instead of the normal $-\pi$ to π, or 0 to 2π. We then have a choice of how to proceed.

For example, if we are told that between $x = 0$ and $x = \pi$, $f(x) = 2x$, then, since the period is 2π, we have no evidence of how the function behaves between $x = -\pi$ and $x = 0$.

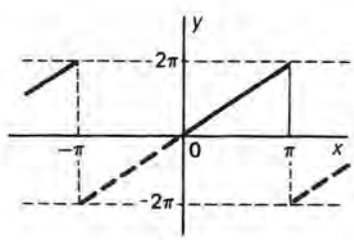


(a)



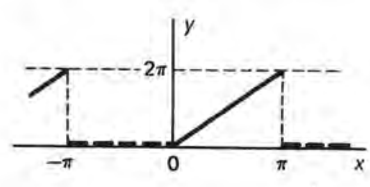
If the waveform were as shown in (a), the function would be an even function, symmetrical about the y-axis and the series would have *only cosine terms* (including possibly a_0).

(b)



On the other hand, if the waveform were as shown in (b), the function would be odd, being symmetrical about the origin and the series would have *only sine terms*.

(c)



Of course, if we choose something quite different for the waveform between $x = -\pi$ and $x = 0$, then $f(x)$ will be neither odd nor even and the series will then contain ....

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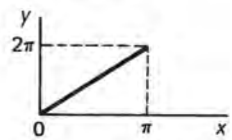
$\boxed{\text{both sine and cosine terms (including } a_0\text{)}}$

In each case, we are making an assumption on how the function behaves between $x = -\pi$ and $x = 0$, and the resulting Fourier series will therefore apply only to $f(x)$ between $x = 0$ and $x = \pi$ for which it is defined. For this reason, such series are called *half-range series*.

Example 1



A function $f(x)$ is defined by

$$f(x) = 2x \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$

Obtain a half-range cosine series to represent the function.

To obtain a cosine series, i.e. a series with no sine terms, we need an .... function.

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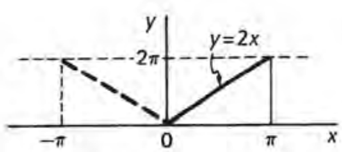
$\boxed{\text{even}}$



Therefore, we assume the waveform between $x = -\pi$ and $x = 0$ to be as shown, making the total graph symmetrical about the y-axis.

Now we can find expressions for the Fourier coefficients as usual.

$$a_0 = \text{............}$$

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$\boxed{a_0 = 2\pi}$

Because

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} 2x\,\mathrm{d}x = \frac{2}{\pi}\left[x^2\right]_0^{\pi} = 2\pi \qquad \therefore\ a_0 = 2\pi$$

Then we need a_n which is ....

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$$a_n = 0 \quad (n \text{ even}) \qquad = -\frac{8}{\pi n^2} \quad (n \text{ odd})$$

Because

$$a_n = \frac{2}{\pi}\int_0^\pi 2x\cos nx\,dx = \frac{4}{\pi}\int_0^\pi x\cos nx\,dx$$

$$= \frac{4}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx\right\}$$

$$= \frac{4}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^\pi\right\} = \frac{4}{\pi n^2}(\cos n\pi - 1)$$

$$\cos n\pi = 1 \quad (n \text{ even}) \qquad = -1 \quad (n \text{ odd})$$

$$\therefore\ a_n = 0 \quad (n \text{ even}) \quad \text{and} \quad a_n = -\frac{8}{\pi n^2} \quad (n \text{ odd})$$

All that now remains is b_n which is ....

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zero, since $f(x)$ is an even function, i.e. $b_n = 0$

So $a_0 = 2\pi$, $a_n = 0$ (n even) or $-\dfrac{8}{\pi n^2}$ (n odd), $b_n = 0$.

Therefore $f(x) =$ ....

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$$f(x) = \pi - \frac{8}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right\}$$

Let us look at a further example, so move on to the next frame

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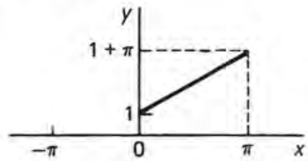
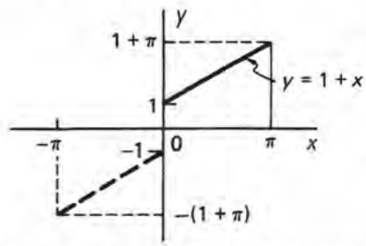
Example 2



Determine a half-range sine series to represent the function $f(x)$ defined by

$$f(x) = 1 + x \qquad 0 < x < \pi$$
$$f(x + 2\pi) = f(x).$$



We choose the waveform between $x = -\pi$ and $x = 0$ so that the graph is symmetrical about the origin. The function is then an odd function and the series will contain only sine terms.

$$\therefore\ a_0 = 0 \quad \text{and} \quad a_n = 0$$

b_n can now easily be determined and the required series obtained.

$$f(x) = \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} f(x) = \left(\frac{4}{\pi} + 2\right)&\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\} \\ &-2\left\{\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x + \dots\right\}\end{aligned}}$$

Check the working.

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1+x)\sin nx\,dx = \frac{2}{\pi}\left\{\left[(1+x)\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$$
$$= \frac{2}{\pi}\left\{-\frac{1+\pi}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\frac{1}{n} - \frac{1+\pi}{n}\cos n\pi\right\} = \frac{2}{\pi n}\{1 - (1+\pi)\cos n\pi\}$$
$$\cos n\pi = 1 \quad (n \text{ even}) \qquad = -1 \quad (n \text{ odd})$$
$$\therefore\ b_n = -\frac{2}{n} \quad (n \text{ even}) \qquad = \frac{4+2\pi}{\pi n} \quad (n \text{ odd})$$

Substituting in the general expression $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ we have

$$\begin{aligned} f(x) = \frac{4+2\pi}{\pi}&\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\} \\ &-2\left\{\frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x + \dots\right\}\end{aligned}$$

So a knowledge of odd and even functions and of half-range series saves a deal of unnecessary work on occasions.

Now let us consider the presence of odd or even harmonics, so move on

74 Series containing only odd harmonics or only even harmonics

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

If we replace x by $(x + \pi)$, this becomes

$$f(x+\pi) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos n(x+\pi) + b_n \sin n(x+\pi)\}$$

Now $\cos(nx + n\pi) = \cos nx \cos n\pi - \sin nx \sin n\pi$.

But for $n = 1, 2, 3, \dots \quad \sin n\pi = 0$

$\therefore \quad \cos n(x+\pi) = \cos nx \cos n\pi$

Also for $n = 1, 2, 3, \dots \quad \cos n\pi = 1 \quad (n \text{ even}) \quad = -1 \ (n \text{ odd})$.

$\therefore \quad \cos n(x+\pi) = \cos nx \quad (n \text{ even}) \quad = -\cos nx \quad (n \text{ odd}) \qquad (1)$

Similarly, $\sin(nx + n\pi) = \sin nx \cos n\pi + \cos nx \sin n\pi$.

Therefore, as before

$\sin n(x+\pi) = \sin nx \quad (n \text{ even}) \quad = -\sin nx \quad (n \text{ odd}) \qquad (2)$

$$\therefore \ f(x+\pi) = \tfrac{1}{2}a_0 - a_1 \cos x + a_2 \cos 2x - a_3 \cos 3x + \dots$$
$$- b_1 \sin x + b_2 \sin 2x - b_3 \sin 3x + \dots$$

But $$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

If $f(x) = f(x+\pi)$, these two series are equal and the odd harmonics that you see differ in sign must be zero.

$$\therefore \ f(x) = f(x+\pi) = \tfrac{1}{2}a_0 + a_2 \cos 2x + a_4 \cos 4x + \dots$$
$$+ b_2 \sin 2x + b_4 \sin 4x + \dots$$

$\therefore$ If $f(x) = f(x+\pi)$, the Fourier series for $f(x)$ contains even harmonics only.

Similarly, from the same two series above

if $f(x) = -f(x+\pi)$, the Fourier series for $f(x)$ contains odd harmonics only.

$$\therefore \ f(x) = a_1 \cos x + a_3 \cos 3x + \dots + b_1 \sin x + b_3 \sin 3x + \dots$$

Make a note of these two results: you will find them useful

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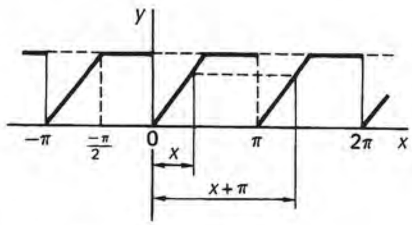
Example 1



Here $f(x) = f(x + \pi)$

Therefore, the series contains ....

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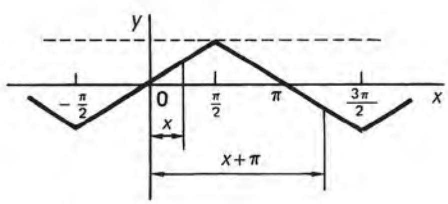
even harmonics only

Example 2



Here we see that $f(x) = -f(x + \pi)$.

Therefore, the series contains ....

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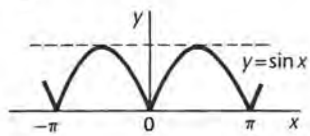
odd harmonics only

Now we can apply our knowledge to date to the following exercise.

Exercise

From each of the following waveforms, we can describe the nature of the terms in the relevant Fourier series.

1



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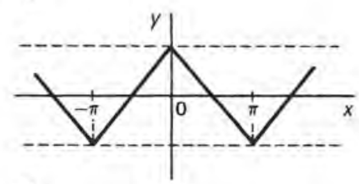
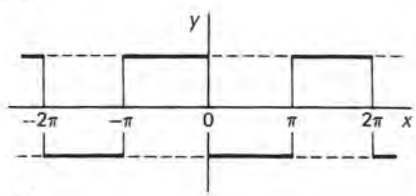
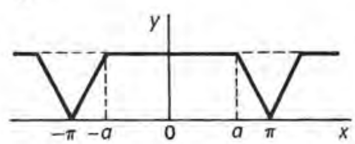


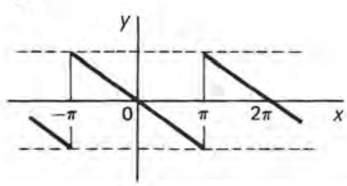
2



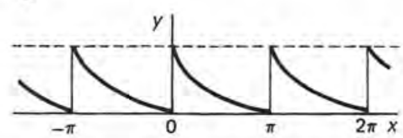
5



3



6



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1 cosine terms ($+a_0$) only; even harmonics only
2 sine terms only; odd harmonics only
3 sine terms only; all harmonics
4 cosine terms ($+a_0$) only; odd harmonics only
5 cosine terms ($+a_0$) only; all harmonics
6 a_0, sine and cosine terms; even harmonics only.

On we go

Significance of the constant term $\frac{1}{2}a_0$

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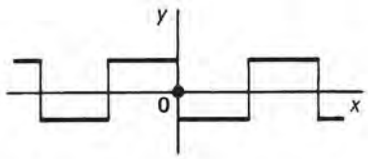
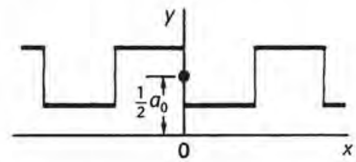
We might, at this point, note that the effect of the constant term $\frac{1}{2}a_0$ is to raise, or lower, the whole waveform on the y-axis.

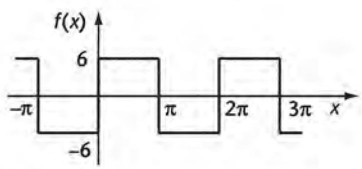




In electrical applications to alternating currents, the constant term $\frac{1}{2}a_0$ of the Fourier series indicates the d.c. component.

For example, from Frames 58–61 we found that the odd square wave

$$f(x) = \begin{cases} -6 & -\pi < x < 0 \\ 6 & 0 < x < \pi \end{cases} \qquad f(x + 2\pi) = f(x)$$

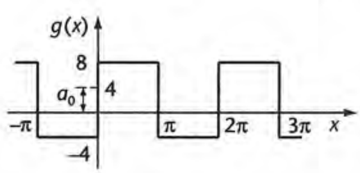


has the Fourier series expansion

$$f(x) = \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\}$$

The function $g(x) = 2 + f(x)$ has the Fourier series expansion

$$g(x) = 2 + \frac{24}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right\}$$



Here $a_0 = 4$.

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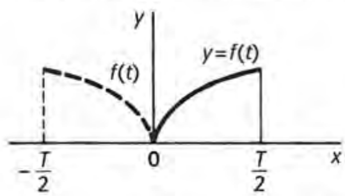
Half-range series with arbitrary period

We now extend the work on half-range sine and cosine series to functions with arbitrary period.

(a) *Even function* Half-range cosine series



$y = f(t) \quad 0 < t < \dfrac{T}{2}$

$f(t+T) = f(t)$

symmetrical about the y-axis.

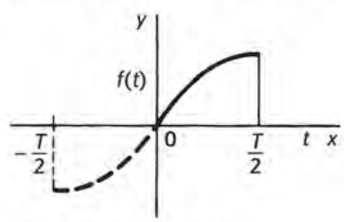
With an even function, we know that $b_n = 0$

$$\therefore\ f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

where $$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{d}t$$

and $$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,\mathrm{d}t$$

(b) *Odd function* Half-range sine series



$y = f(t) \quad 0 < t < \dfrac{T}{2}$

$f(t+T) = f(t)$

symmetrical about the origin.

$\therefore\ a_0 = 0$ and $a_n = 0$

Then $f(t) = \dots\dots\dots\dots$

and $b_n = \dots\dots\dots\dots$

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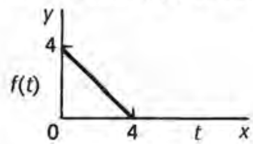
$$\boxed{f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t; \quad b_n = \frac{4}{T}\int_0^{T/2} f(t)\,\sin n\omega t\,\mathrm{d}t}$$

Now for an example or two.

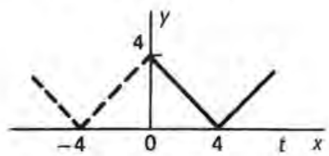
So move on

Example 1 82

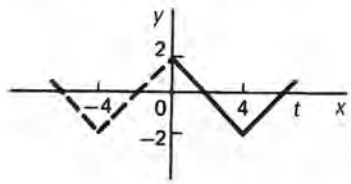
A function $f(t)$ is defined by $f(t) = 4 - t, \quad 0 < t < 4$.



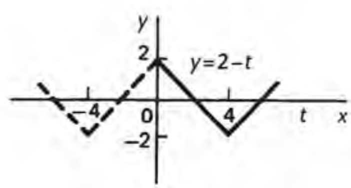
We have to form a half-range cosine series to represent the function in this interval.



First we form an even function, i.e. symmetrical about the y-axis.



Now for a useful little trick. If we lower the waveform 2 units, i.e. to its 'average' position, balanced above and below the x-axis, then in this new position $\frac{1}{2}a_0 = 0$ and we have been saved one set of calculations.



The function is now $y = f_1(t) = 2 - t$ and, for the moment $\frac{1}{2}a_0 = 0$. Also, being an even function $b_n = 0$. All we need to do is to evaluate a_n.

$$\text{So } a_n = \frac{4}{T}\int_0^{T/2} f_1(t)\cos n\omega t \,\mathrm{d}t = \frac{4}{8}\int_0^4 (2-t)\cos n\omega t \,\mathrm{d}t$$

$$= \ldots\ldots\ldots\ldots$$

83

$$\boxed{a_n = 0 \quad (n \text{ even}) \qquad = \frac{1}{n^2\omega^2} \quad (n \text{ odd})}$$

Simple integration by parts gives

$$a_n = \frac{1}{2}\left\{-\frac{2\sin 4n\omega}{n\omega} - \frac{1}{n^2\omega^2}(\cos 4n\omega - 1)\right\}$$

$$\text{But } \omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$a_n = \frac{1}{2}\left\{-\frac{2\sin n\pi}{n\omega} - \frac{1}{n^2\omega^2}(\cos n\pi - 1)\right\} \qquad n = 1, 2, 3, \ldots$$

$$\sin n\pi = 0; \quad \cos n\pi = 1 \quad (n \text{ even}); \quad \cos n\pi = -1 \quad (n \text{ odd})$$

$$\therefore\ a_n = 0 \quad (n \text{ even}) \quad \text{and} \quad a_n = \frac{1}{n^2\omega^2} \quad (n \text{ odd})$$

$$\therefore\ f_1(t) = \ldots\ldots\ldots\ldots$$

84

$$\boxed{f_1(t) = \frac{1}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \dots\right\}}$$

Now if we finally lift the waveform back to its original position by restoring the 2 units (i.e. $\frac{1}{2}a_0 = 2$), the original function is regained with $f(t) = f_1(t) + 2$.

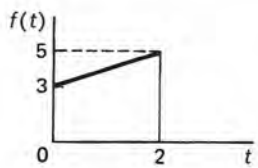
$$\therefore\ f(t) = 2 + \frac{1}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \dots\right\}$$

where $\omega = \dfrac{\pi}{4}$.

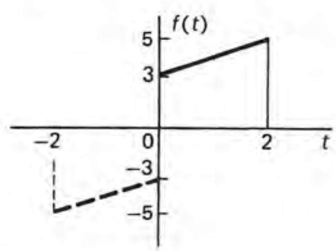
Example 2

A function $f(t)$ is defined by $\quad f(t) = 3 + t \qquad 0 < t < 2$

$$f(t+4) = f(t).$$



Obtain the half-range sine series for the function in this range.



Sine series required. Therefore, we form an odd function, symmetrical about the origin

$a_0 = 0;\ a_n = 0;\ T = 4$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$\therefore\ b_n = \frac{4}{T}\int_0^2 f(t)\ \sin n\omega t\,\mathrm{d}t = \int_0^2 (3+t)\sin n\omega t\,\mathrm{d}t$$

This you can easily evaluate and then, putting $n = 1, 2, 3, \dots$ obtain the series $f(t) = \dots\dots\dots\dots$

85

$$\boxed{f(t)=\frac{2}{\omega}\left\{4\sin\omega t-\frac{1}{2}\sin 2\omega t+\frac{4}{3}\sin 3\omega t-\frac{1}{4}\sin 4\omega t\dots\right\}}$$

Straightforward integration by parts gives

$$b_n=\frac{1}{n\omega}(3-5\cos 2n\omega)+\frac{1}{n^2\omega^2}(\sin 2n\omega)$$

But $T=\dfrac{2\pi}{\omega}\quad \therefore\ \omega=\dfrac{2\pi}{T}=\dfrac{\pi}{2}$

$$\therefore\ b_n=\frac{1}{n\omega}(3-5\cos n\pi)+\frac{1}{n^2\omega^2}\sin n\pi$$

$$=-\frac{2}{n\omega}\quad (n\text{ even})\qquad =\frac{8}{n\omega}\quad (n\text{ odd})$$

$$\therefore\ f(t)=\frac{2}{\omega}\left\{4\sin\omega t-\frac{1}{2}\sin 2\omega t+\frac{4}{3}\sin 3\omega t-\frac{1}{4}\sin 4\omega t\dots\right\}$$

And that just about brings this particular Programme to an end. Fourier series have wide applications so it is very worthwhile paying considerable attention to them.

The **Revision summary** and **Can You?** checklist now follow, after which you will have no trouble with the **Test exercise**. The **Further problems** provide additional practice.

Revision summary 6

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1 *Graphs of $y=A\sin nx$ and $A\cos nx$*

Amplitude $=A$; period $=\dfrac{360°}{n}=\dfrac{2\pi}{n}$ radians

2 *Harmonics*

$y=A_1\sin x$ is the first harmonic or fundamental
$y=A_n\sin nx$ is the nth harmonic.

3 *Periodic function*

$f(x+P)=f(x)\qquad P=\text{period}$

4 *Fourier series – functions of period 2π*

$$f(x)=\tfrac{1}{2}a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\dots+a_n\cos nx\dots$$
$$+b_1\sin x+b_2\sin 2x+b_3\sin 3x+\dots+b_n\sin nx\dots$$
$$=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n\cos nx+b_n\sin nx\}$$

5 *Dirichlet conditions*

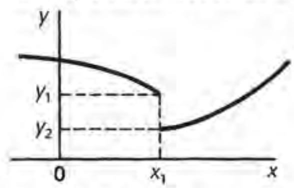
(a) The function $f(x)$ must be defined, single-valued and periodic.
(b) $f(x)$ and $f'(x)$ must be piecewise continuous in the periodic interval.

6 *Fourier coefficients*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

where, in each case, $n = 1, 2, 3, \ldots$

7 *Sum of Fourier series at a finite discontinuity*



At $x = x_1$, series for $f(x)$ converges to the value

$$\tfrac{1}{2}\{f(x_1 - 0) + f(x_1 + 0)\} = \tfrac{1}{2}(y_1 + y_2)$$

8 *Odd and even functions*

(a) Even function: $f(-x) = f(x)$; symmetrical about y-axis.
(b) Odd function: $f(-x) = -f(x)$; symmetrical about the origin.
Product of odd and even functions

(even) × (even) = (even)
(odd) × (odd) = (even)
(odd) × (even) = (odd).

9 *Sine series and cosine series*

If $f(x)$ is *even*, the series contains *cosine terms only* (including a_0).
If $f(x)$ is *odd*, the series contains *sine terms only*.

10 *Half-range series*

Function of period 2π, defined over the range 0 to π. Can be considered as half of an even function, or half of an odd function.

11 *Series containing only odd harmonics or only even harmonics*

If $f(x) = f(x + \pi)$ the Fourier series contains *even harmonics only*.
If $f(x) = -f(x + \pi)$ the Fourier series contains *odd harmonics only*.

12 *Functions with period T*

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos\, n\omega t + b_n\sin\, n\omega t\}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t \qquad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\,\mathrm{d}t$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,\mathrm{d}t \qquad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\,\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,\mathrm{d}t \qquad = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,\mathrm{d}t$$

where $\omega = \dfrac{2\pi}{T}$ i.e. $T = \dfrac{2\pi}{\omega}$.

▶

13 *Half-range series – period T*

(a) Even function: half-range cosine series

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,dt$$
$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt$$
$$b_n = 0.$$

(b) Odd function: half-range sine series

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\, n\omega t\,dt.$$

Can You?

Checklist 6

87

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Determine the period and amplitude of a periodic function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 and 2
• Write down the harmonics of a periodic trigonometric function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	3
• Give an analytic description of a non-sinusoidal periodic function? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	4 to 9
• Evaluate integrals with periodic integrands? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	10 to 15

►

- Demonstrate the orthogonality of the trigonometric functions $\sin nx$ and $\cos nx$ for $n = 0, 1, 2, \dots$? 16 to 20

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe a periodic function as a Fourier series subject to the Dirichlet conditions? 21 and 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the Fourier coefficients and hence the Fourier series of a periodic function? 23 to 25

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe the effects of the harmonics in the construction of the Fourier series? 35

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Find the value of the Fourier series at a point of discontinuity of the periodic function? 36 and 37

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Fourier series of non-sinusoidal periodic functions? 38 to 44

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise even and odd functions and their products? 45 to 51

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Fourier sine and cosine series for odd and even functions respectively? 52 to 64

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive half-range Fourier series? 65 to 73

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the condition for the Fourier series to contain only odd or only even harmonics? 74 to 78

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Explain the significance of the term $a_0/2$? 79 to 85

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 6

88

1 If $f(x)$ is defined in the interval $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$, state whether or not each of the following functions can be represented by a Fourier series.

(a) $f(x) = x^4$
(b) $f(x) = 3 - 2x$
(c) $f(x) = \dfrac{1}{x}$
(d) $f(x) = e^{2x}$
(e) $f(x) = \operatorname{cosec} x$
(f) $f(x) = \pm\sqrt{4x}$.

2 Determine the Fourier series for the function defined by

$$f(x) = 2x \qquad 0 < x < 2\pi$$
$$f(x+2\pi) = f(x).$$

3 State whether each of the following products is odd, even, or neither.

(a) $x^3 \cos 2x$
(b) $x^2 \sin 3x$
(c) $\sin 2x \sin 3x$
(d) $x^2 e^{2x}$
(e) $(x+5)\cos 2x$
(f) $\sin^2 x \cos x$.

4 A function $f(x)$ is defined by $f(x) = \pi - x \qquad 0 < x < \pi$

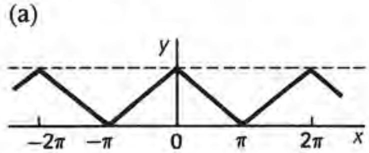
$$f(x+2\pi) = f(x).$$

Express the function

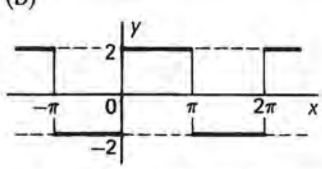
(a) as a half-range cosine series
(b) as a half-range sine series.

5 Comment on the nature of the terms in the Fourier series for the following functions.

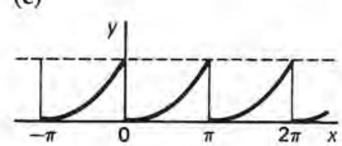
(a)



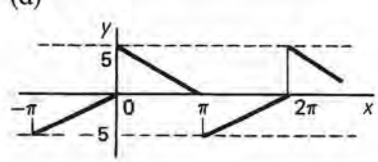
(b)



(c)



(d)



6 A function $f(t)$ is defined by

$$f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases}$$
$$f(t+4) = f(t).$$

Determine its Fourier series.

Further problems 6

89

1 A periodic function $f(x)$ is defined by

$$f(x) = 1 - \frac{x}{\pi} \qquad 0 < x < 2\pi$$
$$f(x + 2\pi) = f(x).$$

Determine the Fourier series up to and including the third harmonic.

2 Determine the Fourier series representation of the function $f(t)$ defined by

$$f(t) = \begin{cases} 3 & -2 < t < 0 \\ -5 & 0 < t < 2 \end{cases}$$
$$f(t + 4) = f(t).$$

3 Determine the half-range cosine series for the function $f(x) = \sin x$ defined in the range $0 < x < \pi$.

4 A function is defined by

$$f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$
$$f(x + 2\pi) = f(x).$$

Obtain the Fourier series.

5 A periodic function is defined by

$$f(x) = \begin{cases} A\sin x & 0 < x < \pi \\ -A\sin x & \pi < x < 2\pi \end{cases}$$
$$f(x + 2\pi) = f(x).$$

Determine its Fourier series up to and including the fourth harmonic.

6 If $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$

obtain the Fourier series.

7 Determine the Fourier series to represent a half-wave rectifier output current, i amperes, defined by

$$i = f(t) = \begin{cases} A\sin\omega t & 0 < t < \dfrac{T}{2} \\ 0 & \dfrac{T}{2} < t < T \end{cases}$$
$$f(t + T) = f(t).$$

▶

8 A function $f(x)$ is defined by

$$f(x) = \begin{cases} a & 0 < x < \dfrac{\pi}{3} \\ 0 & \dfrac{\pi}{3} < x < \dfrac{2\pi}{3} \\ -a & \dfrac{2\pi}{3} < x < \pi \end{cases}$$

$f(x + \pi) = f(x)$.

Obtain the Fourier series to represent the function.

9 If $f(x)$ is defined by $f(x) = x(\pi - x) \quad 0 < x < \pi$, express the function as

(a) a half-range cosine series

(b) a half-range sine series.

10 Determine the Fourier cosine series to represent the function $f(x)$ where

$$f(x) = \begin{cases} \cos x & 0 < x < \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x < \pi \end{cases}$$

$f(x + 2\pi) = f(x)$.

11 If

$$f(x) = \begin{cases} 0 & 0 < x < \dfrac{\pi}{2} \\ \cos x & \dfrac{\pi}{2} < x < \pi \end{cases} \qquad f(x + 2\pi) = f(x),$$

obtain the Fourier cosine series for $f(x)$ in the range $x = 0$ to $x = \pi$.

12 A function $f(x)$ is defined over the interval $0 < x < \pi$ by

$$f(x) = \begin{cases} x & 0 < x < \dfrac{\pi}{2} \\ \pi - x & \dfrac{\pi}{2} < x < \pi \end{cases}$$

For the range $x = 0$ to $x = \pi$, determine the Fourier sine series.

13 A function $f(t)$ is defined by

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 2t & 0 < t < 1 \end{cases}$$

$f(t + 2) = f(t)$.

Obtain the Fourier series up to and including the third harmonic.

14 If $f(x) = x^2 \quad -\pi < x < \pi$ and $f(x + 2\pi) = f(x)$, determine the Fourier series for $f(x)$.

15 A function $f(x)$ is given by $f(x) = 7 - \dfrac{3x}{\pi}$ for $-\pi < x < \pi$ with $f(x + 2\pi) = f(x)$. Obtain the Fourier series up to and including the fourth harmonic.

▶

16 A function $f(t)$ is defined by

$$f(t) = 1 - t^2 \qquad -1 < t < 1$$
$$f(t+2) = f(t).$$

Determine its Fourier series.

17 A function $f(x)$ is such that

$$f(x) = \begin{cases} \dfrac{x+\pi}{2} & -\pi < x < 0 \\ \dfrac{x-\pi}{2} & 0 < x < \pi \end{cases}$$
$$f(x+2\pi) = f(x).$$

Obtain the Fourier series.

18 Determine the Fourier series for a periodic function such that

$$f(t) = \begin{cases} 1 & -2 < t < -1 \\ 0 & -1 < t < 1 \\ -1 & 1 < t < 2 \end{cases}$$
$$f(t+4) = f(t).$$

19 A function is defined by

$$f(x) = x^2 \qquad 0 < x < 2\pi$$
$$f(x+2\pi) = f(x).$$

Determine its Fourier series.

20 Determine the Fourier series for the function $f(t)$ defined by

$$f(t) = \begin{cases} 0 & -2 < t < 0 \\ \dfrac{3t}{4} & 0 < t < 4 \end{cases}$$
$$f(t+6) = f(t).$$

Programme 7

Introduction to the Fourier transform

Frames 1 to 53

Learning outcomes

When you have completed this Programme you will be able to:

- Convert a trigonometric Fourier series into a doubly infinite sum of complex exponentials
- Derive the complex Fourier series of a function that satisfies Dirichlet's conditions
- Recognise the function sinc (t)
- Separate a discrete complex spectrum into an amplitude spectrum and a phase spectrum
- State Fourier's integral theorem in terms of complex exponentials
- Define and derive the Fourier transform of a function satisfying Dirichlet's conditions
- Separate a continuous complex spectrum into an amplitude spectrum and a phase spectrum
- Recognise the functions $\Pi_a(t)$ and $\Lambda_a(t)$ and derive their Fourier transforms along with those of the Dirac delta and the Heaviside unit step
- Recognise alternative forms of the function–transform pair
- Reproduce a collection of properties of the Fourier transform
- Evaluate the convolution of two functions and describe its Fourier transform
- Derive the Fourier sine and cosine transformations.

Complex Fourier series

1

Introduction

In the previous Programme we saw how a periodic function can be represented by an infinite sum of periodic, trigonometric harmonics. Each harmonic has a definite frequency which is an integer multiple of the fundamental frequency. A non-periodic function can be similarly represented, not as a sum but as an integral over a continuous range of frequencies. Before we do this, however, we shall convert the infinite Fourier series in terms of sines and cosines into a doubly infinite series involving complex exponentials.

Complex exponentials

Recall the exponential form of a complex number and its relationship to the polar form, namely

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

From this equation we can see that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

and so

$$\cos(-\theta) + j\sin(-\theta) = e^{-j\theta} = \cos\theta - j\sin\theta$$

Using these two equations we can find the complex exponential form of the trigonometric functions as

$\cos\theta = \ldots\ldots\ldots\ldots$ and $\sin\theta = \ldots\ldots\ldots\ldots$

2

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

Because

$$\cos\theta + j\sin\theta = e^{j\theta} \text{and } \cos\theta - j\sin\theta = e^{-j\theta}$$

so adding these two equations gives

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \text{ that is } \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad (1)$$

and subtracting the two equations gives

$$2j\sin\theta = e^{j\theta} - e^{-j\theta} \text{ that is } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad (2)$$

These two equations permit us to develop an alternative representation of a Fourier series.

▶

In the previous Programme we found that the Fourier series of the piecewise continuous function $f(t)$ with piecewise continuous derivative and where $f(t+T)=f(t)$ is given as

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t) \qquad (3)$$

where $\omega_0=\dfrac{2\pi}{T}$ and where $a_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}f(t)\cos n\omega_0 t\,\mathrm{d}t$

and $b_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}f(t)\sin n\omega_o t\,\mathrm{d}t$

Now, if we substitute the right-hand sides of equations (1) and (2) into equation (3) we obtain

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\{\ldots\ldots\ldots\ldots\}e^{jn\omega_0 t}+\{\ldots\ldots\ldots\ldots\}e^{-jn\omega_0 t}\right)$$

3

$$\boxed{f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\frac{a_n-jb_n}{2}\right\}e^{jn\omega_0 t}+\left\{\frac{a_n+jb_n}{2}\right\}e^{-jn\omega_0 t}\right)}$$

Because

$$\begin{aligned}f(t)&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2}+b_n\frac{e^{jn\omega_0 t}-e^{-jn\omega_0 t}}{2j}\right)\\&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\frac{a_n+b_n/j}{2}\right\}e^{jn\omega_0 t}+\left\{\frac{a_n-b_n/j}{2}\right\}e^{-jn\omega_0 t}\right)\\&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\left\{\frac{a_n-jb_n}{2}\right\}e^{jn\omega_0 t}+\left\{\frac{a_n+jb_n}{2}\right\}e^{-jn\omega_0 t}\right)\end{aligned}$$

In the next frame we shall make some notational changes to simplify this expression

4

If we now define $c_n = \dfrac{a_n - jb_n}{2}$ so that the complex conjugate of c_n is $c_n^* = \dfrac{a_n + jb_n}{2}$ we can write this sum as

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\omega_0 t} + c_n^* e^{-jn\omega_0 t}\right)$$

Note that we have taken $b_0 = 0$. There is no problem about this. There is no term $\sin 0\omega_0 t$ in the Fourier series and so $b_0 = 0$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_n^* e^{-jn\omega_0 t}$$

For notational convenience we denote c_n^* by c_{-n}. This means that $a_{-n} = a_n$ and $b_{-n} = -b_n$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

As n ranges from 1 to ∞ so $-n$ ranges from -1 to $-\infty$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t}$$

Notice the reversed order of summation in the first sum

$$= \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t}$$

Combining all three terms into the *doubly infinite sum*

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $c_n = \dfrac{a_n - jb_n}{2} = \dfrac{2}{2T}\displaystyle\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)\,dt$. That is $c_n = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$.

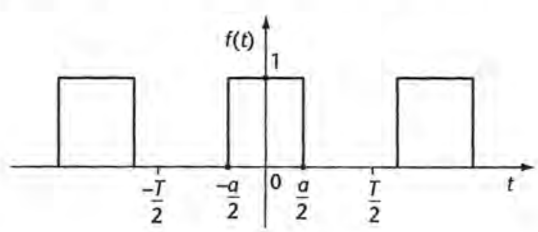
In the next frame we shall look at some examples

5

Example 1

To find the complex Fourier series for the function

$$f(t) = \begin{cases} 0 & -T/2 < t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t < T/2 \end{cases} \quad \text{where } f(t+T) = f(t)$$



we proceed as on the next page.

▶

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} && \text{where } \omega_0 = \frac{2\pi}{T} \text{ and} \\
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} \\
&= \frac{1}{T}\int_{-a/2}^{a/2} e^{-jn\omega_0 t}\,\mathrm{d}t && \text{Because } f(t) = 1 \text{ for } -a/2 < t < a/2 \\
&= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-a/2}^{a/2} && \text{Provided } n \neq 0 \\
&= \left(\frac{e^{-jn\omega_0 a/2} - e^{jn\omega_0 a/2}}{-j2n\pi}\right) && \text{Since } \omega_0 = \frac{2\pi}{T} \\
&= \frac{\sin n\omega_0 a/2}{n\pi} && \text{Recall that } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \\
&= \frac{\sin n\pi a/T}{n\pi} && \text{Since } \omega_0 = \frac{2\pi}{T} \\
&= \frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) && \text{Provided } n \neq 0
\end{aligned}$$

When $n = 0$

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \frac{1}{T}\int_{-a/2}^{a/2}\mathrm{d}t = \frac{a}{T}$$

Therefore

$$f(t) = \frac{a}{T} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)e^{jn\omega_0 t}$$

In the next frame we shall look at the same function retarded by half the width of the peak

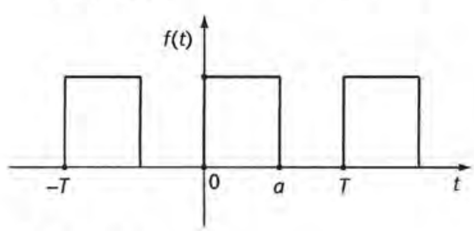
---

6

Example 2

To find the complex Fourier series for the function

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & a < t < T \end{cases} \qquad \text{where } f(t+T) = f(t)$$



We find that, for $n \neq 0$,

$c_n = \ldots\ldots\ldots\ldots$

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$$\boxed{c_n = e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)}$$

Because

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad \text{where } \omega = \frac{2\pi}{T} \text{ and}$$

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^a e^{-jn\omega_0 t}\,\mathrm{d}t \\ &= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^a \qquad \text{Provided } n \neq 0 \\ &= \left(\frac{e^{-jn\omega_0 a} - 1}{-j2n\pi}\right) \\ &= e^{-jn\omega_0 a/2}\left(\frac{e^{-jn\omega_0 a/2} - e^{jn\omega_0 a/2}}{-j2n\pi}\right) \\ &= e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) \qquad \text{Provided } n \neq 0 \end{aligned}$$

To finish

$$c_0 = \ldots\ldots\ldots\ldots$$

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$$\boxed{c_0 = \frac{a}{T}}$$

Because

$$\begin{aligned} c_0 &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^a \mathrm{d}t = \frac{a}{T} \end{aligned}$$

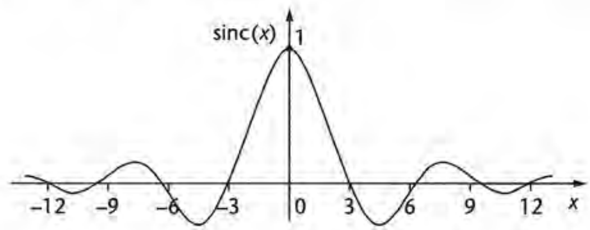
Therefore

$$f(t) = \frac{a}{T} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)e^{jn\omega_0 t}$$

Next frame

9

Before we move on, consider the expression $\dfrac{\sin n\pi a/T}{n\pi a/T}$ that occurs in both of these examples. This is an example of a commonly occurring expression $\dfrac{\sin x}{x}$ which has the special name $\operatorname{sinc}(x)$. Notice that $\operatorname{sinc}(0)$ is not defined. However, because $\displaystyle\lim_{x\to 0}\operatorname{sinc}(x) = \lim_{x\to 0}\frac{\sin x}{x} = 1$ we define $\operatorname{sinc}(0) = 1$.



This means that c_0 can be incorporated into the summations so the solutions to Examples 1 and 2 become

$$f(t) = \sum_{n=-\infty}^{\infty} (a/T)\operatorname{sinc}(n\pi a/T)e^{jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} (a/T)e^{-jn\pi a/T}\operatorname{sinc}(n\pi a/T)e^{jn\omega_0 t} \quad \text{respectively.}$$

Now let's compare these two results

Complex spectra

10

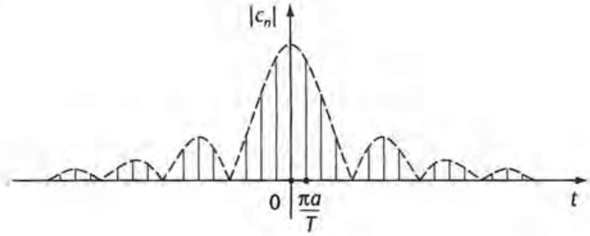
The coefficients c_n in the first example are real numbers whereas in the second example they are complex numbers. In general, the c_n are complex numbers and can be written as

$c_n = |c_n|e^{j\phi_n}$ where, in the last example $|c_n| = \dfrac{a}{T}\left|\dfrac{\sin n\pi aT}{n\pi T}\right|$

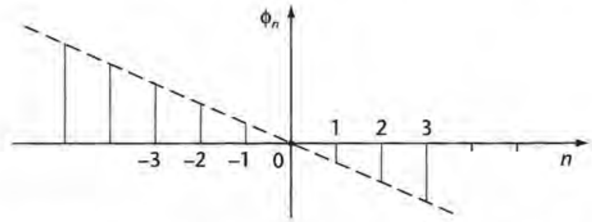
and $\phi_n = -n\pi a/T$.

These complex coefficients constitute a **discrete complex spectrum** where c_n represents the *spectral coefficient* of the nth harmonic. Each spectral coefficient couples an **amplitude spectrum** value $|c_n|$ and a **phase spectrum** value ϕ_n. The amplitude spectrum tells us the magnitude of each of the harmonic components and has, for both examples, the graph shown on the next page.

▶



The phase spectrum $\phi_n = -n\pi a/T$ tells us the phase of each harmonic relative to the fundamental harmonic frequency ω_0.



The phase spectrum of the first example is zero for all n and tells us that each harmonic is in phase with the fundamental harmonic. The phase spectrum of the second example, which is a retarded form of the first example, tells us that the nth harmonic is shifted out of phase from the fundamental harmonic by $n\omega_0$.

Next frame

The two domains

11

A periodic waveform and its spectrum are described in different terms. The waveform is described in terms of behaviour in time whereas the spectrum is described in terms of behaviour relative to frequency. Thus time and frequency form two domains of definition of our functions and whatever information can be gleaned from within one domain can equally be gleaned from within the other. For example, the *power content* of a periodic function $f(t)$ of period T is defined in the time domain as the mean square value of $f(t)$

$$\frac{1}{T}\int_{-T/2}^{T/2}(f(t))^2\,dt$$

Within the frequency domain the power content is given as

....

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$$\boxed{\sum_{n=-\infty}^{\infty} |c_n|^2}$$

Because

$$\begin{aligned}
\frac{1}{T}\int_{-T/2}^{T/2}(f(t))^2\,dt &= \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right)f(t)\,dt \\
&= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}\,dt \\
&= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j(-n)\omega_0 t}\,dt \\
&= \sum_{n=-\infty}^{\infty} c_n c_{-n} = \sum_{n=-\infty}^{\infty} c_n c_n^* \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

So the power content can be obtained from either domain.

Next frame

Continuous spectra

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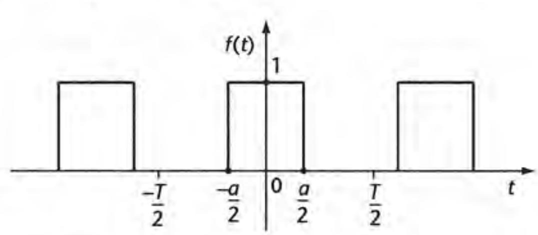
Of interest in the analysis of periodic functions is the behaviour of the Fourier series as the period increases without limit. Consider Example 1 from Frame 5

$$f(t) = \begin{cases} 0 & -T/2 < t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t < T/2 \end{cases} \quad \text{where } f(t+T) = f(t)$$



which has the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T} \text{ and where } c_n = \left(\frac{a}{T}\right)\frac{\sin\left(\frac{n\pi a}{T}\right)}{\frac{n\pi a}{T}}$$

As the period increases the separation between the pulses increases and in the limit as $T \to \infty$ only .... remains and the resulting function is no longer ....

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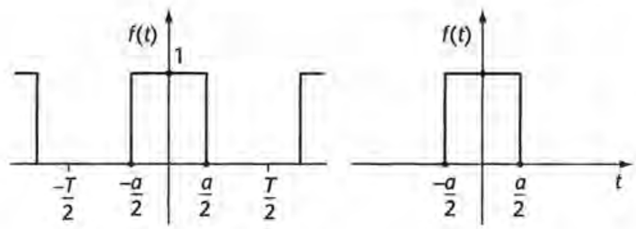
only a single pulse remains and the resulting function is no longer periodic



In the Fourier series the distance between neighbouring harmonics in the complex spectra is the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ and, in the limit as $T \to \infty$, so $\omega_0 \to 0$. This means that as the period increases the space between lines in the spectrum decreases so the spectrum lines come closer together and in the limit merge into a continuous spectrum. That is, for large T

$n\omega_0 = n\delta\omega$ and as $T \to \infty$ so $n\delta\omega \to \omega$

where ω is the continuous frequency variable. To see the effect of this on the general form of the Fourier series we start with

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T}$$

$$\text{and where } c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

Substituting the integral form of c_n into the sum gives

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(u)e^{-jn\omega_0 u}\,du\right]e^{jn\omega_0 t}$$

where u is a dummy variable in place of the variable t.

Now, $\omega_0 = \frac{2\pi}{T}$ and so

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-T/2}^{T/2} f(u)e^{-jn\omega_0 u}\,du\right]\omega_0 e^{jn\omega_0 t}$$

If T is large then $\omega_0 = \delta\omega$ and

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-T/2}^{T/2} f(u)e^{-jn\delta\omega u}\,du\right]e^{jn\delta\omega t}\delta\omega$$

▶

In the limit as $T \to \infty$ so $n\delta\omega \to \omega$, the sum becomes an integral and $\delta\omega$ becomes the differential $\mathrm{d}\omega$ giving

$$\begin{aligned} f(t) &= \int_{\omega=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{u=-\infty}^{\infty} f(u)e^{-j\omega u}\,\mathrm{d}u\right]e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{\omega=-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{u=-\infty}^{\infty} f(u)e^{-j\omega u}\,\mathrm{d}u\right]e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{u=-\infty}^{\infty} f(u)e^{-j\omega u}\,\mathrm{d}u \end{aligned}$$

These two integrals form the conclusion of *Fourier's integral theorem*.

Next frame

Fourier's integral theorem

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Given function $f(t)$ with derivative $f'(t)$ where

(a) $f(t)$ and $f'(t)$ are piecewise continuous in every finite interval

(b) $f(t)$ is absolutely integrable in $(\infty, -\infty)$, that is $\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t$ is finite

then

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \text{ where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

The discrete harmonic values $n\omega_0$ of the periodic function are now replaced by the continuous harmonic variable ω and the discrete spectra $c_n = |c_n|e^{j\phi_n}$ are replaced by the *continuous spectra* $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$. $F(\omega)$ is referred to as the **Fourier transform** of $f(t)$ and can also be written as $\mathscr{F}(f(t))$. Deriving the Fourier transform of a function is then a matter of applying the second of these two integrals. The expressions $f(t)$ and $F(\omega)$ form a Fourier transform pair where $f(t)$ can be referred to as the inverse Fourier transform of $F(\omega)$. That is, $f(t) = \mathscr{F}^{-1}[F(\omega)]$.

Next frame

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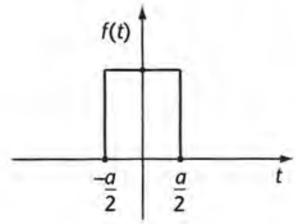
Example 3

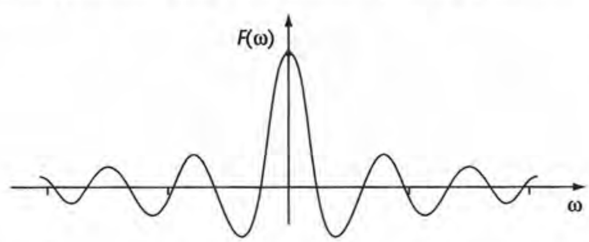
Find the Fourier transform of

$$f(t) = \begin{cases} 0 & t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t \end{cases}$$



$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-a/2}^{a/2} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-j\omega}\right) \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-2j\omega}\right) \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin \omega a/2}{\omega} \\ &= \frac{a}{\sqrt{2\pi}} \frac{\sin \omega a/2}{\omega a/2} \\ &= \frac{a}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2) \end{aligned}$$

A plot of $F(\omega)$ produces the *continuous amplitude spectrum* of $f(t)$



Notice the similarity between the plots of $F(\omega)$ and the discrete spectrum of Frame 10. The lines in the discrete spectrum have merged to form a continuous spectrum while retaining the envelope of the discrete spectrum.

Now you try one

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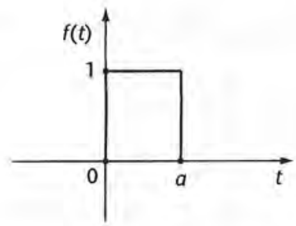
Example 4

The function of the previous example time delayed by $t = a/2$ units is

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & \text{otherwise} \end{cases}$$



And has the Fourier transform

$$F(\omega) = \ldots\ldots\ldots\ldots$$

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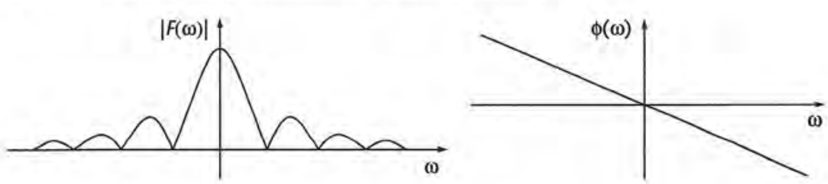
$$\boxed{F(\omega) = \frac{ae^{-j\omega a/2}}{\sqrt{2\pi}}\operatorname{sinc}(\omega a/2)}$$

Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_0^a e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^a \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-j\omega a}-1}{-j\omega}\right) \\
&= \frac{2}{\sqrt{2\pi}}e^{-j\omega a/2}\left(\frac{e^{-j\omega a/2}-e^{j\omega a/2}}{-2j\omega}\right) \\
&= \frac{2}{\sqrt{2\pi}}e^{-j\omega a/2}\left(\frac{\sin \omega a/2}{\omega}\right) \\
&= \frac{a}{\sqrt{2\pi}}e^{-j\omega a/2}\left(\frac{\sin \omega a/2}{\omega a/2}\right) \\
&= \frac{ae^{-j\omega a/2}}{\sqrt{2\pi}}\operatorname{sinc}(\omega a/2)
\end{aligned}$$

Here $F(\omega)$ is a complex function so we write $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ where $|F(\omega)| = \left(a/\sqrt{2\pi}\right)\operatorname{sinc}(\omega a/2)|$ is the *continuous amplitude spectrum* and $\phi(\omega) = -a\omega/2$ is the *continuous phase spectrum.*



Again, notice the similarity between the plots of $\phi(\omega)$ and the discrete phase spectrum of Frame 10. The lines in the discrete spectrum have merged to form a continuous spectrum while retaining the envelope of the discrete spectrum.

Next frame

Some special functions and their transforms

19

Even functions

If $f(t)$ is an even function then

$$f(-t) = f(t) \text{ and } f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

where

$$F(\omega) = \dots\dots\dots\dots \int_0^{\infty} f(t) \dots\dots\dots\dots \, dt$$

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$$\boxed{F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt}$$

Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(t) e^{-j\omega t} \, dt + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) e^{-j\omega t} \, dt \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^{-\infty} f(t) e^{-j\omega t} \, dt + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) e^{-j\omega t} \, dt \\
&\quad \text{reversing the limits on the first integral} \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(-t) e^{j\omega t} \, d(-t) + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) e^{-j\omega t} \, dt \\
&\quad \text{changing the variable of integration in the first integral} \\
&\quad \text{from } t \text{ to } -t \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) \left[e^{j\omega t} + e^{-j\omega t} \right] d(t) \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt
\end{aligned}$$

Notice that if $f(t)$ is even then $F(\omega)$ is real.

Odd functions

If $f(t)$ is an odd function then

$$f(-t) = -f(t) \text{ and } f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

where

$$F(\omega) = \dots\dots\dots\dots \int_0^{\infty} f(t) \dots\dots\dots\dots \, dt$$

$$\boxed{F(\omega) = -j\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\,\mathrm{d}t}$$

Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(t)e^{-j\omega t}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)e^{-j\omega t}\,\mathrm{d}t\\
&= -\frac{1}{\sqrt{2\pi}}\int_0^{-\infty} f(t)e^{-j\omega t}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)e^{-j\omega t}\,\mathrm{d}t
\end{aligned}$$

reversing the limits on the first integral

$$= -\frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(-t)e^{j\omega t}\,\mathrm{d}(-t) + \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)e^{-j\omega t}\,\mathrm{d}t$$

changing the variable of integration in the first integral from t to $-t$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)\left[-e^{j\omega t} + e^{-j\omega t}\right]\mathrm{d}t\\
&= \frac{-2j}{\sqrt{2\pi}}\int_0^\infty f(t)\sin\omega t\,\mathrm{d}t\\
&= -j\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\,\mathrm{d}t
\end{aligned}$$

Notice that if $f(t)$ is odd then $F(\omega)$ is imaginary. An example will show the converse of these two results.

Example

Given that $\mathscr{F}f(t) = F(\omega) = A(\omega) + jB(\omega)$ where $A(\omega)$ and $B(\omega)$ are real functions of ω, then if

(a) $A(\omega) \neq 0$ and $B(\omega) = 0$ then $f(t)$ is an .... function
(b) $A(\omega) = 0$ and $B(\omega) \neq 0$ then $f(t)$ is an .... function

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(a) $A(\omega) \neq 0$ and $B(\omega) = 0$ then $f(t)$ is an even function
(b) $A(\omega) = 0$ and $B(\omega) \neq 0$ then $f(t)$ is an odd function

Because

The Fourier transform is given as

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)[\cos\omega t - j\sin\omega t]\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t - j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t \\ &= A(\omega) + jB(\omega) \end{aligned}$$

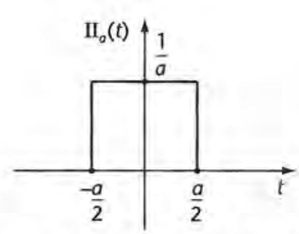
(a) If $\int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t = 0$ then $f(t)\sin\omega t$ is odd. But $\sin\omega t$ is odd, so $f(t)$ must be even.

(b) If $\int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t = 0$ then $f(t)\cos\omega t$ is odd. But $\cos\omega t$ is even, so $f(t)$ must be odd.

Top-hat function

This function is a special form of the function met in Example 3 in Frame 16, and is defined by

$$f(t) = \begin{cases} 0 & t < -a/2 \\ 1/a & -a/2 < t < a/2 \\ 0 & a/2 < t \end{cases}$$



It is, because of its shape, referred to as the *top-hat* function and is denoted by the symbol $\Pi_a(t)$. It is a special form of the function in Example 3 because it has a unit area – width $\times$ height $= a \times (1/a) = 1$, or

$$\int_{-\infty}^{\infty} \Pi_a(t)\,\mathrm{d}t = \int_{-a/2}^{a/2} (1/a)\,\mathrm{d}t = \left[\frac{t}{a}\right]_{-a/2}^{a/2} = 1$$

The Fourier transform of the top-hat function is

$$F(\omega) = \ldots\ldots\ldots\ldots$$

$$\boxed{F(\omega) = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)}$$

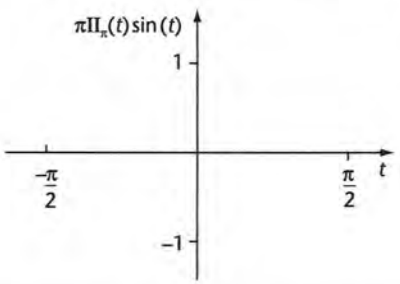
Because

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{II}_a(t) e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} (1/a) e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{a\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)
\end{aligned}$$

This function is useful in that it can be used to select any segment of any function. For example

$\pi \mathrm{II}_\pi(t) \sin t$

selects the segment of $\sin t$ between $\pm\pi/2$ and reduces the rest to zero.



So $\pi \mathrm{II}_\pi(t - \pi) \cos t$ selects the segment of $\cos t$ between .... and ....

24

$$\boxed{\pi/2 \text{ and } 3\pi/2}$$

Because

$$\Pi_\pi(t-\pi) = \begin{cases} 0 & t-\pi < -\pi/2 \\ 1/\pi & -\pi/2 < t-\pi < \pi/2 \\ 0 & \pi/2 < t-\pi \end{cases}$$

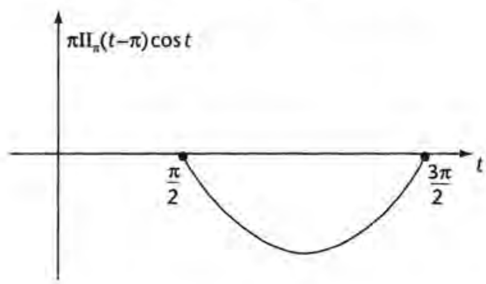
that is

$$\Pi_\pi(t-\pi) = \begin{cases} 0 & t < \pi/2 \\ 1/\pi & \pi/2 < t < 3\pi/2 \\ 0 & \pi/2 < t \end{cases}$$

and so

$$\pi\Pi_\pi(t-\pi)\cos t = \begin{cases} \cos t & \pi/2 < t < 3\pi/2 \\ 0 & \text{otherwise} \end{cases}$$

selects the segment of $\cos t$ between $\pi/2$ and $3\pi/2$.



The Dirac delta (refer to Programme 4, Frames 29*ff*)

In science and technology we often require to use the notion of a force that acts for a very brief interval of time. To simulate this mathematically we can use the unit-area pulse – the top-hat function. If we take the duration of this pulse to decrease while at the same time retaining a unit-area then in the limit we are led to the notion of the Dirac delta. That is

$$\underset{a\to 0}{Lim} \int_{-\infty}^{\infty} \{\Pi_a(t)\}\,dt = \underset{a\to 0}{Lim}\, 1 = 1$$

▶

Here as $a \to 0$ the width of the top-hat decreases as the height increases but all the while retaining the area beneath the top-hat as unity. It is this limit that we can use to justify the integral definition of the Dirac delta because

$$\underset{a\to 0}{Lim}\int_{-\infty}^{\infty}\{\Pi_a(t)\}\,dt = \int_{-\infty}^{\infty}\underset{a\to 0}{Lim}\{\Pi_a(t)\}\,dt = \int_{-\infty}^{\infty}\delta(t)\,dt = 1$$

and it is also in this sense that we accept the validity of the integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

because, like the top-hat function, it selects only that part of $f(t)$ over which it is non-zero, namely at $t = t_0$.

So if $f(t) = \delta(t)$ then $F(\omega) = \ldots\ldots\ldots\ldots$

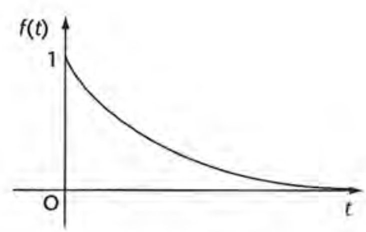
25

$$\boxed{\frac{1}{\sqrt{2\pi}}}$$

Because

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}\,dt \\ &= \frac{e^{-j\omega 0}}{\sqrt{2\pi}} \qquad \text{because } \delta(t) = \delta(t-0) \\ &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

Try another.



The truncated exponential function

$$f(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases}$$

where $a > 0$ can be also expressed in the form $f(t) = e^{-at}u(t)$ and has the Fourier transform

$F(\omega) = \ldots\ldots\ldots\ldots$

26

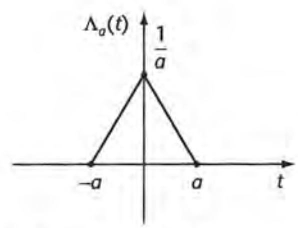
$$\boxed{F(\omega)=\frac{1}{\sqrt{2\pi}(a+j\omega)}}$$

Because

$$\begin{aligned}F(\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-at}u(t)e^{-j\omega t}\,dt\\&=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-(a+j\omega)t}\,dt\\&=\frac{1}{\sqrt{2\pi}(a+j\omega)}\end{aligned}$$

The triangle function

$$\Lambda_a(t)=\begin{cases}(a+t)/a^2 & -a<t<0\\(a-t)/a^2 & 0<t<a\\0 & |t|>a\end{cases}$$ Notice that this also has unit area



The Fourier transform of $\Lambda_1(t)$ is $F(\omega)=$ ....

27

$$\boxed{F(\omega)=\frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega/2)}$$

Because

$$\begin{aligned}F(\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Lambda_1(t)e^{-j\omega t}\,dt\\&=\frac{1}{\sqrt{2\pi}}\int_{-1}^{0}(1+t)e^{-j\omega t}\,dt+\frac{1}{\sqrt{2\pi}}\int_{0}^{1}(1-t)e^{-j\omega t}\,dt\\&=-\frac{1}{\sqrt{2\pi}}\int_{1}^{0}(1-t)e^{j\omega t}\,dt+\frac{1}{\sqrt{2\pi}}\int_{0}^{1}(1-t)e^{-j\omega t}\,dt\end{aligned}$$

changing the variable of integration in the first integral from t to $-t$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{1}(1-t)\cos\omega t\,dt$$ and integration by parts yields

$$\begin{aligned}&=\frac{2}{\sqrt{2\pi}}\left(\frac{1}{2}\frac{\sin^2(\omega/2)}{(\omega/2)^2}\right)\\&=\frac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega/2)\end{aligned}$$

Alternative forms

28

It should be noted that there are a number of alternative forms for the Fourier transform – each dealing with a different location for the constant 2π. Other forms are

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

or

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

or, by absorbing the 2π in the exponential by defining $\omega = 2\pi\nu$

$$f(t) = \int_{-\infty}^{\infty} F(\nu)e^{j2\pi\nu t}\,d\nu \text{ where } F(\nu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\nu t}\,dt$$

We shall remain with our original form because it has the simplest exponential factor and we do not need to remember which integral has the constant in front of it and which does not.

Next frame

Properties of the Fourier transform

29

We now list a number of properties of the Fourier transform that are useful in their manipulation.

Linearity

If the Fourier transforms $\mathscr{F}(f_1(t)) = F_1(\omega)$ and $\mathscr{F}(f_2(t)) = F_2(\omega)$ then

$$\mathscr{F}(\alpha_1 f_1(t) + \alpha_2 f_2(t)) = \alpha_1\mathscr{F}(f_1(t)) + \alpha_2\mathscr{F}(f_2(t)) = \alpha_1 F_1(\omega) + \alpha_2 F_2(\omega)$$

where α_1 and α_2 are constants.

Example

The Fourier transform of $f(t) = 2\Pi_2(t) - 6\Lambda_2(t)$ is

$$F(\omega) = \ldots\ldots\ldots\ldots$$

30

$$\boxed{\sqrt{\frac{2}{\pi}}\operatorname{sinc}(\omega)(1 - 3\operatorname{sinc}(\omega))}$$

Because

If $f(t) = \text{II}_2(t)$ then $F(\omega) = \dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}(\omega)$ and if $f(t) = \Lambda_2(t)$ then $F(\omega) = \dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega)$. Since $f(t) = 2\text{II}_2(t) - 6\Lambda_2(t)$ then

$$\begin{aligned} F(\omega) &= \frac{2}{\sqrt{2\pi}}\operatorname{sinc}(\omega) - \frac{6}{\sqrt{2\pi}}\operatorname{sinc}^2(\omega) \\ &= \sqrt{\frac{2}{\pi}}\operatorname{sinc}(\omega)(1 - 3\operatorname{sinc}(\omega)) \end{aligned}$$

Time shifting

If $\mathscr{F}(f(t)) = F(\omega)$ then $\mathscr{F}(f(t - t_0)) = e^{j\omega t_0}F(\omega)$

Example

The Fourier transform of $\text{II}_2(t)$ is $\dfrac{1}{\sqrt{2\pi}}\operatorname{sinc}(\omega)$ so, by the time shifting property, the Fourier transform of

$\text{II}_2(t-5)$ is ………… and of $\text{II}_2(t+3)$ is …………

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$$\boxed{\frac{e^{j5\omega}}{\sqrt{2\pi}}\operatorname{sinc}(\omega) \quad \text{and} \quad \frac{e^{-j3\omega}}{\sqrt{2\pi}}\operatorname{sinc}(\omega)}$$

Frequency shifting

If $\mathscr{F}(f(t)) = F(\omega)$ then $\mathscr{F}\left(f(t)e^{j\omega_0 t}\right) = F(\omega - \omega_0)$

Example

If the Fourier transform of $f(t)$ is $F(\omega)$ then the transform of $f(t)\cos 4t$ is

…………

32

$$\boxed{\frac{1}{2}(F(\omega+4)+F(\omega-4))}$$

Because

$$\begin{aligned} f(t)\cos 4t &= f(t)\frac{e^{j4t}+e^{-j4t}}{2} \\ &= \frac{1}{2}f(t)e^{j4t}+\frac{1}{2}f(t)e^{-j4t} \\ &= \frac{1}{2}\left(f(t)e^{j4t}+f(t)e^{-j4t}\right) \end{aligned}$$

and so the Fourier transform is $\frac{1}{2}(F(\omega-4)+F(\omega+4))$ by the linearity and the frequency shifting properties.

Time scaling

If $\mathscr{F}(f(t))=F(\omega)$ then

$$\mathscr{F}(f(kt))=\frac{1}{|k|}F\left(\frac{\omega}{k}\right)$$

So, for example, given $f(t)=\text{II}_a(t)$ with Fourier transform $F(\omega)$, if $f(t)$ is shrunk to half its width then $F(\omega)$ is stretched to twice its width but shrunk to half its height.

Example

If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $f(-t)$ is ....

33

$$\boxed{F(-\omega)}$$

Because

$$|k|^{-1}F(\omega/k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(kt)e^{-j\omega t}\,\mathrm{d}t \text{ and when } k=-1 \text{ then}$$

$$|-1|^{-1}F(\omega/[-1])=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(-t)e^{-j\omega t}\,\mathrm{d}t=F(-\omega)$$

Symmetry

If $\mathscr{F}(f(t))=F(\omega)$ then $\mathscr{F}(F(t))=f(-\omega)$

Example

The Fourier transform of $f(t)=\text{II}_2(t)$ is $F(\omega)=\frac{1}{\sqrt{2\pi}}\text{sinc}\,(\omega)$, so the Fourier transform of

$$F(t)=\frac{1}{\sqrt{2\pi}}\text{sinc}\,(t) \text{ is } \ldots\ldots\ldots\ldots$$

34

$$\boxed{f(-\omega) = -\Pi_2(\omega)}$$

Because

The Fourier transform of $F(t) = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}(t)$

is $f(-\omega) = \Pi_2(-\omega) = -\Pi_2(\omega)$

Try one yourself.

Example

The Fourier transform of the unit constant function $f(t) = 1$ is

$$\mathcal{F}[1] = \ldots\ldots\ldots\ldots$$

35

$$\boxed{\sqrt{2\pi}\delta(\omega)}$$

Because

$$\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}} \text{ so } \mathcal{F}\left[\frac{1}{\sqrt{2\pi}}\right] = \delta(\omega), \text{ therefore } \mathcal{F}[1] = \sqrt{2\pi}\delta(\omega)$$

Differentiation

If $f(t) \to 0$ as $t \to \pm\infty$ and if $\mathcal{F}(f(t)) = F(\omega)$ then

$$\mathcal{F}(f'(t)) = \ldots\ldots\ldots\ldots$$

36

$$\boxed{j\omega F(\omega)}$$

Because

$$\begin{aligned}\mathcal{F}[f'(t)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\left[f(t)e^{-j\omega t}\right]_{-\infty}^{\infty} + \frac{j\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t \\ &= 0 + j\omega F(\omega)\end{aligned}$$

In general, if $f(t) \to 0$ as $t \to \pm\infty$ and if $\mathcal{F}(f(t)) = F(\omega)$ then

If $\mathcal{F}(f(t)) = F(\omega)$ then $\mathcal{F}\left(f^{(n)}(t)\right) = (j\omega)^n F(\omega)$

where the superscript (n) indicates the nth derivative.

Example

The differential equation for unforced and undamped harmonic motion is of the form $mf''(t) + kf(t) = 0$. If we take the Fourier transform of this equation we immediately find that the permitted frequencies of oscillation are

$$\omega = \ldots\ldots\ldots\ldots$$

37

$$\boxed{\omega = \pm\sqrt{\frac{k}{m}}}$$

Because

If $F(\omega)$ is the Fourier transform of $f(t)$ then taking the Fourier transform of both sides of the equation $mf''(t) + kf(t) = 0$ gives by the differentiation property

$$m(j\omega)^2 F(\omega) + kF(\omega) = (-m\omega^2 + k)F(\omega) = 0$$

so if $F(\omega) \neq 0$ then $m\omega^2 = k$ and so the permitted frequencies are

$$\omega = \pm\sqrt{\frac{k}{m}}.$$

Next frame

The Heaviside unit step function

38

The Heaviside unit step function is defined as $u(t)$ where

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

If we follow the definition of the Fourier transform we find that

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) e^{-j\omega t}\, dt$$

So that $F(\omega) = \dots\dots\dots\dots$

39

$$\boxed{F(\omega) = \frac{1}{\sqrt{2\pi} j\omega} - \left\{1 - \underset{t\to\infty}{Lim}\left[e^{-j\omega t}\right]\right\}}$$

Because

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) e^{-j\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-j\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi} j\omega} - \left\{1 - \underset{t\to\infty}{Lim}\left[e^{-j\omega t}\right]\right\} \end{aligned}$$

Because $e^{-j\omega t} = \cos\omega t - j\sin\omega t$ we cannot say what happens to the exponential at $t \to \infty$. So how do we resolve the problem?

Next frame

40

Let $\mathscr{F}u(t) = F(\omega)$ and so, by the scaling property, $\mathscr{F}u(-t) = F(-\omega)$. Now, $u(t) + u(-t) = 1$, therefore $\mathscr{F}[u(t)] + \mathscr{F}u[(-t)] = \mathscr{F}[1]$. That is, from Frame 35

$$F(\omega) + F(-\omega) = \sqrt{2\pi}\delta(\omega)$$

We now assume that $F(\omega)$ consists of a combination of the Dirac delta and an arbitrary function $G(\omega)$

$$F(\omega) = \alpha\delta(\omega) + G(\omega) \text{ so that}$$

$$\begin{aligned} F(\omega) + F(-\omega) &= \alpha\delta(\omega) + G(\omega) + \alpha\delta(-\omega) + G(-\omega) \\ &= 2\alpha\delta(\omega) + G(\omega) + G(-\omega) \qquad \text{since } \delta(-\omega) = \delta(\omega) \\ &= \sqrt{2\pi}\delta(\omega) \end{aligned}$$

Therefore $\alpha = \sqrt{\dfrac{\pi}{2}}$ and $G(\omega) + G(-\omega) = 0$. That is, $G(\omega) = -G(-\omega)$.

Consequently $\mathscr{F}[u(t)] = F(\omega) = \sqrt{\dfrac{\pi}{2}}\delta(\omega) + G(\omega)$.

Now, $\mathscr{F}[u'(t)] = j\omega F(\omega) = j\omega\left\{\sqrt{\dfrac{\pi}{2}}\delta(\omega) + G(\omega)\right\}$ and since $u'(t) = \delta(t)$

then $\mathscr{F}[u'(t)] = \mathscr{F}[\delta(t)] = \dfrac{1}{\sqrt{2\pi}}$ giving $j\omega\left\{\sqrt{\dfrac{\pi}{2}}\delta(\omega) + G(\omega)\right\} = \dfrac{1}{\sqrt{2\pi}}$

Since $\omega\delta(\omega) = 0$, then $j\omega G(\omega) = \dfrac{1}{\sqrt{2\pi}}$ and so $G(\omega) = \dfrac{1}{j\omega\sqrt{2\pi}}$ thereby giving

$$\mathscr{F}[u(t)] = \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\}$$

The next property deals with the Fourier transform of a **product of functions** but before we go any further we need to discuss what is meant by the **convolution of two functions**.

Next frame

Convolution

41

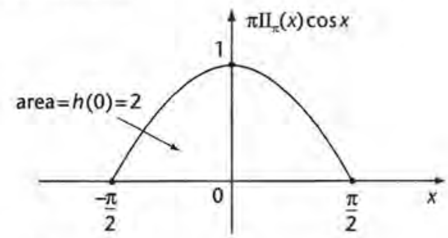
The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx = h(t)$$

where the $*$ denotes the operation of convolution. You will note that this is a function of variable t and here we have denoted it by $h(t)$. To illustrate an interpretation of this operation, consider the two functions $f(t) = \pi\Pi_\pi(t)$ and $g(t) = \cos t$. Then

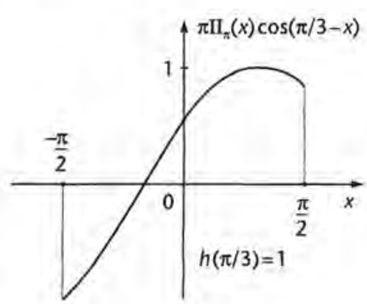
$$f(x)g(-x) = \pi\Pi_\pi(x)\cos(-x) = \pi\Pi_\pi(x)\cos x$$

is that part of the cosine function that lies between $\pm\pi/2$ and is zero elsewhere.



The integral $\int_{-\infty}^{\infty} f(x)g(-x)\,dx = \int_{-\infty}^{\infty} \pi\Pi_\pi(x)\cos x\,dx = 2$. This is the area beneath the single loop of the cosine curve. We shall call this value $h(0)$ because the loop is centred on the origin. That is $h(0) = 2$. Now, the graph of $\cos(\pi/3 - x)$ has the same shape as $\cos(-x) = \cos x$ but it is shifted to the right by $\pi/3$ radians. Consequently, $f(x)g(\pi/3 - x) = \pi\Pi_\pi(x)\cos(\pi/3 - x)$ is that part of the *shifted* cosine function that lies between $\pm\pi/2$ and zero elsewhere, so now $\int_{-\infty}^{\infty} f(x)g(\pi/3 - x)\,dx = 1$. We shall call this value $h(\pi/3)$ because $\pi/3$ measures the amount of the shift of the cosine curve. That is $h(\pi/3) = 1$. Proceeding in this manner to define values of $h(t)$ we see that the function formed from these integrals is

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx = h(t)$$



▶

Example

To find the convolution $f(t) * g(t)$ where

$$f(t) = u(t) \text{ and } g(t) = \begin{cases} \sec^2 t & |t| < \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

where $u(t)$ is the Heaviside function

then

$$h(t) = f(t) * g(t) = \dots\dots\dots\dots$$

42

$$\boxed{\frac{1 + \tan^2 t}{1 + \tan t}}$$

Because

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x &= \int_{-\infty}^{\infty} u(x)g(t-x)\,\mathrm{d}x \\ &= \int_{0}^{\pi/4} \sec^2(t-x)\,\mathrm{d}x \quad \text{because } u(t) = 0 \text{ for } t < 0 \text{ and } g(t) = 0 \text{ for } t > \pi/4 \\ &= \Big[-\tan(t-x)\Big]_0^{\pi/4} \\ &= \{-\tan(t-\pi/4) + \tan t\} \\ &= -\frac{\tan t - 1}{1 + \tan t} + \tan t \\ &= \frac{1 + \tan^2 t}{1 + \tan t} \end{aligned}$$

Next frame

The convolution theorem

43

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively then

(a) The Fourier transform of the convolution of $f(t)$ and $g(t)$ is equal to the product of the individual Fourier transforms. That is

$$\mathcal{F}[f(t) * g(t)] = \sqrt{2\pi}F(\omega)G(\omega) \text{ and so}$$

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = \frac{1}{\sqrt{2\pi}}[f(t) * g(t)]$$

(b) The Fourier transform of the product $f(t)g(t)$ is equal to the convolution of the individual Fourier transforms. That is

$$\mathcal{F}[f(t)g(t)] = \frac{1}{\sqrt{2\pi}}F(\omega) * G(\omega) \text{ and so}$$

$$\mathcal{F}^{-1}[F(\omega) * G(\omega)] = \sqrt{2\pi}f(t)g(t)$$

These provide useful methods of finding inverse transforms. ▶

Example

To find the inverse transform of

$$F(\omega) = \frac{1}{2\pi(a + j\omega)^2} = \frac{1}{\sqrt{2\pi}(a + j\omega)} \times \frac{1}{\sqrt{2\pi}(a + j\omega)} \text{ where } a > 0$$

we note that if $F_1(\omega) = \dfrac{1}{\sqrt{2\pi}(a + j\omega)}$ then from Frame 26

$$f_1(t) = \mathscr{F}^{-1}[F_1(\omega)] = \ldots\ldots\ldots\ldots$$

44

$$\boxed{f_1(t) = e^{-at}u(t)}$$

Now, because

$$F(\omega) = F_1(\omega)F_1(\omega)$$

then

$$\begin{aligned} f(t) &= \mathscr{F}^{-1}[F(\omega)] = \mathscr{F}^{-1}[F_1(\omega)F_1(\omega)] = \frac{1}{\sqrt{2\pi}}[f_1(t) * f_1(t)] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x)f_1(t - x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax}u(x)e^{-a(t-x)}u(t - x)\,\mathrm{d}x \\ &= \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax}u(x)e^{ax}u(t - x)\,\mathrm{d}x \\ &= \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x)u(t - x)\,\mathrm{d}x \end{aligned}$$

Now, $u(x)u(t - x) = 0$ when $x < 0$ or when $t - x < 0$, that is when $x > t$.

Therefore, $u(x)u(t - x) = \begin{cases} 1 & \text{if } 0 < x < t \\ 0 & \text{otherwise} \end{cases}$ so

$$\begin{aligned} f(t) &= \frac{e^{-at}}{\sqrt{2\pi}} \int_0^t \mathrm{d}x \\ &= \begin{cases} \dfrac{te^{-at}}{\sqrt{2\pi}} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \text{that is, } f(t) = \frac{te^{-at}}{\sqrt{2\pi}}u(t) \end{aligned}$$

Now you try one.

The inverse Fourier transform of $F(\omega) = \dfrac{5}{6 + 5j\omega - \omega^2}$ is

$$f(t) = \ldots\ldots\ldots\ldots$$

45

$$\boxed{f(t) = \sqrt{50\pi}\left[e^{-2t} - e^{-3t}\right]u(t)}$$

Because

$$F(\omega) = \frac{5}{6 + 5j\omega - \omega^2}$$

$$= \frac{5}{(2 + j\omega)(3 + j\omega)}$$

Let $F_1(\omega) = \dfrac{1}{\sqrt{2\pi}(2 + j\omega)}$ so that $f_1(t) = e^{-2t}u(t)$ and

$F_2(\omega) = \dfrac{1}{\sqrt{2\pi}(3 + j\omega)}$ so that $f_2(t) = e^{-3t}u(t)$ so that

$$F(\omega) = 10\pi[F_1(\omega)F_2(\omega)]$$

By the convolution theorem

$$f(t) = \frac{10\pi}{\sqrt{2\pi}}[f_1(t) * f_2(t)]$$

$$= \sqrt{50\pi}\int_{-\infty}^{\infty} f_1(x)f_2(t - x)\,\mathrm{d}x$$

$$= \sqrt{50\pi}\int_{-\infty}^{\infty} e^{-2x}u(x)e^{-3(t-x)}u(t - x)\,\mathrm{d}x$$

$$= \sqrt{50\pi}e^{-3t}\int_{-\infty}^{\infty} e^{x}u(x)u(t - x)\,\mathrm{d}x$$

$$= \sqrt{50\pi}e^{-3t}\int_{0}^{t} e^{x}\,\mathrm{d}x \text{ since } u(x)u(t - x) = \begin{cases} 1 & \text{if } 0 < x < t \\ 0 & \text{otherwise} \end{cases}$$

$$= \sqrt{50\pi}e^{-3t}\left[e^{t} - 1\right]u(t) \text{ since } \int_{0}^{t} e^{x}\,\mathrm{d}x = \begin{cases} e^{t} - 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$= \sqrt{50\pi}\left[e^{-2t} - e^{-3t}\right]u(t)$$

Move to the next frame

Fourier cosine and sine transforms

46

Given that

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,\mathrm{d}\omega \text{ where}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,\mathrm{d}t$$

if $f(t)$ is an even function so that $f(-t) = f(t)$ then

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,\mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t \quad \text{since } f(t)\sin\omega t \text{ is odd}$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos\omega t\,\mathrm{d}t$$

This is referred to as the Fourier cosine transformation and is denoted by $F_c(\omega)$. That is

$$F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos\omega t\,\mathrm{d}t$$

Similarly if $f(t)$ is an odd function so that $f(-t) = -f(t)$ then

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,\mathrm{d}t$$

$$= \frac{j}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t \quad \text{since } f(t)\cos\omega t \text{ is odd}$$

$$= j\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t\,\mathrm{d}t$$

This gives rise to the Fourier sine transformation, denoted by $F_s(\omega)$ where

$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin\omega t\,\mathrm{d}t$$

Example 1

The Fourier cosine transformation of $f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ 0 & \text{if } t \geq a \end{cases}$ is

$$F_c(\omega) = \ldots\ldots\ldots\ldots$$

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$$\boxed{F_c(\omega) = \sqrt{\frac{2}{\pi}}a\ \mathrm{sinc}\,(\omega a)}$$

Because

$$\begin{aligned}
F_c(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\int_0^a \cos\omega t\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{\sin\omega t}{\omega}\right]_0^a \\
&= \sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega a}{\omega}\right) = \sqrt{\frac{2}{\pi}}a\mathrm{sinc}\,(\omega a)
\end{aligned}$$

Example 2

The Fourier sine transformation of $f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ 0 & \text{if } t \geq a \end{cases}$ is

$$F_s(\omega) = \dots\dots\dots\dots$$

48

$$\boxed{F_s(\omega) = \sqrt{\frac{2}{\pi}}2a^2\omega\ \mathrm{sinc}^2(\omega a)}$$

Because

$$\begin{aligned}
F_s(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\int_0^a \sin\omega t\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos\omega t}{\omega}\right]_0^a \\
&= \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos\omega a}{\omega}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{2\sin^2\omega a}{\omega}\right) = \sqrt{\frac{2}{\pi}}2a^2\omega\ \mathrm{sinc}^2(\omega a)
\end{aligned}$$

The Fourier cosine and sine transforms are useful when $f(t)$ is only defined for $t \geq 0$ and where an extension can be added to $f(t)$ for $t < 0$ that makes the extended $f(t)$ into an even or odd function respectively.

Table of transforms

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$f(t) = \begin{cases} 1 & \text{if } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$	$F(\omega) = \dfrac{a}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)$
$f(t) = \begin{cases} 1 & \text{if } 0 < t < a \\ 0 & \text{otherwise} \end{cases}$	$F(\omega) = \dfrac{ae^{-j\omega a/2}}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)$
$\Pi_a(t) = \begin{cases} 1/a & \text{if } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \operatorname{sinc}(\omega a/2)$
$f(t) = u(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \left\{ \pi\delta(\omega) + \dfrac{1}{j\omega} \right\}$
$f(t) = e^{-at}u(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \left\{ \pi\delta(\omega + a) + \dfrac{1}{j\omega} \right\}$
$f(t) = te^{-at}u(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}(a + j\omega)^2}$
$f(t) = \delta(t)$	$F(\omega) = \dfrac{1}{\sqrt{2\pi}}$

The main points of the Programme are listed in the **Revision summary** that follows. Read it in conjunction with the **Can You?** checklist and refer back to the relevant parts of the Programme, if necessary. You will then have no trouble with the **Test exercise** and the **Further problems** provide valuable additional practice.

Revision summary 7 50

1 *Complex Fourier series*

The Fourier series of the piecewise continuous function $f(t)$ with piecewise continuous derivative and where $f(t + T) = f(t)$ is given as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $c_n = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$.

2 *Discrete complex spectra*

The c_n are complex numbers and can be written as

$$c_n = |c_n|e^{j\phi_n}$$

These complex coefficients constitute a discrete complex spectrum where c_n represents the *spectral coefficient* of the nth harmonic. Each spectral coefficient couples an amplitude spectrum value $|c_n|$ and a phase spectrum value ϕ_n.

3 *Fourier's integral theorem*

If (a) $f(t)$ and $f'(t)$ are piecewise continuous in every finite interval

(b) $f(t)$ is absolutely integrable in $(\infty, -\infty)$, that is $\int_{-\infty}^{\infty} |f(t)|\,dt$ is finite then

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt.$$

4 *Continuous complex spectra*

The Fourier transform $F(\omega)$ is a complex function so we write $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ where $|F(\omega)| = \left|\left(a/\sqrt{2\pi}\right)\text{sinc}\,(\omega a/2)\right|$ is the *continuous amplitude spectrum* and $\phi(\omega) = -a\omega/2$ is the *continuous phase spectrum*.

5 *Transforms of special functions*

Top-hat function

$$\Pi_a(t) = \begin{cases} 1/a & -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$$

with Fourier transform $F(\omega) = \dfrac{1}{\sqrt{2\pi}}\,\text{sinc}\,(\omega a/2)$.

The Dirac delta

If $f(t) = \delta(t)$ then $F(\omega) = \dfrac{1}{\sqrt{2\pi}}$.

The Heaviside unit step function

$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$ has the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\pi\delta(\omega + a) + \frac{1}{j\omega}\right\}.$$

The triangle function

$\Lambda(t) = \begin{cases} 0 & |t| > 1 \\ 1 & |t| < 1 \end{cases}$ has the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\,\text{sinc}^2(\omega/2).$$

▶

6 *Alternative forms*

There are a number of alternative forms for the Fourier transform – each dealing with a different location for the constant 2π. Other forms are

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \text{ or}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \text{ or}$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j2\pi\omega t}\,d\omega \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\omega t}\,dt.$$

7 *Properties of the Fourier transform*

Time shifting

$$\text{If } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt \text{ then}$$

$$e^{j\omega t_0}F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-t_0)e^{j\omega t}\,dt.$$

Linearity

If $F_1(\omega)$ and $F_2(\omega)$ are the Fourier transforms of $f_1(t)$ and $f_2(t)$ respectively then $\alpha_1 F_1(\omega) + \alpha_2 F_2(\omega)$ is the Fourier transform of $\alpha_1 f_1(t) + \alpha_2 f_2(t)$ where α_1 and α_2 are constants.

Frequency shifting

If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $f(t)e^{-j\omega_0 t}$ is $F(\omega - \omega_0)$.

Time scaling

$$\text{If } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt \text{ then}$$

$$|k|^{-1}F(\omega/k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(kt)e^{j\omega t}\,dt.$$

Symmetry

If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $F(t)$ is $f(-\omega)$.

Differentiation

$$\text{If } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \text{ then}$$

$$(j\omega)^n F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^{(n)}(t)e^{-j\omega t}\,dt \text{ and}$$

$$F^{(n)}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-j\omega)^n f(t)e^{-j\omega t}\,dt.$$

▶

8 *Convolution*

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx = h(t).$$

The convolution theorem

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively then

(a) The Fourier transform of the convolution of $f(t)$ and $g(t)$ is equal to the product of the individual Fourier transforms. That is

$$\mathcal{F}[f(t) * g(t)] = F(\omega)G(\omega).$$

(b) The Fourier transform of the product $f(t)g(t)$ is equal to the convolution of the individual Fourier transforms. That is

$$\mathcal{F}[f(t)g(t)] = F(\omega) * G(\omega).$$

9 *Fourier cosine and sine transforms*

Given that $f(t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,dt$ where

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,d\omega$$

where $f(t)$ is even then

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(\omega)\cos\omega t\,d\omega \text{ where } F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\omega t\,dt$$

and where $F_c(\omega)$ is called the *Fourier cosine transformation*. This transformation is useful when $f(t)$ is defined only for $t \geq 0$ and where an extension can be added to $f(t)$ for $t < 0$ that makes the extended $f(t)$ into an even function.

If $f(t)$ is odd then

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(\omega)\sin\omega\,d\omega \text{ where } F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\omega t\,dt$$

and where $F_s(\omega)$ is called the *Fourier sine transformation*. This transformation is useful when $f(t)$ is defined only for $t \geq 0$ and where an extension can be added to $f(t)$ for $t < 0$ that makes the extended $f(t)$ into an odd function.

☑ Can You?

Checklist 7 51

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Convert a trigonometric Fourier series into a doubly infinite sum of complex exponentials? 1 to 4

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Derive the complex Fourier series of a function that satisfies Dirichlet's conditions? 5 to 8

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognise the function $\mathrm{sinc}\,(t)$? 9

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Separate a discrete complex spectrum into an amplitude spectrum and a phase spectrum? 10 to 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- State Fourier's integral theorem in terms of complex exponentials? 13 to 15

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Define and derive the Fourier transform of a function satisfying Dirichlet's conditions? 16 and 17

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Separate a continuous complex spectrum into an amplitude spectrum and a phase spectrum? 18

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognise the functions $\Pi_a(t)$ and $\Lambda_a(t)$ and derive their Fourier transforms along with those of the Dirac delta and the Heaviside unit step? 19 to 27

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Recognise alternative forms of the function–transform pair? 28

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Reproduce a collection of properties of the Fourier transform? 29 to 40

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate the convolution of two functions and describe its Fourier transform? 41 to 45

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Derive the Fourier sine and cosine transformations? 46 to 48

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 7

52

1. Find the complex Fourier series of the sawtooth wave
$f(t) = t, \quad 0 < t < 1 \quad$ and where $\quad f(t+1) = f(t)$.

2. Find the Fourier transform of
$$f(t) = \begin{cases} e^{-at} & |t| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad a > 0$$

3. Given that the Dirac delta $\delta(t)$ has the Fourier transform $F(\omega) = \dfrac{1}{\sqrt{2\pi}}$, show, by considering the inverse Fourier transform, that
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\,\mathrm{d}\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega t\,\mathrm{d}\omega.$$

4. If $f(t)$ and $F(\omega)$ form a Fourier transform pair, find the Fourier transform of $f(t)\sin\omega_0 t$ where ω_0 is a constant.

5. Find the inverse transform of $F(\omega) = \dfrac{6}{\omega^2 + 5j\omega - 4}$.

6. Find the Fourier sine and cosine transformations of $f(t) = e^{-kt}$ for $t > 0$ and $k > 0$.

Further problems 7

53

1. By comparing the trigonometric Fourier series of a periodic function with its complex exponential counterpart show that
$$|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \text{ and } \phi_n = \arctan\left\{-\frac{b_n}{a_n}\right\} \text{ where } c_n = |c_n|e^{j\phi_n}.$$

2. Prove Parseval's identity for the periodic function with period T
$$\frac{1}{T}\int_{-T/2}^{T/2} \{f(t)\}^2\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=-\infty}^{\infty} (a_n^2 + b_n^2)$$
and show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

3. Draw the graph and find the complex Fourier series of the rectified sine wave
$f(t) = \sin\pi t, \quad 0 < t < 1 \quad$ where $f(t+1) = f(t)$.

4. Draw the graph and find the complex Fourier series of the rectified cosine wave
$f(t) = \cos\pi t, \quad -1/2 < t < 1/2 \quad$ where $f(t+1) = f(t)$.

5 Draw the graph and find the complex Fourier series of

$$f(t) = e^{\pi t}, \quad 0 < t < 2 \quad \text{where } f(t+2) = f(t).$$

6 Draw the graph and find the complex Fourier series of the sawtooth wave

$$f(t) = -\frac{t}{T} + \frac{1}{2}, \quad 0 < t < T \quad \text{where } f(t+T) = f(t).$$

7 If $f_1(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$ and $f_2(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega_0 t}$ where $\omega_0 = 2\pi/T$, show that the convolution

$$f_1(t) * f_2(t) = \sum_{n=-\infty}^{\infty} c_n d_n e^{jn\omega_0 t}.$$

8 Find the Fourier transform of

$$f(t) = \begin{cases} \cosh t & \text{for } |t| < 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

9 Find the Fourier transform of

$$f(t) = \begin{cases} \sinh t & \text{for } |t| < 1 \\ 0 & \text{for } |t| > 1. \end{cases}$$

10 Find the Fourier transform of

$$f(t) = \begin{cases} \sin \pi t & \text{for } 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

11 Find the Fourier transform of

$$f(t) = \begin{cases} \cos \pi t & \text{for } |t| < 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

12 Draw the graph and find the Fourier transform of

$$f(t) = e^{-a|t|}, \quad a > 0.$$

13 Given that

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 0 \\ -1 & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Draw the graph of $f(t)$

(b) Express $f(t)$ in terms of the Heaviside unit step function

(c) Find the Fourier transform of $f(t)$.

14 Draw the graph and find the Fourier transform of

$$f(t) = (u(t) - u(t - \pi)) \cos kt.$$

15 Show that if $f(t)$ is real then the corresponding Fourier transform $F(\omega) = |F(\omega)|e^{j\phi(\omega)}$ is such that $|F(\omega)|$ is even and $\phi(\omega)$ is odd.

16 Show that if the Fourier transform of a real function is real then $f(t)$ is even, and if the Fourier transform of a real function is imaginary then $f(t)$ is odd.

17 Defining the squared modulus of the Fourier transform $|F(\omega)|^2 = F(\omega)F^*(\omega)$ where $F^*(\omega)$ is the complex conjugate of $F(\omega)$, prove Parseval's theorem

$$\int_{-\infty}^{\infty} [f(t)]^2 \, dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \, dt.$$

18 Show that the convolution of a top-hat function with itself is the triangle function. That is

$$\text{II}_a(t) * \text{II}_a(t) = \Lambda_a(t).$$

19 Show that $\operatorname{sinc}(t) * \operatorname{sinc}(t) = \operatorname{sinc}(t)$.

20 Find the Fourier sine and cosine transforms of

$$f(t) = \begin{cases} e^{at} & \text{for } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

21 Find the Fourier sine and cosine transforms of

$$f(t) = \begin{cases} \cosh t & \text{for } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Power series solutions of ordinary differential equations

Frames 1 to 128

Learning outcomes

When you have completed this Programme you will be able to:

- Obtain the nth derivative of the exponential and circular and hyperbolic functions
- Apply the Leibnitz theorem to derive the nth derivative of a product of expressions
- Use the Leibnitz–Maclaurin method of obtaining a series solution to a second-order homogeneous differential equation with constant coefficients
- Use Frobenius' method of obtaining a series solution to a second-order homogeneous differential equation for different cases of the indicial equation
- Apply Frobenius' method to Bessel's equation to derive Bessel functions of the first kind
- Apply Frobenius' method to Legendre's equation to derive Legendre polynomials
- Use Rodrigue's formula to derive Legendre polynomials and the generating function to obtain some of their properties
- Recognise a Sturm–Liouville system and the orthogonality properties of its eigenfunctions
- Write a polynomial in x as a finite series of Legendre polynomials

Prerequisite: Engineering Mathematics (Fifth Edition)

Programmes 13 Series 1, 14 Series 2 and 25 Second-order differential equations

Higher derivatives

1

If $y = \sin x \quad \dfrac{dy}{dx} = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$

$$\frac{d^2y}{dx^2} = -\sin x = \sin(x + \pi) = \sin\left(x + \frac{2\pi}{2}\right)$$

$$\frac{d^3y}{dx^3} = -\cos x = \sin\left(x + \frac{3\pi}{2}\right) \quad \text{etc.}$$

We see a pattern developing. In general $\dfrac{d^ny}{dx^n} = \sin\left(x + \dfrac{n\pi}{2}\right)$. Before we go further, we introduce a shorthand notation for the nth derivative of y as $y^{(n)} = \dfrac{d^ny}{dx^n}$. Note, however, we still use the 'prime' notation y', y'' and y''' to represent the first, second and third derivatives respectively.

The results above can therefore be written

If $y = \sin x \quad \therefore\ y' = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$

$$y'' = -\sin x = \sin\left(x + \frac{2\pi}{2}\right)$$

$$y''' = -\cos x = \sin\left(x + \frac{3\pi}{2}\right)$$

and, in general, $y^{(n)} = \sin\left(x + \dfrac{n\pi}{2}\right)$

It is therefore possible to write down any particular derivative of $\sin x$ without calculating all the previous derivatives. For example

$$\frac{d^7y}{dx^7} = y^{(7)} = \sin\left(x + \frac{7\pi}{2}\right) = -\cos x$$

Similarly, starting with $y = \cos x$, we can determine an expression for the nth derivative of y which is ....

2

$$\boxed{y^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)}$$

Because

$y = \cos x \quad \therefore\ y' = -\sin x = \cos\left(x + \dfrac{\pi}{2}\right)$

$$y'' = -\cos x = \cos\left(x + \frac{2\pi}{2}\right)$$

$$y''' = \sin x \quad = \cos\left(x + \frac{3\pi}{2}\right) \quad \text{etc.}$$

$$\therefore\ y^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

Many of the standard functions can be treated in a similar manner. For example, if $y = e^{ax}$, then $y^{(n)} =$ ....

3

$$\boxed{y^{(n)} = a^n e^{ax}}$$

Because

$$y = e^{ax}, \quad y' = ae^{ax}, \quad y'' = a^2 e^{ax}, \quad y''' = a^3 e^{ax}, \quad \text{etc.}$$

In general, $y^{(n)} = a^n e^{ax}$.

With no great effort, we can now write down expressions for the following

If $y = \sin ax$, $y^{(n)} = \dots\dots\dots\dots$

If $y = \cos ax$, $y^{(n)} = \dots\dots\dots\dots$

4

$$\boxed{\begin{aligned} y &= \sin ax, \quad y^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right) \\ y &= \cos ax, \quad y^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right) \end{aligned}}$$

Now one more.

If $y = \ln x$, $y^{(n)} = \dots\dots\dots\dots$

5

$$\boxed{y^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}}$$

Because

$$\begin{aligned} y = \ln x \quad \therefore\ y' &= \frac{1}{x} \\ y'' &= -\frac{1}{x^2} \\ y''' &= \frac{2}{x^3} \\ y^{(4)} &= -\frac{3!}{x^4} \qquad \therefore\ y^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n} \end{aligned}$$

We already know that, if $y = \ln x$, $\dfrac{dy}{dx} = y' = \dfrac{1}{x} = x^{-1}$.

Therefore, if the result obtained for $y^{(n)}$ is to be valid for $n = 1$, then

$$y' = (-1)^0 \cdot \frac{0!}{x} = \frac{0!}{x}$$

But $y' = x^{-1}$ $\qquad \therefore\ 0! = \dots\dots\dots\dots$

6

$$\boxed{0! = 1}$$

Now let us consider the derivatives of $\sinh ax$ and $\cosh ax$.

Next frame

7

If $y = \sinh ax$, $\quad y' = a\cosh ax$

$$y'' = a^2 \sinh ax$$
$$y''' = a^3 \cosh ax \qquad \text{etc.}$$

Because $\sinh ax$ is not periodic, we cannot proceed as we did with $\sin ax$. We need to find a general statement for $y^{(n)}$ containing terms in $\sinh ax$ and in $\cosh ax$, such that, when n is even, the term in $\cosh ax$ disappears and, when n is odd, the term in $\sinh ax$ disappears.

This we can do by writing $y^{(n)}$ in the form

$$y^{(n)} = \frac{a^n}{2}\{[1 + (-1)^n]\sinh ax + [1 - (-1)^n]\cosh ax\}$$

In very much the same way, we can determine the nth derivative of $y = \cosh ax$ as ....

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$$\boxed{y^{(n)} = \frac{a^n}{2}\{[1 - (-1)^n]\sinh ax + [1 + (-1)^n]\cosh ax\}}$$

Finally, let us deal with $y = x^a$.

$$y = x^a \quad \therefore \quad y' = ax^{a-1}$$
$$y'' = a(a-1)\,x^{a-2}$$
$$y''' = a(a-1)(a-2)\,x^{a-3}$$
$$\dots\dots\dots\dots$$
$$\therefore \quad y^{(n)} = a(a-1)(a-2)\dots(a-n+1)\,x^{a-n}$$
$$\therefore \quad y^{(n)} = \frac{a!}{(a-n)!}\,x^{a-n} \qquad (a \text{ is a positive integer})$$

So, collecting our results together, we have

$$y = x^a \qquad y^{(n)} = \frac{a!}{(a-n)!}x^{a-n}$$
$$y = e^{ax} \qquad y^{(n)} = a^n e^{ax}$$
$$y = \sin ax \qquad y^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right)$$
$$y = \cos ax \qquad y^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right)$$
$$y = \sinh ax \qquad y^{(n)} = \frac{a^n}{2}\{[1 + (-1)^n]\sinh ax + [1 - (-1)^n]\cosh ax\}$$
$$y = \cosh ax \qquad y^{(n)} = \frac{a^n}{2}\{[1 - (-1)^n]\sinh ax + [1 + (-1)^n]\cosh ax\}$$

Make a note of these, as a set, and then move on to the next frame

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Exercise

Determine the following derivatives

1 $y = \sin 4x$ $\quad y^{(5)} = \ldots\ldots\ldots\ldots$

2 $y = e^{x/2}$ $\quad y^{(8)} = \ldots\ldots\ldots\ldots$

3 $y = \cosh 3x$ $\quad y^{(12)} = \ldots\ldots\ldots\ldots$

4 $y = \cos(x\sqrt{2})$ $\quad y^{(10)} = \ldots\ldots\ldots\ldots$

5 $y = x^8$ $\quad y^{(6)} = \ldots\ldots\ldots\ldots$

6 $y = \sinh 2x$ $\quad y^{(7)} = \ldots\ldots\ldots\ldots$

Finish them all; then check with the next frame

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Here are the solutions

1 $y^{(5)} = 4^5 \sin\left(4x + \dfrac{5\pi}{2}\right) = 1024 \sin\left(4x + \dfrac{\pi}{2}\right) = 1024 \cos 4x$

2 $y^{(8)} = \left(\dfrac{1}{2}\right)^8 e^{x/2} = \dfrac{1}{256} e^{x/2} = e^{x/2}/256$

3 $y^{(12)} = \dfrac{3^{12}}{2}\{0 \sinh 3x + 2 \cosh 3x\} = 3^{12} \cosh 3x$

4 $y^{(10)} = (\sqrt{2})^{10} \cos\left(x\sqrt{2} + \dfrac{10\pi}{2}\right)$

$= 32 \cos(x\sqrt{2} + 5\pi) = -32 \cos(x\sqrt{2})$

5 $y^{(6)} = \dfrac{8!}{2!} x^2 = 20\,160\, x^2$

6 $y^{(7)} = \dfrac{2^7}{2}\left\{[1 + (-1)^7] \sinh 2x + [1 - (-1)^7] \cosh 2x\right\}$

$= 2^7 \cosh 2x$

Leibnitz theorem – *n*th derivative of a product of two functions

If $y = uv$, where u and v are functions of x, then

$$y' = uv' + vu' \quad \text{where} \quad v' = \frac{dv}{dx} \quad \text{and} \quad u' = \frac{du}{dx}$$

$$\text{and} \quad y'' = uv'' + v'u' + vu'' + u'v' = u''v + 2u'v' + uv''$$

If we differentiate the last result and collect like terms, we obtain

$$y''' = \ldots\ldots\ldots\ldots$$

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$$\boxed{y''' = u'''v + 3u''v' + 3u'v'' + uv'''}$$

A further stage of differentiation would give

$$y^{(4)} = u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}$$

These results can therefore be written

$$\begin{aligned}
y &= uv \\
y' &= u'v + uv' \\
y'' &= u''v + 2u'v' + uv'' \\
y''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\
y^{(4)} &= u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}
\end{aligned}$$

Notice that in each case

(a) the superscript of u decreases regularly by 1

(b) the superscript of v increases regularly by 1

(c) the numerical coefficients are the normal binomial coefficients.

Indeed, $(uv)^{(n)}$ can be obtained by expanding $(u+v)^{(n)}$ using the binomial theorem where the 'powers' are interpreted as derivatives. So the expression for the nth derivative can therefore be written as

$$\begin{aligned}
y^{(n)} &= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{1\times 2}u^{(n-2)}v^{(2)} \\
&\quad + \frac{n(n-1)(n-2)}{1\times 2\times 3}u^{(n-3)}v^{(3)} + \dots \\
&= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} \\
&\quad + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots
\end{aligned}$$

i.e. $$\begin{aligned}
y^{(n)} &= u^{(n)}v + {}^nC_1\, u^{(n-1)}v^{(1)} + {}^nC_2 u^{(n-2)}v^{(2)} + \dots \\
&\quad + {}^nC_{n-1}u^{(1)}v^{(n-1)} + uv^{(n)}
\end{aligned}$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

If $y = uv$ $$y^{(n)} = \sum_{r=0}^{n} {}^nC_r\, u^{(n-r)}v^{(r)} \quad \text{where } u^{(0)} \equiv u$$

This is the *Leibnitz theorem*. We shall certainly be using it often in the work ahead, so make a note of it for future reference. Then we can see it in use.

Choice of function for u and v 12

For the product $y = uv$ the function taken as

(a) u is the one whose nth derivative can readily be obtained
(b) v is the one whose derivatives reduce to zero after a small number of stages of differentiation.

Example 1

To find $y^{(n)}$ when $y = x^3 e^{2x}$.

Here we choose $v = x^3$ – whose fourth derivative is zero

$u = e^{2x}$ – because we know that the nth derivative

$u^{(n)} = \dots\dots\dots\dots$

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$$\boxed{u^{(n)} = 2^n e^{2x}}$$

Using the Leibnitz theorem:

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots$$

$v = x^3;\quad v^{(1)} = 3x^2;\quad v^{(2)} = 6x;\quad v^{(3)} = 6;\quad v^{(4)} = 0$

$u = e^{2x};\quad u^{(n)} = 2^n e^{2x}$

$\therefore\ y^{(n)} = \dots\dots\dots\dots$

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$$\boxed{y^{(n)} = e^{2x}\,2^{n-3}\{8x^3 + 12nx^2 + n(n-1)\,6x + n(n-1)(n-2)\}}$$

Example 2

If $x^2y'' + xy' + y = 0$, show that

$x^2y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2+1)\,y^{(n)} = 0.$

We take the given equation $x^2y'' + xy' + y = 0$ and differentiate n times, treating each term in turn.

If $w = x^2y''$ $\quad w^{(n)} = \dots\dots\dots\dots$

If $w = xy'$ $\quad w^{(n)} = \dots\dots\dots\dots$

If $w = y$ $\quad w^{(n)} = \dots\dots\dots\dots$

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$$\begin{array}{lll} w = x^2y'' & \therefore & w^{(n)} = y^{(n+2)}x^2 + ny^{(n+1)}2x + \dfrac{n(n-1)}{2!}y^{(n)}2 + 0\dots \\ w = xy' & \therefore & w^{(n)} = y^{(n+1)}x + ny^{(n)}1 + 0 + \dots \\ w = y & \therefore & w^{(n)} = y^{(n)} \end{array}$$

Then $[x^2y'' + xy' + y]^{(n)} = 0$ becomes

....

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$$\boxed{x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0}$$

which is what we had to show.

Example 3

Differentiate n times

$(1+x^2)y'' + 2xy' - 5y = 0.$

The result ....

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$$\boxed{(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2+n-5)y^{(n)} = 0}$$

Because, by the Leibnitz theorem

$$\left\{y^{(n+2)}(1+x^2) + ny^{(n+1)}2x + \frac{n(n-1)}{2!}y^{(n)}2\right\}$$
$$+2\left\{xy^{(n+1)} + ny^{(n)}\cdot 1\right\} - 5y^{(n)} = 0$$
$$(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + \{n(n-1) + 2n - 5\}y^{(n)} = 0$$
$$(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2+n-5)y^{(n)} = 0$$

We shall be using the Leibnitz theorem in the rest of this Programme, so let us move on to see some of its applications.

Power series solutions

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Second-order linear differential equations with constant coefficients of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ can be solved by algebraic methods giving solutions in terms of the normal elementary functions such as exponentials, trigonometric and polynomial functions.

▶

In general, equations of the form $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$, where $P(x)$ and $Q(x)$ are functions of x, cannot be solved in this way. However, it is often possible to obtain solutions in the form of infinite series of powers of x – and the next section of work investigates some of the methods which make this possible.

1 Leibnitz–Maclaurin method

As the title suggests, for this we need to be familiar with the Leibnitz theorem and with Maclaurin's series.

The *Leibnitz theorem* states that, if $y = uv$, where u and v are functions of x, then

$$y^{(n)} = \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} y^{(n)} &= u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \dots \\ &\quad + \frac{n(n-1)\dots(n-r+1)}{r!}u^{(n-r)}v^{(r)} + \dots + uv^{(n)} \end{aligned}}$$

where $u^{(r)}$ and $v^{(r)}$ denote $\dfrac{d^ru}{dx^r}$ and $\dfrac{d^rv}{dx^r}$ respectively.

Maclaurin's series for $y = f(x)$ can be stated as

$$y = \dots\dots\dots\dots$$

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$$\boxed{y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \dots + \frac{x^n}{n!}\left(y^{(n)}\right)_0 + \dots}$$

where $\left(y^{(n)}\right)_0$ denotes the value of the nth derivative of y at $x = 0$.

On to the next frame

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Example 1

Find the power series solution of the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 1.$$

The equation can be written

$$xy'' + y' + xy = 1$$

In the first product term xy'', treat y'' as u and x as v. Then, differentiating the equation n times by the Leibnitz theorem, gives

$$\dots\dots\dots\dots$$

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$$\left(xy^{(n+2)} + n \cdot 1 \cdot y^{(n+1)}\right) + y^{(n+1)} + \left(xy^{(n)} + n \cdot 1 \cdot y^{(n-1)}\right) = 0$$
$$\text{i.e.} \quad xy^{(n+2)} + (n+1)y^{(n+1)} + xy^{(n)} + ny^{(n-1)} = 0$$

At $x = 0$, this becomes

$$(n+1)\left(y^{(n+1)}\right)_0 + n\left(y^{(n-1)}\right)_0 = 0$$

$$\therefore \; \left(y^{(n+1)}\right)_0 = -\frac{n}{n+1}\left(y^{(n-1)}\right)_0 \qquad n \geq 1$$

This relationship is called a *recurrence relation.*

We can now substitute $n = 1, 2, 3, \ldots$ and get a set of relationships between the various coefficients.

$n = 1 \qquad (y'')_0 = -\frac{1}{2}(y)_0$

$n = 2 \qquad (y''')_0 = -\frac{2}{3}(y')_0$

$n = 3 \qquad (y^{(4)})_0 = -\frac{3}{4}(y'')_0 = (-\frac{3}{4})(-\frac{1}{2})(y)_0$

Continuing in the same way,

$(y^{(5)})_0 = \ldots\ldots\ldots\ldots$

$(y^{(6)})_0 = \ldots\ldots\ldots\ldots$

$(y^{(7)})_0 = \ldots\ldots\ldots\ldots$

$(y^{(8)})_0 = \ldots\ldots\ldots\ldots$

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$n = 4 \qquad (y^{(5)})_0 = -\frac{4}{5}(y^{(3)})_0 = (-\frac{4}{5})(-\frac{2}{3})(y^{(1)})_0$

$n = 5 \qquad (y^{(6)})_0 = -\frac{5}{6}(y^{(4)})_0 = (-\frac{5}{6})(-\frac{3}{4})(-\frac{1}{2})(y)_0$

$n = 6 \qquad (y^{(7)})_0 = -\frac{6}{7}(y^{(5)})_0 = (-\frac{6}{7})(-\frac{4}{5})(-\frac{2}{3})(y^{(1)})_0$

$n = 7 \qquad (y^{(8)})_0 = -\frac{7}{8}(y^{(6)})_0 = (-\frac{7}{8})(-\frac{5}{6})(-\frac{3}{4})(-\frac{1}{2})(y)_0$

Notice that, by this means, the values of all the derivatives at $x = 0$ can be expressed in terms of $(y)_0$ and $(y')_0$.

If we now substitute these values for $(y^{(r)})_0$ in the Maclaurin series

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \ldots + \frac{x^r}{r!}(y^{(r)})_0 + \ldots$$

we obtain $\ldots\ldots\ldots\ldots$

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$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}\left(-\frac{1}{2}\right)(y)_0 + \frac{x^3}{3!}\left(-\frac{2}{3}\right)(y')_0 + \frac{x^4}{4!}\left(-\frac{3}{4}\right)\left(-\frac{1}{2}\right)(y)_0 + \frac{x^5}{5!}\left(-\frac{4}{5}\right)\left(-\frac{2}{3}\right)(y')_0 + \frac{x^6}{6!}\left(-\frac{5}{6}\right)\left(-\frac{3}{4}\right)\left(-\frac{1}{2}\right)(y)_0 + \ldots\ldots\ldots\ldots$$

Simplifying, this gives

$$y = (y)_0\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2} + \ldots\right\} + (y')_0\left\{x - \frac{x^3}{3^2} + \frac{x^5}{3^2 \times 5^2} + \ldots\right\}$$

The values of $(y)_0$ and $(y')_0$ provide the two arbitrary constants for the second-order equation and are obtained from the given initial conditions.

For example, if at $x = 0$, $y = 2$ and $\dfrac{dy}{dx} = 1$, then the relevant particular solution is ....

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$$y = 2\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} - \frac{x^6}{2^2 \times 4^2 \times 6^2} + \ldots\right\} + \left\{x - \frac{x^3}{3^2} + \frac{x^5}{3^2 \times 5^2} + \ldots\right\}$$

Because at $x = 0$, $y = 2$ i.e. $(y)_0 = 2$

$\dfrac{dy}{dx} = 1$ i.e. $(y')_0 = 1$.

To be a valid solution, the series obtained must converge. Application of the ratio test will normally indicate any restrictions on the values that x may have.

The Leibnitz–Maclaurin (power series) method therefore involves the following main steps:

(a) Differentiate the given equation n times, using the Leibnitz theorem.

(b) Rearrange the result to obtain the recurrence relation at $x = 0$.

(c) Determine the values of the derivatives at $x = 0$, usually in terms of $(y)_0$ and $(y')_0$.

(d) Substitute in the Maclaurin expansion for $y = f(x)$.

(e) Simplify the result where possible and apply boundary conditions if provided.

That is all there is to it. Let us go through the various steps with another example.

Example 2

Determine a series solution of the equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

The equation can be written $y'' + xy' + y = 0$

(a) Differentiate n times using the Leibnitz theorem, which gives

....

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$$\boxed{y^{(n+2)} + xy^{(n+1)} + (n+1)y^{(n)} = 0}$$

Because $y'' + xy' + y = 0$

$$\therefore\ y^{(n+2)} + \left\{xy^{(n+1)} + n \cdot 1 \cdot y^{(n)}\right\} + y^{(n)} = 0$$

$$\therefore\ y^{(n+2)} + xy^{(n+1)} + (n+1)y^{(n)} = 0.$$

(b) Determine the recurrence relation at $x = 0$, which is

....

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$$\boxed{y^{(n+2)} = -(n+1)\,y^{(n)}}$$

(c) Now taking $n = 0, 1, 2, 3, 4, 5$, determine the derivatives at $x = 0$ in terms of $(y)_0$ and $(y')_0$. List them, as we did before, in table form.

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$n = 0$	$(y'')_0 = -(y)_0$	$= -(y)_0$
1	$(y''')_0 = -2\,(y')_0$	$= -2\,(y')_0$
2	$(y^{(4)})_0 = -3\,(y'')_0 = (-3)[-(y)_0]$	$= 3\,(y)_0$
3	$(y^{(5)})_0 = -4\,(y''')_0 = (-4)[-2(y')_0]$	$= 2 \times 4\,(y')_0$
4	$(y^{(6)})_0 = -5\,(y^{(4)})_0 = (-5)[-3(y'')_0]$	$= -3 \times 5\,(y)_0$
5	$(y^{(7)})_0 = -6\,(y^{(5)})_0 = (-6)[-4(y''')_0]$	$= -2 \times 4 \times 6\,(y')_0$

(d) Substitute these expressions for the derivatives in terms of $(y)_0$ and $(y')_0$ in Maclaurin's expansion

$$y = (y)_0 + x\,(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y^{(4)})_0 + \dots$$

Then $\quad y = \dots\dots\dots\dots$

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$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(-y)_0 + \frac{x^3}{3!}(-2y')_0 + \frac{x^4}{4!}(3y)_0 + \frac{x^5}{5!}(8y')_0 + \frac{x^6}{6!}(-15y)_0 + \frac{x^7}{7!}(-48y')_0 + \dots$$

Collecting now the terms in $(y)_0$ and $(y')_0$, we finally obtain

$$y = (y)_0\left\{1 - \frac{x^2}{2} + \frac{x^4}{2\times 4} - \frac{x^6}{2\times 4\times 6} + \dots\right\} + (y')_0\left\{x - \frac{x^3}{3} + \frac{x^5}{3\times 5} - \frac{x^7}{3\times 5\times 7} + \dots\right\}$$

They are all done in very much the same way. Here is another.

Example 3

Solve the equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 2xy = 0$ given that at $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 1$.

First write the equation as $y'' + y' + 2xy = 0$, differentiate n times by the Leibnitz theorem and obtain the recurrence relation at $x = 0$, which is ....

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$$\boxed{y^{(n+2)} = -\left\{y^{(n+1)} + 2ny^{(n-1)}\right\} \quad n \geq 1}$$

Because $y'' + y' + 2xy = 0$

$$\therefore\ y^{(n+2)} + y^{(n+1)} + 2xy^{(n)} + n2y^{(n-1)} = 0$$

At $x = 0$, $\qquad y^{(n+2)} + y^{(n+1)} + 2ny^{(n-1)} = 0$

$$\therefore\ y^{(n+2)} = -\left\{y^{(n+1)} + 2ny^{(n-1)}\right\}$$

Since we have a term in $y^{(n-1)}$, then n must start at 1 to give $(y)_0$. Therefore the recurrence relation applies for $n \geq 1$.

We now take $n = 1, 2, 3, \dots$ to obtain the relationships between the coefficients up to $(y^{(6)})_0$. Complete the table and check with the next frame.

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$$\begin{array}{|ll|}\hline n=1 & (y^{(3)})_0 = -\{(y^{(2)})_0 + 2(y)_0\} \\ n=2 & (y^{(4)})_0 = -\{(y^{(3)})_0 + 4(y')_0\} \\ n=3 & (y^{(5)})_0 = -\{(y^{(4)})_0 + 6(y^{(2)})_0\} \\ n=4 & (y^{(6)})_0 = -\{(y^{(5)})_0 + 8(y^{(3)})_0\} \\ \hline\end{array}$$

We therefore have expressions for $(y''')_0, (y^{(4)})_0, (y^{(5)})_0, (y^{(6)})_0$, but what about $(y'')_0$?

If we refer to the initial conditions, we know that at $x = 0$, $y = 0$ and $y' = 1$. $\therefore\ (y)_0 = 0$ and $(y')_0 = 1$.

We can find $(y'')_0$ by reference to the given equation itself, because

$$y'' + y' + 2xy = 0$$

Therefore, at $x = 0$, $(y'')_0 + (y')_0 = 0 \quad \therefore\ (y'')_0 = -(y')_0 = -1$.

So now we have

$$\begin{aligned} (y)_0 &= 0 \\ (y')_0 &= 1 \\ (y'')_0 &= -1 \\ (y''')_0 &= -\{(y'')_0 + 2(y)_0\} &&= -\{(-1) + 0\} = 1 \\ (y^{(4)})_0 &= -\{(y''')_0 + 4(y')_0\} &&= -\{1 + 4\} \quad = -5 \\ (y^{(5)})_0 &= -\{(y^{(4)})_0 + 6(y'')_0\} &&= -\{(-5) - 6\} = 11 \\ (y^{(6)})_0 &= -\{(y^{(5)})_0 + 8(y''')_0\} &&= -\{11 + 8\} \quad = -19 \end{aligned}$$

The required series solution is therefore

$y = \dots\dots\dots\dots$

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$$\boxed{y = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{5x^4}{4!} + \frac{11x^5}{5!} - \frac{19x^6}{6!} + \dots}$$

Because

$$\begin{aligned} y &= (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y^{(4)})_0 + \dots \\ &= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-5) + \frac{x^5}{5!}(11) + \frac{x^6}{6!}(-19) \end{aligned}$$

$$\therefore\ y = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{5x^4}{4!} + \frac{11x^5}{5!} - \frac{19x^6}{6!} + \dots$$

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One more of the same kind.

Example 4

Determine the general series solution of the equation

$$(x^2 + 1)y'' + xy' - 4y = 0$$

As usual, establish the recurrence relation at $x = 0$, which is

$\dots\dots\dots\dots$

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$$\boxed{y^{(n+2)} = (4 - n^2)y^{(n)}}$$

Because

$(x^2 + 1)y'' + xy' - 4y = 0$ therefore

$$\left\{(x^2+1)y^{(n+2)} + 2xny^{(n+1)} + 2\frac{n(n-1)}{2!}y^{(n)}\right\} + \left\{xy^{(n+1)} + ny^{(n)}\right\} - 4y^{(n)}$$
$$= 0$$

At $x = 0$, this becomes

$y^{(n+2)} + n(n-1)y^{(n)} + ny^{(n)} - 4y^{(n)} = 0$ that is $y^{(n+2)} = (4 - n^2)y^{(n)}$

Then, starting with $n = 0$, determine expressions for $(y^{(n)})_0$ as far as $n = 7$.

They are ....

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$n = 0$	$(y'')_0 = 4(y)_0$	$= 4(y)_0$
$n = 1$	$(y''')_0 = 3(y')_0$	$= 3(y')_0$
$n = 2$	$(y^{(4)})_0 = 0$	$= 0$
$n = 3$	$(y^{(5)})_0 = -5(y''')_0$	$= -15(y')_0$
$n = 4$	$(y^{(6)})_0 = -12(y^{(4)})_0$	$= 0$
$n = 5$	$(y^{(7)})_0 = -21(y^{(5)})_0$	$= (-21)(-15)(y')_0$

Now substitute in Maclaurin's expansion and simplify the result.

$y =$ ....

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$$\boxed{y = A(1 + 2x^2) + B\left\{x + \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{16} + \dots\right\}}$$

Because

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y^{(4)})_0 + \dots$$
$$= (y)_0 + x(y')_0 + \frac{x^2}{2!}4(y)_0 + \frac{x^3}{3!}3(y')_0 + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(-15)(y')_0 + \text{etc.}$$
$$= (y)_0\{1 + 2x^2\} + (y')_0\left\{x + \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{16} + \dots\right\}$$

Putting $(y)_0 = A$ and $(y')_0 = B$, we have the result stated.

Now to something slightly different

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2 Frobenius' method

In each of the previous examples, we established the solution as a power series in integral powers of x. Such a solution is not always possible and a more general method is to assume a trial solution of the form

$$y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots\}$$

where a_0 is the first coefficient that is not zero.

The type of equation that can be solved by this method is of the form

$$y'' + Py' + Qy = 0$$

where P and Q are functions of x.

However, certain conditions have to be satisfied.

(a) If the functions P and Q are such that both are finite when x is put equal to zero, $x = 0$ is called an *ordinary point* of the equation.
(b) If xP and x^2Q remain finite at $x = 0$, then $x = 0$ is called a *regular singular point* of the equation.

In both of these cases, the method of Frobenius can be applied.

(c) If, however, P and Q do not satisfy either of these conditions stated in (a) or (b), then $x = 0$ is called an *irregular singular point* of the equation and the method of Frobenius cannot be applied.

Solution of differential equations by the method of Frobenius

To solve a given equation, we have to find the coefficients $a_0, a_1, a_2, \ldots$ and also the index c in the trial solution. Basically, the steps in the method are as follows

(a) Differentiate the trial series as required.
(b) Substitute the results in the given differential equation.
(c) Equate coefficients of corresponding powers of x on each side of the equation.

The following examples will demonstrate the method – so move on

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Example 1

Find a series solution for the equation

$$2x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$$

The equation can be written as $2xy''+y'+y=0$.

Assume a solution of the form

$$y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_rx^r+\ldots\} \qquad a_0\neq 0.$$
$$\therefore\; y=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+\ldots+a_rx^{c+r}+\ldots$$

Differentiating term by term, we get

$$y'=\ldots\ldots\ldots\ldots$$

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$$\boxed{y'=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\ldots+a_r(c+r)x^{c+r-1}+\ldots}$$

Repeating the process one stage further, we have

$y''=\ldots\ldots\ldots\ldots$ (give yourself plenty of room)

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$$\boxed{\begin{aligned}y''=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\ldots\\+a_r(c+r-1)(c+r)x^{c+r-2}+\ldots\end{aligned}}$$

So far, we have $\qquad 2xy''+y'+y=0$

$$\begin{aligned}y&=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+\ldots+a_rx^{c+r}+\ldots\\
y'&=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\ldots\\
&\quad+a_r(c+r)x^{c+r-1}+\ldots\\
y''&=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\ldots\\
&\quad+a_r(c+r-1)(c+r)x^{c+r-2}+\ldots\end{aligned}$$

Considering each term of the equation in turn

$$\begin{aligned}2xy''&=2a_0c(c-1)x^{c-1}+2a_1c(c+1)x^c+2a_2(c+1)(c+2)x^{c+1}\\
&\quad+\ldots+a_r(c+r-1)(c+r)x^{c+r-1}+\ldots\\
y'&=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\ldots\\
&\quad+a_r(c+r)x^{c+r-1}+\ldots\\
y&=a_0x^c+a_1x^{c+1}+\ldots+a_rx^{c+r}+\ldots\end{aligned}$$

Adding these three lines to form the left-hand side of the equation, we can equate the total coefficient of each power of x to zero, since the right-hand side is zero.

$$[x^{c-1}] \text{ gives } \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} [x^{c-1}]: \quad & 2a_0c(c-1) + a_0c = 0 \\ \therefore\ & a_0c(2c-1) = 0 \end{aligned}}$$

So, $[x^{c-1}]$ gives $\quad a_0c(2c-1) = 0 \qquad (1)$

Similarly, $[x^c]$ gives ....

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$$\boxed{2a_1c\,(c+1) + a_1(c+1) + a_0 = 0}$$

Simplifying, this becomes

$$a_1(2c^2 + 3c + 1) + a_0 = 0$$

i.e. $\quad a_1(c+1)(2c+1) + a_0 = 0 \qquad (2)$

Also $[x^{c+1}]$ gives ....

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$$\boxed{2a_2(c+1)(c+2) + a_2(c+2) + a_1 = 0}$$

and this simplifies straight away to

$a_2(c+2)(2c+3) + a_1 = 0 \qquad (3)$

Note that the coefficient of x^c involves all three lines of the expressions and, from then on, a general relationship can be obtained for x^{c+r}, $r \geq 0$.

In the expression for $2xy''$ and y' we have terms in x^{c+r-1}. If we replace r by $(r+1)$, we shall obtain the corresponding terms in x^{c+r}.

In the series for $2xy''$, this is $2a_{r+1}(c+r)(c+r+1)x^{c+r}$

In the series for y', this is $a_{r+1}(c+r+1)x^{c+r}$

In the series for y, this is a_rx^{c+r}

Therefore, equating the total coefficient of x^{c+r} to zero, we have

....

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$$\boxed{2a_{r+1}(c+r)(c+r+1) + a_{r+1}(c+r+1) + a_r = 0}$$

and this tidies up to

$a_{r+1}\{(c+r+1)(2c+2r+1)\} + a_r = 0 \qquad (4)$

Make a note of results (1), (2), (3) and (4): we shall return to them in due course.

Then move on

Indicial equation

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Equation (1), formed from the coefficient of the lowest power of x, that is x^{c-1}, is called the *indicial equation* from which the values of c can be obtained. In the present example $a_0c(2c-1)=0$

$$\therefore \; c = \ldots\ldots\ldots\ldots$$

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$$\boxed{c = 0 \;\text{ or }\; \frac{1}{2}, \text{ since } a_0 \neq 0, \text{ by definition}}$$

Both values of c are valid, so that we have two possible solutions of the given equation. We will consider each in turn.

(a) *Using* $c = 0$

(2) gives $a_1(1)(1) + a_0 = 0 \quad \therefore \; a_1 = -a_0$

Similarly

(3) gives $\ldots\ldots\ldots\ldots$

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$$\boxed{a_2(2)(3) + a_1 = 0}$$

$$a_1 = -a_0 \quad \text{and} \quad a_2 = -\frac{a_1}{2\times 3} = \frac{a_0}{2\times 3}$$

and from (4) $\qquad a_{r+1} = \dfrac{-a_r}{(r+1)(2r+1)} \qquad r \geq 0$

From the combined series, the term in x^c and all subsequent terms involve all three lines and the coefficient of the general term can be used.

So we have $a_1 = -a_0$ and $a_{r+1} = \dfrac{-a_r}{(r+1)(2r+1)}$ for $r = 0, 1, 2, \ldots$

$$\therefore \; a_2 = \frac{-a_1}{2\times 3} = \frac{a_0}{2\times 3}$$

$$a_2 = \frac{-a_2}{3\times 5} = \frac{-a_0}{(2\times 3)(3\times 5)}$$

$$a_4 = \frac{-a_3}{4\times 7} = \frac{a_0}{(2\times 3\times 4)(3\times 5\times 7)} \qquad \text{etc.}$$

$$\therefore \; y = x^0\left\{a_0 - a_0x + \frac{a_0}{(2\times 3)}x^2 - \frac{a_0}{(2\times 3)(3\times 5)}x^3 + \ldots\right\}$$

$$\therefore \; y = a_0\left\{1 - x + \frac{x^2}{(2)(3)} - \frac{x^3}{(2\times 3)(3\times 5)} + \frac{x^4}{(2\times 3\times 4)(3\times 5\times 7)} + \ldots\right\}$$

Now we go through the same steps using our second value for c, i.e. $c = \frac{1}{2}$.

Next frame

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(b) *Using* $c = \frac{1}{2}$

Our equations relating the coefficients were

$$a_0 c(2c-1) = 0 \quad \text{which gave} \quad c = 0 \text{ or } c = \tfrac{1}{2} \qquad (1)$$

$$a_1(c+1)(2c+1) + a_0 = 0 \qquad (2)$$

$$a_2(c+2)(2c+3) + a_1 = 0 \qquad (3)$$

$$a_{r+1}(c+r+1)(2c+2r+1) + a_r = 0 \qquad (4)$$

Putting $c = \frac{1}{2}$ in (2) gives ....

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$$\boxed{a_1 = -\frac{a_0}{3}}$$

Similarly (3) gives $\quad a_2 = -\dfrac{a_1}{10} = \dfrac{a_0}{3 \times 10}$

and from the general relationship, (4), we have ....

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$$\boxed{a_{r+1} = \frac{-a_r}{(r+1)(2r+3)}}$$

So
$$a_1 = -\frac{a_0}{3}$$

$$a_2 = -\frac{a_1}{2 \times 5} = \frac{a_0}{(1 \times 2)(3 \times 5)}$$

$$a_3 = -\frac{a_2}{3 \times 7} = \frac{-a_0}{(1 \times 2 \times 3)(3 \times 5 \times 7)}$$

$$a_4 = -\frac{a_3}{4 \times 9} = \frac{a_0}{(1 \times 2 \times 3 \times 4)(3 \times 5 \times 7 \times 9)} \quad \text{etc.}$$

$$y = x^c\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_r x^r + \ldots\}$$

i.e. $\quad y = $ ....

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$$y = x^{\frac{1}{2}}\left\{a_0 - \frac{a_0}{3}x + \frac{a_0}{(1 \times 2)(3 \times 5)}x^2 - \frac{a_0}{(1 \times 2 \times 3)(3 \times 5 \times 7)}x^3 + \dots\right\}$$

i.e. $$y = a_0 x^{\frac{1}{2}}\left\{1 - \frac{x}{(1 \times 3)} + \frac{x^2}{(1 \times 2)(3 \times 5)} - \frac{x^3}{(1 \times 2 \times 3)(3 \times 5 \times 7)} + \dots\right\}$$

Since a_0 is an arbitrary (non-zero) constant in each solution, its values may well be different, A and B say. If we denote the first solution by $u(x)$ and the second by $v(x)$, then

$$u = A\left\{1 - x + \frac{x^2}{(2 \times 3)} - \frac{x^3}{(2 \times 3)(3 \times 5)} + \frac{x^4}{(2 \times 3 \times 4)(3 \times 5 \times 7)} + \dots\right\}$$

and

$$v = Bx^{\frac{1}{2}}\left\{1 - \frac{x}{(1 \times 3)} + \frac{x^2}{(1 \times 2)(3 \times 5)} - \frac{x^3}{(1 \times 2 \times 3)(3 \times 5 \times 7)} + \dots\right\}$$

The general solution $y = u + v$ is therefore ....

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$$y = A\left\{1 - x + \frac{x^2}{(2 \times 3)} - \frac{x^3}{(2 \times 3)(3 \times 5)} + \dots\right\} + Bx^{\frac{1}{2}}\left\{1 - \frac{x}{(1 \times 3)} + \frac{x^2}{(1 \times 2)(3 \times 5)} - \frac{x^3}{(1 \times 2 \times 3)(3 \times 5 \times 7)} + \dots\right\}$$

The method may seem somewhat lengthy, but we have set it out in detail. It is a straightforward routine. Here is another example with the same steps.

Example 2

Find the series solution for the equation

$$3x^2y'' - xy' + y - xy = 0.$$

We proceed in just the same way as in the previous example.

Assume $\quad y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_rx^r + \dots\}$

i.e. $\quad y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \dots + a_rx^{c+r} + \dots$

$$\therefore\ y' = a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \dots + a_r(c+r)x^{c+r-1} + \dots$$

and $\quad y'' = \dots\dots\dots\dots$

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$$y'' = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \dots + a_r(c+r)(c+r-1)x^{c+r-2} + \dots$$

Now we build up the terms in the given equation.

$$3x^2y'' = 3a_0c(c-1)x^c + 3a_1(c+1)cx^{c+1} + 3a_2(c+2)(c+1)x^{c+2} + \dots + 3a_r(c+r)(c+r-1)x^{c+r} + \dots$$

$$-xy' = -a_0cx^c - a_1(c+1)x^{c+1} - a_2(c+2)x^{c+2} - \dots - a_r(c+r)x^{c+r} - \dots$$

$$y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \dots + a_rx^{c+r} + \dots$$

$$-xy = -a_0x^{c+1} - a_1x^{c+2} - \dots - a_rx^{c+r+1} \dots$$

The *indicial equation*, i.e. equating the coefficient of the lowest power of x to zero, gives the values of c. Thus, in this case

$c = \dots\dots\dots\dots$

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$$c = 1 \text{ or } \frac{1}{3}$$

Because the lowest power is x^c and the coefficient of x^c equated to zero gives

$$3a_0c(c-1) - a_0c + a_0 = 0$$

$$\therefore\ a_0(3c^2 - 4c + 1) = 0 \quad \therefore\ (3c-1)(c-1) = 0 \text{ since } a_0 \neq 0$$

$$\therefore\ c = 1 \text{ or } \frac{1}{3}$$

The coefficient of the general term, i.e. x^{c+r} gives

$$3a_r(c+r)(c+r-1) - a_r(c+r) + a_r - a_{r-1} = 0$$

$\therefore\ a_r = \dots\dots\dots\dots$

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$$a_r = \frac{a_{r-1}}{3(c+r)^2 - 4(c+r) + 1} = \frac{a_{r-1}}{(c+r-1)(3c+3r-1)}$$

(a) *Using* $c = 1$ the recurrence relation becomes

$$a_r = \frac{a_{r-1}}{r(3r+2)}$$

$$\therefore\ r = 1 \qquad a_1 = \frac{a_0}{1 \times 5}$$

$$r = 2 \qquad a_2 = \frac{a_1}{2 \times 8} = \frac{a_0}{(1 \times 2)(5 \times 8)}$$

$$r = 3 \qquad a_3 = \frac{a_2}{3 \times 11} = \frac{a_0}{(1 \times 2 \times 3)(5 \times 8 \times 11)}$$

Our first solution is therefore

$y = \dots\dots\dots\dots$

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$$y = x^1\left\{a_0 + \frac{a_0 x}{(1\times 5)} + \frac{a_0 x^2}{(1\times 2)(5\times 8)} + \frac{a_0 x^3}{(1\times 2\times 3)(5\times 8\times 11)} + \dots\right\}$$
$$\therefore\ y = Ax\left\{1 + \frac{x}{(1\times 5)} + \frac{x^2}{(1\times 2)(5\times 8)} + \frac{x^3}{(1\times 2\times 3)(5\times 8\times 11)} + \dots\right\}$$

(b) *For the second solution,* we put $c = \frac{1}{3}$. The recurrence relation then becomes

$$a_r = \dots\dots\dots\dots$$

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$$a_r = \frac{a_{r-1}}{r(3r-2)}$$

Therefore we can now determine the coefficients for $r = 1, 2, 3, \dots$ and complete the second solution.

$$y = \dots\dots\dots\dots$$

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$$y = Bx^{\frac{1}{3}}\left\{1 + x + \frac{x^2}{(2\times 4)} + \frac{x^3}{(2\times 3)(4\times 7)} + \frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)} + \dots\right\}$$

Because

$$a_1 = \frac{a_0}{1\times 1};\qquad a_2 = \frac{a_1}{2\times 4} = \frac{a_0}{(1\times 2)(2\times 4)}$$
$$a_3 = \frac{a_2}{3\times 7} = \frac{a_0}{(2\times 3)(4\times 7)}$$
$$a_4 = \frac{a_3}{4\times 10} = \frac{a_0}{(2\times 3\times 4)(4\times 7\times 10)}$$
$$\therefore\ y = a_0 x^{\frac{1}{3}}\left\{1 + x + \frac{x^2}{(2\times 4)} + \frac{x^3}{(2\times 3)(4\times 7)} + \frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)} + \dots\right\}$$

Therefore, the general solution is

$$y = \dots\dots\dots\dots$$

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$$y = Ax\left\{1 + \frac{x}{(1\times 5)} + \frac{x^2}{(1\times 2)(5\times 8)} + \frac{x^3}{(1\times 2\times 3)(5\times 8\times 11)} + \dots\right\}$$
$$+ Bx^{\frac{1}{3}}\left\{1 + x + \frac{x^2}{(2\times 4)} + \frac{x^3}{(2\times 3)(4\times 7)} + \frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)} + \dots\right\}$$

Example 3

Find the series solution for the equation

$$\frac{d^2y}{dx^2} - y = 0 \quad \text{i.e.} \quad y'' - y = 0.$$

As usual, we start off with the assumed solution

$$y = x^c\{a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots\}$$
$$\text{i.e.} \quad y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + \dots + a_rx^{c+r} + \dots$$
$$\therefore \quad y' = a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \dots$$
$$+ a_r(c+r)x^{c+r-1} + \dots$$
$$y'' = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \dots$$
$$+ a_r(c+r)(c+r-1)x^{c+r-2} + \dots$$

These three expansions are required regularly, so make a note of them

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Now we build up the terms in the left-hand side of the equation.

$$y'' = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + a_2(c+2)(c+1)x^c + \dots$$
$$+ a_r(c+r)(c+r-1)x^{c+r-2} + \dots$$
$$y = a_0x^c + a_1x^{c+1} + \dots + a_rx^{c+r} + \dots$$

The term in x^{c+r} in the first of these expansions is

. . . .

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$$\boxed{a_{r+2}(c+r+2)(c+r+1)x^{c+r}}$$

Because replacing r by $(r+2)$ in $a_r(c+r)(c+r+1)x^{c+r-2}$ gives this result.

Then $y'' - y =$ . . . .

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$$\boxed{\begin{aligned} y''-y = a_0c(c-1)x^{c-2} + a_1(c+1)cx^{c-1} + [a_2(c+2)(c+1)-a_0]x^c \\ + \ldots + [a_{r+2}(c+r+2)(c+r+1)-a_r]x^{c+r} + \ldots \end{aligned}}$$

We now equate each coefficient in turn to zero, since the right-hand side of the equation is zero. The coefficient of the lowest power of x gives the *indicial equation* from which we obtain the values of c.

So, in this case, $c =$ ....

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$$\boxed{c = 0 \quad \text{or} \quad 1}$$

For the term in x^{c-1}, we have

$[x^{c-1}]$: $a_1(c+1)c = 0.$

With $c = 1$, $a_1 = 0$.

But with $c = 0$, a_1 is indeterminate, because any value of a_1 combined with the zero value of c would make the product zero.

$[x^c]$: $a_2(c+2)(c+1) - a_0 = 0 \quad \therefore \; a_2 = \dfrac{a_0}{(c+1)(c+2)}$

For the general term

$[x^{c+r}]$: ....

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$$\boxed{a_{r+2} = \frac{a_r}{(c+r+1)(c+r+2)}}$$

Because $a_{r+2}(c+r+2)(c+r+1) - a_r = 0$. Hence the result above.

From the indicial equation, $c = 0$ or $c = 1$.

(a) When $c = 0$ $\quad a_1$ is indeterminate

$$a_2 = \frac{a_0}{2}$$

In general $\quad a_{r+2} = \dfrac{a_r}{(r+1)(r+2)}$

$r = 1 \quad \therefore \; a_3 = \dfrac{a_1}{2 \times 3}$

$r = 2 \quad a_4 = \dfrac{a_2}{3 \times 4} = \dfrac{a_0}{4!}$

Therefore, one solution is ....

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$$\boxed{y = x^0\left\{a_0 + a_1x + \frac{a_0}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 \ldots\right\}}$$

i.e. $y = a_0\left\{1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\right\} + a_1\left\{x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\right\}$

a_0 and a_1 are arbitrary constants depending on the boundary conditions.

$$\therefore\; y = A\left\{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right\} + B\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right\}$$

Notice that these two series are the Maclaurin series expansions of the hyperbolic functions, so that

$$y = A\cosh x + B\sinh x$$

It is not very often the case that the series solution is so easily expressible in terms of known functions.

(b) Similarly,

when $c = 1$ $\qquad a_1 = 0$

$$a_2 = \frac{a_0}{2 \times 3}$$

$$a_{r+2} = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_{r+2} = \frac{a_r}{(r+2)(r+3)}}$$

$$\therefore\; a_1 = 0$$

$$a_2 = \frac{a_0}{3!}$$

$r = 1 \qquad a_3 = \dfrac{a_1}{3 \times 4} = 0$

$r = 2 \qquad a_4 = \dfrac{a_2}{4 \times 5} = \dfrac{a_0}{5!}$

$r = 3 \qquad a_5 = \dfrac{a_3}{5 \times 6} = 0 \quad$ etc.

A second solution with $c = 1$ is therefore

$$y = \ldots\ldots\ldots\ldots$$

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$$\boxed{y = a_0\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right\}}$$

and, because a_0 is an arbitrary constant

$$y = C\left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\right\}$$

Note: This is not, in fact, a separate solution, since it already forms the second series in the solution for $c = 0$ obtained previously. Therefore, the first solution, with its two arbitrary constants, A and B, gives the general solution. This happens when the two values of c differ by an integer.

Make a note of the following:

> If the two values of c, i.e. c_1 and c_2, differ by an integer, and if $c = c_1$ results in a_1 being indeterminate, then this value of c gives the general solution.
>
> The solution resulting from $c = c_2$ is then merely a multiple of one of the series forming the first solution.

Our last problem was an example of this.
So far, we have met two distinct cases concerning the two roots $c = c_1$ and $c = c_2$ of the indicial equation.

(a) *If c_1 and c_2 differ by a quantity NOT an integer* then two independent solutions, $y = u(x)$ and $y = v(x)$, are obtained. The general solution is then $y = Au + Bv$.

(b) *If c_1 and c_2 differ by an integer*, i.e. $c_2 = c_1 + n$, and if one coefficient (a_r) is indeterminate when $c = c_1$, the complete general solution is given by using this value of c. Using $c = c_1 + n$ gives a series which is a simple multiple of one of the series in the first solution.

Make a note of these two points in your record book. Then move on

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There is a third category to be added to (a) and (b) above.

(c) If the roots $c = c_1$ and $c = c_1 + n$ of the indicial equation differ by an integer and one coefficient (a_r) becomes infinite when $c = c_1$, the series is rewritten with a_0 replaced by $k(c - c_1)$.

Putting $c = c_1$ in the rewritten series and that of its derivative with respect to c gives two independent solutions.

Add this to the previous two. Then we will see how it works in practice

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Example 4

Find the series solution of the equation

$$xy'' + (2 + x)y' - 2y = 0.$$

Using $y = x^c(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots)$ and its first two derivatives, the expansions for

$$xy'' = \ldots\ldots\ldots\ldots$$
$$2y' = \ldots\ldots\ldots\ldots$$
$$xy' = \ldots\ldots\ldots\ldots$$
$$-2y = \ldots\ldots\ldots\ldots$$

Method as before.

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$$\begin{aligned} xy'' &= a_0c(c-1)x^{c-1} + a_1(c+1)cx^c + a_2(c+2)(c+1)x^{c+1} + \ldots \\ &\quad + a_r(c+r)(c+r-1)x^{c+r-1} + \ldots \\ 2y' &= 2a_0cx^{c-1} + 2a_1(c+1)x^c + 2a_2(c+2)x^{c+1} + 2a_3(c+3)x^{c+2} \\ &\quad + \ldots + 2a_r(c+r)x^{c+r-1} + \ldots \\ xy' &= a_0cx^c + a_1(c+1)x^{c+1} + a_2(c+2)c^{c+2} + \ldots \\ &\quad + a_r(c+r)x^{c+r} + \ldots \\ -2y &= -2a_0x^c - 2a_1x^{c+1} - 2a_2x^{c+2} - 2a_3x^{c+3} - \ldots \\ &\quad - 2a_rx^{c+r} - \ldots \end{aligned}$$

From which, the indicial equation is ....

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$$\boxed{a_0(c^2 + c) = 0}$$

i.e. equating the coefficient of the lowest power of x, (x^{c-1}), to zero.

$a_0 \neq 0 \qquad \therefore\ c = 0 \ \text{ or } \ -1$

Also, from the expansions, the total coefficient of x^c gives

$$a_1 = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_1 = \frac{-a_0(c-2)}{(c+1)(c+2)}}$$

From the terms in x^c, all four expansions are involved, so we can form the recurrence relation from the coefficient of x^{c+r}.

$$a_{r+1} = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_{r+1} = \frac{-a_r(c+r-2)}{(c+r+1)(c+r+2)}}$$

Because

$$a_{r+1}(c+r+1)(c+r) + 2a_{r+1}(c+r+1) + a_r(c+r) - 2a_r = 0$$
$$a_{r+1}(c+r+1)(c+r+2) + a_r(c+r-2) = 0$$
$$\therefore\ a_{r+1} = \frac{-a_r(c+r-2)}{(c+r+1)(c+r+2)} \qquad r \geq 0$$
$$\therefore\ a_2 = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_2 = \frac{a_0(c-1)(c-2)}{(c+1)(c+2)^2(c+3)}}$$

and, from the recurrence relation, when $r = 2$

$$a_3 = \ldots\ldots\ldots\ldots$$

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$$\boxed{a_3 = \frac{-a_0c(c-1)(c-2)}{(c+1)(c+2)^2(c+3)^2(c+4)}}$$

$$\therefore\ y = a_0x^c\left\{1 - \frac{c-2}{(c+1)(c+2)}x + \frac{(c-1)(c-2)}{(c+1)(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+1)(c+2)^2(c+3)^2(c+4)}x^3 + \ldots\right\}$$

From the indicial equation above, the values of c are 0 and -1.

Putting $c = 0$, we have one solution

$$y = u = \ldots\ldots\ldots\ldots$$

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$$\boxed{y = u = a_0\left\{1 + x + \frac{x^2}{6}\right\}}$$

Note that coefficients after the x^2 term are zero, because of the factor c in the numerator.

Putting $c = -1$, we soon find that $\ldots\ldots\ldots\ldots$

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coefficients become infinite, because of the factor $(c+1)$ in the denominator.

Therefore, we substitute $a_0 = k(c - c_1) = k(c - [-1]) = k(c+1)$.

$$\therefore\ y = k(c+1)x^c\left\{1 - \frac{c-2}{(c+1)(c+2)}x + \frac{(c-1)(c-2)}{(c+1)(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+1)(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

$$= kx^c\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

Now, putting $c = -1$:

$$y = \dots\dots\dots\dots$$

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$$y = kx^{-1}\left\{3x + 3x^2 + \frac{x^3}{2}\right\}$$

All subsequent terms are zero, since the numerators all contain a factor $(c+1)$.

$$\therefore\ y = v = \left\{3 + 3x + \frac{x^2}{2}\right\}$$

is a solution.

A solution is also given by $\dfrac{\partial y}{\partial c} = 0$.

So, starting from

$$y = kx^c\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$

$$\frac{\partial y}{\partial c} = kx^c \ln x\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \frac{c(c-1)(c-2)}{(c+2)^2(c+3)^2(c+4)}x^3 + \dots\right\}$$
$$+ kx^c\frac{\partial}{\partial c}\left\{(c+1) - \frac{c-2}{c+2}x + \frac{(c-1)(c-2)}{(c+2)^2(c+3)}x^2 - \dots\right\}$$

▶

We now have to determine the partial derivative of each term.

$$\frac{\partial}{\partial c}(c+1) = 1$$
$$\frac{\partial}{\partial c}\left\{\frac{c-2}{c+2}\right\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial}{\partial c}\left\{\frac{c-2}{c+2}\right\} = \frac{4}{(c+2)^2}}$$

Now we have to differentiate $\dfrac{(c-1)(c-2)}{(c+2)^2(c+3)}$

Let $t = \dfrac{(c-1)(c-2)}{(c+2)^2(c+3)}$

$$\therefore \ \ln t = \ln(c-1) + \ln(c-2) - 2\ln(c+2) - \ln(c+3)$$
$$\therefore \ \frac{1}{t}\frac{\partial t}{\partial c} = \frac{1}{c-1} + \frac{1}{c-2} - \frac{2}{c+2} - \frac{1}{c+3}$$
$$\therefore \ \frac{\partial t}{\partial c} = \frac{(c-1)(c-2)}{(c+2)^2(c+3)}\left\{\frac{1}{c-1} + \frac{1}{c-2} - \frac{2}{c+2} - \frac{1}{c+3}\right\}$$

$\therefore$ when $c = -1$, $\dfrac{\partial}{\partial c}(c+1) = 1$

$$\frac{\partial}{\partial c}\left\{\frac{c-2}{c+2}\right\} = 4$$
$$\frac{\partial}{\partial c}\left\{\frac{(c-1)(c-2)}{(c+2)^2(c+3)}\right\} = \ldots\ldots\ldots\ldots$$

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$$\boxed{-10}$$

Therefore, when $c = -1$:

$$\frac{\partial y}{\partial c} = kx^{-1}\ln x\left\{0 + 3x + 3x^2 + \frac{x^3}{2} + \ldots\right\}$$
$$+ kx^{-1}\{1 - 4x - 10x^2 + \ldots\}$$

$\therefore$ Another solution is

$$y = w = C\left\{\ln x\left(3 + 3x + \frac{x^2}{2} + \ldots\right) + x^{-1}(1 - 4x - 10x^2 + \ldots)\right\}$$

▶

Now we have a problem, for we seem to have three separate series solutions for a second-order differential equation.

(a) $y = u = A\left(1 + x + \dfrac{x^2}{6}\right)$

(b) $y = v = B\left(3 + 3x + \dfrac{x^2}{2}\right)$

(c) $y = w = C\left\{\ln x\left(3 + 3x + \dfrac{x^2}{2} + \ldots\right) + x^{-1}(1 - 4x - 10x^3 + \ldots)\right\}$

But (b) is clearly a simple multiple of (a) and thus not a distinct solution. So finally, we have just (a) and (c).

i.e. $\quad y = u = A\left(1 + x + \dfrac{x^2}{6}\right)$

and $\quad y = w = B\left\{\ln\, x\left(3 + 3x + \dfrac{x^2}{2} + \ldots\right) + x^{-1}(1 - 4x - 10x^3 + \ldots)\right\}$

The complete solution is then $\quad y = u + w$

In general if $c_1 - c_2 = n$ where n is a non-zero integer the solution is of the form:

$$y = (1 + k\ln x)x^{c_1}\{a_0 + a_1x + a_2x^2 + \ldots\} + x^{c_2}\{b_0 + b_1x + b_2x^2 + \ldots\}$$

Finally we have just one more variation to the list in Frames 67 and 68, so move on

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Example 5

Solve the equation $\quad xy'' + y' - xy = 0.$

Start off as before and build up expansions for the terms in the left-hand side of the equation.

$$xy'' = \ldots\ldots\ldots\ldots$$
$$y' = \ldots\ldots\ldots\ldots$$
$$-xy = \ldots\ldots\ldots\ldots$$

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$$\begin{aligned}
xy'' &= a_0c(c-1)x^{c-1} + a_1(c+1)cx^c + a_2(c+2)(c+1)x^{c+1} + \ldots \\
&\qquad + a_r(c+r)(c+r-1)x^{c+r-1} + \ldots \\
y' &= a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \ldots \\
&\qquad + a_r(c+r)x^{c+r-1} + \ldots \\
-xy &= \qquad -a_0x^{c+1} - a_1x^{c+2} - \ldots \\
&\qquad - a_rx^{c+r+1} - \ldots
\end{aligned}$$

The indicial equation, therefore, gives $c = \ldots\ldots\ldots\ldots$

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$$\boxed{c = 0 \quad \text{(twice)}}$$

Because $a_0\{c(c-1)+c\} = 0 \quad a_0 \neq 0 \quad \therefore\ c^2 = 0 \quad \therefore\ c = 0$ (twice)

Coefficient of x^c gives ....

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$$\boxed{a_1 = 0}$$

$[x^c]$: $\quad a_1(c^2 + c + c + 1) = 0 \quad \therefore\ a_1(c+1)^2 = 0 \quad \therefore\ a_1 = 0$

$[x^{c+1}]$: This involves all three expansions and from this point, we can use the general recurrence relation.

$[x^{c+r-1}]$: $\quad a_r\{(c+r)(c+r-1) + (c+r)\} - a_{r-2} = 0$

$$\therefore\ a_r(c+r)^2 = a_{r-2} \quad \therefore\ a_r = \frac{a_{r-2}}{(c+r)^2}$$

$$\therefore\ y = \text{............}$$

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$$\boxed{y = x^c\left\{a_0 + \frac{a_0}{(c+2)^2}x^2 + \frac{a_0}{(c+2)^2(c+4)^2}x^4 + \dots\right\}}$$

i.e. $\quad y = a_0x^c\left\{1 + \dfrac{x^2}{(c+2)^2} + \dfrac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$

$\therefore$ *When* $c = 0$

$$y = u = A\left\{1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} + \dots\right\} \tag{1}$$

▶

This is one solution. Another is given by $v = \dfrac{\partial y}{\partial c}$

$$\frac{\partial y}{\partial c} = a_0 x^c \ln x \left\{1 + \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$$

$$+ a_0 x^c \frac{\partial}{\partial c}\left\{1 + \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$$

Now $\quad \dfrac{\partial}{\partial c}(1) = 0; \quad \dfrac{\partial}{\partial c}\left\{\dfrac{1}{(c+2)^2}\right\} = \dfrac{-2}{(c+2)^3}$

Let $t = \dfrac{1}{(c+2)^2(c+4)^2} \quad \therefore \quad \ln t = -2\ln(c+2) - 2\ln(c+4)$

$\therefore \quad \dfrac{1}{t}\dfrac{\partial t}{\partial c} = \dfrac{-2}{c+2} - \dfrac{2}{c+4} \quad \therefore \quad \dfrac{\partial t}{\partial c} = \dfrac{-2}{(c+2)^2(c+4)^2}\left\{\dfrac{1}{c+2} + \dfrac{1}{c+4}\right\}$

$$\therefore \quad \frac{\partial y}{\partial c} = a_0 x^c \ln x \left\{1 + \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots\right\}$$

$$+ a_0 x^c \left\{0 - \frac{2x^2}{(c+2)^3} - \frac{4x^4(c+3)}{(c+2)^3(c+4)^3} + \dots\right\}$$

$\therefore$ When $c = 0$

$$y = v = \dots\dots\dots\dots$$

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$$\boxed{y = v = B\left\{\ln x\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \times 4^2} + \dots\right) - \frac{x^2}{2^2} - \frac{3x^4}{2^3 \times 4^2} + \dots\right\}} \qquad (2)$$

So our two solutions are $y = u$ (at 1) and $y = v$ (at 2). The complete solution is therefore $y = u + v$.

In general if $c_1 = c_2 = c$ the solution is of the form

$$y = (1 + k\ln x)x^c\{a_0 + a_1 x + a_2 x^2 + \dots\} + x^c\{b_1 x + b_2 x^2 + \dots\}$$

Summary

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Let us now summarise the four types of procedures in the method of Frobenius that we have covered.

(a) Assume a series of the form

$$y = x^c(a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots)$$

(b) Indicial equation gives $c = c_1$ and $c = c_2$.

(c) *Case 1.* c_1 and c_2 differ by a quantity *not an integer*. Substitute $c = c_1$ and $c = c_2$ in the series for y.

(d) *Case 2.* c_1 and c_2 differ by *an integer* and make a coefficient *indeterminate* with $c = c_1$. Substitution of $c = c_1$ gives the complete solution.

(e) *Case 3.* c_1 and c_2 $(c_1 < c_2)$ differ by *an integer* and make a coefficient *infinite* for $c = c_1$. Replace a_0 by $k\,(c - c_1)$. Put $c = c_1$ in the new series for y and for $\dfrac{\partial y}{\partial c}$.

In general if $c_1 - c_2 = n$ where n is a non-zero integer, the solution is of the form

$$y = (1 + k\ln x)x^{c_1}\{a_0 + a_1x + a_2x^2 + \ldots\} + x^{c_2}\{b_0 + b_1x + b_2x^2 + \ldots\}$$

(f) *Case 4.* c_1 and c_2 *equal*. Substitute $c = c_1$ in the series for y and for $\dfrac{\partial y}{\partial c}$. Make the substitution after differentiating.

In general if $c_1 = c_2 = c$, the solution is of the form

$$y = (1 + k\ln x)x^c\{a_0 + a_1x + a_2x^2 + \ldots\} + x^c\{b_1x + b_2x^2 + \ldots\}$$

Make a note of this summary for future reference

Bessel's equation

88

A second-order differential equation that occurs frequently in branches of technology is of the form

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

where v is a real constant.

Starting with $y = x^c(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots)$ and proceeding as before, we obtain

$$c = \pm v \quad \text{and} \quad a_1 = 0$$

The recurrence relation is $\quad a_r = \dfrac{a_{r-2}}{v^2 - (c+r)^2}$ for $r \geq 2$.

It follows that $a_1 = a_3 = a_5 = a_7 = \ldots = 0$

and that $\quad a_2 = \ldots\ldots\ldots\ldots; \quad a_4 = \ldots\ldots\ldots\ldots; \quad a_6 = \ldots\ldots\ldots\ldots$

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$$a_2 = \frac{a_0}{v^2 - (c+2)^2}; \quad a_4 = \frac{a_0}{\left[v^2 - (c+2)^2\right]\left[v^2 - (c+4)^2\right]};$$
$$a_6 = \frac{a_0}{\left[v^2 - (c+2)^2\right]\left[v^2 - (c+4)^2\right]\left[v^2 - (c+6)^2\right]}$$

$\therefore$ *When* $c = +v$ $\quad a_2 = \ldots\ldots\ldots\ldots;\quad a_4 = \ldots\ldots\ldots\ldots$

$a_6 = \ldots\ldots\ldots\ldots;\quad a_r = \ldots\ldots\ldots\ldots$

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$$a_2 = \frac{-a_0}{2^2(v+1)}; \qquad a_4 = \frac{a_0}{2^4 \times 2(v+1)(v+2)}$$
$$a_6 = \frac{-a_0}{2^6 \times 3!(v+1)(v+2)(v+3)}$$
$$a_r = \frac{(-1)^{r/2}a_0}{2^r \times (r/2)!(v+1)(v+2)\ldots(v+r/2)} \text{ for } r \text{ even}$$

The resulting series solution is therefore

$$y = u = \ldots\ldots\ldots\ldots$$

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$$y = u = Ax^v\left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \ldots\right\}$$

This is valid provided v is not a negative integer.

Similarly, *when* $c = -v$

$$y = w = Bx^{-v}\left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \ldots\right\}$$

This is valid provided v is not a positive integer.

Except for these two restrictions, the complete solution of Bessel's equation is therefore $y = u + w$ with the two arbitrary constants A and B.

Bessel functions

92

It is convenient to present the two results obtained above in terms of gamma functions, remembering that for $x > 0$

$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(x+2) = (x+1)\Gamma(x+1) = (x+1)x\Gamma(x)$$
$$\Gamma(x+3) = (x+2)\Gamma(x+2) = (x+2)(x+1)x\Gamma(x), \quad \text{etc.}$$

If, at the same time, we assign to the arbitrary constant a_0 the value $\dfrac{1}{2^{\nu}\Gamma(\nu+1)}$, then we have, for $c = \nu$

$$a_2 = \frac{a_0}{\nu^2 - (c+2)^2} = \frac{a_0}{(\nu - c - 2)(\nu + c + 2)} = \frac{a_0}{-2(2\nu+2)}$$
$$= \frac{-1}{2^2(\nu+1)} \cdot \frac{1}{2^{\nu}\Gamma(\nu+1)} = \frac{-1}{2^{\nu+2}(1!)\Gamma(\nu+2)}$$

Similarly $\qquad a_4 = \ldots\ldots\ldots\ldots$

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$$\boxed{a_4 = \frac{1}{2^{\nu+4}(2!)\Gamma(\nu+3)}}$$

Because

$$a_4 = \frac{a_2}{\nu^2 - (c+4)^2} = \frac{a_2}{(\nu - c - 4)(\nu + c + 4)} = \frac{a_2}{-4(2\nu+4)}$$
$$= \frac{-1}{2^3(\nu+2)} \cdot \frac{-1}{2^{\nu+2}(1!)\Gamma(\nu+2)} = \frac{1}{2^{\nu+4}(2!)\Gamma(\nu+3)}$$

and $\quad a_6 = \ldots\ldots\ldots\ldots$

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$$\boxed{a_6 = \frac{-1}{2^{\nu+6}(3!)\Gamma(\nu+4)}}$$

We can see the pattern taking shape.

$$a_r = \frac{(-1)^{r/2}}{2^{\nu+r}\left(\frac{r}{2}!\right)\Gamma\left(\nu + \frac{r}{2} + 1\right)} \quad \text{for } r \text{ even.} \qquad \therefore \text{ Put } r = 2k$$

The result then becomes

$$a_{2k} = \ldots\ldots\ldots\ldots$$

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$$a_{2k} = \frac{(-1)^k}{2^{\nu+2k}(k!)\Gamma(\nu+k+1)} \quad k = 1, 2, 3, \ldots$$

Therefore, we can write the new form of the series for y as

$$y = x^\nu \left\{ \frac{1}{2^\nu \Gamma(\nu+1)} - \frac{x^2}{2^{\nu+2}(1!)\Gamma(\nu+2)} + \frac{x^4}{2^{\nu+4}(2!)\Gamma(\nu+3)} - \ldots \right\}$$

This is called the *Bessel function of the first kind of order* ν and is denoted by $J_\nu(x)$.

$$\therefore \; J_\nu(x) = \left(\frac{x}{2}\right)^\nu \left\{ \frac{1}{\Gamma(\nu+1)} - \frac{x^2}{2^2(1!)\Gamma(\nu+2)} + \frac{x^4}{2^4(2!)\Gamma(\nu+3)} - \ldots \right\}$$

This is valid provided ν is not . . . .

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a negative integer

– otherwise some of the terms would become infinite.

If we take the other value for c, i.e. $c = -\nu$, the corresponding result becomes

$$J_{-\nu}(x) = \ldots\ldots\ldots$$

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$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \left\{ \frac{1}{\Gamma(1-\nu)} - \frac{x^2}{2(1!)\Gamma(2-\nu)} + \frac{x^4}{2^2(2!)\Gamma(3-\nu)} - \ldots \right\}$$

provided that ν is not a positive integer.

In general terms

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(\nu+k+1)}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-\nu+1)}$$

The convergence of the series for all values of x can be established by the normal ratio test.

$J_\nu(x)$ and $J_{-\nu}(x)$ are two independent solutions of the original equation. Hence, the complete solution is

$$y = AJ_\nu(x) + BJ_{-\nu}(x)$$

where A and B are constants.

Make a note of the expressions for $J_\nu(x)$ and $J_{-\nu}(x)$.
Then on to the next frame

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Some Bessel functions are commonly used and are worthy of special mention. This arises when v is a positive integer, denoted by n.

$$\therefore \; J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(n+k+1)}$$

From our work on gamma functions, $\Gamma(k+1) = k!$ for $k = 0, 1, 2, \ldots$

$$\therefore \; \Gamma(n+k+1) = (n+k)!$$

and the result above then becomes

$$J_n(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)(n+k)!}}$$

We have seen that $J_v(x)$ and $J_{-v}(x)$ are two solutions of Bessel's equation. When v and $-v$ are not integers, the two solutions are independent of each other. Then $y = AJ_v(x) + BJ_{-v}(x)$.

When, however, $v = n$ (integer), then $J_n(x)$ and $J_{-n}(x)$ are not independent, but are related by $J_{-n}(x) = (-1)^n J_n(x)$. This can be shown by referring once again to our knowledge of gamma functions.

$$\Gamma(x+1) = x\Gamma(x) \qquad \therefore \; \Gamma(x) = \frac{\Gamma(x+1)}{x}$$

and for negative integral values of x, or zero, $\Gamma(x)$ is infinite.

From the previous result:

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-v+1)} \qquad k = 0, 1, 2, \ldots$$

Let us consider the gamma function $\Gamma(k-v+1)$ in the denominator and let v approach closely to a positive integer n.

Then $\quad \Gamma(k-v+1) \to \Gamma(k-n+1)$.

When $k-n+1 \leq 0$, i.e. when $k \leq (n-1)$, then $\Gamma(k-n+1)$ is infinite.

The first finite value of $\Gamma(k-n+1)$ occurs for $k = n$.

When values of $\Gamma(k-v+1)$ are infinite the coefficients of $J_{-v}(x)$ are

$\ldots\ldots\ldots\ldots$

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$\boxed{\text{zero}}$

The series, therefore, starts at $k = n$

$$\therefore\ J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-n+1)}$$

$$= \sum_{k=n}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n}(k!)\Gamma(k-n+1)} \qquad \text{Put } k = p+n$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^{p+n} x^{2p+n}}{2^{2p+n}(k!)(k-n)!}$$

$$= (-1)^n \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p+n}}{2^{2p+n}(p!)(p+n)!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{p=0}^{\infty} \frac{(-1)^p x^{2p}}{2^{2p}(p!)(p+n)!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)(k+n)!}$$

$$\therefore\ J_{-n}(x) = (-1)^n J_n(x)$$

So, after all that, the series for $J_n(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \ldots\ldots\ldots\ldots\right\}}$$

From this we obtain two commonly used functions

$$J_0(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots}$$

and $\qquad J_1(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 + \ldots\right\}}$$

Bessel functions for a range of values of n and x are tabulated in published lists of mathematical data. Of these, $J_0(x)$ and $J_1(x)$ are most commonly used.

Graphs of Bessel functions $J_0(x)$ and $J_1(x)$

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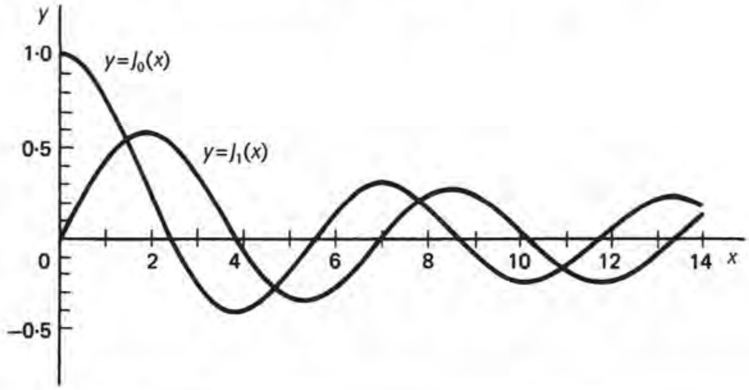

Legendre's equation

105

Another equation of special interest in engineering applications is Legendre's equation of the form

$$(1-x^2)y'' - 2xy' + k(k+1)y = 0$$

where k is a real constant.

This may be solved by the Frobenius method as before. In this case, the indicial equation gives $c = 0$ and $c = 1$, and the two corresponding solutions are

(a) $c = 0$: $\quad y = a_0\left\{1 - \dfrac{k(k+1)}{2!}x^2 + \dfrac{k(k-2)(k+1)(k+3)}{4!}x^4 - \dots\right\}$

(b) $c = 1$: $\quad y = a_1\left\{x - \dfrac{(k-1)(k+2)}{3!}x^3 + \dfrac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \dots\right\}$

where a_0 and a_1 are the usual arbitrary constants

Legendre polynomials

When k is an integer (n), one of the solution series terminates after a finite number of terms. The resulting polynomial in x, denoted by $P_n(x)$, is called a *Legendre polynomial*, with a_0 or a_1 being chosen so that the polynomial has unit value when $x = 1$.

For example $\qquad P_2(x) = \dots\dots\dots\dots$

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$$\boxed{P_2(x) = \frac{1}{2}(3x^2 - 1)}$$

Because, in $P_2(x)$, $n = k = 2$

$$\therefore\ y = a_0\left\{1 - \frac{2 \times 3}{2!}x^2 + 0 + 0 + \dots\right\}$$
$$= a_0\{1 - 3x^2\}$$

The constant a_0 is then chosen to make $y = 1$ when $x = 1$

i.e. $1 = a_0(1 - 3)$ $\quad\therefore\ a_0 = -\frac{1}{2}$

$$\therefore\ P_2(x) = -\tfrac{1}{2}(1 - 3x^2) = \tfrac{1}{2}(3x^2 - 1)$$

Similarly $\qquad P_3(x) = \dots\dots\dots\dots$

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$$\boxed{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$$

Here $n = k = 3$

$$\therefore\ y = a_1\left\{x - \frac{2 \times 5}{3!}x^3 + 0 + 0 + \dots\right\}$$
$$= a_1\left\{x - \frac{5x^3}{3}\right\}$$

$y = 1$ when $x = 1$ $\quad\therefore\ a_1\left(1 - \frac{5}{3}\right) = 1 \quad\therefore\ a_1 = -\frac{3}{2}$

$$\therefore\ P_3(x) = -\frac{3}{2}\left(x - \frac{5x^3}{3}\right) = \frac{1}{2}(5x^3 - 3x)$$

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Rodrigue's formula and the generating function

Legendre polynomials can be derived by using *Rodrigue's formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$$

so using this formula

$$P_4(x) = \dots\dots\dots\dots$$

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$$\boxed{P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)}$$

Because

$$\begin{aligned}
P_4(x) &= \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2 - 1)^4 \\
&= \frac{1}{384}\frac{d^4}{dx^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1) \\
&= \frac{1}{384}\frac{d^3}{dx^3}(8x^7 - 24x^5 + 24x^3 - 8x) \\
&= \frac{1}{384}\frac{d^2}{dx^2}(56x^6 - 120x^4 + 72x^2 - 8) \\
&= \frac{1}{384}\frac{d}{dx}(336x^5 - 480x^3 + 144x) \\
&= \frac{1}{384}(1680x^4 - 1440x^2 + 144) \\
&= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}$$

In addition to Rodrigue's formula, the function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |t| < 1$$

is called the *generating function* for Legendre polynomials and can be used to obtain some of their properties. For example using this generating function we find that

$$P_n(1) = \ldots\ldots\ldots\ldots$$

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$$\boxed{P_n(1) = 1}$$

Because

When $x = 1$ the generating function becomes

$$\frac{1}{\sqrt{1-2t+t^2}} = \sum_{n=0}^{\infty} P_n(1)t^n, \qquad |t| < 1$$

Noting that $\dfrac{1}{\sqrt{1-2t+t^2}} = \dfrac{1}{\sqrt{(1-t)^2}} = \dfrac{1}{1-t} = (1-t)^{-1}$, the left-hand side is expanded by the binomial theorem to give

$$(1-t)^{-1} = 1 + t + t^2 + t^3 + \ldots = \sum_{n=0}^{\infty} t^n.$$

Therefore $\displaystyle\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n$ and so $P_n(1) = 1$

By a similar reasoning

$$P_n(-1) = \ldots\ldots\ldots\ldots$$

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$$\boxed{P_n(-1) = (-1)^n}$$

Because

When $x = -1$ the generating function becomes

$$\frac{1}{\sqrt{1+2t+t^2}} = \sum_{n=0}^{\infty} P_n(-1)t^n$$

Noting that $\dfrac{1}{\sqrt{1+2t+t^2}} = \dfrac{1}{\sqrt{(1+t)^2}} = \dfrac{1}{1+t} = (1+t)^{-1}$, the left-hand side is expanded by the binomial theorem to give

$(1+t)^{-1} = 1 - t + t^2 - t^3 + \ldots = \displaystyle\sum_{n=0}^{\infty}(-1)^n t^n$. Therefore

$\displaystyle\sum_{n=0}^{\infty}(-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1)t^n$ and so $P_n(-1) = (-1)^n$

Legendre's equation, whose solutions are expressed in terms of Legendre polynomials, is an example of a particular class of differential equations referred to as Sturm–Liouville systems. In the following frames we shall look at such systems more closely.

So on to the next frame

Sturm–Liouville systems

112

A boundary value problem that is described by a differential equation of the general form

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0 \quad \text{for} \quad a \le x \le b \text{ and } r(x) > 0$$

where the boundary conditions can be written in the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

is called a **Sturm–Liouville** system. Solutions of such a system are in the form of an infinite sequence of *eigenfunctions* y_n, each corresponding to an *eigenvalue* λ_n of the system for $n = 0, 1, 2, \ldots$.

For example, consider the differential equation

$$y'' + \lambda y = 0 \quad \text{for} \quad 0 \le x \le 5$$

where here, $a = 0$ and $b = 5$. Also

$$y(0) = 0 \quad \text{and} \quad y(5) = 0$$

By comparing this equation with the general form given above we can see that

$$p(x) = \ldots\ldots\ldots\ldots; \quad q(x) = \ldots\ldots\ldots\ldots; \quad r(x) = \ldots\ldots\ldots\ldots;$$
$$\alpha_2 = \ldots\ldots\ldots\ldots; \quad \beta_2 = \ldots\ldots\ldots\ldots$$

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$$\boxed{p(x) = 1; \quad q(x) = 0; \quad r(x) = 1; \quad \alpha_2 = 0; \quad \beta_2 = 0}$$

Because

By performing the differentiation on the left-hand term of $(p(x)y')' + (q(x) + \lambda r(x))y = 0$ we find that the differential equation can be written as

$$p(x)y'' + p'(x)y' + (q(x) + \lambda r(x))y = 0$$

By inspection, comparing this form with the differential equation $y'' + \lambda y = 0$ it is easily seen that $p(x) = 1$, $q(x) = 0$, $r(x) = 1$ and comparing boundary conditions gives $\alpha_2 = 0$ and $\beta_2 = 0$.

To solve the equation $y'' + \lambda y = 0$ we use the auxiliary equation $m^2 + \lambda = 0$ which has solutions $m = \pm j\sqrt{\lambda}$ (refer to *Engineering Mathematics (Fifth Edition)*, page 1077). This means that the solution can be written in the form

$$y = A \sin \ldots\ldots\ldots\ldots + B \cos \ldots\ldots\ldots\ldots$$

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$$\boxed{y = A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x}$$

Because

When the solutions to the auxiliary equation are of the form $m = \alpha \pm j\beta$ the solution to the differential equation is of the form

$$y = e^{\alpha x}(A \sin \beta x + B \cos \beta x) \quad \text{and here} \quad \alpha = 0 \quad \text{and} \quad \beta = \sqrt{\lambda}$$

Applying the boundary condition $y(0) = 0$ then $B = \ldots\ldots\ldots\ldots$

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$$\boxed{B = 0}$$

Because

$y = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$ and so $y(0) = A\sin 0 + B\cos 0 = B = 0$. Therefore $y = A\sin\sqrt{\lambda}x$

Applying the boundary condition $y(5) = 0$ then

$$\lambda = \ldots\ldots\ldots\ldots$$

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$$\boxed{\lambda = \frac{n^2\pi^2}{25}}$$

Because

$y = A\sin\sqrt{\lambda}x$ therefore $y(5) = A\sin\sqrt{\lambda}5 = 0$. If $A = 0$ the solution reduces to the trivial solution $y = 0$. For a non-trivial solution $\sin\sqrt{\lambda}5 = 0$ and so $\sqrt{\lambda}5 = n\pi$, $n = 0, 1, 2, 3, \ldots$. This means that

$$\sqrt{\lambda} = \frac{n\pi}{5} \quad \text{and so} \quad \lambda = \frac{n^2\pi^2}{25}$$

There is an infinity of eigenvalues, the nth eigenvalue being denoted by λ_n where $\lambda_n = \dfrac{n^2\pi^2}{25}$ and to each eigenvalue there is an eigenvector solution $y_n = A_n\sin\dfrac{n\pi x}{5}$.

117 Orthogonality

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$ and

$$\int_a^b f(x)g(x)\,dx = 0$$

then we say that the two functions are mutually **orthogonal**. If, on the other hand, a third function $w(x) > 0$ exists such that

$$\int_a^b f(x)g(x)w(x)\,dx = 0$$

then we say that $f(x)$ and $g(x)$ are mutually orthogonal *with respect to the weight function* $w(x)$.

One important property of the solutions to a Sturm–Liouville system is that the solutions are all mutually orthogonal with respect to the weight function $r(x)$. For instance, in the previous example the individual solutions were given as

$$y_n = A_n\sin\frac{n\pi x}{5} \quad \text{where} \quad r(x) = 1$$

and so if $m \ne n$

$$\int_0^5 y_m(x)y_n(x)r(x)\,dx = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_0^5 y_m(x)y_n(x)r(x)\,\mathrm{d}x = 0}$$

Because

$$\begin{aligned}\int_0^5 y_m(x)y_n(x)r(x)\,\mathrm{d}x &= \int_0^5 A_m \sin\frac{m\pi x}{5} A_n \sin\frac{n\pi x}{5}\,\mathrm{d}x \qquad \text{where } r(x)=1\\ &= A_m A_n \int_0^5 \sin\frac{m\pi x}{5}\sin\frac{n\pi x}{5}\,\mathrm{d}x\\ &= \frac{A_m A_n}{2}\int_0^5 \left(\cos\frac{(m-n)\pi x}{5} - \cos\frac{(m+n)\pi x}{5}\right)\mathrm{d}x\\ &= \frac{A_m A_n}{2}\left[-\frac{5}{(m-n)\pi}\sin\frac{(m-n)\pi x}{5}\right.\\ &\qquad \left. + \frac{5}{(m+n)\pi}\sin\frac{(m+n)\pi x}{5}\right]_0^5 \quad \text{provided } m \neq n\\ &= 0\end{aligned}$$

Summary

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1 A Sturm–Liouville system is a differential equation of the form

$$p(x)y'' + p'(x)y' + (q(x) + \lambda r(x))y = 0 \quad \text{for}$$
$$a \le x \le b \text{ and } r(x) > 0$$

where the boundary conditions can be written in the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0$$

2 Solutions y_n to a Sturm–Liouville system are called eigenvectors, each corresponding to an eigenvalue λ_n for $n = 0, 1, 2, \ldots$.

3 The solutions y_n are mutually orthogonal with respect to the weighting $r(x)$. That is

$$\int_a^b y_m(x)y_n(x)r(x)\,\mathrm{d}x = 0 \qquad (m \neq n)$$

Keep going

Legendre's equation revisited

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The equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is Legendre's equation and has Legendre polynomials as solutions. That is

$$y_n = P_n(x) \qquad \text{where } P_n(1) = 1 \text{ and } P_n(-1) = (-1)^n$$

This equation is an example of a Sturm–Liouville system $p(x)y'' + p'(x)y' + (q(x) + \lambda r(x))y = 0$ with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0 \quad \text{where}$$

$$p(x) = \ldots\ldots\ldots\ldots; \quad q(x) = \ldots\ldots\ldots\ldots; \quad r(x) = \ldots\ldots\ldots\ldots;$$
$$\alpha_1, \alpha_2 = \ldots\ldots\ldots\ldots; \quad \beta_1, \beta_2 = \ldots\ldots\ldots\ldots$$

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$$p(x) = 1 - x^2; \quad q(x) = 0; \quad r(x) = 1; \quad \alpha_1, \alpha_2 = 1, 0; \quad \beta_1, \beta_2 = 1, 0$$

Consequently, Legendre polynomials are mutually orthogonal. That is, if $m \neq n$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \ldots\ldots\ldots\ldots$$

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$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

Polynomials as a finite series of Legendre polynomials

Many differential equations cannot be solved by the normal analytical means and solution by power series provides a powerful tool in many situations. Furthermore, any polynomial can be written as a finite series of Legendre polynomials.

Example 1

Show that $f(x) = x^2$ can be written as a series of Legendre polynomials. Assume that

$$f(x) = x^2 = \sum_{n=0}^{\infty} a_n P_n(x), \text{ then}$$

$$\begin{aligned} x^2 &= a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \ldots \\ &= a_0(1) + a_1(x) + a_2\frac{3x^2-1}{2} + a_3\frac{5x^3-3x}{2} + \ldots \end{aligned}$$

Since the left-hand side is a polynomial of degree 2 then any Legendre polynomial on the right-hand side containing powers of x greater than 2 must be excluded. Therefore $a_3 = a_4 = \ldots = 0$, so that

$$x^2 = a_0 - \frac{a_2}{2} + a_1x + \frac{3}{2}a_2x^2 \quad \text{giving} \quad a_2 = \frac{2}{3},\ a_1 = 0,\ a_0 - \frac{a_2}{2} = 0$$

therefore $a_0 = \frac{1}{3}$, and

$$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

Now you try one

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Example 2

The polynomial $1 + x + x^3$ can be written as a series of Legendre polynomials in the form

$$1 + x + x^3 = \ldots\ldots\ldots\ldots$$

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$$\boxed{1+x+x^3 = P_0(x)+\frac{8}{5}P_1(x)+\frac{2}{5}P_3(x)}$$

Because

$$\begin{aligned}1+x+x^3 &= a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+\dots\\ &= a_0(1)+a_1(x)+a_2\frac{3x^2-1}{2}+a_3\frac{5x^3-3x}{2}+\dots\end{aligned}$$

Since the left-hand side is a polynomial of degree 3 then any Legendre polynomial on the right-hand side containing powers of x greater than 3 must be excluded. Therefore $a_4 = a_5 = \dots = 0$, so that

$$1+x+x^3 = a_0-\frac{a_2}{2}+\left(a_1-\frac{3}{2}a_3\right)x+\frac{3}{2}a_2x^2+\frac{5}{2}a_3x^3$$

This gives $a_3 = \frac{2}{5}$, $a_2 = 0$, $a_1-\frac{3}{2}a_3 = 1$, $a_0-\frac{a_2}{2} = 1$ therefore $a_0 = 1$, and $a_1 = \frac{8}{5}$ so

$$1+x+x^3 = P_0(x)+\frac{8}{5}P_1(x)+\frac{2}{5}P_3(x)$$

As usual, the main points that we have covered in this Programme are listed in the **Revision summary** that follows. Read this in conjunction with the **Can You?** checklist and note any sections that may need further attention: refer back to the relevant parts of the Programme, if necessary. There will then be no trouble with the **Test exercise**. The set of **Further problems** provides an opportunity for further practice.

Revision summary 8 125

1 *Higher derivatives*

y	$y^{(n)}$
x^a	$\dfrac{a!}{(a-n)!}x^{a-n}$
e^{ax}	a^ne^{ax}
$\sin ax$	$a^n\sin\left(ax+\dfrac{n\pi}{2}\right)$
$\cos ax$	$a^n\cos\left(ax+\dfrac{n\pi}{2}\right)$
$\sinh ax$	$\dfrac{a^n}{2}\{[1+(-1)^n]\sinh ax+[1-(-1)^n]\cosh ax\}$
$\cosh ax$	$\dfrac{a^n}{2}\{[1-(-1)^n]\sinh ax+[1+(-1)^n]\cosh ax\}$

2 *Leibnitz theorem* — nth derivative of a product of functions.

If $y = uv$

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots$$
$$\dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}u^{(n-r)}v^{(r)} + \dots$$

i.e. $y^{(n)} = \sum_{r=0}^{\infty} {}^nC_r u^{(n-r)}v^{(r)}$.

$(uv)^{(n)}$ can be obtained by expanding $(u+v)^{(n)}$ using the binomial theorem where the 'powers' are interpreted as derivatives.

3 *Power series solution of second-order differential equations*

(a) *Leibnitz–Maclaurin method*

(1) Differentiate the equation n times by the Leibnitz theorem.
(2) Put $x = 0$ to establish a recurrence relation.
(3) Substitute $n = 1, 2, 3, \dots$ to obtain $y', y'', y''', \dots$ at $x = 0$.
(4) Substitute in Maclaurin's series and simplify where possible.

(b) *Frobenius' method*

Assume a series solution of the form

$$y = x^c\{a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots\} \quad a_0 \neq 0$$

(1) Differentiate the assumed series to find y' and y''.
(2) Substitute in the equation.
(3) Equate coefficients of corresponding powers of x on each side of the equation – usually written with zero on the right-hand side.
(4) Coefficient of the lowest power of x gives the *indicial equation* from which values of c are obtained, $c = c_1$ and $c = c_2$.

Case 1: c_1 and c_2 differ by a quantity *not an integer*. Substitute $c = c_1$ and $c = c_2$ in the series for y.

Case 2: c_1 and c_2 differ by an *integer* and make a coefficient *indeterminate* when $c = c_1$. Substitute $c = c_1$ to obtain the complete solution.

Case 3: c_1 and c_2 $(c_1 < c_2)$ differ by an *integer* and make a coefficient *infinite* when $c = c_1$. Replace a_0 by $k(c - c_1)$. Two independent solutions then obtained by putting $c = c_1$ in the new series for y and for $\frac{\partial y}{\partial c}$.

▶

In general if $c_1 - c_2 = n$ where n is a non-zero integer, the solution is of the form

$$y = (1 + k\ln x)x^{c_1}\{a_0 + a_1x + a_2x^2 + \dots\}$$
$$+ x^{c_2}\{b_0 + b_1x + b_2x^2 + \dots\}$$

Case 4: c_1 and c_2 are *equal*. Substitute $c = c_1$ in the series for y and for $\dfrac{\partial y}{\partial c}$. Make the substitution after differentiating. The second solution will consist of the product of the first solution and $\ln x$, together with a further series.

In general if $c_1 = c_2 = c$, the solution is of the form

$$y = (1 + k\ln x)x^{c}\{a_0 + a_1x + a_2x^2 + \dots\}$$
$$+ x^{c}\{b_1x + b_2x^2 + \dots\}$$

4 *Bessel's equation*

$$x^2y'' + xy' + (x^2 - \nu^2) = 0$$

where ν is a real constant.

Bessel functions: Express the two solutions obtained in terms of gamma functions.

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu\left\{\frac{1}{\Gamma(\nu+1)} - \frac{x^2}{2^2(1!)\Gamma(\nu+2)} + \frac{x^4}{2^4(2!)\Gamma(\nu+3)} - \dots\right\}$$

This is the *Bessel function of the first kind of order* ν – valid for ν not a negative integer.

Also $J_{-\nu}(x) = \left(\dfrac{x}{2}\right)^{-\nu}\left\{\dfrac{1}{\Gamma(1-\nu)} - \dfrac{x^2}{2(1!)\Gamma(2-\nu)} + \dfrac{x^4}{2^2(2!)\Gamma(3-\nu)} - \dots\right\}$

provided that ν is not a positive integer.

Complete solution is therefore $y = AJ_\nu(x) + BJ_{-\nu}(x)$.

When $\nu = n$ *(an integer)* $\qquad J_{-n}(x) = (-1)^nJ_n(x)$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left\{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)(n+3)!}\left(\frac{x}{2}\right)^6 + \dots\right\}$$

In particular

$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \dots$$

and

$$J_1(x) = \frac{x}{2}\left\{1 - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)(4!)}\left(\frac{x}{2}\right)^6 + \dots\right\}$$

▶

5 *Legendre's equation*

$$(1 - x^2)y'' - 2xy' + k(k+1)y = 0$$

where k is a real constant.

Solution by Frobenius gives

$$c = 0: \quad y = a_0\left\{1 - \frac{k(k+1)}{2!}x^2 + \frac{k(k-2)(k+1)(k+3)}{4!}x^4 - \dots\right\}$$

$$c = 1: \quad y = a_1\left\{x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \dots\right\}$$

When k *is an integer*, one series terminates. The resulting polynomial in x, $P_n(x)$, is a *Legendre polynomial*, with a_0 or a_1 being chosen so that the polynomial has unit value when $x = 1$.

6 *Rodrigue's formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n$$

Generating function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

7 *Sturm–Liouville systems*

$(p(x)y')' + (q(x) + \lambda r(x))y = 0$ for $a \le x \le b$ and $r(x) > 0$ with boundary conditions $\alpha_1 y(a) + \alpha_2 y'(a) = 0$ and $\beta_1 y(b) + \beta_2 y(b) = 0$

Solutions y_n to a Sturm–Liouville system are called eigenvectors, each corresponding to an eigenvalue λ_n for $n = 0, 1, 2, \dots$

8 *Orthogonality*

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$ and

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0$$

then the two functions are **orthogonal** to each other. If a function $w(x) > 0$ exists such that

$$\int_a^b f(x)g(x)w(x)\,\mathrm{d}x = 0$$

then $f(x)$ and $g(x)$ are orthogonal to each other *with respect to the weight function* $w(x)$.

The solutions of a Sturm–Liouville system y_n are mutually orthogonal with respect to the weighting $r(x)$. That is

$$\int_a^b y_m(x)y_n(x)r(x)\,\mathrm{d}x = 0 \qquad (m \ne n)$$

▶

9 *Legendre polynomials are mutually orthogonal*
If $m \neq n$ then

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

The orthogonality of the Legendre polynomials permits any polynomial to be written as a finite series of Legendre polynomials.

Can You?

Checklist 8

126

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	**Frames**
• Obtain the nth derivative of the exponential and circular and hyperbolic functions?	1 to 9
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply the Leibnitz theorem to derive the nth derivative of a product of expressions?	10 to 17
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply the Leibnitz–Maclaurin method of obtaining a series solution to a second-order homogeneous differential equation with constant coefficients?	18 to 36
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply Frobenius' method of obtaining a series solution to a second-order homogeneous differential equation for different cases of the indicial equation?	37 to 87
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply Frobenius' method to Bessel's equation to derive Bessel functions of the first kind?	88 to 104
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Apply Frobenius' method to Legendre's equation to derive Legendre polynomials?	104 to 107
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use Rodrigue's formula to derive Legendre polynomials and the generating function to obtain some of their properties?	108 to 111
Yes ☐ ☐ ☐ ☐ ☐ *No*	

- Recognise a Sturm–Liouville system and the orthogonality properties of its eigenfunctions? **112** to **121**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Write a polynomial in x as a finite series of Legendre polynomials? **122** to **124**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 8

127

1. If $y = e^{x^2+x}$, show that $y'' = y'(2x+1) + 2y$ and hence prove that $y^{(n+2)} = (2x+1)y^{(n+1)} + 2(n+1)y^{(n)}$.
2. Obtain a power series solution of the equation
   $(1+x^2)y'' - 3xy' - 5y = 0$
   up to and including the term in x^6.
3. Determine a series solution for each of the following.
   (a) $3xy'' + 2y' + y = 0$
   (b) $y'' + x^2y = 0$
   (c) $xy'' + 3y' - y = 0$.
4. Use Rodrigue's formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ to derive the Legendre polynomials $P_2(x)$ and $P_3(x)$, and show that $P_2(x)$ and $P_3(x)$ are orthogonal on $(-1, 1)$.
5. Write $f(x) = 1 - 2x^2$ as a series of Legendre polynomials.

Further problems 8

128

(a) *Use the Leibnitz theorem* for the following.

1. If $y = x^3e^{4x}$, determine $y^{(5)}$.
2. Find the nth derivative of $y = x^3e^{-x}$ for $n > 3$.
3. If $y = x^3(2x+1)^2$, find $y^{(4)}$.
4. Find the 6th derivative of $y = x^4 \cos x$.
5. If $y = e^{-x}\sin x$, obtain an expression for $y^{(4)}$.

▶

6 Determine $y^{(3)}$ when $y = x^4 \ln x$.

7 If $x^2y'' + xy' + y = 0$, show that

$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$$

8 If $y = (2x - \pi)^4 \sin\left(\frac{x}{2}\right)$, evaluate $y^{(6)}$ when $x = \pi/2$.

9 If $y = e^{-x}\cos x$, show that $y^{(4)} + 4y = 0$.

10 Find the $(2n)$th derivative of (a) $y = x^2 \sinh x$
(b) $y = x^3 \cosh x$.

11 If $y = (x^3 + 3x^2)e^{2x}$, determine an expression for $y^{(6)}$.

12 Find the nth derivative of $y = e^{-ax}\cos ax$ and hence determine $y^{(3)}$.

13 If $y = \dfrac{\sin x}{1 - x^2}$, show that

(a) $(1 - x^2)y'' - 4xy' - (1 + x^2)y = 0$

(b) $y^{(n+2)} - (n^2 + 3n + 1)y^{(n)} - n(n-1)y^{(n-2)} = 0$ at $x = 0$.

(b) *Use the Leibnitz–Maclaurin method* to determine series solutions for the following.

14 $(1 + x^2)y'' + xy' - 9y = 0$.

15 $(x + 1)y'' + (x - 1)y' - 2y = 0$.

16 $(1 - x^2)y'' - 7xy' - 9y = 0$.

17 $(1 - x^2)y'' - 2xy' + 2y = 0$.

18 $xy'' + y' + 2xy = 0$.

(c) *Use the method of Frobenius* to obtain series solutions of the following.

19 $3xy'' + y' - y = 0$.

20 $y'' + y = 0$.

21 $y'' - xy = 0$.

22 $3xy'' + 4y' + y = 0$.

23 $y'' - xy' + y = 0$.

24 $xy'' - 3y' + y = 0$.

25 $xy'' + y' - 3y = 0$.

▶

26 Verify that $y'' + \lambda y = 0$ where $y'(0) = 0$ and $y(2) = 0$ is a Sturm–Liouville system. Find the eigenvalues and eigenfunctions of the system and prove that they are orthogonal in (0, 2).

27 Series solutions of the equation $y'' - 2xy' + 2ny = 0$ are known as Hermite polynomials, $H_n(x)$, where

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)$$

Derive the first four Hermite polynomials and show that they are orthogonal with respect to the weight e^{-x^2} in $(-\infty, \infty)$.

28 Series solutions of the equation $xy'' + (1 - x)y' + ny = 0$ are known as Laguerre polynomials, $L_n(x)$, where

$$L_n(x) = e^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x})$$

Derive the first four Laguerre polynomials and show that they are orthogonal with respect to the weight e^{-x} in $(0, \infty)$.

29 Given the generating function for Laguerre polynomials $L_n(x)$ as

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n$$

show that $L_n(0) = n!$

30 Given the generating function for Hermite polynomials $H_n(x)$ as

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

show that $H_{2n+1}(0) = 0$.

31 Given the generating function for Legendre polynomials $P_n(x)$ as

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

show that $P_{2n+1}(0) = 0$.

Numerical solutions of ordinary differential equations

Frames 1 to 69

Learning outcomes

When you have completed this Programme you will be able to:

- Derive a form of Taylor's series from Maclaurin's series and from it describe a function increment as a series of first and higher-order derivatives of the function
- Describe and apply by means of a spreadsheet the Euler method, the Euler–Cauchy method and the Runge–Kutta method for first-order differential equations
- Describe and apply by means of a spreadsheet the Euler second-order method and the Runge–Kutta method for second-order ordinary differential equations
- Describe and apply by means of a spreadsheet a simple predictor–corrector method.

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme F.4 (Uses of a spreadsheet)

Introduction

1

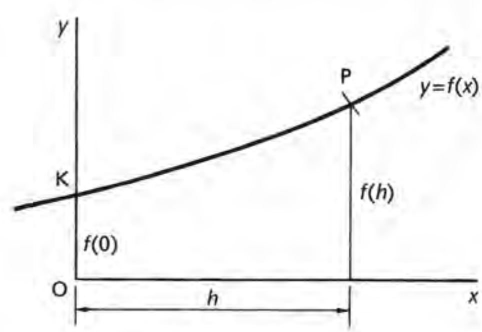
The range of differential equations that can be solved by straightforward analytical methods is relatively restricted. Even solution in series may not always be satisfactory, either because of the slow convergence of the resulting series or because of the involved manipulation in repeated stages of differentiation.

In such cases, where a differential equation and known boundary conditions are given, an approximate solution is often obtainable by the application of numerical methods, where a numerical solution is obtained at discrete values of the independent variable.

The solution of differential equations by numerical methods is a wide subject. The present Programme introduces some of the simpler methods, which nevertheless are of practical use.

Taylor's series

Let us start off by briefly revising the fundamentals of Maclaurin's and Taylor's series.



Maclaurin's series for $f(x)$ is

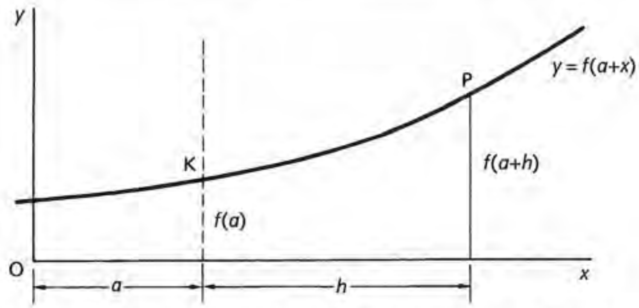
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots \qquad (1)$$

and expresses the function $f(x)$ in terms of its successive derivatives at $x = 0$, i.e. at the point K.

Therefore, at P, $\quad f(h) = \ldots\ldots\ldots\ldots$

2

$$\boxed{f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \ldots + \frac{h^n}{n!}f^n(0) + \ldots} \qquad (2)$$



If the y-axis and origin are moved a units to the left, the equation of the same curve relative to the new axes becomes $y = f(a + x)$ and the function value at K is $f(a)$.

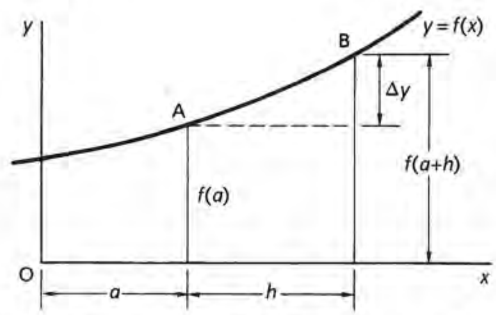
At P, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a) + \ldots$

This is one common form of Taylor's series.

Make a note of it and then move on

Function increment

3



If we know the function value $f(a)$ at A, i.e. at $x = a$, we can apply Taylor's series to determine the function value at a neighbouring point B, i.e. at $x = a + h$.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots \qquad (3)$$

The *function increment* from A to B $= \Delta y = f(a+h) - f(a)$

i.e. $\quad f(a+h) = f(a) + \Delta y$

where $\Delta y = hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots$

This entails evaluation of an infinite number of derivatives at $x = a$: in practice an approximation is accepted by restricting the number of terms that are used in the series.

▶

This approximation of Taylor's series forms the basis of several numerical methods, some of which we shall now introduce. It should be noted that these early examples have been selected because exact solutions can also be found. The purpose of this is to enable a comparison between the results obtained by a particular method with those obtained from an exact solution, and so to demonstrate the accuracy of the method.

On then to the next frame

First-order differential equations

4

Numerical solution of $\dfrac{dy}{dx} = f(x, y)$ with the initial condition that, at $x = x_0$, $y = y_0$.

Euler's method

The simplest of the numerical methods for solving first-order differential equations is *Euler's method*, in which the Taylor's series

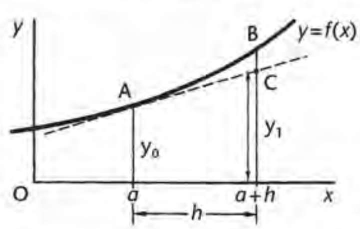
$$f(a+h) = f(a) + hf'(a) \; \boxed{+\frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots}$$

is truncated after the second term to give

$$f(a+h) \approx f(a) + hf'(a) \qquad (4)$$

This is a severe approximation, but in practice the 'approximately equals' sign is replaced by the normal 'equals' sign, in the knowledge that the result we obtain will necessarily differ to some extent from the function value we seek. With this in mind, we write

$$f(a+h) = f(a) + hf'(a)$$



If h is the interval between two near ordinates and if we denote $f(a)$ by y_0, then the relationship

$$f(a+h) = f(a) + hf'(a)$$

becomes

$$y_1 = y_0 + h(y')_0 \qquad (5)$$

Hence, knowing y_0, h and $(y')_0$, we can compute y_1, an approximate value for the function value at B.

Make a note of result (5): we shall be using it quite a lot.

Then move on for an example

5

Example 1

Given that $\dfrac{dy}{dx} = 2(1 + x) - y$ with the initial condition that at $x = 2$, $y = 5$, we can find an approximate value of y at $x = 2{\cdot}2$, as follows.

We have $\quad y' = 2(1 + x) - y \quad$ with $\quad x_0 = 2,\ y_0 = 5$

$$\therefore\ (y')_0 = \ldots\ldots\ldots\ldots$$

6

$$\boxed{(y')_0 = 1}$$

We obtain this by substituting x_0 and y_0 in the given equation:

$(y')_0 = 2(1 + x_0) - y_0 = 2(1 + 2) - 5 \quad \therefore\ (y')_0 = 1$

So we have $x_0 = 2$; $y_0 = 5$; $(y')_0 = 1$; $x_1 = 2{\cdot}2$; $h = 0{\cdot}2$.

By Euler's relationship:

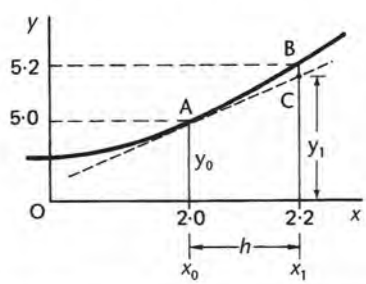
$$y_1 = y_0 + h(y')_0 \quad \therefore\ y_1 = \ldots\ldots\ldots\ldots$$

7

$$\boxed{y_1 = 5{\cdot}2}$$

Because

$y_1 = y_0 + h(y')_0 = 5 + (0{\cdot}2)1 = 5{\cdot}2$



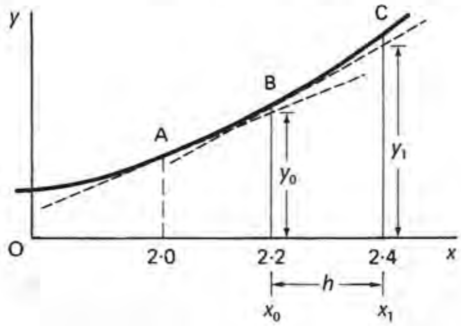
At B, $x_1 = 2{\cdot}2$; $\quad y_1 = 5{\cdot}2$; and

$$(y')_1 = \ldots\ldots\ldots\ldots$$

8

$$\boxed{(y')_1 = 1{\cdot}2}$$

$$(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2{\cdot}2) - 5{\cdot}2 = 1{\cdot}2$$



If we take the values of x, y and y' that we have just found for the point B and treat these as new starter values x_0, y_0, $(y')_0$, we can repeat the process and find values corresponding to the point C.

At B, $x_0 = 2{\cdot}2$; $y_0 = 5{\cdot}2$; $(y')_0 = 1{\cdot}2$; $x_1 = 2{\cdot}4$.

Then at C: $y_1 = \ldots\ldots\ldots\ldots$; $(y')_1 = \ldots\ldots\ldots\ldots$

9

$$\boxed{y_1 = 5{\cdot}44; \quad (y')_1 = 1{\cdot}36}$$

$$y_1 = y_0 + h(y')_0 = 5{\cdot}2 + (0{\cdot}2)1{\cdot}2 = 5{\cdot}44$$
$$(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2{\cdot}4) - 5{\cdot}44 = 1{\cdot}36$$

So we could continue in a step-by-step method. At each stage, the determined values of x_1, y_1 and $(y')_1$ become the new starter values x_0, y_0 and $(y')_0$ for the next stage.

Our results so far can be tabulated thus

x_0	y_0	$(y')_0$	x_1	y_1	$(y')_1$
2·0	5·0	1·0	2·2	5·2	1·2
2·2	5·2	1·2	2·4	5·44	1·36
2·4	5·44	1·36			

Continue the table with a constant interval of $h = 0{\cdot}2$. The third row can be completed to give

$x_1 = \ldots\ldots\ldots\ldots$; $y_1 = \ldots\ldots\ldots\ldots$; $(y')_1 = \ldots\ldots\ldots\ldots$

10

$$x_1 = 2{\cdot}6; \quad y_1 = 5{\cdot}712; \quad (y')_1 = 1{\cdot}488$$

Because

$$x_1 = x_0 + h = 2{\cdot}4 + 0{\cdot}2 = 2{\cdot}6$$
$$y_1 = y_0 + h(y')_0 = 5{\cdot}44 + (0{\cdot}2)1{\cdot}36 = 5{\cdot}712$$
$$(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2{\cdot}6) - 5{\cdot}712 = 1{\cdot}488$$

Now you can continue in the same way and complete the table for

$$x = 2{\cdot}0,\ 2{\cdot}2,\ 2{\cdot}4,\ 2{\cdot}6,\ 2{\cdot}8,\ 3{\cdot}0$$

Finish it off and compare results with the next frame

11

Here is the result.

x_0	y_0	$(y')_0$	x_1	y_1	$(y')_1$
2·0	5·0	1·0	2·2	5·2	1·2
2·2	5·2	1·2	2·4	5·44	1·36
2·4	5·44	1·36	2·6	5·712	1·488
2·6	5·712	1·488	2·8	6·009 6	1·590 4
2·8	6·009 6	1·590 4	3·0	6·327 68	1·672 32
3·0	6·327 68	1·672 32			

In practice, we do not, in fact, enter the values in the right-hand half of the table, but write them in directly as new starter values in the left-hand section of the table.

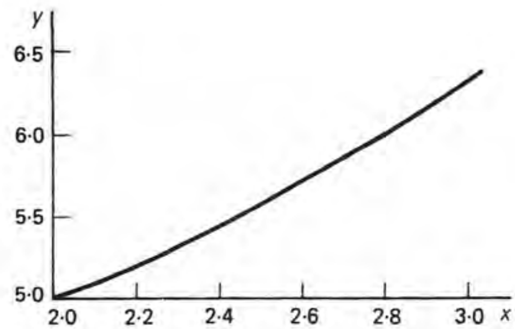
x_0	y_0	$(y')_0$
2·0	5·0	1·0
2·2	5·2	1·2
2·4	5·44	1·36
2·6	5·712	1·488
2·8	6·009 6	1·590 4
3·0	6·327 68	1·672 32

The particular solution is given by the values of y against x and a graph of the function can be drawn.

Draw the graph of the function carefully on graph paper.

12

Graph of the solution of $\frac{dy}{dx} = 2(1 + x) - y$ with $y = 5$ at $x = 2$.



13

It is an advantage to plot the points step-by-step as the results are built up. In that way, one can check that there is a smooth progression and that no apparent errors in the calculations occur at any one stage.

The differential equation $\frac{dy}{dx} = 2(1 + x) - y$ can be solved by the integration factor method (see *Engineering Mathematics, Fifth Edition*, Programme 24) to give the solution

$$y = 2x + e^{2-x}$$

and in the following table we compare our results with the actual values to determine the errors.

x	y (Euler)	y (actual)	Absolute error
2·0	5·0	5·0	0
2·2	5·2	5·218 731	0·018 731
2·4	5·44	5·470 320	0·030 320
2·6	5·712	5·748 812	0·036 812
2·8	6·009 6	6·049 329	0·039 729
3·0	6·327 68	6·367 879	0·040 199

The errors involved in the process are shown. These errors are due mainly to ....

14

the fact that Taylor's series was truncated after the second term

By now you will appreciate the amount of arithmetic manipulation involved in solving these differential equations – a large amount of which is repetitive. To avoid the tedium and to make the computations more efficient we shall resort to the use of a spreadsheet. If the use of a spreadsheet is a totally new experience to you then you are referred to Programme F.4 of *Engineering Mathematics, Fifth Edition*, where the spreadsheet is introduced as a tool for constructing graphs of functions. If you have a limited knowledge then you will be able to follow the text from here. The spreadsheet we shall be using here is Microsoft Excel, though all commercial spreadsheets possess the equivalent functionality. Alternatively, an iteration process can be used in any computer algebra package such as *Derive*, *Maple* or *Mathematica*.

Open your spreadsheet and in cell A1 enter the letter n and press **Enter**. In this first column we are going to enter the iteration numbers. In cell A2 enter the number 0 and press **Enter**. Place the cell highlight in cell A2 and highlight the block of cells A2 to A12 by holding down the mouse button and wiping the highlight down to cell A12. Click the **Edit** command on the Command bar and point at **Fill** from the drop-down menu. Select **Series** from the next drop-down menu and accept the default **Step value** of 1 by clicking **OK** in the Series window.

The cells A3 to A12 fill with ....

15

The numbers 1 to 10

In cell B1 enter the letter x – this column is going to contain the successive x-values for which the y-value is going to be enumerated. In cell B2 enter the number 2 – the initial x-value. We now could fill the column in much the same way as we filled the first column, but we have a better way.

Place the cell highlight in cell F1 and enter the number 0·2 – this is the value of h, the increment in x. Now place the cell highlight in cell B3 and enter the formula

=B2+$F$1 followed by **Enter** (uppercase or lowercase, it does not matter)

The number 2·2 appears in cell B3. Place the cell highlight in cell B3, click the **Edit** command and select **Copy** from the drop-down menu. You have now copied the contents of cell B3 to the clipboard. Now place the cell highlight in cell B4 and highlight the block of cells from B4 to B12. Click the **Edit** command again but this time select **Paste** from the drop-down menu.

The cells B4 to B12 fill with the numbers ....

16

The numbers 2·4 to 4·0 in intervals of 0·2

How has this happened? When you typed in the cell reference B2 into the formula in cell B3, the spreadsheet understood this to mean *the contents of the cell immediately above current cell B3*. When the formula is copied into cell B4 it means *the contents of the cell immediately above current cell B4*. Entered in this way the address B2 is a *relative address*. On the other hand, when you typed in $F$1 the spreadsheet understood this to mean the contents of cell F1 and that meaning remains when it is copied – the dollar signs indicate an *absolute address*. So as you move down the column the contents of a cell contain the contents of the cell immediately above it plus the contents of cell F1. You will shortly see the advantages of all this.

For now, place the cell highlight in cell C1 and enter the letter y – this column is going to contain the computed y-values against the corresponding x-values in column B. Place the cell highlight in cell C2 and enter the number 5 – the initial y-value. Before we can compute the y-values in column C we need to be able to tabulate the values of y' – the derivatives of y. Place the cell highlight in cell D1 and enter y' – this column will contain the values of the derivatives of y against the corresponding x-values. Cell D2 will contain the initial value of y' which can be computed from the equation

$$y' = 2(1+x) - y$$

When $x = x_0 = 2$ and $y = y_0 = 5$ then

$$y'_0 = 2(1+x_0) - y_0 = 2(1+2) - 5 = 1$$

so place the cell highlight in cell D2 and enter the formula

= 2 * (1 + B2) – C2 (B2 contains x_0 and C2 contains y_0)

The number 1 appears in cell D2. We need to copy this formula down the y' column. Place the cell highlight in cell D2, click **Edit** and select **Copy**. Now place the cell highlight in cell D3 and highlight the block of cells D3 to D12. Click the **Edit** command again and select **Paste**.

The cells D3 to D12 fill with ....

17

The numbers 6·4 to 10·0 in intervals of 0·4

Because the cells in the C2 column are currently empty, these values are just 2 ∗ (1 + B2) – 0.

Now, to compute the y-values we use the equation $y_1 = y_0 + h(y')_0$. Place the cell highlight in cell C3 and enter the formula

= C2 + $F$1 ∗ D2 (C2 contains y_0, F1 contains h and D2 contains $(y')_0$)

and the number 5·2 appears. That is, $y_1 = 5 + (0{\cdot}2)(1) = 5{\cdot}2$. This now completes the sequence of operations required to find y_1. To find the values of $y_2 = y(x_2) = y(2{\cdot}4)$ this sequence is repeated and, to ensure this, all that remains is to copy the formula in cell C3 into cells C4 to C12. So do this to reveal the following display

n	**x**	**y**	**y′**	0·2
0	2	5	1	
1	2·2	5·2	1·2	
2	2·4	5·44	1·36	
3	2·6	5·712	1·488	
4	2·8	6·0096	1·5904	
5	3	6·32768	1·67232	
6	3·2	6·662144	1·737856	
7	3·4	7·0097152	1·7902848	
8	3·6	7·36777216	1·83222784	
9	3·8	7·734217728	1·865782272	
10	4	8·107374182	1·892625818	

Now that was a lot easier than all that arithmetic manipulation by hand, wasn't it? We can tidy this display up by using the **Format** command and by using the various options on the tool bars to change the column widths and to display the numbers in a regular format of 10 decimal places to produce a display that is easier to read.

Next frame

18

n	x	y	y'	h=0·2
0	2·0	5·0000000000	1·0000000000	
1	2·2	5·2000000000	1·2000000000	
2	2·4	5·4400000000	1·3600000000	
3	2·6	5·7120000000	1·4880000000	
4	2·8	6·0096000000	1·5904000000	
5	3·0	6·3276800000	1·6723200000	
6	3·2	6·6621440000	1·7378560000	
7	3·4	7·0097152000	1·7902848000	
8	3·6	7·3677721600	1·8322278400	
9	3·8	7·7342177280	1·8657822720	
10	4·0	8·1073741824	1·8926258176	

Notice that we have added ***h*=** in cell E1 and justified it to the right and then justified the number 0·2 in F2 to the left so that together they read as an equation. The advantage of isolating the step value 0·2 in cell F1, as we have done, is that we can change the value and immediately see the effects on the calculations. For example, if the contents of F1 are changed to 0·1 the display changes automatically to

n	x	y	y'	h=0·1
0	2·0	5·0000000000	1·0000000000	
1	2·1	5·1000000000	1·1000000000	
2	2·2	5·2100000000	1·1900000000	
3	2·3	5·3290000000	1·2710000000	
4	2·4	5·4561000000	1·3439000000	
5	2·5	5·5904900000	1·4095100000	
6	2·6	5·7314410000	1·4685590000	
7	2·7	5·8782969000	1·5217031000	
8	2·8	6·0304672100	1·5695327900	
9	2·9	6·1874204890	1·6125795110	
10	3·0	6·3486784401	1·6513215599	

Notice that the different values of h produce different corresponding values in the tables. For example, for $h = 0{\cdot}2$ we find that $y(3{\cdot}0) = 6{\cdot}327\,680\,0000$ whereas for $h = 0{\cdot}1$ we have $y(3{\cdot}0) = 6{\cdot}348\,678\,4401$. The smaller the value of h then, the smaller the errors in the calculation – we shall see this demonstrated explicitly in the next frame.

Go to the next frame

The exact value and the errors

The differential equation

$$y' = 2(1 + x) - y$$

can be solved using the integration factor method (see *Engineering Mathematics, Fifth Edition*, Programme 24) to give the solution

$$y = 2x + e^{2-x}$$

We can programme this into the spreadsheet to compare the exact solution with the solution obtained numerically and compute the actual errors. Place the cell highlight in cell E1 and highlight cells E1 and F1. Click **Insert** on the Command bar and select **Columns**. Immediately two new columns appear. Notice that the numbers in the display do not change despite the fact that the h-value of 0·2 has moved from F1 to H1 – all the formulas in the spreadsheet will have automatically adjusted themselves. You can check this by highlighting a cell with a formula in it to see the change.

In cell E1 enter the word **Exact** and in cell F1 enter **Errors (%)**. In cell E2 enter the right-hand side of the equation $y = 2x + e^{2-x}$ by using the formula

= 2 * B2 + EXP(2 – B2) (the EXP stands for the exponential function)

and copy this into the block of cells E3 to E12. In cell F2 enter the formula for the error

= (E2 – C2) * 100/E2 (the error as a percentage of the exact value)

and copy this into the block of cells F3 to F12 to produce the following display

n	**x**	**y**	**y′**	**Exact**	**Errors (%)**	***h*=0·2**
0	2·0	5·0000000000	1·0000000000	5·0000000000	0·00	
1	2·2	5·2000000000	1·2000000000	5·2187307531	0·36	
2	2·4	5·4400000000	1·3600000000	5·4703200460	0·55	
3	2·6	5·7120000000	1·4880000000	5·7488116361	0·64	
4	2·8	6·0096000000	1·5904000000	6·0493289641	0·66	
5	3·0	6·3276800000	1·6723200000	6·3678794412	0·63	
6	3·2	6·6621440000	1·7378560000	6·7011942119	0·58	
7	3·4	7·0097152000	1·7902848000	7·0465969639	0·52	
8	3·6	7·3677721600	1·8322278400	7·4018965180	0·46	
9	3·8	7·7342177280	1·8657822720	7·7652988882	0·40	
10	4·0	8·1073741824	1·8926258176	8·1353352832	0·34	

▶

Change the value of h to 0·1 and produce the following display

n	x	y	y′	Exact	Errors (%)	h=0·1
0	2·0	5·0000000000	1·0000000000	5·0000000000	0·00	
1	2·1	5·1000000000	1·1000000000	5·1048374180	0·09	
2	2·2	5·2100000000	1·1900000000	5·2187307531	0·17	
3	2·3	5·3290000000	1·2710000000	5·3408182207	0·22	
4	2·4	5·4561000000	1·3439000000	5·4703200460	0·26	
5	2·5	5·5904900000	1·4095100000	5·6065306597	0·29	
6	2·6	5·7314410000	1·4685590000	5·7488116361	0·30	
7	2·7	5·8782969000	1·5217031000	5·8965853038	0·31	
8	2·8	6·0304672100	1·5695327900	6·0493289641	0·31	
9	2·9	6·1874204890	1·6125795110	6·2065696597	0·31	
10	3·0	6·3486784401	1·6513215599	6·3678794412	0·30	

When $h = 0{\cdot}2$ the error in $y(3{\cdot}0)$ is 0·63% whereas when $h = 0{\cdot}1$ the error in $y(3{\cdot}0)$ is 0·30%.

The smaller the value of h the …………

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smaller the error

Having completed your first spreadsheet you can now use it as a template for similar problems.

To avoid losing the work that you have already done, save your spreadsheet under some suitable name. When that is complete, highlight all the cells from A1 to G12 and copy them onto the clipboard using the **Edit-Copy** sequence of commands. Now click the **Sheet 2** tab at the bottom of your spreadsheet to reveal a blank worksheet. Place the cell highlight in cell A1, click **Edit** and select **Paste**. The entire contents of **Sheet 1** are now copied to **Sheet 2** in readiness for editing to accommodate a new problem.

So let's look at another example.

Example 2

Obtain a numerical solution of the equation

$$\frac{dy}{dx} = 1 + x - y$$

with the initial condition that $y = 2$ at $x = 1$, for the range $x = 1{\cdot}0(0{\cdot}2)3{\cdot}0$, that is from $x = 1{\cdot}0$ to $x = 3{\cdot}0$ with step length $x = 0{\cdot}2$.

As initial conditions, we have

$x_0 = \ldots\ldots\ldots\ldots$ and $y_0 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_0 = 1, \quad y_0 = 2}$$

Because

$x_0 = 1$ and $y_0 = 2$ are given initial conditions.

These values can now be inserted into the spreadsheet in cells ....

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$$\boxed{x_1 = 1 \text{ in B2}, \quad y_0 = 2 \text{ in C2}}$$

Notice how the numbers in column B have changed to accommodate the new sequence of x-values. The contents of the cells in column C do not need to be changed as they refer to the equation

$$y_1 = y_0 + h(y')_0$$

which is the same in this spreadsheet as it was in the previous spreadsheet. The contents of column D do have to be changed because they currently refer to the equation to be solved in the previous problem. The equation to be solved here is

$$y' = 1 + x - y$$

so in cell D3 the contents need to be changed to ....

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$$\boxed{= 1 + \text{B2} - \text{C2}}$$

This formula must then be copied into cells C3 to C12. Finally, the **Exact** column needs to be amended to reflect the exact solution to this equation, which is again found by using the integration factor method as

$$y = x + e^{1-x}$$

So, in E2, enter the formula ....

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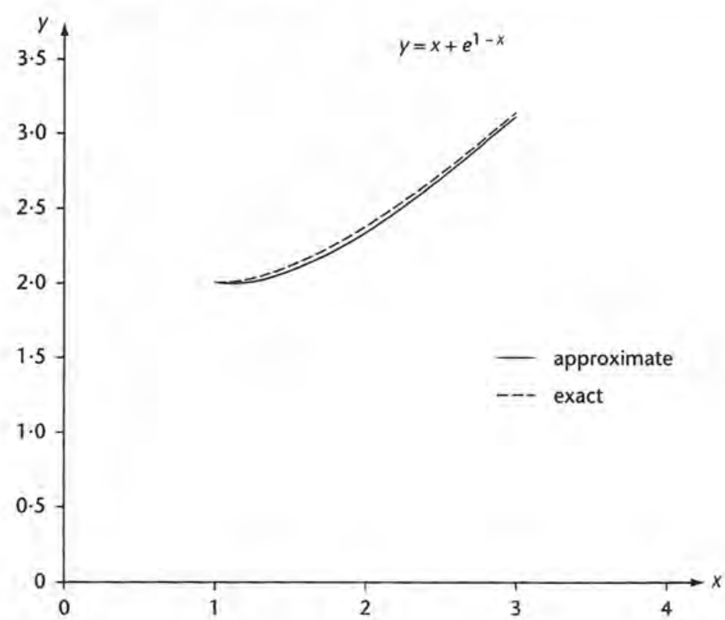
= B2 + EXP(1 – B2)

This formula needs to be copied into cells E3 to E12. This completes the editing of the spreadsheet to reflect the new problem to give the display

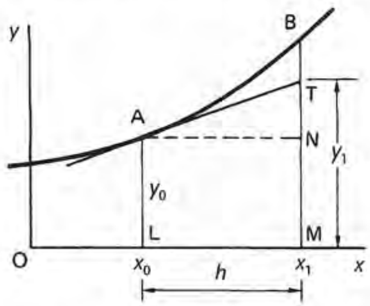
n	x	y	y′	Exact	Errors (%)	h=0·2
0	1·0	2·0000000000	0·0000000000	2·0000000000	0·00	
1	1·2	2·0000000000	0·2000000000	2·0187307531	0·93	
2	1·4	2·0400000000	0·3600000000	2·0703200460	1·46	
3	1·6	2·1120000000	0·4880000000	2·1488116361	1·71	
4	1·8	2·2096000000	0·5904000000	2·2493289641	1·77	
5	2·0	2·3276800000	0·6723200000	2·3678794412	1·70	
6	2·2	2·4621440000	0·7378560000	2·5011942119	1·56	
7	2·4	2·6097152000	0·7902848000	2·6465969639	1·39	
8	2·6	2·7677721600	0·8322278400	2·8018965180	1·22	
9	2·8	2·9342177280	0·8657822720	2·9652988882	1·05	
10	3·0	3·1073741824	0·8926258176	3·1353352832	0·89	

A plot of the graph of y against x for both the computed value and the exact value looks as follows



Graphical interpretation of Euler's method

25



If AT is the tangent to the curve at A,

then $\dfrac{\text{NT}}{\text{AN}} = \left[\dfrac{dy}{dx}\right]_{x=x_0} = (y')_0$

$\dfrac{\text{NT}}{h} = (y')_0 \qquad \therefore\ \text{NT} = h(y')_0$

$\therefore$ At $x = x_1$, $\text{MT} = y_0 + h(y')_0$

By Euler's relationship, $y_1 = y_0 + h(y')_0$ i.e. MT.

The difference between the calculated value of y, i.e. MT, and the actual value of the function y, i.e. MB, at $x = x_1$, is indicated by TB. This error can be considerable, depending on the curvature of the graph and the size of the interval h. It is inherent to the method and corresponds to the truncation of the Taylor's series after the second term.

Euler's method, then

(a) is simple in procedure

(b) is lacking in accuracy, especially away from the starter values of the initial conditions

(c) is of use only for very small values of the interval h.

In spite of its practical limitations, it is the foundation of several more sophisticated methods and hence it is worthy of note.

Here is one more example to work on your own.

Example 3

Obtain the solution of $\dfrac{dy}{dx} = x + y$ with the initial condition that $y = 1$ at $x = 0$, for the range $x = 0(0{\cdot}1)0{\cdot}5$.

By using a previously constructed spreadsheet as a template, the solution is

. . . .

The function values are given in the next frame

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n	x	y	y′	Exact	Errors (%)	$h = 0{\cdot}1$
0	0·0	1·0000000000	1·0000000000	1·0000000000	0·00	
1	0·1	1·1000000000	1·2000000000	1·1103418362	0·93	
2	0·2	1·2200000000	1·4200000000	1·2428055163	1·84	
3	0·3	1·3620000000	1·6620000000	1·3997176152	2·69	
4	0·4	1·5282000000	1·9282000000	1·5836493953	3·50	
5	0·5	1·7210200000	2·2210200000	1·7974425414	4·25	
6	0·6	1·9431220000	2·5431220000	2·0442376008	4·95	
7	0·7	2·1974342000	2·8974342000	2·3275054149	5·59	
8	0·8	2·4871776200	3·2871776200	2·6510818570	6·18	
9	0·9	2·8158953820	3·7158953820	3·0192062223	6·73	
10	1·0	3·1874849202	4·1874849202	3·4365636569	7·25	

Because

The initial conditions are entered as

0 in cell B2 (the initial x-value)
1 in cell C2 (the initial y-value)
0·1 in cell H1 (the x step length)

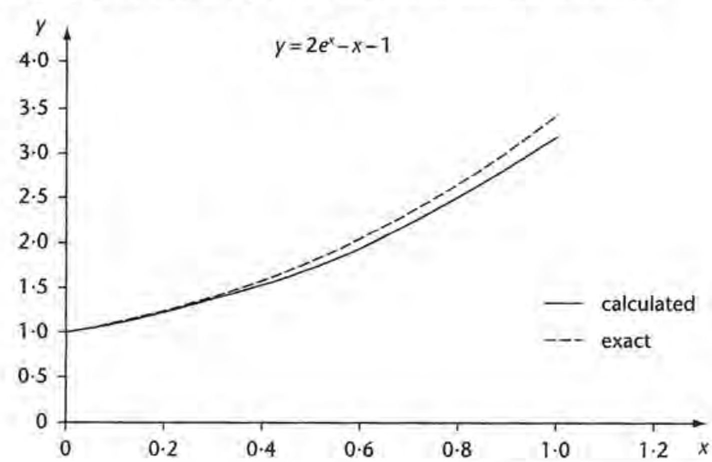
The formulas are entered as

= B2 + C2 in cell D2, copied into cells D3 to D12
(the successive y'-values)
= C2 + $H$1 * D2 in cell C3 copied into cells C4 to C12
(the successive y-values)

The exact solution found by using the integration factor method is $y = 2e^x - x - 1$ and so

= 2 * EXP(B2) – B2 – 1 is entered into cell E2 and copied into cells E3 to E12

Notice how the errors here are significant, which is very evident from the graphs of the computed values and the exact values.



The Euler–Cauchy method – or the improved Euler method

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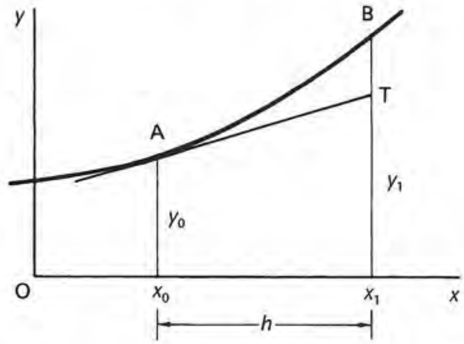
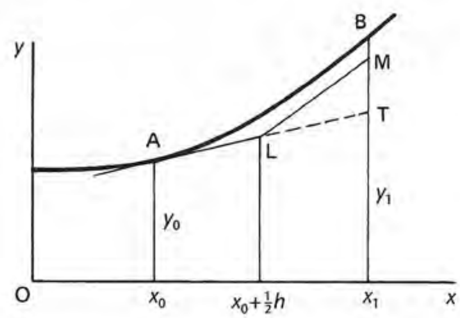


In Euler's method, we use the slope $(y')_0$ at A (x_0, y_0) across the whole interval h to obtain an approximate value of y_1 at B. TB is the resulting error in the result.



In the Euler–Cauchy method, we use the slope at A (x_0, y_0) across half the interval and then continue with a line whose slope approximates to the slope of the curve at x_1.

Let $\bar{\bar{y}}_1$ be the y-value of the point at T.

The error (MB) in the result is now considerably less than the error (TB) associated with the basic Euler method and the calculated results will accordingly be of greater accuracy.

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Euler–Cauchy calculations

The steps in the Euler–Cauchy method are as follows.

1 We start with the given equation $y' = f(x, y)$ with the initial condition that at $x = x_0$, $y = y_0$. We have to determine function values for $x = x_0(h)x_n$.

2 From the equation and the initial condition we obtain $(y')_0 = f(x_0, y_0)$.

3 Knowing x_0, y_0, $(y')_0$ and h, we then evaluate

(a) $x_1 = x_0 + h$

(b) the auxiliary value of y, denoted by $\bar{\bar{y}}$ where $\bar{\bar{y}}_1 = y_0 + h(y')_0$. This is the same step as in Euler's method.

(c) Then $y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\}$

Note that $f(x_1, y_1)$ is the right-hand side of the given equation with x and y replaced by the calculated values of x_1 and $\bar{\bar{y}}_1$.

(d) Finally $(y')_1 = f(x_1, y_1)$.

We have thus evaluated x_1, y_1 and $(y')_1$.

The whole process is then repeated, the calculated values of x_1, y_1 and $(y')_1$ becoming the starter values x_0, y_0, $(y')_0$ for the next stage.

Make a note of the relationships above. We shall be using them quite often.

Then on to the next frame for an example of their use

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Example 1

Apply the Euler–Cauchy method to solve the equation

$y' = x + y$

with the initial condition that at $x = 0$, $y = 1$, for the range $x = 0(0{\cdot}1)1{\cdot}0$.

We proceed as before by copying our template solution to a new worksheet. Before we continue we need to decide what the entries are going to be in our spreadsheet.

1 We are going to have to enter new initial conditions, so

Enter 0 in cell B2	that is $x_0 = 0$
Enter 1 in cell C2	that is $y_0 = 1$
Enter 0·1 in cell H1	this is the x step length

2 The equation to be solved is $y' = x + y$, so enter the formula

=B2+C2 in cell D2 and copy the contents of D2 into cells D3 to D12

3 The Euler–Cauchy method tells us that

$$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x, \bar{\bar{y}}_1)\}$$

where $\bar{\bar{y}}_1 = y_0 + h(y')_0$ so that

$$f(x_1, \bar{\bar{y}}_1) = x_1 + \bar{\bar{y}}_1 = x_1 + y_0 + h(y')_0$$

Therefore $y_1 = \ldots\ldots\ldots\ldots$

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$$\boxed{y_1 = y_0 + \frac{1}{2}h\{x_1 + y_0 + (1+h)(y')_0\}}$$

Because

By replacing $f(x_1, \overline{\overline{y}}_1)$ with $x_1 + y_0 + h(y')_0$ in the expression

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \overline{\overline{y}}_1)\}$

we find that

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + x_1 + y_0 + h(y')_0\}$

$\quad = y_0 + \frac{1}{2}h\{x_1 + y_0 + (1+h)(y')_0\}$

In cell C3 enter the formula ....

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= C2 + (0·5) * $H$1 * (B3 + C2 + (1 + $H$1) * D2)

Because

y_0 is in cell C2, h is in cell H1, x_1 is in cell B3 and $(y')_0$ is in cell D2.

Copy the contents of cell C3 into cells C4 to C12.

4 Finally, for comparison purposes, the exact solution of this equation is $y = 2e^x - x - 1$ and this is

entered into E2 by the formula ....
and copied into cells ....

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= 2 *EXP(B2) – B2 – 1 and copied into cells E3 to E12

The resulting display looks as follows

n	**x**	**y**	**y′**	**Exact**	**Errors (%)**	$h = 0{\cdot}1$
0	0·0	1·0000000000	1·0000000000	1·0000000000	0·00	
1	0·1	1·1100000000	1·2100000000	1·1103418362	0·03	
2	0·2	1·2420500000	1·4420500000	1·2428055163	0·06	
3	0·3	1·3984652500	1·6984652500	1·3997176152	0·09	
4	0·4	1·5818041013	1·9818041013	1·5836493953	0·12	
5	0·5	1·7948935319	2·2948935319	1·7974425414	0·14	
6	0·6	2·0408573527	2·6408573527	2·0442376008	0·17	
7	0·7	2·3231473748	3·0231473748	2·3275054149	0·19	
8	0·8	2·6455778491	3·4455778491	2·6510818570	0·21	
9	0·9	3·0123635233	3·9123635233	3·0192062223	0·23	
10	1·0	3·4281616932	4·4281616932	3·4365636569	0·24	

▶

Comparing these results with the same equation being solved by the Euler method demonstrates how much more accurate the Euler–Cauchy method is, as can be seen from the following table of comparative errors

x	Euler	Euler–Cauchy
0·0	0·00	0·00
0·1	0·93	0·03
0·2	1·84	0·06
0·3	2·69	0·09
0·4	3·50	0·12
0·5	4·25	0·14
0·6	4·95	0·17
0·7	5·59	0·19
0·8	6·18	0·21
0·9	6·73	0·23
1·0	7·25	0·24

Next frame

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Now for another example, but before that, complete the following without reference to your notes – if possible. In the Euler–Cauchy method the relevant relationships are

$$x_1 = \dots\dots\dots\dots$$
$$\bar{\bar{y}}_1 = \dots\dots\dots\dots$$
$$y_1 = \dots\dots\dots\dots$$
$$(y')_1 = \dots\dots\dots\dots$$

Next frame

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$$\boxed{\begin{aligned} x_1 &= x_0 + h \\ \bar{\bar{y}}_1 &= y_0 + h(y')_0 \\ y_1 &= y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\} \\ (y')_1 &= f(x_1, y_1) \end{aligned}}$$

Example 2

Determine a numerical solution of the equation $y' = 2(1 + x) - y$ with the initial condition that $y = 5$ when $x = 2$, for the range 2·0(0·2)4·0. Try this one yourself.

The exact solution is given as $y = 2x + e^{-2x}$
and the final display of results is …………

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n	x	y	y′	Exact	Errors (%)	$h = 0{\cdot}2$
0	2·0	5·0000000000	1·0000000000	5·0000000000	0·00	
1	2·2	5·2200000000	1·1800000000	5·2187307531	−0·02	
2	2·4	5·4724000000	1·3276000000	5·4703200460	−0·04	
3	2·6	5·7513680000	1·4486320000	5·7488116361	−0·04	
4	2·8	6·0521217600	1·5478782400	6·0493289641	−0·05	
5	3·0	6·3707398432	1·6292601568	6·3678794412	−0·04	
6	3·2	6·7040066714	1·6959933286	6·7011942119	−0·04	
7	3·4	7·0492854706	1·7507145294	7·0465969639	−0·04	
8	3·6	7·4044140859	1·7955859141	7·4018965180	−0·03	
9	3·8	7·7676195504	1·8323804496	7·7652988882	−0·03	
10	4·0	8·1374480313	1·8625519687	8·1353352832	−0·03	

Because

1 The initial conditions are entered as

Enter 2 in cell B2 (that is $x_0 = 2$); enter 5 in cell C2 (that is $y_0 = 5$)
Enter 0·2 in cell H1 (this is the x step length)

2 The equation to be solved is $y' = 2(1 + x) - y$, so enter the formula

= 2 ∗ (1 + B2) − C2 in cell D2 and copy the contents of D2 into cells D3 to D12

3 The Euler–Cauchy method tells us that

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\}$ where $\bar{\bar{y}}_1 = y_0 + h(y')_0$ so that
$f(x_1, \bar{\bar{y}}_1) = 2(1 + x_1) - \bar{\bar{y}}_1 = 2(1 + x_1) - y_0 - h(y')_0$ therefore
$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + 2(1 + x_1) - y_0 - h(y')_0\}$ that is
$y_1 = y_0 + \frac{1}{2}h\{2(1 + x_1) - y_0 + (1 - h)(y')_0\}$

This is accommodated by the formula in C3 (copied into cells C4 to C12)

= C2 + (0·5) ∗ \$H\$1 ∗ (2 ∗ (1 + B3) − C2 + (1 − \$H\$1) ∗ D2)

4 Finally the exact solution $y = 2x + e^{-2x}$ is entered into cell E2 as = 2 ∗ B2 + EXP(−2 ∗ B2) and copied into cells E3 to E12.

Refer to Frame 19 for a comparison of errors between this method and the Euler method. Then another example for you to try just to make sure you are clear about the processes involved.

Next frame

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Example 3

Solve the equation $y' = y^2 + xy$ with initial condition that at $x = 1$, $y = 1$, for the range $x = 1{\cdot}0(0{\cdot}1)1{\cdot}7$. Use the Euler–Cauchy method and work to 6 places of decimals.

The solution is ....

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n	x	y	y′	$h = 0{\cdot}1$
0	1·0	1·000000	2·000000	
1	1·1	1·238000	2·894444	
2	1·2	1·591023	4·440583	
3	1·3	2·152410	7·431004	
4	1·4	3·145846	14·300528	
5	1·5	5·251007	35·449581	
6	1·6	11·595613	153·011211	
7	1·7	57·704110	3427·861242	

Because

1 The initial conditions are entered as

Enter 1 in cell B2 (that is $x_0 = 1$); enter 1 in cell C2 (that is $y_0 = 1$)

Enter 0·1 in cell H1 (this is the x step length)

2 The equation to be solved is $y' = y^2 + xy$, so

Enter the formula =C2^2+B2*C2 in cell D2 and copy the contents of D2 into cells D3 to D9. Note that C2^2=C2*C2 – the 'hat' indicates raising to a power.

3 The Euler-Cauchey method tell us that

$y_1 = y_0 + \frac{1}{2}h\{(y')_0 + f(x_1, \bar{\bar{y}}_1)\}$ where $\bar{\bar{y}}_1 = y_0 + h(y')_0$ so that

$f(x_1, \bar{\bar{y}}_1) = \bar{\bar{y}}_1^{\,2} + x_1\bar{\bar{y}}_1 = (y_0 + h(y')_0)^2 + x_1(y_0 + h(y')_0)$ therefore

$$y_1 = y_0 + \frac{1}{2}h\left\{(y')_0 + (y_0 + h(y')_0)^2 + x_1(y_0 + h(y')_0)\right\}$$

This is accommodated by the formula in C3 (copied into cells C4 to C9)

=C2+(0·5)*$F$1*(D2+(C2+$F$1*D2))^2+
B3*(C2+$F$1*D2))

The table shows that as x increases, the computed values of y and its derivative increase dramatically. This is an indication that the exact solution increases without bound near to the larger values of x considered, so bringing the accuracy of these computed values into question. This emphasises the importance of checking every method against a known solution so as to form some idea of the method's accuracy. However, all numerical methods produce significant accuracies whenever the exact solution diverges in this way.

Runge–Kutta method

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The Runge–Kutta method for solving first-order differential equations is widely used and affords a high degree of accuracy. It is a further step-by-step process where a table of function values for a range of values of x is accumulated. Several intermediate calculations are required at each stage, but these are straightforward and present little difficulty.

In general terms, the method is as follows.

To solve $y' = f(x, y)$ with initial condition $y = y_0$ at $x = x_0$, for a range of values of $x = x_0(h)x_n$.

Starting as usual with $x = x_0$, $y = y_0$, $y' = (y')_0$ and h, we have

$$x_1 = x_0 + h$$

Finding y_1 requires four intermediate calculations

$$k_1 = hf(x_0, y_0) = h(y')_0$$
$$k_2 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$

The increment Δy_0 in the y-values from $x = x_0$ to $x = x_1$ is then

$$\Delta y_0 = \tfrac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

and finally $\quad y_1 = y_0 + \Delta y_0.$

We shall be using these repeatedly, so make a note of them for future reference. Then let us see an example

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Example 1

Find the numerical solution of $y' = x + y$ using the Runge–Kutta method with $y = 1$ and $x = 0$ for values in the range $x = 0(0{\cdot}1)1{\cdot}0$.

We shall proceed with the solution of this differential equation using a spreadsheet in much the same manner as before. However, we are going to require a different structure in order to accommodate the four variables k_i for $i = 1, 2, 3, 4$. The structure we shall use is headed by

	A	B	C	D	E	F	G	H	I
1	**n**	**x**	**k1**	**k2**	**k3**	**k4**	**y**	**y′**	***h*** **=**

where the value of h is held in cell J1.

1 Enter the values 0 to 10 in column A from A2 to A12 using the **Edit-Fill-Series** sequence of commands. These are the iteration numbers.

2 Enter the x step value of 0·1 in cell J1.

3 Enter the initial value of x in cell B2 as 0 and in B3 enter the formula =B2+$J$1. Now copy the contents of B3 into cells B4 to B12.

4 Enter the initial value of y in cell G2 as 1.

We can now progressively enter the table of values from the left.

5 $k_1 = hf(x_0, y_0) = h(y')_0$ – the y'-values are in column H, so in cell C2 enter the formula =$J$1 * H2. Copy the contents of C2 into cells C3 to C12.

6 $k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = h\left(x_0 + \frac{1}{2}h + y_0 + \frac{1}{2}k_1\right)$, so in cell D2 enter the formula =$J$1 * (B2+0·5 * $J$1+G2+0·5 * C2). Copy the contents of D2 into cells D3 to D12.

7 $k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = h(x_0 + \frac{1}{2}h + y_0 + \frac{1}{2}k_2)$, so in cell E2 enter the formula =$J$1 * (B2+0·5 * $J$1+G2+0·5 * D2). Copy the contents of E2 into cells E3 to E12.

8 $k_4 = hf(x_0 + h, y_0 + k_3) = h(x_0 + h + y_0 + k_3)$, so in cell F2 enter the formula =$J$1 * (B2+$J$1+G2+E2). Copy the contents of F2 into cells F3 to F12.

9 $y_1 = y_0 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$, so in cell G3 enter the formula =G2+(1/6) * (C2+2 * D2+2 * E2+F2). Copy the contents of G3 into cells G4 to G12.

10 $y' = x + y$, so in H2 enter the formula =B2+G2. Copy the contents of H2 into cells H3 to H12.

The results are displayed in the next frame

n	x	k1	k2	k3	k4	y	y'	$h = 0{\cdot}1$
0	0·0	0·1000000	0·1100000	0·1105000	0·1210500	1·0000000	1·0000000	
1	0·1	0·1210342	0·1320859	0·1326385	0·1442980	1·1103417	1·2103417	
2	0·2	0·1442805	0·1564945	0·1571052	0·1699910	1·2428051	1·4428051	
3	0·3	0·1699717	0·1834703	0·1841452	0·1983862	1·3997170	1·6997170	
4	0·4	0·1983648	0·2132831	0·2140290	0·2297677	1·5836485	1·9836485	
5	0·5	0·2297441	0·2462313	0·2470557	0·2644497	1·7974413	2·2974413	
6	0·6	0·2644236	0·2826448	0·2835558	0·3027792	2·0442359	2·6442359	
7	0·7	0·3027503	0·3228878	0·3238947	0·3451398	2·3275033	3·0275033	
8	0·8	0·3451079	0·3673633	0·3684761	0·3919555	2·6510791	3·4510791	
9	0·9	0·3919203	0·4165163	0·4177461	0·4436949	3·0192028	3·9192028	
10	1·0	0·4436559	0·4708387	0·4721979	0·5008757	3·4365595	4·4365595	

with the following errors

n	x	Exact	Error (%)	Error (%)
0	0·0	1·0000000	0·0000000	0·00
1	0·1	1·1103418	0·0000153	0·93
2	0·2	1·2428055	0·0000301	1·84
3	0·3	1·3997176	0·0000444	2·69
4	0·4	1·5836494	0·0000578	3·50
5	0·5	1·7974425	0·0000703	4·25
6	0·6	2·0442376	0·0000820	4·95
7	0·7	2·3275054	0·0000929	5·59
8	0·8	2·6510819	0·0001030	6·18
9	0·9	3·0192062	0·0001124	6·73
10	1·0	3·4365637	0·0001213	7·25

The column to the far right contains the errors using the Euler method and, as you can see, the Runge–Kutta method provides a significant improvement in accuracy.

Now, without reference to your notes, complete the following expressions for

$$k_1 = \ldots\ldots\ldots\ldots$$
$$k_2 = \ldots\ldots\ldots\ldots$$
$$k_3 = \ldots\ldots\ldots\ldots$$
$$k_4 = \ldots\ldots\ldots\ldots$$
$$\Delta y_0 = \ldots\ldots\ldots\ldots$$
$$y_1 = \ldots\ldots\ldots\ldots$$

It speeds up your working if you can remember them.

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$$\begin{aligned} k_1 &= h(y')_0 \\ k_2 &= hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1) \\ k_3 &= hf(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2) \\ k_4 &= hf(x_0 + h,\ y_0 + k_3) \\ \Delta y_0 &= \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ y_1 &= y_0 + \Delta y_0 \end{aligned}$$

With those in mind, let us move on to a further example. Next frame

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Example 2

Solve $y' = \sqrt{x^2 + y}$ for $x = 0(0{\cdot}2)2{\cdot}0$ given that at $x = 0$, $y = 0{\cdot}8$.

Using the spreadsheet for the previous example as a template for this example. The solution is …………

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n	x	k1	k2	k3	k4	y	y′	h=0·2
0	0·0	0·1788854	0·1896779	0·1902460	0·2030021	0·8000000	0·8944272	
1	0·2	0·2030063	0·2174206	0·2180825	0·2339548	0·9902892	1·0150316	
2	0·4	0·2339473	0·2510185	0·2516977	0·2698134	1·2082838	1·1697366	
3	0·6	0·2698011	0·2887709	0·2894271	0·3091435	1·4598160	1·3490055	
4	0·8	0·3091304	0·3294604	0·3300769	0·3509482	1·7490394	1·5456518	
5	1·0	0·3509358	0·3722562	0·3728285	0·3945492	2·0788983	1·7546790	
6	1·2	0·3945381	0·4165946	0·4171237	0·4394829	2·4515074	1·9726904	
7	1·4	0·4394732	0·4620889	0·4625781	0·4854274	2·8684170	2·1973659	
8	1·6	0·4854190	0·5084682	0·5089213	0·5321545	3·3307894	2·4270948	
9	1·8	0·5321472	0·5555390	0·5559599	0·5794989	3·8395148	2·6607358	
10	2·0	0·5794925	0·6031595	0·6035518	0·6273385	4·3952888	2·8974625	

Because

1. The initial conditions are entered as $x_0 = 0$ and $y_0 = 0{\cdot}8$. The x step length is entered as 0·2
2. The formula for the variable k_1 remains the same as = $J$1 * H2
3. The formula for the variable k_2 is changed to = $J$1 * (((B2 + 0·5 * $J$1)^2 + G2 + 0·5 * C2)^0·5)
4. The formula for the variable k_3 is changed to = $J$1 * (((B2 + 0·5 * $J$1)^2 + G2 + 0·5 * D2)^0·5)
5. The formula for the variable k_4 is changed to = $J$1 * (((B2 + $J$1)^2 + G2 + E2)^0·5)
6. The formula for y remains the same as = G2+(1/6) * (C2 + 2 * D2 + 2 * E2 + F2)
7. The formula for y' is changed to = (B2^2+G2)^0·5

That is it. Now move on to the next frame where we make a new start and apply similar methods to the solution of second-order differential equations by numerical methods.

Second-order differential equations

Euler second-order method

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The first method we will deal with is really an extension of the Euler method for the first-order equations and is a direct application of a truncated form of Taylor's series. We anticipate, therefore, that the method will be relatively easy, but the results will not be accurate to a high degree.

Taylor's series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

Differentiating term by term with respect to x, we obtain

$$f'(x+h) = f'(x) + hf''(x) + \frac{h^2}{2!}f'''(x) + \frac{h^3}{3!}f''''(x) + \dots$$

If we neglect terms in $f'''(x)$ and subsequent terms in each of these two series, we have the approximations

$$f(x+h) \approx \dots\dots\dots\dots$$
$$f'(x+h) \approx \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned} f(x+h) &\approx f(x) + hf'(x) + \frac{h^2}{2!}f''(x) \\ f'(x+h) &\approx f'(x) + hf''(x) \end{aligned}}$$

Although these are approximations, in practice we tend to write them with the 'equals' sign. Therefore, at $x = a$, these become

....

and ....

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$$\boxed{\begin{aligned} f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) \\ f'(a+h) &= f'(a) + hf''(a) \end{aligned}}$$

and these, with the notation we have previously used, can be written

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$
$$(y')_1 = (y')_0 + h(y'')_0$$

Thus, if x_0, y_0, $(y')_0$ and $(y'')_0$ are known, we can find an approximate value of y_1 at $x_1 = x_0 + h$.

Make a note of these two relationships: then we can apply them.

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Example

Solve the equation $y'' = xy' + y$ for $x = 0(0{\cdot}2)2{\cdot}0$ given that at $x = 0$, $y = 1$ and $y' = 0$.

We shall set about finding the numerical solution to this equation as we have done previously by using a spreadsheet. The headings for the sheet will be

	A	B	C	D	E	F	G	H
1	**n**	**x**	**y**	**y′**	**y″**	**Exact**	**Errors (%)**	***h*=**

The entries will then be

1 Column A contains the iteration number from 0 in A2 to 10 in A12.

2 Cell I1 contains the x step length which is 0·2.

3 Column B contains the successive x-values from 0·0 to 2·0 in steps of 0·2. The initial value of $x_0 = 0$ is entered into cell B2 and the formula =B2+\$I\$1 is entered into cell B3 and copied into cells B4 to B12.

4 Column C contains the computed y-values. The initial value of $y_0 = 1$ is entered into cell C2 and the equation

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

is represented in cell C3 by the formula

=C2+\$I\$1 * D2+(\$I\$1^2) * E2/2

copied into cells C4 to C12.

5 Column D contains the computed y'-values. The initial value of $(y')_0 = 0$ is entered into cell D2 and the equation

$$(y')_1 = (y')_0 + h(y'')_0$$

is represented in cell D3 by the formula =D2+\$I\$1 * E2 copied into cells D4 to D12.

6 Column E contains the y''-values which are obtained from the equation $y'' = xy' + y$ which is represented in cell E2 by the formula =B2 * D2+C2 copied into cells E3 to E12.

7 Column F contains the values obtained from the exact solution which can be shown to be $y = e^{x^2/2}$. This is represented in cell F2 by the formula =EXP((B2^2)/2) copied into cells F3 to F12.

8 Column G contains the percentage errors. In cell G2 enter the formula =(F2 – C2) * 100/F2 copied into cells G3 to G12.

Your spreadsheet should now look like the one on the next page (with the appropriate formatting to make it easier to read).

▶

n	x	y	y′	y″	Exact	Errors (%)	$h = 0{\cdot}2$
0	0·0	1·0000000	0·0000000	1·0000000	1·0000000	0·00	
1	0·2	1·0200000	0·2000000	1·0600000	1·0202013	0·02	
2	0·4	1·0812000	0·4120000	1·2460000	1·0832871	0·19	
3	0·6	1·1885200	0·6612000	1·5852400	1·1972174	0·73	
4	0·8	1·3524648	0·9782480	2·1350632	1·3771278	1·79	
5	1·0	1·5908157	1·4052606	2·9960763	1·6487213	3·51	
6	1·2	1·9317893	2·0044759	4·3371604	2·0544332	5·97	
7	1·4	2·4194277	2·8719080	6·4400989	2·6644562	9·20	
8	1·6	3·1226113	4·1599278	9·7784957	3·5966397	13·18	
9	1·8	4·1501667	6·1156269	15·1582952	5·0530903	17·87	
10	2·0	5·6764580	9·1472859	23·9710299	7·3890561	23·18	

You will notice that the errors are significant and grow dramatically as the value of x increases. The main cause of errors is …………

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the truncation of the Taylor's series on which the method is based

A greater degree of accuracy can be obtained by using the Runge–Kutta method for second-order differential equations, which is an extension of the method we have already used for first-order equations. As before, more intermediate calculations are required, but the reliability of results reflects the extra work involved.

Runge–Kutta method for second-order differential equations

Starting with the given equation $y'' = f(x, y, y')$ and initial conditions that at $x = x_0$, $y = y_0$ and $y' = (y')_0$, we can obtain the value of y_1 at $x_1 = x_0 + h$ as follows.

(a) We evaluate

$$k_1 = \tfrac{1}{2}h^2 f\{x_0,\ y_0,\ (y')_0\} = \tfrac{1}{2}h^2 (y'')_0$$

$$k_2 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_1}{h}\right\}$$

$$k_3 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_2}{h}\right\}$$

$$k_4 = \tfrac{1}{2}h^2 f\left\{x_0 + h,\ y_0 + h(y')_0 + k_3,\ (y')_0 + \frac{2k_3}{h}\right\}$$

(b) From these results, we then determine

$$P = \tfrac{1}{3}\{k_1 + k_2 + k_3\}$$

$$Q = \tfrac{1}{3}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

▶

(c) Finally, we have

$$x_1 = x_0 + h$$
$$y_1 = y_0 + h(y')_0 + P$$
$$(y')_1 = (y')_0 + \frac{Q}{h}$$

It is not as complicated as it looks at first sight. Copy down this list of relationships for reference when dealing with some examples that follow.

Then move on

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Note the following

1. Four evaluations for k are required to determine a single new point on the solution curve.
2. The method is self-starting in that no preliminary calculations are required. The equation and initial conditions are sufficient to provide the next point on the curve.
3. As with the Runge–Kutta method for first-order equations, the method contains no self-correcting element or indication of any error involved.

Example 1

Use the Runge–Kutta method to solve the equation $y'' = xy' + y$ for $x = 0{\cdot}0(0{\cdot}2)2{\cdot}0$ given that at $x = 0$, $y = 1$ and $y' = 0$.

This is the same problem that we have just encountered and in due course we shall compare results. As expected, we shall use a spreadsheet to derive the solution. The headings for the sheet this time will be

	A	B	C	D	E	F	G	H	I	J	K	L
1	**n**	**x**	**k1**	**k2**	**k3**	**k4**	**P**	**Q**	**y**	**y′**	**y″**	***h*=**

The entries will then be

1. Column A contains the iteration number from 0 in A2 to 10 in A12.
2. Cell M1 contains the x step length which is 0·2.
3. Column B contains the successive x-values from 0·0 to 2·0 in steps of 0·2. The initial value of $x_0 = 0$ is entered into cell B2 and the formula =B2+$M$1 is entered into cell B3 and copied into cells B4 to B12.
4. Column C contains the computed k_1-values and the equation $k_1 = \frac{1}{2}h^2(y'')_0$ is represented in cell C2 by the formula ....

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=(0·5) * ($M$1^2) * K2

The contents of cell C2 are then copied into cells C3 to C12.

5 Column D contains the computed k_2-values and the equation

$$\begin{aligned}k_2 &= \tfrac{1}{2}h^2 f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + k_1/h)\\ &= \tfrac{1}{2}h^2\big((x_0 + \tfrac{1}{2}h)\big((y')_0 + k_1/h\big) + y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1\big)\end{aligned}$$

is represented in cell D2 by the formula ....

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=(0·5) * ($M$1^2) * ((B2 + 0·5 * $M$1) * (J2 + C2/$M$1)
+ I2 + 0·5 * $M$1 * J2+0·25 *C2)

The contents of cell D2 are then copied into cells D3 to D12.

6 Column E contains the computed k_3-values and the equation

$$\begin{aligned}k_3 &= \tfrac{1}{2}h^2 f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + k_2/h)\\ &= \tfrac{1}{2}h^2\big((x_0 + \tfrac{1}{2}h)\big((y')_0 + k_2/h\big) + y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1\big)\end{aligned}$$

is represented in cell E2 by the formula ....

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=(0·5) * ($M$1^2) * ((B2 + 0·5 * $M$1) * (J2 + D2/$M$1)
+ I2 + 0·5 * $M$1 * J2+0·25 *C2)

The contents of cell E2 are then copied into cells E3 to E12.

7 Column F contains the computed k_4-values and the equation

$$\begin{aligned}k_4 &= \tfrac{1}{2}h^2 f(x_0 + h,\ y_0 + h(y')_0 + k_3,\ (y')_0 + 2k_3/h)\\ &= \tfrac{1}{2}h^2\big((x_0 + h)\big((y')_0 + 2k_3/h\big) + y_0 + h(y')_0 + k_3\big)\end{aligned}$$

is represented in cell F2 by the formula ....

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=(0·5) * ($M$1^2) * ((B2 + $M$1) * (J2 + 2 * E2/$M$1)
+ I2 + $M$1 * J2 + E2)

The contents of cell F2 are then copied into cells F3 to F12.

8 Column G contains the computed P-values and the equation $P = \frac{1}{3}(k_1 + k_2 + k_3)$ is represented in cell G2 by the formula ....

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$$\boxed{=(1/3)*(\text{C2}+\text{D2}+\text{E2})}$$

The contents of cell G2 are then copied into cells G3 to G12.

9 Column H contains the computed Q-values and the equation $Q = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$ is represented in cell H2 by the formula ....

55

$$\boxed{=(1/3)*(\text{C2}+2*\text{D2}+2*\text{E2}+\text{F2})}$$

The contents of cell H2 are then copied into cells H3 to H12.

10 Column I contains the computed y-values. The initial value of $y_0 = 1$ is entered into cell I2 and the equation

$$y_1 = y_0 + h(y')_0 + P$$

is represented in cell I3 by the formula ....

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$$\boxed{=\text{I2}+\$\text{M}\$1*\text{J2}+\text{G2}}$$

The contents of cell I3 are then copied into cells I4 to I12.

11 Column J contains the computed y'-values. The initial value of $(y')_0 = 0$ is entered into cell J2 and the equation $(y')_1 = (y')_0 + Q/h$ is represented in cell J3 by the formula ....

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$$\boxed{=\text{J2}+\text{H2}/\$\text{M}\$1}$$

The contents of cell J3 are then copied into cells J4 to J12.

12 Column K contains the y''-values which are obtained from the equation $y'' = xy' + y$ which is represented in cell K2 by the formula ....

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$$\boxed{=\mathrm{B2}*\mathrm{J2}+\mathrm{I2}}$$

The contents of cell K2 are then copied into cells K3 to K12 and the final spreadsheet looks like the following

n	x	k1	k2	k3	k4	P	Q	y	y′	y″	$h = 0{\cdot}2$
0	0·0	0·0200000	0·0203000	0·0203030	0·0212182	0·0202010	0·0408081	1·0000000	0·0000000	1·0000000	
1	0·2	0·0212202	0·0227790	0·0228258	0·0251351	0·0222750	0·0458550	1·0202010	0·2040403	1·0610091	
2	0·4	0·0251322	0·0282477	0·0284035	0·0325752	0·0272612	0·0570033	1·0832841	0·4333153	1·2566102	
3	0·6	0·0325641	0·0378798	0·0382519	0·0451961	0·0362319	0·0766745	1·1972083	0·7183318	1·6282074	
4	0·8	0·0451694	0·0538673	0·0546501	0·0660061	0·0512289	0·1094035	1·3771066	1·1017045	2·2584702	
5	1·0	0·0659480	0·0801269	0·0816865	0·1003762	0·0759205	0·1633170	1·6486764	1·6487218	3·2973982	
6	1·2	0·1002542	0·1236497	0·1266912	0·1579840	0·1168650	0·2529733	2·0543413	2·4653068	5·0127095	
7	1·4	0·1577302	0·1970991	0·2030044	0·2565931	0·1859446	0·4048434	2·6642677	3·7301734	7·8865105	
8	1·6	0·2560654	0·3238945	0·3354254	0·4295622	0·3051284	0·6680891	3·5962469	5·7543906	12·8032719	
9	1·8	0·4284592	0·5483881	0·5711745	0·7411112	0·5160073	1·1362318	5·0522535	9·0948361	21·4229585	
10	2·0	0·7387844	0·9567270	1·0024949	1·3181400	0·8993354	1·9917894	7·3872279	14·7759954	36·9392186	

The errors have been dramatically reduced, as can be seen from the following table in comparison with those in Frame 47.

n	x	Exact	Error (%)
0	0·0	1·0000000	0·00
1	0·2	1·0202013	0·00
2	0·4	1·0832871	0·00
3	0·6	1·1972174	0·00
4	0·8	1·3771278	0·00
5	1·0	1·6487213	0·00
6	1·2	2·0544332	0·00
7	1·4	2·6644562	0·01
8	1·6	3·5966397	0·01
9	1·8	5·0530903	0·02
10	2·0	7·3890561	0·02

Next frame

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Now here is one for you to do entirely on your own. The method is exactly the same as before and there are no snags. Use the spreadsheet that you created for the previous example as a template for this one.

Example 2

Solve the equation

$$y'' = x - y^2$$

for $x = 0{\cdot}0(0{\cdot}2)2{\cdot}0$ where at $x = 0$, $y = 0$ and $y' = 0$.

When you have finished, check the results with the next frame

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n	x	k1	k2	k3	k4	P	Q	y	y′	y″	h=0·2
0	0·0	0·0000000	0·0020000	0·0020000	0·0039999	0·0013333	0·0040000	0·0000000	0·0000000	0·0000000	
1	0·2	0·0040000	0·0059996	0·0059996	0·0079974	0·0053331	0·0119986	0·0013333	0·0199999	0·1999982	
2	0·4	0·0079977	0·0099915	0·0099915	0·0119731	0·0093269	0·0199789	0·0106664	0·0799930	0·3998862	
3	0·6	0·0119741	0·0139351	0·0139351	0·0158524	0·0132814	0·0278556	0·0359919	0·1798875	0·5987046	
4	0·8	0·0158546	0·0177065	0·0177065	0·0194436	0·0170892	0·0353748	0·0852508	0·3191655	0·7927323	
5	1·0	0·0194477	0·0210264	0·0210264	0·0223594	0·0205002	0·0419709	0·1661731	0·4960396	0·9723865	
6	1·2	0·0223654	0·0233782	0·0233782	0·0239421	0·0230406	0·0466068	0·2858812	0·7058940	1·1182719	
7	1·4	0·0239482	0·0239504	0·0239504	0·0232394	0·0239497	0·0476631	0·4501006	0·9389280	1·1974094	
8	1·6	0·0232395	0·0216639	0·0216639	0·0191107	0·0221891	0·0430019	0·6618359	1·1772436	1·1619732	
9	1·8	0·0190914	0·0153806	0·0153806	0·0105578	0·0166175	0·0303905	0·9194737	1·3922530	0·9545681	
10	2·0	0·0104978	0·0043750	0·0043750	−0·0026809	0·0064159	0·0084390	1·2145418	1·5442053	0·5248882	

Because

The only items that need amending from the previous spreadsheet are the references to the actual differential equation. Consequently

The formula in D2 for k_2 now reads as

= (0·5) * ($M$1^2) * (B2 + 0·5 * $M$1 − (I2 + 0·5 * $M$1 * J2 + 0·25 * C2)^2)

The formula in E2 for k_3 now reads as

= 0·5 * ($M$1^2) * (B2 + 0·5 * $M$1 − (I2 + 0·5 * $M$1 * J2 + 0·25 * C2)^2)

The formula in F2 for k_4 now reads as

= 0·5 *($M$1^2) * (B2+$M$1 − (I2 + $M$1 * J2 + E2)^2)

The formula in K2 for y'' now reads as

= B2 − I2^2

Predictor–corrector methods

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So far, all the methods that we have used for the numerical solution of differential equations have been *single-step* methods. By this is meant that, given the differential equation $y' = f(x, y)$, a set of starting values (x_0 and y_0) and a step length (h), we can then find the value of y_1. The values of x_1 and y_1 become the starting values for the next iteration and so the procedure goes on, one step at a time. More accurate methods employ a *multi-step* procedure where, instead of starting with just a single set of initial values, we use a collection of previously calculated values.

A very simple multi-step method is given by the equations

$$\bar{y}_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + \tfrac{1}{2}h\big(f(x_0, y_0) + f(x_1, \bar{y}_1)\big)$$

Here we calculate $\bar{y}_1$ first from the given initial conditions x_0 and y_0. We call this equation the *predictor* because it gives $\bar{y}_1$ as a first estimate of y_1. Using $\bar{y}_1$ in the second equation then gives a more accurate value for y_1. We call this equation the *corrector*.

▶

An even better pair of predictor–corrector equations is given by

$$\bar{y}_{i+1} = y_i + \tfrac{1}{2}h(3f(x_i, y_i) - f(x_{i-1}, y_{i-1}))$$
$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})\big) \quad \text{for } i = 0, 1, 2, 3, \ldots$$

Here, in order to use the predictor for the first time when $i = 0$ we need to know the value of $f(x_{0-1}, y_{0-1}) = f(x_{-1}, y_{-1})$, which we do not. Instead we shall use the equation $\bar{y}_1 = y_0 + hf(x_0, y_0)$ when $i = 0$.

In the next frame we shall look at an example

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Example

Solve the equation $y' = x + y$ for $x = 0{\cdot}0(0{\cdot}1)1{\cdot}0$ where $y = 1$ when $x = 0$.

We have solved this equation before in Frame 32 using the Euler–Cauchy method and have viewed the accuracy of this method when compared with the exact solution. Here we shall see that this predictor–corrector method is even more accurate. Set up the following heading on your spreadsheet

	A	B	C	D	E	F	G
1	**n**	**x**	**y***	**y**	**Exact**	**Errors (%)**	***h*=**

As usual, column A contains the iteration numbers 0 to 10 in cells A2 to A12 and column B contains the x-values stepped according to the step length $h = 0{\cdot}1$ which is in cell H1. The initial value of $y = 1$ must be entered into cell D2.

Column C contains the predictor values given by the equations

$$\bar{y}_1 = y_0 + hf(x_0, y_0)$$
$$\bar{y}_{i+1} = y_i + \tfrac{1}{2}h(3f(x_i, y_i) - f(x_{i-1}, y_{i-1})) \quad \text{for } i > 0$$

To accommodate these equations in cell C3 enter the formula ....

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=D2 + $H$1 * (B2 + D2)

And in cell C4 enter the formula ....

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=D3 + 0·5 * $H$1 * (3 * B3 + 3 * D3 − B2 − D2)

And copy into cells C5 to C12.

Column D contains the corrector values given by the equation

$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})\big)$$

To accommodate this equation in cell D3 enter the formula ....

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$$=D2+0{\cdot}5*\$H\$1*(B2+D2+B3+C3)$$

And copy into cells D4 to D12.

We have seen that the exact solution to this equation is $2e^x - x - 1$, so this can be programmed into the sheet entering the formula

$=2*EXP(B2) - B2 - 1$ in cell E2 and then copying it into cells E3 to E12.

The final table looks as follows

n	x	y*	y	Exact	Error (%)	$h = 0{\cdot}1$
0	0·0		1·0000000	1·0000000	0·00	
1	0·1	1·1000000	1·1100000	1·1103418	0·03	
2	0·2	1·2415000	1·2425750	1·2428055	0·02	
3	0·3	1·3984613	1·3996268	1·3997176	0·01	
4	0·4	1·5824421	1·5837303	1·5836494	−0·01	
5	0·5	1·7963085	1·7977322	1·7974425	−0·02	
6	0·6	2·0432055	2·0447791	2·0442376	−0·03	
7	0·7	2·3266093	2·3283485	2·3275054	−0·04	
8	0·8	2·6503618	2·6522840	2·6510819	−0·05	
9	0·9	3·0187092	3·0208337	3·0192062	−0·05	
10	1·0	3·4363445	3·4386926	3·4365637	−0·06	

Here the errors are significantly reduced, as seen from the comparisons below.

1	2	3
0·00	0·00	0·00
0·93	0·03	0·03
1·84	0·06	0·02
2·69	0·09	0·01
3·50	0·12	−0·01
4·25	0·14	−0·02
4·95	0·17	−0·03
5·59	0·19	−0·04
6·18	0·21	−0·05
6·73	0·23	−0·05
7·25	0·24	−0·06

Here **1** refers to Euler, **2** refers to Euler–Cauchy and **3** refers to the predictor–corrector method just used.

And that is it. There are many other more sophisticated methods for the solution of ordinary differential equations by numerical methods and a detailed study of these is a course in itself. The methods we have used give an introduction to the processes and are practical in application.

The **Revision summary** and **Can You?** checklist now follow as usual. Check them carefully and refer back to the Programme for any points that may need further brushing up. Then you will be ready for the **Test exercise**, and the **Further problems** provide further practice.

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Revision summary 9

1 *Taylor's series*

$$f(a+h)=f(a)+hf'(a)+\frac{h}{2!}f''(a)+\frac{h}{3!}f'''(a)+\ldots$$

2 *Solution of first-order differential equations*
Equation $y'=f(x,y)$ with $y=y_0$ at $x=x_0$ for $x_0(h)x_n$.

(a) *Euler's method*

$$y_1=y_0+h(y')_0.$$

(b) *Euler–Cauchy method*

$$\begin{aligned}x_1&=x_0+h\\ \bar{\bar{y}}_1&=y_0+h(y')_0\\ y_1&=y_0+\tfrac{1}{2}h\{(y')_0+f(x_1,\bar{\bar{y}}_1)\}\\ (y')_1&=f(x_1,\ y_1).\end{aligned}$$

(c) *Runge–Kutta method*

$$\begin{aligned}x_1&=x_0+h\\ k_1&=hf(x_0,\ y_0)=h(y')_0\\ k_2&=hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1)\\ k_3&=hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_2)\\ k_4&=hf(x_0+h,\ y_0+k_3)\\ \Delta y_0&=\tfrac{1}{6}(k_1+2k_2+2k_3+k_4)\\ y_1&=y_0+\Delta y_0\\ (y')_1&=f(x_1,\ y_1).\end{aligned}$$

3 *Solution of second-order differential equations*
Equation $y''=f(x,y,y')$ with $y=y_0$ and $y'=(y')_0$ at $x=x_0$ for $x=x_0(h)x_n$.

(a) *Euler's second-order method*

$$\begin{aligned}y_1&=y_0+h(y')_0+\frac{h^2}{2!}(y'')_0\\ (y')_1&=(y')_0+h(y'')_0.\end{aligned}$$

▶

(b) *Runge–Kutta method*

$$x_1 = x_0 + h$$
$$k_1 = \tfrac{1}{2}h^2 f\{x_0,\, y_0,\, (y')_0\} = \tfrac{1}{2}h^2 (y'')_0$$
$$k_2 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_1}{h}\right\}$$
$$k_3 = \tfrac{1}{2}h^2 f\left\{x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}h(y')_0 + \tfrac{1}{4}k_1,\ (y')_0 + \frac{k_2}{h}\right\}$$
$$k_4 = \tfrac{1}{2}h^2 f\left\{x_0 + h,\ y_0 + h(y')_0 + k_3,\ (y')_0 + \frac{2k_3}{h}\right\}$$
$$P = \tfrac{1}{3}(k_1 + k_2 + k_3)$$
$$Q = \tfrac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_1 = y_0 + h(y')_0 + P$$
$$(y')_1 = (y')_0 + \frac{Q}{h}$$
$$(y'')_1 = f\{x_1,\, y_1,\, (y')_1\}.$$

4 *Predictor–corrector*

Equation $y' = f(x, y)$ with $y = y_0$ and $y' = (y')_0$ at $x = x_0$ for $x = x_0(h)x_n$, then

Predictor

$$\overline{y}_{i+1} = y_i + \tfrac{1}{2}h(3f(x_i, y_i) - f(x_{i-1}, y_{i-1})) \quad \text{for } i = 1, 2, 3, \ldots$$
$$\overline{y}_1 = y_0 + hf(x_0, y_0) \quad \text{for } i = 0$$

Corrector

$$y_{i+1} = y_i + \tfrac{1}{2}h\big(f(x_i, y_i) + f(x_{i+1}, \overline{y}_{i+1})\big) \quad \text{for } i = 0, 1, 2, 3, \ldots$$

Can You?

Checklist 9

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Derive a form of Taylor's series from Maclaurin's series and from it describe a function increment as a series of first and higher-order derivatives of the function? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply by means of a spreadsheet the Euler method, the Euler–Cauchy method and the Runge–Kutta method for first-order differential equations? 4 to 43

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply by means of a spreadsheet the Euler second-order method and the Runge–Kutta method for second-order ordinary differential equations? 44 to 60

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply by means of a spreadsheet a simple predictor–corrector method? 61 to 65

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 9

68

1 Apply Euler's method to solve the equation

$$\frac{dy}{dx} = 1 + xy \quad \text{for} \quad x = 0(0{\cdot}1)0{\cdot}5$$

given that at $x = 0$, $y = 1$.

2 The equation $\frac{dy}{dx} = x^2 - 2y$ is subject to the initial condition $y = 0$ at $x = 1$. Use the Euler–Cauchy method to obtain function values for $x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$.

3 Using the Runge–Kutta method, solve the equation

$$\frac{dy}{dx} = 1 + y - x \quad \text{for} \quad x = 0(0{\cdot}1)0{\cdot}5$$

given that $y = 1$ when $x = 0$.

4 Apply Euler's second-order method to solve the equation

$$y'' = y - x \quad \text{for} \quad x = 2{\cdot}0(0{\cdot}1)2{\cdot}5$$

given that at $x = 2$, $y = 3$ and $y' = 0$.

5 Use the Runge–Kutta method to solve the equation

$$y'' = (y'/x) + y \quad \text{for} \quad x = 1{\cdot}0(0{\cdot}1)1{\cdot}5$$

given the initial conditions that at $x = 1{\cdot}0$, $y = 0$ and $y' = 1{\cdot}0$.

6 Use the predictor–corrector method in the text to solve the equation

$$y' = 1 + xy \quad \text{for } x = 0(0{\cdot}1)1$$

given that $x = 0$ when $y = 0$.

Further problems 9

69 Solve the following differential equations by the methods indicated.

Euler's method

1	$y' = 2x - y$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
2	$y' = 2x + y^2$	$x = 0,\ y = 1{\cdot}4$	$x = 0(0{\cdot}1)0{\cdot}5$

Euler–Cauchy method

3	$y' = 2 - y/x$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$
4	$y' = x^2 - 2x + y$	$x = 0,\ y = 0{\cdot}5$	$x = 0(0{\cdot}1)0{\cdot}5$
5	$y' = (y - x^2)^{\frac{1}{2}}$	$x = 0,\ y = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
6	$y' = \dfrac{x + y}{xy}$	$x = 1,\ y = 1$	$x = 1{\cdot}0(0{\cdot}1)1{\cdot}5$
7	$y' = y\sin x + \cos x$	$x = 0,\ y = 0$	$x = 0(0{\cdot}1)0{\cdot}5$

Runge Kutta method

8	$y' = 2x - y$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
9	$y' = x - y^2$	$x = 0,\ y = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
10	$y' = y^2 - xy$	$x = 0,\ y = 0{\cdot}4$	$x = 0(0{\cdot}2)1{\cdot}0$
11	$y' = \sqrt{2x + y}$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$
12	$y' = 1 - x^3/y$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
13	$y' = \dfrac{y - x}{y + x}$	$x = 0,\ y = 1$	$x = 0(0{\cdot}2)1{\cdot}0$

Euler second-order method

14	$y'' = (x+1)y' + y$	$x = 0,\ y = 1,\ y' = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
15	$y'' = 2(xy' - 4y)$	$x = 0,\ y = 3,\ y' = 0$	$x = 0(0{\cdot}1)0{\cdot}5$

Runge–Kutta second-order method

16	$y'' = x - y - xy'$	$x = 0,\ y = 0,\ y' = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
17	$y'' = (1 - x)y' - y$	$x = 0,\ y = 1,\ y' = 1$	$x = 0(0{\cdot}2)1{\cdot}0$
18	$y'' = 1 + x - y^2$	$x = 0,\ y = 2,\ y' = 1$	$x = 0(0{\cdot}1)0{\cdot}5$
19	$y'' = (x+2)y - 2y'$	$x = 0,\ y = 1,\ y' = 0$	$x = 0(0{\cdot}2)1{\cdot}0$
20	$y'' = \dfrac{y - xy'}{x^2}$	$x = 1,\ y = 0,\ y' = 1$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$

Predictor–corrector

21	$y' = 2 - y/x$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$
22	$y' = 2x - y$	$x = 0,\ y = 1$	$x = 0{\cdot}0(0{\cdot}2)1{\cdot}0$
23	$y' = \sqrt{2x + y}$	$x = 1,\ y = 2$	$x = 1{\cdot}0(0{\cdot}2)2{\cdot}0$

Partial differentiation

Frames 1 to 81

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the expression for a small increment in an expression of two real variables using Taylor's theorem
- Apply the notion of small increments in expressions in two and three real variables to a variety of problems
- Determine the rate of change with respect to time of an expression involving two or three real variables
- Differentiate implicit functions
- Determine first and second derivatives involving change of variables in expressions of two real variables
- Use the Jacobian to obtain the derivatives of inverse functions of two real variables
- Locate and identify maxima, minima and saddle points of functions of two real variables
- Solve problems where the independent variables are constrained by using the method of Lagrange undetermined multipliers for functions of two and three real variables

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 9 Differentiation applications 2, 10 Partial differentiation 1 and 11 Partial differentiation 2

Small increments

Taylor's theorem for one independent variable

1

Taylor's theorem expands $f(x+h)$ in terms of $f(x)$, powers of h and successive derivatives of $f(x)$, and can be stated as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

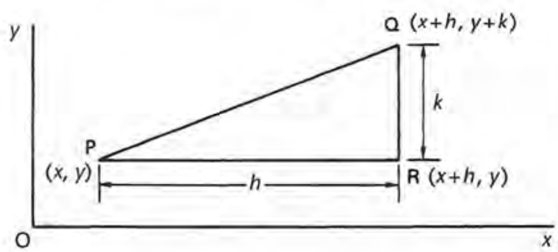
where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$. You will also, no doubt, remember that, by putting $x = 0$ in the result and then letting $h = x$, we obtain Maclaurin's series

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \ldots + \frac{h^n}{n!}f^{(n)}(0) + \ldots$$

Taylor's theorem for two independent variables

If we consider $z = f(x, y)$ where z is a function of two independent variables x and y, then, in general, increases in x and y will produce a combined increase in z.

So, if $z = f(x, y)$ then $z + \delta z = f(x+h,\ y+k)$



h = increase in x
k = increase in y.

For R: $$f(x+h,\ y) = f(x,\ y) + hf_x(x,\ y) + \frac{h^2}{2!}f_{xx}(x,\ y) + \ldots \qquad (1)$$

where $f_x(x, y)$ denotes $\dfrac{\partial}{\partial x}f(x, y)$; $f_{xx}(x, y)$ denotes $\dfrac{\partial^2}{\partial x^2}f(x, y)$ etc.

From R to Q: $(x+h)$ is constant; y changes to $(y+k)$

$$\therefore\ f(x+h,\ y+k) = f(x+h,\ y) + kf_y(x+h,\ y) + \frac{k^2}{2!}f_{yy}(x+h,\ y) + \ldots \qquad (2)$$

To express (2) in terms of $f(x, y)$ we can substitute result (1) for the first term $f(x+h, y)$ and similar expressions which we shall obtain for $f_y(x+h, y)$, $f_{yy}(x+h, y)$ and so on.

If we differentiate (1) with respect to y, we have

$$f_y(x+h,\ y) = \ldots\ldots\ldots\ldots$$

2

$$f_y(x+h,\,y) = f_y(x,\,y) + hf_{yx}(x,\,y) + \frac{h^2}{2!}f_{yxx}(x,\,y) + \dots$$

If we now differentiate this result again with respect to y

$$f_{yy}(x+h,\,y) = \dots\dots\dots\dots$$

3

$$f_{yy}(x+h,y) = f_{yy}(x,y) + hf_{yyx}(x,y) + \frac{h^2}{2!}f_{yyxx}(x,y) + \dots$$

Then our previous expansion (2), i.e.

$$f(x+h,y+k) = f(x+h,y) + kf_y(x+h,y) + \frac{k^2}{2!}f_{yy}(x+h,y) + \dots$$

now becomes

$$\begin{aligned} f(x+h,\,y+k) = f(x,\,y) + hf_x(x,\,y) + \frac{h^2}{2!}f_{xx}(x,\,y) + \dots \\ + k\left\{f_y(x,\,y) + hf_{yx}(x,\,y) + \frac{h^2}{2!}f_{yxx}(x,\,y) + \dots\right\} \\ + \frac{k^2}{2!}\left\{f_{yy}(x,\,y) + hf_{yyx}(x,\,y) + \frac{h^2}{2!}f_{yyxx}(x,\,y) + \dots\right\} \\ + \dots \end{aligned}$$

Rearranging the terms by collecting together all the first derivatives, and then all the second derivatives, and so on, we get

$$f(x+h,\,y+k) = \dots\dots\dots\dots$$

4

$$\begin{aligned} f(x+h,y+k) = f(x,y) + \{hf_x(x,y) + kf_y(x,y)\} \\ + \frac{1}{2!}\{h^2 f_{xx}(x,y) + 2hkf_{xy}(x,y) + k^2 f_{yy}(x,y)\} + \dots \end{aligned}$$

This is Taylor's theorem for two independent variables.

Small increments

If $z = f(x, y)$, $h = \delta x$, $k = \delta y$, then Taylor's theorem can be written as

$$z + \delta z = z + \left\{ h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} \right\} + \frac{1}{2!}\left\{ h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial y\,\partial x} + k^2\frac{\partial^2 z}{\partial y^2} \right\} + \ldots$$

Subtracting z from each side:

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \frac{1}{2!}\left\{ \frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial y\,\partial x}(\delta x\,\delta y) + \frac{\partial^2 z}{\partial y^2}(\delta y)^2 \right\} + \ldots$$

Since δx and δy are small, the expression in the brackets is of the next order of smallness and can be discarded for our purposes. Therefore, we arrive at the result

$$\text{If} \quad z = f(x, y) \quad \text{then} \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

As already explained above, this result is, in fact, an approximation since the smaller terms in the series have been neglected. For practical purposes, however, the result can be used as stated. **Be sure to make a note of the result, for it is the foundation of much that follows.**

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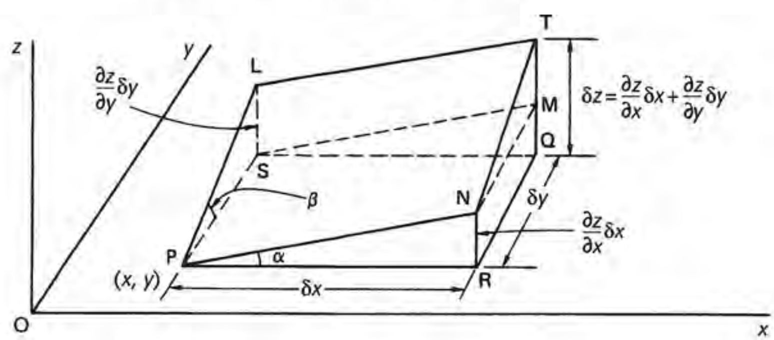
$$\boxed{z = f(x, y); \qquad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y}$$

The following diagram illustrates the result.



$$\frac{\partial z}{\partial x} \text{ is the slope of PN} \quad \therefore\ \text{RN} = \frac{\partial z}{\partial x}\delta x = \text{QM}$$

$$\frac{\partial z}{\partial y} \text{ is the slope of PL} \quad \therefore\ \text{SL} = \frac{\partial z}{\partial y}\delta y = \text{MT}$$

$$\text{QT} = \text{QM} + \text{MT} \quad \therefore\ \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

This is the total increment of $z = f(x, y)$ from P to Q.

▶

It is worth noting at this stage that the result can be extended to the case of three independent variables, i.e. if $u = f(x, y, z)$

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

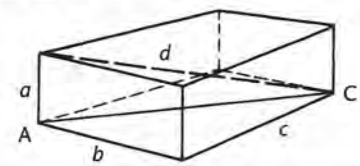
One or two straightforward applications will lay the foundations for future development.

Example

A rectangular box has sides measured as 30 mm, 40 mm and 60 mm. If these measurements are liable to be in error by $\pm 0{\cdot}5$ mm, $\pm 0{\cdot}8$ mm and $\pm 1{\cdot}0$ mm respectively, calculate the length of the diagonal of the box and the maximum possible error in the result.

First build up an expression for the diagonal d in terms of the sides, a, b and c.

$d = \ldots\ldots\ldots\ldots$



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$$\boxed{d = \sqrt{a^2 + b^2 + c^2}}$$

Because

$$d^2 = a^2 + \mathrm{AC}^2 = a^2 + b^2 + c^2 \text{ and so } d = \sqrt{a^2 + b^2 + c^2}$$

Then $\delta d = \dfrac{\partial d}{\partial a}\delta a + \dfrac{\partial d}{\partial b}\delta b + \dfrac{\partial d}{\partial c}\delta c$

We now determine the partial differential coefficients and obtain an expression for δd, but all in terms of a, b and c. Do not yet insert numerical values.

$\delta d = \ldots\ldots\ldots\ldots$

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$$\boxed{\delta d = \frac{1}{\sqrt{a^2 + b^2 + c^2}}\{a\delta a + b\delta b + c\delta c\}}$$

Now, substituting the values $a = 30$, $b = 40$, $c = 60$

$\delta a = \pm 0{\cdot}5$, $\delta b = \pm 0{\cdot}8$, $\delta c = \pm 1{\cdot}0$

the calculated length of the diagonal $= \ldots\ldots\ldots\ldots$

the maximum possible error $= \ldots\ldots\ldots\ldots$

$$\text{diagonal} = \sqrt{a^2 + b^2 + c^2} = 78{\cdot}10 \text{ mm}$$
$$\text{maximum error} = \pm\, 1{\cdot}37 \text{ mm}$$

Because

$$\delta d = \frac{1}{78{\cdot}10}\{30(\pm\, 0{\cdot}5) + 40(\pm\, 0{\cdot}8) + 60(\pm\, 1{\cdot}0)\}$$

Greatest error when the signs are the same

$$\therefore\ \delta d = \frac{1}{78{\cdot}10}\{\pm\,(15 + 32 + 60)\} = \pm 1{\cdot}37 \text{ mm}$$

Rates of change

If $z = f(x, y)$, then we have seen that $\delta z = \dfrac{\partial z}{\partial x}\delta x + \dfrac{\partial z}{\partial y}\delta y$

Dividing through by δt: $\quad \dfrac{\delta z}{\delta t} = \dfrac{\partial z}{\partial x}\dfrac{\delta x}{\delta t} + \dfrac{\partial z}{\partial y}\dfrac{\delta y}{\delta t}$

Then if $\delta t \to 0$: $\quad \dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\partial z}{\partial x}\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{\partial z}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}t}$

Note the result. Then on to an example.

Example

The base radius r of a right circular cone is increasing at the rate of 1·5 mm/s while the perpendicular height is decreasing at 6·0 mm/s. Determine the rate at which the volume V is changing when $r = 12$ mm and $h = 24$ mm.

Find an expression for $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ in terms of r and h which is …………

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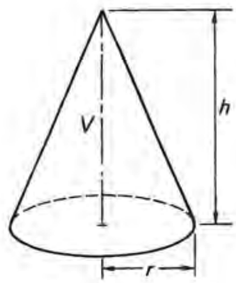
$$\boxed{\frac{dV}{dt} = \frac{2\pi rh}{3} \cdot \frac{dr}{dt} + \frac{\pi r^2}{3} \cdot \frac{dh}{dt}}$$



$$V = \frac{1}{3}\pi r^2 h; \quad \frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

$$\frac{\partial V}{\partial r} = \frac{2\pi rh}{3}; \quad \frac{\partial V}{\partial h} = \frac{\pi r^2}{3}$$

$$\therefore \frac{dV}{dt} = \frac{2\pi rh}{3} \cdot \frac{dr}{dt} + \frac{\pi r^2}{3} \cdot \frac{dh}{dt}$$

Finally, we insert the numerical values:

$r = 12$; $h = 24$; $\dfrac{dr}{dt} = 1{\cdot}5$; $\dfrac{dh}{dt} = -6{\cdot}0$ (h is decreasing)

$$\frac{dV}{dt} = 288\pi - 288\pi = 0$$

$\therefore$ At the instant when $r = 12$ mm and $h = 24$ mm,

the volume is unchanging.

Implicit functions

The same initial result, $\delta z = \dfrac{\partial z}{\partial x}\delta x + \dfrac{\partial z}{\partial y}\delta y$ enables us to determine the derivative of an implicit function $f(x, y) = 0$, i.e. in a case where y is not defined explicitly in terms of x.

If $f(x, y) = 0$ is an implicit function, we let $z = f(x, y)$.

Then, as before: $$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

Dividing through by δx: $$\frac{\delta z}{\delta x} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{\delta y}{\delta x}$$

Then, if $\delta x \to 0$: $$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

But $z = 0 \quad \therefore \dfrac{dz}{dx} = 0$ $$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{\partial z}{\partial x} \Big/ \frac{\partial z}{\partial y}\right)$$

So, if $x^2 - xy - y^2 = 0$, $\dfrac{dy}{dx} = \dots\dots\dots\dots$

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$$\boxed{\frac{dy}{dx} = \frac{2x - y}{x + 2y}}$$

Putting $z = x^2 - xy - y^2$, $\dfrac{\partial z}{\partial x} = 2x - y$ and $\dfrac{\partial z}{\partial y} = -x - 2y$

The rest follows immediately.

Now on to the next frame

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The work so far, important though it is, is largely by way of revision of the more basic ideas of partial differentiation. We now extend these same ideas to further applications.

Change of variables

If $z = f(x, y)$ and x and y are themselves functions of two new independent variables, u and v, then we need expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Yet again, we start from the result we established at the beginning of this Programme.

If $z = f(x, y)$ then $\delta z = \dfrac{\partial z}{\partial x}\delta x + \dfrac{\partial z}{\partial y}\delta y$

Dividing in turn by δu and δv:

$$\frac{\delta z}{\delta u} = \frac{\partial z}{\partial x} \cdot \frac{\delta x}{\delta u} + \frac{\partial z}{\partial y} \cdot \frac{\delta y}{\delta u}$$

$$\frac{\delta z}{\delta v} = \frac{\partial z}{\partial x} \cdot \frac{\delta x}{\delta v} + \frac{\partial z}{\partial y} \cdot \frac{\delta y}{\delta v}$$

Then, as $\delta u \to 0$ and $\delta v \to 0$, these become

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

▶

Example 1

If $z = x^2 - y^2$ and $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

We now need the various partial derivatives

$$\frac{\partial z}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial x}{\partial r} = \dots\dots\dots\dots; \quad \frac{\partial y}{\partial r} = \dots\dots\dots\dots$$

$$\frac{\partial z}{\partial y} = \dots\dots\dots\dots; \quad \frac{\partial x}{\partial \theta} = \dots\dots\dots\dots; \quad \frac{\partial y}{\partial \theta} = \dots\dots\dots\dots$$

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$$\boxed{\begin{array}{lll} \dfrac{\partial z}{\partial x} = 2x; & \dfrac{\partial x}{\partial r} = \cos\theta; & \dfrac{\partial y}{\partial r} = \sin\theta \\ \dfrac{\partial z}{\partial y} = -2y; & \dfrac{\partial x}{\partial \theta} = -r\sin\theta; & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{array}}$$

Substituting in the two equations and simplifying:

$$\frac{\partial z}{\partial r} = \dots\dots\dots\dots; \quad \frac{\partial z}{\partial \theta} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial z}{\partial r} = 2x\cos\theta - 2y\sin\theta; \quad \frac{\partial z}{\partial \theta} = -(2xr\sin\theta + 2yr\cos\theta)}$$

Finally, we can express x and y in terms of r and θ as given, so that, after tidying up, we obtain

$$\frac{\partial z}{\partial r} = \dots\dots\dots\dots; \quad \frac{\partial z}{\partial \theta} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial z}{\partial r} = 2r(\cos^2\theta - \sin^2\theta); \quad \frac{\partial z}{\partial \theta} = -4r^2\sin\theta\cos\theta}$$

Of course, we could express these as

$$\frac{\partial z}{\partial r} = 2r\cos 2\theta \quad \text{and} \quad \frac{\partial z}{\partial \theta} = -2r^2\sin 2\theta$$

From these results, we can, if necessary, find the second partial derivatives in the normal manner.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}(2r\cos 2\theta) = 2\cos 2\theta$$

Similarly $\dfrac{\partial^2 z}{\partial \theta^2} = \dots\dots\dots\dots$ and $\dfrac{\partial^2 z}{\partial r\partial \theta} = \dots\dots\dots\dots$

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$$\boxed{\frac{\partial^2 z}{\partial \theta^2} = -4r^2 \cos 2\theta; \quad \frac{\partial^2 z}{\partial r \partial \theta} = -4r \sin 2\theta}$$

Because

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(-2r^2 \sin 2\theta) = -4r^2 \cos 2\theta$$

and $$\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial r}(-2r^2 \sin 2\theta) = -4r \sin 2\theta$$

Example 2

If $z = f(x, y)$ and $x = \frac{1}{2}(u^2 - v^2)$ and $y = uv$, show that

$$u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right)$$

Although this is much the same as the previous example, there is, at least, one difference. In this case, we are not told the precise nature of $f(x, y)$. We must remember that z is a function of x and y and, therefore, of u and v. With that in mind, we set off with the usual two equations.

$$\frac{\partial z}{\partial u} = \dots\dots\dots\dots$$
$$\frac{\partial z}{\partial v} = \dots\dots\dots\dots$$

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$$\boxed{\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}}$$

From the given information:

$$\frac{\partial x}{\partial u} = \dots\dots\dots\dots; \quad \frac{\partial y}{\partial u} = \dots\dots\dots\dots$$
$$\frac{\partial x}{\partial v} = \dots\dots\dots\dots; \quad \frac{\partial y}{\partial v} = \dots\dots\dots\dots$$

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$$\frac{\partial x}{\partial u} = u; \quad \frac{\partial y}{\partial u} = v$$
$$\frac{\partial x}{\partial v} = -v; \quad \frac{\partial y}{\partial v} = u$$

Whereupon $\dfrac{\partial z}{\partial u} = \ldots\ldots\ldots\ldots$

$\dfrac{\partial z}{\partial v} = \ldots\ldots\ldots\ldots$

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$$\frac{\partial z}{\partial u} = u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}$$
$$\frac{\partial z}{\partial v} = -v\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}$$

If we now multiply the first of these by $(-v)$ and the second by u and add the two equations, we get the desired result.

$$-v\frac{\partial z}{\partial u} = -uv\frac{\partial z}{\partial x} - v^2\frac{\partial z}{\partial y}$$
$$u\frac{\partial z}{\partial v} = -uv\frac{\partial z}{\partial x} + u^2\frac{\partial z}{\partial y}$$

Adding $\quad u\dfrac{\partial z}{\partial v} - v\dfrac{\partial z}{\partial u} = -2uv\dfrac{\partial z}{\partial x} + (u^2 - v^2)\dfrac{\partial z}{\partial y}$

$$= -2y\frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial y}$$
$$\therefore\ u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right)$$

With the same given data, i.e.

$z = f(x, y)$ with $x = \dfrac{1}{2}(u^2 - v^2)$ and $y = uv$

we can now show that $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = (u^2 + v^2)\left(\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}\right)$.

In determining the second partial derivatives, keep in mind that z is a function of u and v and that both of these variables also occur in $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial^2 z}{\partial u^2} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial^2 z}{\partial u^2}=\frac{\partial z}{\partial x}+u^2\frac{\partial^2 z}{\partial x^2}+2uv\frac{\partial^2 z}{\partial x\partial y}+v^2\frac{\partial^2 z}{\partial y^2}}$$

Because

$$\frac{\partial}{\partial u}=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\quad\text{and}\quad\frac{\partial}{\partial v}=\left(-v\frac{\partial}{\partial x}+u\frac{\partial}{\partial y}\right)$$

$$\therefore\ \frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(u\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y}\right)=\frac{\partial z}{\partial x}+u\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)+v\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)$$

$$=\frac{\partial z}{\partial x}+u\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial x}+v\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial y}$$

$$=\frac{\partial z}{\partial x}+u^2\frac{\partial^2 z}{\partial x^2}+uv\frac{\partial^2 z}{\partial x\partial y}+uv\frac{\partial^2 z}{\partial x\partial y}+v^2\frac{\partial^2 z}{\partial y^2}$$

$$\therefore\ \frac{\partial^2 z}{\partial u^2}=\frac{\partial z}{\partial x}+u^2\frac{\partial^2 z}{\partial x^2}+2uv\frac{\partial^2 z}{\partial x\partial y}+v^2\frac{\partial^2 z}{\partial y^2}\qquad(1)$$

Likewise, $\dfrac{\partial^2 z}{\partial v^2}=\dfrac{\partial}{\partial v}\left(\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial v}\left(-v\dfrac{\partial z}{\partial x}+u\dfrac{\partial z}{\partial y}\right)$

$= \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{\partial^2 z}{\partial v^2}=\frac{\partial z}{\partial x}+v^2\frac{\partial^2 z}{\partial x^2}-2uv\frac{\partial^2 z}{\partial x\partial y}+u^2\frac{\partial^2 z}{\partial y^2}}$$

Because

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial}{\partial v}\left(-v\frac{\partial z}{\partial x}+u\frac{\partial z}{\partial y}\right)=-\frac{\partial z}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)+u\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right)$$

$$=\frac{\partial z}{\partial x}-v\left(-v\frac{\partial}{\partial x}+u\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial x}+u\left(-v\frac{\partial}{\partial x}+u\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial y}$$

$$=\frac{\partial z}{\partial x}+v^2\frac{\partial^2 z}{\partial x^2}-uv\frac{\partial^2 z}{\partial x\partial y}-uv\frac{\partial^2 z}{\partial x\partial y}+u^2\frac{\partial^2 z}{\partial y^2}$$

$$\therefore\ \frac{\partial^2 z}{\partial v^2}=\frac{\partial z}{\partial x}+v^2\frac{\partial^2 z}{\partial x^2}-2uv\frac{\partial^2 z}{\partial x\partial y}+u^2\frac{\partial^2 z}{\partial y^2}\qquad(2)$$

Adding together results (1) and (2), we get

$\ldots\ldots\ldots\ldots$

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$$\boxed{\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)}$$

and that is it.

Now, for something slightly different, move on to the next frame

Inverse functions

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If $z = f(x, y)$ and x and y are functions of two independent variables u and v defined by $u = g(x, y)$ and $v = h(x, y)$, we can theoretically solve these two equations to obtain x and y in terms of u and v. Hence we can determine $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ and then $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ as required.

In practice, however, the solution of $u = g(x, y)$ and $v = h(x, y)$ may well be difficult or even impossible by normal means. The following example shows how we can get over this difficulty.

Example 1

If $z = f(x, y)$ and $u = e^x \cos y$ and $v = e^{-x} \sin y$, we have to find $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

We start off once again with our standard relationships

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \qquad (1)$$

$$\delta v = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y \qquad (2)$$

Now $u = e^x \cos y$ and $v = e^{-x} \sin y$

So $\dfrac{\partial u}{\partial x} = \dots\dots\dots\dots;\quad \dfrac{\partial u}{\partial y} = \dots\dots\dots\dots$

$\dfrac{\partial v}{\partial x} = \dots\dots\dots\dots;\quad \dfrac{\partial v}{\partial y} = \dots\dots\dots\dots$

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$$\boxed{\begin{array}{ll} \dfrac{\partial u}{\partial x} = e^x \cos y; & \dfrac{\partial u}{\partial y} = -e^x \sin y \\ \dfrac{\partial v}{\partial x} = -e^{-x} \sin y; & \dfrac{\partial v}{\partial y} = e^{-x} \cos y \end{array}}$$

Substituting in equations (1) and (2) above, we have

$$\delta u = e^x \cos y\, \delta x - e^x \sin y\, \delta y \qquad (3)$$

$$\delta v = -e^{-x} \sin y\, \delta x + e^{-x} \cos y\, \delta y \qquad (4)$$

Eliminating δy from (3) and (4), we get

$$\delta x = \dots\dots\dots\dots$$

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$$\boxed{\delta x = \frac{e^{-x}\cos y}{\cos 2y}\delta u + \frac{e^{x}\sin y}{\cos 2y}\delta v}$$

Because

$$(3) \times e^{-x}\cos y: \qquad e^{-x}\cos y\,\delta u = \cos^2 y\,\delta x - \sin y\cos y\,\delta y$$
$$(4) \times e^{x}\sin y: \qquad e^{x}\sin y\,\delta v = -\sin^2 y\,\delta x + \sin y\cos y\,\delta y$$

Adding: $e^{-x}\cos y\,\delta u + e^{x}\sin y\,\delta v = (\cos^2 y - \sin^2 y)\,\delta x$

$$\therefore\ \delta x = \frac{e^{-x}\cos y}{\cos 2y}\delta u + \frac{e^{x}\sin y}{\cos 2y}\delta v$$

But $\qquad \delta x = \dfrac{\partial x}{\partial u}\delta u + \dfrac{\partial x}{\partial v}\delta v$

$$\therefore\ \frac{\partial x}{\partial u} = \frac{e^{-x}\cos y}{\cos 2y} \quad \text{and} \quad \frac{\partial x}{\partial v} = \frac{e^{x}\sin y}{\cos 2y}$$

which are, of course, two of the expressions we have to find.
Starting again with equations (3) and (4), we can obtain

$$\delta y = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta y = \frac{e^{-x}\sin y}{\cos 2y}\delta u + \frac{e^{x}\cos y}{\cos 2y}\delta v}$$

Because

$$(3) \times e^{-x}\sin y: \qquad e^{-x}\sin y\,\delta u = \sin y\cos y\,\delta x - \sin^2 y\,\delta y$$
$$(4) \times e^{x}\cos y: \qquad e^{x}\cos y\,\delta v = -\sin y\cos y\,\delta x + \cos^2 y\,\delta y$$

Adding: $e^{-x}\sin y\,\delta u + e^{x}\cos y\,\delta v = (\cos^2 y - \sin^2 y)\,\delta y$

$$\therefore\ \delta y = \frac{e^{-x}\sin y}{\cos 2y}\delta u + \frac{e^{x}\cos y}{\cos 2y}\delta v$$

But, $\delta y = \ldots\ldots\ldots\ldots$ Finish it off.

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$$\delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v$$
$$\therefore \quad \frac{\partial y}{\partial u} = \frac{e^{-x}\sin y}{\cos 2y} \quad \text{and} \quad \frac{\partial y}{\partial v} = \frac{e^{x}\cos y}{\cos 2y}$$

So, collecting our four results together:

$$\frac{\partial x}{\partial u} = \frac{e^{-x}\cos y}{\cos 2y}; \quad \frac{\partial x}{\partial v} = \frac{e^{x}\sin y}{\cos 2y}$$
$$\frac{\partial y}{\partial u} = \frac{e^{-x}\sin y}{\cos 2y}; \quad \frac{\partial y}{\partial v} = \frac{e^{x}\cos y}{\cos 2y}$$

We can tackle most similar problems in the same way, but it is more efficient to investigate a general case and to streamline the results. Let us do that.

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General case

If $z = f(x, y)$ with $u = g(x, y)$ and $v = h(x, y)$, then we have

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \tag{1}$$
$$\delta v = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y \tag{2}$$

We now solve these for δx and δy. Eliminating δy, we have

$(1) \times \dfrac{\partial v}{\partial y}$: $\quad \dfrac{\partial v}{\partial y}\delta u = \dfrac{\partial v}{\partial y}\cdot\dfrac{\partial u}{\partial x}\delta x + \dfrac{\partial v}{\partial y}\cdot\dfrac{\partial u}{\partial y}\delta y$

$(2) \times \dfrac{\partial u}{\partial y}$: $\quad \dfrac{\partial u}{\partial y}\delta v = \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x}\delta x + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial y}\delta y$

Subtracting: $\quad \dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v = \left(\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}\right)\delta x$

$$\therefore \quad \delta x = \frac{\dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}}$$

Starting afresh from (1) and (2) and eliminating δx, we have

$$\delta y = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta y = \dfrac{\dfrac{\partial u}{\partial x}\delta v - \dfrac{\partial v}{\partial x}\delta u}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}}}$$

The two results so far are therefore

$$\delta x = \frac{\dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}} \quad \text{and} \quad \delta y = \frac{\dfrac{\partial u}{\partial x}\delta v - \dfrac{\partial v}{\partial x}\delta u}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}}$$

You will notice that the denominator is the same in each case and that it can be expressed in determinant form

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

This determinant is called the *Jacobian* of u, v with respect to x, y and is denoted by the symbol J:

i.e. $\quad J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$ and is often written as $\dfrac{\partial(u,v)}{\partial(x,y)}$

So $\quad J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

Our last two results can therefore be written

$$\delta x = \ldots\ldots\ldots\ldots; \quad \delta y = \ldots\ldots\ldots\ldots$$

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$$\delta x = \frac{\dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v}{J} = \frac{\begin{vmatrix} \delta u & \delta v \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}}, \quad \delta y = \frac{\dfrac{\partial u}{\partial x}\delta v - \dfrac{\partial v}{\partial x}\delta u}{J} = \frac{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \delta u & \delta v \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}}$$

We can now get a number of useful relationships.

(a) *If v is kept constant, $\delta v = 0$* $\quad \therefore\ \delta x = \dfrac{\partial v}{\partial y}\delta u \Big/ J$

Dividing by δu and letting $\delta u \to 0$ $\quad \dfrac{\partial x}{\partial u} = \dfrac{\partial v}{\partial y}\Big/ J$

Similarly $\quad \dfrac{\partial y}{\partial u} = -\dfrac{\partial v}{\partial x}\Big/ J$

(b) *If u is kept constant, $\delta u = 0$* $\quad \therefore\ \delta x = -\dfrac{\partial u}{\partial y}\delta v \Big/ J$

Dividing by δv and letting $\delta v \to 0$ $\quad \dfrac{\partial x}{\partial v} = -\dfrac{\partial u}{\partial y}\Big/ J$

Similarly $\quad \dfrac{\partial y}{\partial v} = \dfrac{\partial u}{\partial x}\Big/ J$

So, at this stage, we had better summarise the results.

Summary

If $z = f(x, y)$ and $u = g(x, y)$ and $v = h(x, y)$ then

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\Big/ J \qquad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}\Big/ J$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\Big/ J \qquad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}\Big/ J$$

where, in each case

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Let us put this into practice by doing again the same example that we started with (Example 1 on page 382), but by the new method. First of all, however, make a note of the important summary listed above for future reference.

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Example 1A

If $z = f(x, y)$ and $u = e^x \cos y$ and $v = e^{-x} \sin y$, find the derivatives $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

$$u = e^x \cos y \qquad v = e^{-x} \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^{-x} \cos y$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^{-x} \sin y \\ -e^x \sin y & e^{-x} \cos y \end{vmatrix}$$

$$= (e^x \cos y)(e^{-x} \cos y) \quad - \quad (-e^x \sin y)(-e^{-x} \sin y)$$

$$= \quad \cos^2 y \quad - \quad \sin^2 y \quad = \cos 2y$$

Then $\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y} \bigg/ J = \frac{e^{-x} \cos y}{\cos 2y}$; $\qquad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y} \bigg/ J = \frac{e^x \sin y}{\cos 2y}$

$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x} \bigg/ J = \frac{e^{-x} \sin y}{\cos 2y}$; $\qquad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} \bigg/ J = \frac{e^x \cos y}{\cos 2y}$

which is a lot shorter than our first approach.

Move on for a further example

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Example 2

If $z = f(x, y)$ with $u = x^2 - y^2$ and $v = xy$, find expressions for $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

First we need

$\frac{\partial u}{\partial x} = \ldots\ldots\ldots\ldots$; $\frac{\partial u}{\partial y} = \ldots\ldots\ldots\ldots$; $\frac{\partial v}{\partial x} = \ldots\ldots\ldots\ldots$; $\frac{\partial v}{\partial y} = \ldots\ldots\ldots\ldots$

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$$\boxed{\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = -2y; \quad \frac{\partial v}{\partial x} = y; \quad \frac{\partial v}{\partial y} = x}$$

Then we calculate J which, in this case, is ....

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$$\boxed{J = 2(x^2 + y^2)}$$

Because

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & y \\ -2y & x \end{vmatrix} = 2x^2 + 2y^2$$

Finally, we have the four relationships

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\Big/ J = \ldots\ldots\ldots\ldots; \qquad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}\Big/ J = \ldots\ldots\ldots\ldots$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\Big/ J = \ldots\ldots\ldots\ldots; \qquad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}\Big/ J = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{array}{ll} \dfrac{\partial x}{\partial u} = \dfrac{x}{2(x^2 + y^2)}; & \dfrac{\partial x}{\partial v} = \dfrac{y}{x^2 + y^2} \\ \dfrac{\partial y}{\partial u} = \dfrac{-y}{2(x^2 + y^2)}; & \dfrac{\partial y}{\partial v} = \dfrac{x}{x^2 + y^2} \end{array}}$$

And that is all there is to it.

If we know the details of the function $z = f(x, y)$ then we can go one stage further and use the results $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ to find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Let us see this in a further example.

Example 3

If $z = 2x^2 + 3xy + 4y^2$ and $u = x^2 + y^2$ and $v = x + 2y$, determine

(a) $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ (b) $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Section (a) is just like the previous example. Complete that on your own.

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$$\boxed{\frac{\partial x}{\partial u} = \frac{1}{2x - y}; \quad \frac{\partial x}{\partial v} = \frac{-y}{2x - y}; \quad \frac{\partial y}{\partial u} = \frac{-1}{2(2x - y)}; \quad \frac{\partial y}{\partial v} = \frac{x}{2x - y}}$$

Because if $u = x^2 + y^2$ and $v = x + 2y$

$$\frac{\partial u}{\partial x} = 2x; \ \frac{\partial u}{\partial y} = 2y; \ \frac{\partial v}{\partial x} = 1; \ \frac{\partial v}{\partial y} = 2$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 1 \\ 2y & 2 \end{vmatrix} = 4x - 2y = 2(2x - y)$$

Then $\dfrac{\partial x}{\partial u} = \dfrac{\partial v}{\partial y} \Big/ J = 2 \Big/ 2(2x - y) = \dfrac{1}{2x - y}$

$$\frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y} \Big/ J = -2y \Big/ 2(2x - y) = \frac{-y}{2x - y}$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x} \Big/ J = -1 \Big/ 2(2x - y) = \frac{-1}{2(2x - y)}$$

$$\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} \Big/ J = 2x \Big/ 2(2x - y) = \frac{x}{2x - y}$$

$$\therefore \ \frac{\partial x}{\partial u} = \frac{1}{2x - y}; \quad \frac{\partial x}{\partial v} = \frac{-y}{2x - y}; \quad \frac{\partial y}{\partial u} = \frac{-1}{2(2x - y)}; \quad \frac{\partial y}{\partial v} = \frac{x}{2x - y}$$

Now for part (b).

Since z is also a function of u and v, the expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ are

$$\frac{\partial z}{\partial u} = \ldots\ldots\ldots\ldots$$

$$\frac{\partial z}{\partial v} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \end{aligned}}$$

The only remaining items of information we need are the expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ which we obtain from $z = 2x^2 + 3xy + 4y^2$

$$\frac{\partial z}{\partial x} = 4x + 3y \quad \text{and} \quad \frac{\partial z}{\partial y} = 3x + 8y$$

Using these and the previous set of derivatives, we now get

$$\frac{\partial z}{\partial u} = \ldots\ldots\ldots\ldots; \quad \frac{\partial z}{\partial v} = \ldots\ldots\ldots\ldots$$

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$$\frac{\partial z}{\partial u} = \frac{5x - 2y}{2(2x - y)}; \quad \frac{\partial z}{\partial v} = \frac{3x^2 + 4xy - 3y^2}{2x - y}$$

Because

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\therefore \quad \frac{\partial z}{\partial u} = (4x + 3y)\left\{\frac{1}{2x - y}\right\} + (3x + 8y)\left\{\frac{-1}{2(2x - y)}\right\}$$

$$= \frac{5x - 2y}{2(2x - y)} \qquad \therefore \quad \frac{\partial z}{\partial u} = \frac{5x - 2y}{2(2x - y)}$$

and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial v}$

$$\therefore \quad \frac{\partial z}{\partial v} = (4x + 3y)\left\{\frac{-y}{2x - y}\right\} + (3x + 8y)\left\{\frac{x}{2x - y}\right\}$$

$$= \frac{3x^2 + 4xy - 3y^2}{2x - y} \qquad \therefore \quad \frac{\partial z}{\partial v} = \frac{3x^2 + 4xy - 3y^2}{2x - y}$$

They are all done in the same general way.

Now on to the next topic

Stationary values of a function

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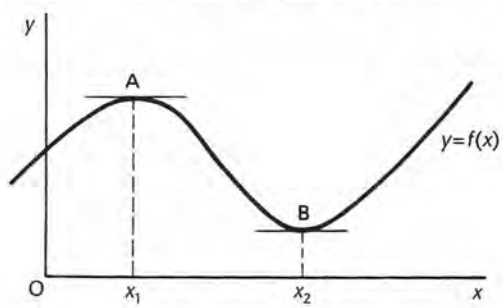
You will doubtless remember that in earlier work you established the characteristics of *stationary points* on a plane curve and derived the conditions that enable these critical points to be calculated.



At A and B $\quad \dfrac{dy}{dx} = 0$

For maximum $\quad \dfrac{d^2y}{dx^2}$ is negative $(x = x_1)$

For minimum $\quad \dfrac{d^2y}{dx^2}$ is positive $(x = x_2)$

▶

We now progress to the application of these same considerations to three dimensions, where $z = f(x, y)$. The function is now represented by a surface and stationary values of the function $z = f(x, y)$ occur when the tangent plane to the surface at a point P (a, b) is parallel to the plane $z = 0$, i.e. to the x–y plane.

Let us take a closer look at this.

Maximum and minimum values

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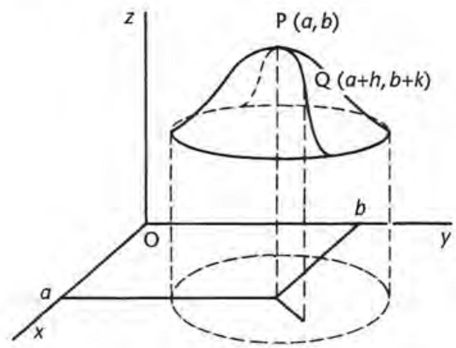
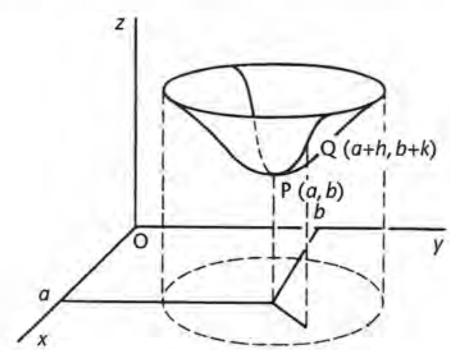


A function $z = f(x, y)$ is said to have *maximum* value at P (a, b) if $f(a, b)$ is greater than the value at a near-by point Q $(a + h, b + k)$ for all values of h and k however small, positive or negative, i.e. in all directions from P.



Similarly, $z = f(x, y)$ is said to have a *minimum* value at P (a, b) if $f(a, b)$ is less than the value at a neighbouring point Q $(a + h, b + k)$ in any direction from P.

To establish maximum and minimum values, we must therefore investigate the sign of the value of $f(a + h, b + k) - f(a, b)$.

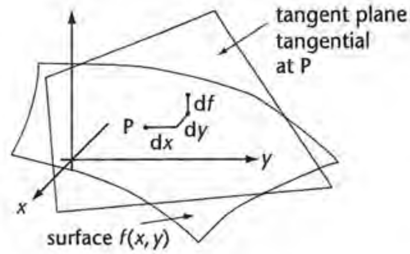
If $f(a + h, b + k) - f(a, b) < 0$ we have a maximum value at P (a, b).
If $f(a + h, b + k) - f(a, b) > 0$ we have a minimum value at P (a, b).

▶

To pursue this further we turn to the total differential

$$df(x, y) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

The total differential measures the rise or fall in the tangent plane from the point of its contact with the surface at (x, y) to the point $(x + dx, y + dy)$.



If the point of contact is a maximum or a minimum then

$$\frac{\partial f}{\partial x} = \dots\dots\dots\dots \quad \text{and} \quad \frac{\partial f}{\partial y} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0}$$

Because

The tangent plane is parallel with the x–y plane and so the tangent plane neither rises nor falls, so that

$$df(x, y) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

Also because $dx \neq 0$ and $dy \neq 0$ then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Notice the logic here. If there is a maximum or a minimum, then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. However, just because $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at a point does not imply that a maximum or a minimum exists at that point. What we can say is that a *stationary* point exists at that point and, as we shall see later, not all stationary points are maxima or minima.

Example 1

Determine the values of x and y at which the stationary values of

$$f(x, y) = x^2 + xy + y^2 + 5x - 5y + 3$$

occur.

All we need to do is to obtain expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, equate each to zero and then solve the pair of simultaneous equations so obtained. In which case

$$x = \dots\dots\dots\dots \quad \text{and} \quad y = \dots\dots\dots\dots$$

$$\boxed{x = -5 \quad \text{and} \quad y = 5}$$

Because

$\dfrac{\partial f}{\partial x} = 2x + y + 5$ and $\dfrac{\partial f}{\partial y} = x + 2y - 5$ giving the pair of simultaneous equations

$$2x + y + 5 = 0 \qquad (1)$$

$$x + 2y - 5 = 0 \qquad (2)$$

Adding (1) + (2) gives $3x + 3y = 0$, that is $y = -x$

Substitution in (1) gives $x = -5$ and so $y = 5$

Although a stationary value occurs at $(-5, 5)$ we have no evidence as to whether it is a maximum or a minimum value. Let us investigate further.

From the previous definitions

$f(a, b)$ will be a maximum value if $f(a + h, b + k) - f(a, b) < 0$

$f(a, b)$ will be a minimum value if $f(a + h, b + k) - f(a, b) > 0$

Now, from Taylor's theorem

$$f(a + h, b + k) = f(a, b) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \dots$$

and we have already seen that at a stationary value $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$. So, at a stationary point, Taylor's theorem becomes

$$f(a + h, b + k) - f(a, b) = \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \dots$$

where subsequent terms are of higher orders of h and k and are neglected.

The expression in the brackets on the right-hand side can be written as

$$\frac{1}{\dfrac{\partial^2 f}{\partial x^2}}\left\{\left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial x \partial y}\right)^2 + k^2\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2\right)\right\}$$

Take a moment and expand the brackets and confirm that this is so.

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So $h^2\dfrac{\partial^2 f}{\partial x^2}+2hk\dfrac{\partial^2 f}{\partial x\partial y}+k^2\dfrac{\partial^2 f}{\partial y^2}$

$$=\frac{1}{\dfrac{\partial^2 f}{\partial x^2}}\left\{\left(h\frac{\partial^2 f}{\partial x^2}+k\frac{\partial^2 f}{\partial x\partial y}\right)^2+k^2\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left[\frac{\partial^2 f}{\partial x\partial y}\right]^2\right)\right\}$$

Now $\left(h\dfrac{\partial^2 f}{\partial x^2}+k\dfrac{\partial^2 f}{\partial x\partial y}\right)^2$, being a square, is always positive and if $\dfrac{\partial^2 f}{\partial x^2}\cdot\dfrac{\partial^2 f}{\partial y^2}>\left[\dfrac{\partial^2 f}{\partial x\partial y}\right]^2$ the second term will also be positive. In that case the sign of the whole expression is given by that of $\dfrac{\partial^2 f}{\partial x^2}$ at the front.

Furthermore, if $\dfrac{\partial^2 f}{\partial x^2}\cdot\dfrac{\partial^2 f}{\partial y^2}>\left(\dfrac{\partial^2 f}{\partial x\partial y}\right)^2$, i.e. $\dfrac{\partial^2 f}{\partial x^2}\cdot\dfrac{\partial^2 f}{\partial y^2}-\left(\dfrac{\partial^2 f}{\partial x\partial y}\right)^2>0$, this can be so only if $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ have the same sign. Therefore,

for $f(a,b)$ to be a maximum, $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ are both negative

and for $f(a,b)$ to be a minimum, $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ are both positive.

So, to determine whether a known stationary value is a maximum or a minimum value, we must find the second derivatives $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x\partial y}$.

Then

(a) If $\dfrac{\partial^2 f}{\partial x^2}\cdot\dfrac{\partial^2 f}{\partial y^2}-\left(\dfrac{\partial^2 f}{\partial x\partial y}\right)^2>0$, the stationary value is a true maximum or minimum value.

(b) In that case

(1) if $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ are both *negative*, $f(a, b)$ is a *maximum*

(2) if $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ are both *positive*, $f(a, b)$ is a *minimum*.

Make a careful note of the conclusions (a) and (b): then let us apply them.

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Example 2

Investigate further the stationary value of the function

$$z = x^2 + xy + y^2 + 5x - 5y + 3$$

We have already seen that this function has a stationary point at

$$x = \dots\dots\dots\dots; \quad y = \dots\dots\dots\dots$$

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$$\boxed{x = -5; \quad y = 5}$$

Next, we investigate the value of $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$. If this is greater than zero at $(-5, 5)$, then either a maximum or a minimum occurs at that point.

Check whether this is so.

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$$\boxed{\text{Yes.} \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0}$$

Because

$$\frac{\partial^2 z}{\partial x^2} = 2; \quad \frac{\partial^2 z}{\partial y^2} = 2; \quad \frac{\partial^2 z}{\partial x \partial y} = 1.$$

This confirms that $(-5, 5)$ is either a maximum or a minimum.

To decide which it is, we note that $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are both *positive*.

$$\therefore \quad \text{at } (-5, 5), z \text{ is a } \dots\dots\dots\dots$$

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$$\boxed{\text{minimum}}$$

Of course to find the actual minimum value of z we substitute $x = -5$ and $y = 5$ into the expression for z. That is really all there is to it. Another example.

Example 3

Determine the stationary values, if any, of the function

$z = x^3 - 6xy + y^3$

The four steps in the routine are:

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ and solve the equations $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$.

(b) Determine whether $\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$.

(c) If so, note the sign of $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ to distinguish between max. and min.

(d) Evaluate the maximum or minimum value of z.

In this example, stationary values occur at …………

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$$\boxed{z = 0 \text{ at } (0, 0) \quad \text{and} \quad z = -8 \text{ at } (2, 2)}$$

Because

$$z = x^3 - 6xy + y^3 \quad \therefore \quad \frac{\partial z}{\partial x} = 3x^2 - 6y \qquad \frac{\partial z}{\partial y} = -6x + 3y^2$$

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0 \;\therefore\; x^2 - 2y = 0 \text{ and } -2x + y^2 = 0$$

A possible stationary point exists when $x^2 - 2y = 0$ and $-2x + y^2 = 0$. From the first equation $y = x^2/2$ and substitution into $-2x + y^2 = 0$ gives $-2x + x^4/4 = 0$.

That is $x^4 - 8x = x(x^3 - 8) = 0$ and so $x = 0$ or $x = 2$.

When $x = 0$ then $y = 0$ and when $x = 2$ then $y = 2$.

$\therefore$ There are stationary values at (0, 0) and (2, 2)

Next we determine whether $\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$

Result …………

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$\boxed{\text{No max. or min. at } (0, 0); \quad \text{Either max. or min. at } (2, 2)}$

Because

$$\frac{\partial z}{\partial x} = 3x^2 - 6y \qquad \therefore \frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -6$$
$$\frac{\partial z}{\partial y} = -6x + 3y^2 \qquad \therefore \frac{\partial^2 z}{\partial y^2} = 6y$$

$$\therefore \text{ at } (0, 0) \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (0)(0) - 36 < 0$$

$\therefore$ No max. or min. at (0, 0)

$$\text{At } (2, 2) \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (12)(12) - 36 > 0$$

$\therefore$ Either max. or min. at (2, 2)

We see that at (2, 2) both $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are positive. Therefore the stationary value at (2, 2) is a ....

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$\boxed{\text{minimum}}$

Finally, the minimum value of z is ....

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$\boxed{-8}$

Therefore, $z_{\min} = -8$ and occurs at (2, 2)

Before doing a further example, let us consider one other aspect of stationary values.

On to a new frame

51

Saddle point

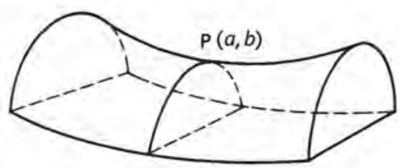
In the previous example, when we substituted the coordinates (0, 0) in the expression $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ we found that this did not satisfy the condition that for a maximum or minimum value

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2>0$$

In fact, if $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$

and $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2<0$

this is an indication of a form of stationary value described as a *saddle point*, as shown at P below.



A saddle point is, in effect, a combined maximum and minimum configuration in different directions. Its name is obvious from the shape.

Add this then to the list of conditions for stationary values that we have built up.

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At this stage, one naturally asks, what is implied if

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2=0$$

In such a case, further detailed study of the function is necessary.

Now for an example to see it all in practice.

Example 4

Determine the stationary values of $z=5xy-4x^2-y^2-2x-y+5$.

Stationary values (or turning points) occur where

$$\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial z}{\partial y}=0, \quad \text{i.e. at } \dots\dots\dots\dots$$

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$$\boxed{x = 1, \quad y = 2}$$

Because

$$\frac{\partial z}{\partial x} = 5y - 8x - 2 \qquad \frac{\partial z}{\partial y} = 5x - 2y - 1$$

$$\therefore \quad \left.\begin{aligned} 8x - 5y + 2 &= 0 \\ 5x - 2y - 1 &= 0 \end{aligned}\right\} \text{ gives } x = 1,\ y = 2$$

Therefore, the only stationary value occurs at (1, 2).
Next we substitute these x and y values in

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \quad \text{and find } \ldots\ldots\ldots\ldots$$

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$$\boxed{\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0}$$

Because

$$\frac{\partial^2 z}{\partial x^2} = -8; \quad \frac{\partial^2 z}{\partial y^2} = -2; \quad \frac{\partial^2 z}{\partial x \partial y} = 5$$

$$\therefore \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (-8)(-2) - 25 = -9 \quad \text{i.e.} < 0$$

The stationary value at (1, 2) is therefore a ....

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$$\boxed{\text{saddle point}}$$

Example 5

Determine stationary values of $z = x^3 - 3x + xy^2$ and their nature.

We go through the same routine as before.

First find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$.

Possible stationary values therefore occur at ....

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$$\boxed{x = 0,\ y = \pm\sqrt{3};\quad x = \pm 1,\ y = 0}$$

Because

$$\frac{\partial z}{\partial x} = 3x^2 - 3 + y^2 \quad \frac{\partial z}{\partial y} = 2xy \quad \therefore\ x = 0 \text{ or } y = 0$$

If $x = 0$, $\quad y^2 = 3 \quad \therefore\ y = \pm\sqrt{3} \quad x = 0,\ y = \pm\sqrt{3}$

If $y = 0$, $\quad 3x^2 = 3 \quad \therefore\ x = \pm 1 \quad x = \pm 1,\ y = 0.$

Now we need the second derivatives and the usual tests. Finish if off. The nature of the stationary values:

$(0, \sqrt{3})$ ....; $\quad (0, -\sqrt{3})$ ....
$(1, 0)$ ....; $\quad (-1, 0)$ ....

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$(0, \sqrt{3})$ saddle point;	$(0, -\sqrt{3})$ saddle point
$(1, 0)$ minimum;	$(-1, 0)$ maximum

Because

$$\frac{\partial^2 z}{\partial x^2} = 6x; \quad \frac{\partial^2 z}{\partial y^2} = 2x; \quad \frac{\partial^2 z}{\partial x \partial y} = 2y$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

$(0, \sqrt{3})$	$(0)(0) - 12$	i.e. < 0	$\therefore$ saddle point
$(0, -\sqrt{3})$	$(0)(0) - 12$	i.e. < 0	$\therefore$ saddle point
$(1, 0)$	$(6)(2) - 0$	i.e. > 0	$\therefore$ minimum
$(-1, 0)$	$(-6)(-2) - 0$	i.e. > 0	$\therefore$ maximum

and that just about does everything.

Substitution of $(1, 0)$ and $(-1, 0)$ in $z = x^3 - 3x + xy^2$ gives the minimum and maximum values of z.

$z_{\min} = -2; \quad z_{\max} = 2.$

The value of z at each of the saddle points is zero.

Let's now look at some examples where the second derivative test fails

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Example 6

Determine the stationary values of $z = x^2 - 6xy + 9y^2$.

Here we see that $\dfrac{\partial z}{\partial x} = 2x - 6y$, $\dfrac{\partial z}{\partial y} = -6x + 18y$ and so these two derivatives vanish when

....

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$$\boxed{y = x/3}$$

Because

$\dfrac{\partial z}{\partial x} = 2x - 6y = 0$ when $2x = 6y$, that is when $y = x/3$ and

$\dfrac{\partial z}{\partial y} = -6x + 18y = 0$ when $6x = 18y$, that is when $y = x/3$, and so there is an infinity of stationary points lying along the line $y = x/3$.

Now

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \ldots\ldots\ldots\ldots$$

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$$\boxed{0}$$

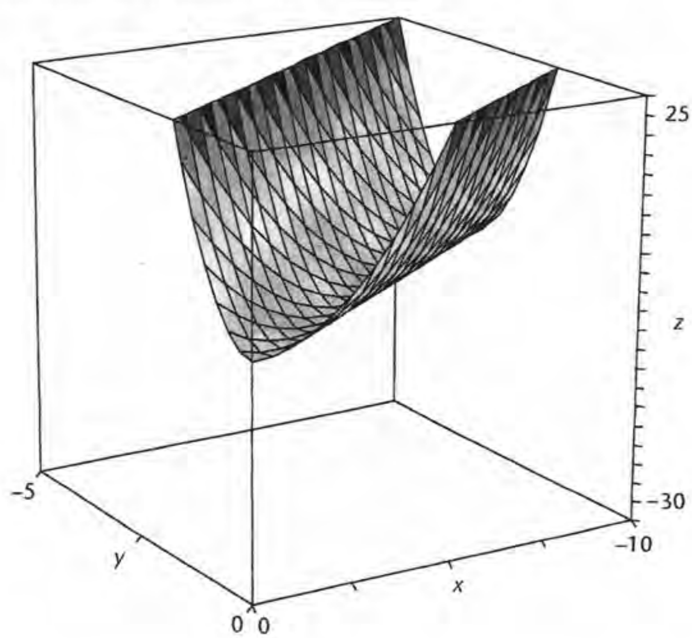
Because

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 18 \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -6 \qquad \text{so that}$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 18 \times 2 - 36 = 0$$

So the second derivative test does not apply and we must look elsewhere to decide the nature of the stationary points.

Since $x^2 - 6xy + 9y^2 = (x - 3y)^2$ then $z \geq 0$ for all values of x and y. Therefore the stationary points are minima – there is an infinity of minimum points along the line $y = x/3$.



▶

Example 7

Find the stationary points of $z = x^4 - y^3$.

Here we see that $\dfrac{\partial z}{\partial x} = 4x^3$, $\dfrac{\partial z}{\partial y} = -3y^2$ and so these two derivatives vanish when $x = \ldots\ldots\ldots\ldots$, $y = \ldots\ldots\ldots\ldots$

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$$\boxed{x = 0, \quad y = 0}$$

Because

$\dfrac{\partial z}{\partial x} = 4x^3 = 0$ when $x = 0$ and $\dfrac{\partial z}{\partial y} = -2y^2 = 0$ when $y = 0$, so there is just one stationary point at (0, 0).

Now, at the stationary point

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \ldots\ldots\ldots\ldots$$

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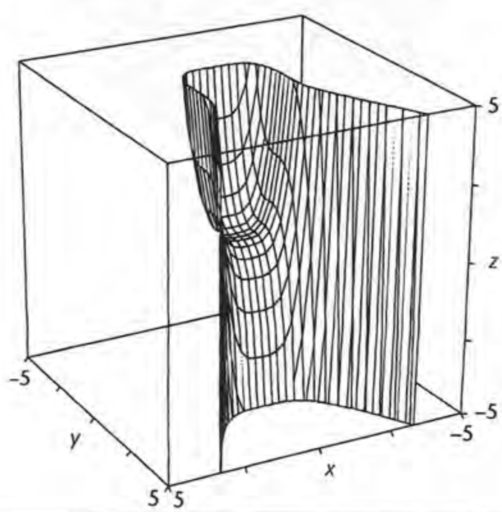
$$\boxed{0}$$

Because

$\dfrac{\partial^2 z}{\partial x^2} = 12x^2$, $\dfrac{\partial^2 z}{\partial y^2} = -4y$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$ so that at (0, 0):

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

So the second derivative test does not apply. However, in the z–x plane $y = 0$ and so $z = x^4$. This means that the line of intersection of the surface with the z–x plane has a minimum at the origin. In the z–y plane $x = 0$ and so $z = -y^3$. This means that the line of intersection of the surface with the z–y plane has a point of inflexion at the origin.



Lagrange undetermined multipliers

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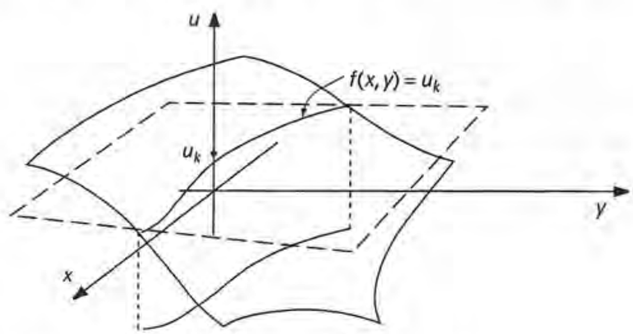
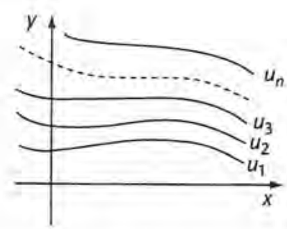
Closely allied to the problem of locating the stationary points of some function $u = f(x, y)$ is the problem of locating points where $u = f(x, y)$ attains its greatest or its least value (an extremal value) subject to the condition that x and y are related to each other via the equation

$$\phi(x, y) = 0 \tag{1}$$

The problem can be clarified if we consider it graphically.

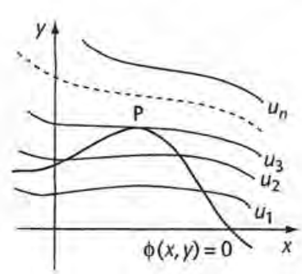
The graph of $u = f(x, y)$ is a surface within the (x, y, u) coordinate system. Selecting a plane parallel to the x–y plane on which the value of u is constant, u_k, we see that the surface intersects the plane in a curve given by the equation $f(x, y) = u_k$.





This line of intersection can now be projected onto the x–y plane to form what is known as a *level curve*. Different values of u_k determine different planes (all parallel to the x–y plane), different lines of intersection and hence different level curves. Accordingly, an alternative graphical description of $u = f(x, y)$ is that of a family of level curves in the x–y plane with each member of the family being associated with a particular value of u_k, where we assume that $u_1 < u_2 < u_3 < \ldots < u_n$ or $u_1 > u_2 > u_3 > \ldots > u_n$.

We now superimpose onto this family of level curves the graph of the constraint equation $\phi(x, y) = 0$.



Clearly, in the figure alongside, u_3 is the extremal value of $f(x, y)$ that coincides with $\phi(x, y) = 0$, and at the point P where they meet they share the same tangent line dy/dx. Now, since $\phi(x, y) = 0$ we see that

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}}$$

Because

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \text{ so that } \frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$$

The same tangent can be found from

$$du = \frac{\partial f}{\partial x}dx = \frac{\partial f}{\partial y}dy$$

by equating the differential $du = 0$. Therefore

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$$

The latter two fractions are equivalent fractions which means that the two numerators and the two denominators each differ by the same multiplicative factor K, enabling us to say that

$$\frac{\partial f}{\partial x} = K\frac{\partial\phi}{\partial x} \text{ and } \frac{\partial f}{\partial y} = K\frac{\partial\phi}{\partial y} \text{ so that}$$

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x} = 0 \qquad (2)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y} = 0 \qquad (3)$$

$\lambda = -K$ is called a Lagrange multiplier and equations (2) and (3), coupled with the constraint equation $\phi(x, y) = 0$, give us three relationships from which the values of x and y at the extremal points – and also the value of λ if required – can be found. Quite often the value of λ is not important.

Let us see how it works in a simple example.

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Example 1

Find the stationary points of the function $u = x^2 + y^2$ subject to the constraint $x^2 + y^2 + 2x - 2y + 1 = 0$.

In this case, $u = x^2 + y^2$

$\phi = x^2 + y^2 + 2x - 2y + 1$

We need to know

$$\frac{\partial u}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial u}{\partial y} = \dots\dots\dots\dots$$

$$\frac{\partial\phi}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial\phi}{\partial y} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = 2y; \quad \frac{\partial \phi}{\partial x} = 2x + 2; \quad \frac{\partial \phi}{\partial y} = 2y - 2}$$

Then we form and solve
$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

together with $\quad \phi = x^2 + y^2 + 2x - 2y + 1 = 0$

which gives $\quad x = \dots\dots\dots\dots; \; y = \dots\dots\dots\dots; \; \lambda = \dots\dots\dots\dots$

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$$\boxed{x = -1 \pm \frac{\sqrt{2}}{2}; \; y = 1 \mp \frac{\sqrt{2}}{2}; \; \lambda = \sqrt{2} - 1}$$

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \therefore \; 2x + \lambda(2x + 2) = 0 \quad \therefore \; x + \lambda(x + 1) = 0$$

$$\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \therefore \; 2y + \lambda(2x - 2) = 0 \quad \therefore \; y + \lambda(y - 1) = 0$$

$$\therefore \; \frac{x}{y} = \frac{-\lambda(x + 1)}{-\lambda(y - 1)} \quad \therefore \; xy - x = xy + y \quad \therefore \; y = -x$$

Substituting this in ϕ

$$x^2 + x^2 + 2x + 2x + 1 = 0 \qquad 2x^2 + 4x + 1 = 0$$
$$\therefore \; x = -1 \pm \frac{\sqrt{2}}{2}$$

But $y = -x$ $\qquad \therefore \; y = 1 \mp \dfrac{\sqrt{2}}{2}$

To find λ, we have $x + \lambda(x + 1) = 0 \quad \therefore \; \lambda = \sqrt{2} \mp 1$

As we have already said, we do not really need to find the value of λ.

On to the next

Functions with three independent variables

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The argument is very much the same as before.

To find stationary points of the function $\quad u = f(x, y, z) \qquad (1)$

subject to the constraint $\quad \phi(x, y, z) = 0 \qquad (2)$

Again we have, at stationary points

$$\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z = 0 \qquad (3)$$

and since $\phi(x, y, z) = 0$

then $$\frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = 0 \qquad (4)$$

Multiplying each term in (4) by λ and adding (4) to (3), we have

$\dots\dots\dots\dots$

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$$\boxed{\left(\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)\delta y+\left(\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)\delta z=0}$$

from which $\quad \dfrac{\partial u}{\partial x}+\lambda\dfrac{\partial \phi}{\partial x}=0 \qquad (5)$

$$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \qquad (6)$$

$$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \qquad (7)$$

Equations (5), (6), (7), together with the constraint (2), provide all the information to determine x, y, z, and, if necessary, λ.

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Example 2

To find the stationary points of the function

$u = x^2 + 2y^2 + z$

subject to the constraint $\phi(x, z) = x^2 - z^2 - 2 = 0$.

So $\dfrac{\partial u}{\partial x} = \ldots\ldots\ldots\ldots;\quad \dfrac{\partial u}{\partial y} = \ldots\ldots\ldots\ldots;\quad \dfrac{\partial u}{\partial z} = \ldots\ldots\ldots\ldots$

$\dfrac{\partial \phi}{\partial x} = \ldots\ldots\ldots\ldots;\quad \dfrac{\partial \phi}{\partial y} = \ldots\ldots\ldots\ldots;\quad \dfrac{\partial \phi}{\partial z} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{array}{lll} \dfrac{\partial u}{\partial x}=2x; & \dfrac{\partial u}{\partial y}=4y; & \dfrac{\partial u}{\partial z}=1 \\ \dfrac{\partial \phi}{\partial x}=2x; & \dfrac{\partial \phi}{\partial y}=0; & \dfrac{\partial \phi}{\partial z}=-2z \end{array}}$$

Now compile the equations

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0;\quad \frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0;\quad \frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

and, together with the constraint $\phi = x^2 - z^2 - 2 = 0$, establish that stationary points occur at ....

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$$\boxed{\left(\tfrac{3}{2},\ 0,\ -\tfrac{1}{2}\right)\ \text{and}\ \left(-\tfrac{3}{2},\ 0,\ -\tfrac{1}{2}\right)}$$

Because

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0\quad \therefore\ 2x+\lambda 2x=0\quad \therefore\ \lambda=-1$$

$$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0\quad 4y+\lambda(0)=0\quad \therefore\ y=0$$

$$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0\quad 1-\lambda 2z=0\quad \therefore\ z=\frac{1}{2\lambda}=-\frac{1}{2}$$

$$\phi = x^2 - z^2 - 2 = 0\quad \therefore\ x^2 - \tfrac{1}{4} - 2 = 0\quad \therefore\ x = \pm\tfrac{3}{2}.$$

Therefore, stationary points at $\left(\tfrac{3}{2}, 0, -\tfrac{1}{2}\right)$ and $\left(-\tfrac{3}{2}, 0, -\tfrac{1}{2}\right)$.

The method of Lagrange multipliers does not lend itself easily to give a distinction between the various types of stationary points. In many practical applications, however, whether a result is a maximum or a minimum value will be apparent from the physical consideration of the problem.

Let us finish with one further example.

So move on

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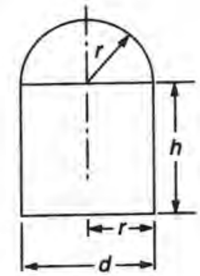
Example 3

A hot water storage tank is a vertical cylinder surmounted by a hemispherical top of the same diameter. The tank is designed to hold 400 m^3 of liquid. Determine the total height and the diameter of the tank if the surface heat loss is to be a minimum.



We first write down the function for the total surface area, A.

$$A = \dots\dots\dots\dots$$

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$$\boxed{A = 3\pi r^2 + 2\pi rh}$$

Because

The surface area of the hemisphere is $2\pi r^2$, the area of the base of the tank is πr^2 and the area of the cylindrical side is $2\pi rh$, giving a total area of $3\pi r^2 + 2\pi rh$.

This is the function which has to be a minimum. The constraint in this problem is that ....

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$$\boxed{\text{the volume is } 400 \text{ m}^3}$$

So we have

$$A = 3\pi r^2 + 2\pi rh \qquad (1)$$

constraint $\quad V = \pi r^2 h + \dfrac{2}{3}\pi r^3 = 400$

So let $\quad \phi = \pi r^2 h + \dfrac{2}{3}\pi r^3 - 400 = 0 \qquad (2)$

We now want $\quad \dfrac{\partial A}{\partial r} = \dots\dots\dots\dots; \quad \dfrac{\partial A}{\partial h} = \dots\dots\dots\dots$

$$\frac{\partial \phi}{\partial r} = \dots\dots\dots\dots; \quad \frac{\partial \phi}{\partial h} = \dots\dots\dots\dots$$

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$$\frac{\partial A}{\partial r} = 6\pi r + 2\pi h; \qquad \frac{\partial \phi}{\partial r} = 2\pi rh + 2\pi r^2$$
$$\frac{\partial A}{\partial h} = 2\pi r; \qquad \frac{\partial \phi}{\partial h} = \pi r^2$$

Now we form $\quad \dfrac{\partial A}{\partial r} + \lambda \dfrac{\partial \phi}{\partial r} = 0$

and $\quad \dfrac{\partial A}{\partial h} + \lambda \dfrac{\partial \phi}{\partial h} = 0$

and, with the constraint, $\phi = \pi r^2 h + \dfrac{2}{3}\pi r^3 - 400 = 0$,

we eventually obtain $r = \ldots\ldots\ldots\ldots$ and $h = \ldots\ldots\ldots\ldots$

Finish it off and hence find the total height and the diameter.

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$$r = 4{\cdot}243 \text{ m}; \quad h = 4{\cdot}243 \text{ m}$$

Check the working:

$$\frac{\partial A}{\partial r} + \lambda \frac{\partial \phi}{\partial r} = 0 \quad \therefore \ 6\pi r + 2\pi h + \lambda(2\pi rh + 2\pi r^2) = 0 \qquad (3)$$
$$\frac{\partial A}{\partial h} + \lambda \frac{\partial \phi}{\partial h} = 0 \quad \therefore \ 2\pi r + \lambda \pi r^2 = 0 \qquad (4)$$

From (4): $\quad \lambda = -\dfrac{2}{r} \qquad$ Substitute this in (3)

$$6\pi r + 2\pi h - \frac{2}{r}(2\pi rh + 2\pi r^2) = 0$$
$$\therefore \ 6r + 2h - 4h - 4r = 0 \qquad \therefore \ h = r$$

Also $\quad \pi r^2 h + \dfrac{2}{3}\pi r^3 = 400 \quad \therefore \ \dfrac{5}{3}\pi r^3 = 400 \quad \therefore \ r = 4{\cdot}243$

$\therefore$ Total height $= h + r = 8{\cdot}49$ m; $\quad$ Diameter $= 8{\cdot}49$ m

That brings us to the end of this particular Programme and to the usual **Revision summary** that follows. Check through the **Can You?** checklist and be sure to revise any section should you feel that is necessary. Then you will find the **Test exercise** straightforward – no tricks. The **Further problems** provide valuable additional practice.

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Revision summary 10

1 *Small increments*

$$z = f(x, y) \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

$$u = f(x, y, z) \quad \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

2 *Rates of change*

$$z = f(x, y) \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

3 *Implicit functions*

$$z = f(x, y) = 0 \quad \frac{dy}{dx} = -\left(\frac{\partial z}{\partial x}\bigg/\frac{\partial z}{\partial y}\right)$$

4 *Change of variables*

$z = f(x, y)$ x and y are functions of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

5 *Inverse functions*

$z = f(x, y)$ $u = g(x, y)$ $v = h(x, y)$

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\bigg/J; \quad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}\bigg/J$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\bigg/J; \quad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}\bigg/J$$

where $J = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

6 *Stationary points*

$z = f(x, y)$ (a) $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$

(b) $\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$ for max. or min.

$\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$ for saddle point

$\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$ no decision without further information

(c) $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ both *negative* for maximum

$\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ both *positive* for minimum.

7 *Lagrange multipliers*
Two independent variables
$u = f(x, y)$ with constraint $\phi(x, y) = 0$

Solve
$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$
with
$$\phi(x, y) = 0.$$

Three independent variables
$u = f(x, y, z)$ with constraint $\phi(x, y, z) = 0$

Solve
$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$
$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$
with
$$\phi(x, y, z) = 0.$$

Can You?

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Checklist 10

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Derive the expression for a small increment in an expression of two real variables using Taylor's theorem? 1 to 4
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Apply the notion of small increments in expressions in two and three real variables to a variety of problems? 5 to 7
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine the rate of change with respect to time of an expression involving two or three real variables? 8 and 9
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Differentiate implicit functions? 9 and 10
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine first and second derivatives involving change of variables in expressions of two real variables? 11 to 21
  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the Jacobian to obtain the derivatives of inverse functions of two real variables? 22 to 37

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Locate and identify maxima, minima and saddle points of functions of two real variables? 38 to 57

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve problems where the independent variables are constrained by using the method of Lagrange undetermined multipliers for functions of two and three real variables? 58 to 77

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 10

80

1 If $z = \dfrac{xy}{x-y}$, show that

(a) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$

(b) $x^2\dfrac{\partial^2 z}{\partial x^2} - y^2\dfrac{\partial^2 z}{\partial y^2} = 0$

(c) $z\dfrac{\partial^2 z}{\partial x \partial y} = 2\dfrac{\partial z}{\partial x}.\dfrac{\partial z}{\partial y}$.

2 Two sides of a triangular plate are measured as 125 mm and 160 mm, each to the nearest millimetre. The included angle is quoted as $60° \pm 1°$. Calculate the length of the remaining side and the maximum possible error in the result.

3 If $z = (x^2 - y^2)^{1/2}$ and x is increasing at 3·5 m/s, determine at what rate y must change in order that z shall be neither increasing nor decreasing at the instant when $x = 5$ m and $y = 3$ m.

4 If $2x^2 + 4xy + 3y^2 = 1$, obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

5 If $u = x^2 + y^2$ and $v = 4xy$, determine

$\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

6 Determine the position and nature of the stationary points of the functions:

(a) $z = 2x^2y^2 + 4xy^2 - 4y^3 + 16y + 5$

(b) $z = 4 - 25x^2 + 20xy - 4y^2$.

▶

7 A rectangular storage tank is to have a capacity of $1{\cdot}0\ \text{m}^3$. If the tank is closed and the top is made of metal half as thick as the sides and base, use Lagrange's method of undetermined multipliers to determine the dimensions of the tank for the total amount of metal used in its construction to be a minimum.

8 Use Lagrange's method of undetermined multipliers to obtain the stationary values of $u = x^2 + y^2 + z^2$ subject to the constraint $\phi = 3x - 2y + z - 4$.

Further problems 10

81

1 If $z = 2x^2 - 3y$ with $u = x^2 \sin y$ and $v = 2y\cos x$, determine expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

2 If $u = x^2 + e^{-3y}$ and $v = 2x + e^{3y}$, determine $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

3 If $z = f(x, y)$ where $x = uv$ and $y = u^2 - v^2$, show that

(a) $2x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y} = u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v}$

(b) $2\dfrac{\partial z}{\partial y} = \dfrac{1}{u^2 + v^2}\left\{u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}\right\}$.

4 If $V = f(x, y)$ and $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

5 If $z = \cosh 2x \sin 3y$ and $u = e^x(1 + y^2)$ and $v = 2ye^{-x}$, determine expressions for $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$, and hence find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

6 If $z = f(u, v)$ where $u = \frac{1}{2}(x^2 - y^2)$ and $v = xy$, prove that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 2u\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) + 4v\frac{\partial^2 z}{\partial u \partial v} + 2\frac{\partial z}{\partial u}.$$

7 Locate the stationary points of the following functions. Determine the nature of the points and calculate the critical function values.

(a) $z = y^2 + xy + x^2 + 4y - 4x + 5$

(b) $z = y^2 + xy + 2x + 3y + 6$

(c) $z = 3xy - 6y^2 - 3x^2 + 6y + 6x + 7$.

▶

8 Find the stationary points of the function

$$z = (x^2 + y^2)^2 - 8(x^2 - y^2)$$

and determine their nature.

9 Verify that the function $z = (x + y - 1)/(x^2 + 2y^2 + 2)$ has stationary values at $(2, 1)$ and $\left(-\frac{2}{3}, -\frac{1}{3}\right)$ and determine their nature.

10 Locate stationary points of the function

$$z = 4x^2 + 10xy + 4y^2 - x^2y^2$$

and determine their nature.

11 Find the stationary points of the following functions and determine their nature.

(a) $z = x(x^2 - 3) + 3y(x - 1)^2 + 18y^2(2y - 3)$

(b) $z = x^2y^2 - x^2 - y^2$.

12 Find the stationary points of the following functions and determine their nature.

(a) $z = (x - y)(x^2 + xy + y^2)$

(b) $z = 6 - x^2 + 8xy - 16y^2$

(c) $z = \cos(x^2 + y^2)$.

13 A metal channel is formed by turning up the sides of width x of a rectangular sheet of metal through an angle θ. If the sheet is 200 mm wide, determine the values of x and θ for which the cross-section of the channel will be a maximum.

14 A container is in the form of a right circular cylinder of length l and diameter d, with equal conical ends of the same diameter and height h. If V is the fixed volume of the container, find the dimensions l, h and d for minimum surface area.

15 A solid consists of a cylinder of length l and diameter d, surmounted at one end by a cone of vertex angle 2θ and base diameter d, and at the other end by a hemisphere of the same diameter. If the volume V of the solid is 50 cm^3, determine the dimensions l, d and θ so that the total surface area shall be a minimum.

16 A rectangular solid of maximum volume is to be cut from a solid sphere of radius r. Determine the dimensions of the solid so formed and its volume.

17 Use Lagrange's method of undetermined multipliers to obtain the stationary values of the following functions u, subject in each case to the constraint ϕ.

(a) $u = x^2y^2z^2 \qquad \phi = x^2 + y^2 + z^2 - 4 = 0$

(b) $u = x^2 + y^2 \qquad \phi = 4x^2 + 6xy + 4y^2 = 9$.

Partial differential equations

Frames 1 to 66

Learning outcomes

When you have completed this Programme you will be able to:

- Summarise the introductory methods of solving ordinary differential equations
- Solve partial differential equations that are amenable to solution by direct integration
- Apply initial and boundary conditions
- Solve the one-dimensional wave and heat equations by separating the variables and obtaining eigenfunctions and corresponding eigenvalues
- Solve the two-dimensional Laplace equation in Cartesian coordinates
- Recognise the need for alternative coordinate systems and solve the two-dimensional Laplace equation in plane polar coordinates

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 24 First-order differential equations and 25 Second-order differential equations

Introduction

1

The formation of ordinary linear differential equations and their solution by various methods were covered in some detail in Programmes 24 and 25 of *Engineering Mathematics (Fifth Edition)*, and reference to these before undertaking the new work of this Programme could be beneficial – especially Programme 25 which dealt with second-order equations. Working through the Test exercise of that Programme would provide worthwhile revision.

The main results obtained are listed here for convenience and easy reference.

1 Equations of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

Auxiliary equation $am^2 + bm + c = 0$. Solutions depend on the roots of this equation.

(a) Real and different roots: $m = m_1$ and $m = m_2$

$$\text{Solution } y = Ae^{m_1x} + Be^{m_2x} \qquad (1)$$

(b) Real and equal roots: $m = m_1$ (twice)

$$\text{Solution } y = e^{m_1x}(A + Bx) \qquad (2)$$

(c) Complex roots: $m = \alpha \pm j\beta$

$$\text{Solution } y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) \qquad (3)$$

2 Equations of the form $\dfrac{d^2y}{dx^2} \pm n^2y = 0$

(a) $\dfrac{d^2y}{dx^2} + n^2y = 0 \quad \therefore\ m^2 + n^2 = 0 \quad \therefore\ m^2 = -n^2 \quad \therefore\ m = \pm jn$

$$\text{Solution } y = A\cos nx + B\sin nx \qquad (4)$$

(b) $\dfrac{d^2y}{dx^2} - n^2y = 0 \quad \therefore\ m^2 - n^2 = 0 \quad \therefore\ m^2 = n^2 \quad \therefore\ m = \pm n$

$$\left.\begin{aligned} \text{Solution } y &= A\cosh nx + B\sinh nx \\ \text{or } y &= Ae^{nx} + Be^{-nx} \\ \text{or } y &= A\sinh n(x + \phi) \end{aligned}\right\} \qquad (5)$$

In each case, A and B are arbitrary constants depending on the initial conditions, and in the last form ϕ is an arbitrary constant.

Partial differential equations

2

A partial differential equation is a relationship between a dependent variable u and two or more independent variables $(x, y, t, \ldots)$ and partial derivatives of u with respect to these independent variables. The solution is therefore of the form $u = f(x, y, t, \ldots)$.

Solution by direct integration

The simplest form of partial differential equation is such that a solution can be determined by direct partial integration.

Example 1

Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$ given that at $x = 0$, $u = \cos\ 2t$ and $\dfrac{\partial u}{\partial x} = \sin t$. Notice that the boundary conditions are functions of t and not just constants. $\dfrac{\partial^2 u}{\partial x^2} = 12x^2(t+1)$. Integrating partially with respect to x, we have $\dfrac{\partial u}{\partial x} = 4x^3(t+1) + \phi(t)$ where the arbitrary function $\phi(t)$ takes the place of the normal arbitrary constant in ordinary integration. Integrating partially again with respect to x gives

$u = \ldots\ldots\ldots\ldots$

3

$$\boxed{u = x^4(t+1) + x\phi(t) + \theta(t)}$$

where $\theta(t)$ is a second arbitrary function.

To find the two arbitrary functions $\phi(t)$ and $\theta(t)$, we apply the given initial conditions that at $x = 0$, $\dfrac{\partial u}{\partial x} = \sin t$ and $u = \cos\ 2t$. Substituting these in the relevant equations gives

$\phi(t) = \ldots\ldots\ldots\ldots;\ \theta(t) = \ldots\ldots\ldots\ldots$

4

$$\boxed{\phi(t) = \sin t;\ \theta(t) = \cos 2t}$$

Because

$u = x^4(t+1) + x\sin t + \cos 2t$

Example 2

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial y} = \sin(x+y)$, given that at $y = 0$, $\dfrac{\partial u}{\partial x} = 1$ and at $x = 0$, $u = (y-1)^2$.

In just the same way as before, $u = \ldots\ldots\ldots\ldots$

5

$$\boxed{u = -\sin(x+y) + x + \sin x + (y-1)^2}$$

Because

$$\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y) \quad \therefore \quad \frac{\partial u}{\partial x} = -\cos(x+y) + \phi(x).$$

At $y = 0$, $\dfrac{\partial u}{\partial x} = 1 \quad \therefore \quad 1 = -\cos x + \phi(x) \quad \therefore \quad \phi(x) = 1 + \cos x$

$$\therefore \quad \frac{\partial u}{\partial x} = -\cos(x+y) + 1 + \cos x$$

Integrating again partially, this time with respect to x, we have

$$u = -\sin(x+y) + x + \sin x + \theta(y)$$

But at $x = 0$, $u = (y-1)^2$. $\therefore \quad (y-1)^2 = -\sin y + \theta(y)$

$$\therefore \quad \theta(y) = (y-1)^2 + \sin y$$

$$\therefore \quad u = -\sin(x+y) + x + \sin x + \sin y + (y-1)^2$$

Initial conditions and boundary conditions

As with any differential equation, the arbitrary constants or arbitrary functions in any particular case are determined from the additional information given concerning the variables of the equation. These extra facts are called the *initial conditions* or, more generally, the *boundary conditions* since they do not always refer to zero values of the independent variables.

Example 3

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial y} = \sin x \cos y$, subject to the boundary conditions that at $y = \dfrac{\pi}{2}$, $\dfrac{\partial u}{\partial x} = 2x$ and at $x = \pi$, $u = 2\sin y$.

Work through it: it is easy enough. $u = \dots\dots\dots\dots$

6

$$\boxed{u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2}$$

Because

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x \cos y \qquad \therefore \quad \frac{\partial u}{\partial x} = \sin x \sin y + \phi(x)$$

$$\text{But } \frac{\partial u}{\partial x} = 2x \text{ at } y = \frac{\pi}{2} \qquad \therefore \quad \phi(x) = 2x - \sin x$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x - \sin x(1 - \sin y) \qquad \therefore \quad u = x^2 + \cos x(1 - \sin y) + \theta(y)$$

But $u = 2\sin y$ at $x = \pi \quad \therefore \quad \theta(y) = 1 - \pi^2 + \sin y$

$$u = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$$

On to the next frame

7

Before we take a closer look at some of the more important partial differential equations occurring in branches of technology, let us recall the fact that if $u = u_1$, $u = u_2$, $u = u_3, \ldots$ are different solutions of a linear partial differential equation, so also is the *linear combination*

$$u = c_1 u_1 + c_2 u_2 + c_3 u_3 + \ldots$$

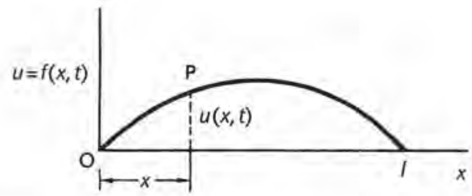
where $c_1, c_2, c_3, \ldots$ are arbitrary constants.

There are many types of partial differential equations, some requiring special treatment in their solution. In this Programme we are concerned with a restricted number of such equations that occur in branches of science and technology, which can be solved by the method of separating the variables, and which also link up with the work we have done on Fourier series techniques.

Let us make a new start

8

The wave equation



Consider a perfectly flexible elastic string stretched between two points at $x = 0$ and $x = l$ with uniform tension T. If the string is displaced slightly from its initial position of rest and released, with the end points remaining fixed, then the string will vibrate. The position of any point P in the string will then depend on its distance from one end and on the instant in time. Its displacement u at any time t can thus be expressed as $u = f(x, t)$ where x is its distance from the left-hand end.

The equation of motion is given by $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$, where $c^2 = \dfrac{T}{\rho}$ in which T is the tension in the string and ρ the mass per unit length of the string. The displacement of the string is regarded as small so that T and ρ remain constant.

Now let us deal with the solution of this equation.

On to the next frame

Solution of the wave equation

The new equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ has a solution $u(x, t)$.

Boundary conditions:

(a) The string is fixed at both ends, i.e. at $x = 0$ and at $x = l$ for all values of time t. Therefore $u\,(x, t)$ becomes

$$\left.\begin{aligned} u(0, t) = 0 \\ u(l, t) = 0 \end{aligned}\right\} \quad \text{for all values of } t \geq 0$$

Initial conditions:

(b) If the initial deflection of P at $t = 0$ is denoted by $f(x)$, then

$$u(x, 0) = f(x)$$

(c) Let the initial velocity of P be $g(x)$, then

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$$

So now we have listed all the information available from the question. Next we turn to solving the equation.

Solution by separating the variables

We assume a trial solution of the form $u(x, t) = X(x)T(t)$ where

$X(x)$ is a function of x only

$T(t)$ is a function of t only.

If we simplify the symbols to $u = XT$ and denote derivatives with respect to their own independent variables by primes, we have

$$u = XT \quad \therefore \quad \frac{\partial u}{\partial x} = X'T \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = XT''$$

The wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ can then be written as

....

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$$\boxed{X''T = \frac{1}{c^2}XT''}$$

and this can be transposed into $\dfrac{X''}{X} = \dfrac{1}{c^2} \cdot \dfrac{T''}{T}$

Notice that the left-hand side expression involves functions of x only and that the right-hand side expression involves functions of t only. Therefore, if these two expressions are to be equal for all values of the separate variables, then both expressions must be equal to

....

11

$$\boxed{\text{a constant}}$$

Denote this arbitrary constant by k. Then we have

$$\frac{X''}{X} = k \quad \text{and} \quad \frac{1}{c^2} \cdot \frac{T''}{T} = k$$

$$\therefore \; X'' - kX = 0 \quad \text{and} \quad T'' - c^2kT = 0$$

Let us consider the first of these two equations for different values of k.

(1) *If* $k = 0$, $X'' = 0 \quad \therefore \; X' = a \quad \therefore \; X = ax + b$.

$$\left.\begin{array}{l} \text{But } X = 0 \text{ at } x = 0 \quad \therefore \; b = 0 \quad \therefore \; X = ax \\ \text{and } X = 0 \text{ at } x = l \quad \therefore \; a = 0 \end{array}\right\} \therefore \; a = b = 0$$

$\therefore \; X = 0$ which is not oscillatory as the problem requires it to be.

(2) *If* k *is positive,* let $k = p^2 \quad \therefore \; X'' - p^2X = 0$.

The auxiliary equation is therefore $m^2 - p^2 = 0 \quad \therefore \; m^2 = p^2$

$$m = \pm p$$

$\therefore \; X = Ae^{px} + Be^{-px}$

But $X = 0$ at $x = 0 \quad \therefore \; 0 = A + B \quad \therefore \; B = -A$

and $X = 0$ at $x = l \quad \therefore \; 0 = Ae^{pl} - Ae^{-pl} \quad \therefore \; 0 = A(e^{pl} - e^{-pl})$

$$\therefore \; A = 0 \quad \therefore \; A = B = 0$$

Here again $X = 0$ which is not oscillatory.

(3) *If* k *is negative,* let $k = -p^2 \quad \therefore \; X'' + p^2X = 0$.

This is one of the standard equations listed at the beginning of the Programme and gives a solution

$$X = A\cos px + B\sin px \qquad (1)$$

which fits the requirements.

The second equation $T'' - c^2kT = 0$ therefore now becomes

....

12

$$\boxed{T'' + c^2p^2T = 0}$$

because the same value for k must apply. This equation is of the same form as before and gives the solution

$$T = C\cos cpt + D\sin cpt \tag{2}$$

So our suggested solution $u = XT$ now becomes

$$u(x,t) = (A\cos px + B\sin px)\,(C\cos cpt + D\sin cpt)$$

and, if we put $cp = \lambda \quad \therefore\ p = \dfrac{\lambda}{c}$, this becomes

$$u(x,t) = \left(A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right)(C\cos\lambda t + D\sin\lambda t) \tag{3}$$

where A, B, C, D are arbitrary constants.

The result, of course, must satisfy the set of boundary conditions which we now turn to.

(a) $u = 0$ when $x = 0$ for all values of t. From this, we get

....

13

$$\boxed{A = 0}$$

Because, substituting $u = 0$ and $x = 0$ in result (3) above

$$0 = A(C\cos\lambda t + D\sin\lambda t) \text{ for all } t \quad \therefore\ A = 0$$

$$\therefore\ u(x,t) = B\sin\frac{\lambda}{c}x(C\cos\lambda t + D\sin\lambda t)$$

(b) $u = 0$ when $x = l$ for all $t \quad \therefore\ 0 = B\sin\dfrac{\lambda l}{c}(C\cos\lambda t + D\sin\lambda t)$

Now $B \neq 0$ or $u(x,t)$ would be identically zero. $\therefore\ \sin\dfrac{\lambda l}{c} = 0$.

$$\therefore\ \frac{\lambda l}{c} = n\pi \text{ where } n = 1, 2, 3, \ldots \quad \therefore\ \lambda = \frac{nc\pi}{l} \text{ for } n = 1, 2, 3, \ldots$$

Note that we exclude $n = 0$ since this would also make $u(x, t)$ identically zero.

▶

As we can see, there is an infinite set of values of λ and each separate value gives a particular solution for $u(x,t)$. The values of λ are called the *eigenvalues* and each corresponding solution the *eigenfunction*.

Putting $n = 1, 2, 3, \ldots$ we therefore have

	Eigenvalues	Eigenfunctions
n	$\lambda = \dfrac{nc\pi}{l}$	$u(x,t) = B\sin\dfrac{\lambda x}{c}\{C\cos\lambda t + D\sin\lambda t\}$
1	$\lambda_1 = \dfrac{c\pi}{l}$	$u_1 = \sin\dfrac{\pi x}{l}\left\{C_1\cos\dfrac{c\pi t}{l} + D_1\sin\dfrac{c\pi t}{l}\right\}$
2	$\lambda_2 = \dfrac{2c\pi}{l}$	$u_2 = \sin\dfrac{2\pi x}{l}\left\{C_2\cos\dfrac{2c\pi t}{l} + D_2\sin\dfrac{2c\pi t}{l}\right\}$
3	$\lambda_3 = \dfrac{3c\pi}{l}$	$u_3 = \sin\dfrac{3\pi x}{l}\left\{C_3\cos\dfrac{3c\pi t}{l} + D_3\sin\dfrac{3c\pi t}{l}\right\}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \dfrac{rc\pi}{l}$	$u_r = \sin\dfrac{r\pi x}{l}\left\{C_r\cos\dfrac{rc\pi t}{l} + D_r\sin\dfrac{rc\pi t}{l}\right\}$

Note that the constant B has been absorbed into the constants C and D so that $BC = C_n$ and $BD = D_n$, where $C_1, C_2, C_3, \ldots$ and $D_1, D_2, D_3, \ldots$ are arbitrary constants.

Since the original wave equation is linear in form, we have already noted that if $u = u_1$, $u = u_2$, $u = u_3 \ldots$ are particular solutions, a more general solution is

. . . .

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$$\boxed{u = u_1 + u_2 + u_3 + \ldots}$$

The more general solution is therefore

$$u(x,t) = \sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty}\left\{\sin\frac{r\pi x}{l}\left(C_r\cos\frac{rc\pi t}{l} + D_r\sin\frac{rc\pi t}{l}\right)\right\} \qquad (4)$$

We still have to find C_r and D_r and for this we use the initial conditions which we have not yet taken into account.

(c) At $t = 0$, $u(x, 0) = f(x)$ for $0 \leq x \leq l$

Therefore from (4), $u(x, 0) = f(x) = \sum_{r=1}^{\infty} C_r \sin\dfrac{r\pi x}{l}$.

(d) Also at $t = 0$, $\left[\dfrac{\partial u}{\partial t}\right]_{t=0} = g(x)$ for $0 \leq x \leq l$

We therefore differentiate (4) with respect to t and put $t = 0$, which gives

. . . .

$$\boxed{g(x) = \frac{c\pi}{l}\sum_{r=1}^{\infty} D_r r \sin\frac{r\pi x}{l}}$$

Because

$$\frac{\partial u}{\partial t} = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{l}\left\{-C_r\frac{rc\pi}{l}\sin\frac{rc\pi t}{l} + D_r\frac{rc\pi}{l}\cos\frac{rc\pi t}{l}\right\}$$

$$\therefore \text{ With } t = 0, \quad \frac{\partial u}{\partial t} = g(x) = \sum_{r=1}^{\infty} D_r\frac{rc\pi}{l}\sin\frac{r\pi x}{l}$$

$$\therefore \; g(x) = \frac{c\pi}{l}\sum_{r=1}^{\infty} D_r r \sin\frac{r\pi x}{l}$$

Finally we can draw on our knowledge of Fourier series techniques to determine the coefficients C_r and D_r.

$C_r = 2 \times$ mean value of $f(x)\sin\frac{r\pi x}{l}$ between $x = 0$ and $x = l$

$$\therefore \; C_r = \frac{2}{l}\int_0^l f(x)\sin\frac{r\pi x}{l}\,dx \qquad r = 1, 2, 3, \ldots$$

and $D_r\frac{rc\pi}{l} = 2 \times$ mean value of $g(x)\sin\frac{r\pi x}{l}$ between $x = 0$ and $x = l$

$$\therefore \; D_r = \frac{2}{rc\pi}\int_0^l g(x)\sin\frac{r\pi x}{l}\,dx \qquad r = 1, 2, 3, \ldots$$

The general solution (4) then becomes

$$u(x, t) = \sum_{r=1}^{\infty}\left\{\left[\frac{2}{l}\int_0^l f(w)\sin\frac{r\pi w}{l}\,dw\right]\cos\frac{rc\pi t}{l}\sin\frac{r\pi x}{l} + \left[\frac{2}{rc\pi}\int_0^l g(w)\sin\frac{r\pi w}{l}\,dw\right]\sin\frac{rc\pi t}{l}\sin\frac{r\pi x}{l}\right\} \quad (5)$$

Notice that the variable of integration has been changed from x to w because we wish to use the variable x in the final expression for $u(x, t)$. The value of a definite integral depends only on the limit points of the integral and we are free to use any symbol that we desire for the variable of integration – we call such a variable a *dummy variable*.

At first sight, the solution seems very involved, but it can be analysed into a definite sequence of logical steps. Given the equation and relevant initial and boundary conditions, we go through the following stages.

(a) Assume a solution of the form $u = XT$ and express the equation in terms of X and T and their derivatives.

(b) Transpose the equation by separation of the variables and equate each side to a constant, so obtaining two separate equations, one in x and the other in t.

(c) Choose $k = -p^2$ to give an oscillatory solution.

(d) The two solutions are of the form

$$X = A\cos px + B\sin px$$
$$T = C\cos cpt + D\sin cpt$$

Then $u(x,t) = \{A\cos px + B\sin px\}\{C\cos cpt + D\sin cpt\}$.

(e) Putting $cp = \lambda$, i.e. $p = \dfrac{\lambda}{c}$, this becomes

$$u(x,t) = \left\{A\cos\frac{\lambda}{c}x + B\sin\frac{\lambda}{c}x\right\}\{C\cos\lambda t + D\sin\lambda t\}.$$

(f) Apply boundary conditions to determine A and B.

(g) List the eigenvalues and eigenfunctions for $n = 1, 2, 3, \ldots$ and determine the general solution as an infinite sum.

(h) Apply the remaining initial or boundary conditions.

(i) Determine the coefficients C_r and D_r by Fourier series techniques.

Make a list of these steps: then we can follow them with an example.

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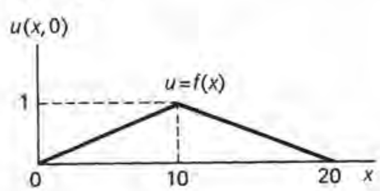
Example



A stretched string of length 20 cm is set oscillating by displacing its mid-point a distance 1 cm from its rest position and releasing it with zero initial velocity. Solve the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\cdot\dfrac{\partial^2 u}{\partial t^2}$ where $c^2 = 1$ to determine the resulting motion, $u(x,t)$.

First we make a list of the boundary conditions from the data given in the question.

$$u(0, t) = \ldots\ldots\ldots\ldots; \quad u(20, t) = \ldots\ldots\ldots\ldots$$
$$u(x, 0) = \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \ldots\ldots\ldots\ldots$$

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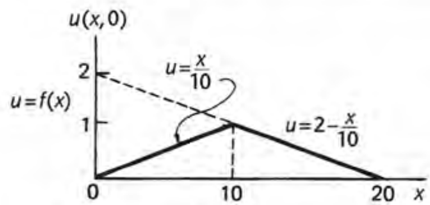
$$\boxed{\begin{aligned} &u(0,t)=0; \quad u(20,t)=0 \quad \text{(fixed end points)}\\ &u(x,0)=f(x)=\frac{x}{10} \qquad 0\le x\le 10\\ &\qquad\qquad\;\;=\frac{20-x}{10} \quad 10\le x\le 20\\ &\left[\frac{\partial u}{\partial t}\right]_{t=0}=0 \quad \text{(zero initial velocity)}\end{aligned}}$$



Now we can apply our sequence of operations which we listed.

So move on

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(a) Assume a solution $u = XT$ where X is a function of x only and T is a function of t only. Then the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ (since $c = 1$) becomes ....

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$$\boxed{X''T = XT''}$$

Because

$$u = XT \qquad \therefore\ \frac{\partial u}{\partial x} = X'T \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\text{and} \qquad \frac{\partial u}{\partial t} = XT' \qquad \frac{\partial^2 u}{\partial t^2} = XT''$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \therefore\ X''T = XT''$$

(b) Next we rearrange the equation to separate the variables, giving

....

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$$\boxed{\frac{X''}{X} = \frac{T''}{T}}$$

(c) Since the two sides are equal for all values of the variables, each must be equal to a constant k and to give an oscillatory solution we put $k = -p^2$. The two separate equations then are written

.... and ....

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$$\boxed{X'' + p^2X = 0 \quad \text{and} \quad T'' + p^2T = 0}$$

(d) These have solution $X = \ldots\ldots\ldots\ldots$

$T = \ldots\ldots\ldots\ldots$

so that $u(x, t) = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{array}{c} X = A\cos px + B\sin px; \quad T = C\cos pt + D\sin pt \\ \therefore\ u(x, t) = \{A\cos px + B\sin px\}\{C\cos pt + D\sin pt\} \end{array}}$$

(e) We normally now put $cp = \lambda$, but in this case $c = 1 \ \therefore\ p = \lambda$ and $u(x, t) = \ldots\ldots\ldots\ldots$

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$$\boxed{u(x, t) = \{A\cos\lambda x + B\sin\lambda x\}\{C\cos\lambda t + D\sin\lambda t\}}$$

(f) Now we determine A and B from the boundary conditions.

(1) $u(0, t) = 0 \quad \therefore\ 0 = A(C\cos\lambda t + D\sin\lambda t) \quad \therefore\ A = 0$

$\therefore\ u(x, t) = B\sin\lambda x(C\cos\lambda t + D\sin\lambda t)$

(2) $u(20, t) = 0 \quad \therefore\ 0 = B\sin 20\lambda(C\cos\lambda t + D\sin\lambda t)$

$B \neq 0$ or u would be identically zero. $\quad \therefore\ \sin 20\lambda = 0.$

$\therefore\ 20\lambda = n\pi \quad \therefore\ \lambda = \dfrac{n\pi}{20}$

$$\therefore\ u(x, t) = \sin\frac{n\pi}{20}x\left\{P\cos\frac{n\pi}{20}t + Q\sin\frac{n\pi}{20}t\right\}$$

where $P = B \times C$ and $Q = B \times D$.

(g) The next step is to list the eigenvalues and eigenfunctions.

	Eigenvalues	Eigenfunctions
n	$\lambda = \dfrac{n\pi}{20}$	$u(x,t) = \sin\lambda x\{P\cos\lambda t + Q\sin\lambda t\}$
1	$\lambda_1 = \dfrac{\pi}{20}$	$u_1 = \sin\dfrac{\pi x}{20}\left\{P_1\cos\dfrac{\pi t}{20} + Q_1\sin\dfrac{\pi t}{20}\right\}$
2	$\lambda_2 = \dfrac{2\pi}{20}$	$u_2 = \sin\dfrac{2\pi x}{20}\left\{P_2\cos\dfrac{2\pi t}{20} + Q_2\sin\dfrac{2\pi t}{20}\right\}$
3	$\lambda_3 = \dfrac{3\pi}{20}$	$u_3 = \sin\dfrac{3\pi x}{20}\left\{P_3\cos\dfrac{3\pi t}{20} + Q_3\sin\dfrac{3\pi t}{20}\right\}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \dfrac{r\pi}{20}$	$u_r = \sin\dfrac{r\pi x}{20}\left\{P_r\cos\dfrac{r\pi t}{20} + Q_r\sin\dfrac{r\pi t}{20}\right\}$

$$u = u_1 + u_2 + u_3 + \ldots \quad \therefore\ u(x, t) = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left\{P_r\cos\frac{r\pi t}{20} + Q_r\sin\frac{r\pi t}{20}\right\}$$

▶

(h) Now we apply the remaining initial conditions

$$(1)\quad u(x,0) = f(x) = \frac{x}{10} \qquad 0 \le x \le 10$$
$$= \frac{20-x}{10} \qquad 10 \le x \le 20$$

Also $u(x,0) = \ldots\ldots\ldots\ldots$

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$$\boxed{u(x,\,0) = \sum_{r=1}^{\infty} P_r \sin\frac{r\pi x}{20}}$$

Then $P_r = 2 \times$ mean value of $f(x)\sin\dfrac{r\pi x}{20}$ between $x = 0$ and $x = 20$

$$= \frac{2}{20}\int_0^{20} f(x)\sin\frac{r\pi x}{20}\,\mathrm{d}x$$

$$\therefore\ 10P_r = \int_0^{10}\frac{x}{10}\sin\frac{r\pi x}{20}\,\mathrm{d}x + \int_{10}^{20}\frac{20-x}{10}\sin\frac{r\pi x}{20}\,\mathrm{d}x$$
$$= \quad I_1 \quad + \quad I_2$$

$$I_1 = \int_0^{10}\frac{x}{10}\sin\frac{r\pi x}{20}\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_1 = -\frac{20}{r\pi}\cos\frac{r\pi}{2} + \frac{40}{r^2\pi^2}\sin\frac{r\pi}{2}}$$

Using integration by parts

$$I_2 = \int_{10}^{20}\frac{20-x}{10}\sin\frac{r\pi x}{20}\,\mathrm{d}x = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_2 = \frac{20}{r\pi}\cos\frac{r\pi}{2} - \frac{40}{r^2\pi^2}\left(\sin r\pi - \sin\frac{r\pi}{2}\right)}$$

Then $10P_r = -\dfrac{20}{r\pi}\cos\dfrac{r\pi}{2} + \dfrac{40}{r^2\pi^2}\sin\dfrac{r\pi}{2} + \dfrac{20}{r\pi}\cos\dfrac{r\pi}{2} - \dfrac{40}{r^2\pi^2}\left(\sin r\pi - \sin\dfrac{r\pi}{2}\right)$

$$\therefore\ \text{For } r = 1,\,2,\,3,\ldots\ P_r = \frac{8}{r^2\pi^2}\sin\frac{r\pi}{2}$$

$$\therefore\ u(x,t) = \sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left\{\frac{8}{r^2\pi^2}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20} + Q_r\sin\frac{r\pi t}{20}\right\}$$

(2) Also at $t = 0$, $\dfrac{\partial u}{\partial t} = 0$.

$$\frac{\partial u}{\partial t} = \ldots\ldots\ldots\ldots$$

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$$\frac{\partial u(x,t)}{\partial t}=\sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}\left\{\left(\frac{8}{r^2\pi^2}\sin\frac{r\pi}{2}\right)\left(-\frac{r\pi}{20}\sin\frac{r\pi t}{20}\right)+Q_r\frac{r\pi}{20}\cos\frac{r\pi t}{20}\right\}$$

$\therefore$ At $t=0$, $\quad 0=\sum_{r=1}^{\infty}\sin\frac{r\pi x}{20}Q_r\frac{r\pi}{20} \qquad \therefore\ Q_r=0$

So finally we have $\quad u(x,t)=$ ....

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$$u(x,t)=\frac{8}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{r\pi x}{20}\sin\frac{r\pi}{2}\cos\frac{r\pi t}{20}$$

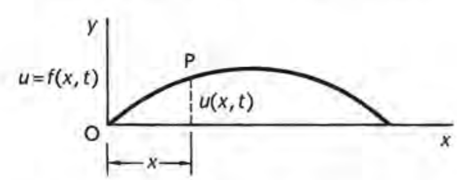
And that is it.

Now let us turn to a slightly different equation, but one for which the method of solution is very much along the same lines.

The heat conduction equation for a uniform finite bar

The conduction of heat in a uniform bar depends on the initial distribution of temperature and on the physical properties of the bar, i.e. the thermal conductivity and specific heat of the material, and the mass per unit length of the bar.

With a uniform bar insulated except at its ends, any heat flow is along the bar and, at any instant, the temperature u at a point P is a function of its distance x from one end and of the time t.



The one-dimensional heat equation is then of the form

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\cdot\frac{\partial u}{\partial t} \qquad (1)$$

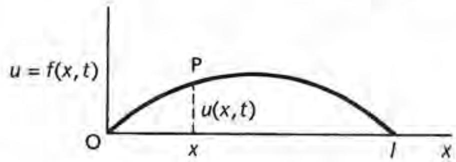
where $c^2=\dfrac{k}{\sigma\rho}$ in which $k=$ thermal conductivity of the material; $\sigma=$ specific heat of the material; $\rho=$ mass per unit length of the bar.

You will already have noticed that the heat equation differs from the wave equation only in the fact that the right-hand side contains the first partial derivative instead of the second. It is not surprising therefore that the method of solution is very much like that of our previous examples.

Solutions of the heat conduction equation

Consider the case where

(a) the bar extends from $x = 0$ to $x = l$
(b) the temperature of the ends of the bar is maintained at zero
(c) the initial temperature distribution along the bar is defined by $f(x)$.



The boundary conditions can be expressed as ....

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$$\boxed{\begin{array}{l} u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0 \quad \text{for all } t \geq 0 \\ u(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq l \end{array}}$$

As before, we assume a solution of the form $u(x, t) = X(x)T(t)$ where

X is a function of x only
T is a function of t only.

Then, starting with $u = XT$ we can write the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$ in terms of X and T, and separating the variables, we obtain

....

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$$\boxed{\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T}}$$

Arguing as before, since the left-hand side is a function of x only and the right-hand side a function of t only, for these to be equal each side must equal the same constant. Let this be $(-p^2)$ as before.

$$\therefore \; \frac{X''}{X} = -p^2 \quad \therefore \; X'' + p^2 X = 0 \;\; \text{giving} \;\; X = A\cos px + B\sin px$$

and $\dfrac{1}{c^2} \cdot \dfrac{T'}{T} = -p^2 \quad \therefore \; T' + p^2 c^2 T = 0$ giving $T = $ ....

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$$\boxed{T = Ce^{-p^2c^2t}}$$

Because

$$\frac{T'}{T} = -p^2c^2 \quad \therefore \ \ln T = -p^2c^2t + c_1 \quad \therefore \ T = Ce^{-p^2c^2t}$$

$$u(x,t) = XT = \{A\cos px + B\sin px\}\,Ce^{-p^2c^2t}$$

$$\therefore \ u(x,t) = \{P\cos px + Q\sin px\}e^{-p^2c^2t} \qquad \text{where } P = AC \text{ and } Q = BC$$

Now put $pc = \lambda \ \ \therefore \ p = \dfrac{\lambda}{c}$

$$\therefore \ u(x,t) = \left\{P\cos\frac{\lambda}{c}x + Q\sin\frac{\lambda}{c}x\right\}e^{-\lambda^2t}$$

Applying the boundary condition $u(0,t) = 0$ gives

.... and ....

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$$\boxed{P = 0 \ \text{ and } \ u(x,t) = Qe^{-\lambda^2t}\sin\frac{\lambda}{c}x}$$

Also $u(l,t) = 0$ and from this we get

....

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$$\boxed{\lambda = \frac{nc\pi}{l} \ \text{ for } \ n = 1,\ 2,\ 3,\ldots}$$

Because

$$\text{if } u = 0 \text{ when } x = l, \qquad 0 = Qe^{-\lambda^2t}\sin\frac{\lambda l}{c}$$

$Q \neq 0$ or $u(x,\ t)$ would be identically zero $\therefore \ \sin\dfrac{\lambda l}{c} = 0$

$$\therefore \ \frac{\lambda l}{c} = n\pi \qquad \therefore \ \lambda = \frac{nc\pi}{l} \qquad n = 1,\ 2,\ 3,\ldots$$

▶

Now we can compile the table of eigenfunctions.

n	$\lambda = \dfrac{nc\pi}{l}$	$u(x,t) = Qe^{-\lambda^2 t}\sin\dfrac{n\pi x}{l}$
1	$\lambda_1 = \dfrac{c\pi}{l}$	$u_1 = Q_1 e^{-\lambda_1^2 t}\sin\dfrac{\pi x}{l}$
2	$\lambda_2 = \dfrac{2c\pi}{l}$	$u_2 = Q_2 e^{-\lambda_2^2 t}\sin\dfrac{2\pi x}{l}$
3	$\lambda_3 = \dfrac{3c\pi}{l}$	$u_3 = Q_3\, e^{-\lambda_3^2 t}\sin\dfrac{3\pi x}{l}$
$\vdots$	$\vdots$	$\vdots$
r	$\lambda_r = \dfrac{rc\pi}{l}$	$u_r = Q_r\, e^{-\lambda_r^2 t}\sin\dfrac{r\pi x}{l}$

$$u = u_1 + u_2 + u_3 + \dots$$

$$\therefore\ u(x,\, t) = \sum_{r=1}^{\infty}\left\{Q_r\, e^{-\lambda_r^2 t}\sin\frac{r\pi x}{l}\right\}$$

The remaining boundary condition still to be applied is that when

$t = 0,\ \ u(x,0) = f(x) \quad 0 \le x \le l$

This gives $f(x) = \dots\dots\dots\dots$

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$$\boxed{f(x) = \sum_{r=1}^{\infty}\left\{Q_r \sin\frac{r\pi x}{l}\right\}}$$

and from our knowledge of Fourier series techniques:

$Q_r = \dots\dots\dots\dots$

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$$\boxed{Q_r = 2 \times \text{mean value of } f(x)\sin\frac{r\pi x}{l} \text{ from } x = 0 \text{ to } x = l}$$

$\therefore\ Q_r = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{r\pi x}{l}\,\mathrm{d}x$ and the final solution becomes

$$u(x,t) = \frac{2}{l}\sum_{r=1}^{\infty}\left\{\left[\int_0^l f(w)\sin\frac{r\pi w}{l}\,\mathrm{d}w\right]e^{-\lambda_r^2 t}\sin\frac{r\pi x}{l}\right\}$$

where $\lambda_r = \dfrac{rc\pi}{l} \qquad r = 1,\, 2,\, 3,\, \dots$

Now on to the next frame for an example

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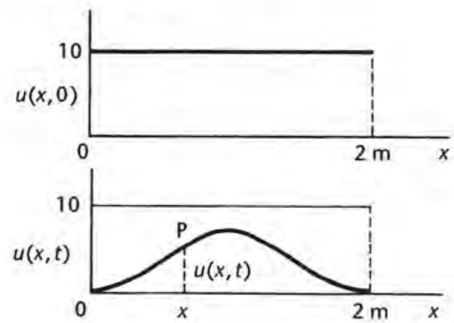
Example

A bar of length 2 m is fully insulated along its sides. It is initially at a uniform temperature of 10°C and at $t = 0$ the ends are plunged into ice and maintained at a temperature of 0°C. Determine an expression for the temperature at a point P a distance x from one end at any subsequent time t seconds after $t = 0$.



We have the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$ with the boundary conditions

............; ....; and ....

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$$\boxed{u(0, t) = 0; \quad u(2, t) = 0; \quad u(x, 0) = 10}$$

Assuming a solution of the form $u = XT$, we know that this gives for this equation $\quad X = A\cos px + B\sin px$

and $\quad T = Ce^{-p^2c^2t}$

so that the general solution is

$$u(x,t) = \{P\cos px + Q\sin px\}e^{-p^2c^2t}$$

If we now write $pc = \lambda$, $\;p = \dfrac{\lambda}{c}$ and the solution becomes

$$u(x,t) = \left\{P\cos\frac{\lambda}{c}x + Q\sin\frac{\lambda}{c}x\right\}e^{-\lambda^2 t}$$

Applying the first two of the boundary conditions gives us

....

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$$\boxed{P = 0 \quad \text{and} \quad u(x,t) = \left\{Q \sin \frac{n\pi x}{2}\right\} e^{-\lambda^2 t}}$$

Because

$u(0, t) = 0 \quad \therefore \; 0 = Pe^{-\lambda^2 t} \quad \therefore \; P = 0$

$$\therefore \; u(x, t) = \left\{Q \sin \frac{\lambda}{c} x\right\} e^{-\lambda^2 t}$$

Also $u(2, t) = 0 \quad \therefore \; 0 = \left\{Q \sin \frac{2\lambda}{c}\right\} e^{-\lambda^2 t}$

$Q \neq 0 \quad \therefore \; \sin \frac{2\lambda}{c} = 0 \quad \therefore \; \frac{2\lambda}{c} = n\pi \quad \therefore \; \lambda = \frac{nc\pi}{2} \quad n = 1, 2, 3, \ldots$

$$\therefore \; u(x, t) = \left\{Q \sin \frac{n\pi x}{2}\right\} e^{-\lambda^2 t}$$

There is, of course, an infinite number of such solutions with different values of n. We can write the solution so far therefore as

$$u(x, t) = \ldots\ldots\ldots\ldots$$

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$$\boxed{u(x,t) = \sum_{r=1}^{\infty} Q_r \sin \frac{r\pi x}{2} e^{-\lambda_r^2 t}}$$

Finally, there is the remaining initial condition that at $t = 0$, $u = 10$.

$$\therefore \; u(x, 0) = f(x) = 10 \quad \therefore \; 10 = \sum_{r=1}^{\infty} Q_r \sin \frac{r\pi x}{2}$$

where $Q_r = 2 \times$ mean value of $10 \sin \frac{r\pi x}{2}$ from $x = 0$ to $x = 2$.

$$\therefore \; Q_r = \ldots\ldots\ldots\ldots$$

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$$\boxed{0 \; (r \text{ even}); \quad \frac{40}{\pi r} \; (r \text{ odd})}$$

Because

$$Q_r = \frac{2}{2}\int_0^2 10 \sin \frac{r\pi x}{2} \, dx = 10 \int_0^2 \sin \frac{r\pi x}{2} \, dx$$
$$= -\frac{20}{\pi r}\left[\cos \frac{r\pi x}{2}\right]_0^2 = \frac{20}{\pi r}\{1 - \cos r\pi\}$$
$$= 0 \; (r \text{ even}) \text{ and } \frac{40}{r\pi} \; (r \text{ odd})$$

Therefore the required solution is

$$u(x, t) = \ldots\ldots\ldots\ldots$$

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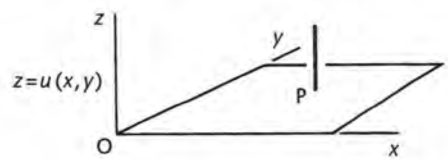
$$u(x,\,t) = \frac{40}{\pi}\sum_{r\,(\text{odd})=1}^{\infty}\frac{1}{r}\sin\frac{r\pi x}{2}e^{-\lambda_r^2 t} \qquad r = 1, 3, 5, \ldots$$

$$\text{where } \lambda_r = \frac{rc\pi}{2}$$

By now you will appreciate that the approach to all these problems is very much the same, as indeed it still is with the next important equation.

Laplace's equation

The Laplace equation concerns the distribution of a field, e.g. temperature, potential, etc., over a plane area subject to certain boundary conditions.

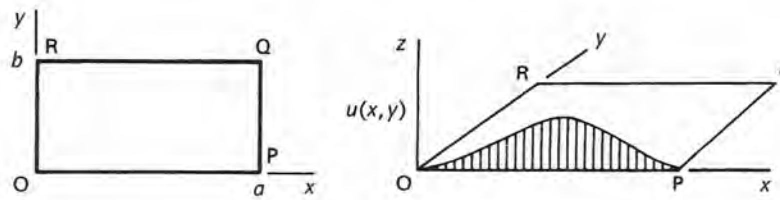


The potential at a point P in a plane can be indicated by an ordinate axis and is a function of its position, i.e. $z = u(x,\,y)$ where $u(x,\,y)$ is the solution of the Laplace two-dimensional equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Let us consider the situation in the next frame

Solution of the Laplace equation

42



We are required to determine a solution of the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the rectangle bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$, subject to the following boundary conditions

$u = 0$ when $x = 0$ $0 \le y \le b$

$u = 0$ when $x = a$ $0 \le y \le b$

$u = 0$ when $y = b$ $0 \le x \le a$

$u = f(x)$ when $y = 0$ $0 \le x \le a$

i.e. $u(0, y) = 0$ and $u(a, y) = 0$ for $0 \le y \le b$
and $u(x, b) = 0$ and $u(x, 0) = f(x)$ for $0 \le x \le a$.

The solution $z = u(x, y)$ will give the potential at any point within the rectangle OPQR.

We start off, as usual, by assuming a solution of the form $u(x, y) = X(x)Y(y)$ where X is a function of x only and Y is a function of y only. We now express the equation in terms of X and Y and separate the variables to give

....

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$$\boxed{\frac{X''}{X} = -\frac{Y''}{Y}}$$

Because

$$u = XY \quad \therefore \quad \frac{\partial u}{\partial x} = X'Y \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''Y$$
$$\frac{\partial u}{\partial y} = XY' \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

The equation is then $X''Y = -XY''$ $\quad \therefore \quad \frac{X''}{X} = -\frac{Y''}{Y}$

Putting each side equal to a constant $(-p^2)$ gives two equations

$$X'' + p^2 X = 0 \quad \text{and} \quad Y'' - p^2 Y = 0$$

$X'' + p^2 X = 0$ has a solution $X =$ ....

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$$\boxed{X = A\cos px + B\sin px}$$

In the introduction to this Programme we said that the equation $Y'' - p^2Y = 0$ has a solution of the form $Y = C\cosh py + D\sinh py$ which can also be expressed as $Y = E\sinh p(y + \phi)$.

$$\therefore\ u(x, y) = \{A\cos px + B\sin px\}E\sinh p(y + \phi)$$
$$\therefore\ u(x, y) = \{P\cos px + Q\sin px\}\sinh p(y + \phi)$$

Now we apply the first of the boundary conditions.

$$u(0, y) = 0 \quad \therefore\ 0 = P\sinh p(y + \phi) \quad \therefore\ P = 0$$
$$\therefore\ u(x, y) = Q\sin px\sinh p(y + \phi)$$

From the second boundary condition, we have

$$u(a, y) = 0 \quad \therefore\ 0 = Q\sin pa\sinh p(y + \phi) \quad \therefore\ \sin pa = 0$$
$$\therefore\ pa = n\pi \qquad \text{for } n = 1, 2, 3, \dots$$

If we write $\lambda = p$ then $\lambda = \dfrac{n\pi}{a}$ and $u(x, y) = Q\sin\lambda x\sinh\lambda(y + \phi)$

Now from the third condition

$u(x, b) = 0$ from which we have ....

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$$\boxed{u(x, y) = Q\sin\lambda x\sinh\lambda(b - y)}$$

Because

$$0 = Q\sin\lambda x\sinh\lambda(b + \phi) \quad \therefore\ \sinh\lambda(b + \phi) = 0 \quad \therefore\ \phi = -b.$$
$$\therefore\ u(x, y) = Q\sin\lambda x\sinh\lambda(y - b)$$
$$\sinh\lambda(y - b) = -\sinh\lambda(b - y) \quad \therefore\ u(x, y) = Q\sin\lambda x\sinh\lambda(b - y),$$

the minus sign being absorbed in the symbol Q whose value has yet to be found. Now $\lambda = \dfrac{n\pi}{a}$ with $n = 1, 2, 3, \dots$ and there is therefore an infinite number of values for λ and hence an infinite number of solutions for $u(x, y)$. Therefore, again using $u = u_1 + u_2 + u_3 + \dots$ we have $u(x, y) = \dots\dots\dots\dots$

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$$\boxed{u(x, y) = \sum_{r=1}^{\infty} Q_r \sin \lambda_r x \sinh \lambda_r (b - y)}$$

Now there remains the fourth boundary condition to be applied.

$$u(x, 0) = f(x) \quad \therefore \; f(x) = \sum_{r=1}^{\infty} Q_r \sin \lambda_r x \; \sinh \; \lambda_r b$$

$$\therefore \; Q_r \sinh \lambda_r b = 2 \; \times \; \text{mean value of } f(x) \sin \lambda_r x \text{ from } x = 0 \text{ to } x = a$$

$$= \frac{2}{a} \int_0^a f(x) \sin \lambda_r x \, \mathrm{d}x$$

$$= \frac{2}{a} \int_0^a f(x) \sin \frac{r\pi x}{a} \, \mathrm{d}x$$

from which the coefficients Q_r can be found.

Let us work through an example with numerical values.

Example

Determine a solution $u(x, y)$ of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the following boundary conditions

$u = 0$ when $x = 0$; $\quad u = 0$ when $x = \pi$
$u \to 0$ when $y \to \infty$; $\quad u = 3$ when $y = 0$

As always, we begin with $u(x, y) = X(x)Y(y)$, rewrite the equation in terms of X and Y and separate the variables. The equation then becomes

. . . .

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$$\boxed{\frac{X''}{X} = -\frac{Y''}{Y}}$$

Equating each side to $-p^2$, we have $X'' + p^2 X = 0$ and $Y'' - p^2 Y = 0$.

$X'' + p^2 X = 0$ has a solution . . . .

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$$\boxed{X = A \cos px + B \sin px}$$

The solution of $Y'' - p^2 Y = 0$ can be stated in three different forms

$Y = C \cosh py + D \sinh py$; $\quad Y = Ce^{py} + De^{-py}$; $\quad Y = C \sinh p(y + \phi)$

On this occasion, we will use the second one

$$Y = Ce^{py} + De^{-py}$$

Then $u(x, y) = \{A \cos px + B \sin px\}\{Ce^{py} + De^{-py}\}$

Application of the first boundary condition $u(0, y) = 0$ gives

. . . . and . . . .

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$$\boxed{A = 0 \text{ and } u(x, y) = \sin px\{Pe^{py} + Qe^{-py}\}}$$

Because

$0 = A\{Ce^{py} + De^{-py}\} \qquad \therefore\ A = 0$

and $u(x, y) = B\sin px\{Ce^{py} + De^{-py}\} = \sin px\{Pe^{py} + Qe^{-py}\}$.

The second boundary condition $u(\pi, y) = 0$ then gives

....

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$$\boxed{u(x, y) = \sin nx\ \{Pe^{ny} + Qe^{-ny}\} \qquad n = 1, 2, 3, \ldots}$$

Because

$u = 0$ when $x = \pi\ \ \therefore\ \ 0 = \sin p\pi\{Pe^{py} + Qe^{-py}\}$

$\therefore\ \sin p\pi = 0\ \ \therefore\ \ p\pi = n\pi\ \ \therefore\ \ p = n \qquad n = 1, 2, 3, \ldots$

$\therefore\ u(x, y) = \sin\ nx\ \{Pe^{ny} + Qe^{-ny}\}$

The third condition is that $u \to 0$ as $y \to \infty$.

Because $e^{-ny} \to 0$ as $y \to \infty$ then $0 = \sin nx\{Pe^{ny}\}$, so that $P = 0$

$\therefore\ u(x, y) = Qe^{-ny}\sin nx$

But n can have an infinite number of values giving an infinite number of solutions

$$\begin{aligned}
u_1 &= Q_1e^{-y}\sin x\\
u_2 &= Q_2e^{-2y}\sin 2x\\
u_3 &= Q_3e^{-3y}\sin 3x\\
&\ \vdots\\
u_r &= Q_re^{-ry}\sin rx
\end{aligned}$$

So the solution at this stage can be written as

$u(x, y) = \ldots\ldots\ldots\ldots$

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$$\boxed{u(x, y) = \sum_{r=1}^{\infty} Q_re^{-ry}\sin rx}$$

Now we turn to the final boundary condition that $u = 3$ when $y = 0$.

$\therefore\ 3 = \sum_{r=1}^{\infty} Q_r\sin rx$ from which we obtain

$Q_r = \ldots\ldots\ldots\ldots$

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$$\boxed{Q_r = 0 \ (r \text{ even}); \quad Q_r = \frac{12}{r\pi}(r \text{ odd})}$$

Because

$$\begin{aligned} Q_r &= 2 \times \text{ mean value of } 3\sin rx \text{ between } x = 0 \text{ and } x = \pi \\ &= \frac{2}{\pi}\int_0^{\pi} 3\sin rx\,\mathrm{d}x = \frac{6}{\pi}\left[-\frac{\cos rx}{r}\right]_0^{\pi} = \frac{6}{r\pi}(1 - \cos r\pi) \end{aligned}$$

$$\therefore\ Q_r = 0 \ \ (r \text{ even) and } \frac{12}{r\pi} \ (r \text{ odd})$$

$$\therefore\ u(x, y) = \sum_{r\,(\text{odd})=1}^{\infty} \frac{12}{r\pi} e^{-ry} \sin rx \qquad r = 1, 3, 5, \ldots$$

$$\therefore\ u(x, y) = \frac{12}{\pi}\left\{e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \ldots\right\}$$

Laplace's equation in plane polar coordinates

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Laplace's equation

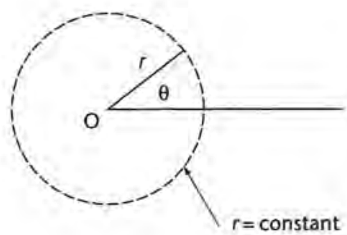
$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

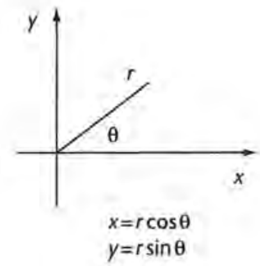
is often referred to as the *potential equation* because such physical entities as the electrostatic and gravitational potentials can be shown to satisfy it. It is an equation that is commonly met in science and engineering. Solving this equation inside a region of the x–y plane subject to some specified condition applied to $u(x, y)$ on the boundary of the region is known as a *Dirichlet problem*. To solve this Dirichlet problem we proceed, as we have seen, by separating the variables to find the general solution and then matching up the general solution to the boundary conditions to find the specific solution. However, the process of finding the specific solution from the general solution is very sensitive to the shape of the boundary, and difficulties can arise if the symmetries of the boundary do not match the symmetries of the coordinate system used. For example, if the region under consideration is bounded by the circle

$$x^2 + y^2 = a^2$$

employing Cartesian coordinates will create difficulties when we come to match up the general solution in Cartesians to the boundary conditions on the circular boundary. To avoid such difficulties we choose a coordinate system that has the same symmetries as the ▶

boundary where the coordinate symmetries are exhibited when we let one variable vary while keeping all the others constant. The Cartesian coordinate system (x, y) produces straight lines $x =$ constant as y varies and $y =$ constant as x varies. The plane polar coordinate system (r, θ), on the other hand, produces circles $r =$ constant when θ varies and so is suitable for dealing with circular boundaries in the plane.





Before we attempt to find the solution we must pose the problem *from the beginning* in terms of the coordinates that are appropriate to the boundary conditions. This means, of course, that Laplace's equation must also be given in the same coordinates. To convert Laplace's equation from its current form in Cartesians (x, y) to a new form in plane polar coordinates (r, θ) where

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

requires manipulations using Frame 11 onwards of Programme 10. We shall not go into this here, suffice it to say that in plane polar coordinates Laplace's equation is

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

where $v(r, \theta)$ is the expression obtained by changing the coordinates in $u(x, y)$ using $x = r\cos\theta$ and $y = r\sin\theta$.

We shall now pose the problem anew in the next frame

54 The problem

Find the solution to

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = a^2$ (that is, for $0 \le r \le a$) of the plane where

1 $v(r, \theta)$ is finite for $0 \le r \le a$ and for all θ

2 $v(a, \theta) = f(\theta)$ – the condition on the boundary of the circular region

3 θ is unbounded but $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \le r \le a$. That is, though θ can take any finite value, the value of $v(r, \theta)$ repeats itself as θ winds round every 2π.

▶

Separating the variables

The variables are r and θ and we assume they are separable and write $v(r, \theta) = R(r)\Theta(\theta)$. This form is then substituted into Laplace's equation and the entire equation multiplied by $\dfrac{r^2}{R(r)\Theta(\theta)}$ to obtain

$$\dots\dots\dots\dots = 0$$

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$$\boxed{\frac{r^2}{R(r)}\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{r}{R(r)}\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \frac{1}{\Theta(\theta)}\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} = 0}$$

Because

Substituting $R(r)\Theta(\theta)$ for $v(r, \theta)$ gives

$$\frac{\partial^2 R(r)\Theta(\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)\Theta(\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 R(r)\Theta(\theta)}{\partial \theta^2} = 0$$

That is

$$\Theta(\theta)\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{\Theta(\theta)}{r}\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \frac{R(r)}{r^2}\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} = 0$$

Multiplying the entire equation by $\dfrac{r^2}{R(r)\Theta(\theta)}$ then gives

$$\frac{r^2}{R(r)}\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{r}{R(r)}\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \frac{1}{\Theta(\theta)}\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} = 0$$

From this result we can say that

$$\frac{r^2}{R(r)}\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{r}{R(r)}\frac{\mathrm{d}R(r)}{\mathrm{d}r} = -\frac{1}{\Theta(\theta)}\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} = k$$

which gives rise to the two uncoupled, second-order ordinary differential equations

$$\frac{r^2}{R(r)}\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{r}{R(r)}\frac{\mathrm{d}R(r)}{\mathrm{d}r} = k \quad \text{so that}$$

$$r^2\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + r\frac{\mathrm{d}R(r)}{\mathrm{d}r} = kR(r) \qquad (1)$$

and

$$\frac{1}{\Theta(\theta)}\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} = -k \quad \text{so that} \quad \frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} = -k\Theta(\theta) \qquad (2)$$

The general solution to equation (2) for $k > 0$ is

$$\dots\dots\dots\dots$$

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$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \text{ where } n = 1, 2, \ldots$$

Because

To solve $\dfrac{d^2\Theta(\theta)}{d\theta^2} = -k\Theta(\theta)$, that is $\dfrac{d^2\Theta(\theta)}{d\theta^2} + k\Theta(\theta) = 0$ we use the auxiliary equation $m^2 + k = 0$ with solutions $m = \pm j\sqrt{k}$. This gives the solution, periodic with period 2π as

$$\Theta(\theta) = A\cos\sqrt{k}\theta + B\sin\sqrt{k}\theta \qquad (3)$$

provided $k > 0$ so that m is pure imaginary. If $k < 0$ then non-periodic solutions would result which would be physically incorrect. To ensure periodicity, that is to ensure that $k > 0$ write $k = n^2$, $n = 1, 2, \ldots$.

$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$ is a solution to equation (2).

We shall look at the case $n = 0$ later.

Substituting $k = n^2$ into equation (1) then gives

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} = n^2R(r) \qquad (4)$$

As a trial solution to equation (4) let $R(r) = pr^q$. Substitution into (4) gives

$$q = \ldots\ldots\ldots\ldots$$

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$$q = \pm n \text{ where } n = 1, 2, \ldots$$

Because

$r\dfrac{dR(r)}{dr} = r\dfrac{d(pr^q)}{dr} = rpqr^{q-1} = pqr^q$. Similarly $r^2\dfrac{d^2R(r)}{dr^2} = pq(q-1)r^q$.

Therefore, substitution into $r^2\dfrac{d^2R(r)}{dr^2} + r\dfrac{dR(r)}{dr} = n^2R(r)$ gives

$[q(q-1) + q]pr^q = n^2pr^q$ and so $[q^2 - n^2]pr^q = 0$ giving $q = \pm n$ where $n = 1, 2, \ldots$

Therefore, a solution to equation (4) is

$R_n(r) = c_nr^n + d_nr^{-n}$ provided $n \neq 0$. The case $n = 0$ is special.

Summary

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To summarise the results so far, we have started to solve Laplace's equation

$$\frac{\partial^2 v(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial v(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2}=0$$

in the circular region $x^2+y^2=a^2$ (that is, for $0\le r\le a$) of the plane where

1 $v(r,\theta)$ is finite for $0\le r\le a$ and for all θ
2 $v(a,\theta)=f(\theta)$
3 θ is unbounded but $v(r,\theta+2\pi)=v(r,\theta)$ for $0\le r\le a$.

We have found that, assuming $v(r,\theta)=R(r)\Theta(\theta)$ then, provided $n\neq 0$

$$\Theta_n(\theta)=a_n\cos n\theta+b_n\sin n\theta$$
$$R_n(r)=c_nr^n+d_nr^{-n}$$

So that

$$v_n(r,\theta)=R_n(r)\Theta_n(\theta)=(c_nr^n+d_nr^{-n})(a_n\cos n\theta+b_n\sin n\theta)$$

If we now apply the boundary condition **1** we find that

$d_n=$ . . . .

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$$\boxed{d_n=0}$$

Because

$v(r,\theta)$ is finite for $0\le r\le a$. In particular, the solution is finite when $r=0$ and so we cannot have a term of the form r^{-n}. Accordingly $d_n=0$, so omitting the r^{-n} term the solution then becomes

$$v_n(r,\theta)=c_nr^n(a_n\cos n\theta+b_n\sin n\theta)$$

There is an infinite number of such solutions (eigenfunctions), one for each eigenvalue n. The complete solution to Laplace's equation is then a linear combination of all these eigenfunctions. That is

$$v(r,\theta)=\sum_{n=1}^{\infty}e_nv_n(r,\theta)=\sum_{n=1}^{\infty}r^n(A_n\cos n\theta+B_n\sin n\theta)$$

And now for the $n=0$ case

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The $n = 0$ case

When $n = 0$ then $k = 0$ and equation (1) becomes

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} = 0$$

and if we let $S(r) = \frac{dR(r)}{dr}$ then this equation becomes

$$r^2\frac{dS(r)}{dr} + rS(r) = 0, \text{ that is } r\left[r\frac{dS(r)}{dr} + S(r)\right] = 0 \text{ and so}$$

$$r\frac{dS(r)}{dr} + S(r) = \frac{d[rS(r)]}{dr} = 0$$

This has the solution

$$rS(r) = \alpha \text{ (constant) and so } S(r) = \frac{dR(r)}{dr} = \frac{\alpha}{r}$$

$$\text{giving } R(r) = \alpha \ln r + \beta \qquad (5)$$

When $n = 0$ then $k = 0$ and equation (2) becomes

$$\frac{d^2\Theta(\theta)}{d\theta^2} = 0 \text{ with solution } \Theta(\theta) = \gamma\theta + \delta \qquad (6)$$

Applying the boundary conditions to the solutions (5) and (6) gives

$\alpha = \ldots\ldots\ldots\ldots$ and $\gamma = \ldots\ldots\ldots\ldots$

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$$\boxed{\alpha = 0 \text{ and } \gamma = 0}$$

Because

(a) $v(r, \theta)$ is finite for $0 \le r \le a$, in particular when $r = 0$, and so $\alpha = 0$

(b) $v(r, \theta + 2\pi) = v(r, \theta)$. That is, though θ can take any finite value, the value of $v(r, \theta)$ repeats itself as θ winds round every 2π and this means that $\gamma = 0$.

So, when $n = 0$ the solution is $v_0(r, \theta) =$ constant. We therefore write the complete solution as

$$v(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

where the constant is taken to be in the form $\frac{A_0}{2}$.

Applying the condition on the boundary where $v(a, \theta) = f(\theta)$ we see that

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n(A_n \cos n\theta + B_n \sin n\theta)$$

which is a Fourier series and hence the form of the constant term being taken as $\frac{A_0}{2}$.

The Fourier coefficients are then

$$A_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \quad\text{and}\quad B_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

Example

Solve Laplace's equation

$$\frac{\partial^2 v(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial\theta^2} = 0$$

in the circular region $x^2 + y^2 = a^2$ of the plane where

1 $v(r,\theta)$ is finite for $0 \le r \le a$ and for all θ
2 $v(a,\theta) = \sin\theta$
3 $v(r,\theta + 2\pi) = v(r,\theta)$ for $0 \le r \le a$.

The solution, as we have seen, is

$$v(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta) \quad\text{where}$$

$A_n = \dots\dots\dots\dots$ and $B_n = \dots\dots\dots\dots$

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$$\boxed{A_n = 0 \quad\text{and}\quad B_n = \frac{1}{2a^n}\delta_{1,n}}$$

Because

$$A_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{2\pi a^n}\int_0^{2\pi}\sin\theta\cos n\theta\,d\theta = 0 \quad\text{and}$$

$$B_n = \frac{1}{2\pi a^n}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta = \frac{1}{2\pi a^n}\int_0^{2\pi}\sin\theta\sin n\theta\,d\theta = \frac{1}{2\pi a^n}\pi\delta_{1,n}$$

where $\delta_{1,n}$ is the Kronecker delta

That is, $B_1 = \frac{1}{2a}$, $B_n = 0$ for $n = 2, 3, \dots$. The complete solution is then

$$v(r,\theta) = \frac{r}{a}\sin\theta$$

Notice that all three conditions in Frame 61 are satisfied by this solution, that is

1 $v(r,\theta) = \frac{r}{a}\sin\theta$ is finite for $0 \le r \le a$ and for all θ

2 $v(a,\theta) = \frac{a}{a}\sin\theta = \sin\theta$

3 $v(r,\theta + 2\pi) = \frac{r}{a}\sin(\theta + 2\pi) = \frac{r}{a}\sin\theta = v(r,\theta)$ for $0 \le r \le a$.

That covers the main steps in the method of solving linear, second-order partial differential equations applied specifically to the wave equation, the heat conduction equation and Laplace's equation. The same approach can be made with other similar equations. ▶

The **Revision summary** and the **Can You?** checklist now follow, then the **Test exercise** with problems like those we have considered. Although the solutions take rather more steps than with other forms of equations, the method is straightforward and follows a clear pattern. The **Further problems** give additional practice.

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Revision summary 11

1 *Ordinary second-order linear differential equations*

(a) Equation of the form $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$

Auxiliary equation $am^2+bm+c=0$

(1) Real and different roots: $m=m_1$ and $m=m_2$

$y=Ae^{m_1x}+Be^{m_2x}$

(2) Real and equal roots: $m=m_1$ (twice)

$y=e^{m_1x}(A+Bx)$

(3) Complex roots: $m=\alpha\pm j\beta$

$y=e^{\alpha x}\{A\cos\beta x+B\sin\beta x\}$.

(b) Equations of the form $\dfrac{d^2y}{dx^2}\pm n^2y=0$

(1) $\dfrac{d^2y}{dx^2}+n^2y=0$; $\quad y=A\cos nx+B\sin nx$

(2) $\dfrac{d^2y}{dx^2}-n^2y=0$; $\quad y=A\cosh nx+B\sinh nx$

or $\quad y=Ae^{nx}+Be^{-nx}$

or $\quad y=A\sinh n(x+\phi)$.

2 *Partial differential equations* Solution $u=f(x, y, t, \dots)$

Linear equations: If $u=u_1$, $u=u_2$, $u=u_3$, ... are solutions, so also is $u=u_1+u_2+u_3+\dots+u_r+\dots=\sum_{r=1}^{\infty}u_r$.

(a) *Wave equation* – transverse vibrations of an elastic string

$$\frac{\partial^2u}{\partial x^2}=\frac{1}{c^2}\cdot\frac{\partial^2u}{\partial t^2}\quad\text{where } c^2=\frac{T}{\rho},$$

T = tension of string
ρ = mass per unit length.

(b) *Heat conduction equation* – heat flow in uniform finite bar

$$\frac{\partial^2u}{\partial x^2}=\frac{1}{c^2}\cdot\frac{\partial u}{\partial t}\qquad\text{where } c^2=\frac{k}{\sigma\rho}$$

k = thermal conductivity of material
σ = specific heat of the material
ρ = mass per unit length of bar.

▶

(c) *Laplace equation* – distribution of a field over a plane area

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

3 *Separating the variables*
Let $u(x, y)=X(x)Y(y)$ where $X(x)$ is a function of x only
and $Y(y)$ is a function of y only.

Then $\dfrac{\partial u}{\partial x}=X'Y$; $\quad \dfrac{\partial^2 u}{\partial x^2}=X''Y$

$\dfrac{\partial u}{\partial y}=XY'$; $\quad \dfrac{\partial^2 u}{\partial y^2}=XY''$

Substitute in the given partial differential equation and form separate differential equations to give $X(x)$ and $Y(y)$ by introducing a common constant $(-p^2)$. Determine arbitrary functions by use of the initial and boundary conditions.

4 *Laplace's equation in plane polar coordinates*

$$\frac{\partial^2 v(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial v(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2}=0$$

Separating the variables by $v(r,\theta)=R(r)\Theta(\theta)$ produces two uncoupled, second-order ordinary differential equations

$$r^2\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+r\frac{\mathrm{d}R(r)}{\mathrm{d}r}=kR(r)$$

and $$\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}=-k\Theta(\theta)$$

These two ordinary differential equations can then be solved under the application of appropriate boundary conditions.

☑ Can You?

Checklist 11

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

• Summarise the introductory methods of solving ordinary differential equations? 1

Yes ☐ ☐ ☐ ☐ ☐ *No*

• Solve partial differential equations that are amenable to solution by direct integration? 2 to 7

Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Apply initial and boundary conditions? 5 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve the one-dimensional wave and heat equations by separating the variables and obtaining eigenfunctions and corresponding eigenvalues? 8 to 40

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve the two-dimensional Laplace equation in Cartesian coordinates? 41 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the need for alternative coordinate systems and solve the two-dimensional Laplace equation in plane polar coordinates? 53 to 62

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 11

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1 Solve the following equations

(a) $\dfrac{\partial^2 u}{\partial x^2} = 24x^2(t-2)$, given that at $x = 0$, $u = e^{2t}$ and $\dfrac{\partial u}{\partial x} = 4t$.

(b) $\dfrac{\partial^2 u}{\partial x \partial y} = 4e^y \cos 2x$, given that at $y = 0$, $\dfrac{\partial u}{\partial x} = \cos x$ and at $x = \pi$, $u = y^2$.

2 A perfectly elastic string is stretched between two points 10 cm apart. Its centre point is displaced 2 cm from its position of rest at right angles to the original direction of the string and then released with zero velocity. Applying the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ with $c^2 = 1$, determine the subsequent motion $u(x, t)$.

3 One end A of an insulated metal bar AB of length 2 m is kept at 0°C while the other end B is maintained at 50°C until a steady state of temperature along the bar is achieved. At $t = 0$, the end B is suddenly reduced to 0°C and kept at that temperature. Using the heat conduction equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial u}{\partial t}$, determine an expression for the temperature at any point in the bar distance x from A at any time t.

▶

4 A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$. Apply the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ to determine the potential distribution $u(x, y)$ over the plate, subject to the following boundary conditions.

$$\begin{aligned} u &= 0 \quad \text{when } x = 0 \quad 0 \le y \le 2 \\ u &= 0 \quad \text{when } x = 2 \quad 0 \le y \le 2 \\ u &= 0 \quad \text{when } y = 0 \quad 0 \le x \le 2 \\ u &= 5 \quad \text{when } y = 2 \quad 0 \le x \le 2. \end{aligned}$$

5 Solve Laplace's equation in plane polar coordinates

$$\frac{\partial^2 v(r, \theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r, \theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r, \theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

(1) $v(r, \theta)$ is finite for $0 \le r \le 1$ and for all θ
(2) $v(1, \theta) = 5\cos 3\theta$
(3) $v(r, \theta + 2\pi) = v(r, \theta)$ for $0 \le r \le 1$.

Further problems 11

66

1 Show that the equation $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2} = 0$ is satisfied by $u = f(x + ct) + F(x - ct)$ where f and F are arbitrary functions.

2 If $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$ and $c = 3$, determine the solution $u = f(x, t)$ subject to the boundary conditions

$$u(0, t) = 0 \text{ and } u(2, t) = 0 \text{ for } t \ge 0$$

$$u(x, 0) = x(2 - x) \text{ and } \left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 \qquad 0 \le x \le 2.$$

3 The centre point of a perfectly elastic string stretched between two points A and B, 4 m apart, is deflected a distance 0·01 m from its position of rest perpendicular to AB and released initially with zero velocity. Apply the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$ where $c = 10$ to determine the subsequent motion of a point P distant x from A at time t.

4 An elastic string is stretched between two points 10 cm apart. A point P on the string 2 cm from the left-hand end, i.e. the origin, is drawn aside 1 cm from its position of rest and released with zero velocity. Solve the one-dimensional wave equation to determine the displacement of any point at any instant.

5 An insulated uniform metal bar, 10 units long, has the temperature of its ends maintained at 0°C and at $t = 0$ the temperature distribution $f(x)$ along the bar is defined by $f(x) = x(10 - x)$. Solve the heat conduction equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t}$ with $c^2 = 4$ to determine the temperature u of any point in the bar at time t.

6 The ends of an insulated rod AB, 10 units long, are maintained at 0°C. At $t = 0$, the temperature within the rod rises uniformly from each end reaching 2°C at the mid-point of AB. Determine an expression for the temperature $u(x, t)$ at any point in the rod, distant x from the left-hand end at any subsequent time t.

7 A rectangular plate OPQR is bounded by the lines $x = 0$, $y = 0$, $x = 4$, $y = 2$. Determine the potential distribution $u(x, y)$ over the rectangle using the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the following boundary conditions

$$
\begin{array}{ll}
u(0, y) = 0 & 0 \le y \le 2 \\
u(4, y) = 0 & 0 \le y \le 2 \\
u(x, 2) = 0 & 0 \le x \le 4 \\
u(x, 0) = x(4 - x) & 0 \le x \le 4.
\end{array}
$$

8 Two sides AB and AD of a rectangular plate ABCD lie along the x and y axes respectively. The remaining two sides are the lines $x = 5$ and $y = 2$. The sides BC, CD and DA are maintained at zero temperature. The temperature distribution along AB is defined by $f(x) = x(x - 5)$. Determine an expression for the steady-state temperature at any point in the plate.

9 Solve Laplace's equation in plane polar coordinates

$$\frac{\partial^2 v(r,\,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,\,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r,\,\theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

(1) $v(r,\,\theta)$ is finite for $0 \le r \le 1$ and for all θ
(2) $v(1,\,\theta) = \sin 2\theta - 4\cos\theta$
(3) $v(r,\,\theta + 2\pi) = v(r,\,\theta)$ for $0 \le r \le 1$.

10 Solve Laplace's equation in plane polar coordinates

$$\frac{\partial^2 v(r,\,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,\,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r,\,\theta)}{\partial \theta^2} = 0$$

in the circular region $x^2 + y^2 = 1$ of the plane where

(1) $v(r,\,\theta)$ is finite for $0 \le r \le 1$ and for all θ
(2) $v(1,\,\theta) = 3\sin^2\theta$
(3) $v(r,\,\theta + 2\pi) = v(r,\,\theta)$ for $0 \le r \le 1$.

Programme 12

Matrix algebra

Frames 1 to 112

Learning outcomes

When you have completed this Programme you will be able to:

- Determine whether a matrix is singular or non-singular
- Determine the rank of a matrix
- Determine the consistency of a set of linear equations and hence demonstrate the uniqueness of their solution
- Obtain the solution of a set of simultaneous linear equations by using matrix inversion, by row transformation, by Gaussian elimination and by triangular decomposition
- Obtain the eigenvalues and corresponding eigenvectors of a square matrix
- Demonstrate the validity of the Cayley–Hamilton theorem
- Solve systems of first-order ordinary differential equations using eigenvalue and eigenvector methods
- Construct the modal matrix from the eigenvectors of a matrix and the spectral matrix from the eigenvalues
- Solve systems of second-order ordinary differential equations using diagonalisation
- Use matrices to represent transformations between coordinate systems

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 4 Determinants and 5 Matrices

Singular and non-singular matrices

1

Every square matrix $\mathbf{A}$ has associated with it a number called the determinant of $\mathbf{A}$ and denoted by $|\mathbf{A}|$. If $|\mathbf{A}| \neq 0$ then $\mathbf{A}$ is called a *non-singular* matrix. Otherwise if $|\mathbf{A}| = 0$, then $\mathbf{A}$ is called a *singular* matrix.

Example 1

Is $\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{pmatrix}$ singular or non-singular?

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{vmatrix}$$
$$= 1\begin{vmatrix} 7 & 6 \\ 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 9 & 3 \end{vmatrix} + 8\begin{vmatrix} 4 & 7 \\ 9 & 5 \end{vmatrix}$$
$$= (21 - 30) - 2(12 - 54) + 8(20 - 63)$$
$$= -9 + 84 - 344$$
$$= -269$$

Because $|\mathbf{A}| \neq 0$ then $\mathbf{A}$ is non-singular.

Example 2

Is $\mathbf{A} = \begin{pmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix}$ singular or non-singular?

A is ....

2

singular

Because

$$|\mathbf{A}| = \begin{vmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{vmatrix}$$
$$= 3(20 - 42) - 9(4 - 12) + 2(7 - 10)$$
$$= -66 + 72 - 6$$
$$= 0$$

Because $|\mathbf{A}| = 0$ then $|\mathbf{A}|$ is singular.

▶

Exercise

Determine whether each of the following is singular or non-singular.

1 $|\mathbf{A}| = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ **2** $|\mathbf{B}| = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$

3 $|\mathbf{C}| = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 7 & 3 \\ 5 & 8 & 1 \end{pmatrix}$ **4** $|\mathbf{D}| = \begin{pmatrix} 3 & 2 & 4 \\ 5 & 1 & 6 \\ 2 & 0 & 3 \end{pmatrix}$

3

1 non-singular	**2** singular
3 singular	**4** non-singular

Because

Straightforward evaluation of the relevant determinants gives

1 $|\mathbf{A}| = 2$ **2** $|\mathbf{B}| = 0$

3 $|\mathbf{C}| = 0$ **4** $|\mathbf{D}| = -5$

Closely related to the notion of the singularity or otherwise of a square matrix is the notion of **rank** of a general $n \times m$ matrix.

Rank of a matrix

The rank of an $n \times m$ matrix $\mathbf{A}$ is the order of the largest square, non-singular sub-matrix. That is, the largest square sub-matrix whose determinant is non-zero. If $n = m$, so making $\mathbf{A}$ itself square, then this sub-matrix could be the matrix $\mathbf{A}$ itself.

Example

To find the rank of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ we note that

$$|\mathbf{A}| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \dots\dots\dots\dots$$

4

$$\boxed{0}$$

Because

$$|\mathbf{A}| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 3(12 - 15) - 4(6 - 12) + 5(5 - 8)$$
$$= -9 + 24 - 15 = 0$$

Therefore we can say that the rank of **A** is ....

5

$$\boxed{\text{not } 3}$$

Because

$|\mathbf{A}| = 0$ and therefore **A** is singular.

Now try a sub-matrix of order 2.

$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = -2 \neq 0$. Therefore the rank of **A** is ....

6

$$\boxed{2}$$

Because

The largest square, non-singular sub-matrix of **A** has order 2 therefore **A** has rank 2.

This method of finding the rank of a matrix can be a very hit and miss affair and a better, more systematic method is to use **elementary operations** and the notion of an **equivalent matrix**.

Next frame

Elementary operations and equivalent matrices

7

Each of the following row operations on matrix **A** produces a *row equivalent matrix* **B**, where the order and rank of **B** is the same as that of **A**. We write $\mathbf{A} \sim \mathbf{B}$.

1 Interchanging two rows
2 Multiplying each element of a row by the same non-zero scalar quantity
3 Adding or subtracting corresponding elements from those of another row

are operations called *elementary row operations*. There is a corresponding set of three *elementary column operations* that can be used to form *column equivalent matrices*.

▶

Example 1

Given $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & -4 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$ by subtracting 3 times each element of row 2 from row 1

$$\sim \begin{pmatrix} 0 & -2 & -4 \\ 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$$ by subtracting 4 times each element of row 2 from row 3

$$\sim \begin{pmatrix} 0 & -3 & -6 \\ 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$$ by multiplying each element of row 1 by 3/2

$$\sim \begin{pmatrix} 0 & -3 & -6 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$ by subtracting corresponding elements of row 1 from row 3

$$= \mathbf{B}$$

The row of zeros in matrix **B** means that its determinant is zero and so its rank is not 3. The largest sub-matrix with non-zero determinant has order 2 and so the rank of **B** is 2. Because matrix **B** is row equivalent to matrix **A** we can say that the rank of **A** is also 2.

Example 2

Determine the rank of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{pmatrix}$

By taking 4 times the elements of row 1 from row 2 we obtain the equivalent matrix ....

8

$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & -1 & -26 \\ 9 & 5 & 3 \end{pmatrix}}$$

By taking 9 times the elements of row 1 from row 3 we obtain the equivalent matrix ....

9

$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & -1 & -26 \\ 0 & -13 & -69 \end{pmatrix}}$$

By multiplying the elements of row 2 by -13 we obtain the equivalent matrix ....

10

$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & 13 & 338 \\ 0 & -13 & -69 \end{pmatrix}}$$

By adding corresponding elements of row 2 to row 3 we obtain the equivalent matrix …………

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$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & 13 & 269 \\ 0 & 0 & -69 \end{pmatrix}}$$

Because all the elements below the main diagonal of this matrix are zero we call the matrix an *upper triangular matrix*. By inspection we can see that the determinant of this triangular matrix is non-zero, being the product of its three diagonal elements $1 \times 13 \times (-69) = -897$. Therefore its rank is 3 and so the rank of matrix $\mathbf{A}$ is also 3.

Try another one for yourself.

Example 3

The rank of $\mathbf{A} = \begin{pmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix}$ is …………

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$$\boxed{2}$$

Because

$$\mathbf{A} = \begin{pmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & -6 & -16 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix} \quad \text{Subtracting 3 times row 2 from row 1}$$

$$\sim \begin{pmatrix} 0 & -6 & -16 \\ 1 & 5 & 6 \\ 0 & -3 & -8 \end{pmatrix} \quad \text{Subtracting 2 times row 2 from row 3}$$

$$\sim \begin{pmatrix} 0 & 3 & 8 \\ 1 & 5 & 6 \\ 0 & -3 & -8 \end{pmatrix} \quad \text{Multiplying row 1 by } -1/2$$

$$\sim \begin{pmatrix} 0 & 3 & 8 \\ 1 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Adding row 1 to row 3}$$

$$\sim \begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 8 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Interchanging rows 1 and 2}$$

▶

and $\begin{vmatrix} 1 & 5 & 6 \\ 0 & 3 & 8 \\ 0 & 0 & 0 \end{vmatrix} = 0$. So the rank of this matrix is not 3. The largest square sub-matrix of this matrix with non-zero determinant is, by inspection, of order 2 and so the rank of this matrix, and hence the rank of the equivalent matrix $\mathbf{A}$ is 2.

Finally try a non-square matrix.

Example 4

The rank of $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 & 1 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix}$ is ....

13

$\boxed{3}$

Because

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 & 1 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -12 & -3 & -3 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix} \quad \text{Subtracting 2 times row 3 from row 1}$$

$$\sim \begin{pmatrix} 0 & -8 & -2 & -2 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix} \quad \text{Multiplying row 1 by 2/3}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix} \quad \text{Adding row 2 to row 1}$$

It is possible to find a 3×3 sub-matrix of this matrix that has non-zero determinant, namely

$$\begin{pmatrix} 0 & 0 & 2 \\ 8 & 2 & 4 \\ 7 & 3 & 2 \end{pmatrix} \text{ where } \begin{vmatrix} 0 & 0 & 2 \\ 8 & 2 & 4 \\ 7 & 3 & 2 \end{vmatrix} = 2(24 - 14) = 20.$$

Consequently, this matrix and hence matrix $\mathbf{A}$ has rank 3.

Consistency of a set of equations

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In solving sets of simultaneous equations, we can express the equations in matrix form. For example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

i.e. $\mathbf{A}\mathbf{x} = \mathbf{b}$

The set of three equations is said to be *consistent* if solutions for x_1, x_2, x_3 exist and *inconsistent* if no such solutions can be found.

In practice, we can solve the equations by operating on the *augmented coefficient matrix*, i.e. we write the constant terms as a fourth column of the coefficient matrix to form $\mathbf{A}_b$.

$$\mathbf{A}_b = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$$

which, of course, is a (3×4) matrix.

The general test for consistency is then:

A set of n simultaneous equations in n unknowns is consistent if the rank of the coefficient matrix $\mathbf{A}$ is equal to the rank of the augmented matrix $\mathbf{A}_b$.

If the rank of $\mathbf{A}$ is less than the rank of $\mathbf{A}_b$, then the equations are inconsistent and have no solution.

Make a note of this test. It can save time in working

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Example

If $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ then

$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ and $\mathbf{A}_b = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 5 \end{pmatrix}$

Rank of $\mathbf{A}$: $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0 \quad \therefore$ rank of $\mathbf{A} = 1$

Rank of $\mathbf{A}_b$: $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$ as before

but $\begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix} = 15 - 24 = -9 \quad \therefore$ rank of $\mathbf{A}_b = 2$

In this case, rank of $\mathbf{A} <$ rank of $\mathbf{A}_b$

$\therefore$ ....

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$\boxed{\text{no solution exists}}$

Remember that, for consistency,

rank of $\mathbf{A}$ = ...

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$\boxed{\text{rank of } \mathbf{A}_b}$

Uniqueness of solutions

1 With a set of n equations in n unknowns, the equations are consistent if the coefficient matrix $\mathbf{A}$ and the augmented matrix $\mathbf{A}_b$ are each of rank n. There is then a *unique* solution for the n equations.

Note that if the rank of $\mathbf{A} = n$ then $\mathbf{A}$ is a non-singular sub-matrix of $\mathbf{A}_b$ and so the rank of $\mathbf{A}_b = n$ also. Therefore there is no need to test for the rank of $\mathbf{A}_b$ in this case.

2 If the rank of $\mathbf{A}$ and that of $\mathbf{A}_b$ is m, where $m < n$, then the matrix $\mathbf{A}$ is singular, i.e. $|\mathbf{A}| = \mathbf{0}$, and there will be an *infinite number* of solutions for the equations.

3 As we have already seen, if the rank of $\mathbf{A} <$ the rank of $\mathbf{A}_b$, then *no solution* exists.

Copy these up in your record book; they are important

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Writing the results in a slightly different way:

With a set of n equations in n unknowns, checking the rank of the coefficient matrix $\mathbf{A}$ and that of the augmented matrix $\mathbf{A}_b$ enables us to see whether

(a) a unique solution exists

$$\text{rank } \mathbf{A} = \text{rank } \mathbf{A}_b = n$$

(b) an infinite number of solutions exist

$$\text{rank } \mathbf{A} = \text{rank } \mathbf{A}_b = m < n$$

(c) no solution exists

$$\text{rank } \mathbf{A} < \text{rank } \mathbf{A}_b$$

Example

$$\begin{pmatrix} -4 & 5 \\ -8 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

Finding the rank of $\mathbf{A}$ and of $\mathbf{A}_b$ leads us to the conclusion that

....

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there is an infinite number of solutions

Because

$$\mathbf{A} = \begin{pmatrix} -4 & 5 \\ -8 & 10 \end{pmatrix} \text{ and } \mathbf{A}_b = \begin{pmatrix} -4 & 5 & -3 \\ -8 & 10 & -6 \end{pmatrix}$$

Rank of $\mathbf{A}$: $\begin{vmatrix} -4 & 5 \\ -8 & 10 \end{vmatrix} = -40 + 40 = 0 \quad \therefore$ Rank of $\mathbf{A} = 1$

Rank of $\mathbf{A}_b$: $\begin{vmatrix} -4 & 5 \\ -8 & 10 \end{vmatrix} = 0; \begin{vmatrix} 5 & -3 \\ 10 & -6 \end{vmatrix} = 0; \begin{vmatrix} -4 & -3 \\ -8 & -6 \end{vmatrix} = 0$

$\therefore$ Rank of $\mathbf{A}_b = 1$

$\therefore$ Rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 1$

But there are two equations in two unknowns, i.e. $n = 2$

$\therefore$ Rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 1 < n$

$\therefore$ Infinite number of solutions.

The solutions can be written as x_1 arbitrary and $x_2 = \dfrac{4x_1 - 3}{5}$.

You will recall that, for a unique solution of n equations in n unknowns

....

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rank $\mathbf{A}$ = rank $\mathbf{A}_b = n$

Now for some examples for you to try. In each of the following cases, apply the rank tests to determine the nature of the solutions. Do not solve the sets of equations.

Example 1

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_b = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 4 & 2 & -2 \\ 1 & 4 & 3 & 3 \end{pmatrix}$$

Finish it off and we find that ....

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$$\boxed{\text{a unique solution exists}}$$

Because

$n = 3$; rank of $\mathbf{A} = 3$; rank of $\mathbf{A}_b = 3$.

$\therefore$ rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 3 = n$ $\therefore$ Solution unique

And this one.

Example 2

$$\begin{pmatrix} 2 & -1 & 7 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

This time we find that ....

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$$\boxed{\text{no solution is possible}}$$

Because

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 7 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix}; \quad \mathbf{A}_b = \begin{pmatrix} 2 & -1 & 7 & 2 \\ 4 & 2 & 2 & 5 \\ 3 & 1 & 3 & 1 \end{pmatrix}$$

$n = 3$; rank of $\mathbf{A} = 2$; rank of $\mathbf{A}_b = 3$

$\therefore$ rank of $\mathbf{A}$ < rank of $\mathbf{A}_b$

$\therefore$ No solution exists

and finally

Example 3

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

In this case, we find that ....

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infinite number of solutions possible

Because

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix} \text{ and } \mathbf{A}_b = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 5 & 1 & 3 \end{pmatrix}$$

Rank of **A**:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 2 & 5 & 1 \end{pmatrix} \quad \text{Subtracting row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 0 & 1 & 7 \end{pmatrix} \quad \text{Subtracting 2 times row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Subtracting row 2 from row 3}$$

and so rank of **A** is 2 by inspection.

Rank of $\mathbf{A}_b$:

$$\mathbf{A}_b = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 5 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 7 & 1 \\ 2 & 5 & 1 & 3 \end{pmatrix} \quad \text{Subtracting row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 1 & 7 & 1 \end{pmatrix} \quad \text{Subtracting 2 times row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Subtracting row 2 from row 3}$$

and so rank of $\mathbf{A}_b$ is 2 by inspection.

Therefore rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 2 < n$ (that is 3), therefore there is an infinite number of solutions.

Now let us move on to a new section of the work

Solution of sets of equations

1 *Inverse method*

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Let us work through an example by way of explanation.

Example 1

To solve
$$3x_1 + 2x_2 - x_3 = 4$$
$$2x_1 - x_2 + 2x_3 = 10$$
$$x_1 - 3x_2 - 4x_3 = 5.$$

We first write this in matrix form, which is ....

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$$\boxed{\begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}}$$

Then if $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}$ then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$

where $\mathbf{A}^{-1}$ is the *inverse* of $\mathbf{A}$.

To find $\mathbf{A}^{-1}$

(a) Form the determinant of $\mathbf{A}$ and evaluate it.

$$|\mathbf{A}| = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{vmatrix} = 3(4+6) - 2(-8-2) - 1(-6+1) = 55$$

(b) Form a new matrix $\mathbf{C}$ consisting of the cofactors of the elements in $\mathbf{A}$.

The cofactor of any one element is its minor together with its 'place sign'

i.e. $\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

where A_{11} is the cofactor of a_{11} in $\mathbf{A}$.

$$A_{11} = \begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = 10; \quad A_{12} = -\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} = 10;$$

$$A_{13} = \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -5$$

$$A_{21} = -\begin{vmatrix} 2 & -1 \\ -3 & -4 \end{vmatrix} = 11; \quad A_{22} = \begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = -11;$$

$$A_{23} = -\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = 11$$

$A_{31} =$ ....; $\quad A_{32} =$ ....; $\quad A_{33} =$ ....

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$$A_{31} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3; \quad A_{32} = -\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} = -8; \quad A_{33} = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7$$

So $\mathbf{C} = \begin{pmatrix} 10 & 10 & -5 \\ 11 & -11 & 11 \\ 3 & -8 & -7 \end{pmatrix}$

We now write the transpose of $\mathbf{C}$, i.e. $\mathbf{C}^{\mathrm{T}}$ in which we write rows as columns and columns as rows.

$$\mathbf{C}^{\mathrm{T}} = \ldots\ldots\ldots\ldots$$

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$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

This is called the *adjoint* (adj) of the original matrix $\mathbf{A}$

i.e. $\text{adj}\,\mathbf{A} = \mathbf{C}^{\mathrm{T}}$

Then the inverse of $\mathbf{A}$, i.e. $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

As a check that all the calculations have been done correctly and without error, the product of matrix $\mathbf{A}$ with its adjoint should be equal to the unit matrix multiplied by the determinant of $\mathbf{A}$. That is

$$\mathbf{A} \times \text{adj}\,\mathbf{A} = \det \mathbf{A} \times \mathbf{I}$$

For this case

$$\begin{aligned} \mathbf{A} \times \text{adj}\,\mathbf{A} &= \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 55 & 0 & 0 \\ 0 & 55 & 0 \\ 0 & 0 & 55 \end{pmatrix} \\ &= \det \mathbf{A} \times \mathbf{I} \end{aligned}$$

Thus all is well. We can now continue to find the solution.

So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$ becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix} = \ldots\ldots\ldots\ldots$$

$$\boxed{x_1 = 3; \quad x_2 = -2; \quad x_3 = 1}$$

Because

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$$

$$= \frac{1}{55}\begin{pmatrix} 40 & +110 & +15 \\ 40 & -110 & -40 \\ -20 & +110 & -35 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore\ x_1 = 3; \quad x_2 = -2; \quad x_3 = 1$

The method is the same every time.

To solve $\mathbf{Ax} = \mathbf{b} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

To find $\mathbf{A}^{-1}$

(1) Evaluate $|\mathbf{A}|$

If $|\mathbf{A}| \neq 0$ then proceed to (2)

If $|\mathbf{A}| = 0$ then there is no inverse and hence no unique solution. Later we shall discover how to determine whether there is an infinity of solutions or none.

(2) Form $\mathbf{C}$, the matrix of cofactors of $\mathbf{A}$

(3) Write $\mathbf{C}^{\mathrm{T}}$, the transpose of $\mathbf{C}$

(4) Then $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}}$.

Now apply the method to Example 2.

Example 2

$$4x_1 + 5x_2 + x_3 = 2$$
$$x_1 - 2x_2 - 3x_3 = 7$$
$$3x_1 - x_2 - 2x_3 = 1.$$

$x_1 = \ldots\ldots\ldots\ldots; \qquad x_2 = \ldots\ldots\ldots\ldots; \qquad x_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = -2; \quad x_2 = 3; \quad x_3 = -5}$$

Here is the complete working.

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 1 \\ 1 & -2 & -3 \\ 3 & -1 & -2 \end{pmatrix} \therefore |\mathbf{A}| = \begin{vmatrix} 4 & 5 & 1 \\ 1 & -2 & -3 \\ 3 & -1 & -2 \end{vmatrix} = -26$$

$$\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} -2 & -3 \\ -1 & -2 \end{vmatrix} = 1 \qquad A_{12} = -\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} = -7 \quad A_{13} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$$

$$A_{21} = -\begin{vmatrix} 5 & 1 \\ -1 & -2 \end{vmatrix} = 9 \qquad A_{22} = \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -11 \quad A_{23} = -\begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} = 19$$

$$A_{31} = \begin{vmatrix} 5 & 1 \\ -2 & -3 \end{vmatrix} = -13 \quad A_{32} = -\begin{vmatrix} 4 & 1 \\ 1 & -3 \end{vmatrix} = 13 \quad A_{33} = \begin{vmatrix} 4 & 5 \\ 1 & -2 \end{vmatrix} = -13$$

$$\therefore \mathbf{C} = \begin{pmatrix} 1 & -7 & 5 \\ 9 & -11 & 19 \\ -13 & 13 & -13 \end{pmatrix} \quad \therefore \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}} = -\frac{1}{26}\begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = -\frac{1}{26}\begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{26}\begin{pmatrix} 2 & +63 & -13 \\ -14 & -77 & +13 \\ 10 & +133 & -13 \end{pmatrix}$$

$$= -\frac{1}{26}\begin{pmatrix} 52 \\ -78 \\ 130 \end{pmatrix} = -\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$\therefore\ x_1 = -2; \quad x_2 = 3; \quad x_3 = -5$

With a set of four equations with four unknowns, the method becomes somewhat tedious as there are then sixteen cofactors to be evaluated and each one is a third-order determinant! There are, however, other methods that can be applied – so let us see method 2.

2 Row transformation method

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Elementary row transformations that can be applied are as follows

(a) Interchange any two rows.
(b) Multiply (or divide) every element in a row by a non-zero scalar (constant) k.
(c) Add to (or subtract from) all the elements of any row k times the corresponding elements of any other row.

Equivalent matrices
Two matrices, **A** and **B**, are said to be equivalent if **B** can be obtained from **A** by a sequence of elementary transformations.

Solutions of equations
The method is best described by working through a typical example.

Example 1

Solve
$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 &= 1 \\ 3x_1 - 2x_2 - 4x_3 &= -4. \end{aligned}$$

This can be written $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

and for convenience we introduce the unit matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$$

where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ may be regarded as the coefficient of $\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

We then form the combined coefficient matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{pmatrix}$$

and work on this matrix from now on.

On then to the next frame

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The rest of the working is mainly concerned with applying row transformations to convert the left-hand half of the matrix to a unit matrix and the right-hand side to the inverse, eventually obtaining

$$\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$$

with $a, b, c, \dots g, h, i$ being evaluated in the process.

The following notation will be helpful to denote the transformation used:

$(1) \sim (2)$ denotes 'interchange rows 1 and 2'
$(3) - 2(1)$ denotes 'subtract twice row 1 from row 3', etc.

So off we go.

$$(1) \sim (2) \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} (2) - 2(1) \\ (3) - 3(1) \\ \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -11 & -10 & 0 & -3 & 1 \end{pmatrix}$$

$$(3) - 2(2) \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -1 & -4 & -2 & 1 & 1 \end{pmatrix}$$

$$-(2) \sim -(3) \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 5 & 3 & -1 & 2 & 0 \end{pmatrix}$$

$$(3) - 5(2) \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & -17 & -11 & 7 & 5 \end{pmatrix}$$

$$\begin{matrix} (1) - 3(2) \\ (3) \div (-17) \\ \end{matrix} \begin{pmatrix} 1 & 0 & -10 & -6 & 4 & 3 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{pmatrix}$$

$$\begin{matrix} (1) + 10(3) \\ (2) - 4(3) \\ \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 8/17 & -2/17 & 1/17 \\ 0 & 1 & 0 & -10/17 & 11/17 & 3/17 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{pmatrix}$$

We now have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ 11 & -7 & -5 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$$

$\therefore \ x_1 = \dots\dots\dots\dots; \quad x_2 = \dots\dots\dots\dots; \quad x_3 = \dots\dots\dots\dots$

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$$\boxed{x_1 = 2; \quad x_2 = -3; \quad x_3 = 4}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 40 & -2 & -4 \\ -50 & +11 & -12 \\ 55 & -7 & +20 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 34 \\ -51 \\ 68 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$x_1 = 2; \quad x_2 = -3; \quad x_3 = 4$

Of course, there is no set pattern of how to carry out the row transformations. It depends on one's ingenuity and every case is different. Here is a further example.

Example 2

$$\begin{aligned} 2x_1 - x_2 - 3x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - 2x_2 - 5x_3 &= 2. \end{aligned}$$

First write the set of equations in matrix form – with the unit matrix included. This gives ....

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$$\boxed{\begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}$$

The combined coefficient matrix is now ....

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$$\boxed{\begin{pmatrix} 2 & -1 & -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{pmatrix}}$$

If we start off by interchanging the top two rows, we obtain a 1 at the beginning of the top row which is a help.

$$(1) \sim (2) \quad \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & -3 & 1 & 0 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{pmatrix}$$

Now, if we subtract $2\times$ row 1 from row 2
and $2\times$ row 1 from row 3, we get

....

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$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -5 & -5 & 1 & -2 & 0 \\ 0 & -6 & -7 & 0 & -2 & 1 \end{pmatrix}$$

Continuing with the same line of reasoning, we then have

$$(2)-(3) \quad \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & -6 & -7 & 0 & -2 & 1 \end{pmatrix}$$

$$(3)+6(2) \quad \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 5 & 6 & -2 & -5 \end{pmatrix}$$

$$\begin{matrix}(1)-2(2) \\ (3)\div 5 \\ \end{matrix} \quad \begin{pmatrix} 1 & 0 & -3 & -2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{2}{5} & -1 \end{pmatrix}$$

Notice the three diagonal 1s appearing at the left-hand end

What do you suggest we should do now?

....

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Add three times row 3 to row 1
and subtract twice row 3 from row 2

Right. That gives

$$\begin{matrix}(1)+3(3) \\ (2)-3(3) \\ \end{matrix} \begin{pmatrix} 1 & 0 & 0 & \frac{8}{5} & -\frac{1}{5} & -1 \\ 0 & 1 & 0 & -\frac{7}{5} & \frac{4}{5} & 1 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{2}{5} & -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 & -1 & -5 \\ -7 & 4 & 5 \\ 6 & -2 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Now you can finish it off.

$x_1 =$ ....; $x_2 =$ ....; $x_3 =$ ....

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$x_1 = -1$; $x_2 = 3$; $x_3 = -2$

Because

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 - 3 - 10 \\ -7 + 12 + 10 \\ 6 - 6 - 10 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -5 \\ 15 \\ -10 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

Let us now look at a somewhat similar method with rather fewer steps involved.

So move on

3 *Gaussian elimination method*

Once again we will demonstrate the method by a typical example.

Example 1

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 9 \\ 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 - 4x_2 + 2x_3 &= 6. \end{aligned}$$

We start off as usual

$$\begin{pmatrix} 2 & -3 & 2 \\ 3 & 2 & -1 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 6 \end{pmatrix}$$

We then form the *augmented coefficient matrix* by including the constants as an extra column on the right-hand side of the matrix

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 2 & 6 \end{array}\right)$$

Now we operate on the rows to convert the first three columns into an upper triangular matrix

$$(1) \sim (3) \begin{pmatrix} 1 & -4 & 2 & 6 \\ 3 & 2 & -1 & 4 \\ 2 & -3 & 2 & 9 \end{pmatrix} \qquad (2) \sim (3) \begin{pmatrix} 1 & -4 & 2 & 6 \\ 2 & -3 & 2 & 9 \\ 3 & 2 & -1 & 4 \end{pmatrix}$$

$$\begin{array}{r} (2) - 2(1) \\ (3) - 3(1) \\ \end{array} \begin{pmatrix} 1 & -4 & 2 & 6 \\ 0 & 5 & -2 & -3 \\ 0 & 14 & -7 & -14 \end{pmatrix} \qquad \begin{array}{r} (2) \div 5 \\ (3) \div 7 \\ \end{array} \begin{pmatrix} 1 & -4 & 2 & 6 \\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & 2 & -1 & -2 \end{pmatrix}$$

$$(3) - 2(2) \begin{pmatrix} 1 & -4 & 2 & 6 \\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix} \qquad (3) \times (-5) \begin{pmatrix} 1 & -4 & 2 & 6 \\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

The first three columns now form an upper triangular matrix which has been our purpose. If we now detach the fourth column back to its original position on the right-hand side of the matrix equation, we have

....

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$$\begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -\frac{3}{5} \\ 4 \end{pmatrix}$$

Expanding from the bottom row, working upwards

$$\begin{aligned} x_3 &= 4 & & \therefore\ x_3 = 4 \\ x_2 - \tfrac{2}{5}x_3 &= -\tfrac{3}{5} & \therefore\ x_2 = -\tfrac{3}{5} + \tfrac{8}{5} = 1 & \quad \therefore\ x_2 = 1 \\ x_1 - 4x_2 + 2x_3 &= 6 & \therefore\ x_1 - 4 + 8 = 6 & \quad \therefore\ x_1 = 2 \end{aligned}$$

$\therefore\ x_1 = 2;\quad x_2 = 1;\quad x_3 = 4$

It is a very useful method and entails fewer tedious steps, and can be used to solve efficiently higher-order sets of equations and non-square systems. It can also solve a sequence of problems with the same coefficient matrix $\mathbf{A}$ by using the augmented matrix $(\mathbf{A}\mathbf{b}_1\mathbf{b}_2 \ldots \mathbf{b}_n)$.

Example 2

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 &= -1 \\ 2x_1 - 2x_2 + x_3 - 2x_4 &= 1 \\ x_1 + x_2 - 3x_3 + x_4 &= 6 \\ 3x_1 - x_2 + 2x_3 - x_4 &= 3. \end{aligned}$$

First we write this in matrix form and compile the augmented matrix which is

....

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$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 2 & -2 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 & 6 \\ 3 & -1 & 2 & -1 & 3 \end{array}\right)$$

Next we operate on rows to convert the left-hand side to an upper triangular matrix. There is no set way of doing this. Use any trickery to save yourself unnecessary work.

So now you can go ahead and complete the transformations and obtain

$x_1 = \ldots\ldots\ldots\ldots;\quad x_2 = \ldots\ldots\ldots\ldots$

$x_3 = \ldots\ldots\ldots\ldots;\quad x_4 = \ldots\ldots\ldots\ldots$

$$\boxed{x_1 = 2; \quad x_2 = -3; \quad x_3 = -1; \quad x_4 = 4}$$

Here is one way. You may well have taken quite a different route.

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 2 & -2 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 & 6 \\ 3 & -1 & 2 & -1 & 3 \end{array}\right)$$

$$\begin{array}{r} \\ (2)-2(1) \\ (3)-(1) \\ (4)-[(1)+(2)] \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -8 & 5 & -4 & 3 \\ 0 & -2 & -1 & 0 & 7 \\ 0 & -2 & 3 & 0 & 3 \end{array}\right)$$

$$\begin{array}{r} \\ (2)-4(4) \\ (3)-(4) \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 0 & -7 & -4 & -9 \\ 0 & 0 & -4 & 0 & 4 \\ 0 & -2 & 3 & 0 & 3 \end{array}\right)$$

$$\begin{array}{r} \\ (2)\sim(4) \\ (3)\div 4 \\ \\ \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -2 & 3 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -7 & -4 & -9 \end{array}\right)$$

$$\begin{array}{r} \\ \\ \\ (4)-7(3) \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -2 & 3 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -4 & -16 \end{array}\right)$$

Returning the right-hand column to its original position

$$\begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -16 \end{pmatrix}$$

Expanding from the bottom row, we have

$-4x_4 = -16 \qquad \therefore\ x_4 = 4$

$-x_3 = 1 \qquad \therefore\ x_3 = -1$

$-2x_2 + 3x_3 = 3 \quad \therefore\ -2x_2 = 6 \qquad \therefore\ x_2 = -3$

$x_1 + 3x_2 - 2x_3 + x_4 = -1 \quad \therefore\ x_1 - 9 + 2 + 4 = -1 \qquad \therefore\ x_1 = 2$

$$\therefore\ x_1 = 2; \quad x_2 = -3; \quad x_3 = -1; \quad x_4 = 4$$

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We still have a further method for solving sets of simultaneous equations.

4 Triangular decomposition method

A square matrix **A** can usually be written as a product of a lower-triangular matrix **L** and an upper-triangular matrix **U**, where $\mathbf{A} = \mathbf{LU}$.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{pmatrix}$, **A** can be expressed as

$$\mathbf{A} = \mathbf{LU} = \underset{(\mathbf{L})}{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}} \underset{(\mathbf{U})}{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}$$

$$= \begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{pmatrix}$$

Note that, in **L** and **U**, elements occur in the major diagonal in each case. These are related in the product and whatever values we choose to put for u_{11}, u_{22}, $u_{33}\ldots$ then the corresponding values of l_{11}, l_{22}, $l_{33}\ldots$ will be determined – and vice versa.

For convenience, we put $u_{11} = u_{22} = u_{33} \ldots = 1$

Then $\mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$

In our example, $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{pmatrix}$

$\therefore\ l_{11} = 1;$ $\quad l_{11}u_{12} = 2 \quad \therefore\ u_{12} = 2;$ $\quad l_{11}u_{13} = 3 \quad \therefore\ u_{13} = 3$

$l_{21} = 3;$ $\quad$ Similarly $l_{22} = \ldots\ldots\ldots\ldots;$ $\quad u_{23} = \ldots\ldots\ldots\ldots$

$l_{31} = 4;$ $\quad l_{32} = \ldots\ldots\ldots\ldots;$ $\quad l_{33} = \ldots\ldots\ldots\ldots$

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$$\boxed{l_{22} = -1; \quad u_{23} = 1; \quad l_{32} = 1; \quad l_{33} = -3}$$

Because

$l_{21}u_{12} + l_{22}u_{22} = 5$ that is $3 \times 2 + l_{22} \times 1 = 5$ and so $l_{22} = -1$

$l_{21}u_{13} + l_{22}u_{23} = 8$ that is $3 \times 3 + (-1) \times u_{23} = 8$ and so $u_{23} = 1$

$l_{31}u_{12} + l_{32}u_{22} = 9$ that is $4 \times 2 + l_{32} \times 1 = 9$ and so $l_{32} = 1$

$l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} = 10$ that is $4 \times 3 + 1 \times 1 + l_{33} \times 1 = 10$ and so $l_{33} = -3$

Now we substitute all these values back into the upper and lower triangular matrices and obtain

$$\mathbf{A} = \mathbf{LU} = \ldots\ldots\ldots\ldots$$

$$\boxed{\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

We have thus expressed the given matrix **A** as the product of lower and upper triangular matrices. Let us now see how we use them.

Example 1

$$\begin{aligned} x_1 + 2x_2 + \ \ 3x_3 &= 16 \\ 3x_1 + 5x_2 + \ \ 8x_3 &= 43 \\ 4x_1 + 9x_2 + 10x_3 &= 57. \end{aligned}$$

i.e. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 43 \\ 57 \end{pmatrix}$ i.e. $\mathbf{Ax} = \mathbf{b}$.

We have seen above that **A** can be written as **LU** where

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

To solve $\mathbf{Ax} = \mathbf{b}$, we have $\mathbf{LUx} = \mathbf{b}$ i.e. $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$

Putting $\mathbf{Ux} = \mathbf{y}$, we solve $\mathbf{Ly} = \mathbf{b}$ to obtain $\mathbf{y}$

and then $\mathbf{Ux} = \mathbf{y}$ to obtain $\mathbf{x}$.

(a) Solving $\mathbf{Ly} = \mathbf{b}$ $\begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 43 \\ 57 \end{pmatrix}$

Expanding from the top $y_1 = 16$; $3y_1 - y_2 = 43$ $\therefore$ $y_2 = 5$; and $4y_1 + y_2 - 3y_3 = 57$ $\therefore$ $64 + 5 - 3y_3 = 57$ $\therefore$ $y_3 = 4$

$$\therefore \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 4 \end{pmatrix}$$

(b) Solving $\mathbf{Ux} = \mathbf{y}$ $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 4 \end{pmatrix}$

Expanding from the bottom, we then have

$x_1 = \dots\dots\dots\dots$; $x_2 = \dots\dots\dots\dots$; $x_3 = \dots\dots\dots\dots$

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$$\boxed{x_1 = 2; \quad x_2 = 1; \quad x_3 = 4}$$

Note:

1. If $l_{ii} = 0$, then either decomposition is not possible, or, if $\mathbf{A}$ is singular, i.e. $|\mathbf{A}| = 0$, there is an infinite number of possible decompositions.
2. Instead of putting $u_{11} = u_{22} = u_{33} \ldots = 1$, we could have used the alternative substitution $l_{11} = l_{22} = l_{33} \ldots = 1$ and obtained values of $u_{11}, u_{22}, u_{33} \ldots$ etc. The working is as before.
3. One advantage of employing **LU** decomposition over Gaussian elimination is in the solution of a sequence of problems in which the same coefficient matrix occurs.

Now for another example.

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Example 2

$$x_1 + 3x_2 + 2x_3 = 19$$
$$2x_1 + x_2 + x_3 = 13$$
$$4x_1 + 2x_2 + 3x_3 = 31.$$

$$\therefore \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 13 \\ 31 \end{pmatrix} \quad \text{i.e. } \mathbf{Ax} = \mathbf{b}$$

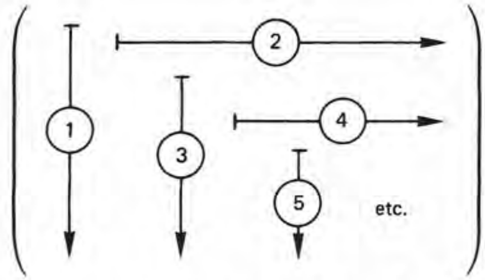
$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{pmatrix}$$

Now we have to find the values of the various elements. The usual order of doing this is shown by the diagram.

▶



That is, first we can write down values for l_{11}, l_{21}, l_{31} from the left-hand column; then follow this by finding u_{12}, u_{13} from the top row; and proceed for the others.

So, completing the two triangular matrices, we have

$$\mathbf{A} = \mathbf{LU} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 4 & -10 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}}$$

As we stated before: $\mathbf{Ax} = \mathbf{b}$; $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. Put $\mathbf{Ux} = \mathbf{y}$

then (a) solve $\mathbf{Ly} = \mathbf{b}$ to obtain $\mathbf{y}$

and (b) solve $\mathbf{Ux} = \mathbf{y}$ to obtain $\mathbf{x}$.

Solving $\mathbf{Ly} = \mathbf{b}$ gives $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \end{pmatrix}$

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$$\boxed{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 5 \\ 5 \end{pmatrix}}$$

Because

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 4 & -10 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 13 \\ 31 \end{pmatrix}$$

Expanding from the top gives

$y_1 = 19$; $y_2 = 5$; $y_3 = 5$.

(b) Now solve $\mathbf{Ux} = \mathbf{y}$ from which $x_1 = \ldots\ldots\ldots\ldots$; $x_2 = \ldots\ldots\ldots\ldots$; $x_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = 3; \quad x_2 = 2; \quad x_3 = 5}$$

Because we have

$$\mathbf{Ux} = y$$

i.e. $$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 5 \\ 5 \end{pmatrix}$$

Expanding from the bottom $x_3 = 5; \; x_2 + \dfrac{3}{5}x_3 = 5 \quad \therefore \; x_2 = 2$

and $x_1 + 3x_2 + 2x_3 = 19 \quad \therefore \; x_1 + 6 + 10 = 19 \quad \therefore \; x_1 = 3$

$$\therefore \; x_1 = 3; \quad x_2 = 2; \quad x_3 = 5$$

We can of course apply the same method to a set of four equations.

Example 3

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 9 \\ 2x_1 - x_2 + 3x_3 + 2x_4 &= 23 \\ 3x_1 + 3x_2 + x_3 + x_4 &= 5 \\ 4x_1 + 5x_2 - 2x_3 + 2x_4 &= -2. \end{aligned}$$

i.e. $$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 2 \\ 3 & 3 & 1 & 1 \\ 4 & 5 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 23 \\ 5 \\ -2 \end{pmatrix} \quad \text{i.e. } \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 2 \\ 3 & 3 & 1 & 1 \\ 4 & 5 & -2 & 2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & l_{11}u_{14} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} & l_{21}u_{14} + l_{22}u_{24} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} & l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} \\ l_{41} & l_{41}u_{12} + l_{42} & l_{41}u_{13} + l_{42}u_{23} + l_{43} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44} \end{pmatrix}$$

Now we have to find the values of the individual elements. It is easy enough if we follow the order indicated in the diagram earlier. So the two triangular matrices are

$$\mathbf{A} = \mathbf{LU} = (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

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$$\boxed{\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -5 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 4 & -3 & -1 & -\frac{66}{5} \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & \frac{4}{5} \\ 0 & 0 & 1 & -\frac{28}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

As usual $\mathbf{Ax} = \mathbf{b}$; $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. Put $\mathbf{Ux} = \mathbf{y}$ $\therefore$ $\mathbf{Ly} = \mathbf{b}$

(a) Solving $\mathbf{Ly} = \mathbf{b}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -5 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 4 & -3 & -1 & -\frac{66}{5} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 23 \\ 5 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \dots \\ \dots \\ \dots \\ \dots \end{pmatrix}$$

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$$\boxed{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ -25 \\ 5 \end{pmatrix}}$$

(b) Solving $\mathbf{Ux} = \mathbf{y}$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & \frac{4}{5} \\ 0 & 0 & 1 & -\frac{28}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ -25 \\ 5 \end{pmatrix}$$

which finally gives

$$x_1 = \dots\dots\dots\dots; \quad x_2 = \dots\dots\dots\dots$$
$$x_3 = \dots\dots\dots\dots; \quad x_4 = \dots\dots\dots\dots$$

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$$x_1 = 1; \quad x_2 = -2; \quad x_3 = 3; \quad x_4 = 5$$

Comparison of methods

Inverse method

This is an elementary method but it is very inefficient when the number of equations to solve increases beyond three.

Row transformation method

An efficient method but each case is different and relies on ingenuity to see the way forward.

Gaussian elimination method

The most efficient method and should be used in most cases. It must be used when there is a singular or non-square system.

Triangular decomposition method

An alternative to Gaussian elimination in some cases.

Now let us proceed to something rather different, so move on to the next frame for a new start

Eigenvalues and eigenvectors

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Matrices commonly appear in technological problems, for example those involving coupled oscillations and vibrations, and give rise to equations of the form

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where $\mathbf{A} = (a_{ij})$ is a square matrix, $\mathbf{x}$ is a column matrix (x_i) and λ is a scalar quantity, i.e. a number.

For non-trivial solutions, i.e. for $\mathbf{x} \neq \mathbf{0}$, the values of λ are called the *eigenvalues*, *characteristic values*, or *latent roots* of the matrix $\mathbf{A}$ and the corresponding solutions of the given equations $\mathbf{Ax} = \lambda\mathbf{x}$ are called the *eigenvectors*, or *characteristic vectors* of $\mathbf{A}$ (refer to *Engineering Mathematics (Fifth Edition)*, pages 558ff).

▶

The set of equations

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

then simplifies to

$$\begin{pmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & (a_{nn}-\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

That is, $\mathbf{Ax} = \lambda\mathbf{x}$ becomes $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$

$$\text{i.e.} \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

the unit matrix $\mathbf{I}$ being introduced since we can subtract only a matrix from another matrix.

For this set of homogeneous linear equations (right-hand side constant terms all zero) to have non-trivial solutions

$|\mathbf{A} - \lambda\mathbf{I}|$ **must be zero**

This is called the *characteristic determinant* of $\mathbf{A}$ and $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the *characteristic equation*, the solution of which gives the values of λ, i.e. the eigenvalues of $\mathbf{A}$.

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Example 1

Find the eigenvalues and corresponding eigenvectors of

$\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

The characteristic equation is $|\mathbf{A} - \lambda\mathbf{I}| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$, which, when expanded, gives

$\lambda_1 = \dots\dots\dots\dots$ and $\lambda_2 = \dots\dots\dots\dots$

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$$\boxed{\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 5}$$

Because

$(2-\lambda)(1-\lambda) - 12 = 0 \quad \therefore\ 2 - 3\lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0 \quad (\lambda - 5)(\lambda + 2) = 0 \qquad \therefore\ \lambda = -2 \text{ or } 5$

Now we substitute each value of λ in turn in the equation

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

With $\lambda = -2$

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - (-2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out the left-hand side, we get

....

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$$\boxed{4x_1 + 3x_2 = 0}$$

from which we get $x_2 = -\frac{4}{3}x_1$ i.e. not specific values for x_1 and x_2, but a relationship between them. Whatever value we assign to x_1 we obtain a corresponding value of x_2.

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ -8 \end{pmatrix} \text{ or } \begin{pmatrix} 9 \\ -12 \end{pmatrix}, \quad \text{etc.}$$

The most convenient way to do this is to choose $x_1 = 1$ and then scale $\mathbf{x}_1$ to obtain integer elements. So here we find for $x_1 = 1$ then $x_2 = -4/3$ so $\mathbf{x}_1$ is of the form

$$\begin{pmatrix} 1 \\ -\frac{4}{3} \end{pmatrix}$$

This is now scaled up by multiplying by 3 to give

$\mathbf{x}_1 = \alpha\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ where α is a constant multiplier.

The simplest result, with $\alpha = 1$, is the one normally quoted.

$$\therefore\ \text{for } \lambda_1 = -2, \quad \mathbf{x}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Similarly, for $\lambda_2 = 5$, the corresponding eigenvector is ....

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$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Because, with $\lambda_2 = 5$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ becomes

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \quad -3x_1 + 3x_2 = 0 \quad$ i.e. $x_2 = x_1$

$\therefore \quad$ with $\lambda_2 = 5$, the corresponding eigenvector is $\mathbf{x_2} = \beta\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Again, taking $\beta = 1$, for $\lambda_2 = 5$, $\quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So the required eigenvectors are

$\mathbf{x}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ corresponding to $\lambda_1 = -2$

$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponding to $\lambda_2 = 5$.

Example 2

Determine the eigenvalues and corresponding eigenvectors of

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}.$$

The characteristic equation is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which in this case can be written as ....

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$$\boxed{\begin{vmatrix} 3-\lambda & 10 \\ 2 & 4-\lambda \end{vmatrix} = 0}$$

Expanding the determinant and solving the equation gives

$$\lambda_1 = \ldots\ldots\ldots\ldots; \quad \lambda_2 = \ldots\ldots\ldots\ldots$$

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$$\boxed{\lambda_1 = -1; \quad \lambda_2 = 8}$$

Because the equation is $(3-\lambda)(4-\lambda) - 20 = 0 \quad \therefore \lambda^2 - 7\lambda - 8 = 0$

$\therefore (\lambda + 1)(\lambda - 8) = 0 \quad \therefore \lambda = -1$ or 8

(a) With $\lambda_1 = -1$, we solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ to obtain an eigenvector, which is ....

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$$\boxed{\mathbf{x}_1 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}}$$

Because

$$\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} \quad \therefore \left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 4x_1 + 10x_2 = 0 \quad \therefore x_2 = -\frac{2}{5}x_1 \quad \mathbf{x_1} = \alpha\begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\therefore \text{with } \alpha = 1 \quad \lambda_1 = -1 \quad \text{and} \quad \mathbf{x_1} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

(b) In the same way the corresponding eigenvector $\mathbf{x_2}$ for $\lambda_2 = 8$ is

....

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$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Because

$$\left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 10 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -5x_1 + 10x_2 = 0 \quad \therefore x_2 = \frac{1}{2}x_1 \quad \mathbf{x}_2 = \beta\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \text{with } \beta = 1, \quad \lambda_2 = 8 \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

▶

The same basic method can similarly be applied to third-order sets of equations.

Example 3

Determine the eigenvalues and eigenvectors of $\mathbf{Ax} = \lambda\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}.$$

As before, we have $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$$\text{i.e.} \begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{vmatrix} = 0$$

Expanding this we have

$$\lambda_1 = \ldots\ldots\ldots\ldots; \quad \lambda_2 = \ldots\ldots\ldots\ldots; \quad \lambda_3 = \ldots\ldots\ldots\ldots$$

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$$\boxed{\lambda_1 = 2; \quad \lambda_2 = 3; \quad \lambda_3 = -5}$$

Because

$$(1-\lambda)\{(2-\lambda)(-3-\lambda) - 0\} + 4\{0 - 3(2-\lambda)\} = 0$$
$$(1-\lambda)(2-\lambda)(-3-\lambda) - 12(2-\lambda) = 0$$
$$\therefore\ (2-\lambda)\{(1-\lambda)(-3-\lambda) - 12\} = 0$$
$$\therefore\ \lambda = 2 \ \text{ or } \ \lambda^2 + 2\lambda - 15 = 0 \quad \therefore\ (\lambda-3)(\lambda+5) = 0$$
$$\therefore\ \lambda = 2,\ 3,\ \text{or } -5$$

(a) With $\lambda_1 = 2$, $\quad(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ becomes

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which a corresponding eigenvector $\mathbf{x}_1$ is ....

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$$\boxed{\mathbf{x}_1 = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}}$$

Because we have $-x_1 + 4x_3 = 0 \qquad \therefore\ x_3 = \frac{1}{4}x_1$

$3x_1 + x_2 - 5x_3 = 0 \quad \therefore\ 3x_1 + x_2 - \frac{5}{4}x_1 = 0 \quad \therefore\ x_2 = -\frac{7}{4}x_1$

$\therefore\ x_1, x_2, x_3$ are in the ratio $1 : -\dfrac{7}{4} : \dfrac{1}{4}$ i.e. $4 : -7 : 1 \quad \therefore\ \mathbf{x_1} = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$

(b) Similarly for $\lambda_2 = 3$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which a corresponding eigenvector is

$\mathbf{x}_2 = \dots\dots\dots\dots$

64

$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}$$

Because

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 3 & 1 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore\ -2x_1 + 4x_3 = 0 \quad \therefore\ x_3 = \frac{1}{2}x_1$

Also $\qquad -x_2 = 0 \quad \therefore\ x_2 = 0 \qquad \therefore\ \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

(c) All that now remains is $\lambda_3 = -5$. A corresponding eigenvector $\mathbf{x}_3$ is

$\mathbf{x}_3 = \dots\dots\dots\dots$

Finish it on your own. Method just the same as before.

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$$\boxed{\mathbf{x_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}}$$

Check the working.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} \text{ and } \lambda_3 = -5 \text{ with } (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 0 & 4 \\ 0 & 7 & 0 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ 6x_1 + 4x_3 = 0 \quad \therefore\ x_3 = -\tfrac{3}{2}x_1$$

$$7x_2 = 0 \quad \therefore\ x_2 = 0 \qquad \therefore\ \mathbf{x_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

Collecting the results together, we finally have

$$\lambda_1 = 2,\ \mathbf{x_1} = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix};\ \lambda_2 = 3,\ \mathbf{x_2} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix};\ \lambda_3 = -5,\ \mathbf{x_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

Cayley–Hamilton theorem

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The Cayley–Hamilton theorem states that every square matrix satisfies its characteristic equation. For example the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

of Frame 54 has the characteristic equation

$$\lambda^2 - 3\lambda - 10 = 0$$

and so the Cayley–Hamilton theorem tells us that

$$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} = \mathbf{0}$$

▶

To verify this we note that

$$\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} \text{ so that}$$

$$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 12 & 3 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

You try one. Verify that the matrix $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$ of Frame 57 with the characteristic equation

$$\lambda^2 - 7\lambda - 8 = 0$$

satisfies the Cayley–Hamilton theorem, that is ....

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$$\boxed{\mathbf{A}^2 - 7\mathbf{A} - 8\mathbf{I} = \mathbf{0}}$$

Because

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 29 & 70 \\ 14 & 36 \end{pmatrix} \text{ so that}$$

$$\mathbf{A}^2 - 7\mathbf{A} - 8\mathbf{I} = \begin{pmatrix} 29 & 70 \\ 14 & 36 \end{pmatrix} - 7\begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 70 \\ 14 & 36 \end{pmatrix} - \begin{pmatrix} 21 & 70 \\ 14 & 28 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Now on to something different

Systems of first-order ordinary differential equations

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Matrix methods involving eigenvalues and their associated eigenvectors can be used to solve systems of coupled differential equations, though we shall only consider cases where the relevant eigenvalues are distinct. We proceed by example.

Example 1

Consider the system of two coupled ordinary differential equations

$$\begin{aligned} f_1'(x) &= 2f_1(x) + 3f_2(x) \\ f_2'(x) &= 4f_1(x) + f_2(x) \end{aligned} \quad \text{where } f_1(0) = 2 \text{ and } f_2(0) = 1$$

These can be written in matrix form as ....

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$$\boxed{\begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$\mathbf{F}'(x) = \mathbf{AF}(x)$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}'(x) = \begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and where $\mathbf{F}(0) = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are the boundary conditions in matrix form.

The matrix differential equation $\mathbf{F}'(x) = \mathbf{AF}(x)$ is similar in form to the single differential equation $f'(x) = af(x)$ (a constant) which has solution $f(x) = \alpha e^{ax}$ (α constant), so to solve the matrix equation we try a solution of the form

$\mathbf{F}(x) = \mathbf{C}e^{kx}$ where the number k and the constants c_1 and c_2 of the matrix $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ are to be determined.

Substituting $\mathbf{F}(x) = \mathbf{C}e^{kx}$ into the matrix equation $\mathbf{F}'(x) = \mathbf{AF}(x)$ gives

....

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$$\boxed{k\mathbf{C}e^{kx} = \mathbf{AC}e^{kx}}$$

Because

$\mathbf{F}(x) = \mathbf{C}e^{kx}$ so $\mathbf{F}'(x) = k\mathbf{C}e^{kx}$. Since $\mathbf{F}'(x) = \mathbf{AF}(x)$ then $k\mathbf{C}e^{kx} = \mathbf{AC}e^{kx}$

Dividing both sides by e^{kx} gives

$k\mathbf{C} = \mathbf{AC}$ that is $\mathbf{AC} = k\mathbf{C}$.

So, from Frame 53, k is an *eigenvalue* of $\mathbf{A}$ and $\mathbf{C}$ is the corresponding *eigenvector*. Therefore, we must first find the eigenvalues of $\mathbf{A}$ and for this matrix these have been found earlier in Frames 54 to 57. They are

$\lambda = -2$ (and so $k = -2$) with corresponding eigenvector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$\lambda = 5$ (and so $k = 5$) with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

To each eigenvalue the matrix $\mathbf{F}(x) = \mathbf{C}e^{kx}$ is a solution. The complete solution to $\mathbf{F}' = \mathbf{AF}$ is then

$$\mathbf{F}_1(x) = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} a_1 e^{\cdots} \quad \text{and} \quad \mathbf{F}_2(x) = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} a_2 e^{\cdots}$$

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$$\mathbf{F}_1(x) = \begin{pmatrix} 3 \\ -4 \end{pmatrix} a_1 e^{-2x} \text{ and } \mathbf{F}_2(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_2 e^{5x}$$

Because

$\mathbf{F}(x) = \mathbf{C}e^{kx}$ is the solution corresponding to the eigenvalue k with associated eigenvector $\mathbf{C}$.

The complete solution to the equation $\mathbf{F}'(x) = \mathbf{AF}(x)$ is then a combination of these two solutions in the form

$$\mathbf{F}(x) = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-2x} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5x}$$

Applying the boundary conditions gives $\mathbf{F}(0) = \ldots\ldots\ldots\ldots$

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$$\mathbf{F}(0) = \begin{pmatrix} 3A + B \\ -4A + B \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Because

$$\mathbf{F}(x) = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-2x} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5x} \text{ and so } \mathbf{F}(0) = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3A + B \\ -4A + B \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore

$$\begin{aligned} 3A + B &= 2 \\ -4A + B &= 1 \end{aligned} \quad \text{with solution } A = 1/7 \text{ and } B = 11/7,$$

giving the final solution as $\mathbf{F}(x) = \ldots\ldots\ldots\ldots$

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$$\mathbf{F}(x) = \begin{pmatrix} 3/7 \\ -4/7 \end{pmatrix} e^{-2x} + \begin{pmatrix} 11/7 \\ 11/7 \end{pmatrix} e^{5x}$$

Summary

To solve an equation of the form

$\mathbf{F}'(x) = \mathbf{AF}(x)$

1 Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ (assuming they are all distinct)
2 Find the associated eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$
3 Write the solution of the equation as $\mathbf{F}(x) = \sum_{r=1}^{n} (A_r e^{\lambda_r x})\mathbf{C}_r$ and use the boundary conditions to find the values of a_r for $r = 1, 2, \ldots, n$.

Now you try one.

Next frame

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Example 2

The system of two coupled ordinary differential equations

$$\begin{aligned} f_1'(x) &= 3f_1(x) + 10f_2(x) \\ f_2'(x) &= 2f_1(x) + 4f_2(x) \end{aligned} \quad \text{where } f_1(0) = 0 \text{ and } f_2(0) = 1$$

has the solution (refer to Frames 57 to 61)

$$f_1(x) = \ldots\ldots\ldots\ldots$$
$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} f_1(x) &= -\frac{10}{9}e^{-x} + \frac{10}{9}e^{8x} \\ f_2(x) &= \frac{4}{9}e^{-x} + \frac{5}{9}e^{8x} \end{aligned}}$$

Because

$$\begin{aligned} f_1'(x) &= 3f_1(x) + 10f_2(x) \\ f_2'(x) &= 2f_1(x) + 4f_2(x) \end{aligned}$$

can be written in matrix form as

$$\ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$$\mathbf{F}'(x) = \mathbf{AF}(x)$$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}'(x) = \begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$ and where $\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

To solve the matrix equation we first need the eigenvalues and associated eigenvectors of the matrix **A**. These have already been found in Frames 57 to 61 and they are

$\lambda = -1$ with corresponding eigenvector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\lambda = 8$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The complete solution of $\mathbf{F}' = \mathbf{AF}$ is then

$$\mathbf{F}(x) = A\begin{pmatrix} 5 \\ -2 \end{pmatrix}e^{-x} + B\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{8x}$$

That is $f_1(x) = \ldots\ldots\ldots\ldots$

$f_2(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned} f_1(x) &= 5Ae^{-x} + 2Be^{8x} \\ f_2(x) &= -2Ae^{-x} + Be^{8x} \end{aligned}}$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = A\begin{pmatrix} 5 \\ -2 \end{pmatrix} e^{-x} + B\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{8x}$$

and so

$$f_1(x) = 5Ae^{-x} + 2Be^{8x}$$
$$f_2(x) = -2Ae^{-x} + Be^{8x}$$

Applying the boundary conditions, we find

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \ldots A + \ldots B \\ \ldots A + \ldots B \end{pmatrix}$$

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$$\boxed{\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 5A + 2B \\ -2A + B \end{pmatrix}}$$

Because

The boundary conditions are $f_1(0) = 0$ and $f_2(0) = 1$ therefore

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 5A + 2B \\ -2A + B \end{pmatrix}$$

This gives the pair of simultaneous equations

$$\begin{aligned} 5A + 2B &= 0 \\ -2A + B &= 1 \end{aligned} \quad \text{which have solution}$$

$$A = \ldots\ldots\ldots\ldots \quad \text{and} \quad B = \ldots\ldots\ldots\ldots$$

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$$\boxed{A = -2/9 \quad \text{and} \quad B = 5/9}$$

This gives the complete solution as

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -10/9 \\ 4/9 \end{pmatrix} e^{-x} + \begin{pmatrix} 10/9 \\ 5/9 \end{pmatrix} e^{8x}$$

$$f_1(x) = -\frac{10}{9}e^{-x} + \frac{10}{9}e^{8x}$$

$$f_2(x) = \frac{4}{9}e^{-x} + \frac{5}{9}e^{8x}$$

Diagonalisation of a matrix

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Modal matrix

We have already discussed the eigenvalues and eigenvectors of a matrix **A** of order n. In this section we shall assume that all the eigenvalues are distinct. If the n eigenvectors $\mathbf{x}_i$ are arranged as columns of a square matrix, the *modal matrix* of **A**, denoted by **M**, is formed

i.e. $\mathbf{M} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n)$

For example, we have seen earlier that if

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} \text{ then } \lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = -5$$

and the corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

Then the modal matrix $\mathbf{M} = \begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix}$

Spectral matrix

Also, we define the *spectral matrix* of **A**, i.e. **S**, as a diagonal matrix with the eigenvalues only on the main diagonal

$$\text{i.e. } \mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

So, in the example above, $\mathbf{S} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{S} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}}$$

Note that the eigenvalues of **S** and **A** are the same.

So, if $\mathbf{A} = \begin{pmatrix} 5 & -6 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ has eigenvalues $\lambda = 1, 2, 4$ and

corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

then $\mathbf{M} = \ldots\ldots\ldots\ldots$ and $\mathbf{S} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{M} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 6 & 3 & 3 \end{pmatrix}; \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$$

Now how are these connected? Let us investigate.

The eigenvectors $\mathbf{x}$ arranged in the modal matrix satisfy the original equation

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Also $\quad \mathbf{M} = (\mathbf{x}_1 \quad \mathbf{x}_2 \quad \dots \quad \mathbf{x}_n)$

Then $\quad \mathbf{AM} = \mathbf{A}(\mathbf{x}_1 \quad \mathbf{x}_2 \quad \dots \quad \mathbf{x}_n)$

$$= (\mathbf{Ax}_1 \quad \mathbf{Ax}_2 \quad \dots \quad \mathbf{Ax}_n)$$

$$= (\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad \dots \quad \lambda_n\mathbf{x}_n) \quad \text{since } \mathbf{Ax} = \lambda\mathbf{x}$$

Now $\quad \mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \quad \therefore (\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad \dots \quad \lambda_n\mathbf{x}_n) = \mathbf{MS}$

$$\therefore \mathbf{AM} = \mathbf{MS}$$

If we now pre-multiply both sides by $\mathbf{M}^{-1}$ we have

$$\mathbf{M}^{-1}\mathbf{AM} = \mathbf{M}^{-1}\mathbf{MS} \qquad \text{But } \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

$$\therefore \mathbf{M}^{-1}\mathbf{AM} = \mathbf{S}$$

Make a note of this result. Then we will consider an example

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Example 1

From the results of a previous example in Frame 65, if

$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}$ then $\lambda_1 = 2, \ \lambda_2 = 3, \ \lambda_3 = -5$ and

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}.$$

Also $\mathbf{M} = \begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix}$.

We can find $\mathbf{M}^{-1}$ by any of the methods we have established previously.

$$\mathbf{M}^{-1} = \dots\dots\dots\dots$$

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$$\boxed{\mathbf{M}^{-1} = \begin{pmatrix} 0 & -1/7 & 0 \\ 3/8 & 1/4 & 1/4 \\ 1/8 & 1/28 & -1/4 \end{pmatrix}}$$

Here is one way of determining the inverse. You may have done it by another.

$$\left(\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ -7 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 7 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \\ 4 & 2 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & -3 & 0 & 1/7 & 1 \\ 0 & 2 & 2 & 1 & 4/7 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & -3 & 0 & 1/7 & 1 \\ 0 & 0 & 8 & 1 & 2/7 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & -3 & 0 & 1/7 & 1 \\ 0 & 0 & 1 & 1/8 & 1/28 & -1/4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & 0 & 3/8 & 7/28 & 1/4 \\ 0 & 0 & 1 & 1/8 & 1/28 & -1/4 \end{array}\right)$$

$$\therefore\ \mathbf{M}^{-1} = \begin{pmatrix} 0 & -1/7 & 0 \\ 3/8 & 1/4 & 1/4 \\ 1/8 & 1/28 & -1/4 \end{pmatrix}$$

So now $\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix}$

$$\therefore\ \mathbf{AM} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}\begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 6 & -10 \\ -14 & 0 & 0 \\ 2 & 3 & 15 \end{pmatrix}$$

Then $\mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 0 & -1/7 & 0 \\ 3/8 & 1/4 & 1/4 \\ 1/8 & 1/28 & -1/4 \end{pmatrix}\begin{pmatrix} 8 & 6 & -10 \\ -14 & 0 & 0 \\ 2 & 3 & 15 \end{pmatrix}$

$= \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}}$$

So we have transformed the original matrix **A** into a diagonal matrix and notice that the elements on the main diagonal are, in fact, the eigenvalues of **A**

i.e. $\mathbf{M}^{-1}\mathbf{AM} = \mathbf{S}$

Therefore, let us list a few relevant facts

1 $\mathbf{M}^{-1}\mathbf{AM}$ transforms the square matrix **A** into a diagonal matrix **S**.
2 A square matrix **A** of order n can be so transformed if the matrix has n independent eigenvectors.
3 A matrix **A** always has n linearly independent eigenvectors if it has n distinct eigenvalues or if it is a symmetric matrix.
4 If the matrix has repeated eigenvalues and is not symmetric, it may or may not have n linearly independent eigenvectors.

Now here is one straightforward example with which to finish.

Example 2

If $\mathbf{A} = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix}$, $\quad \mathbf{M} = \ldots\ldots\ldots\ldots$; $\mathbf{M}^{-1} = \ldots\ldots\ldots\ldots$;

and hence $\mathbf{M}^{-1}\mathbf{AM} = \ldots\ldots\ldots\ldots$

Work through it entirely on your own:

(1) Determine the eigenvalues and corresponding eigenvectors.

(2) Hence form the matrix **M**.

(3) Determine $\mathbf{M}^{-1}$, the inverse of **M**.

(4) Finally form the matrix products **AM** and $\mathbf{M}^{-1}(\mathbf{AM})$.

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$$\mathbf{M}=\begin{pmatrix}1 & 5\\ 2 & -2\end{pmatrix};\quad \mathbf{M}^{-1}=\begin{pmatrix}1/6 & 5/12\\ 1/6 & -1/12\end{pmatrix};\quad \mathbf{M}^{-1}\mathbf{AM}=\begin{pmatrix}4 & 0\\ 0 & -8\end{pmatrix}$$

Here is the working. See whether you agree.

$$\mathbf{A}=\begin{pmatrix}-6 & 5\\ 4 & 2\end{pmatrix}\qquad \therefore \begin{vmatrix}-6-\lambda & 5\\ 4 & 2-\lambda\end{vmatrix}=0$$

$$(-6-\lambda)(2-\lambda)-20=0\qquad \therefore\ \lambda^2+4\lambda-32=0$$

$$(\lambda-4)(\lambda+8)=0\qquad \therefore\ \lambda=4 \text{ or } -8$$

(a) $\lambda_1=4$

$$\left\{\begin{pmatrix}-6 & 5\\ 4 & 2\end{pmatrix}-\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-10 & 5\\ 4 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore\ -10x_1+5x_2=0\quad \therefore\ x_2=2x_1\qquad \mathbf{x}_1=\begin{pmatrix}1\\ 2\end{pmatrix}$$

(b) $\lambda_2=-8$

$$\left\{\begin{pmatrix}-6 & 5\\ 4 & 2\end{pmatrix}+\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}\right\}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}2 & 5\\ 4 & 10\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore\ 2x_1+5x_2=0\quad \therefore\ x_2=-\tfrac{2}{5}x_1\quad \therefore\ \mathbf{x}_2=\begin{pmatrix}5\\ -2\end{pmatrix}$$

$$\therefore\ \mathbf{M}=\begin{pmatrix}1 & 5\\ 2 & -2\end{pmatrix}$$

To find $\mathbf{M}^{-1}$ $\left(\begin{array}{cc|cc}1 & 5 & 1 & 0\\ 2 & -2 & 0 & 1\end{array}\right)$

Operating on rows, we have

$$\left(\begin{array}{cc|cc}0 & 5 & 1 & 0\\ 0 & -12 & -2 & 1\end{array}\right)=\left(\begin{array}{cc|cc}1 & 5 & 1 & 0\\ 0 & 1 & 1/6 & -1/12\end{array}\right)$$

$$=\left(\begin{array}{cc|cc}1 & 0 & 1/6 & 5/12\\ 0 & 1 & 1/6 & -1/12\end{array}\right)$$

$$\therefore\ \mathbf{M}^{-1}=\begin{pmatrix}1/6 & 5/12\\ 1/6 & -1/12\end{pmatrix}$$

$$\therefore\ \mathbf{AM}=\begin{pmatrix}-6 & 5\\ 4 & 2\end{pmatrix}\begin{pmatrix}1 & 5\\ 2 & -2\end{pmatrix}=\begin{pmatrix}4 & -40\\ 8 & 16\end{pmatrix}$$

$$\therefore\ \mathbf{M}^{-1}\mathbf{AM}=\begin{pmatrix}1/6 & 5/12\\ 1/6 & -1/12\end{pmatrix}\begin{pmatrix}4 & -40\\ 8 & 16\end{pmatrix}=\begin{pmatrix}4 & 0\\ 0 & -8\end{pmatrix}$$

$$\therefore\ \mathbf{M}^{-1}\mathbf{AM}=\begin{pmatrix}4 & 0\\ 0 & -8\end{pmatrix}$$

Systems of second-order differential equations

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The process of uncoupling a system of differential equations to obtain their solution can be achieved by diagonalising the matrix of coefficients. For simplicity we shall only consider second-order equations and again, we proceed by example.

Example 1

Consider the system of coupled second-order differential equations

$$f_1''(x) = 2f_1(x) + 3f_2(x)$$
$$f_2''(x) = 4f_1(x) + f_2(x)$$

where $f_1(0) = 2$, $f_2(0) = 1$, $f_1'(0) = 4$ and $f_2'(0) = 3$

These can be written in matrix form as

. . . .

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$$\boxed{\begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$$\mathbf{F}''(x) = \mathbf{AF}(x)$$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}''(x) = \begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and where $\mathbf{F}(0) = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{F}'(0) = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ are the boundary conditions in matrix form.

The matrix differential equation $\mathbf{F}''(x) = \mathbf{AF}(x)$ is similar in form to the single differential equation $f''(x) = af(x)$ (a constant) which has solution $f(x) = \alpha e^{\sqrt{a}x} + \beta e^{-\sqrt{a}x}$ (α, β constants), so to solve the matrix equation we try a solution of this form. We already know from Frames 54 to 57 that the eigenvalues and eigenvectors of matrix $\mathbf{A}$ are

$\lambda = -2$ with corresponding eigenvector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$\lambda = 5$ with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The modal matrix of $\mathbf{A}$ is the matrix $\mathbf{M}$ and the spectral matrix of $\mathbf{A}$ is the matrix $\mathbf{S}$ where

$$\mathbf{M} = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix}$$

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$$\boxed{\mathbf{M} = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}}$$

Because

The modal matrix is formed from the eigenvectors of $\mathbf{A}$. That is

$$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} \text{ where the two eigenvectors are } \begin{pmatrix} 3 \\ -4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The spectral matrix is formed from the eigenvalues of $\mathbf{A}$. That is

$$\mathbf{S} = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \text{ where the two eigenvalues are } -2 \text{ and } 5$$

If we now define the matrix $\mathbf{G}(x)$ by the equation $\mathbf{F}(x) = \mathbf{MG}(x)$, then differentiating gives

$$\mathbf{F}''(x) = [\mathbf{MG}(x)]'' = \mathbf{MG}''(x) \text{ where}$$
$$\mathbf{F}''(x) = \mathbf{AF}(x) = \mathbf{AMG}(x)$$

and so, from Frame 85, $\mathbf{M}^{-1}\mathbf{MG}''(x) = \mathbf{G}''(x) = \mathbf{M}^{-1}\mathbf{AMG}(x) = \mathbf{SG}(x)$. That is

$$\mathbf{G}''(x) = \mathbf{SG}(x)$$

Therefore, in component terms

$$\mathbf{G}''(x) = \begin{pmatrix} g_1''(x) \\ g_2''(x) \end{pmatrix} = \mathbf{SG}(x) = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

and so

$$g_1''(x) = \ldots g_1(x) \text{ with solution } g_1(x) = k_{11}e^{\cdots x} + k_{12}e^{-\cdots x}$$
$$g_2''(x) = \ldots g_2(x) \text{ with solution } g_2(x) = k_{21}e^{\cdots x} + k_{22}e^{-\cdots x}$$

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$$\boxed{\begin{aligned} g_1''(x) &= -2g_1(x) \text{ with solution } g_1(x) = k_{11}e^{j\sqrt{2}x} + k_{12}e^{-j\sqrt{2}x} \\ g_2''(x) &= 5g_2(x) \text{ with solution } g_2(x) = k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x} \end{aligned}}$$

Now, $\mathbf{F}(x) = \mathbf{MG}(x)$ so

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \end{pmatrix}$$

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$$\boxed{\mathbf{F}(x)=\begin{pmatrix}f_1(x)\\f_2(x)\end{pmatrix}=\begin{pmatrix}3k_{11}e^{j\sqrt{2}x}+3k_{12}e^{-j\sqrt{2}x}+k_{21}e^{\sqrt{5}x}+k_{22}e^{-\sqrt{5}x}\\-4k_{11}e^{j\sqrt{2}x}-4k_{12}e^{-j\sqrt{2}x}+k_{21}e^{\sqrt{5}x}+k_{22}e^{-\sqrt{5}x}\end{pmatrix}}$$

Because

$$\mathbf{F}(x)=\begin{pmatrix}f_1(x)\\f_2(x)\end{pmatrix}=\mathbf{MG}(x)=\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}k_{11}e^{j\sqrt{2}x}+k_{12}e^{-j\sqrt{2}x}\\k_{21}e^{\sqrt{5}x}+k_{22}e^{-\sqrt{5}x}\end{pmatrix}$$

and so

$$f_1(x)=3k_{11}e^{j\sqrt{2}x}+3k_{12}e^{-j\sqrt{2}x}+k_{21}e^{j\sqrt{5}x}+k_{22}e^{-\sqrt{5}x}$$

and

$$f_2(x)=-4k_{11}e^{j\sqrt{2}x}-4k_{12}e^{-j\sqrt{2}x}+k_{21}e^{\sqrt{5}x}+k_{22}e^{-\sqrt{5}x}$$

This solution can be written in terms of circular and hyperbolic trigonometric expressions as

$$\mathbf{F}(x)=\begin{pmatrix}\dots&\dots\\\dots&\dots\end{pmatrix}\begin{pmatrix}P\cos\dots x+Q\sin\dots x\\R\cosh\dots x+S\sinh\dots x\end{pmatrix}$$

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$$\boxed{\mathbf{F}(x)=\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}P\cos\sqrt{2}x+Q\sin\sqrt{2}x\\R\cosh\sqrt{5}x+S\sinh\sqrt{5}x\end{pmatrix}}$$

Because

$$\begin{aligned}&3k_{11}e^{j\sqrt{2}x}+3k_{12}e^{-j\sqrt{2}x}\\&\quad=3k_{11}\left(\cos\sqrt{2}x+j\sin\sqrt{2}x\right)+3k_{12}\left(\cos\sqrt{2}x-j\sin\sqrt{2}x\right)\\&\quad=P\cos\sqrt{2}x+Q\sin\sqrt{2}x\end{aligned}$$

where $P=3k_{11}+3k_{12}$ and $Q=(3k_{11}-3k_{12})j$

and

$$\begin{aligned}&k_{21}e^{\sqrt{5}x}+k_{22}e^{-\sqrt{5}x}\\&\quad=k_{21}\left(\cosh\sqrt{5}x+\sinh\sqrt{5}x\right)+k_{22}\left(\cosh\sqrt{5}x-\sinh\sqrt{5}x\right)\\&\quad=R\cosh\sqrt{5}x+S\sinh\sqrt{5}x\text{ where }R=k_{21}+k_{22}\text{ and }S=k_{21}-k_{22}\end{aligned}$$

Therefore

$$\mathbf{F}(x)=\begin{pmatrix}\dots\\\dots\end{pmatrix}$$

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$$\mathbf{F}(x) = \begin{pmatrix} 3P\cos\sqrt{2}x + 3Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \\ -4P\cos\sqrt{2}x - 4Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \end{pmatrix}$$

That is

$$f_1(x) = \ldots\ldots\ldots\ldots$$
$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$f_1(x) = 3P\cos\sqrt{2}x + 3Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x$$
$$f_2(x) = -4P\cos\sqrt{2}x - 4Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

and so

$$f_1(x) = 3P\cos\sqrt{2}x + 3Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x$$
$$f_2(x) = -4P\cos\sqrt{2}x - 4Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x$$

Applying the boundary conditions, we find

$$\mathbf{F}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \ldots P + \ldots R \\ \ldots P + \ldots R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \ldots Q + \ldots S \\ \ldots Q + \ldots S \end{pmatrix}$$

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$$\mathbf{F}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3P + R \\ -4P + R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}Q + \sqrt{5}S \\ -4\sqrt{2}Q + \sqrt{5}S \end{pmatrix}$$

Because

$f_1(0) = 2$, $f_2(0) = 1$, $f_1'(0) = 4$ and $f_2'(0) = 3$ and so

$$\mathbf{F}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 3P + R \\ -4P + R \end{pmatrix} \text{ and}$$

$$\mathbf{F}'(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}Q + \sqrt{5}S \\ -4\sqrt{2}Q + \sqrt{5}S \end{pmatrix}$$

This gives the two sets of simultaneous equations

$$\begin{matrix} 3P + R = 2 \\ -4P + R = 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 3\sqrt{2}Q + \sqrt{5}S = 4 \\ -4\sqrt{2}Q + \sqrt{5}S = 3 \end{matrix} \quad \text{which have solution}$$

$$P = \ldots\ldots\ldots\ldots,\quad R = \ldots\ldots\ldots\ldots,$$
$$Q = \ldots\ldots\ldots\ldots \text{ and } S = \ldots\ldots\ldots\ldots$$

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$$P = 1/7,\ R = 11/7,\ Q = 1/\left(7\sqrt{2}\right) \text{ and } S = 25/\left(7\sqrt{5}\right)$$

This gives the complete solution as

$$f_1(x) = \frac{3}{7}\cos\sqrt{2}x + \frac{3}{7\sqrt{2}}\sin\sqrt{2}x + \frac{11}{7}\cosh\sqrt{5}x + \frac{25}{7\sqrt{5}}\sinh\sqrt{5}x$$

$$f_2(x) = -\frac{4}{7}\cos\sqrt{2}x - \frac{4}{7\sqrt{2}}\sin\sqrt{2}x + \frac{11}{7}\cosh\sqrt{5}x + \frac{25}{7\sqrt{5}}\sinh\sqrt{5}x$$

This method is quite straightforwardly extended to three or more such coupled differential equations.

Summary

To solve the system of coupled second-order differential equations

$$\mathbf{F}''(x) = \mathbf{AF}(x)$$

1 Find the eigenvalues and eigenvectors of matrix **A** and construct the modal matrix **M** and the diagonal spectral matrix **S**

2 Solve the equation $\mathbf{G}'(x) = \mathbf{SG}(x)$
(note that even though $\mathbf{M}^{-1}$ is used there was no need to calculate it)

3 Apply $\mathbf{F}(x) = \mathbf{MG}(x)$ to find $\mathbf{F}(x)$.

Try one yourself.

Next frame

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Example 2

The system of coupled second-order differential equations (refer to Frames 57 to 61)

$$f_1''(x) = 3f_1(x) + 10f_2(x)$$

$$f_2''(x) = 2f_1(x) + 4f_2(x)$$

where $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 1$ and $f_2'(0) = 0$

has the solution (refer to Frames 57 to 61)

$$f_1(x) = \ldots\ldots\ldots\ldots$$

$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$f_1(x) = 10\cos x + \frac{5}{9}\sin x + 10\cosh 2\sqrt{2}x + \frac{2}{9\sqrt{2}}\sinh 2\sqrt{2}x$$
$$f_2(x) = -4\cos x - \frac{2}{9}\sin x + 5\cosh 2\sqrt{2}x + \frac{1}{9\sqrt{2}}\sinh 2\sqrt{2}x$$

Because

$$f_1''(x) = 3f_1(x) + 10f_2(x)$$
$$f_2''(x) = 2f_1(x) + 4f_2(x)$$

can be written in matrix form as

. . . .

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$$\begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

That is

$$\mathbf{F}''(x) = \mathbf{AF}(x)$$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}''(x) = \begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$

and where $\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{F}'(0) = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

To solve the matrix equation we first need the eigenvalues and associated eigenvectors of the matrix **A**. These have already been found in Frames 57 to 61 and they are

$\lambda = -1$ with corresponding eigenvector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\lambda = 8$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The complete solution of $\mathbf{F}'' = \mathbf{AF}$ is then

$$\begin{aligned}\mathbf{F}(x) &= (P\cos x + Q\sin x)\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \left(R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 5P\cos x + 5Q\sin x + 2R\cosh 2\sqrt{2}x + 2S\sinh 2\sqrt{2}x \\ -2P\cos x - 2Q\sin x + R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x \end{pmatrix}\end{aligned}$$

That is

$$f_1(x) = \ldots\ldots\ldots\ldots$$
$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} f_1(x) &= 5P\cos x + 5Q\sin x + 2R\cosh 2\sqrt{2}x + 2S\sinh 2\sqrt{2}x \\ f_2(x) &= -2P\cos x - 2Q\sin x + R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x \end{aligned}}$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

and so

$$f_1(x) = 5P\cos x + 5Q\sin x + 2R\cosh 2\sqrt{2}x + 2S\sinh 2\sqrt{2}x$$
$$f_2(x) = -2P\cos x - 2Q\sin x + R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x$$

Applying the boundary conditions, we find

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dots P + \dots R \\ \dots P + \dots R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dots Q + \dots S \\ \dots Q + \dots S \end{pmatrix}$$

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$$\boxed{\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5P + 2R \\ -2P + R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5Q + 4\sqrt{2}S \\ -2Q + 2\sqrt{2}S \end{pmatrix}}$$

Because

The boundary conditions are $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 1$ and $f_2'(0) = 0$, therefore

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 5P + 2R \\ -2P + R \end{pmatrix} \text{ and}$$
$$\mathbf{F}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 5Q + 4\sqrt{2}S \\ -2Q + 2\sqrt{2}S \end{pmatrix}$$

This gives the two sets of simultaneous equations

$$\begin{aligned} 5P + 2R &= 0 \\ -2P + R &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 5Q + 4\sqrt{2}S &= 1 \\ -2Q + 2\sqrt{2}S &= 0 \end{aligned} \quad \text{which have solution}$$

$$P = \dots\dots\dots\dots, \quad R = \dots\dots\dots\dots,$$
$$Q = \dots\dots\dots\dots \text{ and } S = \dots\dots\dots\dots$$

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$$\boxed{P = -2/9, \quad R = 5/9, \quad Q = 1/9 \text{ and } S = 1/\left(9\sqrt{2}\right)}$$

This gives the complete solution as

$$f_1(x) = -\frac{10}{9}\cos x + \frac{5}{9}\sin x + \frac{10}{9}\cosh 2\sqrt{2}x + \frac{2}{9\sqrt{2}}\sinh 2\sqrt{2}x$$
$$f_2(x) = \frac{4}{9}\cos x - \frac{2}{9}\sin x + \frac{5}{9}\cosh 2\sqrt{2}x + \frac{1}{9\sqrt{2}}\sinh 2\sqrt{2}x$$

Matrix transformation

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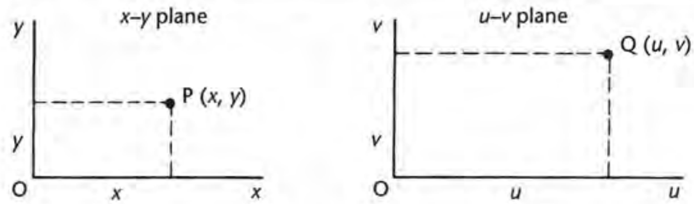


If for every point Q (u, v) in the u–v plane there is a corresponding point P (x, y) in the x–y plane, then there is a relationship between the two sets of coordinates. In the simple case of scaling the coordinate where

$u = ax$ and $v = by$

we have a *linear transformation* and we can combine these in matrix form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then provides the transformation between the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in one set of coordinates and the vector $\begin{pmatrix} u \\ v \end{pmatrix}$ in the other set of coordinates.

Similarly, if we solve the two equations for x and y, we have

$$x = \frac{1}{a}u \quad \text{and} \quad y = \frac{1}{b}v$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

which allows us to transform back from the u–v plane coordinates to the x–y plane coordinates.

Now for an example.

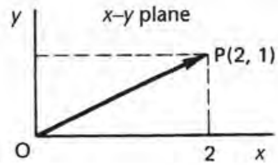
▶

Example

If $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with the transformation $\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}$ determine $\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{TX}$ and show the positions on the x–y and u–v planes.

In this case

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$



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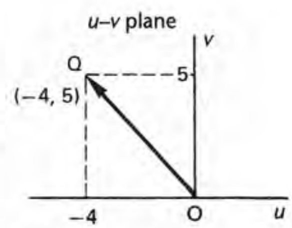

If $\mathbf{T}$ is non-singular and $\mathbf{U} = \mathbf{TX}$ then $\mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$ and since

$$\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix} \text{ then } \mathbf{T}^{-1} = \dots\dots\dots\dots$$

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$$\boxed{\mathbf{T}^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}}$$

There are several ways of finding the inverse of a matrix. One method is as follows.

$$\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$\therefore\ \mathbf{T}^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}$$

So we have $\mathbf{U} = \mathbf{TX}\quad \therefore\ \mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Hence a vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ in the u–v plane transforms into $\begin{pmatrix} x \\ y \end{pmatrix}$ in the x–y plane where $\begin{pmatrix} x \\ y \end{pmatrix} = \dots\dots\dots\dots$

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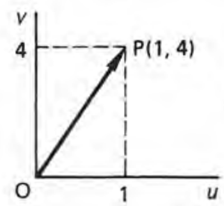
$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 \\ 5 \end{pmatrix}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 5 \end{pmatrix}$$



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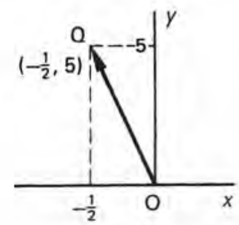
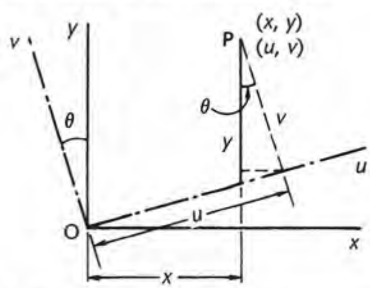


Rotation of axes

A more interesting case occurs with a degree of rotation between the two sets of coordinate axes.



Let P be the point (x, y) in the x–y plane and the point (u, v) in the u–v plane.

Let θ be the angle of rotation between the two systems. From the diagram we can see that

$$\left.\begin{aligned} x &= u\cos\theta - v\sin\theta \\ y &= u\sin\theta + v\cos\theta \end{aligned}\right\} \qquad (1)$$

In matrix form, this becomes $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

which enables us to transform from the u–v plane coordinates to the corresponding x–y plane coordinates.

Make a note of this and then move on

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If we solve equations (1) for u and v, we have

$$x\sin\theta = u\sin\theta\cos\theta - v\sin^2\theta$$
$$y\cos\theta = u\sin\theta\cos\theta + v\cos^2\theta$$
$$\therefore\ y\cos\theta - x\sin\theta = v(\cos^2\theta + \sin^2\theta) = v$$

Also
$$x\cos\theta = u\cos^2\theta - v\sin\theta\cos\theta$$
$$y\sin\theta = u\sin^2\theta + v\sin\theta\cos\theta$$
$$\therefore\ x\cos\theta + y\sin\theta = u(\cos^2\theta + \sin^2\theta) = u$$

So
$$u = x\cos\theta + y\sin\theta$$
$$v = -x\sin\theta + y\cos\theta$$

and written in matrix form, this is ....

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$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$

So we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

and
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $\mathbf{X} = \mathbf{TU}$ and $\mathbf{U} = \mathbf{T}^{-1}\mathbf{X}$

where $\mathbf{T}$ is the matrix of transformation and the equations provide a linear transformation between the two sets of coordinates.

Example

If the u–v plane axes rotate through 30° in an anticlockwise manner from the x–y plane axes, determine the (u, v) coordinates of a point whose (x, y) coordinates are $x = 2$, $y = 3$ in the x–y plane.

This is a straightforward application of the results above.

So $\begin{pmatrix} u \\ v \end{pmatrix} =$ ....

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$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{3} + 3/2 \\ -1 + 3\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 3{\cdot}23 \\ 1{\cdot}60 \end{pmatrix}}$$

Because

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \begin{array}{l} \cos\theta = \sqrt{3}/2 \\ \sin\theta = 1/2 \end{array}$$

$$= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3} + 3/2 \\ -1 + 3\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 3{\cdot}23 \\ 1{\cdot}60 \end{pmatrix}$$

As usual, the Programme ends with the **Revision summary**, to be read in conjunction with the **Can You?** checklist. Go back to the relevant part of the Programme for any points on which you are unsure. The **Test exercise** should then be straightforward and the **Further problems** give valuable additional practice.

Revision summary 12

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1 *Singular* square matrix: $|\mathbf{A}| = 0$

Non-singular square matrix $|\mathbf{A}| \neq 0$.

2 *Rank of a matrix* – order of the largest non-zero determinant that can be formed from the elements of the matrix.

3 *Elementary operations and equivalent matrices*

Each of the following row operations on matrix **A** produces a *row equivalent matrix* **B** where the order and rank of **B** are the same as those of **A**. We write $\mathbf{A} \sim \mathbf{B}$.

(1) Interchanging two rows

(2) Multiplying each element of a row by the same non-zero scalar quantity

(3) Adding or subtracting corresponding elements from those of another row.

These operations are called *elementary row operations*. There is a corresponding set of three *elementary column operations* that can be used to form *column equivalent matrices*.

4 *Consistency* of a set of n equations in n unknowns with coefficient matrix **A** and augmented matrix $\mathbf{A}_b$.

(a) Consistent if rank of $\mathbf{A}$ = rank of $\mathbf{A}_b$

(b) Inconsistent if rank of $\mathbf{A} <$ rank of $\mathbf{A}_b$.

5 *Uniqueness of solutions – n equations with n unknowns.*

(a) rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = n$ *unique solutions*

(b) rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = m < n$ *infinite number of solutions*

(c) rank of $\mathbf{A} <$ rank of $\mathbf{A}_b$ *no solution*

6 *Solution of sets of equations*

(a) *Inverse matrix method* $\mathbf{Ax} = \mathbf{b}$; $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

To find $\mathbf{A}^{-1}$

(1) evaluate $|\mathbf{A}|$

(2) form $\mathbf{C}$, the matrix of cofactors of $\mathbf{A}$

(3) write $\mathbf{C}^{\mathrm{T}}$, the transpose of $\mathbf{A}$

(4) $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}}$.

(b) *Row transformation method* $\mathbf{Ax} = \mathbf{b}$; $\mathbf{Ax} = \mathbf{Ib}$

(1) form the combined coefficient matrix $[\mathbf{A} \vdots \mathbf{I}]$

(2) row transformations to convert to $[\mathbf{I} \vdots \mathbf{A}^{-1}]$

(3) then solve $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

(c) *Gaussian elimination method* $\mathbf{Ax} = \mathbf{b}$

(1) form augmented matrix $[\mathbf{A} \vdots \mathbf{b}]$

(2) operate on rows to convert to $[\mathbf{U} \vdots \mathbf{b}']$ where $\mathbf{U}$ is the upper-triangular matrix.

(3) expand from bottom row to obtain $\mathbf{x}$.

(d) *Triangular decomposition method* $\mathbf{Ax} = \mathbf{b}$

Write $\mathbf{A}$ as the product of upper and lower triangular matrices.

$\mathbf{A} = \mathbf{LU}$, $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. Put $\mathbf{Ux} = \mathbf{y}$ $\therefore$ $\mathbf{Ly} = \mathbf{b}$

(1) solve $\mathbf{Ly} = \mathbf{b}$ to obtain $\mathbf{y}$

(2) solve $\mathbf{Ux} = \mathbf{y}$ to obtain $\mathbf{x}$.

7 *Eigenvalues and eigenvectors* $\mathbf{Ax} = \lambda\mathbf{x}$

Sets of equations of form $\mathbf{Ax} = \lambda\mathbf{x}$, where $\mathbf{A}$ = coefficient matrix, $\mathbf{x}$ = column matrix, λ = scalar quantity.

Equations become $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

For non-trivial solutions, $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the *characteristic equation* and gives values of λ i.e. the *eigenvalues*.

Substitution of each eigenvalue gives a corresponding *eigenvector*.

8 *Cayley–Hamilton theorem*

Every square matrix satisfies its own characteristic equation.

9 *Solving systems of first-order ordinary differential equations*

To solve the system of coupled first-order differential equations

$$\mathbf{F}'(x) = \mathbf{AF}(x)$$

▶

(a) Find the eigenvalues and eigenvectors of matrix **A** and construct the modal matrix **M** and the diagonal spectral matrix **S**

(b) Solve the equation $\mathbf{G}'(x) = \mathbf{SG}(x)$

(c) Apply $\mathbf{F}(x) = \mathbf{MG}(x)$ to find $\mathbf{F}(x)$.

10 *Diagonalisation of a matrix*

Modal matrix of **A**

If **A** has distinct eigenvalues $\mathbf{M} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are eigenvectors of **A**, then $\mathbf{M}^{-1}\mathbf{AM} = \mathbf{S}$ where **S** is the *spectral matrix* of **A**

$$\text{and} \quad \mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of **A**.

11 *Solving systems of second-order ordinary differential equations*

To solve an equation of the form

$$\mathbf{F}''(x) = \mathbf{AF}(x)$$

(a) Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of **A**

(b) Assuming the eigenvectors are all distinct, find the associated eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$

(c) Write the solution of the equation as

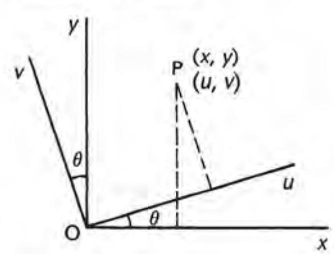
$$\mathbf{F}(x) = \sum_{r=1}^{n} \left(a_r e^{\sqrt{\lambda_r}x} + b_r e^{-\sqrt{\lambda_r}x} \right) \mathbf{C}_r$$

and use the boundary conditions to find the values of a_r and b_r for $r = 1, 2, \ldots, n$.

12 *Matrix transformation*

(a) $\mathbf{U} = \mathbf{TX}$, where **T** is a transformation matrix, transforms a vector in the x–y plane to a corresponding vector in the u–v plane. Similarly, $\mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$ converts a vector in the u–v plane to a corresponding vector in the x–y plane.

(b) *Rotation of axes*



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Can You?

110 Checklist 12

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Determine whether a matrix is singular or non-singular?	1 to 3
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Determine the rank of a matrix?	3 to 13
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Determine the consistency of a set of linear equations and hence demonstrate the uniqueness of their solution?	14 to 23
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Obtain the solution of a set of simultaneous linear equations by using matrix inversion, by row transformation, by Gaussian elimination and by triangular decomposition?	24 to 52
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Obtain the eigenvalues and corresponding eigenvectors of a square matrix?	53 to 65
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Demonstrate the validity of the Cayley–Hamilton theorem?	66 and 67
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Solve systems of first-order ordinary differential equations using eigenvalue and eigenvector methods?	68 to 79
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Construct the modal matrix from the eigenvectors of a matrix and the spectral matrix from the eigenvalues?	80 to 86
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Solve systems of second-order ordinary differential equations using diagonalisation?	87 to 102
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use matrices to represent transformations between coordinate systems?	103 to 108
Yes ☐ ☐ ☐ ☐ ☐ *No*	

Test exercise 12

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1 Determine the rank of **A** and of $\mathbf{A}_b$ for the following sets of equations and hence determine the nature of the solutions. Do *not* solve the equations.

(a) $\begin{aligned} x_1 + 3x_2 - 2x_3 &= 6 \\ 4x_1 + 5x_2 + 2x_3 &= 3 \\ x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 - 4x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= -4 \\ 2x_1 + 4x_2 + x_3 &= -3. \end{aligned}$

2 If $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 2 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 10 \\ 9 \end{pmatrix}$, determine $\mathbf{A}^{-1}$ and hence solve the set of equations.

3 Given that $\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_1 - x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 - 2x_3 &= 0 \end{aligned}$

apply the method of row transformation to obtain the value of x_1, x_2, x_3.

4 By the method of Gaussian elimination, solve the equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$.

5 If $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 5 & 3 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$, express **A** as the product $\mathbf{A} = \mathbf{LU}$ where **L** and **U** are lower and upper-triangular matrices and hence determine the values of x_1, x_2, x_3.

6 Determine the eigenvalues and corresponding eigenvectors of $\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$.

7 If $\mathbf{x_1}$ and $\mathbf{x_2}$ are eigenvectors of $\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ determine

(a) $\mathbf{M} = (\mathbf{x}_1\mathbf{x}_2)$

(b) $\mathbf{M}^{-1}$

(c) $\mathbf{M}^{-1}\mathbf{AM}$.

8 Solve the system of first-order differential equations

$\begin{aligned} f_1'(x) &= 5f_1(x) - 2f_2(x) \\ f_2'(x) &= -f_1(x) + 4f_2(x) \end{aligned}$ where $f_1(0) = -3$ and $f_2(0) = 2$.

▶

9 Solve the system of second-order differential equations

$$\begin{aligned} f_1''(x) &= f_1(x) + 6f_2(x) \\ f_2''(x) &= 3f_1(x) - 2f_2(x) \end{aligned} \quad \text{where } f_1(0) = 1,\ f_2(0) = 0,\ f_1'(0) = 2,\ f_2'(0) = -1.$$

10 (a) Determine the vector in the u–v plane formed by $\mathbf{U} = \mathbf{TX}$, where the transformation matrix is $\mathbf{T} = \begin{pmatrix} -2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is a vector in the x–y plane.

(b) The coordinate axes in the x–y plane and in the u–v plane have the same origin O, but OU is inclined to OX at an angle of 60° in an anticlockwise manner. Transform a vector $\mathbf{X} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ in the x–y plane into the corresponding vector in the u–v plane.

Further problems 12

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1 If $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 5 & 2 & 3 \\ 3 & -2 & -2 \\ 4 & 3 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 5 \\ -5 \end{pmatrix}$, determine $\mathbf{A}^{-1}$ and hence solve the set of equations.

2 Apply the method of row transformation to solve the following sets of equations.

(a)
$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 8 \\ 2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 - 4x_2 + 2x_3 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 8 \\ 2x_1 - x_2 + x_3 &= 9 \\ 3x_1 + 2x_2 + 3x_3 &= 5. \end{aligned}$$

3 Solve the following sets of equations by Gaussian elimination.

(a)
$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 4 \\ 2x_1 + x_2 + x_3 - 4x_4 &= 3 \\ 3x_1 - x_2 - 2x_3 + 2x_4 &= 6 \\ x_1 + 3x_2 - x_3 + x_4 &= 8 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 2x_4 &= 2 \\ 4x_1 + 2x_2 - 3x_3 - x_4 &= 6 \\ x_1 - x_2 + 4x_3 - 2x_4 &= 7 \\ 3x_1 + 2x_2 + x_3 - x_4 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + 5x_3 + x_4 &= 4 \\ 3x_1 - 4x_2 + 3x_3 - 2x_4 &= 7 \\ 4x_1 + 3x_2 + 2x_3 - x_4 &= 1 \\ x_1 - 2x_2 - 4x_3 - x_4 &= 2. \end{aligned}$$

▶

4 Using the method of triangular decomposition, solve the following sets of equations.

(a) $\begin{pmatrix} 1 & 4 & -1 \\ 4 & 2 & 3 \\ 7 & -3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -18 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \\ 6 & -3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 17 \\ 22 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -2 & 3 & -1 \\ 3 & 1 & -3 & 2 \\ 5 & 3 & 2 & 3 \\ 2 & -4 & -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \\ 21 \\ -10 \end{pmatrix}.$

5 If $\mathbf{Ax} = \lambda\mathbf{x}$, determine the eigenvalues and corresponding eigenvectors in each of the following cases.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} -5 & 9 \\ 1 & 3 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} 2 & 7 & 0 \\ 1 & 3 & 1 \\ 5 & 0 & 8 \end{pmatrix}$ (f) $\mathbf{A} = \begin{pmatrix} 5 & -6 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

(g) $\mathbf{A} = \begin{pmatrix} -3 & 0 & 6 \\ 4 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ (h) $\mathbf{A} = \begin{pmatrix} 4 & 10 & -8 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{pmatrix}.$

6 Solve each of the following systems of first-order differential equations.

(a) $f_1'(x) = 2f_1(x) - 5f_2(x)$
$f_2'(x) = f_1(x) - 4f_2(x)$
where $f_1(0) = 1$ and $f_2(0) = 0$

(b) $f_1'(x) = -5f_1(x) + 9f_2(x)$
$f_2'(x) = f_1(x) + 3f_2(x)$
where $f_1(0) = 0$ and $f_2(0) = -2$

(c) $f_1'(x) = 5f_1(x) - 6f_2(x) + f_3(x)$
$f_2'(x) = f_1(x) + f_2(x)$
$f_3'(x) = 3f_1(x) + f_3(x)$
where $f_1(0) = 1$, $f_2(0) = 0$ and $f_3(0) = 2$

(d) $f_1'(x) = 4f_1(x) + 10f_2(x) - 8f_3(x)$
$f_2'(x) = f_1(x) + 2f_2(x) + f_3(x)$
$f_3'(x) = -f_1(x) + 2f_2(x) + 3f_3(x)$
where $f_1(0) = 4$, $f_2(0) = -2$ and $f_3(0) = -1$.

▶

7 If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 10 & -3 \\ 0 & -3 & 9 \end{pmatrix}$, determine the three eigenvalues λ_1, λ_2, λ_3 of $\mathbf{A}$ and verify that if $\mathbf{M} = \begin{pmatrix} -9 & 1 & 1 \\ 3 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$ then $\mathbf{M}^{-1}\mathbf{AM} = \mathbf{S}$, where $\mathbf{S}$ is a diagonal matrix with elements λ_1, λ_2, λ_3.

8 Invert the matrix $\mathbf{A} = \begin{pmatrix} 8 & 10 & 7 \\ 5 & 9 & 4 \\ 9 & 11 & 8 \end{pmatrix}$ and hence solve the equations

$$\begin{aligned} 8I_1 + 10I_2 + 7I_3 &= 0 \\ 5I_1 + 9I_2 + 4I_3 &= -9 \\ 9I_1 + 11I_2 + 8I_3 &= 1. \end{aligned}$$

9 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 5 & 8 & 9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 6 & -4 \\ -1 & -6 & 5 \\ 2 & 2 & -2 \end{pmatrix}$, verify that $\mathbf{AB} = k\mathbf{I}$ where $\mathbf{I}$ is a unit matrix and k is a constant. Hence solve the equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ 4x_1 + 6x_2 + 7x_3 &= 2 \\ 5x_1 + 8x_2 + 9x_3 &= 3. \end{aligned}$$

10 Solve each of the following systems of second-order differential equations.

(a) $f_1''(x) = 4f_1(x) + 3f_2(x)$
$f_2''(x) = 2f_1(x) + 5f_2(x)$
where $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 4$ and $f_2'(0) = 1$

(b) $f_1''(x) = -6f_1(x) + 5f_2(x)$
$f_2''(x) = 4f_1(x) + 2f_2(x)$
where $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 1$ and $f_2'(0) = 0$

(c) $f_1''(x) = 2f_1(x) + 7f_2(x)$
$f_2''(x) = f_1(x) + 3f_2(x) + f_3(x)$
$f_3''(x) = 5f_1(x) + 8f_3(x)$
where $f_1(0) = 1$, $f_2(0) = 1$, $f_3(0) = 0$, $f_1'(0) = 0$, $f_2'(0) = 0$ and $f_3'(0) = 1$

(d) $f_1''(x) = -3f_1(x) + 6f_3(x)$
$f_2''(x) = 4f_1(x) + 5f_2(x) + 3f_3(x)$
$f_3''(x) = f_1(x) + 2f_2(x) + f_3(x)$
where $f_1(0) = 1$, $f_2(0) = 1$, $f_3(0) = 0$, $f_1'(0) = 0$, $f_2'(0) = 0$, $f_3'(0) = 1$.

Numerical solutions of partial differential equations

Frames 1 to 64

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the finite difference formulas for the first partial derivatives of a function of two real variables and construct the central finite difference formula to represent a first-order partial differential equation
- Draw a rectangular grid of points overlaid on the domain of a function of two real variables and evaluate the function at the boundary grid points
- Construct the computational molecule for a first-order partial differential equation in two real variables and use the molecule to evaluate the solutions to the equation at the grid points interior to the boundary
- Describe the solution as a set of simultaneous linear equations and use matrices to represent them
- Invert the coefficient matrix and thereby represent the solution to the partial differential equation as a column matrix
- Take account of a boundary condition in the form of the derivative normal to the boundary
- Obtain the central finite difference formulas for the second derivatives of a function of two real variables and construct finite difference formulas for second-order partial differential equations
- Use the forward difference formula for the first time derivatives in partial differential equations involving time and distance
- Use the Crank–Nicolson procedure for a partial differential equation involving a first time derivative
- Appreciate the use of dimensional analysis in the conversion of a partial differential equation modelling a physical system into a dimensionless equation

Introduction

1

The numerical solution of partial differential equations is a large subject and can form the content of a course in itself. Here we shall just introduce the subject by considering the basic methods of solving some first- and second-order partial differential equations that involve functions of two real variables. The approach that is used is to construct finite difference formulas for the first and second partial derivatives and then to construct a finite difference formula that represents an approximation to the differential equation. However, before we move into the realm of functions of two real variables we shall derive the finite difference formulas for the ordinary first derivative of a function of a single real variable.

Next frame

Numerical approximation to derivatives

2

A function of one real variable $f(x)$ has the Taylor series expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

and, equally, replacing h by $-h$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \dots$$

From the first equation we can see that by dividing through by h, we have

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2!}f''(x) + \frac{h^2}{3!}f'''(x) + \dots$$

and from the second equation

$$\frac{f(x-h) - f(x)}{h} = -f'(x) + \frac{h}{2!}f''(x) - \frac{h^2}{3!}f'''(x) + \dots$$

If we now neglect terms of the order two and higher we see that

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$ [this is the *forward difference formula* for the first derivative of $f(x)$]

and

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$ [this is the *backward difference formula* for the first derivative of $f(x)$]

and both of these are accurate up to terms of order two. A more accurate estimate of the derivative can be obtained by subtracting the two Taylor series expansions from each other to get

$f'(x) \approx$ .... neglecting terms of the order of ....

3

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

neglecting terms of the order two and higher

Because

$$\begin{aligned} f(x+h) - f(x-h) &= \left(f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\right) \\ &\quad - \left(f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \dots\right) \\ &= 2\left(hf'(x) + \frac{h^3}{3!}f'''(x) + \dots\right) \end{aligned}$$

and so

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{3!}f'''(x) + \dots$$

giving

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

neglecting terms of the order two and higher.

The derivative at x is given as the difference between the two values either side of $f(x)$ divided by $2h$.

This is called the *central difference formula* for the derivative of $f(x)$ and because it is the most accurate of the three for small h, it is the one that we shall use in the remainder of the Programme.

Now we need to look at the second derivative. By adding the first two Taylor series expansions in Frame 2 we find that

$f''(x) \approx$ .... neglecting terms of the order ....

4

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$
neglecting terms of the order two and higher

Because

$$\begin{aligned} f(x+h) + f(x-h) &= \left(f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\right) \\ &\quad + \left(f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \dots\right) \\ &= 2\left(f(x) + \frac{h^2}{2!}f''(x) + \frac{h^4}{4!}f^{iv}(x) + \dots\right) \\ &= 2f(x) + h^2 f''(x) + \frac{h^4}{12}f^{iv}(x) + \dots \end{aligned}$$

and so

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{h^2}{12}f^{iv}(x) + \dots$$

Therefore

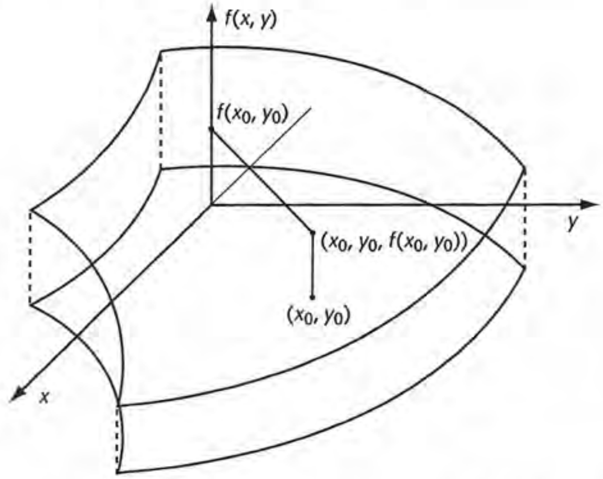
$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$ neglecting terms of the order two and higher

This is the *central difference formula* for the second derivative and, as you see, it possesses the same level of accuracy as the central difference formula for the first derivative.

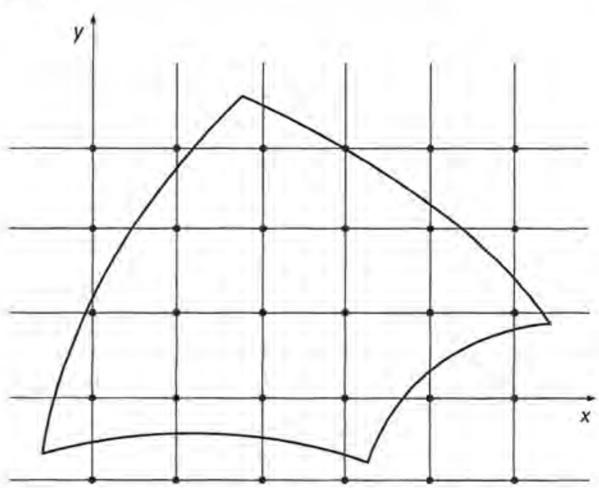
Functions of two real variables

5

A function of two real variables $f(x, y)$ is graphically represented as a surface in three-dimensional space.



If $f(x, y)$ is single-valued, then to every *domain* point (x, y) there corresponds a single range point $f(x, y)$ and hence a single surface point $(x, y, f(x, y))$. If we know the exact form of $f(x, y)$ then we can compute its value at any domain point (x, y) selected at random. If we do not know the exact form of $f(x, y)$ but we do know that it satisfies a given differential equation then to evaluate $f(x, y)$ numerically we have to be more systematic. What we do is to lay a rectangular grid over the domain and evaluate $f(x, y)$ at the grid points – the points of intersection of the lines parallel with the x-axis and the lines parallel with the y-axis.



▶

In this Programme we shall be considering functions of two real variables that satisfy given differential equations and whose domains are restricted to being rectangular. This restriction avoids many of the problems that occur with arbitrary domain shapes where the grid lines can cross the domain boundary.

Grid values

6

The rectangular domain of the function is overlaid by a grid whose mesh size is of h units in the x direction and k units in the y direction. We shall denote the value of $f(x, y)$ at the ijth grid point as

$$f_{i,j} \equiv f(ih, jk)$$

The values of the expression $f(x, y)$ are required to be found at the grid points as shown:

$$\begin{array}{ccccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & f_{i-1,j+1} & f_{i,j+1} & f_{i+1,j+1} & \cdots \\ \cdots & f_{i-1,j} & f_{i,j} & f_{i+1,j} & \cdots \\ \cdots & f_{i-1,j-1} & f_{i,j-1} & f_{i+1,j-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Notice as you move along the jth row of this table that the value of y is constant at $y_j = y_0 + jk$ for all points on that row. Similarly, as you move up and down the ith column that the value of x is constant at $x_i = x_0 + ih$ for all points in that column. These facts now enable us to define the central difference formulas for the partial derivatives of $f(x, y)$.

The first partial derivative of $f(x, y)$ with respect to the variable x is obtained by differentiating $f(x, y)$ with respect to x whilst keeping the value of the variable y constant. Therefore, as with the ordinary derivative

$\left.\dfrac{\partial f(x, y)}{\partial x}\right|_{ij}$ *is equal to the difference between the two adjacent values of $f(x, y)$ in the x-direction divided by twice the mesh size in the x-direction.*

That is

$$\left.\frac{\partial f(x, y)}{\partial x}\right|_{-ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$

This is the central difference formula for the partial derivative with respect to x. Similarly, the central difference formula for the partial derivative with respect to y is

$$\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = \dots\dots\dots\dots$$

7

$$\boxed{\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}}$$

Because

$\left.\dfrac{\partial f(x, y)}{\partial y}\right|_{ij}$ *is equal to the difference between the two adjacent values of $f(x, y)$ in the y-direction divided by twice the mesh size in the y-direction.*

That is

$$\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

Let's try an example so that we can put all this information together.

Example 1

Find the solution to $3\dfrac{\partial f(x, y)}{\partial x} - 4\dfrac{\partial f(x, y)}{\partial y} = 0$, for $0 \le x \le 1$ and $0 \le y \le 1$ given that the boundary conditions are

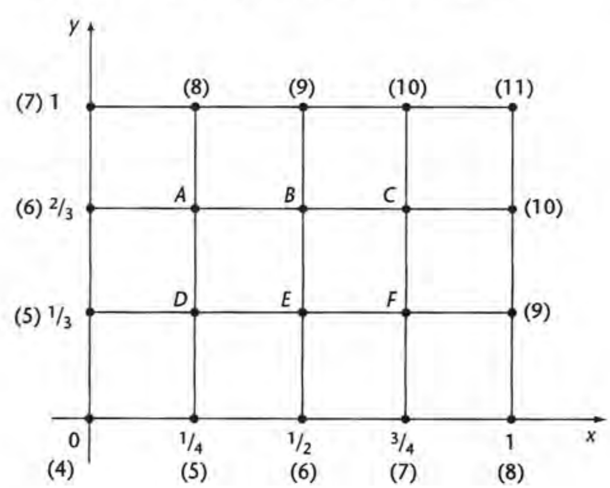
$f(x, 0) = 4x + 4$
$f(x, 1) = 4x + 7$
$f(0, y) = 3y + 4$
$f(1, y) = 3y + 8$

for a mesh of size 1/4 in the x-direction and of size 1/3 in the y-direction.

Next frame

8

The first thing we must do is to make a reasonable drawing of the domain of the function with the grid overlaid. The domain of $f(x, y)$ is the square of side length 1 as shown in the diagram.



Overlaid on the function domain in the x–y plane is a mesh of grid points. The values of $f(x, y)$ that we can compute directly from the boundary conditions are shown in brackets. For example, from $f(x, 0) = 4x + 4$ we obtain $f(1/4, 0) = 5$, $f(1/2, 0) = 6$, $f(3/4, 0) = 7$ and $f(1, 0) = 8$. From $f(1, y) = 3y + 8$ we obtain $f(1, 0) = 8$, $f(1, 1/3) = 9$, $f(1, 2/3) = 10$ and $f(1, 1) = 11$. Notice that the value found at $f(1, 0) = 8$ using $f(x, 0) = 4x + 4$ is the same as the value found using $f(1, y) = 3y + 8$, as of course it must be. The values of $f(x, y)$ that we have to determine are labelled A to F.

The second part of the procedure is to find the central difference formula that describes the differential equation:

We have $\left.\dfrac{\partial f(x, y)}{\partial x}\right|_{ij} = \dfrac{f_{i+1,j} - f_{i-1,j}}{2h} = 2(f_{i+1,j} - f_{i-1,j})$ because $h = 1/4$

$$\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k} = 1{\cdot}5(f_{i,j+1} - f_{i,j-1}) \text{ because } k = 1/3$$

Therefore

$$3\frac{\partial f(x, y)}{\partial x} - 4\frac{\partial f(x, y)}{\partial y} = 0 \text{ becomes } \ldots\ldots\ldots\ldots$$

9

$$\boxed{6(f_{i+1,j} - f_{i-1,j}) - 6(f_{i,j+1} - f_{i,j-1}) = 0}$$

Because

$$3\frac{\partial f(x, y)}{\partial x} - 4\frac{\partial f(x, y)}{\partial y} = 0 \text{ evaluated at the } ij\text{th grid point is}$$

$$3\left.\frac{\partial f(x, y)}{\partial x}\right|_{ij} - 4\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = 0$$

which is

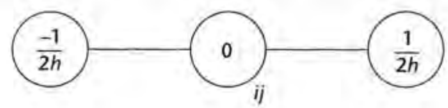
$$3 \times 2(f_{i+1,j} - f_{i-1,j}) - 4 \times 1{\cdot}5(f_{i,j+1} - f_{i,j-1}) = 0, \text{ that is}$$
$$6(f_{i+1,j} - f_{i-1,j}) - 6(f_{i,j+1} - f_{i,j-1}) = 0$$

Computational molecules

The value of the first derivative with respect to x at the point (x_i, y_j) on the grid overlaying the function domain is found by evaluating the right-hand side of the equation

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} = \frac{-f_{i-1,j} + f_{i+1,j}}{2h}$$

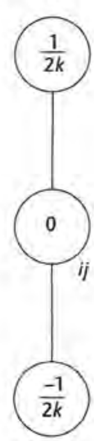
and this process is repeated for every grid point in the function domain. We can construct a graphic template to assist us in this process:



The three circles in a row are used to calculate the contribution of three adjacent row members to the equation. If the circle labelled ij is laid over the ijth grid point then the derivative at that point is given by multiplying the value of the function at the $i-1, j$ grid point (one to the left) by $-1/2h$ and adding the product of the value of the function at the $i+1, j$ grid point (one to the right) by $1/2h$. The number 0 in the centre circle means that we multiply $f_{i,j}$ by zero because it does not enter into the formula. This template is called a *computational molecule*. The horizontal structure reflects the fact that we are evaluating along a row. By a similar reasoning the first derivative with respect to y at the ijth grid point is

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

and this is represented by the computational molecule:



The vertical structure reflects the fact that we are evaluating up and down a column.

By combining such computational molecules we can construct a composite molecule that represents the entire differential equation. For example, the partial differential equation

$$a\frac{\partial f(x, y)}{\partial x} + b\frac{\partial f(x, y)}{\partial y} = c$$

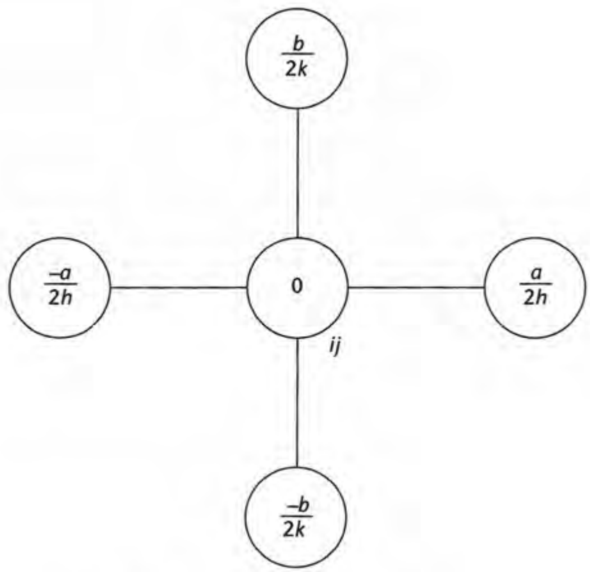
evaluated at the *ij*th grid point is

$$a\left.\frac{\partial f(x, y)}{\partial x}\right|_{ij} + b\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = c$$

and is represented by the central difference formula

$$\frac{a}{2h}(f_{i+1,j} - f_{i-1,j}) + \frac{b}{2k}(f_{i,j+1} - f_{i,j-1}) = c$$

which is in turn represented by the composite computational molecule:



So the equation $3\frac{\partial f(x, y)}{\partial x} - 4\frac{\partial f(x, y)}{\partial y} = 0$ which is represented by the finite difference formula

$$6(f_{i+1,j} - f_{i-1,j}) - 6(f_{i,j+1} - f_{i,j-1}) = 0$$

has the computational molecule ....

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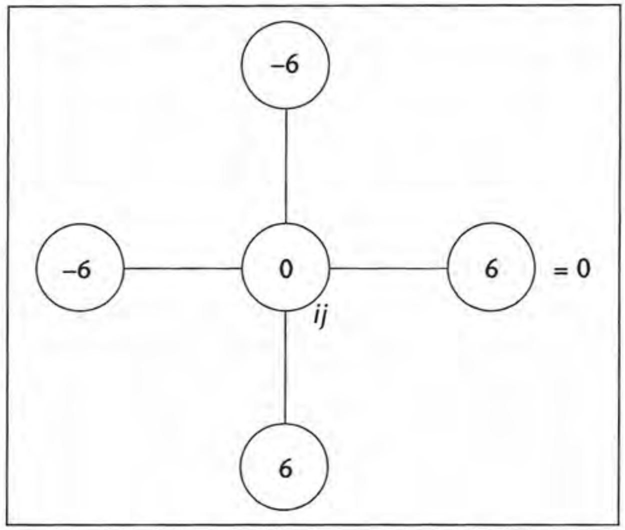

We now place the centre of the molecule, in turn, on each of the grid points at which we need to find the value of $f(x, y)$:

On A	$-36 - 48 + 6B + 6D = 0$
On B	$-6A - 54 + 6C + 6E = 0$
On C	....
On D	....
On E	....
On F	....

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On A	$-36 - 48 + 6B + 6D = 0$
On B	$-6A - 54 + 6C + 6E = 0$
On C	$-6B - 60 + 60 + 6F = 0$
On D	$-30 - 6A + 6E + 30 = 0$
On E	$-6D - 6B + 6F + 36 = 0$
On F	$-6E - 6C + 54 + 42 = 0$

We now have six simultaneous linear equations in six unknowns.

These can be written in matrix form as ....

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$$\begin{pmatrix} 0 & 6 & 0 & 6 & 0 & 0 \\ -6 & 0 & 6 & 0 & 6 & 0 \\ 0 & -6 & 0 & 0 & 0 & 6 \\ -6 & 0 & 0 & 0 & 6 & 0 \\ 0 & -6 & 0 & -6 & 0 & 6 \\ 0 & 0 & -6 & 0 & -6 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 84 \\ 54 \\ 0 \\ 0 \\ -36 \\ -96 \end{pmatrix}$$

That is: $\mathbf{Ax} = \mathbf{b}$ with solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

There are many ways to derive the inverse matrix $\mathbf{A}^{-1}$, many of them time consuming and prone to arithmetic error. An efficient method in terms of time and accuracy is to use a spreadsheet, provided of course that the spreadsheet has the appropriate functionality. Here we shall use the *Microsoft Excel* spreadsheet which possesses matrix functions. If your spreadsheet does not have these functions then you are referred to Programme 12, Matrix algebra.

If you do possess the *Microsoft Excel* spreadsheet then follow the instructions in the next frame.

Next frame

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1 Open your spreadsheet.
2 Place the cell highlight in cell A1 and then enter the values of matrix **A** into the cells A1 to F6.
3 Place the cell highlight in cell H1 and then enter the values of matrix **b** into the cells H1 to H6.
4 Place the cell highlight in cell A8 and drag the mouse to highlight the block of cells A8 to F13 – this is where the inverse of **A** is going to go.
5 With this block of cells highlighted, type the function:

   =MINVERSE(A1:F6) and then press the three keys **Ctrl-Shift-Enter** together

   As you type, the function is entered into cell A8 and when you press the **Ctrl-Shift-Enter** keys together the block of cells A8 to F13 fills with entries. This block of cells is the inverse matrix $\mathbf{A}^{-1}$. (*Note*: You must remember to press the three keys **Ctrl-Shift-Enter** together. If you just press **Enter** it will not work.)

   MINVERSE(array) is the *Excel* function that computes the inverse of the square matrix denoted by **array**.

6 Place the cell highlight in cell H8 and drag the mouse to highlight the block of cells H8 to H13 – this is where the solution **x** is going to go.
7 With this block of cells highlighted type the function:

   =MMULT(A8:F13, H8:H13) and then press the three keys **Ctrl-Shift-Enter** together

   MMULT(array1,array2) is the *Excel* function that multiplies the two matrices denoted by **array1** and **array2**.

▶

As you type, the function is entered into cell H8 and when you press the **Ctrl-Shift-Enter** keys together the block of cells H8 to H13 fills with entries. This block of cells is the product matrix $\mathbf{A}^{-1}\mathbf{b}$, that is, the solution matrix $\mathbf{x}$.

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \\ 6 \\ 7 \\ 8 \end{pmatrix}$$

These values are identical to the values found from the exact solution which is $f(x, y) = 4x + 3y + 4$.

Next frame

Summary of procedures

15

The procedure to solve a first-order partial differential equation requires a number of steps to be completed in a certain order, and the following list describes the sequence:

1 Draw the domain of the function with the grid overlaid.
2 On the drawing enter the values of $f(x, y)$ that can be obtained from the boundary conditions.

Put these values in brackets so that they will be easily distinguished from the x- and y-values on the axes.

3 Label the grid points at which $f(x, y)$ is to be evaluated with capital letters.
4 Construct the central difference equation that represents the numerical approximation to the partial differential equation.
5 Construct the computational molecule for this equation.
6 Lay the centre of the molecule on each of the lettered grid points in turn and derive a set of simultaneous linear equations – the unknowns being represented by the letters at the grid points.
7 Write the simultaneous linear equations in matrix form $\mathbf{Ax} = \mathbf{b}$.
8 Find the inverse matrix $\mathbf{A}^{-1}$ and compute the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Now try one yourself. Just follow the procedure in order and you should have no problems.

▶

Example 2

The solution to $x\dfrac{\partial f(x, y)}{\partial x} - y\dfrac{\partial f(x, y)}{\partial y} = 0$, for $0 \le x \le 1$ and $0 \le y \le 1$ given that

$$f(x, 0) = 2$$
$$f(x, 1) = x + 2$$
$$f(0, y) = 2$$
$$f(1, y) = y + 2$$

for a mesh of $1/4$ in the x-direction and $1/3$ in the y-direction is: ....

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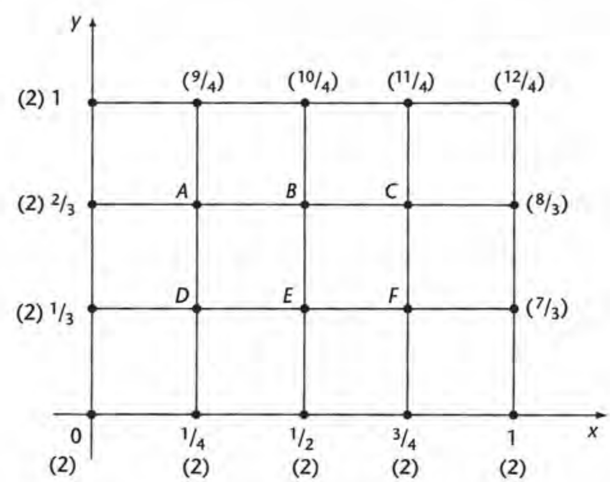
$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 2{\cdot}166\ldots \\ 2{\cdot}33\ldots \\ 2{\cdot}5 \\ 2{\cdot}0833\ldots \\ 2{\cdot}166\ldots \\ 2{\cdot}25 \end{pmatrix} = \begin{pmatrix} 13/6 \\ 7/3 \\ 5/2 \\ 25/12 \\ 13/6 \\ 9/4 \end{pmatrix}$$

Because

The domain of the function $f(x, y)$ with the overlaid grid looks as follows:



where the numbers at the grid points in brackets are the values of $f(x, y)$ obtained by applying the boundary conditions and the letters $A \ldots F$ represent the values of $f(x, y)$ that we have yet to determine.

The central difference formulas for the two first partial derivatives of $f(x, y)$ are

$$\left.\frac{\partial f(x, y)}{\partial x}\right|_{ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} = 2(f_{i+1,j} - f_{i-1,j}) \text{ because } h = 1/4$$

$$\left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k} = 1{\cdot}5(f_{i,j+1} - f_{i,j-1}) \text{ because } k = 1/3$$

Therefore

$$x\frac{\partial f(x, y)}{\partial x} - y\frac{\partial f(x, y)}{\partial y} = 0 \text{ becomes } \ldots\ldots\ldots\ldots$$

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$$\boxed{2(x_i f_{i+1,j} - x_i f_{i-1,j}) - 1{\cdot}5(y_j f_{i,j+1} - y_j f_{i,j-1}) = 0}$$

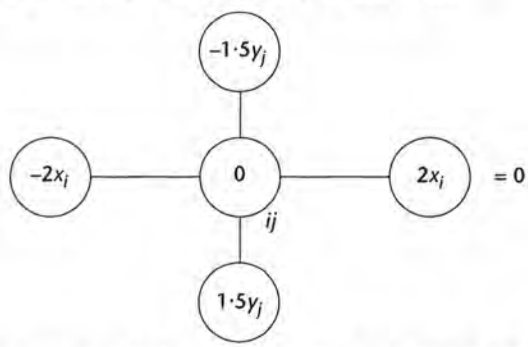
Because

$$x\frac{\partial f(x, y)}{\partial x} - y\frac{\partial f(x, y)}{\partial y} = 0$$

is written using the central difference formulas as

$$\begin{aligned} &2x_i(f_{i+1,j} - f_{i-1,j}) - 1{\cdot}5y_j(f_{i,j+1} - f_{i,j-1}) \\ &= 2(x_i f_{i+1,j} - x_i f_{i-1,j}) - 1{\cdot}5(y_j f_{i,j+1} - y_j f_{i,j-1}) = 0 \end{aligned}$$

This has the following computational molecule:



Placing the centre of the molecule, in turn, on each of the grid points that we need to evaluate, we obtain the six simultaneous equations:

On A at $\left(\frac{1}{4}, \frac{2}{3}\right)$: $-2\left(\frac{1}{4}\right)(2) - \frac{3}{2}\left(\frac{2}{3}\right)\left(\frac{9}{4}\right) + 2\left(\frac{1}{4}\right)B + \frac{3}{2}\left(\frac{2}{3}\right)D = 0$

On B at $\left(\frac{1}{2}, \frac{2}{3}\right)$: $-2\left(\frac{1}{2}\right)A - \frac{3}{2}\left(\frac{2}{3}\right)\left(\frac{10}{4}\right) + 2\left(\frac{1}{2}\right)C + \frac{3}{2}\left(\frac{2}{3}\right)E = 0$

On C at $\left(\frac{3}{4}, \frac{2}{3}\right)$: $-2\left(\frac{3}{4}\right)B - \frac{3}{2}\left(\frac{2}{3}\right)\left(\frac{11}{4}\right) + 2\left(\frac{3}{4}\right)\left(\frac{8}{3}\right) + \frac{3}{2}\left(\frac{2}{3}\right)F = 0$

On D at $\left(\frac{1}{4}, \frac{1}{3}\right)$: $-2\left(\frac{1}{4}\right)(2) - \frac{3}{2}\left(\frac{1}{3}\right)A + 2\left(\frac{1}{4}\right)E + \frac{3}{2}\left(\frac{1}{3}\right)(2) = 0$

On E at $\left(\frac{1}{2}, \frac{1}{3}\right)$: $-2\left(\frac{1}{2}\right)D - \frac{3}{2}\left(\frac{1}{3}\right)B + 2\left(\frac{1}{2}\right)F + \frac{3}{2}\left(\frac{1}{3}\right)(2) = 0$

On F at $\left(\frac{3}{4}, \frac{1}{3}\right)$: $-2\left(\frac{3}{4}\right)E - \frac{3}{2}\left(\frac{1}{3}\right)C + 2\left(\frac{3}{4}\right)\left(\frac{7}{3}\right) + \frac{3}{2}\left(\frac{1}{3}\right)(2) = 0$

These six equations can be simplified as …………

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On A	$B/2 + D = 13/4$
On B	$-A + C + E = 10/4$
On C	$-3B/2 + F = -5/4$
On D	$-A/2 + E/2 = 0$
On E	$-B/2 - D + F = -1$
On F	$-C/2 - 3E/2 = -9/2$

These six simultaneous linear equations can be expressed in matrix form as …………

19

$$\begin{pmatrix} 0 & 0{\cdot}5 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1{\cdot}5 & 0 & 0 & 0 & 1 \\ -0{\cdot}5 & 0 & 0 & 0 & 0{\cdot}5 & 0 \\ 0 & -0{\cdot}5 & 0 & -1 & 0 & 1 \\ 0 & 0 & -0{\cdot}5 & 0 & -1{\cdot}5 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 13/4 \\ 10/4 \\ -5/4 \\ 0 \\ -1 \\ -9/2 \end{pmatrix}$$

That is

$\mathbf{Ax} = \mathbf{b}$ with solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Inverting the matrix $\mathbf{A}$ we find that

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 2{\cdot}166\ldots \\ 2{\cdot}3\ldots \\ 2{\cdot}5 \\ 2{\cdot}0833\ldots \\ 2{\cdot}166\ldots \\ 2{\cdot}25 \end{pmatrix} = \begin{pmatrix} 13/6 \\ 7/3 \\ 5/2 \\ 25/12 \\ 13/6 \\ 9/4 \end{pmatrix}$$

which is identical to the values found from the exact solution $f(x, y) = xy + 2$.

Next frame

Derivative boundary conditions

20

The process of solving a differential equation, either ordinary or partial, involves using indefinite integration and each time we integrate we produce an integration constant. For a differential equation to have a complete solution, where all the integration constants are evaluated, the differential equation must be accompanied by a set of conditions that are sufficient to do this.

If the differential equation involves time t then it is natural for these ▶

conditions to give values of the function and its derivatives at time $t = 0$. Such conditions are known as *initial conditions* and we have met these before when we studied the Laplace transform, for example. Other conditions, like the conditions we met in the previous two examples, are called *boundary conditions* because they gave the values of the function on the boundary of the function domain. We now consider boundary conditions in the form of derivatives normal to the boundary and this we do in the following example.

Example 3

Find the solution to $4\dfrac{\partial f(x, y)}{\partial x} + 2\dfrac{\partial f(x, y)}{\partial y} = 3$, for $0 \le x \le 1$ and $0 \le y \le 1$ given that the boundary conditions are

$$f(x, 0) = f(x, 1) = f(0, y) = 10$$

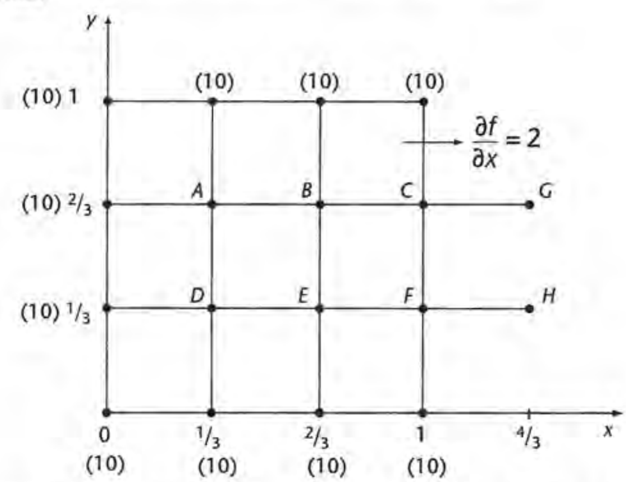
$$\text{and} \quad \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = 2$$

for a mesh of size $1/3$ in both the x-direction and the y-direction.

Next frame

21

The domain of $f(x, y)$ is the square of side length 1 as shown in the diagram:



Overlaid on the function domain in the x–y plane is a mesh of grid points. Because the boundary condition relating to the side $x = 1$ is in the form of a derivative normal to the side, *we extend the grid over the boundary of the function domain by adding two additional points outside the domain* and distant $1/3$ from it, as shown in the figure.

The values of $f(x, y)$ that we can compute from the boundary conditions alone are shown in brackets. The values of $f(x, y)$ that we have to determine are labelled A to F and we shall need the two additional points G and H outside the domain of $f(x, y)$ to do this. ▶

The second part of the procedure is to find the central difference formula that describes the differential equation:

We have $\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{ij} = \dfrac{f_{i+1,j} - f_{i-1,j}}{2h} = 1{\cdot}5\left(f_{i+1,j} - f_{i-1,j}\right)$

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k} = 1{\cdot}5\left(f_{i,j+1} - f_{i,j-1}\right)$$

because both h and $k = 1/3$

Therefore

$$4\frac{\partial f(x,y)}{\partial x} + 2\frac{\partial f(x,y)}{\partial y} = 3 \text{ becomes} \ldots\ldots\ldots\ldots$$

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$$\boxed{6\left(f_{i+1,j} - f_{i-1,j}\right) + 3\left(f_{i,j+1} - f_{i,j-1}\right) = 3}$$

Because

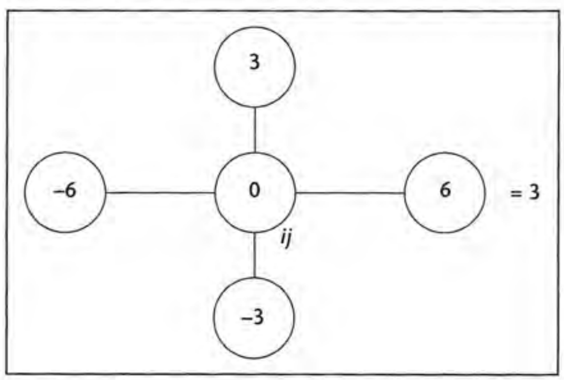
$4\dfrac{\partial f(x,y)}{\partial x} + 2\dfrac{\partial f(x,y)}{\partial y} = 3$ can be written as

$4 \times 1{\cdot}5\left(f_{i+1,j} - f_{i-1,j}\right) + 2 \times 1{\cdot}5\left(f_{i,j+1} - f_{i,j-1}\right) = 3$, that is

$6\left(f_{i+1,j} - f_{i-1,j}\right) + 3\left(f_{i,j+1} - f_{i,j-1}\right) = 3$

This has the computational molecule ....

23



We now place the centre of the molecule, in turn, on each of the grid points that we need to evaluate:

On A	$-60 + 30 + 6B - 3E = 3$
On B	$-6A + 30 + 6C - 3E = 3$
On C	....
On D	....
On E	....
On F	....

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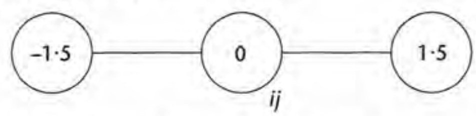
On A	$-60+30+6B-3E=3$
On B	$-6A+30+6C-3E=3$
On C	$-6B+30+6G-3F=3$
On D	$-60+3A+6E-30=3$
On E	$-6D+3B+6F-30=3$
On F	$-6E+3C+6H-30=3$

At the boundary $x=1$ the boundary condition $\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{x=1}=2$ can be written using the central difference formula as

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{\substack{x=1\\y=y_j}}=\frac{f_{i+1,j}-f_{i-1,j}}{2h}=1{\cdot}5(f_{i+1,j}-f_{i-1,j})=2$$

which has the computational molecule:



We now place the centre of this molecule, in turn, on each of the grid points C and F to obtain

On C $\quad -1{\cdot}5B+1{\cdot}5G=2$

On F $\quad$ ....

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On C	$-1{\cdot}5B+1{\cdot}5G=2$
On F	$-1{\cdot}5E+1{\cdot}5H=2$

We can now use these last two equations either to eliminate the points G and H from the six equations in Frame 24 or to form an 8×8 system. We shall eliminate the points G and H to obtain the six equations, with the constant on the right-hand side as

....

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On A	$6B-3E=33$
On B	$-6A+6C-3E=-27$
On C	$-3F=-35$
On D	$3A+6E=93$
On E	$-6D+3B+6F=33$
On F	$3C=25$

These six simultaneous linear equations can be expressed in matrix form as ....

27

$$\begin{pmatrix} 0 & 6 & 0 & 0 & -3 & 0 \\ -6 & 0 & 6 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \\ 3 & 0 & 0 & 0 & 6 & 0 \\ 0 & 3 & 0 & -6 & 0 & 6 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 33 \\ -27 \\ -35 \\ 93 \\ 33 \\ 25 \end{pmatrix}$$

That is

$\mathbf{Ax} = \mathbf{b}$ with solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Inverting the matrix $\mathbf{A}^{-1}$ we find that $\mathbf{x} = \ldots\ldots\ldots\ldots$

28

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 6{\cdot}777\ldots \\ 11{\cdot}555\ldots \\ 8{\cdot}333\ldots \\ 11{\cdot}9444\ldots \\ 12{\cdot}111\ldots \\ 11{\cdot}666\ \ldots \end{pmatrix} = \begin{pmatrix} 61/9 \\ 104/9 \\ 25/3 \\ 215/18 \\ 109/9 \\ 35/3 \end{pmatrix}$$

Next frame

Second-order partial differential equations

29

The most general form of a second-order partial differential equation is

$$a(x,y)\frac{\partial^2 f}{\partial x^2} + b(x,y)\frac{\partial^2 f}{\partial x \partial y} + c(x,y)\frac{\partial^2 f}{\partial y^2} + d(x,y)\frac{\partial f}{\partial x} + e(x,y)\frac{\partial f}{\partial y} + g(x,y) = 0$$

Three types of equation are of particular interest because they feature so prominently in engineering and science.

Elliptic equations

If $b^2 - 4ac < 0$ the partial differential equation is called an *elliptic* equation. Such equations arise out of steady-state problems as occur in potential or flow theory. Two examples are

Poisson's equation

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = g(x,y)$$

Laplace's equation

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0$$

In both cases $a = 1$, $b = 0$ and $c = 1$ and so $b^2 - 4ac < -4$.

▶

Hyperbolic equations

If $b^2 - 4ac > 0$ the partial differential equation is called an *hyperbolic* equation. Such equations arise out of vibrational and radiative problems as occur in wave mechanics. An example is

The wave equation

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} = \frac{1}{\kappa^2}\frac{\partial^2 \phi(x,t)}{\partial t^2}$$

Here $a = 1$, $b = 0$ and $c = -\dfrac{1}{\kappa^2}$ and so $b^2 - 4ac > 0$.

Parabolic equations

If $b^2 - 4ac = 0$ the partial differential equation is called a *parabolic* equation. Such equations arise out of transient flow problems as occur in conduction or consolidation. An example is

The consolidation (or heat conduction) equation

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \phi(x,t)}{\partial t}$$

Here $a = 1$, $b = 0$ and $c = 0$ and so $b^2 - 4ac = 0$.

In the equations above a, b and c are constant but in the general case they depend on x and y and so a given equation may change from one type to another within the same domain.

Next frame

Second partial derivatives

30

In Frame 4 we found that for a function of a single real variable $f(x)$ the central difference formula approximating the second derivative was

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

The second derivative at x is given as the sum of the two adjacent values less twice the value at the point, all divided by h^2.

If we apply this to a function of two real variables $f(x, y)$ and use $f_{i,j} \equiv f(ih, jk)$ to represent the value of $f(x, y)$ at the point (ih, jk) then the central difference formulas for the second partial derivatives with respect to x and y are seen to be ....

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$$\left.\frac{\partial^2 f(x,y)}{\partial x^2}\right|_{ij} \approx \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}$$
$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{ij} \approx \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2}$$

Because

The second derivative at x_i is given as the sum of the two adjacent values on the jth row less twice the value at x_i, all divided by the cell width squared – h^2, and so

$$\left.\frac{\partial^2 f(x,y)}{\partial x^2}\right|_{ij} \approx \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}$$

The second derivative at y_j is given as the sum of the two adjacent values in the jth column less twice the value at y_j, all divided by the cell height squared – k^2, and so

$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{ij} \approx \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2}$$

We are now ready to consider the construction of central difference formulas for second-order partial differential equations. We shall proceed by example.

Example 4

Given a grid with mesh size $h = k = 1/3$, find a numerical solution to the equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0 \text{ for } 0 \le x \le 1,\ 0 \le y \le 1, \text{ given that}$$

$$f(x,0) = f(x,1) = 5 - 3x$$

$$f(0,y) = 9y^2 - 9y + 5 \quad \text{and}$$

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{x=1} = -6$$

The domain with the grid overlaid is ....

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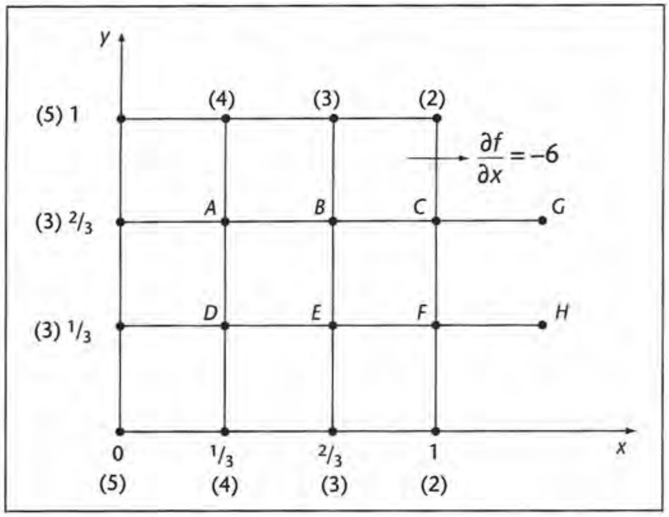

The solution is to be evaluated at the grid points A to F – the external grid points G and H are inserted to accommodate the derivative boundary condition. The numbers in brackets are the values of $f(x, y)$ as found from the boundary conditions.

The central difference formula that represents the partial differential equation is ....

33

$$\boxed{9\left(f_{i+1,j}+f_{i,j+1}-4f_{i,j}+f_{i-1,j}+f_{i,j-1}\right)}$$

Because

$$\left.\frac{\partial^2 f(x,y)}{\partial x^2}\right|_{ij}+\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{ij}\approx\frac{f_{i+1,j}-2f_{i,j}+f_{i-1,j}}{h^2}+\frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{k^2}$$
$$=9\left(f_{i+1,j}-2f_{i,j}+f_{i-1,j}\right)+9\left(f_{i,j+1}-2f_{i,j}+f_{i,j-1}\right)$$
$$=9\left(f_{i+1,j}+f_{i,j+1}-4f_{i,j}+f_{i-1,j}+f_{i,j-1}\right)$$

From this we can construct the computational molecule for this differential equation as ....

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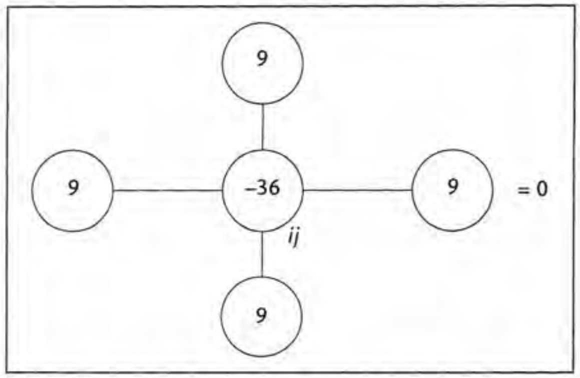
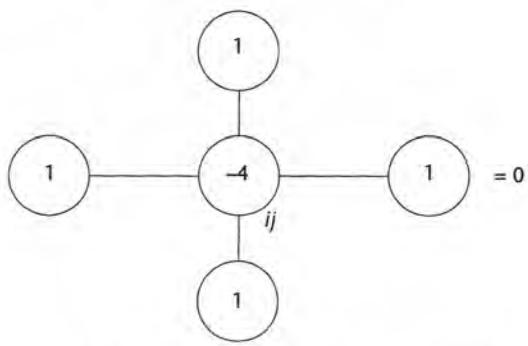


If we applied this computational molecule to the grid points A to F then the six simultaneous linear equations that result would all have a common factor of 9 arising from the 9 in the molecule. If we divided every equation by 9 to remove this common factor we would not change the overall validity of the equations. So, to make the computation simpler we divide each term in the computational molecule by 9 and use the resulting molecule:



We now proceed as we have done before. Laying the centre of the computational molecule on each grid point in turn gives the six simultaneous linear equations:

At *A* $3+4+B+D-4A=0$

At *B* . . . .

At *C* . . . .

At *D* . . . .

At *E* . . . .

At *F* . . . .

35

At A	$3+4+B+D-4A=0$
At B	$A+3+C+E-4B=0$
At C	$B+2+G+F-4C=0$
At D	$3+A+E+4-4D=0$
At E	$D+B+F+3-4E=0$
At F	$E+C+H+2-4F=0$

We now apply the derivative boundary condition at the grid points C and F by using the computational molecule for the first partial derivative with respect to x

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{x=1} = \frac{f_{i+1,j}-f_{i-1,j}}{2h} = \frac{3}{2}(f_{i+1,j}-f_{i-1,j}) = -6$$

This gives ....

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$$\frac{3}{2}(-B+G) = -6$$
$$\frac{3}{2}(-E+H) = -6$$

Because

The computational molecule for the first partial derivative with respect to x is

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{ij} = \frac{-f_{i-1,j}+f_{i+1,j}}{2h} = \frac{3}{2}(-f_{i-1,j}+f_{i+1,j}) \text{ because } h = 1/3$$

Applying this molecule at the boundary points C and F gives the two equations

$$\frac{3}{2}(-B+G) = -6 \text{ so } G = -4+B$$
$$\frac{3}{2}(-E+H) = -6 \text{ so } H = -4+E$$

Substitution of these two equations into the first six eliminates the grid points G and H to produce the six equations in six unknowns.

These are written in matrix form as ....

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$$\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -7 \\ -3 \\ 2 \\ -7 \\ -3 \\ 2 \end{pmatrix}$$

which has solution ....

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$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 28/9 \\ 7/3 \\ 8/9 \\ 28/9 \\ 7/3 \\ 8/9 \end{pmatrix}$$

Next frame

Time-dependent equations

39

Many physical systems have their behaviour modelled by a differential equation. For example, a long thin metal bar of length L, insulated along its length, has its ends maintained at a temperature of 0°C and, at time $t = 0$, the temperature distribution is given by

$$T(x, 0) = x^2 - 2xL + L^2$$

The future distribution of temperature $T(x, t)$ can then be found by solving the partial differential equation (the *heat equation*)

$$\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T(x, t)}{\partial t}$$

subject to the given boundary and initial conditions. The constant $\kappa = \dfrac{K}{\omega}$ is called the *diffusivity* constant where K is the *thermal conductivity* and ω is the *specific heat per unit volume* of the metal that constitutes the rod. Apart from the physical considerations that set up the equation in the first place, the dimensions of κ are $[L^2T^{-1}]$ and are necessary to balance the dimensions on either side of the equation.

If we wished to solve the heat equation numerically as it stands then we would need to know the value of κ, and this would vary depending upon the specific metal used for the bar. We can overcome this problem by absorbing κ using a process of *dimension analysis* when we transform the equation into an equation of the form

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \frac{\partial f(x, t)}{\partial t}$$

where the variables x and t are now dimensionless – they are measured in numbers rather than units of distance and time respectively. How this is done we shall leave to the end of the Programme. For now we are interested in numerically solving such dimensionless equations over a rectangular domain of width 1, and as usual we shall proceed by example.

Next frame

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Example 5

Solve the partial differential equation

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$$

for $0 \le x \le 1$ and $t \ge 0$ where

$$f(0,t) = 1$$

$$f(x,0) = 1 + x \text{ and}$$

$$\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=1} = 0$$

We now have a change in procedure. Hitherto, the first thing we did was to draw the domain of the function with the grid overlaid. We could do this because we knew the step lengths in the x- and y-directions from the beginning. Here, the first thing we must do is to construct the finite difference formula that will represent the differential equation because its structure will dictate the step lengths. We can immediately write down the central difference formula for the second derivative on the left of this equation.

It is ....

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$$\boxed{\left.\frac{\partial^2 f(x,t)}{\partial x^2}\right|_{ij} \approx \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}}$$

To use a central difference formula for the derivative with respect to t would require a knowledge of $f(x, t)$ for values of $t < 0$ and this we do not possess. Consequently, for the derivative with respect to t we use the *forward* difference formula. Do you remember this one?

It is ....

42

$$\boxed{\left.\frac{\partial f(x,t)}{\partial t}\right|_{ij} \approx \frac{f_{i,j+1} - f_{i,j}}{k}}$$

Because

For a function of a single real variable the forward difference formula is given as

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \text{ and so } \left.\frac{\partial f(x,t)}{\partial t}\right|_{ij} \approx \frac{f_{i,j+1} - f_{i,j}}{k}$$

Using these two finite difference formulas we can write down the finite difference representation of the partial differential equation.

The finite difference representation is ....

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$$\boxed{\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2} = \frac{f_{i,j+1} - f_{i,j}}{k}}$$

That is

$$f_{i,j+1} = f_{i,j} + \frac{k}{h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j})$$

It can be shown that there will be no growth of rounding errors when evaluating this equation if $\frac{k}{h^2} \leq \frac{1}{2}$.

In compliance with this condition we shall take $h = 0{\cdot}2$ and $k = 0{\cdot}02$ so that $\frac{k}{h^2} = \frac{1}{2}$. We shall also restrict ourselves to finding solutions for t ranging from 0 to 0·16.

The finite difference equation then reduces to ....

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$$\boxed{f_{i,j+1} = \frac{1}{2}(f_{i-1,j} + f_{i+1,j})}$$

Because

$$f_{i,j+1} = f_{i,j} + \frac{k}{h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) \text{ and so}$$

$$f_{i,j+1} = f_{i,j} + \frac{1}{2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) = \frac{1}{2}(f_{i-1,j} + f_{i+1,j})$$

Notice that this is an equation for stepping forwards in time, so that given the solution is known at $t = 0$ then the solution at $t = k$ can be found from this equation. We can use our spreadsheet to construct the solution from this equation. Open your spreadsheet and

1 Cell A1 enter ***t* \ *x*** to represent the fact that the first column will contain the t-values and the first row the x-values.
2 In cells B1 to H1 enter the values of x from 0 to 1·2 in steps of 0·2.

The column headed 1·2 contains grid points outside the domain of $f(x, t)$ to accommodate the derivative boundary condition.

3 In cells A2 to A10 enter the values of t from 0 to 0·16 in steps of 0·02.
4 In cells B2 to B10 enter the value 1 to represent the boundary condition $f(0, t) = 1$.
5 In cell C2 enter the formula **= 1 + C1** to represent the initial condition $f(x, 0) = 1 + x$. Copy this formula into cells D2 to G2.
6 In cell C3 enter the formula **= 0·5 * (B2 + D2)** to represent the finite difference equation

$$f_{i,j+1} = \frac{1}{2}(f_{i-1,j} + f_{i+1,j})$$

▶

7 Copy the contents of cell C3 into the block of cells C3 to G10.

Because the derivative boundary condition $\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=1} = 0$ is represented by the central difference formula $f_{i+1,j} - f_{i-1,j} = 0$, the values of $f(x, t)$ at the external grid points when $x = 1{\cdot}2$ are equal to the values at the internal grid points when $x = 0{\cdot}8$.

8 In cell H2 enter the formula **=F2** and copy this into cells H3 to H10 to produce the following final display:

$t \backslash x$	0·0	0·2	0·4	0·6	0·8	1·0	1·2
0·00	1·00000	1·20000	1·40000	1·60000	1·80000	2·00000	1·80000
0·02	1·00000	1·20000	1·40000	1·60000	1·80000	1·80000	1·80000
0·04	1·00000	1·20000	1·40000	1·60000	1·70000	1·80000	1·70000
0·06	1·00000	1·20000	1·40000	1·55000	1·70000	1·70000	1·70000
0·08	1·00000	1·20000	1·37500	1·55000	1·62500	1·70000	1·62500
0·10	1·00000	1·18750	1·37500	1·50000	1·62500	1·62500	1·62500
0·12	1·00000	1·18750	1·34375	1·50000	1·56250	1·62500	1·56250
0·14	1·00000	1·17188	1·34375	1·45313	1·56250	1·56250	1·56250
0·16	1·00000	1·17188	1·31250	1·45313	1·50781	1·56250	1·50781

If the diffusion equation in Frame 40 to which this solution refers is taken to represent the temperature distribution along a heated rod then this tableau displays how the temperature is changing both in time and spatially along the rod. Notice how, as the heat diffuses through the rod, the temperature changes faster at points that are further away from the end that is maintained at constant temperature.

Try one yourself. Next frame

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Example 6

The solution of the partial differential equation

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$$

for $0 \le x \le 1$ taken in steps of $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}16$ in steps of $k = 0{\cdot}02$ where

$$f(0, t) = 2$$

$$f(x, 0) = 2 + x \quad \text{and} \quad \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=1} = 0{\cdot}5$$

is ....

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$t\backslash x$	0·0	0·2	0·4	0·6	0·8	1·0	1·2
0·00	2·00000	2·20000	2·40000	2·60000	2·80000	3·00000	3·00000
0·02	2·00000	2·20000	2·40000	2·60000	2·80000	2·90000	3·00000
0·04	2·00000	2·20000	2·40000	2·60000	2·75000	2·90000	2·95000
0·06	2·00000	2·20000	2·40000	2·57500	2·75000	2·85000	2·95000
0·08	2·00000	2·20000	2·38750	2·57500	2·71250	2·85000	2·91250
0·10	2·00000	2·19375	2·38750	2·55000	2·71250	2·81250	2·91250
0·12	2·00000	2·19375	2·37188	2·55000	2·68125	2·81250	2·88125
0·14	2·00000	2·18594	2·37188	2·52656	2·68125	2·78125	2·88125
0·16	2·00000	2·18594	2·35625	2·52656	2·65391	2·78125	2·85391

Because

$$f_{i,j+1} = f_{i,j} + \frac{k}{h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) \text{ and so}$$

$$f_{i,j+1} = f_{i,j} + \frac{1}{2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) = \frac{1}{2}(f_{i-1,j} + f_{i+1,j})$$

We can use our spreadsheet to construct the solution from this equation. Open your spreadsheet and

1 In cell A1 enter **$t\backslash x$** to represent the fact that the first column will contain the t-values and the first row the x-values.

2 In cells B1 to H1 enter the values of x from 0 to 1·2 in steps of 0·2.

The column headed 1·2 contains grid points outside the domain of $f(x, t)$ to accommodate the derivative boundary condition.

3 In cells A2 to A10 enter the values of t from 0 to 0·16 in steps of 0·02.

4 In cells B2 to B10 enter the value 2 to represent the boundary condition $f(0, t) = 2$.

5 In cell C2 enter the formula **=2+C1** to represent the initial condition $f(x, 0) = 2 + x$. Copy this formula into cells D2 to G2.

6 In cell C3 enter the formula to represent the finite difference equation

$$f_{i,j+1} = \frac{1}{2}(f_{i-1,j} + f_{i+1,j})$$

The formula is ....

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=0.5 * (B2 + D2)

7 Copy the contents of cell C3 into the block of cells C3 to G10.

Because the derivative boundary condition $\left.\dfrac{\partial f(x,t)}{\partial x}\right|_{x=1} = 0{\cdot}5$ is represented by the central difference formula $f_{i+1,j} - f_{i-1,j} = 0{\cdot}2$, the values of $f(x, t)$ at the external grid points when $x = 1{\cdot}2$ are equal to

The values at the internal grid points when
$x = \ldots\ldots\ldots\ldots$ plus $\ldots\ldots\ldots\ldots$

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$x = 0{\cdot}8$ plus $0{\cdot}2$

8 In cell H2 enter the formula **= F2 + 0·2** and copy this into cells H3 to H10 to produce the following display:

t\x	0·0	0·2	0·4	0·6	0·8	1·0	1·2
0·00	2·00000	2·20000	2·40000	2·60000	2·80000	3·00000	3·00000
0·02	2·00000	2·20000	2·40000	2·60000	2·80000	2·90000	3·00000
0·04	2·00000	2·20000	2·40000	2·60000	2·75000	2·90000	2·95000
0·06	2·00000	2·20000	2·40000	2·57500	2·75000	2·85000	2·95000
0·08	2·00000	2·20000	2·38750	2·57500	2·71250	2·85000	2·91250
0·10	2·00000	2·19375	2·38750	2·55000	2·71250	2·81250	2·91250
0·12	2·00000	2·19375	2·37188	2·55000	2·68125	2·81250	2·88125
0·14	2·00000	2·18594	2·37188	2·52656	2·68125	2·78125	2·88125
0·16	2·00000	2·18594	2·35625	2·52656	2·65391	2·78125	2·85391

The Crank–Nicolson procedure

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The forward difference formula that we used for the derivative with respect to time is not as accurate as a central difference formula. However, because we do not possess information about $f(x, t)$ for $t < 0$ we were forced to adopt the forward difference formula. To overcome this the Crank–Nicolson procedure makes the assumption that the partial differential equation is satisfied not just at the grid points but also at points in time halfway between two grid points. That is

$$\left.\frac{\partial^2 f(x,t)}{\partial x^2}\right|_{i,j+1/2} = \left.\frac{\partial f(x,t)}{\partial t}\right|_{i,j+1/2}$$

▶

We can then derive a central finite difference formula for the time derivative based on this intermediate point

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{i,j+1/2} = \frac{f_{i,j+1} - f_{i,j}}{2(k/2)} = \frac{f_{i,j+1} - f_{i,j}}{k}$$

Here the two grid points either side of the $i, j+1/2$th point are the i, jth and the $i, j+1$th, each separated by half the grid step in the time direction. You will note that the outcome is identical to the forward difference taken from the i, jth grid point. However, the finite difference formula that represents the partial differential equation will *not* be the same. For the second derivative with respect to x on the left-hand side of the equation we use a finite difference formula that is the average of the central difference formulas for the i, jth grid point and the $i, j+1$th grid point. That is

$$\left.\frac{\partial^2 f(x,t)}{\partial x^2}\right|_{i,j+1/2} = \frac{1}{2}\left(\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2} + \frac{f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}}{h^2}\right)$$

The partial differential equation is then represented by the central difference formula …………

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$$\boxed{\frac{1}{2}\left(\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2} + \frac{f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}}{h^2}\right) = \frac{f_{i,j+1} - f_{i,j}}{k}}$$

That is

$$\begin{aligned} &-f_{i,j+1} + \frac{k}{2h^2}\left(f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}\right) \\ &\qquad = -f_{i,j} - \frac{k}{2h^2}\left(f_{i-1,j} - 2f_{i,j} + f_{i+1,j}\right) \end{aligned}$$

Unlike the previous case there is now no restriction on the value of $\frac{k}{2h^2}$ and different choices of h and k will result in different difference formulas. If we choose $\frac{k}{2h^2} = 1$ this difference formula becomes

$$f_{i-1,j+1} - 3f_{i,j+1} + f_{i+1,j+1} = -f_{i-1,j} + f_{i,j} - f_{i+1,j}$$

So we have three unknown quantities on the left-hand side of this equation given in terms of three known quantities on the right. We shall do an example to see exactly how this procedure operates.

Next frame

Example 7

Use the Crank–Nicolson procedure to solve the partial differential equation

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \frac{\partial f(x, t)}{\partial t}$$

for $0 \leq x \leq 1$ taken in steps of $h = 0{\cdot}25$ and $0 \leq t \leq 0{\cdot}5$ in steps of $k = 0{\cdot}125$ where:

$$f(0, t) = f(1, t) = 0$$
$$f(x, 0) = x(1 - x)$$

We can use our spreadsheet to construct the solution from this equation. Open your spreadsheet and

1. In cell A1 enter ***t**x*** to represent the fact that the first column will contain the t-values and the first row the x-values.
2. In cells B1 to F1 enter the values of x from 0 to 1 in steps of 0·25.
3. In cells A2 to A6 enter the values of t from 0 to 0·5 in steps of 0·125.
4. In cells B2 to B6 enter the value 0 to represent the boundary condition $f(0, t) = 0$.
5. In cells F2 to F6 enter the value 0 to represent the boundary condition $f(1, t) = 0$.
6. In cell C2 enter the formula **=C1*(1–C1)** to represent the boundary condition $f(x, 0) = x(1 - x)$ and copy into cells D2 to F2.

We now want to know the values that are going to go into the block of cells C3 to E6. We shall work on one row at a time and consider cells C3, D3 and E3 – we shall call these values A, B and C respectively.

Applying the central difference formula for the differential equation

$$f_{i-1,j+1} - 3f_{i,j+1} + f_{i+1,j+1} = -f_{i-1,j} + f_{i,j} - f_{i+1,j}$$

we find that by working along rows 2 and 3

From columns B to D: $0 - 3A + B = -0 + 0{\cdot}1875 - 0{\cdot}25$, that is $-3A + B = -0{\cdot}0625$

From columns C to E: $A - 3B + C = -0{\cdot}1875 + 0{\cdot}25 - 0{\cdot}1875$, that is $A - 3B + C = -0{\cdot}125$

From columns D to F: $B - 3C + 0 = -0{\cdot}25 + 0{\cdot}1875 - 0$, that is $B - 3C = -0{\cdot}0625$

These equations have solution

$A = 0{\cdot}044643$, $B = 0{\cdot}071429$ and $C = 0{\cdot}044643$

Enter these values into cells C3 to E3 respectively and repeat the procedure to find the values in cells C4 to E4.

These are C4: ...., D4: .... and E4: ....

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$\boxed{\text{C4: } 0{\cdot}014031, \ \text{D4: } 0{\cdot}015306 \ \text{ and } \ \text{E4: } 0{\cdot}014031}$

Because

From columns B to D: $-3A + B = -0{\cdot}026786$

From columns C to E: $A - 3B + C = -0{\cdot}017857$

From columns D to F: $B - 3C = -0{\cdot}026786$

These equations have solution

$A = 0{\cdot}014031$, $B = 0{\cdot}015306$ and $C = 0{\cdot}014031$

This process is repeated until all the appropriate values have been found, giving the following display:

$t \backslash x$	0·00	0·25	0·50	0·75	1·00
0·000	0·000000	0·187500	0·250000	0·187500	0·000000
0·125	0·000000	0·044643	0·071429	0·044643	0·000000
0·250	0·000000	0·014031	0·015306	0·014031	0·000000
0·375	0·000000	0·002369	0·005831	0·002369	0·000000
0·500	0·000000	0·001328	0·000521	0·001328	0·000000

Try one yourself.

Next frame

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Example 8

Use the Crank–Nicolson procedure to solve the partial differential equation

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \frac{\partial f(x, t)}{\partial t}$$

for $0 \le x \le 1$ taken in steps of $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}2$ in steps of $k = 0{\cdot}04$ where

$$f(0, t) = 2$$
$$f(1, t) = 1$$
$$f(x, 0) = 2 - x^2$$

The very first thing we must do in solving this equation numerically is ....

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Derive the finite difference equation to be used

Because

The Crank–Nicolson procedure tells us that

$$-f_{i,j+1}+\frac{k}{2h^2}\left(f_{i-1,j+1}-2f_{i,j+1}+f_{i+1,j+1}\right)$$
$$=-f_{i,j}-\frac{k}{2h^2}\left(f_{i-1,j}-2f_{i,j}+f_{i+1,j}\right)$$

so for each different ratio $\dfrac{k}{2h^2}$ we have a different finite difference formula.

Here we choose $h=0{\cdot}2$ and $k=0{\cdot}04$ so that $\dfrac{k}{2h^2}=\dfrac{1}{2}$ and the terms in $f_{i,j}$ do not appear.

This gives the finite difference formula …………

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$$\boxed{f_{i-1,j+1}-4f_{i,j+1}+f_{i+1,j+1}=-\left(f_{i-1,j}+f_{i+1,j}\right)}$$

Because

$$-f_{i,j+1}+\frac{k}{2h^2}\left(f_{i-1,j+1}-2f_{i,j+1}+f_{i+1,j+1}\right)$$
$$=-f_{i,j}-\frac{k}{2h^2}\left(f_{i-1,j}-2f_{i,j}+f_{i+1,j}\right)$$

and so

$$-f_{i,j+1}+\frac{1}{2}\left(f_{i-1,j+1}-2f_{i,j+1}+f_{i+1,j+1}\right)=-f_{i,j}-\frac{1}{2}\left(f_{i-1,j}-2f_{i,j}+f_{i+1,j}\right)$$

that is

$$\frac{1}{2}\left(f_{i-1,j+1}-4f_{i,j+1}+f_{i+1,j+1}\right)=-\frac{1}{2}\left(f_{i-1,j}+f_{i+1,j}\right)$$

giving

$$\left(f_{i-1,j+1}-4f_{i,j+1}+f_{i+1,j+1}\right)=-\left(f_{i-1,j}+f_{i+1,j}\right)$$

The complete solution required is …………

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$t\backslash x$	0·00	0·20	0·40	0·60	0·80	1·00
0·000	2·000000	1·960000	1·840000	1·640000	1·360000	1·000000
0·040	2·000000	1·901818	1·767273	1·567273	1·301818	1·000000
0·080	2·000000	1·870083	1·713058	1·513058	1·270083	1·000000
0·120	2·000000	1·847483	1·676875	1·476875	1·247483	1·000000
0·160	2·000000	1·832271	1·65221	1·45221	1·232271	1·000000
0·200	2·000000	1·821919	1·635467	1·435467	1·221919	1·000000

Because

Using your spreadsheet to construct the solution from this equation

1. In cell A1 enter **t\x** to represent the fact that the first column will contain the t-values and the first row the x-values.
2. In cells B1 to G1 enter the values of x from 0 to 1 in steps of 0·2.
3. In cells A2 to A7 enter the values of t from 0 to 0·2 in steps of 0·04.
4. In cells B2 to B7 enter the value 2 to represent the boundary condition $f(0, t) = 2$.
5. In cells G2 to G7 enter the value 1 to represent the boundary condition $f(1, t) = 1$.
6. In cell C2 enter the formula **=2-C1^2** to represent the boundary condition $f(x, 0) = 2 - x^2$ and copy into cells D2 to F2.

We now want to know the values that are going to go into the block of cells C3 to F7. We shall work on one row at a time and consider cells C3, D3, E3 and F3 – we shall call these values A, B, C and D respectively.

Applying the central difference formula for the differential equation

$$f_{i-1,j+1} - 4f_{i,j+1} + f_{i+1,j+1} = -(f_{i-1,j} + f_{i+1,j})$$

Then by working along rows 2 and 3

From columns B to D: $2 - 4A + B = -2 - 1{\cdot}6$, that is

$-4A + B = -5{\cdot}6$

From columns C to E: . . . .

From columns D to F: . . . .

From columns E to G: . . . .

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From columns B to D:	$-4A + B = -5{\cdot}6$
From columns C to E:	$A - 4B + C = -3{\cdot}2$
From columns D to F:	$B - 4C + D = -2{\cdot}8$
From columns E to G:	$C - 4D = -3{\cdot}4$

Because

From columns B to D: $2 - 4A + B = -2 - 1{\cdot}6$, that is
$-4A + B = -5{\cdot}6$

From columns C to E: $A - 4B + C = -1{\cdot}8 - 1{\cdot}4$, that is
$A - 4B + C = -3{\cdot}2$

From columns D to F: $B - 4C + D = -1{\cdot}6 - 1{\cdot}2$, that is
$B - 4C + D = -2{\cdot}8$

From columns E to G: $C - 4D + 1 = -1{\cdot}4 - 1{\cdot}0$, that is
$C - 4D = -3{\cdot}4$

These equations have solution

$A = \ldots\ldots\ldots\ldots$, $B = \ldots\ldots\ldots\ldots$,
$C = \ldots\ldots\ldots\ldots$ and $D = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned} A &= 1{\cdot}901818 \\ B &= 1{\cdot}767273 \\ C &= 1{\cdot}567273 \\ D &= 1{\cdot}301818 \end{aligned}}$$

Enter these values into cells C3 to F3 respectively and repeat the procedure to find the values for cells C4 to F4.

These are C4: ...., D4: ....,
E4: .... and F4: ....

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C4:	1·870083
D4:	1·713058
E4:	1·513058
F4:	1·270083

Continuing in this way we find the complete solution as:

$t\backslash x$	0·00	0·20	0·40	0·60	0·80	1·00
0·000	2·000000	1·960000	1·840000	1·640000	1·360000	1·000000
0·040	2·000000	1·901818	1·767273	1·567273	1·301818	1·000000
0·080	2·000000	1·870083	1·713058	1·513058	1·270083	1·000000
0·120	2·000000	1·847483	1·676875	1·476875	1·247483	1·000000
0·160	2·000000	1·832271	1·65221	1·45221	1·232271	1·000000
0·200	2·000000	1·821919	1·635467	1·435467	1·221919	1·000000

Next frame

Dimensional analysis

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The equation of Frame 39

$$\frac{\partial^2 T(x, t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T(x, t)}{\partial t} \quad \text{for } 0 \leq x \leq L \text{ and } t \geq 0$$

models the temperature distribution $T(x, t)$ along a long thin metal bar of length L. Solutions of this equation will produce values for the temperature distant x along the rod $(0 \leq x \leq L)$ at time t. The dimensions of the left- and right-hand sides of this equation due to the derivatives are

$$\left[\frac{\partial^2}{\partial x^2}\right] \equiv [L^{-2}] \text{ and } \left[\frac{\partial}{\partial t}\right] \equiv [T^{-1}]$$

To ensure that the dimensions of the left-hand side are the same as the dimensions of the right-hand side we find that the dimensions of $\dfrac{1}{\kappa}$ are

$$\left[\frac{1}{\kappa}\right] \equiv [L^{-2}T]$$

This then ensures that the equation compares quantities with the same dimension. To solve this equation numerically would require a knowledge of the value of κ which would be different for different problems. To avoid this we transform the equation into a dimensionless form, so ensuring that the variables are measured in numbers and not in any particular dimensional units. We do this as on the following page.

Define new dimensionless variables as: $X = \dfrac{x}{L}$ (so that $0 \le X \le 1$), $\tau = \dfrac{\kappa t}{L^2}$ and define

$U(X, \tau) = T(x[X], t[\tau])$

then

$$\frac{\partial T}{\partial t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\partial U}{\partial \tau} = \frac{\kappa}{L^2}\frac{\partial U}{\partial \tau} \quad \text{and}$$

$$\frac{\partial T}{\partial x} = \frac{\mathrm{d}X}{\mathrm{d}x}\frac{\partial U}{\partial X} = \frac{1}{L}\frac{\partial U}{\partial X}$$

therefore $$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\frac{1}{L}\frac{\partial U}{\partial X} = \frac{\mathrm{d}X}{\mathrm{d}x}\frac{1}{L}\frac{\partial^2 U}{\partial X^2} = \frac{1}{L^2}\frac{\partial^2 U}{\partial X^2}$$

This means that

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\kappa^2}\frac{\partial T(x,t)}{\partial t} \quad \text{becomes}$$

$$\frac{1}{L^2}\frac{\partial^2 U(X,\tau)}{\partial X^2} = \frac{1}{\kappa}\frac{\kappa}{L^2}\frac{\partial U(X,\tau)}{\partial \tau} = \frac{1}{L^2}\frac{\partial U(X,\tau)}{\partial \tau}$$

so $$\frac{\partial^2 U(X,\tau)}{\partial X^2} = \frac{\partial U(X,\tau)}{\partial \tau}$$

is the required equation in dimensionless form.

This now completes the work for this Programme. Read through the **Revision summary** that follows and then check your understanding against the **Can You?** checklist. When you are satisified that you do understand the contents of the Programme, try the **Test exercises**. There are no tricks and you should find them quite straightforward. Finally there are some **Further problems** to give additional practice.

Revision summary 13

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1 *Numerical approximation to derivatives of $f(x)$*

The *forward difference formula*

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{neglecting terms of the order } h$$

The *backward difference formula*

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \quad \text{neglecting terms of the order } h$$

The *central difference formulas*

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad \text{neglecting terms of the order } h^2$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad \text{neglecting terms of the order } h^2.$$

▶

2 *Functions of two real variables*

If $f(x, y)$ is single-valued, then to every domain point (x, y) there corresponds a single range point $f(x, y)$.

Grid values

The rectangular domain of the function is overlaid by a grid whose mesh size is of h units in the x-direction and k units in the y-direction. The value of $f(x, y)$ at the ijth grid point is denoted by

$$f_{i,j} \equiv f(x_0 + ih, y_0 + jk)$$

The values of the expression $f(x, y)$ are required to be found at the grid points

$$\begin{array}{ccccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & f_{i-1,j+1} & f_{i,j+1} & f_{i+1,j+1} & \cdots \\ \cdots & f_{i-1,j} & f_{i,j} & f_{i+1,j} & \cdots \\ \cdots & f_{i-1,j-1} & f_{i,j-1} & f_{i+1,j-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

3 *Central difference formulas for partial derivatives*

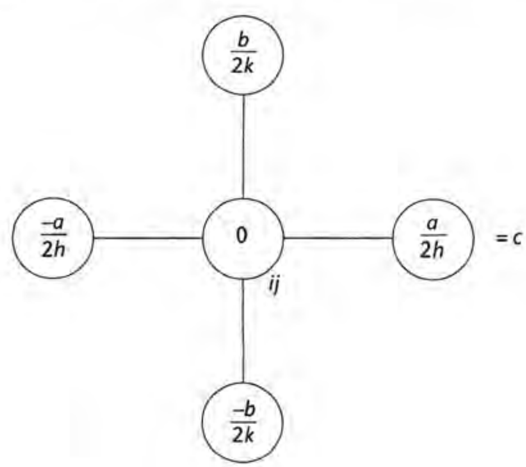
$$\left.\frac{\partial f(x, y)}{\partial x}\right|_{ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} \quad \text{and} \quad \left.\frac{\partial f(x, y)}{\partial y}\right|_{ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

4 *Computational molecules*

The partial differential equation $a\dfrac{\partial f(x, y)}{\partial x} + b\dfrac{\partial f(x, y)}{\partial y} = c$, evaluated at the ijth grid point, is $a\left.\dfrac{\partial f(x, y)}{\partial x}\right|_{ij} + b\left.\dfrac{\partial f(x, y)}{\partial y}\right|_{ij} = c$ and is by the central difference formula

$$\frac{a}{2h}(f_{i+1,j} - f_{i-1,j}) + \frac{b}{2k}(f_{i,j+1} - f_{i,j-1}) = c$$

which is in turn represented by the composite computational molecule:



5 *Numerical solutions*

The solutions are in the form of simultaneous linear equations in that they can be written in matrix form as $\mathbf{Ax} = \mathbf{b}$ with solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Using the *Microsoft Excel* spreadsheet the two functions **MINVERSE(array)** and **MMULT(array1, array2)** are employed.

6 *Derivative boundary conditions*

The grid is extended over the boundary of the function domain by adding additional points outside the domain.

7 *Second-order partial differential equations*

The most general form of a second-order partial differential equation is

$$a(x,y)\frac{\partial^2 f}{\partial x^2} + b(x,y)\frac{\partial^2 f}{\partial x \partial y} + c(x,y)\frac{\partial^2 f}{\partial y^2} + d(x,y)\frac{\partial f}{\partial x} + e(x,y)\frac{\partial f}{\partial y} + g(x,y) = 0$$

Elliptic equations

If $b^2 - 4ac < 0$ then the partial differential equation is called an *elliptic* equation

Hyperbolic equations

If $b^2 - 4ac > 0$ then the partial differential equation is called an *hyperbolic* equation

Parabolic equations

If $b^2 - 4ac = 0$ then the partial differential equation is called a *parabolic* equation.

8 *Second partial derivatives – central difference formulas*

$$\left.\frac{\partial^2 f(x,y)}{\partial x^2}\right|_{ij} \approx \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2}$$

and $$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{ij} \approx \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{k^2}$$

9 *Time-dependent equations*

To use a central difference formula for the derivative with respect to t would require a knowledge of $f(x, t)$ for values of $t < 0$ and this we do not possess. Consequently, for the derivative with respect to t we use the *forward* difference formula

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{ij} \approx \frac{f_{i,j+1} - f_{i,j}}{k}$$

So the partial differential equation $\dfrac{\partial^2 f(x,t)}{\partial x^2} = \dfrac{\partial f(x,t)}{\partial t}$ becomes

$$f_{i,j+1} = f_{i,j} + \frac{k}{h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j})$$

where it can be shown that there will be no growth of rounding errors when evaluating this equation if

$$\frac{k}{h^2} \le \frac{1}{2}.$$

10 *The Crank–Nicolson procedure*

The Crank–Nicolson procedure makes the assumption that the partial differential equation can be satisfied at points in time halfway between two grid points. That is

$$\left.\frac{\partial^2 f(x,t)}{\partial x^2}\right|_{i,j+1/2} = \left.\frac{\partial f(x,t)}{\partial t}\right|_{i,j+1/2}$$

This gives

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{i,j+1/2} = \frac{f_{i,j+1} - f_{i,j}}{2(k/2)} = \frac{f_{i,j+1} - f_{i,j}}{k}$$

$$\left.\frac{\partial^2 f(x,t)}{\partial x^2}\right|_{i,j+1/2} = \frac{1}{2}\left(\frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h^2} + \frac{f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1}}{h^2}\right)$$

So that

$$-f_{i,j+1} + \frac{k}{2h^2}(f_{i-1,j+1} - 2f_{i,j+1} + f_{i+1,j+1})$$
$$= -f_{i,j} - \frac{k}{2h^2}(f_{i-1,j} - 2f_{i,j} + f_{i+1,j})$$

with no restriction on the value of $\dfrac{k}{2h^2}$.

Can You?

Checklist 13

62

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Derive the finite difference formulas for the first partial derivatives of a function of two real variables and construct the central finite difference formula to represent a first-order partial differential equation? 1 to 4

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Draw a rectangular grid of points overlaid on the domain of a function of two real variables and evaluate the function at the boundary grid points? 5 to 9

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Construct the computational molecule for a first-order partial differential equation in two real variables and use the molecule to evaluate the solutions to the equation at the grid points interior to the boundary? 10 and 11

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe the solution as a set of simultaneous linear equations and use matrices to represent them? 12 and 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Invert the coefficient matrix and thereby represent the solution to the partial differential equation as a column matrix? 14 to 19

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Take account of a boundary condition in the form of the derivative normal to the boundary? 20 to 28

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the central finite difference formulas for the second derivatives of a function of two real variables and construct finite difference formulas for second-order partial differential equations? 29 to 38

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the forward difference formula for the first time derivatives in partial differential equations involving time and distance? 39 to 48

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Use the Crank–Nicolson procedure for a partial differential equation involving a first time derivative? 49 to 59

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Appreciate the use of dimensional analysis in the conversion of a partial differential equation modelling a physical system into a dimensionless equation? 60

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 13

63

1 Solve the following equation numerically.

$$5\frac{\partial f(x,y)}{\partial x} - 4\frac{\partial f(x,y)}{\partial y} = -5$$

for $0 \le x \le 1$ with a step length $h = 1/4$ and $0 \le y \le 1$ with a step length $k = 1/3$ where

$f(x,0) = 3x - 4$, $f(x,1) = 3x + 1$, $f(0,y) = 5y - 4$ and $f(1,y) = 5y - 1$

2 Solve the following equation numerically.

$$10\frac{\partial f(x,y)}{\partial x} + 8\frac{\partial f(x,y)}{\partial y} = -10$$

for $0 \le x \le 1$ with a step length $h = 1/3$ and $0 \le y \le 1$ with a step length $k = 1/3$ where

$f(x,0) = 7x + 5$, $f(x,1) = 7x - 5$, $f(0,y) = 5 - 10y$ and $\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{x=1} = 7$

3 Name the type of equation in each of the following.

(a) $2\dfrac{\partial f(x,y)}{\partial x} - 3y\dfrac{\partial f(x,y)}{\partial y} = 4xy$

(b) $\dfrac{\partial f(x,y)}{\partial x} + \dfrac{\partial^2 f(x,y)}{\partial x \partial y} - \dfrac{\partial f(x,y)}{\partial y} = \dfrac{x}{y}$

(c) $\dfrac{\partial^2 f(x,y)}{\partial x^2} - 2\dfrac{\partial^2 f(x,y)}{\partial x \partial y} + \dfrac{\partial^2 f(x,y)}{\partial y^2} = 0$

(d) $\dfrac{\partial}{\partial x}\left[\dfrac{\partial f(x,y)}{\partial x} + \dfrac{\partial f(x,y)}{\partial y}\right] = \dfrac{x^2}{y^3}$

(e) $3\dfrac{\partial^2 f(x,y)}{\partial x^2} - 2\dfrac{\partial^2 f(x,y)}{\partial x \partial y} + \dfrac{\partial^2 f(x,y)}{\partial y^2} = 3xy$

4 Solve the following equation numerically.

$\dfrac{\partial^2 f(x,y)}{\partial x^2} + \dfrac{\partial^2 f(x,y)}{\partial y^2} = -2$ for $0 \le x \le 1$ and $0 \le y \le 1$

with step lengths $h = k = 1/3$ where

$f(x,0) = f(x,1) = x - 2$, $f(0,y) = y^2 - y - 2$ and $\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{x=1} = 1$

▶

5 Solve the following equation numerically using the forward difference approximation for the first derivative with respect to time.

$$\frac{\partial^2 f(x,\,t)}{\partial x^2} = \frac{\partial f(x,\,t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}2$ with step length $k = 0{\cdot}02$ where

$$f(x,\,0) = x^2,\; f(0,t) = 0 \text{ and } \left.\frac{\partial f(x,\,t)}{\partial x}\right|_{x=1} = 0{\cdot}25$$

6 Solve the following equation numerically using the Crank–Nicolson procedure.

$$\frac{\partial^2 f(x,\,t)}{\partial x^2} = \frac{\partial f(x,\,t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}2$ with step length $k = 0{\cdot}04$ where

$$f(x,\,0) = x^2 - x + 1 \text{ and } f(0,t) = f(1,t) = 1$$

Further problems 13

64

1 Solve the following equation numerically.

$$-2\frac{\partial f(x,\,y)}{\partial x} + \frac{\partial f(x,\,y)}{\partial y} = 0$$

for $0 \le x \le 1$ with a step length $h = 1/4$ and $0 \le y \le 1$ with a step length $k = 1/3$ where

$$f(x,\,0) = x - 3,\; f(x,\,1) = x - 1,\; f(0,\,y) = 2y - 3 \text{ and } f(1,\,y) = 2y - 2$$

2 Solve the following equation numerically.

$$9\frac{\partial f(x,\,y)}{\partial x} - 7\frac{\partial f(x,\,y)}{\partial y} = -7$$

for $0 \le x \le 1$ with a step length $h = 1/3$ and $0 \le y \le 1$ with a step length $k = 1/3$ where

$$f(x,\,0) = 7x + 4,\; f(x,\,1) = 7x + 14,\; f(0,\,y) = 10y + 4$$
$$\text{and } f(1,\,y) = 10y + 11$$

3 Solve the following equation numerically.

$$x\frac{\partial f(x,\,y)}{\partial x} + (y+1)\frac{\partial f(x,\,y)}{\partial y} = 0$$

for $0 \le x \le 1$ with a step length $h = 1/3$ and $0 \le y \le 1$ with a step length $k = 1/3$ where

$$f(x,\,0) = x - 1,\; f(x,\,1) = (x-2)/2,\; f(0,\,y) = -1$$
$$\text{and } f(1,\,y) = -y/(y+1)$$

▶

4 Solve the following equation numerically.

$$\frac{\partial f(x, y)}{\partial y} - \frac{\partial f(x, y)}{\partial x} = x^2 + y^2$$

for $0 \leq x \leq 1$ with a step length $h = 1/4$ and $0 \leq y \leq 1$ with a step length $k = 1/3$ where

$$f(x, 0) = 0,\ f(x, 1) = x(x - 1),\ f(0, y) = 0 \text{ and } f(1, y) = y(1 - y)$$

5 Solve the following equation numerically.

$$3\frac{\partial f(x, y)}{\partial x} - 5\frac{\partial f(x, y)}{\partial y} = -4$$

for $0 \leq x \leq 1$ with a step length $h = 1/3$ and $0 \leq y \leq 1$ with a step length $k = 1/3$ where

$$f(x, 0) = 7x + 15,\ f(x, 1) = 7x + 20,\ f(0, y) = 5y + 15$$

$$\text{and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = 7$$

6 Solve the following equation numerically.

$$11\frac{\partial f(x, y)}{\partial x} + 12\frac{\partial f(x, y)}{\partial y} = 19$$

for $0 \leq x \leq 1$ with a step length $h = 1/3$ and $0 \leq y \leq 1$ with a step length $k = 1/3$ where

$$f(x, 0) = 5x + 21,\ f(x, 1) = 5x + 18,\ f(0, y) = 21 - 3y$$

$$\text{and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = 5$$

7 Solve the following equation numerically.

$$2x\frac{\partial f(x, y)}{\partial x} - y\frac{\partial f(x, y)}{\partial y} = 8x^2$$

for $0 \leq x \leq 1$ with a step length $h = 1/3$ and $0 \leq y \leq 1$ with a step length $k = 1/3$ where

$$f(x, 0) = 2x^2 + 4,\ f(x, 1) = 2x^2 - 3x + 4,\ f(0, y) = 4$$

$$\text{and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = 4 - 3y^2$$

8 Solve the following equation numerically.

$$y\frac{\partial f(x, y)}{\partial x} + x\frac{\partial f(x, y)}{\partial y} = x^4 - y^4$$

for $0 \leq x \leq 1$ with a step length $h = 1/3$ and $0 \leq y \leq 1$ with a step length $k = 1/3$ where

$$f(x, 0) = 0,\ f(x, 1) = x(x + 1)(x - 1),\ f(0, y) = 0$$

$$\text{and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = y(3 - y^2)$$

▶

9 Solve the following equation numerically.

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = -4$$

for $0 \le x \le 1$ and $0 \le y \le 1$ with step lengths $h = k = 1/3$ where

$$f(x, 0) = 3x^2,\ f(x, 1) = 3x^2 - 5,\ f(0, y) = -5y^2 \text{ and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = 6$$

10 Solve the following equation numerically.

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 2(x + y)$$

for $0 \le x \le 1$ and $0 \le y \le 1$ with step lengths $h = k = 1/3$ where

$$f(x, 0) = -1,\ f(x, 1) = x^2 + 3x - 1,\ f(0, y) = -1$$

and $\left.\dfrac{\partial f(x, y)}{\partial x}\right|_{x=1} = y^2 + 4y$

11 Solve the following equation numerically.

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = (2 - x^2)\cos y$$

for $0 \le x \le 1$ and $0 \le y \le 1$ with step lengths $h = k = 1/3$ where

$$f(x, 0) = x^2,\ f(x, 1) = 0{\cdot}540302x^2,\ f(0, y) = 0 \text{ and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = 2x\cos y$$

12 Solve the following equation numerically.

$$\frac{\partial^2 f(x, y)}{\partial x^2} - \frac{\partial^2 f(x, y)}{\partial y^2} = 4(x - y)$$

for $0 \le x \le 1$ and $0 \le y \le 1$ with step lengths $h = k = 1/3$ where

$$f(x, 0) = x^3,\ f(x, 1) = (x + 1)(x^2 + 1),\ f(0, y) = y^3$$

and $\left.\dfrac{\partial f(x, y)}{\partial x}\right|_{x=1} = y^2 + 2y + 3$

13 Given the central difference formula

$$\left.\frac{\partial^2 f(x, y)}{\partial x \partial y}\right|_{ij} = \frac{1}{4h^2}\left(f_{i-1,j-1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i+1,j+1}\right)$$

where the step length in both directions is h, construct the computational molecule for this formula.

Solve the equation

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} = 1$$

for $0 \le x \le 1$ and $0 \le y \le 1$ with step lengths $h = 1/3$ where

$$f(x, 0) = 0,\ f(x, 1) = x,\ f(0, y) = 0 \text{ and } \left.\frac{\partial f(x, y)}{\partial x}\right|_{x=1} = y$$

▶

14 Given the central difference formula

$$\left.\frac{\partial^2 f(x,y)}{\partial x \partial y}\right|_{ij} = \frac{1}{4h^2}\left(f_{i-1,j-1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i+1,j+1}\right)$$

where the step length in both directions is h, construct the computational molecule for this formula.

Solve the equation

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = 2(x-y)$$

for $0 \le x \le 1$ and $0 \le y \le 1$ with step lengths $h = 1/3$ where

$$f(x,0) = 0,\ f(x,1) = x(x-1),\ f(0,y) = 0 \text{ and } \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=1} = y(2-y)$$

15 Solve the following equation numerically using the forward difference approximation for the first derivative with respect to time.

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}2$ with a step length $k = 0{\cdot}02$ where

$$f(x,0) = x(x-1),\ f(0,t) = 2t \text{ and } \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=1} = 1$$

16 Solve the following equation numerically using the forward difference approximation for the first derivative with respect to time.

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{0{\cdot}1}\frac{\partial f(x,t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}2$ with a step length $k = 0{\cdot}02$ where

$$f(x,0) = \sin x,\ f(0,t) = 0 \text{ and } \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=1} = 0{\cdot}54e^{-t/10}$$

17 Solve the following equation numerically using the forward difference approximation for the first derivative with respect to time.

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial f(x,t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}2$ with a step length $k = 0{\cdot}02$ where

$$f(x,0) = 3\sin(0{\cdot}64x),\ f(0,t) = 0 \text{ and } \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=1} = 2{\cdot}41e^{-0{\cdot}41t}$$

▶

18 Solve the following equation numerically using the Crank–Nicolson procedure.

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \frac{\partial f(x, t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}2$ and $0 \le t \le 0{\cdot}6$ with a step length $k = 0{\cdot}04$ where

$$f(x, 0) = x^2 + x - 1 \text{ and } f(0, t) = 2t - 1,\ f(1, t) = 1 + 2t$$

19 Solve the following equation numerically using the Crank–Nicolson procedure.

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \frac{\partial f(x, t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}1$ and $0 \le t \le 0{\cdot}14$ with a step length $k = 0{\cdot}02$ where

$$f(x, 0) = 10x(x - 1) \text{ and } f(0, t) = f(1, t) = 20t$$

20 Solve the following equation numerically using the Crank–Nicolson procedure.

$$\frac{\partial^2 f(x, t)}{\partial x^2} = \frac{\partial f(x, t)}{\partial t}$$

for $0 \le x \le 1$ with a step length $h = 0{\cdot}1$ and $0 \le t \le 0{\cdot}6$ with a step length $k = 0{\cdot}04$ where

$$f(x, 0) = 100 \sin \pi x \text{ and } f(0, t) = f(1, t) = 0$$

Multiple integration 1

Frames 1 to 83

Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate double and triple integrals and apply them to the determination of the areas of plane figures and the volumes of solids
- Understand the role of the differential of a function of two or more real variables
- Determine exact differentials in two real variables and their integrals
- Evaluate the area enclosed by a closed curve by contour integration
- Evaluate line integrals and appreciate their properties
- Evaluate line integrals around closed curves within a simply connected region
- Link line integrals to integrals along the x-axis
- Link line integrals to integrals along a contour given in parametric form
- Discuss the dependence of a line integral between two points on the path of integration
- Determine exact differentials in three real variables and their integrals
- Demonstrate the validity and use of Green's theorem

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme 23 Multiple integrals

Introduction

1

The introductory work on double and triple integrals was covered in detail in Programme 23 of *Engineering Mathematics (Fifth Edition)* and another look at the main points before launching forth on the current development could well be worth while.

You will no doubt recognise the following.

1 *Double integrals*

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\,dx\,dy$$

is a double integral and is evaluated from the inside outwards, i.e.

$$\boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\,dx}^{①}\,dy}^{②}$$

A double integral is sometimes expressed in the form

$$\int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x, y)\,dx$$

in which case, we evaluate from the right-hand end, i.e.

$$\int_{y_1}^{y_2} dy\; \boxed{\int_{x_1}^{x_2} f(x, y)\,dx}^{①}$$

then $$\boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\,dx}\,dy}^{②}$$

2 *Triple integrals*

Triple integrals follow the same procedure.

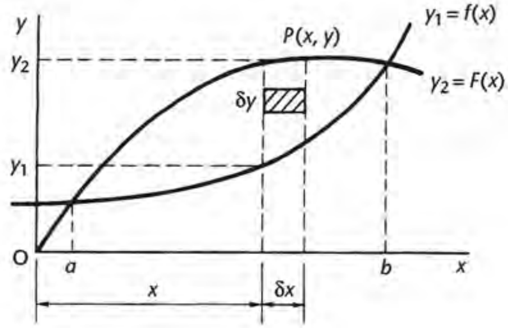
$\displaystyle\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\,dx\,dy\,dz$ is evaluated in the order

$$\boxed{\int_{z_1}^{z_2} \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y,z)\,dx}^{①}\,dy}^{②}\,dz}^{③}$$

▶

3 *Applications*

(a) *Areas of plane figures*



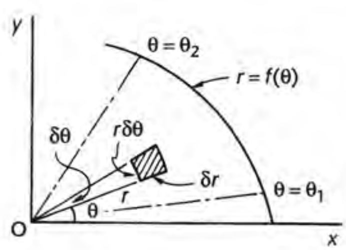
Area of element $\delta A = \delta x \delta y$

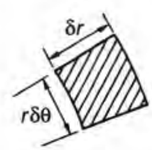
Area of strip $\approx \sum_{y=y_1}^{y=y_2} \delta x \delta y$

Area of all such strips $\approx \sum_{x=a}^{x=b} \left\{ \sum_{y=y_1}^{y=y_2} \delta x \delta y \right\}$

If $\delta x \to 0$ and $\delta y \to 0$, $A = \int_a^b \int_{y_1}^{y_2} \mathrm{d}y \, \mathrm{d}x$

(b) *Areas of plane figures bounded by a polar curve* $r = f(\theta)$ and radius vectors at $\theta = \theta_1$ and $\theta = \theta_2$





Small arc of circle of radius r, subtending angle $\delta\theta$ at centre.

$\therefore$ Arc $= r\delta\theta$

Area of element $\delta A \approx r\delta\theta \, \delta r$

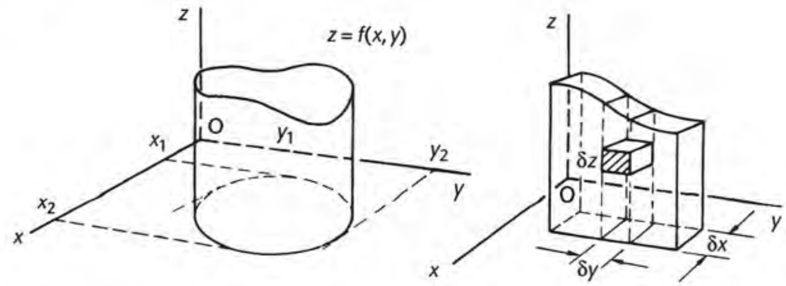
Area of thin sector $\approx \sum_{r=0}^{r=f(\theta)} r \, \delta\theta \, \delta r$

$\therefore$ Total area of all such sectors $\approx \sum_{\theta=\theta_1}^{\theta=\theta_2} \left\{ \sum_{r=0}^{r=f(\theta)} r \, \delta r \, \delta\theta \right\}$

$\therefore$ If $\delta r \to 0$ and $\delta\theta \to 0$, $A = \int_{\theta_1}^{\theta_2} \int_0^{r=f(\theta)} r \, \mathrm{d}r \, \mathrm{d}\theta$

▶

(c) *Volume of solids*



Volume of element $\delta V = \delta x\, \delta y\, \delta z$

Volume of column $\approx \displaystyle\sum_{z=0}^{z=f(x,\,y)} \delta x\, \delta y\, \delta z$

Volume of slice $\approx \displaystyle\sum_{y=y_1}^{y=y_2} \left\{ \sum_{z=0}^{z=f(x,\,y)} \delta x\, \delta y\, \delta z \right\}$

$\therefore$ Total volume $V \approx$ sum of all such slices

i.e. $\quad V \approx \displaystyle\sum_{x=x_1}^{x=x_2} \sum_{y=y_1}^{y=y_2} \sum_{z=0}^{z=f(x,\,y)} \delta x\, \delta y\, \delta z$

Then, if $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$,

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{0}^{z=f(x,\,y)} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

If $z = f(x, y)$, this becomes

$$V = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\, \mathrm{d}y\, \mathrm{d}x$$

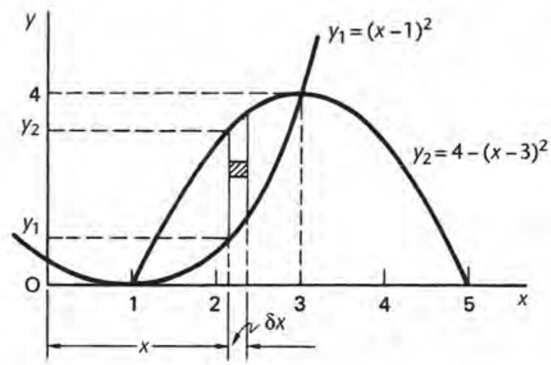
2

4 *Revision examples* As a means of 'warming up', let us work through one or two straightforward examples on the previous work.

Example 1

Find the area of the plane figure bounded by the curves $y_1 = (x-1)^2$ and $y_2 = 4-(x-3)^2$.

The first thing, as always, is to sketch the curves – each of which is a parabola – and to determine their points of intersection.



Points of intersection: $(x-1)^2 = 4-(x-3)^2$

$x^2 - 2x + 1 = 4 - x^2 + 6x - 9$ i.e. $x^2 - 4x + 3 = 0$

$\therefore\ (x-1)(x-3) = 0 \quad \therefore\ x = 1$ or $x = 3$.

Now we have all the information to determine the required area, which is

....

3

$$\boxed{A = 2\tfrac{2}{3} \text{ square units}}$$

Because

$$A = \int_{x=1}^{x=3}\int_{y_1}^{y_2} \mathrm{d}y\,\mathrm{d}x = \int_{x=1}^{x=3}\int_{y=(x-1)^2}^{y=4-(x-3)^2} \mathrm{d}y\,\mathrm{d}x$$

$$= \int_1^3 \{4-(x-3)^2-(x-1)^2\}\,\mathrm{d}x = -2\int_1^3 (x^2-4x+3)\,\mathrm{d}x$$

$$= -2\left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 = 2\tfrac{2}{3} \text{ square units}$$

Now for another.

Example 2

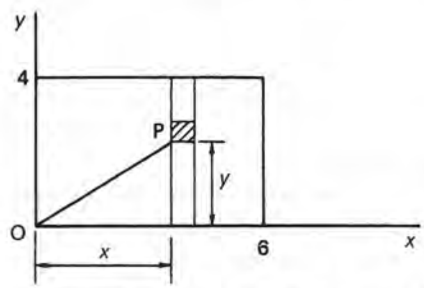
A rectangular plate is bounded by the x and y axes and the lines $x = 6$ and $y = 4$. The thickness t of the plate at any point is proportional to the square of the distance of the point from the origin. Determine the total volume of the plate.

First of all draw the figure and build up the appropriate double integral. Do not evaluate it yet. The expression is therefore

$$V = \dots\dots\dots\dots$$

4

$$\boxed{V = \int_{x=0}^{x=6}\int_{y=0}^{y=4} k(x^2 + y^2)\,dy\,dx}$$



Thickness t of plate at P is

$$t = k\,\mathrm{OP}^2 = k(x^2 + y^2)$$

Element of area $= \delta y\,\delta x$

$\therefore$ Element of volume at P $\approx k(x^2 + y^2)\,\delta y\,\delta x$

$\therefore$ Total volume $V = \displaystyle\int_{x=0}^{x=6}\int_{y=0}^{y=4} k(x^2 + y^2)\,dy\,dx$

Now we can evaluate the integral. We start from the inside with $\displaystyle\int_{y=2}^{y=4} k(x^2 + y^2)\,dy$, remembering that for this integral (volume of the strip) x is constant. This gives ....

5

$$\boxed{k\left(4x^2 + \frac{64}{3}\right)}$$

Because

$$k\int_0^4 (x^2 + y^2)\,dy = k\left[x^2y + \frac{y^3}{3}\right]_{y=0}^{y=4} = k\left(4x^2 + \frac{64}{3}\right)$$

Then $\quad V = k\displaystyle\int_0^6 \left(4x^2 + \frac{64}{3}\right) dx = \dots\dots\dots\dots$

6

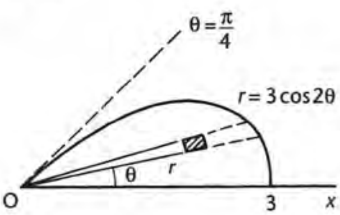
$$\boxed{V = 416k \text{ cubic units}}$$

That was easy enough. Notice that an alternative interpretation of this problem could be that of a uniform lamina with a variable density $\rho = k(x^2 + y^2)$ at any point (x, y). Now for one in polar coordinates.

Example 3

Express as a double integral the area enclosed by one loop of the curve $r = 3\cos 2\theta$ and evaluate the integral (refer to *Engineering Mathematics (Fifth Edition)*, Programme 22, Frame 11).

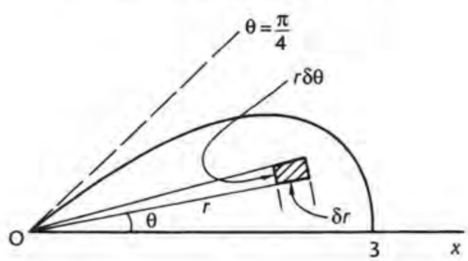
Consider the half loop shown.



First set up the double integral which is ....

7

$$\boxed{A = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=3\cos 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta}$$



Area of element $= r\,\delta r\,\delta\theta$

$\therefore$ Area of sector $\approx \sum_{r=0}^{r=3\cos 2\theta} r\,\delta r\,\delta\theta$

$\therefore$ Area of half loop $\approx \sum_{\theta=0}^{\theta=\pi/4} \sum_{r=0}^{r=3\cos 2\theta} r\,\delta r\,\delta\theta$

If $\delta r \to 0$ and $\delta\theta \to 0$,

$$A = \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=3\cos 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$$

Now finish it off to find the area of the whole loop, which is

....

$$\boxed{\frac{9\pi}{8} \text{ square units}}$$

Because

$$\begin{aligned}
A &= \int_{\theta=0}^{\theta=\pi/4} \int_{r=0}^{r=3\cos 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{3\cos 2\theta} \mathrm{d}\theta \\
&= \frac{9}{2}\int_0^{\pi/4} \cos^2\, 2\theta\,\mathrm{d}\theta \\
&= \frac{9}{4}\int_0^{\pi/4} (1+\cos\, 4\theta)\,\mathrm{d}\theta \\
&= \frac{9}{4}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} \\
&= \frac{9\pi}{16}
\end{aligned}$$

This is the area of a half loop.

$$\text{Required area} = \frac{9\pi}{8} \text{ square units}$$

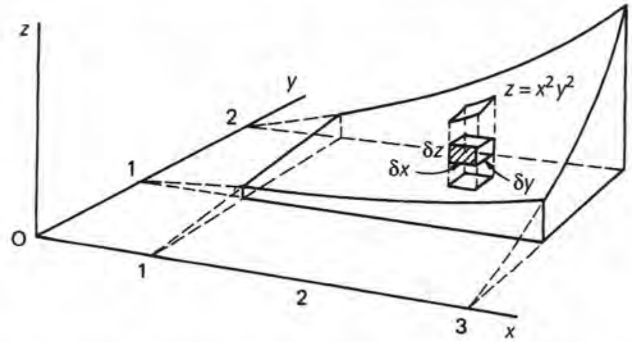
Now here is another.

Example 4

Find the volume of the solid bounded by the planes $z = 0$, $x = 1$, $x = 3$, $y = 1$, $y = 2$ and the surface $z = x^2 y^2$.

As always, we start off by sketching the figure. When you have done that, check the result with the next frame.

9



We now build up the integral which will give us the volume of the solid.

Element of volume $\delta V = \delta x\,\delta y\,\delta z$

$$\text{Volume of column} \approx \sum_{z=0}^{z=x^2y^2} \delta x\,\delta y\,\delta z$$

$$\text{Volume of slice} \approx \sum_{y=1}^{y=2} \left\{ \sum_{z=0}^{z=x^2y^2} \delta x\,\delta y\,\delta z \right\}$$

$$\text{Volume of solid} \approx \sum_{x=1}^{x=3} \left\{ \sum_{y=1}^{y=2} \sum_{z=0}^{z=x^2y^2} \delta x\,\delta y\,\delta z \right\}$$

When $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$,

$$V = \int_{x=1}^{x=3} \int_{y=1}^{y=2} \int_{z=0}^{z=x^2y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Evaluating this, $V = \dots\dots\dots\dots$

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$$\boxed{V = 20\tfrac{2}{9} \text{ cubic units}}$$

Because, starting with the innermost integral

$$V = \int_{x=1}^{x=3} \int_{y=1}^{y=2} \Big[z \Big]_0^{x^2y^2} \mathrm{d}y\,\mathrm{d}x = \int_1^3 \int_1^2 x^2y^2\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_1^3 \left[\frac{x^2y^3}{3} \right]_{y=1}^{y=2} \mathrm{d}x \qquad = \int_1^3 \frac{7x^2}{3}\,\mathrm{d}x = 20\tfrac{2}{9}$$

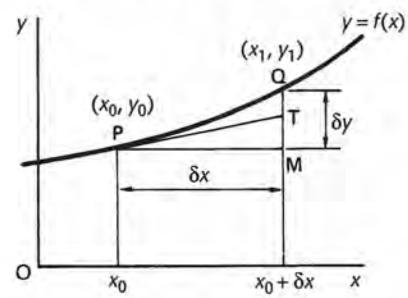
Now that we have revised the basics, let us move on to something rather different

Differentials

11

It is convenient in various branches of the calculus to denote small increases in value of a variable by the use of *differentials*. The method is particularly useful in dealing with the effects of small finite changes and shortens the writing of calculus expressions.

We are already familiar with the diagram from which finite changes δy and δx in a function $y = f(x)$ are depicted.



The increase in y from P to Q $= \text{MQ} = \delta y = f(x_0 + \delta x) - f(x_0)$

If PT is the tangent at P, then MQ = MT + TQ. Also $\dfrac{\text{MT}}{\delta x} = f'(x_0)$

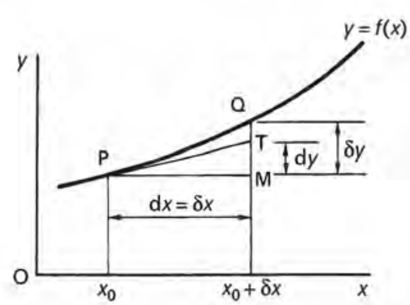
$\therefore\ \text{MT} = f'(x_0)\delta x$

$\therefore\ \text{MQ} = \delta y = f'(x_0)\cdot\delta x + \text{TQ}$

and, if Q is close to P, then $\delta y \approx f'(x_0)\delta x$

We define the differentials $\mathrm{d}y$ and $\mathrm{d}x$ as finite quantities such that

$$\mathrm{d}y = f'(x_0)\,\mathrm{d}x$$



Note that the differentials $\mathrm{d}y$ and $\mathrm{d}x$ are finite quantities – not necessarily zero – and can therefore exist alone.

Note too that $\mathrm{d}x = \delta x$.

▶

From the diagram, we can see that

δy is the increase in y as we move from P to Q along the curve.
$\mathrm{d}y$ is the increase in y as we move from P to T along the tangent.

As Q approaches P, the difference between δy and $\mathrm{d}y$ decreases to zero. The use of differentials simplifies the writing of many relationships and is based on the general statement $\mathrm{d}y = f'(x)\,\mathrm{d}x$.

For example

(a) $y = x^5$ then $\mathrm{d}y = 5x^4\,\mathrm{d}x$
(b) $y = \sin 3x$ then $\mathrm{d}y = 3\cos 3x\,\mathrm{d}x$
(c) $y = e^{4x}$ then $\mathrm{d}y = 4e^{4x}\,\mathrm{d}x$
(d) $y = \cosh 2x$ then $\mathrm{d}y = 2\sinh 2x\,\mathrm{d}x$

Note that when the left-hand side is a differential $\mathrm{d}y$ the right-hand side must also contain a differential. Remember therefore to include the '$\mathrm{d}x$' on the right-hand side.

The product and quotient rules can also be expressed in differentials.

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{becomes} \quad \mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \quad \text{becomes} \quad \mathrm{d}\left(\frac{u}{v}\right) = \frac{v\,\mathrm{d}u - u\,\mathrm{d}v}{v^2}$$

So, if $y = e^{2x}\sin 4x$, $\mathrm{d}y = \ldots\ldots\ldots\ldots$

and if $y = \dfrac{\cos 2t}{t^2}$ $\mathrm{d}y = \ldots\ldots\ldots\ldots$

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$$y = e^{2x}\sin 4x, \quad \mathrm{d}y = 2e^{2x}(2\cos 4x + \sin 4x)\,\mathrm{d}x$$
$$y = \frac{\cos 2t}{t^2}, \quad \mathrm{d}y = -\frac{2}{t^3}\{t\sin 2t + \cos 2t\}\,\mathrm{d}t$$

That was easy enough. Let us now consider a function of two independent variables, $z = f(x, y)$.

If $z = f(x, y)$ then $z + \delta z = f(x + \delta x, y + \delta y)$

$$\therefore \ \delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

Expanding δz in terms of δx and δy, gives

$\delta z = A\delta x + B\delta y +$ higher powers of δx and δy,
where A and B are functions of x and y.

If y remains constant, i.e. $\delta y = 0$, then

$$\delta z = A\,\delta x + \text{higher powers of } \delta x \quad \therefore \ \frac{\delta z}{\delta x} \approx A$$

$$\therefore \ \text{If } \delta x \to 0, \text{ then } A = \frac{\partial z}{\partial x}$$

Similarly, if x remains constant, i.e. $\delta x = 0$, then

$$\delta z = B\,\delta y + \text{higher powers of } \delta y \quad \therefore \; \frac{\delta z}{\delta y} \approx B$$

$$\therefore \text{ If } \delta y \to 0, \text{ then } B = \frac{\partial z}{\partial y}$$

$$\therefore \; \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \text{higher powers of small quantities}$$

$$\therefore \; \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

In terms of differentials, this result can be written

$$\text{If } z = f(x, y). \text{ then } \mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

The result can be extended to functions of more than two independent variables.

$$\text{If } z = f(x, y, w), \quad \mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y + \frac{\partial z}{\partial w}\mathrm{d}w$$

Make a note of these results in differential form as shown.

Exercise

Determine the differential dz for each of the following functions.

1 $z = x^2 + y^2$

2 $z = x^3 \sin 2y$

3 $z = (2x - 1)\,e^{3y}$

4 $z = x^2 + 2y^2 + 3w^2$

5 $z = x^3y^2w$.

Finish all five and then check the results.

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1 $\mathrm{d}z = 2(x\,\mathrm{d}x + y\,\mathrm{d}y)$

2 $\mathrm{d}z = x^2(3\sin 2y\,\mathrm{d}x + 2x\cos 2y\,\mathrm{d}y)$

3 $\mathrm{d}z = e^{3y}\{2\,\mathrm{d}x + (6x - 3)\mathrm{d}y\}$

4 $\mathrm{d}z = 2(x\,\mathrm{d}x + 2y\,\mathrm{d}y + 3w\,\mathrm{d}w)$

5 $\mathrm{d}z = x^2y(3yw\,\mathrm{d}x + 2xw\,\mathrm{d}y + xy\,\mathrm{d}w)$

Now move on

14 Exact differential

We have just established that if $z = f(x, y)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

We now work in reverse.

Any expression $dz = P\,dx + Q\,dy$, where P and Q are functions of x and y, is an *exact differential* if it can be integrated to determine z.

$$\therefore\ P = \frac{\partial z}{\partial x} \quad \text{and} \quad Q = \frac{\partial z}{\partial y}$$

Now $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$ and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial^2 z}{\partial x\,\partial y}$ and we know that $\dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{\partial^2 z}{\partial x\,\partial y}$.

Therefore, for dz to be an exact differential $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and this is the test we apply.

Example 1

$dz = (3x^2 + 4y^2)\,dx + 8xy\,dy$.

If we compare the right-hand side with $P\,dx + Q\,dy$, then

$$P = 3x^2 + 4y^2 \quad \therefore\ \frac{\partial P}{\partial y} = 8y$$

$$Q = 8xy \quad \therefore\ \frac{\partial Q}{\partial x} = 8y$$

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ $\quad\therefore$ dz is an exact differential

Similarly, we can test this one.

Example 2

$dz = (1 + 8xy)\,dx + 5x^2\,dy$.

From this we find ....

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dz is *not* an exact differential

Because $dz = (1 + 8xy)\,dx + 5x^2\,dy$

$$\therefore\ P = 1 + 8xy \quad \therefore\ \frac{\partial P}{\partial y} = 8x$$

$$Q = 5x^2 \quad \therefore\ \frac{\partial Q}{\partial x} = 10x$$

$\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ $\quad\therefore$ dz is not an exact differential.

▶

Exercise

Determine whether each of the following is an exact differential.

1 $\mathrm{d}z = 4x^3y^3\,\mathrm{d}x + 3x^4y^2\,\mathrm{d}y$

2 $\mathrm{d}z = (4x^3y + 2xy^3)\,\mathrm{d}x + (x^4 + 3x^2y^2)\,\mathrm{d}y$

3 $\mathrm{d}z = (15y^2e^{3x} + 2xy^2)\,\mathrm{d}x + (10ye^{3x} + x^2y)\,\mathrm{d}y$

4 $\mathrm{d}z = (3x^2e^{2y} - 2y^2e^{3x})\,\mathrm{d}x + (2x^3e^{2y} - 2ye^{3x})\,\mathrm{d}y$

5 $\mathrm{d}z = (4y^3\cos 4x + 3x^2\cos 2y)\,\mathrm{d}x + (3y^2\sin 4x - 2x^3\sin 2y)\,\mathrm{d}y$.

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1 Yes	**2** Yes	**3** No	**4** No	**5** Yes

We have just tested whether certain expressions are, in fact, exact differentials – and we said previously that, by definition, an exact differential can be integrated. But how exactly do we go about it? The following examples will show.

Integration of exact differentials

$$\mathrm{d}z = P\,\mathrm{d}x + Q\,\mathrm{d}y \quad \text{where} \quad P = \frac{\partial z}{\partial x} \quad \text{and} \quad Q = \frac{\partial z}{\partial y}$$

$$\therefore\ z = \int P\,\mathrm{d}x \quad \text{and also} \quad z = \int Q\,\mathrm{d}y$$

Example 1

$\mathrm{d}z = (2xy + 6x)\,\mathrm{d}x + (x^2 + 2y^3)\,\mathrm{d}y$.

$$P = \frac{\partial z}{\partial x} = 2xy + 6x \qquad \therefore\ z = \int (2xy + 6x)\,\mathrm{d}x$$

$\therefore\ z = x^2y + 3x^2 + f(y)$ where $f(y)$ is an arbitrary function of y only, and is akin to the constant of integration in a normal integral.

$$\text{Also}\ \ Q = \frac{\partial z}{\partial y} = x^2 + 2y^3 \qquad \therefore\ z = \int (x^2 + 2y^3)\,\mathrm{d}y$$

$$\therefore\ z = \ldots\ldots\ldots\ldots$$

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$$\boxed{z = x^2y + \frac{y^4}{2} + F(x) \text{ where } F(x) \text{ is an arbitrary function of } x \text{ only}}$$

So the two results tell us

$$z = x^2y + 3x^2 + f(y) \qquad (1)$$

and $$z = x^2y + \frac{y^4}{2} + F(x) \qquad (2)$$

For these two expressions to represent the same function, then

$f(y)$ in (1) must be $\frac{y^4}{2}$ already in (2)

and $F(x)$ in (2) must be $3x^2$ already in (1)

$$\therefore\ z = x^2y + 3x^2 + \frac{y^4}{2}$$

Example 2

Integrate $\mathrm{d}z = (8e^{4x} + 2xy^2)\,\mathrm{d}x + (4\cos 4y + 2x^2y)\,\mathrm{d}y$.

Argue through the working in just the same way, from which we obtain

$$z = \ldots\ldots\ldots\ldots$$

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$$\boxed{z = 2e^{4x} + x^2y^2 + \sin 4y}$$

Here it is. $\mathrm{d}z = (8e^{4x} + 2xy^2)\,\mathrm{d}x + (4\cos 4y + 2x^2y)\,\mathrm{d}y$

$$P = \frac{\partial z}{\partial x} = 8e^{4x} + 2xy^2 \qquad \therefore\ z = \int (8e^{4x} + 2xy^2)\mathrm{d}x$$

$$\therefore\ z = 2e^{4x} + x^2y^2 + f(y) \qquad (1)$$

$$Q = \frac{\partial z}{\partial y} = 4\cos 4y + 2x^2y \quad \therefore\ z = \int (4\cos 4y + 2x^2y)\,\mathrm{d}y$$

$$\therefore\ z = \sin 4y + x^2y^2 + F(x) \qquad (2)$$

For (1) and (2) to agree, $f(y) = \sin 4y$ and $F(x) = 2e^{4x}$

$$\therefore\ z = 2\,e^{4x} + x^2y^2 + \sin 4y$$

They are all done in the same way, so you will have no difficulty with the short exercise that follows. *On you go.*

Exercise

Integrate the following exact differentials to obtain the function z.

1. $\mathrm{d}z = (6x^2 + 8xy^3)\,\mathrm{d}x + (12x^2y^2 + 12y^3)\,\mathrm{d}y$
2. $\mathrm{d}z = (3x^2 + 2xy + y^2)\,\mathrm{d}x + (x^2 + 2xy + 3y^2)\,\mathrm{d}y$
3. $\mathrm{d}z = 2(y+1)e^{2x}\,\mathrm{d}x + (e^{2x} - 2y)\,\mathrm{d}y$
4. $\mathrm{d}z = (3y^2\cos 3x - 3\sin 3x)\,\mathrm{d}x + (2y\sin 3x + 4)\,\mathrm{d}y$
5. $\mathrm{d}z = (\sinh y + y\sinh x)\mathrm{d}x + (x\cosh y + \cosh x)\,\mathrm{d}y$.

Finish all five before checking with the next frame.

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1 $z = 2x^3 + 4x^2y^3 + 3y^4$
2 $z = x^3 + x^2y + xy^2 + y^3$
3 $z = e^{2x}(1 + y) - y^2$
4 $z = y^2 \sin 3x + \cos 3x + 4y$
5 $z = x \sinh y + y \cosh x$.

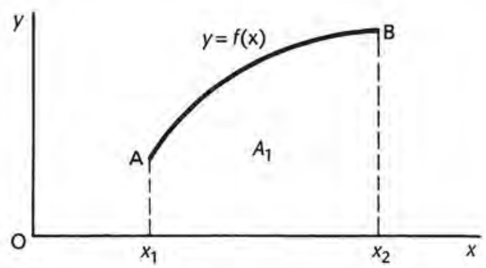
In the last one, of course, we find that the two expressions for z agree without any further addition of $f(y)$ or $F(x)$.

We shall be meeting exact differentials again later on, but for the moment let us deal with something different. On then to the next frame

Area enclosed by a closed curve

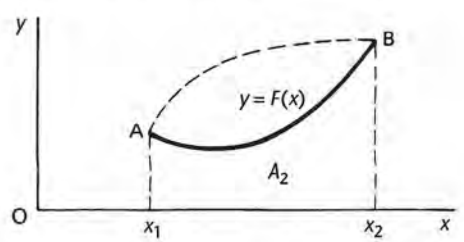
20

One of the earliest applications of integration is finding the area of a plane figure bounded by the x-axis, the curve $y = f(x)$ and ordinates at $x = x_1$ and $x = x_2$.



$$A_1 = \int_{x_1}^{x_2} y \, dx = \int_{x_1}^{x_2} f(x) \, dx$$

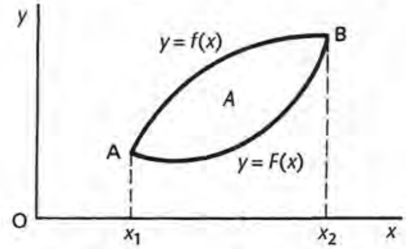
If points A and B are joined by another curve, $y = F(x)$



$$A_2 = \int_{x_1}^{x_2} F(x) \, dx$$

▶

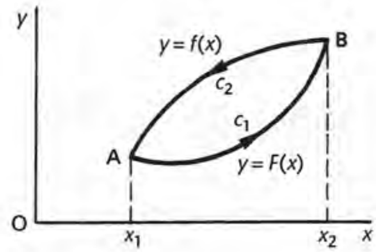
Combining the two figures, we have



$$A = A_1 - A_2$$

$$\therefore\ A = \int_{x_1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} F(x)\,dx$$

It is convenient on occasions to arrange the limits so that the integration follows the path round the enclosed area in a regular order.



For example

$\int_{x_1}^{x_2} F(x)\,dx$ gives A_2 as before, but integrating from B to A along c_2 with $y = f(x)$, i.e. $\int_{x_2}^{x_1} f(x)\,dx$, is the integral for A_1 with the sign changed, i.e. $\int_{x_2}^{x_1} f(x)\,dx = -\int_{x_1}^{x_2} f(x)\,dx$

$\therefore$ The result $A = A_1 - A_2 = \int_{x_1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} F(x)\,dx$ becomes

$$A = \ldots\ldots\ldots\ldots$$

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$$\boxed{A = -\int_{x_1}^{x_2} F(x)\,dx - \int_{x_2}^{x_1} f(x)\,dx}$$

i.e. $$A = -\left\{\int_{x_1}^{x_2} F(x)\,dx + \int_{x_2}^{x_1} f(x)\,dx\right\}$$

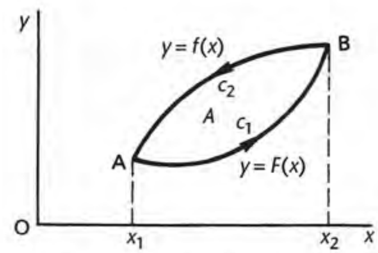
If we proceed round the boundary in an *anticlockwise manner*, the enclosed area is kept on the *left-hand side* and the resulting area is considered *positive*. If we proceed round the boundary in a *clockwise manner*, the enclosed area remains on the *right-hand side* and the resulting area is *negative*.

The final result above can be written in the form

$$A = -\oint y\,dx$$

where the symbol $\oint$ indicates that the integral is to be evaluated round the closed boundary in the positive (i.e. anticlockwise) direction

$$\therefore\ A = -\oint y\,dx = -\left\{\underset{(\text{along } c_1)}{\int_{x_1}^{x_2} F(x)\,dx} + \underset{(\text{along } c_2)}{\int_{x_2}^{x_1} f(x)\,dx}\right\}$$



Let us apply this result to a very simple case.

Example 1

Determine the area enclosed by the graphs of $y = x^3$ and $y = 4x$ for $x \geq 0$.

First we need to know the points of intersection. These are

. . . .

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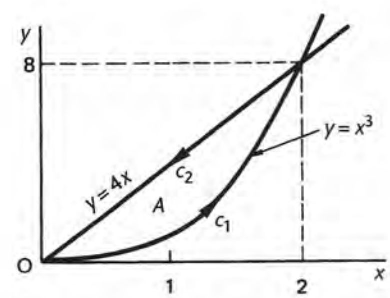
$\boxed{x = 0 \text{ and } x = 2}$



We integrate in an anticlockwise manner

c_1: $y = x^3$, limits $x = 0$ to $x = 2$

c_2: $y = 4x$, limits $x = 2$ to $x = 0$.

$$A = -\oint y\,dx = \ldots\ldots\ldots\ldots$$

23

$$\boxed{A = 4 \text{ square units}}$$

Because

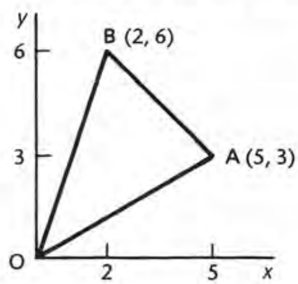
$$A = -\oint y\,dx = -\left\{\int_0^2 x^3\,dx + \int_2^0 4x\,dx\right\}$$

$$= -\left\{\left[\frac{x^4}{4}\right]_0^2 + \left[2x^2\right]_2^0\right\} = 4$$

Another example.

Example 2

Find the area of the triangle with vertices (0, 0), (5, 3) and (2, 6).

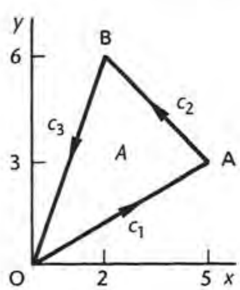


The equation of

OA is ....
BA is ....
OB is ....

24

$$\boxed{\begin{array}{l} \text{OA is } y = \frac{3}{5}x \\ \text{BA is } y = 8 - x \\ \text{OB is } y = 3x \end{array}}$$



Then $A = -\oint y\,dx$

$= \ldots\ldots\ldots\ldots$

Write down the component integrals with appropriate limits.

25

$$\boxed{A = -\oint y\,dx = -\left\{\int_0^5 \frac{3}{5}x\,dx + \int_5^2 (8 - x)\,dx + \int_2^0 3x\,dx\right\}}$$

The limits chosen must progress the integration round the boundary of the figure in an *anticlockwise manner*. Finishing off the integration, we have

$A = \ldots\ldots\ldots\ldots$

26

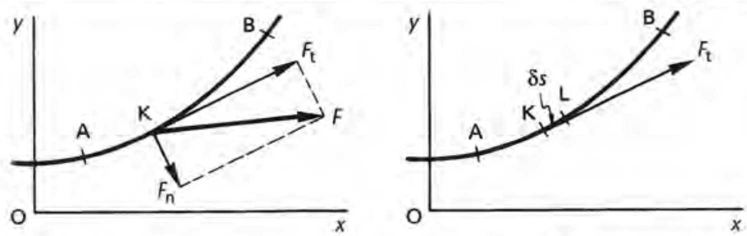
$$\boxed{A = 12 \text{ square units}}$$

The actual integration is easy enough.

The work we have just done leads us on to consider ***line integrals****, so let us make a fresh start in the next frame*

Line integrals

27



If a field exists in the x–y plane, producing a force F on a particle at K, then F can be resolved into two components

F_t along the tangent to the curve AB at K
F_n along the normal to the curve AB at K.

The work done in moving the particle through a small distance δs from K to L along the curve is then approximately $F_t\,\delta s$. So the total work done in moving a particle along the curve from A to B is given by

....

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$$\boxed{\underset{\delta s \to 0}{Lim} \sum F_t\,\delta s = \int F_t\,\text{d}s \text{ from A to B}}$$

This is normally written $\int_{AB} F_t\,\text{d}s$ where A and B are the end points of the curve, or as $\int_c F_t\,\text{d}s$ where the curve c connecting A and B is defined.

Such an integral thus formed is called a *line integral* since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore\ I = \int_{AB} F_t\,\text{d}s = \int_c F_t\,\text{d}s$$

where c is the curve $y = f(x)$ between A (x_1, y_1) and B (x_2, y_2).

There is in fact an alternative form of the integral which is often useful, so let us also consider that

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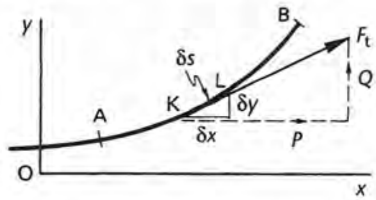
Alternative form of a line integral

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.



If F_t has a component

P in the x-direction
Q in the y-direction

then the work done from K to L can be stated as $P\,\delta x + Q\,\delta y$.

$$\therefore \int_{AB} F_t\,ds = \int_{AB} (P\,dx + Q\,dy)$$

where P and Q are functions of x and y.

In general then, the line integral can be expressed as

$$I = \int_c F_t\,ds = \int_c (P\,dx + Q\,dy)$$

where c is the prescribed curve and F, or P and Q, are functions of x and y.

Make a note of these results – then we will apply them to one or two examples

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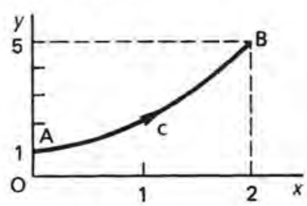
Example 1

Evaluate $\int_c (x+3y)dx$ from A (0, 1) to B (2, 5) along the curve $y = 1 + x^2$.



The line integral is of the form

$$\int_c (P\,dx + Q\,dy)$$

where, in this case, $Q = 0$ and c is the curve $y = 1 + x^2$.

It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.

$$I = \int_c (P\,dx + Q\,dy) = \int_c (x+3y)\,dx = \int_0^2 (x + 3 + 3x^2)\,dx$$
$$= \left[\frac{x^2}{2} + 3x + x^3\right]_0^2 = 16$$

Now for another, so move on

31

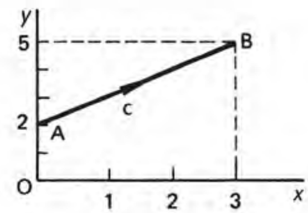
Example 2

Evaluate $I = \int_c (x^2 + y)\,dx + (x - y^2)\,dy$ from A (0, 2) to B (3, 5) along the curve $y = 2 + x$.

$$I = \int_c (P\,dx + Q\,dy)$$

$$P = x^2 + y = x^2 + 2 + x = x^2 + x + 2$$

$$Q = x - y^2 = x - (4 + 4x + x^2)$$
$$= -(x^2 + 3x + 4)$$



Also $y = 2 + x \quad \therefore \; dy = dx$ and the limits are $x = 0$ to $x = 3$.

$$\therefore \; I = \ldots\ldots\ldots\ldots$$

32

$$\boxed{I = -15}$$

Because

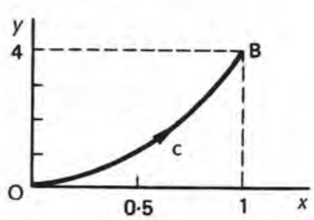
$$I = \int_0^3 \{(x^2 + x + 2)\,dx - (x^2 + 3x + 4)\,dx\}$$

$$\int_0^3 -(2x + 2)dx = \left[x^2 - 2x\right]_0^3 = -15$$

Here is another.

Example 3

Evaluate $I = \int_c \{(x^2 + 2y)\,dx + xy\,dy\}$ from O (0, 0) to B (1, 4) along the curve $y = 4x^2$.



In this case, c is the curve $y = 4x^2$.

$\therefore \; dy = 8x\,dx$

Substitute for y in the integral and apply the limits.

Then $I = \ldots\ldots\ldots\ldots$

Finish it off: it is quite straightforward.

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$$\boxed{I = 9{\cdot}4}$$

Because

$$I = \int_c \{(x^2 + 2y)\,dx + xy\,dy\} \qquad y = 4x^2 \qquad \therefore\ dy = 8x\,dx$$

Also $x^2 + 2y = x^2 + 8x^2 = 9x^2$; $\qquad xy = 4x^3$

$$\therefore\ I = \int_0^1 \{9x^2\,dx + 32x^4\,dx\} = \int_0^1 (9x^2 + 32x^4)\,dx = 9.4$$

They are all done in very much the same way.

Move on for Example 4

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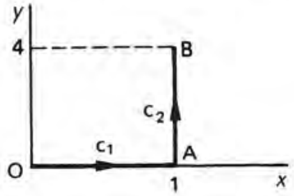
Example 4

Evaluate $I = \int_c \{(x^2 + 2y)\,dx + xy\,dy\}$ from O (0, 0) to A (1, 0) along line $y = 0$ and then from A (1, 0) to B (1, 4) along the line $x = 1$.



(1) OA: c_1 is the line $y = 0$ $\therefore\ dy = 0$. Substituting $y = 0$ and $dy = 0$ in the given integral gives

$$I_{OA} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(2) AB: Here c_2 is the line $x = 1$ $\qquad \therefore\ dx = 0$

$$\therefore\ I_{AB} = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_{AB} = 8}$$

Because

$$I_{AB} = \int_0^4 \{(1 + 2y)(0) + y\,dy\}$$

$$= \int_0^4 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^4 = 8$$

Then $I = I_{OA} + I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3}$ $\quad \therefore\ I = 8\frac{1}{3}$

If we now look back to Examples 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but ....

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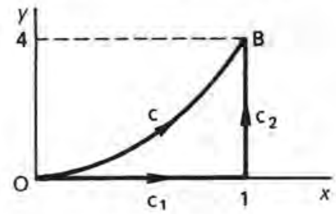
along different paths of integration

If we combine the two diagrams, we have



where c is the curve $y = 4x^2$ and $c_1 + c_2$ are the lines $y = 0$ and $x = 1$. The results obtained were

$I_c = 9\frac{2}{5}$ and $I_{c_1+c_2} = 8\frac{1}{3}$

Notice therefore that integration along two distinct paths joining the same two end points does not necessarily give the same results.

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Let us pause here a moment and list the main properties of line integrals.

Properties of line integrals

1 $\int_c F\mathrm{d}s = \int_c \{P\,\mathrm{d}x + Q\,\mathrm{d}y\}$

2 $\int_{AB} F\mathrm{d}s = -\int_{BA} F\,\mathrm{d}s$ and $\int_{AB} \{P\,\mathrm{d}x + Q\,\mathrm{d}y\} = -\int_{BA} \{P\,\mathrm{d}x + Q\,\mathrm{d}y\}$

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3 (a) For a path of integration parallel to the y-axis, i.e. $x = k$,

$$\mathrm{d}x = 0. \quad \therefore \int_c P\,\mathrm{d}x = 0 \quad \therefore I_c = \int_c Q\,\mathrm{d}y.$$

(b) For a path of integration parallel to the x-axis, i.e. $y = k$,

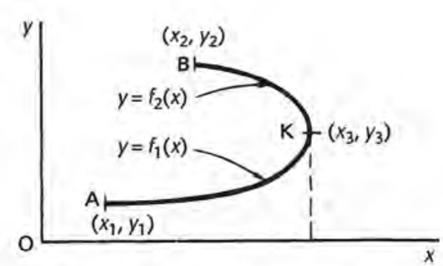
$$\mathrm{d}y = 0. \quad \therefore \int_c Q\,\mathrm{d}y = 0 \quad \therefore I_c = \int_c P\,\mathrm{d}x.$$

4 If the path of integration c joining A to B is divided into two parts AK and KB, then $I_c = I_{AB} = I_{AK} + I_{KB}$.

5 In all cases, the function $y = f(x)$ that describes the path of integration involved must be continuous and single-valued – or dealt with as in item **6** below.

6 If the function $y = f(x)$ that describes the path of integration c is not single-valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$ from A to K
$y = f_2(x)$ from K to B.



Make a note of this list for future reference and revision

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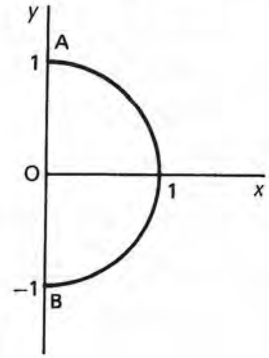
Example

Evaluate $I = \int_c (x+y)\,dx$ from A (0, 1) to B (0, -1) along the semicircle $x^2 + y^2 = 1$ for $x \geq 0$.



The first thing we notice is that

....

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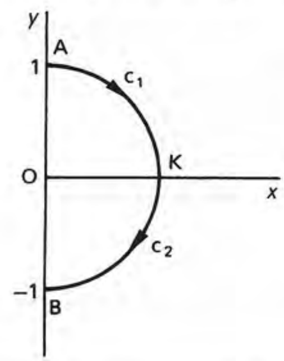
the function $y = f(x)$ that describes the path of integration c is *not* single-valued

For any value of x, $y = \pm\sqrt{1-x^2}$. Therefore, we divide c into two parts



(1) $y = \sqrt{1-x^2}$ from A to K

(2) $y = -\sqrt{1-x^2}$ from K to B.

As usual, $I = \int_c (P\,dx + Q\,dy)$ and in this particular case, Q = ...

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$\boxed{Q = 0}$

$$\therefore\ I = \int_c P\,dx = \int_0^1 \left(x + \sqrt{1-x^2}\right) dx + \int_1^0 (x - \sqrt{1-x^2})\,dx$$

$$= \int_0^1 (x + \sqrt{1-x^2} - x + \sqrt{1-x^2})\,dx = 2\int_0^1 \sqrt{1-x^2}\,dx$$

Now substitute $x = \sin\theta$ and finish it off.

$I =$ ....

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$$\boxed{I = \frac{\pi}{2}}$$

Because

$$I = 2\int_0^1 \sqrt{1 - x^2}\,dx \quad x = \sin\theta \quad \therefore\ dx = \cos\theta\,d\theta$$

$$\sqrt{1 - x^2} = \cos\theta$$

Limits: $x = 0,\ \theta = 0;\ x = 1,\ \theta = \dfrac{\pi}{2}$

$$\therefore\ I = 2\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

Now let us extend this line of development a stage further.

Regions enclosed by closed curves

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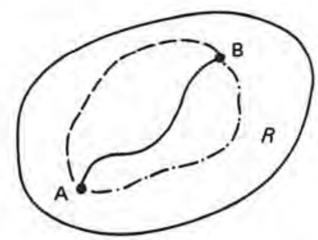
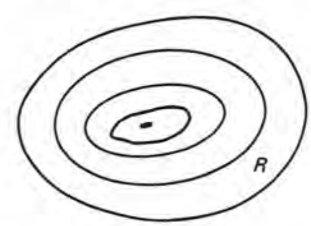
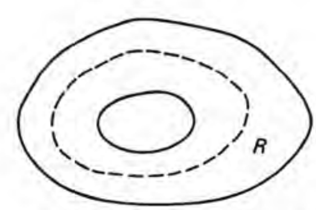
A region is said to be *simply connected* if a path joining A and B can be deformed to coincide with any other line joining A and B without going outside the region.



Another definition is that a region is simply connected if any closed path in the region can be contracted to a single point without leaving the region.



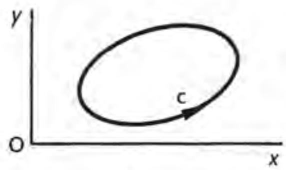
Clearly, this would not be satisfied in the case where the region R contains one or more 'holes'.



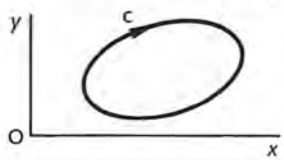
The closed curves involved in problems in this Programme all relate to simply connected regions, so no difficulties will arise.

43 Line integrals round a closed curve

We have already introduced the symbol $\oint$ to indicate that an integral is to be evaluated round a closed curve in the positive (anticlockwise) direction.

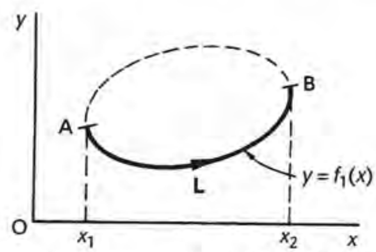


Positive direction (anticlockwise) line integral denoted by $\oint$.

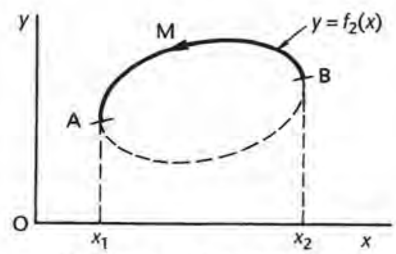


Negative direction (clockwise) line integral denoted by $-\oint$.

With a closed curve, the y-values on the path c cannot be single-valued. Therefore, we divide the path into two or more parts and treat each separately.



(1) Use $y = f_1(x)$ for ALB



(2) Use $y = f_2(x)$ for BMA.

Unless specially required otherwise, we always proceed round the closed curve in an …………

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anticlockwise direction

Example 1

Evaluate the line integral $I = \oint_c (x^2\,dx - 2xy\,dy)$ where c comprises the three sides of the triangle joining O (0, 0), A (1, 0) and B (0, 1).

First draw the diagram and mark in c_1, c_2 and c_3, the proposed directions of integration. Do just that.

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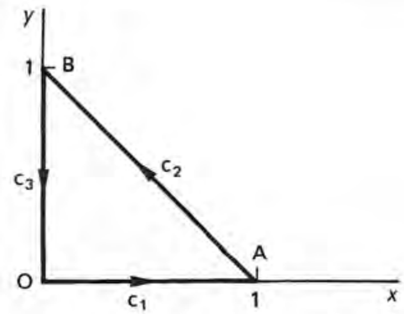

The three sections of the path of integration must be arranged in an anticlockwise manner round the figure. Now we deal with each part separately.

(a) OA: c_1 is the line $y = 0 \quad \therefore \; dy = 0$.

Then $I = \oint (x^2\,dx - 2xy\,dy)$ for this part becomes

$$I_1 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \qquad \therefore \; I_1 = \frac{1}{3}$$

(b) AB: c_2 is the line $y = 1 - x \quad \therefore \; dy = -dx$

$I_2 = \ldots\ldots\ldots\ldots$ (evaluate it)

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$$\boxed{I_2 = -\tfrac{2}{3}}$$

Because c_2 is the line $y = 1 - x \quad \therefore \; dy = -dx$.

$$I_2 = \int_1^0 \{x^2\,dx + 2x(1-x)\,dx\} = \int_1^0 (x^2 + 2x - 2x^2)\,dx$$

$$= \int_1^0 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_1^0 = -\frac{2}{3} \quad \therefore \; I_2 = -\frac{2}{3}$$

Note that anticlockwise progression is obtained by arranging the limits in the appropriate order.

Now we have to determine I_3 for BO.

(c) BO: c_3 is the line $x = 0$

$I_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{I_3 = 0}$$

Because for c_3, $\quad x = 0 \quad \therefore \; dx = 0 \quad \therefore \; I_3 = \int 0\,dy = 0 \quad \therefore \; I_3 = 0$

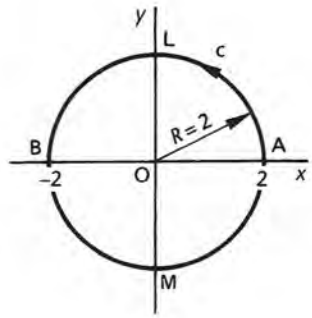
Finally, $I = I_1 + I_2 + I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3} \qquad \therefore \; I = -\frac{1}{3}$

Let us work through another example.

▶

Example 2

Evaluate $\oint_c y\,dx$ when c is the circle $x^2 + y^2 = 4$.



$$x^2 + y^2 = 4 \quad \therefore \; y = \pm\sqrt{4 - x^2}$$

y is thus not single-valued. Therefore use $y = \sqrt{4 - x^2}$ for ALB between $x = 2$ and $x = -2$ and $y = -\sqrt{4 - x^2}$ for BMA between $x = -2$ and $x = 2$.

$$\therefore \; I = \int_2^{-2} \sqrt{4 - x^2}\,dx + \int_{-2}^{2} \{-\sqrt{4 - x^2}\}\,dx$$

$$= 2\int_2^{-2} \sqrt{4 - x^2}\,dx \; = \; -2\int_{-2}^{2} \sqrt{4 - x^2}\,dx$$

$$= -4\int_0^2 \sqrt{4 - x^2}\,dx.$$

To evaluate this integral, substitute $x = 2\sin\theta$ and finish it off.

$I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = -4\pi}$$

Because

$$x = 2\sin\theta \quad \therefore \; dx = 2\cos\theta\,d\theta \quad \therefore \; \sqrt{4 - x^2} = 2\cos\theta$$

limits: $x = 0,\; \theta = 0;\; x = 2,\; \theta = \dfrac{\pi}{2}$

$$\therefore \; I = -4\int_0^{\pi/2} 2\cos\theta\, 2\cos\theta\,d\theta = -16\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= -8\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta = -8\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = -4\pi$$

Now for one more

Example 3

Evaluate $I = \oint_c \{xy\,dx + (1 + y^2)\,dy\}$ where c is the boundary of the rectangle joining A (1, 0), B (3, 0), C (3, 2) and D (1, 2).

First draw the diagram and insert c_1, c_2, c_3, c_4.

That gives $\ldots\ldots\ldots\ldots$

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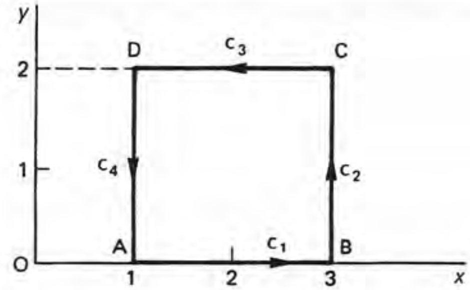


Now evaluate I_1 for AB; I_2 for BC; I_3 for CD; I_4 for DA; and finally I.

Complete the working and then check with the next frame

50

$$I_1 = 0; \quad I_2 = 4\tfrac{2}{3}; \quad I_3 = -8; \quad I_4 = -4\tfrac{2}{3}; \quad I = -8$$

Here is the complete working.

$$I = \oint_c \{xy\,dx + (1+y^2)\,dy\}$$

(a) AB: c_1 is $y = 0 \quad \therefore\ dy = 0 \quad \therefore\ I_1 = 0$

(b) BC: c_2 is $x = 3 \quad \therefore\ dx = 0$

$$\therefore\ I_2 = \int_0^2 (1+y^2)dy = \left[y + \frac{y^3}{3}\right]_0^2 = 4\tfrac{2}{3} \qquad \therefore\ I_2 = 4\tfrac{2}{3}$$

(c) CD: c_3 is $y = 2 \quad \therefore\ dy = 0$

$$\therefore\ I_3 = \int_3^1 2x\,dx = \left[x^2\right]_3^1 = -8 \qquad \therefore\ I_3 = -8$$

(d) DA: c_4 is $x = 1 \quad \therefore\ dx = 0$

$$\therefore\ I_4 = \int_2^0 (1+y^2)\,dy = \left[y + \frac{y^3}{3}\right]_2^0 = -4\tfrac{2}{3} \quad \therefore\ I_4 = -4\tfrac{2}{3}$$

Finally

$$I = I_1 + I_2 + I_3 + I_4$$
$$= 0 + 4\tfrac{2}{3} - 8 - 4\tfrac{2}{3} = -8 \qquad \therefore\ I = -8$$

Remember that, unless we are directed otherwise, we always proceed round the closed boundary in an anticlockwise manner.

On now to the next piece of work

51 Line integral with respect to arc length

We have already established that

$$I = \int_{AB} F_t \, ds = \int_{AB} \{P\,dx + Q\,dy\}$$

where F_t denoted the tangential force along the curve c at the sample point K (x, y).

The same kind of integral can, of course, relate to any function $f(x, y)$ which is a function of the position of a point on the stated curve, so that $I = \int_c f(x, y)\,ds$.

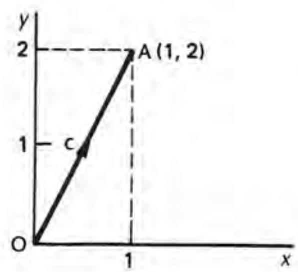
This can readily be converted into an integral in terms of x. (Refer to *Engineering Mathematics (Fifth Edition)*, Programme 19, Frame 30.)

$$I = \int_c f(x, y)\,ds = \int_c f(x, y)\frac{ds}{dx}\,dx \quad \text{where} \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \int_c f(x, y)\,ds = \int_{x_1}^{x_2} f(x, y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad (1)$$

Example

Evaluate $I = \int_c (4x + 3xy)\,ds$ where c is the straight line joining O (0, 0) to A (1, 2).



c is the line $y = 2x \quad \therefore \dfrac{dy}{dx} = 2$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{5}$$

$$\therefore I = \int_{x=0}^{x=1} (4x + 3xy)\,ds = \int_0^1 (4x + 3xy)(\sqrt{5})\,dx. \quad \text{But } y = 2x$$

$$\therefore I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = 4\sqrt{5}}$$

Because

$$I = \int_0^1 (4x + 6x^2)(\sqrt{5})\,dx = 2\sqrt{5}\int_0^1 (2x + 3x^2)\,dx = 4\sqrt{5}$$

Try another.

The path length of the parabola defined by $y = x^2$ betwen the values $x = 0$ and $x = 2$ is given by the integral

$$I = \int_c ds = \ldots\ldots\ldots\ldots \text{ to 3 dp}$$

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$$\boxed{3{\cdot}393 \text{ to 3 dp}}$$

Because

$$\begin{aligned} I &= \int_c ds = \int_{x=0}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)}\,dx \\ &= \int_{x=0}^{2} \sqrt{1 + 2x}\,dx \end{aligned}$$

Let $u = 1 + 2x$ so that $du = 2dx$ and so

$$\begin{aligned} I &= \int_{u=1}^{5} u^{1/2}\frac{du}{2} \\ &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^5 \\ &= \frac{1}{3}\left(125^{1/2} - 1\right) \\ &= 3{\cdot}393 \text{ to 3 dp} \end{aligned}$$

Parametric equations

54

When x and y are expressed in parametric form, e.g. $x = f(t)$, $y = g(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \therefore\ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

and result (1) above becomes

$$I = \int_c f(x, y)\,ds = \int_{t_1}^{t_2} f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad (2)$$

Make a note of results (1) and (2) for future use

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Example

Evaluate $I = \oint_c 4xy\,ds$ where c is defined as the curve $x = \sin t$, $y = \cos t$ between $t = 0$ and $t = \dfrac{\pi}{4}$.

$$\text{We have } x = \sin t \quad \therefore \quad \frac{dx}{dt} = \cos t$$
$$y = \cos t \quad \therefore \quad \frac{dy}{dt} = -\sin t$$
$$\therefore \quad \frac{ds}{dt} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{ds}{dt} = 1}$$

Because

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\therefore \ I = \int_{t_1}^{t_2} f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\pi/4} 4\sin t\ \cos t\,dt$$

$$= 2\int_0^{\pi/4} \sin 2t\,dt = -2\left[\frac{\cos 2t}{2}\right]_0^{\pi/4} = 1 \quad \therefore \ I = 1$$

Dependence of the line integral on the path of integration

We saw earlier in the Programme that integration along two separate paths joining the same two end points does not necessarily give identical results.

With this in mind, let us investigate the following problem.

Example

Evaluate $I = \oint_c \{3x^2y^2\,dx + 2x^3y\,dy\}$ between O (0, 0) and A (2, 4)

(a) along c_1 i.e. $y = x^2$

(b) along c_2 i.e. $y = 2x$

(c) along c_3 i.e. $x = 0$ from (0, 0) to (0, 4) and $y = 4$ from (0, 4) to (2, 4).

Let us concentrate on section (a).

First we draw the figure and insert relevant information.

This gives ....

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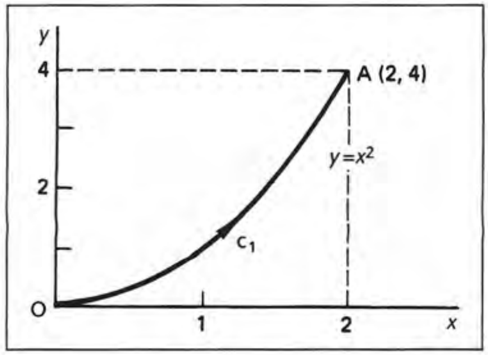


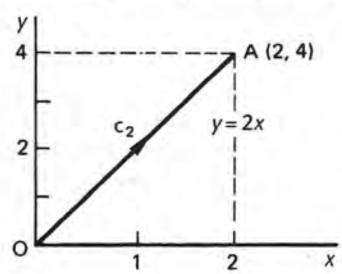
(a) $I = \int_c \{3x^2y^2\,dx + 2x^3y\,dy\}$

The path c_1 is $y = x^2 \quad \therefore\ dy = 2x\,dx$

$$\therefore\ I_1 = \int_0^2 \{3x^2x^4\,dx + 2x^3x^2 2x\,dx\} = \int_0^2 (3x^6 + 4x^4)\,dx$$

$$= \left[x^7\right]_0^2 = 128 \qquad \therefore\ I_1 = 128$$

(b) In (b), the path of integration changes to c_2, i.e. $y = 2x$



So, in this case,

$I_2 = \dots\dots\dots\dots$

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$$\boxed{I_2 = 128}$$

Because with c_2, $\quad y = 2x \quad \therefore\ dy = 2\,dx$

$$\therefore\ I_2 = \int_0^2 \{3x^2\,4x^2\,dx + 2x^3\,2x2\,dx\} = \int_0^2 20x^4\,dx$$

$$= 4\left[x^5\right]_0^2 = 128 \qquad \therefore\ I_2 = 128$$

(c) In the third case, the path c_3 is split

$x = 0$ from (0, 0) to (0, 4)

$y = 4$ from (0, 4) to (2, 4)

Sketch the diagram and determine I_3.

$I_3 = \dots\dots\dots\dots$

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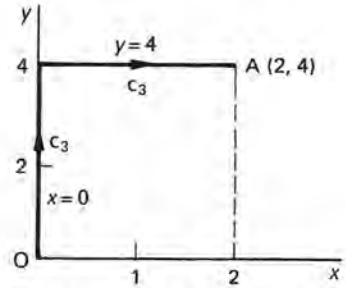
$$\boxed{I_3 = 128}$$



From (0, 0) to (0, 4) $x = 0 \quad \therefore \ dx = 0 \quad \therefore \ I_{3a} = 0$

From (0, 4) to (2, 4) $y = 4 \quad \therefore \ dy = 0 \quad \therefore \ I_{3b} = 48\int_0^2 x^2 dx = 128$

$$\therefore \ I_3 = 128$$

On to the next frame

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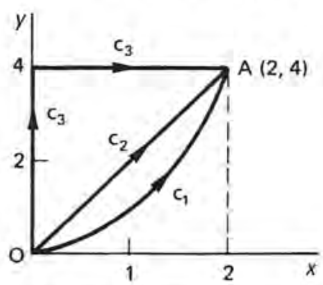
In the example we have just worked through, we took three different paths and in each case, the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.



How then does this integral perhaps differ from those of previous cases?

Let us investigate

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We have been dealing with $I = \int_c \{3x^2y^2\, dx + 2x^3y\, dy\}$

On reflection, we see that the integrand $3x^2y^2\, dx + 2x^3y\, dy$ is of the form $P\, dx + Q\, dy$ which we have met before and that it is, in fact, an *exact differential* of the function $z = x^3y^2$, because

$$\frac{\partial z}{\partial x} = 3x^2y^2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 2x^3y$$

Provided P, Q and their first partial derivatives are finite and continuous at all points inside and on any closed curve, this always happens. If the integrand of the given integral is seen to be an *exact differential*, then the value of the line integral is *independent of the path taken and depends only on the coordinates of the two end points*

Make a note of this. It is important

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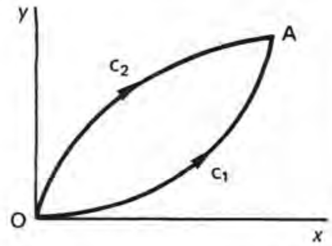
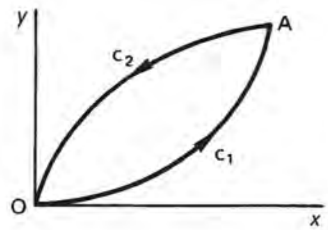


If $I = \int_c \{P\,dx + Q\,dy\}$ and $(P\,dx + Q\,dy)$ is an exact differential, then

$$I_{c_1} = I_{c_2}$$



If we reverse the direction of c_2, then

$$I_{c_1} = -I_{c_2}$$

i.e. $I_{c_1} + I_{c_2} = 0$

Hence, *if $(P\,dx + Q\,dy)$ is an exact differential, then the integration taken round a closed curve is zero.*

$\therefore$ If $(P\,dx + Q\,dy)$ is an exact differential, $\oint (P\,dx + Q\,dy) = 0$

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Example 1

Evaluate $I = \int_c \{3y\,dx + (3x + 2y)\,dy\}$ from A (1, 2) to B (3, 5).

No path is given, so the integrand is probably an exact differential of some function $z = f(x, y)$. In fact $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$.

We have already dealt with the integration of exact differentials, so there is no difficulty. Compare with $I = \int_c \{P\,dx + Q\,dy\}$.

$$P = \frac{\partial z}{\partial x} = 3y \qquad \therefore\ z = \int 3y\,dx = 3xy + f(y) \qquad (1)$$

$$Q = \frac{\partial z}{\partial y} = 3x + 2y \quad \therefore\ z = \int (3x + 2y)\,dy = 3xy + y^2 + F(x) \qquad (2)$$

For (1) and (2) to agree

$f(y) = \dots\dots\dots\dots$ and $F(x) = \dots\dots\dots\dots$

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$$\boxed{f(y) = y^2; \quad F(x) = 0}$$

Hence $z = 3xy + y^2$

$$\therefore \; I = \int_c \{3y\,dx + (3x + 2y)\,dy\} = \int_{(1,\,2)}^{(3,\,5)} d(3xy + y^2)$$
$$= \Big[3xy + y^2\Big]_{(1,\,2)}^{(3,\,5)}$$
$$= (45 + 25) - (6 + 4)$$
$$= 60$$

Example 2

Evaluate $I = \int_c \{(x^2 + ye^x)\,dx + (e^x + y)\,dy\}$ between A (0, 1) and B (1, 2).

As before, compare with $\int_c \{P\,dx + Q\,dy\}$.

$$P = \frac{\partial z}{\partial x} = x^2 + ye^x \quad \therefore \; z = \ldots\ldots\ldots\ldots$$
$$Q = \frac{\partial z}{\partial y} = e^x + y \quad \therefore \; z = \ldots\ldots\ldots\ldots$$

Continue the working and complete the evaluation.

When you have finished, check the result with the next frame

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$$\boxed{\begin{array}{l} z = \dfrac{x^3}{3} + ye^x + f(y) \\ z = ye^x + \dfrac{y^2}{2} + F(x) \end{array}}$$

For these expressions to agree, $\quad f(y) = \dfrac{y^2}{2}; \quad F(x) = \dfrac{x^3}{3}$

$$\text{Then } I = \left[\frac{x^3}{3} + ye^x + \frac{y^2}{2}\right]_{(0,\,1)}^{(1,\,2)}$$
$$= \frac{5}{6} + 2e$$

So the main points are that, if $(P\,dx + Q\,dy)$ is an exact differential

(a) $I = \int_c (P\,dx + Q\,dy)$ is independent of the path of integration

(b) $I = \int_c (P\,dx + Q\,dy)$ is zero when c is a closed curve.

On to the next frame

Exact differentials in three independent variables

66

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$\mathrm{d}w = P\mathrm{d}x + Q\mathrm{d}y + R\,\mathrm{d}z$ is an exact differential of $w = f(x, y, z)$

if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$; $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$; $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$

If the test is successful, then

(a) $\displaystyle\int_c (P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z)$ is independent of the path of integration

(b) $\displaystyle\oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z)$ is zero when c is a closed curve.

Example

Verify that $\mathrm{d}w = (3x^2yz + 6x)\mathrm{d}x + (x^3z - 8y)\mathrm{d}y + (x^3y + 1)\mathrm{d}z$ is an exact differential and hence evaluate $\displaystyle\int_c \mathrm{d}w$ from A (1, 2, 4) to B (2, 1, 3).

First check that $\mathrm{d}w$ is an exact differential by finding the partial derivatives above, when $P = 3x^2yz + 6x$; $Q = x^3z - 8y$; and $R = x^3y + 1$.

We have ....

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$$\frac{\partial P}{\partial y} = 3x^2z;\quad \frac{\partial Q}{\partial x} = 3x^2z\quad \therefore\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial P}{\partial z} = 3x^2y;\quad \frac{\partial R}{\partial x} = 3x^2y\quad \therefore\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$
$$\frac{\partial R}{\partial y} = x^3;\quad \frac{\partial Q}{\partial z} = x^3\quad \therefore\ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$

$\therefore$ $\mathrm{d}w$ is an exact differential

Now to find w. $P = \dfrac{\partial z}{\partial x}$; $Q = \dfrac{\partial z}{\partial y}$; $R = \dfrac{\partial w}{\partial z}$

$$\therefore\ \frac{\partial w}{\partial x} = 3x^2yz + 6x\qquad \therefore\ w = \int (3x^2yz + 6x)\mathrm{d}x$$
$$= x^3yz + 3x^2 + f(y, z)$$
$$\frac{\partial w}{\partial y} = x^3z - 8y\qquad \therefore\ w = \int (x^3z - 8y)\,\mathrm{d}y$$
$$= x^3zy - 4y^2 + F(x, z)$$
$$\frac{\partial w}{\partial z} = x^3y + 1\qquad \therefore\ w = \int (x^3y + 1)\,\mathrm{d}z$$
$$= x^3yz + z + g(x, y)$$

For these three expressions for z to agree

$f(y, z) =$ ....; $F(x, z) =$ ....; $g(x, y) =$ ....

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$$\boxed{f(y,z) = -4y^2; \quad F(x,z) = z; \quad g(x,y) = 3x^2}$$

$$\therefore \; w = x^3yz + 3x^2 - 4y^2 + z$$

$$\therefore \; I = \Big[x^3yz + 3x^2 - 4y^2 + z\Big]_{(1,2,4)}^{(2,1,3)}$$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{I = 36}$$

Because

$$I = \Big[x^3yz + 3x^2 - 4y^2 + z\Big]_{(1,2,4)}^{(2,1,3)}$$

$$= (24 + 12 - 4 + 3) - (8 + 3 - 16 + 4) = 36$$

The extension to line integrals in space is thus quite straightforward.
Finally, we have a theorem that can be very helpful on occasions and which links up with the work we have been doing.

It is important, so let us start a new section

Green's theorem

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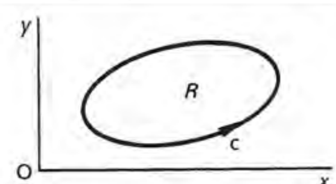
Let P and Q be two functions of x and y that are, along with their first partial derivatives, finite and continuous inside and on the boundary c of a region R in the x–y plane.



If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that

$$\int_R\!\!\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \mathrm{d}x\,\mathrm{d}y = -\oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

That is, a double integral over the plane region R can be transformed into a line integral over the boundary c of the region – and the action is reversible.
Let us see how it works.

Example 1

Evaluate $I = \oint_c \{(2x - y)\,dx + (2y + x)\,dy\}$ around the boundary c of the ellipse $x^2 + 9y^2 = 16$.

The integral is of the form $I = \oint_c \{P\,dx + Q\,dy\}$ where

$$P = 2x - y \quad \therefore \quad \frac{\partial P}{\partial y} = -1$$

$$\text{and } Q = 2y + x \quad \therefore \quad \frac{\partial Q}{\partial x} = 1.$$

$$\therefore \; I = -\int_R\int\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$= -\int_R\int(-1 - 1)\,dx\,dy$$

$$= 2\int_R\int dx\,dy$$

But $\int_R\int dx\,dy$ over any closed region gives ....

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the area of the figure

In this case, then, $I = 2A$ where A is the area of the ellipse

$$x^2 + 9y^2 = 16 \quad \text{i.e.} \quad \frac{x^2}{16} + \frac{9y^2}{16} = 1$$

$$\therefore \; a = 4;\; b = \frac{4}{3}$$

$$\therefore \; A = \pi ab = \frac{16\pi}{3}$$

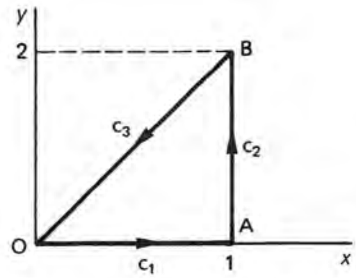
$$\therefore \; I = 2A = \frac{32\pi}{3}$$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the method of line integrals, and (b) by applying Green's theorem.

▶

Example 2

Evaluate $I = \oint_c \{(2x + y)\,dx + (3x - 2y)\,dy\}$ taken in anticlockwise manner round the triangle with vertices at O (0, 0), A (1, 0) and B (1, 2).



$$I = \oint_c \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

(a) *By the method of line integrals*

There are clearly three stages with c_1, c_2, c_3. Work through the complete evaluation to determine the value of I. It will be good revision.

When you have finished, check the result with the solution in the next frame

---

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$$\boxed{I = 2}$$

(a) (1) c_1 is $y = 0 \quad \therefore\ dy = 0$

$$\therefore\ I_1 = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1 \quad \therefore\ I_1 = 1$$

(2) c_2 is $x = 1 \quad \therefore\ dx = 0$

$$\therefore\ I_2 = \int_0^2 (3 - 2y)\,dy = \left[3y - y^2\right]_0^2 = 2 \quad \therefore\ I_2 = 2$$

(3) c_3 is $y = 2x \quad \therefore\ dy = 2\,dx$

$$\therefore\ I_3 = \int_1^0 \{4x\,dx + (3x - 4x)2\,dx\}$$

$$= \int_1^0 2x\,dx = \left[x^2\right]_1^0 = -1 \qquad \therefore\ I_3 = -1$$

$$I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2 \quad \therefore\ I = 2$$

Now we will do the same problem by applying Green's theorem, so move on

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(b) *By Green's theorem*

$$I = \oint_c \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

$$P = 2x + y \quad \therefore \quad \frac{\partial P}{\partial y} = 1; \quad Q = 3x - 2y \quad \therefore \quad \frac{\partial Q}{\partial x} = 3$$

$$I = -\int_R\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

Finish it off. $I = \dots\dots\dots\dots$

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$$\boxed{I = 2}$$

Because

$$\begin{aligned} I &= -\int_R\int (1 - 3)\,dx\,dy \\ &= 2\int_R\int dx\,dy = 2A \\ &= 2 \times \text{ the area of the triangle} \\ &= 2 \times 1 = 2 \quad \therefore \ I = 2 \end{aligned}$$

Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available. If you have not already done so, make a note of Green's theorem.

$$\int_R\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_c (P\,dx + Q\,dy)$$

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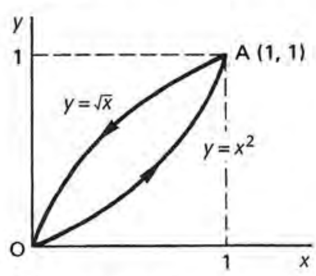
Example 3

Evaluate the line integral $I = \oint_c \{xy\,dx + (2x - y)\,dy\}$ round the region bounded by the curves $y = x^2$ and $x = y^2$ by the use of Green's theorem.



Points of intersection are O (0, 0) and A (1, 1). P and Q are known, so there is no difficulty.

Complete the working.

$I = \dots\dots\dots\dots$

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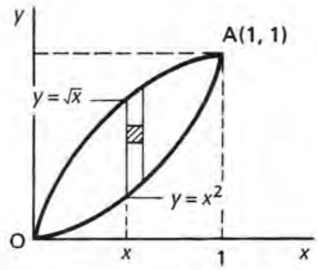
$$I = \frac{31}{60}$$

Here is the working.

$$I = \oint_c \{xy\,dx + (2x - y)\,dy\}$$

$$\oint_c \{P\,dx + Q\,dy\} = -\int_R\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$P = xy \quad \therefore \quad \frac{\partial P}{\partial y} = x; \quad Q = 2x - y \quad \therefore \quad \frac{\partial Q}{\partial x} = 2$$



$$I = -\int_R\int (x - 2)\,dx\,dy$$

$$= -\int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x - 2)\,dy\,dx$$

$$= -\int_0^1 (x - 2)\Big[y\Big]_{x^2}^{\sqrt{x}} dx$$

$$\therefore \; I = -\int_0^1 (x - 2)(\sqrt{x} - x^2)\,dx$$

$$= -\int_0^1 (x^{3/2} - x^3 - 2x^{1/2} + 2x^2)\,dx$$

$$= -\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 - \frac{4}{3}x^{3/2} + \frac{2}{3}x^3\right]_0^1 = \frac{31}{60}$$

Before we finally leave this section of the work, there is one more result to note.

In the special case when $P = y$ and $Q = -x$

$$\frac{\partial P}{\partial y} = 1 \quad \text{and} \quad \frac{\partial Q}{\partial x} = -1$$

Green's theorem then states

$$\int_R\int \{1 - (-1)\}\,dx\,dy = -\oint_c (P\,dx + Q\,dy)$$

i.e.
$$2\int_R\int dx\,dy = -\oint_c (y\,dx - x\,dy)$$

$$= \oint_c (x\,dy - y\,dx)$$

Therefore, the area of the closed region

$$A = \int_R\int dx\,dy = \frac{1}{2}\oint_c (x\,dy - y\,dx)$$

Note this result in your record book. Then let us see an example

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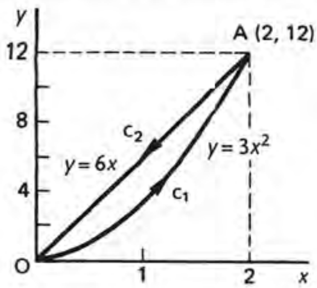
Example 1

Determine the area of the figure enclosed by $y = 3x^2$ and $y = 6x$.



Points of intersection:

$$3x^2 = 6x \quad \therefore \; x = 0 \text{ or } 2$$

$$\text{Area } A = \frac{1}{2}\oint_c (x\,dy - y\,dx)$$

We evaluate the integral in two parts, i.e. OA along c_1

and AO along c_2

$$2A = \int_{c_1 \text{ (along OA)}} (x\,dy - y\,dx) + \int_{c_2 \text{ (along AO)}} (x\,dy - y\,dx) = I_1 + I_2$$

I_1: c_1 is $y = 3x^2 \quad \therefore \; dy = 6x\,dx$

$$\therefore \; I_1 = \int_0^2 (6x^2\,dx - 3x^2\,dx) = \int_0^2 3x^2\,dx = \left[x^3\right]_0^2 = 8$$

$$\therefore \; I_1 = 8$$

Similarly, $I_2 = \ldots\ldots\ldots\ldots$

78

$$\boxed{I_2 = 0}$$

Because

c_2 is $y = 6x \quad \therefore \; dy = 6\,dx$

$$\therefore \; I_2 = \int_2^0 (6x\,dx - 6x\,dx) = 0 \quad \therefore \; I_2 = 0$$

$$\therefore \; I = I_1 + I_2 = 8 + 0 = 8 \quad \therefore \; A = 4 \text{ square units}$$

Finally, here is one for you to do entirely on your own.

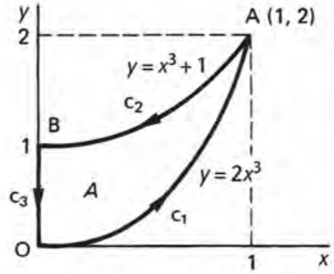
Example 2

Determine the area bounded by the curves $y = 2x^3$, $y = x^3 + 1$ and the axis $x = 0$ for $x \geq 0$.

Complete the working and see if you agree with the working in the next frame

79

Here it is.



$y = 2x^3$; $y = x^3 + 1$; $x = 0$

Point of intersection

$2x^3 = x^3 + 1 \quad \therefore\ x^3 = 1 \quad \therefore\ x = 1$

Area $A = \dfrac{1}{2}\oint_c (x\,dy - y\,dx)$

$\therefore\ 2A = \oint_c (x\,dy - y\,dx)$

(a) OA: c_1 is $y = 2x^3 \quad \therefore\ dy = 6x^2\,dx$

$$\therefore\ I_1 = \int_{c_1}(x\,dy - y\,dx) = \int_0^1 (6x^3\,dx - 2x^3\,dx)$$

$$= \int_0^1 4x^3\,dx = \Big[x^4\Big]_0^1 = 1 \qquad \therefore\ I_1 = 1$$

(b) AB: c_2 is $y = x^3 + 1 \quad \therefore\ dy = 3x^2\,dx$

$$\therefore\ I_2 = \int_1^0 \{3x^3\,dx - (x^3+1)\,dx\} = \int_1^0 (2x^3 - 1)\,dx$$

$$= \left[\frac{x^4}{2} - x\right]_1^0 = -(\tfrac{1}{2} - 1) = \tfrac{1}{2} \qquad \therefore\ I_2 = \tfrac{1}{2}$$

(c) BO: c_3 is $x = 0 \quad \therefore\ dx = 0$

$$I_3 = \int_{y=1}^{y=0}(x\,dy - y\,dx) = 0 \qquad \therefore\ I_3 = 0$$

$\therefore\ 2A = I = I_1 + I_2 + I_3 = 1 + \tfrac{1}{2} + 0 = 1\tfrac{1}{2}$

$\therefore\ A = \tfrac{3}{4}$ square units

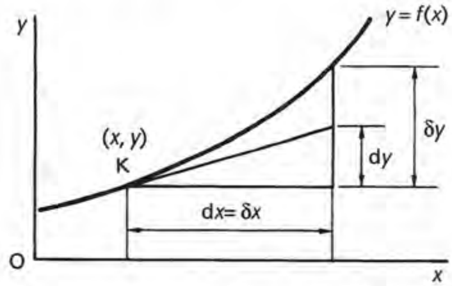
And that brings this Programme to an end. We have covered some important topics, so check down the **Revision summary** and the **Can You?** checklist that follow and revise any part of the text if necessary, before working through the **Test exercise**. The **Further problems** provide an opportunity for additional practice.

80

Revision summary 14

1 *Differentials* dy and dx

(a)



$$\mathrm{d}y = f'(x)\,\mathrm{d}x$$

(b) If $z = f(x, y)$, $\quad \mathrm{d}z = \dfrac{\partial z}{\partial x}\,\mathrm{d}x + \dfrac{\partial z}{\partial y}\,\mathrm{d}y$

If $z = f(x, y, w)$, $\quad \mathrm{d}z = \dfrac{\partial z}{\partial x}\,\mathrm{d}x + \dfrac{\partial z}{\partial y}\,\mathrm{d}y + \dfrac{\partial z}{\partial w}\,\mathrm{d}w.$

(c) $\mathrm{d}z = P\,\mathrm{d}x + Q\,\mathrm{d}y$, where P and Q are functions of x and y, is an *exact differential* if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

2 *Line integrals* – definition

$$I = \int_c f(x, y)\,\mathrm{d}s = \int_c (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

3 *Properties of line integrals*

(a) Sign of line integral is reversed when the direction of integration along the path is reversed.

(b) Path of integration parallel to y-axis, $\mathrm{d}x = 0 \quad \therefore\ I_c = \int_c Q\,\mathrm{d}y$.

Path of integration parallel to x-axis, $\mathrm{d}y = 0 \quad \therefore\ I_c = \int_c P\,\mathrm{d}x$.

(c) The y-values on the path of integration must be continuous and single-valued.

4 *Line of integral round a closed curve* $\oint$

Positive direction $\circlearrowleft\!\!\oint$ anticlockwise

Negative direction $\circlearrowright\!\!\oint$ clockwise, i.e. $\circlearrowright\!\!\oint = -\circlearrowleft\!\!\oint$.

▶

5 *Line integral related to arc length*

$$I = \int_{AB} F\,ds = \int_{AB} (P\,dx + Q\,dy)$$

$$= \int_{\substack{x_1 \\ c}}^{x_2} f(x, y)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

With parametric equations, x and y in terms of t,

$$I = \int_c f(x, y)\,ds = \int_{t_1}^{t_2} f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

6 *Dependence of line integral on path of integration*

In general, the value of the line integral depends on the particular path of integration.

7 *Exact differential*

If $P\,dx + Q\,dy$ is an exact differential where P, Q and their first derivatives are finite and continuous inside the simply connected region R

(a) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

(b) $I = \int_c (P\,dx + Q\,dy)$ is independent of the path of integration where c lies entirely within R

(c) $I = \oint_c (P\,dx + Q\,dy)$ is zero when c is a closed curve lying entirely within R.

8 *Exact differentials in three variables*

If $P\,dx + Q\,dy + R\,dz$ is an exact differential where P, Q, R and their first partial derivatives are finite and continuous inside a simply connected region containing path c

(a) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$; $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$; $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$

(b) $\int_c (P\,dx + Q\,dy + R\,dz)$ is independent of the path of integration

(c) $\oint_c (P\,dx + Q\,dy + R\,dz)$ is zero when c is a closed curve.

9 *Green's theorem*

$$\oint_c (P\,dx + Q\,dy) = -\int_R\int \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} dx\,dy$$

and, for a simple closed curve

$$\oint_c (x\,dy - y\,dx) = 2\int_R\int dx\,dy = 2A$$

where A is the area of the enclosed figure.

Can You?

Checklist 14

81

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Evaluate double and triple integrals and apply them to the determination of the areas of plane figures and the volumes of solids? 1 to 10

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Understand the role of the differential of a function of two or more real variables? 11 to 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine exact differentials in two real variables and their integrals? 14 to 19

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate the area enclosed by a closed curve by contour integration? 20 to 26

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate line integrals and appreciate their properties? 27 to 41

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Evaluate line integrals around closed curves within a simply connected region? 42 to 50

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Link line integrals to integrals along the x-axis? 51 to 53

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Link line integrals to integrals along a contour given in parametric form? 54 to 56

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Discuss the dependence of a line integral between two points on the path of integration? 56 to 65

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine exact differentials in three real variables and their integrals? 66 to 69

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Demonstrate the validity and use of Green's theorem? 70 to 79

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 14

82

1 Determine the differential dz of each of the following.
(a) $z = x^4 \cos 3y$; (b) $z = e^{2y} \sin 4x$; (c) $z = x^2yw^3$.

2 Determine which of the following are exact differentials and integrate where appropriate to determine z.
(a) $dz = (3x^2y^4 + 8x)\,dx + (4x^3y^3 - 15y^2)\,dy$
(b) $dz = (2x\cos 4y - 6\sin 3x)dx - 4(x^2 \sin 4y - 2y)\,dy$
(c) $dz = 3e^{3x}(1 - y)\,dx + (e^{3x} + 3y^2)\,dy$.

3 Calculate the area of the triangle with vertices at O (0, 0), A (4, 2) and B (1, 5).

4 Evaluate the following.
(a) $I = \int_c \{(x^2 - 3y)\,dx + xy^2\,dy\}$ from A (1, 2) to B (2, 8) along the curve $y = 2x^2$.
(b) $I = \int_c (2x + y)\,dx$ from A (0, 1) to B (0, −1) along the semicircle $x^2 + y^2 = 1$ for $x \geq 0$.
(c) $I = \oint_c \{(1 + xy)\,dx + (1 + x^2)\,dy\}$ where c is the boundary of the rectangle joining A (1, 0), B (4, 0), C (4, 3) and D (1, 3).
(d) $I = \int_c 2xy\,ds$ where c is defined by the parametric equations $x = 4\cos\theta$, $y = 4\sin\theta$ between $\theta = 0$ and $\theta = \dfrac{\pi}{3}$.
(e) $I = \int_c \{(8xy + y^3)\,dx + (4x^2 + 3xy^2)\,dy\}$ from A(1, 3) to B(2, 1).
(f) $I = \oint_c \{(3x + y)\,dx + (y - 2x)\,dy\}$ round the boundary of the ellipse $x^2 + 4y^2 = 36$.

5 Apply Green's theorem to determine the area of the plane figure bounded by the curves $y = x^3$ and $y = \sqrt{x}$.

6 Verify that $dw = (2xyz + 2z - y^2)dx + (x^2z - 2yx)dy + (x^2y + 2x)dz$ is an exact differential and find the value of

$\int_c dw$ where

(a) c is the straight line joining (0, 0, 0) to (1, 1, 1)
(b) c is the curve of intersection of the unit sphere centred on the origin and the plane $x + y + z = 1$.

Further problems 14

83

1 Show that $I = \int_c \{xy^2w^2\,dx + x^2yw^2\,dy + x^2y^2w\,dw\}$ is independent of the path of integration c and evaluate the integral from A (1, 3, 2) to B (2, 4, 1).

2 Determine whether $dz = 3x^2(x^2 + y^2)\,dx + 2y(x^3 + y^4)\,dy$ is an exact differential. If so, determine z and hence evaluate $\int_c dz$ from A (1, 2) to B (2, 1).

3 Evaluate the line integral $I = \oint_c \left\{\dfrac{x\,dy - y\,dx}{x^2 + y^2 + 4}\right\}$ where c is the boundary of the segment formed by the arc of the circle $x^2 + y^2 = 4$ and the chord $y = 2 - x$ for $x \geq 0$.

4 Show that

$$I = \int_c \{(3x^2 \sin y + 2\sin 2x + y^3)\,dx + (x^3 \cos y + 3xy^2)\,dy\}$$

is independent of the path of integration and evaluate it from A (0, 0) to B $\left(\dfrac{\pi}{2}, \pi\right)$.

5 Evaluate the integral $I = \int_c xy\,ds$ where c is defined by the parametric equations $x = \cos^3 t$, $y = \sin^3 t$ from $t = 0$ to $t = \dfrac{\pi}{2}$.

6 Verify that $dz = \dfrac{x\,dx}{x^2 - y^2} - \dfrac{y\,dy}{x^2 - y^2}$ for $x^2 > y^2$ is an exact differential and evaluate $z = f(x, y)$ from A (3, 1) to B (5, 3).

7 The parametric equations of a circle, centre (1, 0) and radius 1, can be expressed as $x = 2\cos^2\theta$, $y = 2\cos\theta\sin\theta$.

Evaluate $I = \int_c \{(x + y)\,dx + x^2\,dy\}$ along the semicircle for which $y \geq 0$ from O (0, 0) to A (2, 0).

8 Evaluate $\oint_c \{x^3y^2\,dx + x^2y\,dy\}$ where c is the boundary of the region enclosed by the curve $y = 1 - x^2$, $x = 0$ and $y = 0$ in the first quadrant.

9 Use Green's theorem to evaluate

$$I = \oint_c \{(4x + y)\,dx + (3x - 2y)\,dy\}$$

where c is the boundary of the trapezium with vertices A (0, 1), B (5, 1), C (3, 3) and D (1, 3).

10 Evaluate $I = \int_c \{(3x^2y^2 + 2\cos 2x - 2xy)\,dx + (2x^3y + 8y - x^2)\,dy\}$

(a) along the curve $y = x^2 - x$ from A (0, 0) to B (2, 2)

(b) round the boundary of the quadrilateral joining the points (1, 0), (3, 1), (2, 3) and (0, 3)

▶

11 Verify that $dw = \frac{y}{z}dx + \frac{x}{z}dy - \frac{xy}{z^2}dz$ is an exact differential and find the value of

$$\int_c dw$$

where c is the straight line joining (0, 0, 1) to (1, 2, 3) for either region $z > 0$ or $z < 0$.

Multiple integration 2

Frames 1 to 77

Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate double integrals and surface integrals
- Relate three-dimensional Cartesian coordinates to cylindrical and spherical polar forms
- Evaluate volume integrals in Cartesian coordinates and in cylindrical and spherical polar coordinates
- Use the Jacobian to convert integrals given in Cartesian coordinates into general curvilinear coordinates in two and three dimensions

Double integrals

1

Let us start off with an example with which we are already familiar.

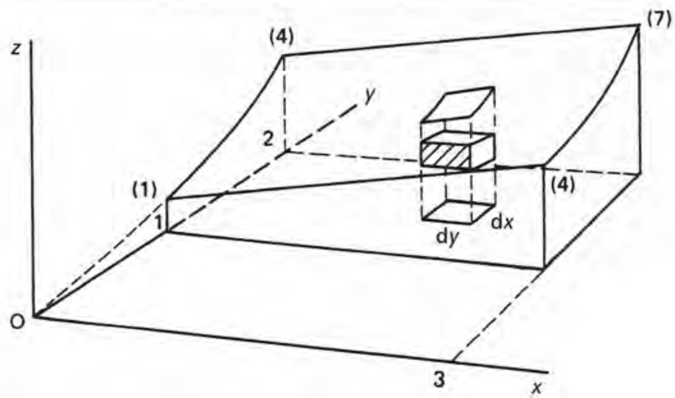
Example 1

A solid is enclosed by the planes $z = 0$, $y = 1$, $y = 2$, $x = 0$, $x = 3$ and the surface $z = x + y^2$. We have to determine the volume of the solid so formed.

First take some care in sketching the figure, which is

....

2



In the plane $y = 1$, $z = x + 1$, i.e. a straight line joining (0, 1, 1) and (3, 1, 4)

In the plane $y = 2$, $z = x + 4$, i.e. a straight line joining (0, 2, 4) and (3, 2, 7)

In the plane $x = 0$, $z = y^2$, i.e. a parabola joining (0, 1, 1) and (0, 2, 4)

In the plane $x = 3$, $z = 3 + y^2$, i.e. a parabola joining (3, 1, 4) and (3, 2, 7).

Consideration like this helps us to visualise the problem and the time involved is well spent.

Now we can proceed.

The element of volume $\delta v = \delta x \, \delta y \, \delta z$

Then the total volume $V = \iiint dx\, dy\, dz$ between appropriate limits in each case.

▶

We could also have said that the element of area on the $z = 0$ plane

$\delta a = \delta y\,\delta x$

and that the volume of the column $\quad \delta v_c = z\,\delta a = z\,\delta x\,\delta y$

Then, since $z = x + y^2$, this becomes $\quad \delta v_c = (x + y^2)\,\delta x\,\delta y$

Summing in the usual way then gives

$$V = \int z\,\mathrm{d}a$$
$$= \int_R \int (x + y^2)\,\mathrm{d}x\,\mathrm{d}y$$

where R is the region bounded in the x–y plane.

Now we insert the appropriate limits and complete the integration

$V = \ldots\ldots\ldots\ldots$

3

$\boxed{V = 11{\cdot}5 \text{ cubic units}}$

Because

$$V = \int_{y=1}^{y=2} \int_{x=0}^{x=3} (x + y^2)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_1^2 \left[\frac{x^2}{2} + xy^2\right]_{x=0}^{x=3} \mathrm{d}y$$
$$= \int_1^2 \left(\frac{9}{2} + 3y^2\right) \mathrm{d}y$$
$$= \left[\frac{9}{2}y + y^3\right]_1^2$$
$$= 11{\cdot}5$$
$$\therefore \; V = 11{\cdot}5 \text{ cubic units}$$

Although we have found a volume, this is, in fact, an example of a *double integral* since the expression for z was a function of position in the x–y plane within the closed region

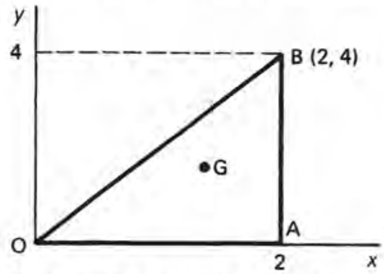
i.e. $$I = \int_R \int f(x, y)\,\mathrm{d}a$$
$$= \int_R \int f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

In this particular case, R is the region in the x–y plane bounded by $x = 0$, $x = 3$, $y = 1$, $y = 2$.

▶

Example 2

A triangular thin plate has the dimensions shown and a variable density ρ where $\rho = 1 + x + xy$.

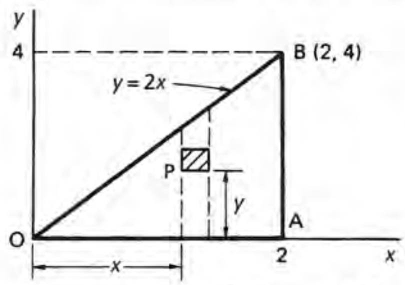


We have to determine

(a) the mass of the plate

(b) the position of its centre of gravity G.

(a) Consider an element of area at the point P (x, y) in the plate



$\delta a = \delta x\, \delta y$

The mass δm of the element is then

$\delta m = \rho\, \delta x\, \delta y$

$$\therefore \text{ Total mass } M = \int_R\int \mathrm{d}m = \int_R\int \rho\, \mathrm{d}x\, \mathrm{d}y$$

Now we insert the limits and complete the integration, remembering that $\rho = (1 + x + xy)$

$M = \ldots\ldots\ldots\ldots$

4

$$\boxed{M = 17\frac{1}{3}}$$

Because we have

$$M = \int_R\int \rho\, \mathrm{d}x\, \mathrm{d}y = \int_{x=0}^{x=2}\int_{y=0}^{y=2x}(1 + x + xy)\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_0^2\left[y + xy + \frac{xy^2}{2}\right]_{y=0}^{y=2x}\mathrm{d}x$$

$$= \int_0^2\{2x + 2x^2 + 2x^3\}\mathrm{d}x$$

$$= \left[x^2 + \frac{2x^3}{3} + \frac{x^4}{2}\right]_0^2 = 17\frac{1}{3}$$

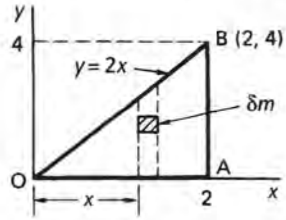
(b) To find the position of the centre of gravity, we need to know

$\ldots\ldots\ldots\ldots$

5

the sum of the moments of mass about OY and OX

(1) To find $\bar{x}$, we take moments about OY.



Moment of mass of element about OY

$$= x\,\delta m$$
$$= x(1 + x + xy)\,\delta x\,\delta y$$

$$\therefore \text{ Sum of first moments} = \int_R\int (x + x^2 + x^2y)\,dx\,dy$$
$$= \ldots\ldots\ldots\ldots$$

6

$$\boxed{26\frac{2}{15}}$$

$$\text{Because sum of first moments} = \int_{x=0}^{x=2}\int_{y=0}^{y=2x}(x + x^2 + x^2y)\,dy\,dx$$
$$= \int_0^2 \left[xy + x^2y + \frac{x^2y^2}{2}\right]_{y=0}^{y=2x} dx$$
$$= \int_0^2 \{2x^2 + 2x^3 + 2x^4\}dx$$
$$= 2\int_0^2 (x^2 + x^3 + x^4)\,dx$$
$$= 2\left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right]_0^2 = 26\frac{2}{15}$$

Now $M\bar{x} =$ sum of moments $\therefore\ \bar{x} = \ldots\ldots\ldots\ldots$

7

$$\boxed{\bar{x} = 1{\cdot}508}$$

We found previously that $M = 17\frac{1}{3}$ $\therefore\ \left(17\frac{1}{3}\right)\bar{x} = 26\frac{2}{15}$

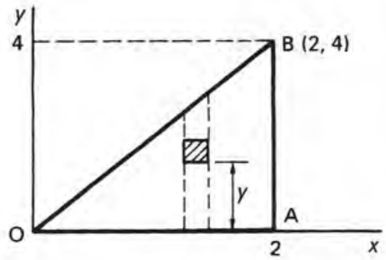
which gives $\bar{x} = 1\frac{33}{65} = 1{\cdot}508$

(2) To find $\bar{y}$ we proceed in just the same way, this time taking moments about OX. Work right through it on your own.

$\bar{y} = \ldots\ldots\ldots\ldots$

8

$$\boxed{\bar{y} = 1{\cdot}754}$$



Moment of element of mass δm about OX

$$= y\,\delta m = y(1 + x + xy)\,\delta x\,\delta y$$

$$\therefore \text{ Sum of first moments about OX} = \int_R\int (y + xy + xy^2)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2x} (y + xy + xy^2)\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^2 \left[\frac{y^2}{2} + \frac{xy^2}{2} + \frac{xy^3}{3}\right]_{y=0}^{y=2x} \mathrm{d}x$$

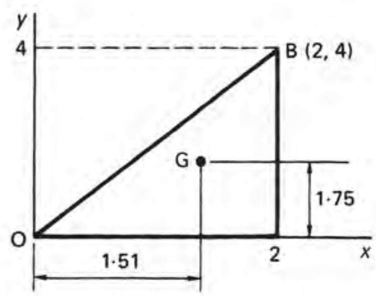
$$= \int_0^2 \left\{2x^2 + 2x^3 + \frac{8x^4}{3}\right\} \mathrm{d}x$$

$$= \left[\frac{2x^3}{3} + \frac{x^4}{2} + \frac{8x^5}{15}\right]_0^2$$

$$= 30\frac{2}{5}$$

$$\therefore\ M\bar{y} = 30\frac{2}{5} \quad \therefore\ \bar{y} = 30\frac{2}{5}\Big/17\frac{1}{3} = 1{\cdot}754$$

So we finally have:



Note that this again referred to a plane figure in the x–y plane.

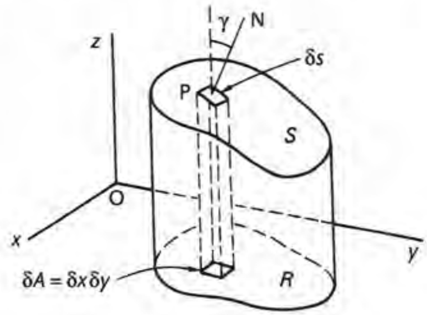
Now let us move on to something slightly different

Surface integrals

9

When the area over which we integrate is not restricted to the x–y plane, matters become rather more involved, but also more interesting.

If S is a two-sided surface in space and R is its projection on the x–y plane, then the equation of S is of the form $z = f(x, y)$ where f is a single-valued function and continuous throughout R.



Let δA denote an element of R and δS the corresponding element of area of S at the point $P(x, y, z)$ in S.

Let also $\phi(x, y, z)$ be a function of position on S (e.g. potential) and let γ denote the angle between the outward normal PN to the surface at P and the positive z-axis.

Then $\delta A \approx \delta S \cos\gamma$ i.e. $\delta S \approx \dfrac{\delta A}{\cos\gamma} = \delta A \sec\gamma$ and

$\sum \phi(x, y, z)\delta S$ is the total value of $\phi(x, y, z)$ taken over the surface S.

As $\delta S \to 0$, this sum becomes the integral

$$I = \int_S \phi(x, y, z)\,\mathrm{d}S$$

and, since $\delta S \approx \delta A \sec\gamma$, the result can be written

$$I = \int_R\int \phi(x, y, z)\sec\gamma\,\mathrm{d}x\,\mathrm{d}y \qquad \left(\gamma < \frac{\pi}{2}\right)$$

Notice that $\cos\gamma = \hat{\mathbf{n}}\cdot\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction and $\hat{\mathbf{n}}$ is the unit normal to the surface at P.

With limits inserted for x and y, the integral seems straightforward, except for the factor $\sec\gamma$, which naturally varies over the surface S.

We can, in fact, show that $\sec\gamma = \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}$

(see Programme 17, Frames 69 *ff*)

Therefore, the *surface integral* of $\phi(x, y, z)$ over the surface S is given by

(a) $I = \int_S \phi(x, y, z)\,\mathrm{d}S$ (1)

or (b) $I = \int_R\int \phi(x, y, z)\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$ (2)

where $z = f(x, y)$

▶

Note that, when $\phi(x, y, z) = 1$, then $I = \int_S \mathrm{d}S$ gives the area of the surface S.

$$\therefore\ S = \int_S \mathrm{d}S = \int_R \int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}x\, \mathrm{d}y \qquad (3)$$

Make a note of these three important results.

Then we will apply them to a few examples.

10

Example 1

Find the area of the surface $z = \sqrt{x^2 + y^2}$ over the region bounded by $x^2 + y^2 = 1$.

$$S = \int_R \int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}x\, \mathrm{d}y$$

So we now find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ and determine $\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}$ which is ....

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$$\boxed{\sqrt{2}}$$

Because

$$z = (x^2 + y^2)^{1/2} \quad \therefore\ \frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2} 2y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\therefore\ 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2 + y^2}{x^2 + y^2} = 2$$

$$\therefore\ \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2}$$

$$\therefore\ S = \sqrt{2} \int_R \int \mathrm{d}x\, \mathrm{d}y = \sqrt{2} \times \ldots\ldots\ldots\ldots$$

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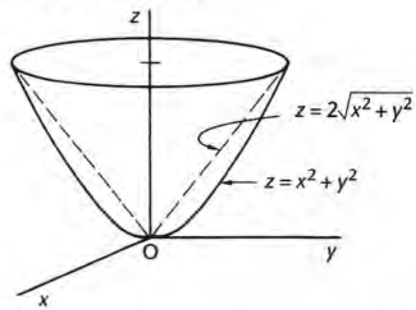
$\boxed{\text{the area of the region } R}$

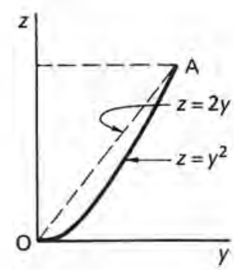
But R is bounded by $x^2 + y^2 = 1$, i.e. a circle, centre the origin and radius 1. $\therefore$ area $= \pi$

$$\therefore \; S = \sqrt{2}\int_R\int \mathrm{d}x\,\mathrm{d}y = \sqrt{2}\pi$$

Example 2

Find the area of the surface S of the paraboloid $z = x^2 + y^2$ cut off by the cone $z = 2\sqrt{x^2 + y^2}$.





We can find the point of intersection A by considering the y–z plane, i.e. put $x = 0$.

Coordinates of A are ....

13

$\boxed{\text{A } (2, 4)}$

The projection of the surface S on the x–y plane is

....

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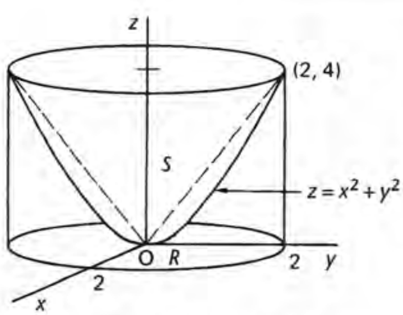
$\boxed{\text{the circle } x^2 + y^2 = 4}$



$$S = \int_R\int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$$

For this we use the equation of the surface S. The information from the projection R on the x–y plane will later provide the limits of the two stages of integration.

For the time being, then, $S =$ ....

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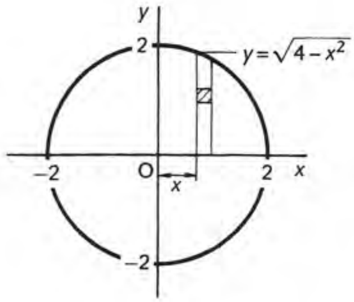
$$S = \int_R \int \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$



Using Cartesian coordinates, we could integrate with respect to y from $y = 0$ to $y = \sqrt{4 - x^2}$ and then with respect to x from $x = 0$ to $x = 2$. Finally, we should multiply by four to cover all four quadrants.

i.e. $$S = 4 \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

But how do we carry out the actual integration?

It becomes a lot easier if we use polar coordinates.

The same integral in polar coordinates is ....

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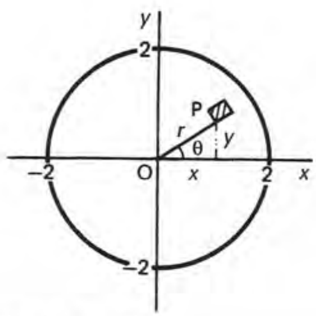
$$S = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$



$x = r\cos\theta$; $\quad y = r\sin\theta$

$x^2 + y^2 = r^2 \quad dx\, dy = r\, dr\, d\theta$

(refer to Frame 67)

$$S = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$\therefore\ S = \ldots\ldots\ldots\ldots$

Finish it off.

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$$\boxed{S = 36{\cdot}18 \text{ square units}}$$

Because

$$S = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}(1+4r^2)^{1/2} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^2 d\theta$$
$$= \frac{1}{12}\int_0^{2\pi}\{17^{3/2}-1\}\,d\theta = 5{\cdot}7577\left[\theta\right]_0^{2\pi} = 36{\cdot}18$$

Now on to Example 3.

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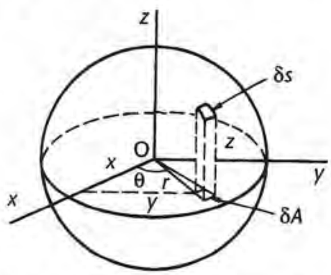
Example 3

To determine the moment of inertia of a thin spherical shell of radius a about a diameter as axis. The mass per unit area of shell is ρ.



Equation of sphere

$$x^2 + y^2 + z^2 = a^2$$

Mass of element $= m = \rho\,\delta S$

$$I \approx \Sigma m r^2 \approx \Sigma \rho\,\delta S r^2$$

Let us deal with the upper hemisphere

$$\therefore\ I_H = \int_S \rho r^2\,dS$$
$$= \int_R\int \rho r^2\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

Now determine the partial derivatives and simplify the integral as far as possible in Cartesian coordinates.

$$I_H = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_H = \int_R\int \rho r^2 \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy}$$

In this particular example, R is, of course, the region bounded by the circle $x^2+y^2=a^2$ in the x–y plane.

Converting to polar coordinates

$$x = r\cos\theta;\quad y = r\sin\theta;\quad dx\,dy = r\,dr\,d\theta$$

the integral becomes $I_H = \ldots\ldots\ldots\ldots$

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$$\boxed{I_{\mathrm{H}} = \rho a \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r \, \mathrm{d}\theta}$$

Because for $x^2 + y^2 = r^2$: limits of r: $r = 0$ to $r = a$
limits of θ: $\theta = 0$ to $\theta = 2\pi$

$$I_{\mathrm{H}} = \int_R \int \rho r^2 \frac{a}{\sqrt{a^2 - r^2}} r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$= \rho a \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r \, \mathrm{d}\theta$$

First we have to evaluate

$$I_r = \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r$$

If we substitute $u = a^2 - r^2$ then the integral is evaluated as

$I_r = \ldots\ldots\ldots\ldots$

21

$$\boxed{I_r = \frac{2a^3}{3}}$$

Because

When $u = a^2 - r^2$ then $\mathrm{d}u = -2r \, \mathrm{d}r$ so that $r^2 = a^2 - u$ and $r \, \mathrm{d}r = -\dfrac{\mathrm{d}u}{2}$. Therefore

$$I_r = \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} \, \mathrm{d}r = \int_{r=0}^a \frac{r^2}{\sqrt{a^2 - r^2}} r \, \mathrm{d}r$$

$$= -\int_{u=a^2}^0 \frac{a^2 - u}{\sqrt{u}} \frac{\mathrm{d}u}{2}$$

$$= -\frac{a^2}{2} \int_{u=a^2}^0 u^{-1/2} \, \mathrm{d}u + \frac{1}{2} \int_{u=a^2}^0 u^{1/2} \, \mathrm{d}u$$

$$= -\frac{a^2}{2} \left[2u^{1/2} \right]_{u=a^2}^0 + \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{u=a^2}^0$$

$$= a^3 - \frac{a^3}{3}$$

$$= \frac{2a^3}{3}$$

Now, to complete I_{H} we have

$$I_{\mathrm{H}} = \rho a \int_0^{2\pi} \frac{2a^3}{3} \, \mathrm{d}\theta$$

$= \ldots\ldots\ldots\ldots$

22

$$\boxed{I_{\mathrm{H}} = \frac{4\pi\rho a^4}{3}}$$

Because

$$I_{\mathrm{H}} = \rho a \int_0^{2\pi} \frac{2a^3}{3}\,\mathrm{d}\theta = \frac{2a^4\rho}{3}\Big[\theta\Big]_0^{2\pi} = \frac{4\pi a^4\rho}{3}$$

Therefore, the moment of inertia for the complete spherical shell is

$$I_s = \frac{8\pi a^4\rho}{3}$$

The total mass of the shell $M = 4\pi a^2\rho \quad \therefore\; I = \dfrac{2Ma^2}{3}$

Now let us turn our attention towards *volume integrals* and in preparation review systems of space coordinates.

Space coordinate systems

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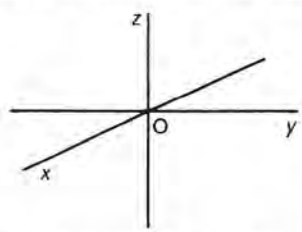
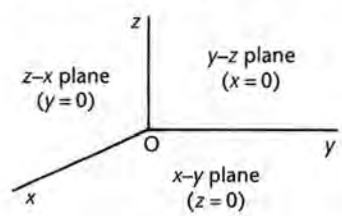
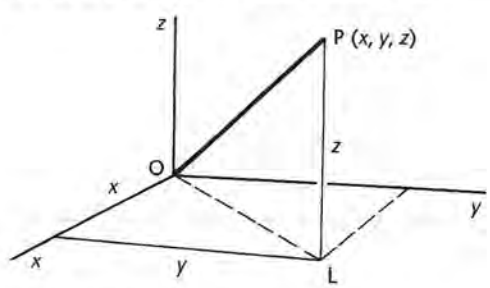
1 *Cartesian coordinates* (x, y, z) – referred to three coordinate axes OX, OY, OZ at right angles to each other. These are arranged in a *right-handed* manner, i.e. turning from OX to OY gives a right-handed screw action in the positive direction of OZ.





The three coordinate planes, $x = 0$, $y = 0$, $z = 0$, divide the space into eight sections called *octants*. The section containing $x \geq 0$, $y \geq 0$, $z \geq 0$ is called the *first octant*.



For a point P (x, y, z)

$\mathrm{OL}^2 = x^2 + y^2$

$\mathrm{OP}^2 = x^2 + y^2 + z^2$

Note that this is Pythagoras' theorem in three dimensions.

We are all familiar with this system of coordinates.

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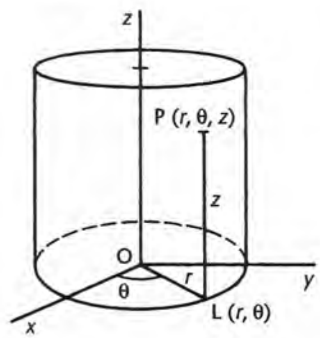
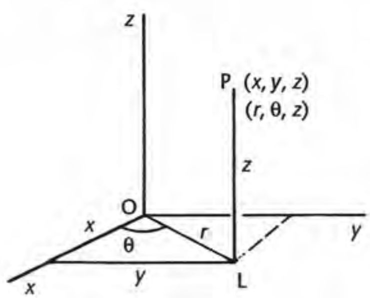
2 *Cylindrical coordinates* (r, θ, z) are useful where an axis of symmetry occurs.



Any point P is considered as having a position on a cylinder. If L is the projection of P on the x–y plane, then (r, θ) are the usual polar coordinates of L. The cylindrical coordinates of P then merely require the addition of the z-coordinate.

$r \geq 0$

Relationship between Cartesian and cylindrical coordinates



If we consider a combined figure, we can easily relate the two systems. Expressing each of the following in terms of the alternative system,

$x = \ldots\ldots\ldots\ldots$ $r = \ldots\ldots\ldots\ldots$
$y = \ldots\ldots\ldots\ldots$ $\theta = \ldots\ldots\ldots\ldots$
$z = \ldots\ldots\ldots\ldots$ $z = \ldots\ldots\ldots\ldots$

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$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\theta = \arctan(y/x)$
$z = z$	$z = z$

So, in cylindrical coordinates, the surface defined by

(1) $r = 5$ is ....
(2) $\theta = \pi/6$ is ....
(3) $z = 4$ is ....

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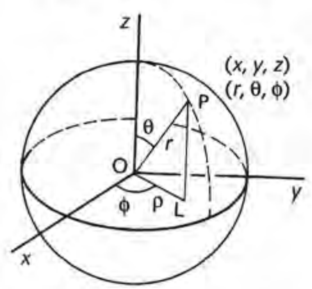
(1) $r = 5$ is a right cylinder, radius 5, with OZ as axis.
(2) $\theta = \pi/6$ is a plane through OZ, making an angle $\pi/6$ with OX.
(3) $z = 4$ is a plane parallel to the x–y plane cutting OZ at 4 units above the origin.

So position P (2, 3, 4) in Cartesian coordinates
= .... in cylindrical coordinates
and position Q (2·5, $\pi/3$, 6) in cylindrical coordinates
= .... in Cartesian coordinates.

27

P (2, 3, 4) = ($\sqrt{13}$, 0·983, 4) in cylindrical coordinates
Q (2·5, $\pi/3$, 6) = (1·25, 2·165, 6) in Cartesian coordinates.

3 *Spherical coordinates* (r, θ, ϕ) are appropriate where a centre of symmetry occurs. The position of a point is considered as being a point on a sphere.



r is the distance of P from the origin and is always taken as positive.

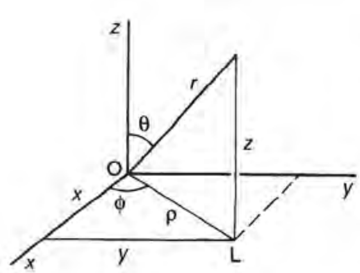
L is the projection of P on the x–y plane

θ is the angle between OP and the positive OZ axis

ϕ is the angle between OL and the OX axis.

Note that (a) ϕ may be regarded as the longitude of P from OX
(b) θ may be regarded as the complement of the latitude of P.

Relationship between Cartesian and spherical coordinates



The combined figure shows the connection between the two systems, so

$x =$ .... $r =$ ....
$y =$ .... $\theta =$ ....
$z =$ .... $\phi =$ ....

28

$$\boxed{\begin{array}{ll} x = r\sin\theta\cos\phi & r = \sqrt{x^2+y^2+z^2} \\ y = r\sin\theta\sin\phi & \theta = \arccos(z/r) \\ z = r\cos\theta & \phi = \arctan(y/x) \end{array}}$$

For the spherical coordinates of any point in space

$r \geq 0$; $\quad 0 \leq \theta \leq \pi$; $\quad 0 \leq \phi \leq 2\pi$

So, converting Cartesian coordinates (2, 3, 4) to spherical coordinates gives ....

29

$$\boxed{\text{P}\,(r,\,\theta,\,\phi) = (5{\cdot}385,\ 0{\cdot}734,\ 0{\cdot}983)}$$

Because

$$x = 2,\ y = 3,\ z = 4$$

$$\therefore\ r = \sqrt{x^2+y^2+z^2} = \sqrt{4+9+16} = \sqrt{29} = 5{\cdot}385$$

$$\theta = \arccos(z/r) = \arccos(4/\sqrt{29}) = 0{\cdot}734$$

$$\phi = \arctan(y/x) = \arctan 1{\cdot}5 = 0{\cdot}983$$

And, in reverse, spherical coordinates $(5,\ \pi/4,\ \pi/3)$ transform into Cartesian coordinates ....

30

$$\boxed{\text{P}\,(x,\,y,\,z) = (1{\cdot}768,\ 3{\cdot}061,\ 3{\cdot}536)}$$

Because

$$\begin{aligned} x &= r\sin\theta\cos\phi = 5\sin\frac{\pi}{4}\cos\frac{\pi}{3} = 5(0{\cdot}707)(0{\cdot}5) &&= 1{\cdot}768 \\ y &= r\sin\theta\sin\phi = 5\sin\frac{\pi}{4}\sin\frac{\pi}{3} = 5(0{\cdot}707)(0{\cdot}866) &&= 3{\cdot}061 \\ z &= r\cos\theta = 5\cos\frac{\pi}{4} = 5(0{\cdot}707) &&= 3{\cdot}536. \end{aligned}$$

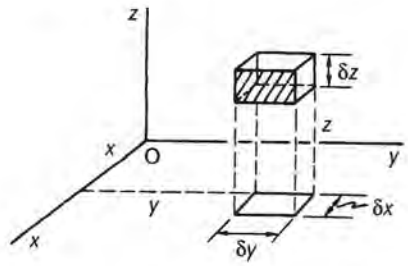
One of the main uses of cylindrical and spherical coordinates occurs in integrals dealing with volumes of solids. In preparation for this, let us consider the next important section of the work.

So move on

Element of volume in space in the three coordinate systems

31

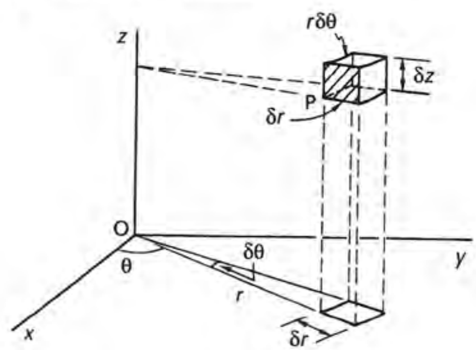
1 *Cartesian coordinates*

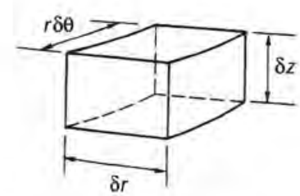


We have already used this many times.

$$\delta v = \delta x\,\delta y\,\delta z$$

2 *Cylindrical coordinates*

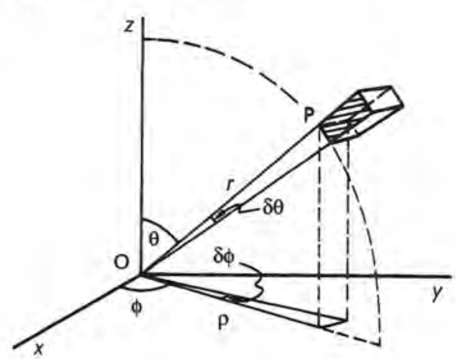


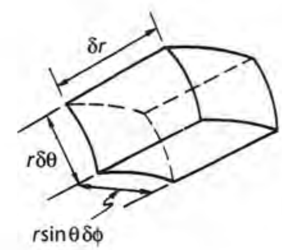


$$\delta v = r\delta\theta\,\delta r\,\delta z$$

$$\therefore\ \delta v = r\,\delta r\,\delta\theta\,\delta z$$

3 *Spherical coordinates*





$$\delta v = \delta r\,r\delta\theta\,r\,\sin\theta\,\delta\phi$$

$$\therefore\ \delta v = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

It is important to make a note of these results, since they are required when we change the variables in various types of integrals. We shall meet them again before long, so be sure of them now.

Volume integrals

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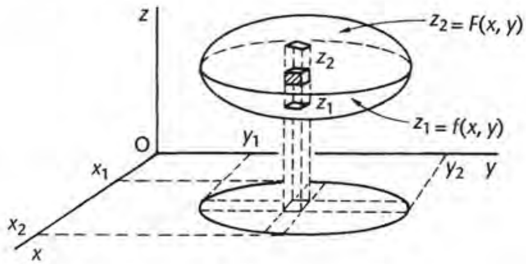

A solid is enclosed by a lower surface $z_1 = f(x, y)$ and an upper surface $z_2 = F(x, y)$.

Then, in general, using Cartesian coordinates, the element of volume is $\delta v = \delta x\,\delta y\,\delta z$.

The approximate value of the total volume V is then found

(a) by summing δv from $z = z_1$ to $z = z_2$ to obtain the volume of the column

(b) by summing all such columns from $y = y_1$ to $y = y_2$ to obtain the volume of the slice

(c) by summing all such slices from $x = x_1$ to $x = x_2$ to obtain the total volume V.

Then, when $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$, the summation becomes an integral

$$V = \int_{x=x_1}^{x=x_2} \int_{y=y_1}^{y=y_2} \int_{z=z_1}^{z=z_2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Example 1

Find the volume of the solid bounded by the planes $z = 0$, $x = 0$, $y = 0$, $x^2 + y^2 = 4$ and $z = 6 - xy$ for $x \geq 0$, $y \geq 0$, $z \geq 0$.

First sketch the figure, so that we can see what we are doing. Take your time over it.

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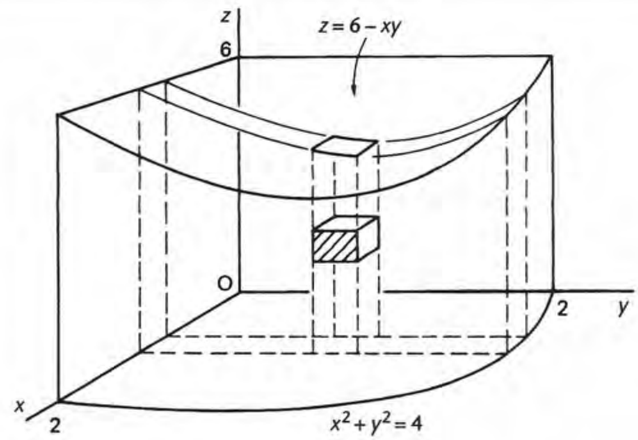


$$\delta v = \delta x\,\delta y\,\delta z$$

$$\text{Volume of column} \approx \sum_{z=0}^{z=6-xy} \delta x\,\delta y\,\delta z$$

$$\text{Volume of slice} \approx \sum_{y=0}^{\sqrt{4-x^2}} \left\{ \sum_{z=0}^{6-xy} \delta x\,\delta y\,\delta z \right\}$$

$$\text{Total volume} \approx \sum_{x=0}^{2} \sum_{y=0}^{\sqrt{4-x^2}} \sum_{z=0}^{6-xy} \delta x\,\delta y\,\delta z$$

If $\delta x \to 0$, $\delta y \to 0$, $\delta z \to 0$, then

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{6-xy} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Starting with the innermost integral

$$\int_0^{6-xy} \mathrm{d}z = \Big[z\Big]_0^{6-xy}$$
$$= 6 - xy$$

Then $\displaystyle\int_0^{\sqrt{4-x^2}} (6 - xy)\,\mathrm{d}y = \ldots\ldots\ldots\ldots$

34

$$\boxed{6\sqrt{4-x^2}-\frac{x}{2}(4-x^2)}$$

Because

$$\int_0^{\sqrt{4-x^2}}(6-xy)\,dy=\left[6y-\frac{xy^2}{2}\right]_{y=0}^{y=\sqrt{4-x^2}}$$
$$=6\sqrt{4-x^2}-\frac{x}{2}(4-x^2)$$

Then finally $V=\int_0^2\left\{6(4-x^2)^{1/2}-2x+\frac{x^3}{2}\right\}dx$

Now we are faced with $\int(4-x^2)^{1/2}\,dx$. You may remember that this is a standard form $\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\left\{x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right\}$.

If not, to evaluate $\int_0^2\sqrt{4-x^2}\,dx$, put $x=2\sin\theta$ and proceed from there.

Finish off the main integral, so that we have

$V=$ ....

35

$$\boxed{V=6\pi-2\approx 16{\cdot}8 \text{ cubic units}}$$

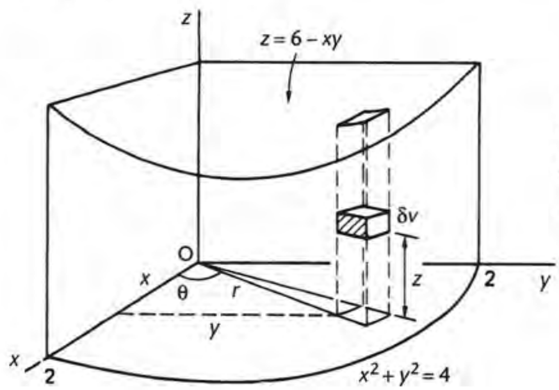
Because we had

$$V=\int_0^2\left\{6(4-x^2)^{1/2}-2x+\frac{x^3}{2}\right\}dx$$
$$=3\left[x\sqrt{4-x^2}+4\arcsin\frac{x}{2}\right]_0^2-\left[x^2-\frac{x^4}{8}\right]_0^2$$
$$=3\{4\arcsin 1-4\arcsin 0\}-4+2$$
$$=3\{2\pi\}-2=6\pi-2$$
$$\approx 16{\cdot}8$$

▶

Alternative method

We could, of course, have used cylindrical coordinates in this problem.



$$\delta v = r\,\delta r\,\delta\theta\,\delta z$$
$$x = r\cos\theta;\; y = r\sin\theta$$
$$\therefore\; z = 6 - xy$$
$$= 6 - r^2\sin\theta\cos\theta$$
$$= 6 - \frac{r^2}{2}\sin 2\theta$$

$$\therefore\; V = \int_{r=0}^{2}\int_{\theta=0}^{\pi/2}\int_{z=0}^{6-(r^2/2)\sin 2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$
$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{z=0}^{6-(r^2/2)\sin 2\theta} \mathrm{d}z\,r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \ldots\ldots\ldots\ldots$$

Finish it

36

$$\boxed{V = 6\pi - 2 \text{ (as before)}}$$

$$V = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\left(6 - \frac{r^2}{2}\sin 2\theta\right) r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\left(6r - \frac{r^3}{2}\sin 2\theta\right)\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_{0}^{\pi/2}\left[3r^2 - \frac{r^4}{8}\sin 2\theta\right]_{r=0}^{r=2}\mathrm{d}\theta$$
$$= \int_{0}^{\pi/2}(12 - 2\sin 2\theta)\,\mathrm{d}\theta$$
$$= \Big[12\theta + \cos 2\theta\Big]_{0}^{\pi/2}$$
$$= (6\pi - 1) - 1$$
$$\therefore\; V = 6\pi - 2$$

In this case, the use of cylindrical coordinates facilitates the evaluation.

Let us consider another example.

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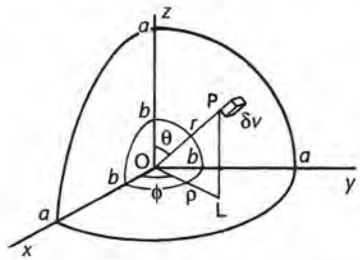
Example 2

To find the moment of inertia and radius of gyration of a thick hollow sphere about a diameter as axis. Outer radius $= a$; inner radius $= b$; density of material $= c$.

It is convenient to deal with one-eighth of the sphere in the first octant.



$\therefore$ Total mass of the solid $M_1 = \dfrac{1}{8}M$

$$M_1 = \frac{1}{8}\cdot\frac{4}{3}\pi(a^3 - b^3)c = \frac{\pi}{6}(a^3 - b^3)c$$

Using spherical coordinates, the element of volume

$$\delta v = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta v = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi}$$

Also the element of mass $m = c\delta v$

Second moment of mass of the element about OZ

$$\begin{aligned} &= m\rho^2 = m(r\sin\theta)^2 \\ &\qquad = c\,r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi\,r^2\sin^2\theta \\ &\qquad = c\,r^4\sin^3\theta\,\delta r\,\delta\theta\,\delta\phi \end{aligned}$$

$\therefore$ Total second moment for the solid

$$I_1 \approx \sum_{\phi=0}^{\pi/2}\sum_{\theta=0}^{\pi/2}\sum_{r=b}^{a} c\,r^4\,\delta r\ \sin^3\theta\,\delta\theta\,\delta\phi$$

Then, as usual, if $\delta r \to 0$, $\delta\theta \to 0$, $\delta\phi \to 0$, we finally obtain

$$I_1 = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=b}^{a} c\,r^4\,\mathrm{d}r\ \sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

which you can evaluate without any difficulty and obtain

$$I_1 = \ldots\ldots\ldots\ldots$$

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$$\boxed{I_1 = \frac{\pi}{15}(a^5 - b^5)c}$$

Because

$$\begin{aligned}
I_1 &= \int_0^{\pi/2}\int_0^{\pi/2}\left[c\frac{r^5}{5}\right]_b^a \sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \int_0^{\pi/2}\int_0^{\pi/2}\frac{c}{5}(a^5 - b^5)\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \frac{c}{5}(a^5 - b^5)\int_0^{\pi/2}\int_0^{\pi/2}(1 - \cos^2\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \frac{c}{5}(a^5 - b^5)\int_0^{\pi/2}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi/2}\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
&= \frac{c}{5}(a^5 - b^5)\int_0^{\pi/2}\left(1 - \frac{1}{3}\right)\mathrm{d}\phi\\
&= \frac{2c}{15}(a^5 - b^5)\Big[\phi\Big]_0^{\pi/2} = \frac{c\pi}{15}(a^5 - b^5)
\end{aligned}$$

Therefore, the moment of inertia for the whole sphere I is

$$I = 8I_1 \qquad \text{i.e.} \quad I = \frac{8\pi}{15}(a^5 - b^5)c$$

Radius of gyration (k) $\quad Mk^2 = I$

$$\therefore\ k = \ldots\ldots\ldots\ldots$$

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$$\boxed{k = \sqrt{\frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)}}$$

We had already calculated the total mass $M = \frac{4\pi}{3}(a^3 - b^3)c$ and since $I = \frac{8\pi}{15}(a^5 - b^5)c$ then

$$\frac{4\pi}{3}(a^3 - b^3)ck^2 = \frac{8\pi}{15}(a^5 - b^5)c$$

$$\therefore\ k^2 = \frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right) \qquad \therefore\ k = \sqrt{\frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)}$$

We have set the working out in considerable detail, since spherical coordinates may be a new topic. Many of the statements can be streamlined when one is familiar with the system.

Now move on for another example

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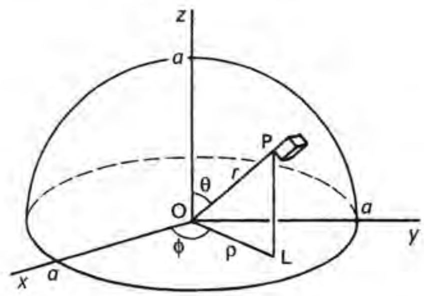
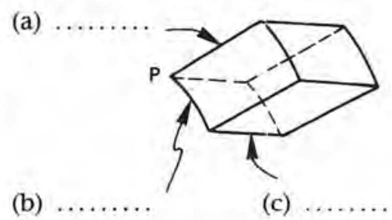
Example 3

Find the total mass of a solid sphere of radius a, enclosed by the surface $x^2 + y^2 + z^2 = a^2$ and having variable density c where $c = 1 + r\,|z|$ and r is the distance of any point from the origin.

This is a case where spherical coordinates can clearly be used with advantage.



In the element of volume, the three dimensions are



42

$$\boxed{\text{(a)}\ \delta r \quad \text{(b)}\ r\,\delta\theta \quad \text{(c)}\ \rho\,\delta\phi = r\sin\theta\,\delta\phi}$$

so that $\quad \delta v = \ldots\ldots\ldots\ldots$

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$$\boxed{\delta v = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi}$$

Then the mass of the element $= c\,\delta v = (1 + r\,|z|)\,\delta v$

and $\qquad z = r\cos\theta$

$$\therefore\ m = c\,\delta v = (1 + r^2\cos\theta)\,r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

Since the density uses $|\,z\,| = 1$ we must only consider the region where $\cos\theta \geq 0$ and so we consider the *upper hemisphere* only. The integral for the total mass M_1 is

$$M_1 = \ldots\ldots\ldots\ldots$$

Write out the integral and insert the limits.

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$$\boxed{M_1 = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=a}(1 + r^2\cos\theta)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

$$\text{i.e.}\quad M_1 = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}\{r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi + r^4\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,d\phi\}$$

$$= \qquad I_1 \qquad + \qquad I_2$$

$$I_1 = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \text{ gives } \ldots\ldots\ldots\ldots$$

Do *not* work it out. You can doubtless recognise what the result would represent.

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$$\boxed{\text{The volume of the hemisphere}}$$

Because the integral is simply the summation of elements of volume throughout the region of the hemisphere.

Thus, without more ado, $I_1 = \dfrac{2}{3}\pi a^3$.

Now for I_2.

$$I_2 = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a r^4\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= \ldots\ldots\ldots\ldots \text{ Evaluate the triple integral.}$$

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$$\boxed{I_2 = \frac{\pi a^5}{5}}$$

Because

$$I_2 = \int_0^{2\pi}\int_0^{\pi/2}\frac{a^5}{5}\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= \frac{a^5}{5}\int_0^{2\pi}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\mathrm{d}\phi$$
$$= \frac{a^5}{10}\int_0^{2\pi}1\mathrm{d}\phi$$
$$= \frac{a^5}{10}\Big[\phi\Big]_0^{2\pi} = \frac{\pi a^5}{5}$$
$$\therefore\ I_2 = \frac{\pi a^5}{5}$$

So now finish it off. For the complete sphere

$$M = \ldots\ldots\ldots\ldots$$

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$$\boxed{M = \frac{2\pi a^3}{15}(10 + 3a^2)}$$

Because

$$M_1 = I_1 + I_2 = \frac{2}{3}\pi a^3 + \frac{\pi a^5}{5} = \frac{\pi a^3}{15}(10 + 3a^2)$$

Then, for the whole sphere, $M = 2M_1 = \dfrac{2\pi a^3}{15}(10 + 3a^2)$

Each problem, then, is tackled in much the same way.

(a) Draw a careful sketch diagram, inserting all relevant information.
(b) Decide on the most appropriate coordinate system to use.
(c) Build up the multiple integral and insert correct limits.
(d) Evaluate the integral.

And now we can apply the general guide lines to a final problem.

Example 4

Determine the volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = x$, $z = 2$ and $y = 4 - x^2$ in the first quadrant.

First we sketch the diagram.

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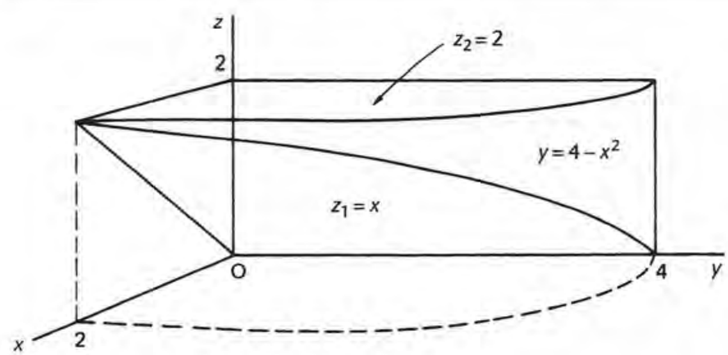

There is no axis of symmetry and no spherical centre. We shall therefore use .... coordinates.

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$\boxed{\text{Cartesian}}$

So off you go on your own. There are no snags.

$$V = \text{...........}$$

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$$\boxed{V = 6\frac{2}{3} \text{ cubic units}}$$

Here is the complete solution.

$$V \approx \sum_{x=0}^{2} \sum_{y=0}^{4-x^2} \sum_{z=x}^{2} \delta x\, \delta y\, \delta z$$

$$\therefore \; V = \int_{x=0}^{2} \int_{y=0}^{4-x^2} \int_{z=x}^{2} \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_{0}^{2} \int_{0}^{4-x^2} (2-x)\, \mathrm{d}y\, \mathrm{d}x$$

$$= \int_{0}^{2} \Big[2y - xy\Big]_{y=0}^{4-x^2} \mathrm{d}x$$

$$= \int_{0}^{2} \{8 - 2x^2 - 4x + x^3\} \mathrm{d}x$$

$$= \left[8x - \frac{2x^3}{3} - 2x^2 + \frac{x^4}{4}\right]_0^2$$

$$= 6\frac{2}{3}$$

And that is it. Now we move to the next section of work

Change of variables in multiple integrals

51

In Cartesian coordinates, we use the variables (x, y, z); in cylindrical coordinates, we use the variables (r, θ, z); in spherical coordinates, we use the variables (r, θ, ϕ); and we have established relationships connecting these systems of variables, permitting us to transfer from one system to another. These relationships, you will remember, were obtained geometrically in Frames 23 to 30 of this Programme.

There are occasions, however, when it is expedient to make other transformations beside those we have used and it is worth looking at the problem in a rather more general manner.

This we will now do

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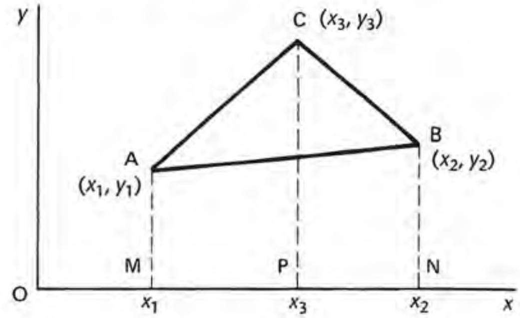
First, however, let us revise a result from an earlier Programme on determinants to find the area of the triangle ABC.



If we arrange the vertices A (x_1, y_1)
B (x_2, y_2)
C (x_3, y_3)

in an anticlockwise manner then

area triangle ABC = trapezium AMPC + trapezium CPNB − trapezium AMNB

$$= \tfrac{1}{2}\{(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)\}$$

$$= \tfrac{1}{2}\{x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 + x_2y_2 + x_2y_3 - x_3y_2 - x_3y_3 - x_2y_1 - x_2y_2 + x_1y_1 + x_1y_2\}$$

$$= \tfrac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\}$$

$$= \tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

The determinant is positive if the points A, B, C are taken in an anticlockwise manner.

We shall need to use this result in a short while, so keep it in mind.

On to the next frame

Curvilinear coordinates

53

Consider the double integral $\int_R \int \phi(x, y) \mathrm{d}A$ where $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ in Cartesian coordinates. Let u and v be two new independent variables defined by $u = F(x, y)$ and $v = G(x, y)$ where these equations can be simultaneously solved to obtain $x = f(u, v)$ and $y = g(u, v)$. Furthermore, these transformation equations are such that every point (x, y) is mapped to a unique point (u, v) and vice versa.

Let us see where this leads us, so on to the next frame

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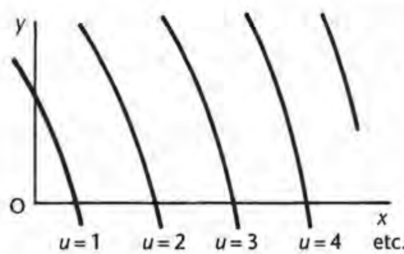
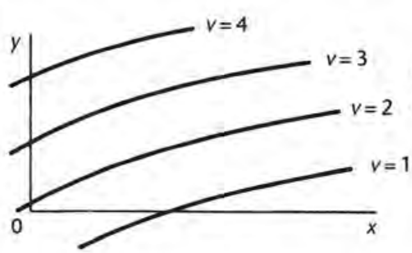
The equation $u = F(x, y)$ will be a family of curves depending on the particular constant value given to u in each case.



Curves $u = F(x, y)$ for different constant values of u.

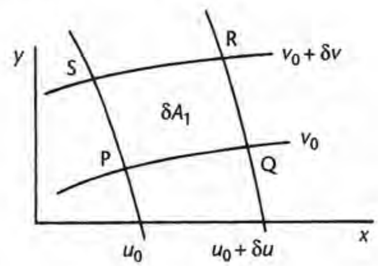
Similarly, $v = G(x, y)$ will be a family of curves depending on the particular constant value assigned to v in each case.



Curves $v = G(x, y)$ for different constant values of v.

▶

These two sets of curves will therefore cover the region R and form a network, and to any point P (x_0, y_0) there will be a pair of curves $u = u_0$ (constant) and $v = v_0$ (constant) that intersect at that point.



The u- and v-values relating to any particular point are known as its *curvilinear coordinates* and $x = f(u, v)$ and $y = g(u, v)$ are the *transformation equations* between the two systems.

In the Cartesian coordinates (x, y) system, the element of area $\delta A = \delta x \delta y$ and is the area bounded by the lines $x = x_0$, $x = x_0 + \delta x$, $y = y_0$, and $y = y_0 + \delta y$.

In the new system of *curvilinear coordinates* (u, v) the element of area δA_1 can be taken as that of the figure P, Q, R, S, i.e. the area bounded by the curves $u = u_0$, $u = u_0 + \delta u$, $v = v_0$ and $v = v_0 + \delta v$.

Since δA_1 is small, PQRS may be regarded as a parallelogram

i.e. $\delta A_1 \approx 2 \times$ area of triangle PQS

and this is where we make use of the result previously revised that the area of a triangle ABC with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) can be expressed in determinant form as

Area = ...

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$$\boxed{\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}$$

Before we can apply this, we must find the Cartesian coordinates of P, Q and S in the diagram on page 646 where we omit the subscript $_0$ on the coordinates.

If $x = f(u, v)$, then a small increase δx in x is given by

$$\delta x = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta x = \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v}$$

i.e. $\delta x = \dfrac{\partial x}{\partial u}\delta u + \dfrac{\partial x}{\partial v}\delta v$

and, for $y = g(u, v)$

$$\delta y = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v}$$

Now

(a) P is the point (x, y)

(b) Q corresponds to small changes from P.

$$\delta x = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v \quad \text{and} \quad \delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v$$

But along PQ v is constant. $\therefore\ \delta v = 0$.

$$\therefore\ \delta x = \frac{\partial x}{\partial u}\delta u \quad \text{and} \quad \delta y = \frac{\partial y}{\partial u}\delta u$$

i.e. Q is the point $\left(x + \dfrac{\partial x}{\partial u}\delta u,\ y + \dfrac{\partial y}{\partial u}\delta u\right)$.

(c) Similarly for S, since u is constant along PS $\delta u = 0$ and

$\therefore$ S is the point $\left(x + \dfrac{\partial x}{\partial v}\delta v,\ y + \dfrac{\partial y}{\partial v}\delta v\right)$

So the Cartesian coordinates of P, Q, S are

$$\text{P}\ (x, y);\quad \text{Q}\ \left(x + \frac{\partial x}{\partial u}\delta u,\ y + \frac{\partial y}{\partial u}\delta u\right);\quad \text{S}\ \left(x + \frac{\partial x}{\partial v}\delta v,\ y + \frac{\partial y}{\partial v}\delta v\right)$$

$\therefore$ The determinant for the area PQS is ....

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$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x & x + \dfrac{\partial x}{\partial u}\delta u & x + \dfrac{\partial x}{\partial v}\delta v \\ y & y + \dfrac{\partial y}{\partial u}\delta u & y + \dfrac{\partial y}{\partial v}\delta v \end{vmatrix}$$

Subtracting column 1 from columns 2 and 3 gives

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ x & \dfrac{\partial x}{\partial u}\delta u & \dfrac{\partial x}{\partial v}\delta v \\ y & \dfrac{\partial y}{\partial u}\delta u & \dfrac{\partial y}{\partial v}\delta v \end{vmatrix}$$

which simplifies immediately to

....

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$$\text{Area} = \frac{1}{2}\begin{vmatrix} \dfrac{\partial x}{\partial u}\delta u & \dfrac{\partial x}{\partial v}\delta v \\ \dfrac{\partial y}{\partial u}\delta u & \dfrac{\partial y}{\partial v}\delta v \end{vmatrix}$$

Then, taking out the factor δu from the first column and the factor δv from the second column, this becomes

Area = ...

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$$\frac{1}{2}\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\delta u \delta v$$

The area of the approximate parallelogram is twice the area of the triangle.

$$\therefore \text{ Area of parallelogram} = \delta A_1 = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\delta u\, \delta v$$

Expressing this in differentials

$$\mathrm{d}A = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathrm{d}u\, \mathrm{d}v$$

and, for convenience, this is often written

$$\mathrm{d}A = \frac{\partial(x, y)}{\partial(u, v)}\, \mathrm{d}u\, \mathrm{d}v$$

▶

$\dfrac{\partial(x, y)}{\partial(u, v)}$ is called the *Jacobian of the transformation* from the Cartesian coordinates (x, y) to the curvilinear coordinates (u, v).

$$\therefore\ J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

So, if the transformation equations are

$$x = u(u + v) \quad \text{and} \quad y = uv^2$$
$$J(u, v) = \dots\dots\dots\dots$$

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$$\boxed{J(u, v) = uv(4u + v)}$$

Because

$$\frac{\partial x}{\partial u} = 2u + v \quad \frac{\partial x}{\partial v} = u$$
$$\frac{\partial y}{\partial u} = v^2 \quad \frac{\partial y}{\partial v} = 2uv$$
$$\therefore\ J(u, v) = \begin{vmatrix} 2u + v & u \\ v^2 & 2uv \end{vmatrix} = 4u^2v + 2uv^2 - uv^2$$
$$= 4u^2v + uv^2 = uv(4u + v)$$

Next frame

62

Sometimes the transformation equations are given the other way round. That is, where u and v are given as expressions in x and y. In such a case $J(u, v)$ can be found using the fact that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\left(\dfrac{\partial(u, v)}{\partial(x, y)}\right)}$$

For example, if the transformation equations are given as $u = x^2 + y^2$ and $v = 2xy$ then

$$J(u, v) = \dots\dots\dots\dots$$

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$$\boxed{J(u,\ v) = \frac{1}{4\sqrt{u^2 - v^2}}}$$

Because

$$\frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} = 4x^2 - 4y^2$$

and so

$$J(u,\ v) = \frac{\partial(x,\ y)}{\partial(u,\ v} = \frac{1}{\left(\dfrac{\partial(u,\ v)}{\partial(x,\ y)}\right)} = \frac{1}{4(x^2 - y^2)}$$

Now $u - v = x^2 - 2xy + y^2 = (x - y)^2$
and $u + v = x^2 + 2xy + y^2 = (x + y)^2$
and so $x^2 - y^2 = (x - y)(x + y) = \sqrt{u - v}\sqrt{u + v} = \sqrt{u^2 - v^2}$ giving

$$J(u,\ v) = \frac{1}{4\sqrt{u^2 - v^2}}$$

There is one further point to note in this piece of work, so move on

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Note: In the transformation, it is possible for the order of the points P, Q, R, S to be reversed with the result that δA may give a negative result when the determinant is evaluated. To ensure a positive element of area, the result is finally written

$$dA = \left|\frac{\partial(x,\ y)}{\partial(u,\ v)}\right| du\,dv$$

where the 'modulus' lines indicate the absolute value of the Jacobian.

Therefore, to rewrite the integral $\int_R\int F(x,\ y)\,dx\,dy$ in terms of the new variables, u and v, where $x = f(u, v)$ and $y = g(u, v)$, we substitute for x and y in $F(x,\ y)$ and replace $dx\,dy$ with $\left|\dfrac{\partial(x,\ y)}{\partial(u, v)}\right| du\,dv$.

The integral then becomes

$$\int_R\int F\{f(u,\ v),\ g(u, v)\}\left|\frac{\partial(x,\ y)}{\partial(u,\ v)}\right| du\,dv$$

Make a note of this result

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Example 1

Express $I = \int_R \int xy^2 \, dx \, dy$ in polar coordinates, making the substitutions $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \; J(r,\theta) = \ldots\ldots\ldots\ldots$$

66

$$\boxed{J(r,\theta) = r}$$

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Then $I = \int_R \int xy^2 \, dx \, dy$ becomes............

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$$\boxed{I = \int_R \int r^3 \sin^2\theta \cos\theta \, r \, dr \, d\theta}$$

Because $xy^2 = r\cos\theta \, r^2 \sin^2\theta = r^3 \sin^2\theta \cos\theta$

$$\left| \frac{\partial(x, y)}{\partial(r,\theta)} \right| dr \, d\theta = r \, dr \, d\theta$$

$$\therefore \; I = \int_R \int r^3 \sin^2\theta \cos\theta \, r \, dr \, d\theta = \int_R \int r^4 \sin^2\theta \cos\theta \, dr \, d\theta$$

Now this one.

Example 2

Express $I = \int_R \int (x^2 + y^2) dx \, dy$ in terms of u and v, given that $x = u^2 - v^2$ and $y = 2uv$.

First of all, the expression for $\dfrac{\partial(x, y)}{\partial(u,v)}$ gives ....

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$$\boxed{4(u^2+v^2)}$$

Because

$$x = u^2 - v^2 \quad \therefore \quad \frac{\partial x}{\partial u} = 2u \quad \frac{\partial x}{\partial v} = -2v$$

$$y = 2uv \quad \therefore \quad \frac{\partial y}{\partial u} = 2v \quad \frac{\partial y}{\partial v} = 2u$$

$$\therefore \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4(u^2+v^2)$$

Also $x^2 + y^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2$

$$= u^4 + 2u^2v^2 + v^4 = (u^2+v^2)^2$$

Then $I = \int\int_R (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$ becomes $I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = 4\int_R\int (u^2+v^2)^3\,\mathrm{d}u\,\mathrm{d}v}$$

One more.

Example 3

By substituting $x = 2uv$ and $y = u(1-v)$ where $u > 0$ and $v > 0$, express the integral $I = \int_R\int x^2y\,\mathrm{d}x\,\mathrm{d}y$ in terms of u and v.

Complete it: there are no snags. $I = \ldots\ldots\ldots\ldots$

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$$\boxed{I = 8\int_R\int u^4v^2(1-v)\,\mathrm{d}u\,\mathrm{d}v}$$

Working:

$$x = 2uv \qquad \therefore \quad \frac{\partial x}{\partial u} = 2v \qquad \frac{\partial x}{\partial v} = 2u$$

$$y = u - uv \qquad \frac{\partial y}{\partial u} = 1 - v \qquad \frac{\partial y}{\partial v} = -u$$

$$\therefore \; J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2v & 1-v \\ 2u & -u \end{vmatrix}$$

$$= 2u\begin{vmatrix} v & 1-v \\ 1 & -1 \end{vmatrix} = 2u\begin{vmatrix} v & 1 \\ 1 & 0 \end{vmatrix} = -2u$$

$$\therefore \; \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = 2u$$

$$x^2y = 4u^2v^2(u - uv) = 4u^3v^2(1 - v)$$

$$\therefore \; I = \int_R\int 4u^3v^2(1-v)\,2u\,\mathrm{d}u\,\mathrm{d}v$$

$$I = 8\int_R\int u^4v^2(1-v)\,\mathrm{d}u\,\mathrm{d}v$$

Transformation in three dimensions

If we extend the previous results to convert variables (x, y, z) to (u, v, w), we proceed in just the same way.

If $x = f(u, v, w)$; $y = g(u, v, w)$; $z = h(u, v, w)$

Then $$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

and the element of volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ becomes

$$\mathrm{d}V = |J(u, v, w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Also $\int\int\int F(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is transformed into

$$\int\int\int G(u, v, w)\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Now for an example, so move on

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Example 4

To transform a triple integral $I = \iiint F(x, y, z)\,dx\,dy\,dz$ in Cartesian coordinates to spherical coordinates by the transformation equations

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

First we need the partial derivatives, from which to build up the Jacobian.

These are ....

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$$\boxed{\begin{array}{lll} \dfrac{\partial x}{\partial r} = \sin\theta\cos\phi & \dfrac{\partial y}{\partial r} = \sin\theta\sin\phi & \dfrac{\partial z}{\partial r} = \cos\theta \\ \dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi & \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi & \dfrac{\partial z}{\partial \theta} = -r\sin\theta \\ \dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi & \dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi & \dfrac{\partial z}{\partial \phi} = 0 \end{array}}$$

$$\therefore\ J(r, \theta, \phi) = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & r\cos\theta\sin\phi \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$+ r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{r^2\sin\theta}$$

Because

$$J(r,\ \theta,\ \phi) = r^2\cos^2\theta\sin\theta \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix}$$

$$+ r^2\sin^3\theta \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix}$$

$$= (r^2\sin^3\theta + r^2\sin\theta\cos^2\theta) \begin{vmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{vmatrix}$$

$$= r^2\sin\theta(\sin^2\theta + \cos^2\theta)(\cos^2\phi + \sin^2\phi) = r^2\sin\theta$$

$$\therefore\ I = \iiint G(u, v, w)r^2\sin\theta\,dr\,d\theta\,d\phi$$

which agrees, of course, with the result we had previously obtained by a geometric consideration.

▶

And that is about it. Check carefully down the **Revision summary** and the **Can You?** checklist that now follow, before working through the **Test exercise**. The **Further problems** give additional practice.

Revision summary 15

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1 *Surface integrals*

$$I=\int_R f(x,y)\,\mathrm{d}a=\int_R\int f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

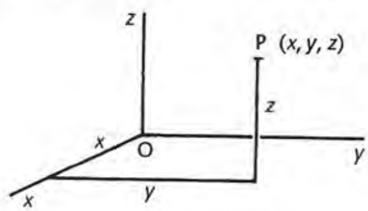
2 *Surface in space*

$$I=\int_S \phi(x,y,z)\,\mathrm{d}S=\int_R\int \phi(x,y,z)\sec\gamma\,\mathrm{d}x\,\mathrm{d}y \qquad (\gamma<\pi/2)$$

$$=\int_R\int \phi(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$$

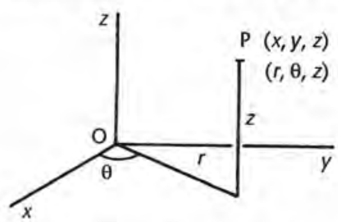
3 *Space coordinate systems*

(a) *Cartesian coordinates* (x, y, z)



First octant:
$x\geq 0;\ \ y\geq 0;\ \ z\geq 0$

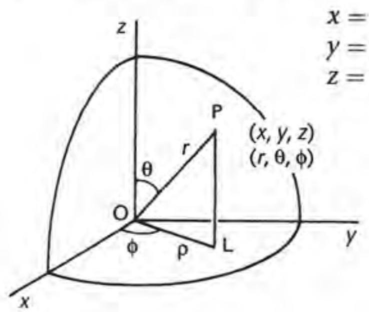
(b) *Cylindrical coordinates* (r, θ, z) $\quad r\geq 0$



$$x=r\cos\theta \qquad r=\sqrt{x^2+y^2}$$
$$y=r\sin\theta \qquad \theta=\arctan(y/x)$$
$$z=z \qquad z=z$$

(c) *Spherical coordinates* (r, θ, ϕ) $\quad r\geq 0$

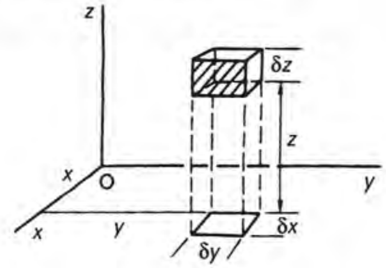
$$x=r\sin\theta\cos\phi \qquad r=\sqrt{x^2+y^2+z^2}$$
$$y=r\sin\theta\sin\phi \qquad \theta=\arccos(z/r)$$
$$z=r\cos\theta \qquad \phi=\arctan(y/x)$$



▶

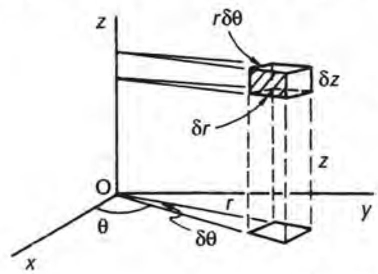
4 *Elements of volume*

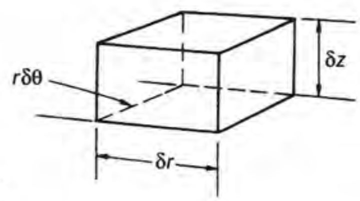
(a) *Cartesian coordinates*



$\delta v = \delta x\, \delta y\, \delta z$

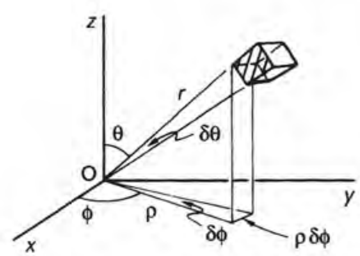
(b) *Cylindrical coordinates* $r \geq 0$

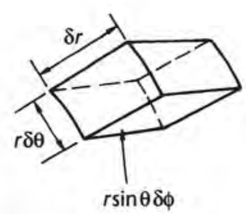




$\delta v = r\, \delta r\, \delta\theta\, \delta z$

(c) *Spherical coordinates*





$\delta v = r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi$

5 *Volume integrals*

$$V = \iiint \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

$$I = \iiint f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

▶

6 *Change of variables* in multiple integrals

(a) *Double integrals* $x = f(u, v)$; $\quad y = g(u, v)$

$$\mathrm{d}A = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v; \qquad J(u,v) = \frac{\partial(x, y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$I = \int_R \int F(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_R \int F\{f(u, v), g(u, v)\} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$$

(b) *Triple integrals* $\quad x = f(u, v, w)$; $y = g(u, v, w)$; $z = h(u, v, w)$

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$\text{Then } I = \iiint F(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \iiint G(u, v, w) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Can You?

Checklist 15

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5, how confident are you that you can:	**Frames**
• Evaluate double integrals and surface integrals?	1 to 22
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Relate three-dimensional Cartesian coordinates to cylindrical and spherical polar forms?	23 to 31
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Evaluate volume integrals in Cartesian coordinates and in cylindrical and spherical polar coordinates?	32 to 50
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use the Jacobian to convert integrals given in Cartesian coordinates into general curvilinear coordinates in two and three dimensions?	51 to 73
Yes ☐ ☐ ☐ ☐ ☐ *No*	

Test exercise 15

76

1 Determine the area of the surface $z = \sqrt{x^2 + y^2}$ over the region bounded by $x^2 + y^2 = 4$.

2 Evaluate the surface integral $I = \int_S \phi \, dS$ where $\phi = \dfrac{1}{\sqrt{x^2 + y^2}}$ over the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

3 (a) Transform the Cartesian coordinates
   (1) (4, 2, 3) to cylindrical coordinates (r, θ, z)
   (2) (3, 1, 5) to spherical coordinates (r, θ, ϕ).
   (b) Express in Cartesian coordinates (x, y, z)
   (1) the cylindrical coordinates $(5, \pi/4, 3)$
   (2) the spherical coordinates $(4, \pi/6, 2)$.

4 Determine the volume of the solid bounded by the plane $z = 0$ and the surfaces $x^2 + y^2 = 4$ and $z = x^2 + y^2 + 1$.

5 Determine the total mass of a solid hemisphere bounded by the plane $z = 0$ and the surface $x^2 + y^2 + z^2 = a^2 \quad (z \geq 0)$ if the density at any point is given by $\rho = 1 - z \quad (z < a)$.

6 (a) Express the integral $I = \int_R \int (x - y) \, dx \, dy$ in terms of u and v, where $x = u(1 + v)$ and $y = u - v$.
   (b) Express the triple integral $I = \iiint \left(\dfrac{x + z}{y} \right) dx \, dy \, dz$ in terms of u, v, w using the transformation equations $x = u + v + w; \quad y = v^2 w; \quad z = u - w$.

Further problems 15

77

1 Evaluate the surface integral $I = \int_S (x^2 + y^2) \, dS$ over the surface of the cone $z^2 = 4(x^2 + y^2)$ between $z = 0$ and $z = 4$.

2 Find the position of the centre of gravity of that part of a thin spherical shell $x^2 + y^2 + z^2 = a^2$ which exists in the first octant.

3 Determine the surface area of the plane $6x + 3y + 4z = 60$ cut off by $x = 0$, $y = 0$, $x = 5$, $y = 8$.

4 Find the surface area of the plane $3x + 2y + 3z = 12$ cut off by the planes $x = 0$, $y = 0$, and the cylinder $x^2 + y^2 = 16$ for $x \geq 0$, $y \geq 0$.

5 Determine the area of the paraboloid $z = 2(x^2 + y^2)$ cut off by the cone $z = \sqrt{x^2 + y^2}$.

6 Find the area of the cone $z^2 = 4(x^2 + y^2)$ which is inside the paraboloid $z = 2(x^2 + y^2)$.

7 Cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ intersect. Determine the total external surface area of the common portion.

8 Determine the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by the cylinder $x^2 + y^2 = ax$.

9 A cylinder of radius b, with the z-axis as its axis of symmetry, is removed from a sphere of radius a, $a > b$, with centre at the origin. Calculate the total curved surface area of the ring so formed, including the inner cylindrical surface.

10 Find the volume enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 5 - x$.

11 Determine the volume of the solid bounded by the surfaces $y = x^2$, $x = y^2$, $z = 2$ and $x + y + z = 4$.

12 Find the volume of the solid bounded by the plane $z = 0$, the cylinder $x^2 + y^2 = a^2$ and the surface $z = x^2 + y^2$.

13 A solid is bounded by the planes $x = 0$, $y = 0$, $z = 2$, $z = x$ and the surface $x^2 + y^2 = 4$. Determine the volume of the solid.

14 Find the position of the centre of gravity of the part of the solid sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

15 A solid is bounded by the cone $z = 2\sqrt{x^2 + y^2}$, $z \geq 0$, and the sphere $x^2 + y^2 + (z - a)^2 = 2a^2$. Determine the volume of the solid so formed.

16 Determine the volume enclosed by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

17 Find the volume of the solid in the first octant bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = x + y$ and the surface $x^2 + y^2 = a^2$.

18 Express the integral $\displaystyle\iint (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y$ in terms of u and v, using the transformations $u = x + y$, $v = x - y$.

19 Determine an expression for the element of volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in terms of u, v, w using the transformations $x = u(1 - v)$, $y = uv$, $z = uvw$.

20 A solid sphere of radius a has variable density c at any point (x, y, z) given by $c = k(a - z)$ where k is a constant. Determine the position of the centre of gravity of the sphere.

21 Calculate $\displaystyle\iint x^2y^2\,\mathrm{d}x\,\mathrm{d}y$ over the triangular region in the x–y plane with vertices $(0, 0)$, $(1, 1)$, $(1, 2)$.

►

22 Evaluate the integral $I = \int_0^2 \int_{\sqrt{y(2-y)}}^{\sqrt{4-y^2}} \frac{y}{x^2+y^2}\,dx\,dy$ by transforming to polar coordinates.

23 Evaluate $I = \int_0^1 \int_0^y \frac{xy^2}{\sqrt{x^2+y^2}}\,dx\,dy$.

24 Find the volume bounded by the cylinder $x^2+y^2=a^2$, the plane $z=0$ and the surface $z=x^2+y^2$. Convert to polar coordinates and show that $V=\frac{\pi a^4}{2}$.

25 By changing the order of integration in the integral

$$I = \int_0^a \int_x^a \frac{y^2\,dy\,dx}{\sqrt{x^2+y^2}}$$

show that $I = \frac{1}{3}a^3 \ln(1+\sqrt{2})$.

Programme 16

Integral functions

Frames
1 to 79

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the recurrence relation for the gamma function and evaluate the gamma function for certain rational arguments
- Evaluate integrals that require the use of the gamma function in their solution
- Identify the beta function and evaluate integrals that require the use of the beta function in their solution
- Derive the relationship between the gamma function and the beta function
- Use the duplication formula to evaluate the gamma function for half integer arguments
- Recognise the error function and its relation to the Gaussian probability distribution
- Recognise elliptic functions of the first and second kind
- Evaluate integrals that require the use of elliptic functions in their solution
- Use alternative forms of the elliptic functions

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 15 Integration 1, 16 Integration 2 and 17 Reduction formulas

Integral functions

1

Some functions are most conveniently defined in the form of integrals and we shall deal with one or two of these in the present Programme.

The gamma function

The gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \qquad (1)$$

and is convergent for $x > 0$.

From (1): $\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt$

Integrating by parts

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\,dt$$

$$= \{0-0\} + x\Gamma(x)$$

$$\therefore\ \Gamma(x+1) = x\Gamma(x) \qquad (2)$$

This is a fundamental recurrence relation for gamma functions. It can also be written as $\Gamma(x) = (x-1)\Gamma(x-1)$

With it we can derive a number of other results.

For instance, when $x = n$, a positive integer ≥ 1, then

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) && \text{But } \Gamma(n) = (n-1)\Gamma(n-1)\\ &= n(n-1)\Gamma(n-1) && \Gamma(n-1) = (n-2)\Gamma(n-2)\\ &= n(n-1)(n-2)\Gamma(n-2) \\ &\quad ------ \\ &= n(n-1)(n-2)(n-3)\dots 1\Gamma(1) = n!\Gamma(1) \end{aligned}$$

But, from the original definition $\Gamma(1) = \dots\dots\dots\dots$

2

$$\boxed{\Gamma(1) = 1}$$

Because

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}dt = \left[-e^{-t}\right]_0^\infty = 0 + 1 = 1$$

Therefore, we have $\Gamma(1) = 1$ (3)

and $\Gamma(n+1) = n!$ provided n is a positive integer.

$$\therefore\ \Gamma(7) = \dots\dots\dots\dots$$

3

$$\boxed{\Gamma(7) = 720}$$

Because

$\Gamma(7) = \Gamma(6+1) = 6! = 720.$

Knowing $\Gamma(7) = 720$, $\Gamma(8) = \dots\dots\dots\dots$ and $\Gamma(9) = \dots\dots\dots\dots$

4

$$\boxed{\Gamma(8) = 5040; \quad \Gamma(9) = 40\,320}$$

Because

$$\Gamma(8) = \Gamma(7+1) = 7\Gamma(7) = 7(720) \quad = 5040$$
$$\Gamma(9) = \Gamma(8+1) = 8\Gamma(8) = 8(5040) = 40\,320$$

We can also use the recurrence relation in reverse

$$\Gamma(x+1) = x\Gamma(x)$$
$$\therefore \ \Gamma(x) = \frac{\Gamma(x+1)}{x} \tag{4}$$

For example, given that $\Gamma(7) = 720$, we can determine $\Gamma(6)$

$$\Gamma(6) = \frac{\Gamma(6+1)}{6} = \frac{\Gamma(7)}{6} = \frac{720}{6} = 120$$

and then $\Gamma(5) = \dots\dots\dots\dots$

5

$$\boxed{\Gamma(5) = 24}$$

$$\Gamma(5) = \frac{\Gamma(5+1)}{5} = \frac{\Gamma(6)}{5} = \frac{120}{5} = 24.$$

So far, we have used the original definition

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

for cases where x is a positive integer n.

What happens when $x = \frac{1}{2}$? We will investigate.

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}\,dt$$

Putting $t = u^2$, $dt = 2u\,du$, then

$$\Gamma(\tfrac{1}{2}) = \dots\dots\dots\dots$$

6

$$\boxed{\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du}$$

Because

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty u^{-1}e^{-u^2}2u\,du = 2\int_0^\infty e^{-u^2}\,du.$$

Unfortunately, $\int_0^\infty e^{-u^2}\,du$ cannot easily be determined by normal means. It is, however, important, so we have to find a way of getting round the difficulty.

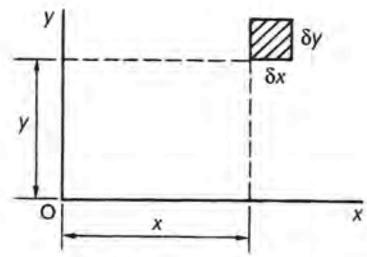
▶

Evaluation of $\int_0^\infty e^{-x^2}\,dx$

$$\text{Let } I = \int_0^\infty e^{-x^2}\,dx, \text{ then also } I = \int_0^\infty e^{-y^2}\,dy$$

$$\therefore\ I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

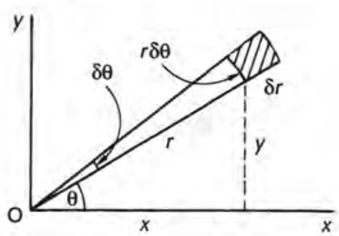
$\delta a = \delta x\,\delta y$ represents an element of area in the x–y plane and the integration with the stated limits covers the whole of the first quadrant.



Converting to polar coordinates, the element of area $\delta a = r\,\delta\theta\,\delta r$. Also, $x^2 + y^2 = r^2$

$$\therefore\ e^{-(x^2+y^2)} = e^{-r^2}$$

For the integration to cover the same region as before,



the limits of r are $r = 0$ to $r = \infty$
the limits of θ are $\theta = 0$ to $\theta = \pi/2$.

$$\therefore\ I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^\infty d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1}{2}\right)d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$\therefore\ I = \frac{\sqrt{\pi}}{2}$$

$$\therefore\ \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad (5)$$

This result opens the way for others, so make a note of it and then move on to the next frame

7

Before that diversion, we had established that

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du$$

We now know that $\displaystyle\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$ $\quad\therefore\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

From this, using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we can obtain the following

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\ \Gamma(\tfrac{1}{2}) = \tfrac{1}{2}(\sqrt{\pi}) \qquad \therefore\ \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\ \Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) \qquad \therefore\ \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4}$$

$\Gamma(\tfrac{7}{2}) =$ ....

8

$$\boxed{\Gamma(\tfrac{7}{2}) = \frac{15\sqrt{\pi}}{8}}$$

Because

$$\Gamma(\tfrac{7}{2}) = \Gamma(\tfrac{5}{2}+1) = \tfrac{5}{2}\ \Gamma(\tfrac{5}{2}) = \frac{5}{2}\left(\frac{3\sqrt{\pi}}{4}\right) = \frac{15\sqrt{\pi}}{8}$$

Using the recurrence relation in reverse, i.e. $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, we can also obtain

$$\Gamma(-\tfrac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{(-\frac{3}{2})(-\frac{1}{2})} = \tfrac{4}{3}\sqrt{\pi}$$

Negative values of x

Since $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$, then as $x \to 0$, $\Gamma(x) \to \infty$ $\ \therefore\ \Gamma(0) = \infty$.

The same result occurs for all negative integral values of x – which does not follow from the original definition, but which is obtainable from the recurrence relation.

Because at $x = -1$, $\quad \Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty$

$x = -2$, $\quad \Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty$ etc.

Also, at $x = -\tfrac{1}{2}$, $\quad \Gamma(-\tfrac{1}{2}) = \dfrac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$

and at $x = -\tfrac{3}{2}$, $\quad \Gamma(-\tfrac{3}{2}) = \dfrac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \dfrac{4}{3}\sqrt{\pi}$

Similarly $\quad \Gamma(-\tfrac{5}{2}) =$ ....

and $\quad \Gamma(-\tfrac{7}{2}) =$ ....

9

$$\Gamma(-\tfrac{5}{2}) = -\frac{8}{15}\sqrt{\pi};\quad \Gamma(-\tfrac{7}{2}) = \frac{16}{105}\sqrt{\pi}$$

So we have

(a) For n a positive integer

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(1) = 1;\quad \Gamma(0) = \infty;\quad \Gamma(-n) = \pm\infty$

(b) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi};\qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$

$\Gamma(\tfrac{3}{2}) = \dfrac{\sqrt{\pi}}{2};\qquad \Gamma(-\tfrac{3}{2}) = \dfrac{4}{3}\sqrt{\pi}$

$\Gamma(\tfrac{5}{2}) = \dfrac{3\sqrt{\pi}}{4};\qquad \Gamma(-\tfrac{5}{2}) = -\dfrac{8}{15}\sqrt{\pi}$

$\Gamma(\tfrac{7}{2}) = \dfrac{15\sqrt{\pi}}{8};\qquad \Gamma(-\tfrac{7}{2}) = \dfrac{16}{105}\sqrt{\pi}$

This is quite a useful list. Make a note of it for future use

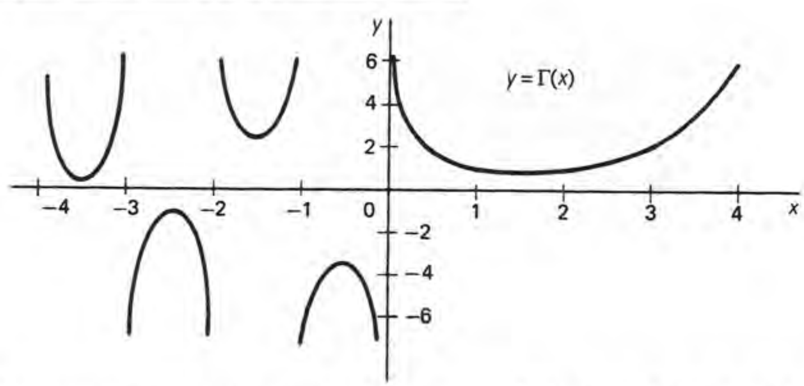
10

Graph of $y = \Gamma(x)$

Values of $\Gamma(x)$ for a range of positive values of x are available in tabulated form in various sets of mathematical tables. These, together with the results established above, enable us to draw the graph of $y = \Gamma(x)$.

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270



For large n it can be shown that $\Gamma(n+1) \approx \sqrt{2\pi n}\, n^n e^{-n}$ which gives rise to Stirling's formula for an approximation to the factorial of a large number

$n! \approx \sqrt{2\pi n}\, n^n e^{-n}$

11

Revision

Let us now revise the main points before we move on to some examples.

The definition of $\Gamma(x)$ is that $\Gamma(x) = \ldots\ldots\ldots\ldots$

12

$$\boxed{\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t}$$

The recurrence relation states that

$$\Gamma(x+1) = \ldots\ldots\ldots\ldots$$

13

$$\boxed{\Gamma(x+1) = x\Gamma(x)}$$

When x is a positive integer, i.e. $x = n$, then

$$\Gamma(n+1) = \ldots\ldots\ldots\ldots$$

14

$$\boxed{\Gamma(n+1) = n!}$$

Then we have a number of specific results

$$\Gamma(1) = \ldots\ldots\ldots\ldots; \ \Gamma(0) = \ldots\ldots\ldots\ldots; \ \Gamma(\tfrac{1}{2}) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\Gamma(1) = 1; \quad \Gamma(0) = \infty; \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$$

and finally, for all negative integral values of n

$$\Gamma(n) = \ldots\ldots\ldots\ldots$$

16

$$\boxed{\Gamma(n) = \pm\infty}$$

Listing them together, we have

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(n+1) = n! \qquad \text{for } n \text{ a positive integer}$$

$$\Gamma(1) = 1; \qquad \Gamma(0) = \infty; \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(n) = \pm\infty \quad \text{for } n \text{ a negative integer.}$$

17

Now for a few examples of evaluation of integrals.

Example 1

Evaluate $\int_0^\infty x^7 e^{-x} \mathrm{d}x$.

We recognise this as the standard form of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t \qquad \text{with the variables changed.}$$

It is often convenient to write the gamma function as

$$\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} \mathrm{d}x$$

Our example then becomes

$$I = \int_0^\infty x^7 e^{-x} \mathrm{d}x = \int_0^\infty x^{v-1} e^{-x}\, \mathrm{d}x \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

18

$$\boxed{v = 8}$$

$$\therefore\ I = \Gamma(v) = \Gamma(8) = \ldots\ldots\ldots\ldots$$

19

$$\boxed{\Gamma(8) = 7! = 5040}$$

$$\text{i.e. } \int_0^\infty x^7\, e^{-x}\, \mathrm{d}x = \Gamma(8) = 7! = 5040$$

Example 2

Evaluate $\int_0^\infty x^3\, e^{-4x}\, \mathrm{d}x$.

If we compare this with $\Gamma(v) = \int_0^\infty x^{v-1}\, e^{-x}\, \mathrm{d}x$, we must reduce the power of e to a single variable, i.e. put $y = 4x$, and we use this substitution to convert the whole integral into the required form.

$$y = 4x \qquad \therefore\ \mathrm{d}y = 4\,\mathrm{d}x \qquad \text{Limits remain unchanged.}$$

The integral now becomes …………

20

$$\boxed{I = \int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y} \frac{\mathrm{d}y}{4}}$$

$$\therefore\ I = \frac{1}{4^4} \int_0^\infty y^3\, e^{-y}\, \mathrm{d}y = \frac{1}{4^4}\Gamma(v) \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

21

$$\boxed{v = 4}$$

Because

$$\int_0^\infty y^{v-1}e^{-y}\mathrm{d}y = \int_0^\infty y^3 e^{-y}\mathrm{d}y \qquad \therefore\ v = 4$$

$$\therefore\ I = \frac{1}{4^4}\,\Gamma(4) = \ldots\ldots\ldots\ldots$$

22

$$\boxed{I = \frac{3}{128}}$$

Because

$$I = \frac{1}{256}\Gamma(4) = \frac{1}{256}(3!) = \frac{6}{256} = \frac{3}{128}$$

One more.

Example 3

Evaluate $\displaystyle\int_0^\infty x^{1/2}\,e^{-x^2}\,\mathrm{d}x$.

The substitution here is to put …………

23

$$\boxed{y = x^2}$$

Work through it as before. When you have completed it, check with the next frame.

24

Here is the working.

$y = x^2 \quad \therefore\ \mathrm{d}y = 2x\,\mathrm{d}x \quad$ Limits $x = 0, y = 0; \quad x = \infty, y = \infty.$

$x = y^{1/2} \quad \therefore\ x^{1/2} = y^{1/4}$

$$\therefore\ I = \int_0^\infty y^{1/4}\,e^{-y}\,\mathrm{d}y/2x = \int_0^\infty \frac{y^{1/4}\,e^{-y}\,\mathrm{d}y}{2y^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty y^{-1/4}\,e^{-y}\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^\infty y^{v-1}\,e^{-y}\mathrm{d}y \quad \text{where } v = \frac{3}{4} \qquad \therefore\ I = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

From tables, $\Gamma(0{\cdot}75) = 1{\cdot}2254$

$\therefore\ I = 0{\cdot}613$

▶

Here is part of a table that may be useful.

x	$\Gamma(x)$
0·25	3·6256
0·50	1·7725
0·75	1·2254
1·00	1·0000
1·25	0·9064
1·50	0·8862
1·75	0·9191
2·00	1·0000
2·25	1·1330
2·50	1·3293

x	$\Gamma(x)$
2·75	1·6084
3·00	2·0000
3·25	2·5493
3·50	3·3234
3·75	4·4230
4·00	6·0000
4·25	8·2851
4·50	11·6318
4·75	16·5862
5·00	24·0000

Now we will move on to another set of functions closely related to gamma functions.

Let us start a new frame

25 The beta function

The beta function $\mathrm{B}(m, n)$, is defined by

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x \qquad (1)$$

which converges for $m > 0$ and $n > 0$.

Putting $(1-x) = u \quad \therefore\ x = 1-u \quad \therefore\ \mathrm{d}x = -\,\mathrm{d}u$

Limits: when $x = 0,\ u = 1;\quad$ when $x = 1,\ u = 0$

$$\therefore\ \mathrm{B}(m, n) = -\int_1^0 (1-u)^{m-1}\,u^{n-1}\,\mathrm{d}u = \int_0^1 (1-u)^{m-1}\,u^{n-1}\mathrm{d}u$$

$$= \int_0^1 u^{n-1}(1-u)^{m-1}\mathrm{d}u = \mathrm{B}(n, m)$$

$$\therefore\ \mathrm{B}(m, n) = \mathrm{B}(n, m) \qquad (2)$$

Alternative form of the beta function

We had

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$$

If we put $x = \sin^2\theta$, the result then becomes …………

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$$\boxed{\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ \mathrm{d}\theta}$$

Because if $x = \sin^2\theta$, $\mathrm{d}x = 2\sin\theta\ \cos\theta\ \mathrm{d}\theta$.

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/2$. $1 - x = 1 - \sin^2\theta = \cos^2\theta$

$$\therefore\ \mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\ \cos\theta\ \mathrm{d}\theta$$

$$\therefore\ \mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\ \mathrm{d}\theta \qquad (3)$$

Make a note of this result. We shall need to use it later.

27

Reduction formulas

In Programme 17 of *Engineering Mathematics (Fifth Edition)* we established useful reduction formulas relating to integrals of powers of sines and cosines, particularly when the integral limits are 0 and $\pi/2$.

(a) $$\int_0^{\pi/2} \sin^n x\ \mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\ \mathrm{d}x \quad \text{i.e. } S_n = \frac{n-1}{n}S_{n-2} \qquad (4)$$

(b) $$\int_0^{\pi/2} \cos^n x\ \mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\ \mathrm{d}x \quad \text{i.e. } C_n = \frac{n-1}{n}C_{n-2} \qquad (5)$$

A third reduction formula for products of powers of sines and cosines is

(c) $$\int_0^{\pi/2} \sin^m x\ \cos^n x\ \mathrm{d}x = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x\ \cos^n x\ \mathrm{d}x$$

If we denote $\int_0^{\pi/2} \sin^m x\ \cos^n x\ \mathrm{d}x$ by $I_{m,n}$, the last result can be written

$$I_{m,n} = \frac{m-1}{m+n}I_{m-2,n} \qquad (6)$$

Alternatively, $\int_0^{\pi/2} \sin^m x\ \cos^n x\ \mathrm{d}x$ can be expressed as

$$\frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x\ \cos^{n-2} x\ \mathrm{d}x$$

i.e. $$I_{m,n} = \frac{n-1}{m+n}I_{m,n-2} \qquad (7)$$

▶

Now $\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ and if we apply (6) to the integral, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$
$$= \frac{(2m-1)-1}{(2m-1)+(2n-1)} \int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-1}\theta \, d\theta$$
$$= \frac{m-1}{m+n-1} \int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-1}\theta \, d\theta$$

Now, using (7) with the right-hand integral

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{m-1}{m+n-1} \cdot \frac{(2n-1)-1}{(2m-3)+(2n-1)} \times \int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-3}\theta \, d\theta$$
$$= \frac{m-1}{m+n-1} \cdot \frac{n-1}{m+n-2} \times \int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-3}\theta \, d\theta$$

$$\therefore \quad \mathrm{B}(m, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \cdot 2\int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-3}\theta \, d\theta$$

i.e. $$\mathrm{B}(m, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \mathrm{B}(m-1, n-1) \qquad (8)$$

This is obviously a reduction formula for $\mathrm{B}(m, n)$ and the process can be repeated as required.

For example $\mathrm{B}(4, 3) = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathrm{B}(4, 3) = \frac{(3)(2)}{(6)(5)} \frac{(2)(1)}{(4)(3)} \mathrm{B}(2, 1)}$$

Because, applying (8)

$$\mathrm{B}(4, 3) = \frac{(3)(2)}{(6)(5)} \mathrm{B}(3, 2) = \frac{(3)(2)}{(6)(5)} \frac{(2)(1)}{(4)(3)} \mathrm{B}(2, 1)$$

Now we must evaluate $\mathrm{B}(2, 1)$ for we can go no further in the reduction process, since, from the definition of $\mathrm{B}(m, n)$, m and n must be

$\ldots\ldots\ldots\ldots$

29

$$\boxed{>0}$$

But $B(2,\ 1) = 2\int_0^{\pi/2} \sin^3\theta\ \cos\theta\ d\theta = 2\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} = \frac{1}{2}$

$$\therefore\ B(4,\ 3) = \frac{(3)(2)}{(6)(5)}\frac{(2)(1)}{(4)(3)}\frac{1}{2}$$
$$= \frac{(3)(2)(1)\times(2)(1)}{(6)(5)(4)(3)(2)(1)} = \frac{(3!)(2!)}{(6!)}$$

Similarly, $B(5,\ 3) =$ ....

30

$$\boxed{B(5,3) = \frac{(4!)(2!)}{(7!)}}$$

Because

$$B(5,3) = \frac{(4)(2)}{(7)(6)}B(4,2) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}B(3,1)$$
$$B(3,1) = 2\int_0^{\pi/2}\sin^5\theta\ \cos\theta\ d\theta = 2\left[\frac{\sin^6\theta}{6}\right]_0^{\pi/2} = \frac{1}{3}$$
$$\therefore\ B(5,3) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}\frac{1}{3}\frac{(2)}{(2)} = \frac{(4!)(2!)}{(7!)}$$

In general $B(m,\ n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$ (9)

Note that
$$B(k,\ 1) = 2\int_0^{\pi/2}\sin^{2k-1}\theta\ \cos\theta\ d\theta$$
$$= 2\int_0^{\pi/2}\sin^{2k-1}\theta\ d(\sin\theta)$$
$$= 2\left[\frac{\sin^{2k}\theta}{2k}\right]_0^{\pi/2} = \frac{1}{k}$$
$$\therefore\ B(k,\ 1) = \frac{1}{k}$$
$$\therefore\ B(k,\ 1) = B(1,\ k) = \frac{1}{k} \tag{10}$$

We can also use the trigonometrical definition (3) to evaluate $B(\frac{1}{2},\frac{1}{2})$

$B(\frac{1}{2},\frac{1}{2}) =$ ....

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$$\boxed{\mathrm{B}(\tfrac{1}{2},\tfrac{1}{2}) = \pi}$$

Because

$$\mathrm{B}(m,\, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, \mathrm{d}\theta$$

$$\therefore\ \mathrm{B}(\tfrac{1}{2},\, \tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^0\theta\, \cos^0\theta\, \mathrm{d}\theta$$

$$= 2\int_0^{\pi/2} 1\, \mathrm{d}\theta = 2\Big[\theta\Big]_0^{\pi/2} = \pi \qquad (11)$$

Now let us summarise our various results so far.

Next frame

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Revision

$$\mathrm{B}(m,\, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, \mathrm{d}x \qquad m > 0,\ n > 0$$

$$\mathrm{B}(m,\, n) = \mathrm{B}(n, m)$$

$$\mathrm{B}(m,\, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, \mathrm{d}\theta$$

$$\mathrm{B}(m,\, n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\mathrm{B}(m-1,\, n-1)$$

$$\mathrm{B}(m,\, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \qquad m \text{ and } n \text{ positive integers}$$

$$\mathrm{B}(k,\, 1) = \mathrm{B}(1, k) = \frac{1}{k} \qquad \therefore\ \mathrm{B}(1, 1) = 1$$

$$\mathrm{B}(\tfrac{1}{2},\, \tfrac{1}{2}) = \pi$$

Be sure that you are familiar with all these. We shall be using them all in due course.

33

Relation between the gamma and beta functions

If m and n are positive integers

$$\mathrm{B}(m,\, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Also, we have previously established that, for n a positive integer,

$$n! = \Gamma(n+1)$$

$$\therefore\ (m-1)! = \Gamma(m) \quad \text{and} \quad (n-1)! = \Gamma(n)$$

and also $(m+n-1)! = \Gamma(m+n)$

$$\therefore\ \mathrm{B}(m,\, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad (12)$$

The relation $\mathrm{B}(m,\, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ holds good even when m and n are not necessarily integers.

We will prove this in the next frame, so move on

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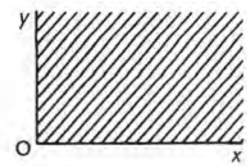
Proof that $\mathrm{B}(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Let $\Gamma(m) = \displaystyle\int_0^\infty x^{m-1}\, e^{-x}\, dx$ and $\Gamma(n) = \displaystyle\int_0^\infty y^{n-1}\, e^{-y}\, dy$

$$\therefore\ \Gamma(m)\Gamma(n) = \int_0^\infty x^{m-1}\, e^{-x}\, dx \int_0^\infty y^{n-1}\, e^{-y}\, dy$$
$$= \int_0^\infty \int_0^\infty x^{m-1}\, y^{n-1}\, e^{-(x+y)}\, dx\, dy$$

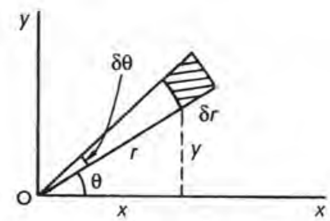


Note that the integration is carried out over the first quadrant of the x–y plane.

Putting $x = u^2$ and $y = v^2$ $dx = 2u\,du$ and $dy = 2v\,dv$

$$\therefore\ \Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^\infty u^{2m-2}\, v^{2n-2}\ e^{-(u^2+v^2)} uv\, du\, dv$$
$$= 4\int_0^\infty \int_0^\infty u^{2m-1}\, v^{2n-1}\, e^{-(u^2+v^2)}\, du\, dv$$

If we now convert to polar coordinates,
$u = r\cos\theta$; $v = r\sin\theta$; $du\,dv = r\,dr\,d\theta$
$u^2 + v^2 = r^2$ $0 < r < \infty$ $0 < \theta < \pi/2$



$$\therefore\ \Gamma(m)\Gamma(n) = 4\int_0^{\pi/2} \int_0^\infty r^{2m-1}\, \cos^{2m-1}\theta\, r^{2n-1}\, \sin^{2n-1}\theta\ e^{-r^2}\, r\, dr\, d\theta$$
$$= 4\int_0^{\pi/2} \int_0^\infty r^{2m+2n-2}\, e^{-r^2}\, \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, r\, dr\, d\theta$$

Then, writing $w = r^2$ $\therefore\ dw = 2r\,dr$

$$\Gamma(m)\Gamma(n) = 2\int_0^\infty w^{m+n-1}\, e^{-w}\, dw \int_0^{\pi/2} \sin^{2n-1}\theta\, \cos^{2m-1}\theta\, d\theta$$
$$= \Gamma(m+n) \times \mathrm{B}(m, n)$$

$$\therefore\ \mathrm{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad (13)$$

So $\mathrm{B}(\frac{3}{2}, \frac{1}{2}) = \ldots\ldots\ldots\ldots$

35

$$\boxed{\mathrm{B}(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{\pi}{2}}$$

Because

$$\mathrm{B}(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{\sqrt{\pi}/2 \times \sqrt{\pi}}{1} = \frac{\pi}{2}$$

Now for some examples.

36

Application of gamma and beta functions

The use of gamma and beta functions in the evaluation of definite integrals depends largely on the ability to change the variables to express the integral in the basic form of the beta function $\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

or its trigonometrical form $2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\mathrm{d}\theta$.

Example 1

Evaluate $I = \int_0^1 x^5(1-x)^4\,\mathrm{d}x$.

Compare this with $\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

Then $m - 1 = 5 \quad \therefore\ m = 6$ and $n - 1 = 4 \quad \therefore\ n = 5$

$$\therefore\ I = \mathrm{B}(6, 5) = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \mathrm{B}(6, 5) = \frac{5!\,4!}{10!} = \frac{1}{1260}}$$

Example 2

Evaluate $I = \int_0^1 x^4\sqrt{1-x^2}\mathrm{d}x$.

Comparing this with $\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x$

we see that we have x^2 in the root, instead of a single x.

Therefore, put $x^2 = y \quad \therefore\ x = y^{\frac{1}{2}} \quad \mathrm{d}x = \frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y$

The limits remain unchanged. $\therefore\ I = \ldots\ldots\ldots\ldots$

38

$$\boxed{I = \tfrac{1}{2}\mathrm{B}\left(\tfrac{5}{2}, \tfrac{3}{2}\right)}$$

Because

$$I = \int_0^1 y^2(1-y)^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y = \frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}\mathrm{d}y$$

$$m - 1 = \tfrac{3}{2} \quad \therefore\ m = \tfrac{5}{2} \quad \text{and} \quad n - 1 = \tfrac{1}{2} \quad \therefore\ n = \tfrac{3}{2}$$

$$\therefore\ I = \tfrac{1}{2}\mathrm{B}\left(\tfrac{5}{2}, \tfrac{3}{2}\right)$$

Expressing this in gamma functions

$$I = \ldots\ldots\ldots\ldots$$

39

$$\boxed{I = \frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}}$$

From our previous work on gamma functions

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2};\ \ \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4};\ \ \Gamma(4) = 3!$$

$$\therefore\ I = \ldots\ldots\ldots\ldots$$

40

$$\boxed{I = \frac{\pi}{32}}$$

Because

$$I = \frac{1}{2}\cdot\frac{(3\sqrt{\pi}/4)(\sqrt{\pi}/2)}{3!} = \frac{\pi}{32}.$$

Now you can work through this one in much the same way. There are no tricks.

Example 3

Evaluate $I = \displaystyle\int_0^3 \frac{x^3\,\mathrm{d}x}{\sqrt{3-x}}$.

You need to compare this with $\mathrm{B}(m, n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x$ so bring everything up on to the top line and then make the necessary change in the variables. Finish it off and then compare the results with the next frame.

41

$$\boxed{I = \frac{864\sqrt{3}}{35} = 42{\cdot}76}$$

Here is the working; see whether you agree.

$$I = \int_0^3 \frac{x^3\,\mathrm{d}x}{\sqrt{3-x}} = \int_0^3 x^3(3-x)^{-\frac{1}{2}}\,\mathrm{d}x = 3^{-\frac{1}{2}}\int_0^3 x^3\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}\,\mathrm{d}x$$

Put $\dfrac{x}{3} = y$, i.e. $x = 3y \quad \therefore\ \mathrm{d}x = 3\,\mathrm{d}y$

Limits: $x = 0,\ y = 0;\quad x = 3,\ y = 1$

$$\therefore\ I = 27\sqrt{3}\int_0^1 y^3(1-y)^{-\frac{1}{2}}\,\mathrm{d}y \qquad \begin{array}{ll} m-1 = 3 & \therefore\ m = 4 \\ n-1 = -\frac{1}{2} & \therefore\ n = \frac{1}{2} \end{array}$$

$$\therefore\ I = 27\sqrt{3}\,\mathrm{B}(4, \tfrac{1}{2}) = 27\sqrt{3}\,\frac{\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma\,(9/2)}$$

Now $\Gamma(\frac{1}{2}) = \sqrt{\pi};\ \ \Gamma(9/2) = \dfrac{105\sqrt{\pi}}{16};\ \ \Gamma(4) = 3!$

$$\therefore\ I = 27\sqrt{3}\times 6\times\sqrt{\pi}\times\frac{16}{105\sqrt{\pi}} = \frac{864\sqrt{3}}{35} = 42{\cdot}76$$

▶

Example 4

Evaluate $I = \int_0^{\pi/2} \sin^5\theta\cos^4\theta\,d\theta$.

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$\therefore\ 2m-1=5 \quad \therefore\ m=3; \quad 2n-1=4 \quad \therefore\ n=5/2$$

$$\therefore\ I = \tfrac{1}{2}B(3, 5/2) = \ldots\ldots\ldots\ldots$$

Finish it off

42

$$\boxed{I = \frac{8}{315}}$$

$$I = \frac{1}{2}B(3, 5/2) = \frac{1}{2}\cdot\frac{\Gamma(3)\Gamma(5/2)}{\Gamma(11/2)}$$

$$= \frac{1}{2}\cdot\frac{2!(3\sqrt{\pi})/4}{(945\sqrt{\pi})/32} = \frac{3\sqrt{\pi}}{4}\cdot\frac{32}{945\sqrt{\pi}} = \frac{8}{315}$$

Finally, one more.

Example 5

Evaluate $I = \int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$.

Somehow, we need to turn this into the form

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

So off you go; express the result in gamma functions

$$I = \ldots\ldots\ldots\ldots$$

43

$$\boxed{I = \frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}}$$

Because

$$I = \int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta$$

$$\therefore\ 2m-1=\frac{1}{2} \quad \therefore\ m=\frac{3}{4}; \quad 2n-1=-\frac{1}{2} \quad \therefore\ n=\frac{1}{4}$$

$$\therefore\ I = \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2}\cdot\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}$$

and, unless we have appropriate tables to evaluate $\Gamma(\frac{3}{4})$ and $\Gamma(\frac{1}{4})$, we cannot proceed much further. However, we do have such a table in Frame 24 so refer to it to evaluate the integral of our example.

$$I = \ldots\ldots\ldots\ldots$$

44

$$\boxed{I = 2{\cdot}2214}$$

Because

$$\Gamma(0{\cdot}25) = 3{\cdot}6256 \quad \text{and} \quad \Gamma(0{\cdot}75) = 1{\cdot}2254$$

$$\therefore\ I = \frac{1}{2}\cdot\frac{(1{\cdot}2254)(3{\cdot}6256)}{1{\cdot}0000} = 2{\cdot}2214$$

Duplication formula for gamma functions

We already know that, when n is a positive integer

$$\Gamma(n) = (n-1)!$$

A useful formula enables us to calculate the gamma functions for values of n halfway between the integers. This is the *duplication formula* which can be stated as

$$\Gamma(n+\tfrac{1}{2}) = \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)} \qquad (14)$$

Thus, to find $\Gamma(3{\cdot}5)$ $\quad \Gamma(n) = \Gamma(3) = 2!$

$$\Gamma(2n) = \Gamma(6) = 5!$$

$$\therefore\quad \Gamma(3{\cdot}5) = \Gamma(3+\tfrac{1}{2}) = \frac{5!\sqrt{\pi}}{2^5\ 2!} = 3{\cdot}3234$$

The formula is quoted here without proof, but it is useful to have on occasions.

So $\Gamma(6{\cdot}5) = \ldots\ldots\ldots\ldots$

45

$$\boxed{\Gamma(6{\cdot}5) = 287{\cdot}9}$$

$$\Gamma(6{\cdot}5) = \Gamma(6+\tfrac{1}{2}) = \frac{\Gamma(12)\sqrt{\pi}}{2^{11}\Gamma(6)}$$

$$\Gamma(6) = 5!;\ \ \Gamma(12) = 11!;\ \ 2^{11} = 2048$$

$$\therefore\ \Gamma(6{\cdot}5) = \frac{11!\sqrt{\pi}}{2048\times 5!} = 287{\cdot}9$$

Now let us consider another function represented by an integral.

On then to the next frame

The error function

46

The error function erf (x) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

and occurs in statistics and various studies in physics and engineering. This integral, for arbitrary x, can only be evaluated numerically and values of erf (x) for various values of x are obtained from tables.

Where the limits of $\int_a^b e^{-t^2}\,dt$ are zero or $\pm\infty$, however, an exact result is possible. We have already considered the integral $I = \int_0^\infty e^{-t^2}\,dt$ in Frame 6 when dealing with gamma functions and we established then that

$$\int_0^\infty e^{-t^2}\,dt = \ldots\ldots\ldots\ldots$$

47

$$\boxed{\int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}}$$

Consequently

$$\underset{x\to\infty}{Lim}\,(\operatorname{erf}(x)) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt = 1$$

By representing the exponential function in the integral by its Maclaurin series we see that

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\ldots\ldots\ldots\ldots$$

48

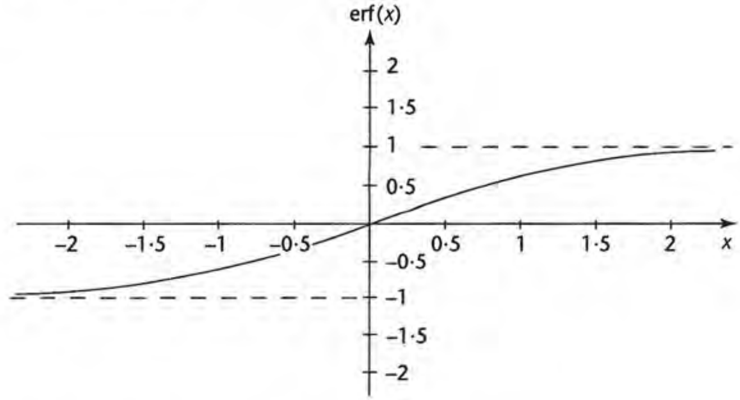
$$\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}}$$

Because

$$\begin{aligned}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\\ &= \frac{2}{\sqrt{\pi}}\int_0^x\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}\right)dt\\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left(\int_0^x\frac{(-1)^n t^{2n}}{n!}\,dt\right)\\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\end{aligned}$$

Consequently erf $(-x) = -$erf (x) and so erf (x) is an odd function. ▶

The graph of erf (x)



The complementary error function erfc (x)

The complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

which is related to the error function by the relation

$$\operatorname{erfc}(x) = \dots\dots\dots\dots$$

49

$$\boxed{\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)}$$

Because

$$\begin{aligned}\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt\\ &= \frac{2}{\sqrt{\pi}}\left(\int_0^\infty e^{-t^2}\,dt - \int_0^x e^{-t^2}\,dt\right)\\ &= 1 - \operatorname{erf}(x)\end{aligned}$$

Example 1

In terms of the complementary error function, for $0 < a < b$

$$\int_a^b e^{-t^2}\,dt = \dots\dots\dots\dots$$

50

$$\boxed{\frac{\sqrt{\pi}}{2}[\text{erfc}\,(a) - \text{erfc}\,(b)]}$$

Because

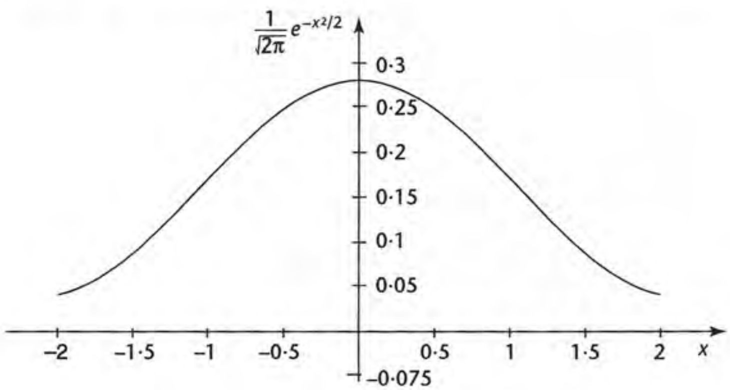
$$\begin{aligned}\int_a^b e^{-t^2}\,\mathrm{d}t &= \int_0^b e^{-t^2}\,\mathrm{d}t - \int_0^a e^{-t^2}\,\mathrm{d}t \\ &= \frac{\sqrt{\pi}}{2}\text{erf}\,(b) - \frac{\sqrt{\pi}}{2}\text{erf}\,(a) \\ &= \frac{\sqrt{\pi}}{2}[1 - \text{erfc}\,(b)] - \frac{\sqrt{\pi}}{2}[1 - \text{erfc}\,(a)] \\ &= \frac{\sqrt{\pi}}{2}[\text{erfc}\,(a) - \text{erfc}\,(b)]\end{aligned}$$

Example 2

In statistics the integral

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t$$

is the area beneath the Gaussian or normal probability distribution $\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ for the values $-\infty < t \leq x$.



The area beneath the complete Gaussian curve is then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

51

$$\boxed{1}$$

Because

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t &= \frac{1}{\sqrt{2\pi}}\left(2\int_0^{\infty} e^{-t^2/2}\,\mathrm{d}t\right) && \text{because the integrand is even}\\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t^2/2}\,\mathrm{d}t\\
&= \sqrt{\frac{2}{\pi}}\times\sqrt{2}\int_0^{\infty} e^{-u^2}\,\mathrm{d}u && \text{where } u = t/\sqrt{2}\\
&= 1 && \text{from Frame 47}
\end{aligned}$$

For positive x, $\Phi(x)$ is related to the error function

$$\Phi(x) = \ldots\ldots\ldots\ldots$$

52

$$\boxed{\Phi(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)}$$

Because

$$\begin{aligned}
\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-t^2/2}\,\mathrm{d}t + \frac{1}{\sqrt{2\pi}}\int_0^{x} e^{-t^2/2}\,\mathrm{d}t\\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\times\sqrt{2}\int_0^{x/\sqrt{2}} e^{-u^2}\,\mathrm{d}u \quad \text{where } u = t/\sqrt{2}\\
&= \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)
\end{aligned}$$

Now let us consider a new set of integral funtions.

Elliptic functions

53

The use of *elliptic functions* provides a means of evaluating a further range of definite integrals, provided that the integrals can be converted by various appropriate substitutions into certain standard forms.

If an integrand is a rational function of x and of $\sqrt{P(x)}$ where $P(x)$ is a polynomial in x of degree 3 or 4, then the integral is said to be *elliptic*.

For example, $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-2x^2)(4-3x^2)}}$ is an elliptic integral. The name is derived from such an integral occurring in the determination of the arc length of part of an ellipse.

▶

Standard forms of elliptic functions

(a) *Of the first kind*

$$F(k,\phi)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad (1)$$

where $0 \le \phi \le \pi/2$ and $0<k<1$.

(b) *Of the second kind*

$$E(k,\phi)=\int_0^{\phi}\sqrt{1-k^2\sin^2\theta}\,d\theta \qquad (2)$$

where $0 \le \phi \le \dfrac{\pi}{2}$ and $0<k<1$.

Make a careful note of these two standard forms: then we can apply them to some examples.

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Example 1

Evaluate $\displaystyle\int_0^{\pi/2}\sqrt{4-\sin^2\theta}\,d\theta$ in terms of an elliptic function.

Taking out a factor 4 to reduce the first term to 1

$$I=2\int_0^{\pi/2}\sqrt{1-\frac{1}{4}\sin^2\theta}\,d\theta$$

The integral now agrees with the standard form, where $k^2=\frac{1}{4}$, i.e. $k=\frac{1}{2}$ and $\phi=\pi/2$.

$$\therefore\ I=\ldots\ldots\ldots\ldots$$

55

$$\boxed{I=2E(\tfrac{1}{2},\ \pi/2)}$$

Complete elliptic functions

In each of the cases (1) and (2) listed above, if $\phi=\pi/2$, the integral is said to be *complete* and then

$F(k,\pi/2)$ is denoted by $K(k)$
and $E(k,\pi/2)$ is denoted by $E(k)$.

The method, then, rests on making suitable substitutions in a given integral to transform the integrand into one of the standard forms stated above. For various values of k and ϕ, values of the functions $F(k,\phi)$, $E(k,\phi)$, $K(k)$ and $E(k)$ are obtainable from published tables. These tables, which are quite extensive, are not reproduced here and so many required values will be given in the text.

Incidentally, the result of Example 1 above, i.e. $I=2E(\frac{1}{2},\ \pi/2)$ could also be written as

$$I=\ldots\ldots\ldots\ldots$$

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$$\boxed{I = 2E(\tfrac{1}{2})}$$

because, in this case, $\phi = \pi/2$.

From tables, we find that $E(\frac{1}{2}) = 1{\cdot}4675 \quad \therefore\ I = 2{\cdot}935$

Example 2

Evaluate $I = \displaystyle\int_0^{\pi/6} \frac{d\theta}{\sqrt{1 - 4\sin^2\theta}}$.

At first sight, this seems to be in standard form, but notice that the value of k^2 is 4, i.e. $k = 2$ – and this does not comply with the requirement that $0 < k < 1$. We therefore proceed as follows.

$$I = \int_0^{\pi/6} \frac{d\theta}{\sqrt{1 - 4\sin^2\theta}}$$

Put $\quad 4\sin^2\theta = \sin^2\psi$

i.e. $\quad 2\sin\theta = \sin\psi$

$$\therefore\ 2\cos\theta\, d\theta = \cos\psi\, d\psi \quad \therefore\ d\theta = \frac{\cos\psi\, d\psi}{2\cos\theta}$$

Also, for the new limits, when $\theta = 0$, $\quad \psi = \ldots\ldots\ldots\ldots$

and when $\theta = \pi/6$, $\quad \psi = \ldots\ldots\ldots\ldots$

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$$\boxed{\theta = 0,\ \psi = 0;\quad \theta = \pi/6,\ \psi = \pi/2}$$

$$\therefore\ I = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \sin^2\psi}} \cdot \frac{\cos\psi\, d\psi}{2\ \cos\theta}$$

We now transform the $\cos\theta$

$$\sin\theta = \tfrac{1}{2}\sin\psi \quad \therefore\ 1 - \cos^2\theta = \tfrac{1}{4}\sin^2\psi \quad \therefore\ \cos\theta = \sqrt{1 - \tfrac{1}{4}\sin^2\psi}$$

$$\therefore\ I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\cos\psi} \cdot \frac{\cos\psi\, d\psi}{\sqrt{1 - \frac{1}{4}\sin^2\psi}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \frac{1}{4}\sin^2\psi}} \quad \text{which is now in standard form}$$

$$\therefore\ I = \ldots\ldots\ldots\ldots$$

58

$$\boxed{I = \tfrac{1}{2}F(\tfrac{1}{2}, \pi/2) = \tfrac{1}{2}K(\tfrac{1}{2})}$$

From the appropriate tables, $K(\frac{1}{2}) = 1{\cdot}6858 \quad \therefore\ I = 0{\cdot}8429$
Now for another

Example 3

Evaluate $I = \displaystyle\int_0^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{3 - 4\sin^2\theta}}$.
The first step is to ....

59

take out a factor 3 to reduce the first term to 1

$$\therefore\ I = \frac{1}{\sqrt{3}}\int_0^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{1 - \frac{4}{3}\sin^2\theta}}$$

Next, we see that $k^2 > 1$. Therefore, we put ....

60

$$\boxed{\tfrac{4}{3}\sin^2\theta = \sin^2\psi}$$

$$\frac{2}{\sqrt{3}}\sin\theta = \sin\psi \quad \therefore\ \frac{2}{\sqrt{3}}\cos\theta\,\mathrm{d}\theta = \cos\psi\,\mathrm{d}\psi \quad \therefore\ \mathrm{d}\theta = \frac{\sqrt{3}\cos\psi\,\mathrm{d}\psi}{2\cos\theta}$$

Then, so far, we have $I = \ldots\ldots\ldots\ldots$

61

$$\boxed{I = \frac{1}{\sqrt{3}}\int_{\theta=0}^{\theta=\pi/3} \frac{1}{\sqrt{1 - \sin^2\psi}} \cdot \frac{\sqrt{3}\cos\psi\,\mathrm{d}\psi}{2\cos\theta}}$$

$$\frac{2}{\sqrt{3}}\sin\theta = \sin\psi$$

Limits: when $\theta = 0$, $\psi = 0$

$$\theta = \frac{\pi}{3},\ \frac{2}{\sqrt{3}}\sin\theta = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = 1 \quad \therefore\ \psi = \pi/2$$

Also $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{3}{4}\sin^2\psi}$

$$\therefore\ I = \ldots\ldots\ldots\ldots$$

62

$$\boxed{I = \frac{1}{2}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \frac{3}{4}\sin^2\psi}}}$$

which is now in standard form with $k = \dfrac{\sqrt{3}}{2}$ and $\phi = \pi/2$

$$\therefore\ I = \frac{1}{2}F\left(\frac{\sqrt{3}}{2},\ \pi/2\right) = \frac{1}{2}K\left(\frac{\sqrt{3}}{2}\right)$$

From tables $K\left(\dfrac{\sqrt{3}}{2}\right) = 2{\cdot}1565 \qquad \therefore\ I = 1{\cdot}078$

Now, what about this one?

Example 4

Evaluate $I = \displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + 4\sin^2\theta}}$.

The trouble here is the *plus* sign in the denominator. Were it a minus sign as in Example 2, the integral could be converted into standard form and would present no difficulty.

In this case, the key is to put $\theta = \pi/2 - \psi$, i.e. $\sin\theta = \cos\psi$.

Expressing the integral in terms of ψ, we have

$$I = \ldots\ldots\ldots\ldots$$

63

$$\boxed{I = \int_{\pi/2}^{0} \frac{-d\psi}{\sqrt{5 - 4\sin^2\psi}}}$$

Because

$$\theta = \pi/2 - \psi \qquad \therefore\ d\theta = -d\psi$$

$$1 + 4\sin^2\theta = 1 + 4(1 - \cos^2\theta) = 5 - 4\cos^2\theta = 5 - 4\sin^2\psi$$

Limits: when $\theta = 0$, $\psi = \pi/2$; when $\theta = \pi/2$, $\psi = 0$ and the expression above immediately follows.

Move on

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So we have $I = \displaystyle\int_{\pi/2}^{0} \frac{-d\psi}{\sqrt{5 - 4\sin^2\psi}}$

The minus sign in the numerator can be absorbed by …………

65

$$\boxed{\text{changing the order of the limits}}$$

$$\therefore \; I = \int_0^{\pi/2} \frac{\mathrm{d}\psi}{\sqrt{5 - 4\sin^2\psi}}$$

Finally, taking out a factor 5 from the denominator, the integral becomes

$$I = \frac{1}{\sqrt{5}} \int_0^{\pi/2} \frac{\mathrm{d}\psi}{\sqrt{1 - \frac{4}{5}\sin^2\psi}}$$

and this can then be written ....

66

$$\boxed{I = \frac{1}{\sqrt{5}} F\left(\frac{2}{\sqrt{5}}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{5}} K\left(\frac{2}{\sqrt{5}}\right)}$$

From tables $K\left(\frac{2}{\sqrt{5}}\right) = K(0{\cdot}8944) = 2{\cdot}2435 \quad \therefore \; I = 1{\cdot}003$

Alternative forms of elliptic functions

(a) *Of the first kind*

$$F(k, x) = \int_0^x \frac{\mathrm{d}u}{\sqrt{(1 - u^2)(1 - k^2u^2)}} \tag{3}$$

where $0 \leq x \leq 1$ and $0 < k < 1$.

(b) *Of the second kind*

$$E(k, x) = \int_0^x \sqrt{\frac{1 - k^2u^2}{1 - u^2}}\, \mathrm{d}u \tag{4}$$

where $0 \leq x \leq 1$ and $0 < k < 1$.

Note these two new forms and then we can deal with a few examples. As before, it is a case of transforming the given integrand into the required form by suitable substitutions.

67

Example 1

Evaluate $I = \displaystyle\int_0^{1/\sqrt{2}} \sqrt{\frac{4 - 3u^2}{1 - u^2}}\, \mathrm{d}u$.

Here we remove a factor 4 from the numerator to reduce the first term to 1.

$$I = 2\int_0^{1/\sqrt{2}} \sqrt{\frac{1 - \frac{3}{4}u^2}{1 - u^2}}\, \mathrm{d}u$$

This is now in standard form with $k =$ .... and $x =$ ....

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$$\boxed{k = \frac{\sqrt{3}}{2}; \quad x = \frac{1}{\sqrt{2}}}$$

$$\therefore \; I = 2E\left(\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right) = 2(0{\cdot}7282) \text{ from tables}$$

$$\therefore \; I = 1{\cdot}4564$$

Example 2

Evaluate $I = \displaystyle\int_0^{1/2} \frac{\mathrm{d}u}{\sqrt{5 - 6u^2 + u^4}}$.

Factorising the denominator gives $I = \ldots\ldots\ldots\ldots$

69

$$\boxed{I = \int_0^{1/2} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(5-u^2)}}}$$

Taking out a factor 5

$$I = \frac{1}{\sqrt{5}} \int_0^{1/2} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\frac{1}{5}u^2)}}$$

which is in standard form with $k = 1/\sqrt{5}$ and $x = 1/2$

$$\therefore \; I = \ldots\ldots\ldots\ldots$$

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$$\boxed{I = \frac{1}{\sqrt{5}} F\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)}$$

In some tables, k is quoted as $\sin\theta$, i.e. $\sin\theta = \dfrac{1}{\sqrt{5}}$ $\quad \therefore \; \theta = 26^\circ\, 34'$

and x is quoted as $\sin\phi$, i.e. $\sin\phi = \dfrac{1}{2}$ $\quad \therefore \; \phi = 30^\circ$.

Then $F(1/\sqrt{5}, 1/2) = 0{\cdot}528$

$$\therefore \; I = 0{\cdot}236$$

Now move on for Example 3

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Example 3

Evaluate $I = \int_0^{\sqrt{3}/4} \sqrt{\dfrac{2-x^2}{1-4x^2}}\,dx$.

We have to convert this into the form $\int \sqrt{\dfrac{1-k^2u^2}{1-u^2}}\,du$, so first concentrate on the denominator. Any suggestions?

72

$$\boxed{\text{Put } 4x^2 = u^2 \quad \text{i.e. } 2x = u}$$

$4x^2 = u^2 \quad \therefore\ 2x = u \quad \therefore\ 2\,dx = du$

Limits: when $x = 0$, $u = 0$ and when $x = \sqrt{3}/4$, $u = \sqrt{3}/2$

Also $\quad 2 - x^2 = 2 - u^2/4$

The integral now becomes ....

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$$\boxed{I = \int_0^{\sqrt{3}/2} \sqrt{\frac{2-u^2/4}{1-u^2}} \cdot \frac{du}{2}}$$

Finally, taking out the factor 2 in the numerator

$I =$ ....

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$$\boxed{I = \frac{1}{\sqrt{2}} \int_0^{\sqrt{3}/2} \frac{\sqrt{1-u^2/8}}{1-u^2}\,du}$$

i.e. $\quad k^2 = \dfrac{1}{8} \quad \therefore\ k = \dfrac{\sqrt{2}}{4} \quad$ and $\quad x = \dfrac{\sqrt{3}}{2}$

So $\quad I =$ ....

75

$$\boxed{I = \frac{1}{\sqrt{2}} E\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right)}$$

Then $\sin\theta = \dfrac{\sqrt{2}}{4} \quad \therefore\ \theta = 20^\circ\,42'$ and $\sin\phi = \dfrac{\sqrt{3}}{2} \quad \therefore\ \phi = 60^\circ$

From tables, $E\left(\dfrac{\sqrt{2}}{4}, \dfrac{\sqrt{3}}{2}\right) = 1{\cdot}029 \quad \therefore\ I = 0{\cdot}728$

So it is all just a question of manipulation to transform the given integral into the required standard forms, and then of reference to the appropriate tables.

▶

The **Revision summary** follows, to be read in conjunction with the **Can You?** checklist, checking with the relevant parts of the Programme any points of which you are unsure. You will then find the **Test exercise** straightforward. Finally the **Further problems** provide additional practice.

Revision summary 16

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1 *Gamma functions*

(a) $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \qquad x > 0$

$\Gamma(x+1) = x\Gamma(x)$

(b) If $x = n$, a positive integer

$$\Gamma(n+1) = n!$$
$$\Gamma(1) = 1$$
$$\Gamma(0) = \infty \qquad \Gamma(-n) = \pm\infty$$

(c) $\int_0^\infty e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$

(d) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad \Gamma(\tfrac{3}{2}) = \dfrac{\sqrt{\pi}}{2}$

$\Gamma(\tfrac{5}{2}) = \dfrac{3\sqrt{\pi}}{4} \qquad \Gamma(\tfrac{7}{2}) = \dfrac{15\sqrt{\pi}}{8}$

$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi} \qquad \Gamma(-\tfrac{3}{2}) = \dfrac{4\sqrt{\pi}}{3}$

(e) Duplication formula $\quad \Gamma(n+\tfrac{1}{2}) = \dfrac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\cdot\Gamma(n)}$

2 *Beta functions*

(a) $\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m > 0;\ n > 0$

$\mathrm{B}(m, n) = \mathrm{B}(n, m)$

$\mathrm{B}(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

(b) $\mathrm{B}(m, n) = \dfrac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\mathrm{B}(m-1, n-1)$

$\mathrm{B}(k, 1) = \mathrm{B}(1, k) = \dfrac{1}{k}$

$\mathrm{B}(1, 1) = 1; \qquad \mathrm{B}(\tfrac{1}{2}, \tfrac{1}{2}) = \pi$

$\mathrm{B}(m, n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

(c) m and n positive integers

$\mathrm{B}(m, n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

▶

3 *Error function*

(a) $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,\mathrm{d}t$

(b) $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}; \qquad \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}$$

Complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,\mathrm{d}t = 1 - \operatorname{erf}(x)$$

4 *Elliptic functions*

(a) *Standard forms*

(1) of the first kind: $F(k, \phi) = \displaystyle\int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}}$

(2) of the second kind: $E(k, \phi) = \displaystyle\int_0^\phi \sqrt{1 - k^2 \sin^2\theta}\,\mathrm{d}\theta$

In each case, $0 \le \phi \le \pi/2$; $0 < k < 1$.

(b) *Complete elliptic integrals* $\phi = \dfrac{\pi}{2}$

$$F\left(k, \frac{\pi}{2}\right) = K(k)$$
$$E\left(k, \frac{\pi}{2}\right) = E(k)$$

(c) *Alternative forms of elliptic functions*

(1) of the first kind: $F(k, x) = \displaystyle\int_0^x \frac{\mathrm{d}u}{\sqrt{(1 - u^2)(1 - k^2u^2)}}$

(2) of the second kind: $E(k, x) = \displaystyle\int_0^x \sqrt{\frac{1 - k^2u^2}{1 - u^2}}\,\mathrm{d}u$

In each case $0 \le x \le 1$; $0 < k < 1$.

☑ Can You?

Checklist 16

77

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: | **Frames**
• Derive the recurrence relation for the gamma function and evaluate the gamma function for certain rational arguments?	1 to 16
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Evaluate integrals that require the use of the gamma function in their solution?	17 to 24
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Identify the beta function and evaluate integrals that require the use of the beta function in their solution?	25 to 32
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Derive the relationship between the gamma function and the beta function?	33 to 44
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use the duplication formula to evaluate the gamma function for half integer arguments?	44 and 45
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Recognise the error function and its relation to the Gaussian probability distribution?	46 to 52
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Recognise elliptic functions of the first and second kind?	53
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Evaluate integrals that require the use of elliptic functions in their solution?	54 to 66
Yes ☐ ☐ ☐ ☐ ☐ *No*	
• Use alternative forms of the elliptic functions?	66 to 75
Yes ☐ ☐ ☐ ☐ ☐ *No*	

Test exercise 16

78

1 Evaluate (a) $\dfrac{\Gamma(6)}{3\Gamma(4)}$ (b) $\dfrac{\Gamma(1{\cdot}5)}{\Gamma(2{\cdot}5)}$ (c) $\dfrac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2})}$

(d) $\displaystyle\int_0^{\infty} x^5\, e^{-x}\, dx$ (e) $\displaystyle\int_0^{\infty} x^6\, e^{-4x^2}\, dx.$

2 Determine (a) $\displaystyle\int_0^1 x^5(2-x)^4 dx$

(b) $\displaystyle\int_0^{\pi/2} \sin^7\theta\cos^3\theta\, d\theta$

(c) $\displaystyle\int_0^{\pi/8} \sin^2 4\theta\cos^5 4\theta\, d\theta.$

3 Show that

(a) $\displaystyle\int_{-a}^{a} e^{-t^2}\, dt = \sqrt{\pi}\,\text{erf}\,(a)$

(b) $\displaystyle\int_0^{\infty} e^{-k^2t^2}\, dt = \frac{\sqrt{\pi}}{2k}, \quad k > 0.$

4 Evaluate

(a) $\text{erfc}\,(\infty)$

(b) $\text{erfc}\,(0)$.

5 Express the following in elliptic functions.

(a) $\displaystyle\int_0^{\pi/4} \frac{d\theta}{\sqrt{1-2\sin^2\theta}}$

(b) $\displaystyle\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{4-5u^2+u^4}}.$

Further problems 16

79

1 Evaluate (a) $\dfrac{\Gamma(5)}{2\Gamma(3)}$; (b) $\dfrac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})}$; (c) $\dfrac{\Gamma(2{\cdot}5)}{\Gamma(3{\cdot}5)}$;

(d) $\displaystyle\int_0^{\infty} x^4 e^{-x}\, dx$; (e) $\displaystyle\int_0^{\infty} x^8 e^{-2x}\, dx.$

2 Determine (a) $\displaystyle\int_0^{\infty} x^3 e^{-x}\, dx$; (b) $\displaystyle\int_0^{\infty} x^4 e^{-3x}\, dx$;

(c) $\displaystyle\int_0^{\infty} x^2 e^{-2x^2} dx$; (d) $\displaystyle\int_0^{\infty} \sqrt{x}\cdot e^{-\sqrt{x}}\, dx.$

3 If m and n are positive constants, show that $\int_0^\infty x^m e^{-ax^n}\,dx$ can be expressed in the form $\dfrac{1}{n \cdot a^{(m+1)/n}}\Gamma\left(\dfrac{m+1}{n}\right)$.

4 Evaluate the following.

(a) $\int_0^{1/2} x^4(1-2x)^3\,dx$

(b) $\int_0^{1/\sqrt{2}} x^2\sqrt{1-2x^2}\,dx$

(c) $\int_0^{\pi/2} \sin^5\theta\cos^4\theta\,d\theta$

(d) $\int_0^{\pi/2} \sin\theta\sqrt{\cos^5\theta}\,d\theta$

(e) $\int_0^{\pi/4} \sin^3 2\theta\cos^6 2\theta\,d\theta$

(f) $\int_0^{1/3} x^2\sqrt{1-9x^2}\,dx.$

5 Show that $\dfrac{d}{dx}\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}e^{-x^2}$.

6 Show that the Laplace transform of the error function is given as

$$F(s) = \int_0^\infty \operatorname{erf}(t)\,e^{-st}\,dt = \frac{e^{-s^2/4}}{s}\operatorname{erfc}\left(\frac{s}{2}\right) \text{ for } s > 0.$$

7 The Fresnel integrals are defined as

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt \text{ and } S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$$

Show that

$$\frac{1}{\sqrt{2j}}\operatorname{erf}\left(x\sqrt{\frac{j\pi}{2}}\right) = C(x) - jS(x)$$

▶

8 Express the following in elliptic functions.

(a) $\displaystyle\int_0^{\pi/2} \sqrt{1+4\sin^2\theta}\,\mathrm{d}\theta$

(b) $\displaystyle\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta}}$

(c) $\displaystyle\int_0^{1} \sqrt{\frac{4-x^2}{1-x^2}}\,\mathrm{d}x$

(d) $\displaystyle\int_0^{2} \frac{\mathrm{d}x}{\sqrt{(9-x^2)(16-x^2)}}$

(e) $\displaystyle\int_0^{2} \frac{\mathrm{d}x}{\sqrt{(4-x^2)(5-x^2)}}$

(f) $\displaystyle\int_0^{\pi/6} \frac{\mathrm{d}\theta}{\sqrt{\sin^2\theta+2\cos^2\theta}}$

(g) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{\sin^2\theta+2\cos^2\theta}}.$

9 Using the substitution $x=\tan\theta$ prove that the integral

$$\int_0^{1} \frac{\mathrm{d}x}{\sqrt{(1+x^2)(1+4x^2)}}$$

can be expressed in the form

$$\frac{1}{2}\int_0^{\pi/4} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{3}{4}\cos^2\theta}}$$

Hence, using $\theta=\dfrac{\pi}{2}-\phi$, evaluate the integral in terms of elliptic functions.

10 Evaluate the following.

(a) $\displaystyle\int_0^{0{\cdot}5} \frac{\mathrm{d}x}{\sqrt{3-4x^2+x^4}}$

(b) $\displaystyle\int_{0{\cdot}5}^{1{\cdot}0} \frac{\mathrm{d}x}{\sqrt{3-4x^2+x^4}}$

(c) $\displaystyle\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{25+9\sin^2\theta}}$

(d) $\displaystyle\int_0^{\pi/3} \frac{\mathrm{d}\theta}{\sqrt{4+3\sin^2\theta}}.$

Programme 17

Vector analysis 1

Frames 1 to 95

Learning outcomes

When you have completed this Programme you will be able to:

- Obtain the scalar and vector product of two vectors
- Reproduce the relationships between the scalar and vector products of the Cartesian coordinate unit vectors
- Obtain the scalar and vector triple products and appreciate their geometric significance
- Differentiate a vector field and derive a unit vector tangential to the vector field at a point
- Integrate a vector field
- Obtain the gradient of a scalar field, the directional derivative and a unit normal to a surface
- Obtain the divergence of a vector field and recognise a solenoidal vector field
- Obtain the curl of a vector field
- Obtain combinations of div, grad and curl acting on scalar and vector fields as appropriate

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme 6 Vectors

Introduction

1

The initial work on vectors was covered in detail in Programme 6 of *Engineering Mathematics (Fifth Edition)* and, if you are in any doubt, spend some time reviewing that section of the work before proceeding further.

The current Programmes on vector analysis build on these early foundations, so, for quick reference, the essential results of the previous work are summarised in the following list.

Summary of prerequisites

1 A *scalar* quantity has magnitude only; a *vector* quantity has both magnitude and direction.

2 The axes of reference, OX, OY, OZ, form a right-handed set. The symbols $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote *unit vectors* in the directions OX, OY, OZ, respectively.

If $\overline{\mathrm{OP}} = \mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ then $\mathrm{OP} = |\mathbf{r}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ where $|\mathbf{r}|$ is the modulus of $\mathbf{r}$.

3 The *direction cosines* $[l, m, n]$ are the cosines of the angles between the vector $\mathbf{r}$ and the axes OX, OY, OZ, respectively. For any vector $\mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$l = \frac{a_x}{|\mathbf{r}|}; \quad m = \frac{a_y}{|\mathbf{r}|}; \quad n = \frac{a_z}{|\mathbf{r}|}$$

and $\quad l^2 + m^2 + n^2 = 1.$

4 *Scalar product* ('dot product')

$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ and where A and B are the moduli of $\mathbf{A}$ and $\mathbf{B}$.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ then

$$\mathbf{A} \cdot \mathbf{B} = a_xb_x + a_yb_y + a_zb_z \quad \text{and} \quad \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

5 *Vector product* ('cross product')

$\mathbf{A} \times \mathbf{B} = AB\sin\theta$ in a direction perpendicular to $\mathbf{A}$ and $\mathbf{B}$ so that $\mathbf{A}$, $\mathbf{B}$, $(\mathbf{A} \times \mathbf{B})$ form a right-handed set.

Therefore $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$

Also $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$ where $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

6 *Angle between two vectors*

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively.

For perpendicular vectors $\quad l_1l_2 + m_1m_2 + n_1n_2 = 0$

For parallel vectors $\quad l_1l_2 + m_1m_2 + n_1n_2 = 1.$

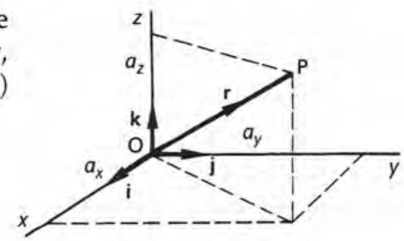
One or two examples will no doubt help to recall the main points.

Example 1 (Direction cosines)

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the directions OX, OY, OZ, respectively, then any position vector $\overline{\mathrm{OP}}\ (=\mathbf{r})$ can be represented in the form

$$\overline{\mathrm{OP}} = \mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

Then $|\mathbf{r}| = \dots\dots\dots\dots$



2

$$\boxed{|\mathbf{r}| = \sqrt{a_x^2 + a_y^2 + a_z^2}}$$

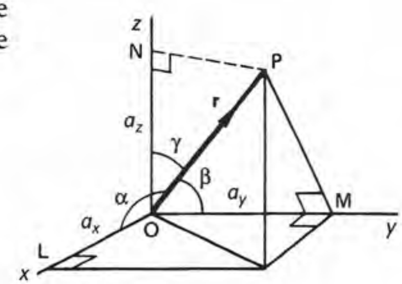
The direction of OP is denoted by stating the direction cosines of the angles made by OP and the three coordinate axes.

$$l = \cos\alpha = \frac{\mathrm{OL}}{\mathrm{OP}} = \frac{a_x}{|\mathbf{r}|}$$

$$m = \cos\beta = \frac{\mathrm{OM}}{\mathrm{OP}} = \frac{a_y}{|\mathbf{r}|}$$

$$n = \cos\gamma = \frac{\mathrm{ON}}{\mathrm{OP}} = \frac{a_z}{|\mathbf{r}|}$$

$$\therefore\ l,\ m,\ n = \cos\alpha,\quad \cos\beta,\quad \cos\gamma$$



So, if P is the point (3, 2, 6), then

$$|\mathbf{r}| = \dots\dots\dots\dots;$$
$$l = \dots\dots\dots\dots;\quad m = \dots\dots\dots\dots;\quad n = \dots\dots\dots\dots$$

3

$$\boxed{\begin{gathered}|\mathbf{r}| = 7;\\ l = 0{\cdot}429;\quad m = 0{\cdot}286;\quad n = 0{\cdot}857\end{gathered}}$$

Because

$$(|\,\mathbf{r}\,|)^2 = 9 + 4 + 36 = 49 \qquad \therefore\quad |\,\mathbf{r}\,| = 7$$
$$l = \cos\alpha = \tfrac{3}{7} = 0{\cdot}4286$$
$$m = \cos\beta = \tfrac{2}{7} = 0{\cdot}2857$$
$$n = \cos\gamma = \tfrac{6}{7} = 0{\cdot}8571.$$

▶

Example 2 (Angle between two vectors)

If the direction cosines of **A** are l_1, m_1, n_1 and those of **B** are l_2, m_2, n_2, then the angle between the vectors is given by

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2. \qquad (1)$$

If $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, we can find the direction cosines of each and hence θ which is ....

4

$$\boxed{\theta = 66^\circ\, 36'}$$

Because

For **A**: $|\,\mathbf{r}_1\,| = \sqrt{4+9+16} = \sqrt{29}$

$$\therefore\ l_1 = \frac{2}{\sqrt{29}};\quad m_1 = \frac{3}{\sqrt{29}};\quad n_1 = \frac{4}{\sqrt{29}}$$

For **B**: $|\,\mathbf{r}_2\,| = \sqrt{1+4+9} = \sqrt{14}$

$$\therefore\ l_2 = \frac{1}{\sqrt{14}};\quad m_2 = \frac{-2}{\sqrt{14}};\quad n_2 = \frac{3}{\sqrt{14}}$$

Then $\cos\theta = \dfrac{1}{\sqrt{14 \times 29}}\{2 - 6 + 12\} = 0{\cdot}3970$

$\therefore\ \theta = 66^\circ\, 36'$

Let us now look at the question of scalar and vector products.

On to the next frame

5

Example 3 (Scalar product)

If **A** and **B** are two vectors, the scalar product of **A** and **B** is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta \qquad (2)$$

where θ is the angle between the two vectors. If $\mathbf{A} \cdot \mathbf{B} = 0$ then $\mathbf{A} \perp \mathbf{B}$.

If we consider the *scalar products of the unit vectors* **i**, **j**, **k**, which are mutually perpendicular, then

$$\mathbf{i} \cdot \mathbf{j} = (1)(1)\cos 90^\circ = 0 \qquad \therefore\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

and $$\mathbf{i} \cdot \mathbf{i} = (1)(1)\cos 0^\circ = 1 \qquad \therefore\ \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1.$$

In general, if $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ then $\mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z$ which is, of course, a scalar quantity.

So, if $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, then

$\mathbf{A} \cdot \mathbf{B} =$ ....

6

$$\boxed{\mathbf{A} \cdot \mathbf{B} = 2 - 6 + 20 = 16}$$

Also, since $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$, we can determine the angle θ between the vectors. In this case $\theta =$ ....

7

$$\boxed{\theta = 57^\circ\, 9'}$$

$\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \qquad \therefore\ A = |\,\mathbf{A}\,| = \sqrt{4+9+16} = \sqrt{29}$

$\mathbf{B} = \ \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} \qquad \therefore\ B = |\,\mathbf{B}\,| = \sqrt{1+4+25} = \sqrt{30}$

We have already found that $\mathbf{A}\cdot\mathbf{B} = 16$ and $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$

$\therefore\ 16 = \sqrt{29}\,\sqrt{30}\cos\theta \quad \therefore\ \cos\theta = 0{\cdot}5425 \quad \therefore\ \theta = 57^\circ\, 9'$

So, the *scalar product* of $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$

is $\quad \mathbf{A}\cdot\mathbf{B} = a_xb_x + a_yb_y + a_zb_z$

and $\quad \mathbf{A}\cdot\mathbf{B} = AB\cos\theta$ where θ is the angle between the vectors.

It can also be shown that

(a) $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

and (b) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

Make a note of these results

8

Example 4 (Vector product)

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ the vector product $\mathbf{A}\times\mathbf{B}$ has magnitude $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$ in the direction perpendicular to $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}$, $\mathbf{B}$ and $(\mathbf{A}\times\mathbf{B})$ form a right-handed set.

We can write this as

$$\mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{n} \qquad (3)$$

where $\mathbf{n}$ is defined as a unit vector in the positive normal direction to the plane of $\mathbf{A}$ and $\mathbf{B}$, i.e. forming a right-handed set.

Also
$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (4)$$

If we consider the *vector products of the unit vectors*, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, then

$\mathbf{i}\times\mathbf{j} = (1)(1)\sin 90^\circ\mathbf{k} = \mathbf{k}$

$\mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$

Note that

$\mathbf{j}\times\mathbf{i} = -(\mathbf{i}\times\mathbf{j}) = -\mathbf{k}$

$\mathbf{k}\times\mathbf{j} = -\mathbf{i}, \quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}$

Also

$\mathbf{i}\times\mathbf{i} = (1)(1)\sin 0^\circ\mathbf{n} = \mathbf{0}$

$\mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$

It can also be shown that

(a) $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

and (b) $\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ (5)

Make a note of these results (3), (4) and (5).

Then, if $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

$\mathbf{A}\times\mathbf{B} = \ldots\ldots\ldots\ldots$

9

$$\boxed{\mathbf{A} \times \mathbf{B} = 16\mathbf{i} + 14\mathbf{j} - 5\mathbf{k}}$$

We simply evaluate the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ 2 & -3 & -2 \end{vmatrix}$$

$$= \mathbf{i}(4 + 12) - \mathbf{j}(-6 - 8) + \mathbf{k}(-9 + 4) = 16\mathbf{i} + 14\mathbf{j} - 5\mathbf{k}$$

Move on to the next frame

10

We have seen therefore that

the scalar product of two vectors is a scalar

but that the vector product of two vectors is a vector.

We know also that $|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$

Therefore, the angle between the vectors **A** and **B** given in Example 4 is

$\theta = \ldots\ldots\ldots\ldots$

11

$$\boxed{\theta = 79^\circ\ 40'}$$

Because

$$\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k};\quad \mathbf{B} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k};\quad \text{and}\quad \mathbf{A} \times \mathbf{B} = 16\mathbf{i} + 14\mathbf{j} - 5\mathbf{k}$$

$$\therefore\quad |\,\mathbf{A} \times \mathbf{B}\,| = \sqrt{16^2 + 14^2 + 5^2} = \sqrt{477} = 21{\cdot}84$$

$$A = |\,\mathbf{A}\,| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{29} = 5{\cdot}385$$

$$B = |\,\mathbf{B}\,| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{17} = 4{\cdot}123$$

$$\therefore\quad 21{\cdot}84 = (5{\cdot}385)(4{\cdot}123)\sin\theta$$

$$\therefore\quad \sin\theta = 0{\cdot}9838 \quad \therefore\quad \theta = 79^\circ\ 40'$$

So, to recapitulate:

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ and θ is the angle between them

(a) Scalar product $= \mathbf{A} \cdot \mathbf{B} = a_xb_x + a_yb_y + a_zb_z$

$$= AB\cos\theta$$

(b) Vector product $= \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

and $|\,\mathbf{A} \times \mathbf{B}\,| = AB\sin\theta.$

Make a note of these fundamental results: we shall certainly need them. Then, in the next frame, we can set off on some new work

Triple products

12

We now deal with the various products that we form with three vectors.

Scalar triple product of three vectors

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three vectors, the scalar formed by the product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is called the scalar triple product.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; $\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$;

then $$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Multiplying the top row by the external bracket and remembering that

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \quad\text{and}\quad \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

we have $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad (6)$$

Example

If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; $\mathbf{B} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{C} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$;

then $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -2 & -3 \\ 2 & 1 & 2 \end{vmatrix}$$
$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = 42}$$

Because

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -2 & -3 \\ 2 & 1 & 2 \end{vmatrix}$$
$$= 2(-4+3) + 3(2+6) + 4(1+4) = 42$$

As simple as that.

14 Properties of scalar triple products

(a) $$\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix}=-\begin{vmatrix} a_x & a_y & a_z \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix}$$

since interchanging two rows in a determinant reverses the sign. If we now interchange rows 2 and 3 and again change the sign, we have

$$\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

$$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \qquad (7)$$

i.e. the scalar triple product is unchanged by a cyclic change of the vectors involved.

(b) $$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})=\begin{vmatrix} b_x & b_y & b_z \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix}=-\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\therefore\ \mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})=-\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \qquad (8)$$

i.e. a change of vectors not in cyclic order, changes the sign of the scalar triple product.

(c) $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{A})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ a_x & a_y & a_z \end{vmatrix}=0 \text{ since two rows are identical.}$$

$$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{A})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{B})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{C})=0 \qquad (9)$$

Example

If $\mathbf{A}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$; $\mathbf{B}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$; $\mathbf{C}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=$ .... $\quad \mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})=$ ....

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = 52; \quad \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = -52}$$

Because

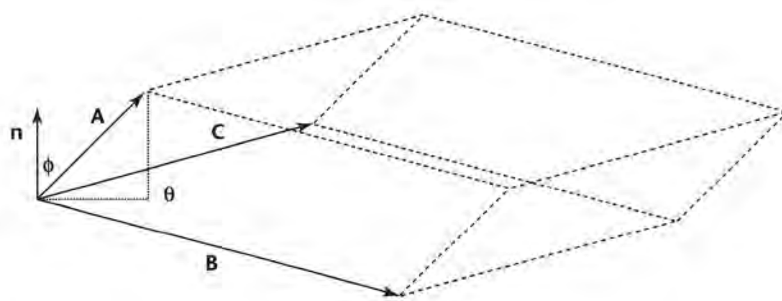
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 1(6-1) - 2(-4-3) + 3(2+9) = 52$$

$\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})$ is not a cyclic change from the above. Therefore

$$\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = -52$$

Coplanar vectors

The magnitude of the scalar triple product $|\,\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\,|$ is equal to the volume of the parallelepiped with three adjacent sides defined by **A**, **B** and **C**.



The scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot(BC\sin\theta\,\mathbf{n}) = ABC\sin\theta\cos\phi$ where **n** is a unit vector perpendicular to the plane containing **B** and **C**, θ is the angle between **B** and **C** and ϕ is the angle between **A** and **n**. Therefore

$$|\,\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\,| = ABC\,|\,\sin\theta\cos\phi\,|$$

Notice that in the figure both θ and ϕ are drawn as acute but in the general case this may not be so. Now, $BC\,|\,\sin\theta\,|$ is the area of the parallelogram defined by **B** and **C**. The altitude of the parallelepiped is $A\,|\,\cos\phi\,|$ and so $ABC\,|\,\sin\theta\cos\phi\,|$ is the *volume* of the parallelepiped with three adjacent sides defined by **A**, **B** and **C**.

Consequently if $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{0}$ then the volume of the parallelepiped is zero and the three vectors **A**, **B** and **C** are coplanar.

Example 1

Show that $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$; and $\mathbf{C} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ are coplanar.

We just evaluate $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \ldots\ldots\ldots\ldots$ and apply the test.

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$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0}$$

Because

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}1 & 2 & -3\\ 2 & -1 & 2\\ 3 & 1 & -1\end{vmatrix}=1(1-2)-2(-2-6)-3(2+3)\ =0.$$

Therefore $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are coplanar.

Example 2

If $\mathbf{A}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$; $\mathbf{B}=3\mathbf{i}+2\mathbf{j}+\mathbf{k}$; $\mathbf{C}=\mathbf{i}+p\mathbf{j}+4\mathbf{k}$ are coplanar, find the value of p.

The method is clear enough. We merely set up and evaluate the determinant and solve the equation $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0$.

$p=$ ....

17

$$\boxed{p=-3}$$

Because

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0 \qquad \therefore \begin{vmatrix}2 & -1 & 3\\ 3 & 2 & 1\\ 1 & p & 4\end{vmatrix}=0$$

$\therefore\ 2(8-p)+1(12-1)+3(3p-2)=0 \quad \therefore\ 7p=-21 \quad \therefore\ p=-3$

One more.

Example 3

Determine whether the three vectors $\mathbf{A}=3\mathbf{i}+2\mathbf{j}-\mathbf{k}$; $\mathbf{B}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$; $\mathbf{C}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ are coplanar.

Work through it on your own. The result shows that

....

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$$\boxed{\mathbf{A},\ \mathbf{B},\ \mathbf{C} \text{ are not coplanar}}$$

Because

$$\text{in this case } \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}3 & 2 & -1\\ 2 & -1 & 3\\ 1 & -2 & 2\end{vmatrix}=13$$

$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\neq 0 \qquad \therefore\ \mathbf{A},\mathbf{B},\mathbf{C}$ are not coplanar.

Now on to something different

Vector triple products of three vectors

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If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are three vectors, then

$$\left.\begin{array}{rr} & \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \\ \text{and} & (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \end{array}\right\} \text{ are called the vector triple products.} \qquad (10)$$

Consider $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ where $\mathbf{A}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$; $\mathbf{B}=b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}$ and $\mathbf{C}=c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k}$.

Then $(\mathbf{B}\times\mathbf{C})$ is a vector perpendicular to the plane of $\mathbf{B}$ and $\mathbf{C}$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is a vector perpendicular to the plane containing $\mathbf{A}$ and $(\mathbf{B}\times\mathbf{C})$, i.e. coplanar with $\mathbf{B}$ and $\mathbf{C}$.

Note that, similarly, $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ is coplanar with $\mathbf{A}$ and $\mathbf{B}$ and so in general $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\neq(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

Now

$$(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}=\mathbf{i}\begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix}-\mathbf{j}\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix}+\mathbf{k}\begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}$$

$$\text{Then}\quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ \begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix} & -\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix} & \begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}\end{vmatrix}$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ \begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix} & \begin{vmatrix}b_z & b_x\\ c_z & c_x\end{vmatrix} & \begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}\end{vmatrix}$$

In symbolic form, further expansion of the determinant becomes somewhat tedious. However a numerical example will clarify the method.

So make a note of the definition (10) above and then go on to the next frame

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Example 1

If $\mathbf{A}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$; $\mathbf{B}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$; $\mathbf{C}=3\mathbf{i}+\mathbf{j}+3\mathbf{k}$; determine the vector triple product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

We start off with $\mathbf{B}\times\mathbf{C}=$ ....

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$$\boxed{\mathbf{B} \times \mathbf{C} = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}}$$

Because

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & 1 & 3 \end{vmatrix} \begin{aligned} &= \mathbf{i}(6+1) - \mathbf{j}(3+3) + \mathbf{k}(1-6) \\ &= 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k} \end{aligned}$$

Then $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 21\mathbf{i} + 17\mathbf{j} + 9\mathbf{k}}$$

Because

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 7 & -6 & 5 \end{vmatrix} \\ &= \mathbf{i}(15+6) - \mathbf{j}(-10-7) + \mathbf{k}(-12+21) \\ &= 21\mathbf{i} + 17\mathbf{j} + 9\mathbf{k} \end{aligned}$$

That is fundamental enough. There is, however, an even easier way of determining a vector triple product. It can be proved that

$$\begin{aligned} &\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ \text{and} \quad &(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \end{aligned} \tag{11}$$

The proof of this is given in the Appendix. For the moment, make a careful note of the expressions: then we will apply the method to the example we have just completed.

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$\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$; $\mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and we have

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (6 - 3 + 3)(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (2 - 6 - 1)(3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= 6\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + 5(3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= 21\mathbf{i} + 17\mathbf{j} + 9\mathbf{k} \end{aligned}$$

which is, of course, the result we achieved before.

Here is another.

Example 2

If $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; $\mathbf{C} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ determine $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ using the relationship $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A}$.

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \ldots\ldots\ldots\ldots$$

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$$\boxed{-50\mathbf{i} - 26\mathbf{j} + 22\mathbf{k}}$$

Because

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ &= (6 - 6 - 2)(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - (8 + 3 + 3)(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \\ &= -2(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - 14(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \\ &= -50\mathbf{i} - 26\mathbf{j} + 22\mathbf{k}\end{aligned}$$

Now one more.

Example 3

If $\mathbf{A} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$; $\mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \ldots\ldots\ldots\ldots$$
$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \ldots\ldots\ldots\ldots$$

Finish them both.

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$$\boxed{\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= 11\mathbf{i} + 35\mathbf{j} - 58\mathbf{k} \\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= 17\mathbf{i} + 38\mathbf{j} - 31\mathbf{k}\end{aligned}}$$

Because

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (1 + 6 + 6)(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - (2 + 15 - 2)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 13(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - 15(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 11\mathbf{i} + 35\mathbf{j} - 58\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ &= (1 + 6 + 6)(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - (2 + 10 - 3)(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \\ &= 13(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - 9(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = 17\mathbf{i} + 38\mathbf{j} - 31\mathbf{k}\end{aligned}$$

These two results clearly confirm that

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ *so beware!*

Before we proceed, note the following concerning the unit vectors.

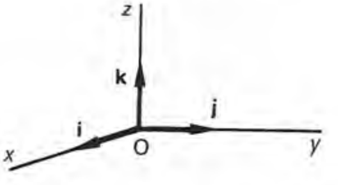
(a) $(\mathbf{i} \times \mathbf{j}) = \mathbf{k}$

$\therefore\ \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

$\therefore\ \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j}$

(b) $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = (0) \times \mathbf{j} = 0$

$\therefore\ (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0$



and once again, we see that

$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$

On to the next

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Finally, by way of revision:

Example 4

If $\mathbf{A} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; determine

(a) the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

(b) the vector triple products (1) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
(2) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

Finish all these and then check with the next frame

27

$$\boxed{\begin{aligned} &\text{(a)}\ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -12 \\ &\text{(b)}\ (1)\ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 62\mathbf{i} + 44\mathbf{j} - 74\mathbf{k} \\ &\qquad (2)\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = 109\mathbf{i} + 7\mathbf{j} - 22\mathbf{k} \end{aligned}}$$

Here is the working.

$$\text{(a)}\ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 5 & -2 & 3 \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= 5(4 - 6) + 2(12 + 2) + 3(-9 - 1) = -12$$

$$\begin{aligned} \text{(b)}\ (1)\ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (5 + 6 + 12)(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \\ &\qquad - (15 - 2 - 6)(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \\ &= 23(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 7(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \\ &= 62\mathbf{i} + 44\mathbf{j} - 74\mathbf{k} \end{aligned}$$

$$\begin{aligned} (2)\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ &= 23(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-8)(5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= 109\mathbf{i} + 7\mathbf{j} - 22\mathbf{k} \end{aligned}$$

Let us now move to the next topic

28

Differentiation of vectors

In many practical problems, we often deal with vectors that change with time, e.g. velocity, acceleration, etc. If a vector $\mathbf{A}$ depends on a scalar variable t, then $\mathbf{A}$ can be represented as $\mathbf{A}(t)$ and $\mathbf{A}$ is then said to be a function of t.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ then a_x, a_y, a_z will also be dependent on the parameter t.

i.e. $\mathbf{A}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k}$

Differentiating with respect to t gives ....

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$$\boxed{\frac{d}{dt}\{\mathbf{A}(t)\} = \mathbf{i}\frac{d}{dt}\{a_x(t)\} + \mathbf{j}\frac{d}{dt}\{a_y(t)\} + \mathbf{k}\frac{d}{dt}\{a_z(t)\}}$$

In short $\dfrac{d\mathbf{A}}{dt} = \mathbf{i}\dfrac{da_x}{dt} + \mathbf{j}\dfrac{da_y}{dt} + \mathbf{k}\dfrac{da_z}{dt}$.

The independent scalar variable is not, of course, restricted to t. In general, if u is the parameter, then

$$\frac{d\mathbf{A}}{du} = \ldots\ldots\ldots\ldots$$

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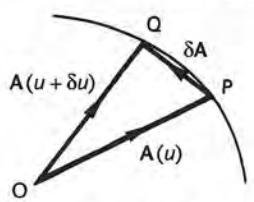
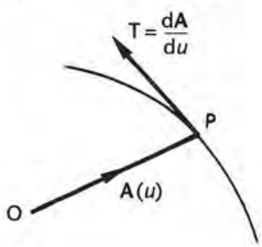
$$\boxed{\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{da_x}{du} + \mathbf{j}\frac{da_y}{du} + \mathbf{k}\frac{da_z}{du}}$$





If a position vector $\overline{OP}$ moves to $\overline{OQ}$ when u becomes $u + \delta u$, then as $\delta u \to 0$, the direction of the chord $\overline{PQ}$ becomes that of the tangent to the curve at $\mathbf{P}$, i.e. the direction of $\dfrac{d\mathbf{A}}{du}$ is along the tangent to the locus of $\mathbf{P}$.

Example 1

If $\mathbf{A} = (3u^2 + 4)\mathbf{i} + (2u - 5)\mathbf{j} + 4u^3\mathbf{k}$, then

$$\frac{d\mathbf{A}}{du} = \ldots\ldots\ldots\ldots$$

31

$$\boxed{\frac{d\mathbf{A}}{du} = 6u\mathbf{i} + 2\mathbf{j} + 12u^2\mathbf{k}}$$

If we differentiate this again, we get $\dfrac{d^2\mathbf{A}}{du^2} = 6\mathbf{i} + 24u\mathbf{k}$

When $u = 2$, $\dfrac{d\mathbf{A}}{du} = 12\mathbf{i} + 2\mathbf{j} + 48\mathbf{k}$ and $\dfrac{d^2\mathbf{A}}{du^2} = 6\mathbf{i} + 48\mathbf{k}$

Then $\left|\dfrac{d\mathbf{A}}{du}\right| = \ldots\ldots\ldots\ldots$ and $\left|\dfrac{d^2\mathbf{A}}{du^2}\right| = \ldots\ldots\ldots\ldots$

32

$$\left|\frac{d\mathbf{A}}{du}\right| = 49{\cdot}52; \quad \left|\frac{d^2\mathbf{A}}{du^2}\right| = 48{\cdot}37$$

Because

$$\left|\frac{d\mathbf{A}}{du}\right| = \{12^2 + 2^2 + 48^2\}^{1/2} = \{2452\}^{1/2} = 49{\cdot}52$$

and $$\left|\frac{d^2\mathbf{A}}{du^2}\right| = \{6^2 + 48^2\}^{1/2} = \{2340\}^{1/2} = 48{\cdot}37$$

Example 2

If $\mathbf{F} = \mathbf{i}\sin 2t + \mathbf{j}e^{3t} + \mathbf{k}(t^3 - 4t)$, then when $t = 1$

$$\frac{d\mathbf{F}}{dt} = \ldots\ldots\ldots\ldots; \quad \frac{d^2\mathbf{F}}{dt^2} = \ldots\ldots\ldots\ldots$$

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$$\frac{d\mathbf{F}}{dt} = 2\cos 2\mathbf{i} + 3e^3\mathbf{j} - \mathbf{k}$$
$$\frac{d^2\mathbf{F}}{dt^2} = -4\sin 2\mathbf{i} + 9e^3\mathbf{j} + 6\mathbf{k}$$

From these, we could if required find the magnitudes of $\dfrac{d\mathbf{F}}{dt}$ and $\dfrac{d^2\mathbf{F}}{dt^2}$.

$$\left|\frac{d\mathbf{F}}{dt}\right| = \ldots\ldots\ldots\ldots; \quad \left|\frac{d^2\mathbf{F}}{dt^2}\right| = \ldots\ldots\ldots\ldots$$

34

$$\left|\frac{d\mathbf{F}}{dt}\right| = 60{\cdot}27; \quad \left|\frac{d^2\mathbf{F}}{dt^2}\right| = 180{\cdot}9$$

Because

$$\left|\frac{d\mathbf{F}}{dt}\right| = \{(2\cos 2)^2 + 9e^6 + 1\}^{1/2}$$
$$= \{0{\cdot}6927 + 3631 + 1\}^{1/2} = 60{\cdot}27$$

and $$\left|\frac{d^2\mathbf{F}}{dt^2}\right| = \{(-4\sin 2)^2 + 81e^6 + 36\}^{1/2}$$
$$= \{13{\cdot}23 + 32\,678 + 36\}^{1/2} = 180{\cdot}9$$

One more example.

Example 3

If $\mathbf{A} = (u+3)\mathbf{i} - (2+u^2)\mathbf{j} + 2u^3\mathbf{k}$, determine

(a) $\dfrac{d\mathbf{A}}{du}$ (b) $\dfrac{d^2\mathbf{A}}{du^2}$ (c) $\left|\dfrac{d\mathbf{A}}{du}\right|$ (d) $\left|\dfrac{d^2\mathbf{A}}{du^2}\right|$ at $u = 3$.

Work through all sections and then check with the next frame

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Here is the working. $\mathbf{A} = (u+3)\mathbf{i} - (2+u^2)\mathbf{j} + 2u^3\mathbf{k}$

(a) $\dfrac{d\mathbf{A}}{du} = \mathbf{i} - 2u\mathbf{j} + 6u^2\mathbf{k}$ At $u = 3$, $\dfrac{d\mathbf{A}}{du} = \mathbf{i} - 6\mathbf{j} + 54\mathbf{k}$

(b) $\dfrac{d^2\mathbf{A}}{du^2} = -2\mathbf{j} + 12u\mathbf{k}$ At $u = 3$, $\dfrac{d^2\mathbf{A}}{du^2} = -2\mathbf{j} + 36\mathbf{k}$

(c) $\left|\dfrac{d\mathbf{A}}{du}\right| = \{1 + 36 + 2916\}^{1/2} = (2953)^{1/2} = 54{\cdot}34$

(d) $\left|\dfrac{d^2\mathbf{A}}{du^2}\right| = \{4 + 1296\}^{1/2} = (1300)^{1/2} = 36{\cdot}06$

The next example is of a rather different kind, so move on

36

Example 4

A particle moves in space so that at time t its position is stated as $x = 2t + 3$, $y = t^2 + 3t$, $z = t^3 + 2t^2$. We are required to find the components of its velocity and acceleration in the direction of the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ when $t = 1$.

First we can write the position as a vector $\mathbf{r}$

$$\mathbf{r} = (2t + 3)\mathbf{i} + (t^2 + 3t)\mathbf{j} + (t^3 + 2t^2)\mathbf{k}$$

Then, at $t = 1$

$$\frac{d\mathbf{r}}{dt} = \ldots\ldots\ldots\ldots; \qquad \frac{d^2\mathbf{r}}{dt^2} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}; \qquad \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j} + 10\mathbf{k}}$$

Because

$$\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + (2t + 3)\mathbf{j} + (3t^2 + 4t)\mathbf{k}$$

$$\therefore \text{ At } t = 1, \quad \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and $$\frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j} + (6t + 4)\mathbf{k}$$

$$\therefore \text{ At } t = 1, \quad \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j} + 10\mathbf{k}$$

Now, a unit vector parallel to $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ is …………

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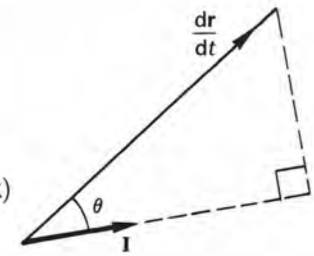
$$\boxed{\frac{2\mathbf{i}+3\mathbf{j}+4\mathbf{k}}{\sqrt{4+9+16}}=\frac{1}{\sqrt{29}}(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})}$$

Denote this unit vector by **I**. Then the component of $\dfrac{d\mathbf{r}}{dt}$ in the direction of **I**

$$\begin{aligned}&=\frac{d\mathbf{r}}{dt}\cos\theta\\&=\frac{d\mathbf{r}}{dt}\cdot\mathbf{I}\\&=\frac{1}{\sqrt{29}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\\&=\ldots\ldots\ldots\ldots\end{aligned}$$



39

$$\boxed{8{\cdot}73}$$

Because

$$\begin{aligned}\frac{1}{\sqrt{29}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})&=\frac{1}{\sqrt{29}}(4+15+28)\\&=\frac{47}{\sqrt{29}}\\&=8{\cdot}73\end{aligned}$$

Similarly, the component of $\dfrac{d^2\mathbf{r}}{dt^2}$ in the direction of **I** is

....

40

$$\boxed{8{\cdot}54}$$

Because

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2}\cos\theta&=\frac{d^2\mathbf{r}}{dt^2}\cdot\mathbf{I}\\&=\frac{1}{\sqrt{29}}(2\mathbf{j}+10\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\\&=\frac{1}{\sqrt{29}}(6+40)\\&=\frac{46}{\sqrt{29}}\\&=8{\cdot}54\end{aligned}$$

▶

Differentiation of sums and products of vectors

If $\mathbf{A} = \mathbf{A}(u)$ and $\mathbf{B} = \mathbf{B}(u)$, then

(a) $\dfrac{d}{du}\{c\mathbf{A}\} = c\dfrac{d\mathbf{A}}{du}$

(b) $\dfrac{d}{du}\{\mathbf{A}+\mathbf{B}\} = \dfrac{d\mathbf{A}}{du} + \dfrac{d\mathbf{B}}{du}$

(c) $\dfrac{d}{du}\{\mathbf{A}\cdot\mathbf{B}\} = \mathbf{A}\cdot\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}$

(d) $\dfrac{d}{du}\{\mathbf{A}\times\mathbf{B}\} = \mathbf{A}\times\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\times\mathbf{B}$.

These are very much like the normal rules of differentiation.

However, if $\mathbf{A}(u)\cdot\mathbf{A}(u) = a_x^2 + a_y^2 + a_z^2 = |\mathbf{A}|^2 = A^2$ is a constant then

$$\begin{aligned}\frac{d}{du}\{\mathbf{A}(u)\cdot\mathbf{A}(u)\} &= \mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\} + \mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\}\\ &= 2\mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\} = \frac{d}{du}\{\mathbf{A}^2\} = 0\end{aligned}$$

Assuming that $\mathbf{A}(u) \neq 0$, then since $\mathbf{A}(u)\cdot\dfrac{d}{du}\{\mathbf{A}(u)\} = 0$ it follows that $\mathbf{A}(u)$ and $\dfrac{d}{du}\{\mathbf{A}(u)\}$ are perpendicular vectors because

. . . .

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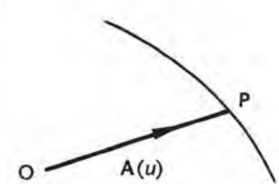
$$\boxed{\begin{array}{l}\mathbf{A}(u)\cdot\dfrac{d}{du}\{\mathbf{A}(u)\} = |\mathbf{A}(u)|\left|\dfrac{d}{du}\{\mathbf{A}(u)\}\right|\cos\theta = 0\\ \therefore\ \cos\theta = 0 \qquad \therefore\ \theta = \dfrac{\pi}{2}\end{array}}$$

Now let us deal with unit tangent vectors.

Unit tangent vectors

We have already established in Frame 30 of this Programme that if $\overline{OP}$ is a position vector $\mathbf{A}(u)$ in space, then the direction of the vector denoting $\dfrac{d}{du}\{\mathbf{A}(u)\}$ is



. . . .

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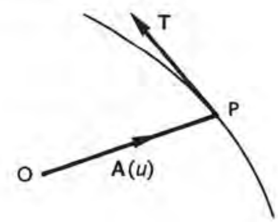
parallel to the tangent to the curve at P

Then the unit tangent vector $\mathbf{T}$ at P can be found from

$$\mathbf{T}=\frac{\dfrac{\mathrm{d}}{\mathrm{d}u}\{\mathbf{A}(u)\}}{\left|\dfrac{\mathrm{d}}{\mathrm{d}u}\{\mathbf{A}(u)\}\right|}$$



In simpler notation, this becomes:

If $\mathbf{r}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$ then the unit tangent vector $\mathbf{T}$ is given by

$$\mathbf{T}=\frac{\mathrm{d}\mathbf{r}/\mathrm{d}u}{|\mathrm{d}\mathbf{r}/\mathrm{d}u|}$$

Example 1

Determine the unit tangent vector at the point (2, 4, 7) for the curve with parametric equations $x=2u$; $y=u^2+3$; $z=2u^2+5$.

First we see that the point (2, 4, 7) corresponds to $u=1$.

The vector equation of the curve is

$$\mathbf{r}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}=2u\mathbf{i}+(u^2+3)\mathbf{j}+(2u^2+5)\mathbf{k}$$

$$\therefore\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}=\ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}=2\mathbf{i}+2u\mathbf{j}+4u\mathbf{k}}$$

and at $u=1$, $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}=2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$

Hence $\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right|=\ldots\ldots\ldots\ldots$ and $\mathbf{T}=\ldots\ldots\ldots\ldots$

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$$\boxed{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right|=2\sqrt{6};\quad \mathbf{T}=\frac{1}{\sqrt{6}}\{\mathbf{i}+\mathbf{j}+2\mathbf{k}\}}$$

Because

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right|=\{4+4+16\}^{1/2}=24^{1/2}=2\sqrt{6}$$

$$\mathbf{T}=\frac{\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}}{\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right|}=\frac{2\mathbf{i}+2\mathbf{j}+4\mathbf{k}}{2\sqrt{6}}=\frac{1}{\sqrt{6}}\{\mathbf{i}+\mathbf{j}+2\mathbf{k}\}$$

▶

Let us do another.

Example 2

Find the unit tangent vector at the point $(2, 0, \pi)$ for the curve with parametric equations $x = 2\sin\theta$; $y = 3\cos\theta$; $z = 2\theta$.

We see that the point $(2, 0, \pi)$ corresponds to $\theta = \pi/2$.

Writing the curve in vector form $\mathbf{r} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{r} = 2\sin\theta\,\mathbf{i} + 3\cos\theta\,\mathbf{j} + 2\theta\,\mathbf{k}}$$

$$\text{Then, at } \theta = \pi/2, \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta} = \ldots\ldots\ldots\ldots$$

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}\right| = \ldots\ldots\ldots\ldots$$

$$\mathbf{T} = \ldots\ldots\ldots\ldots$$

Finish it off

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$$\boxed{\begin{array}{l} \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta} = -3\mathbf{j} + 2\mathbf{k}; \quad \left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}\theta}\right| = \sqrt{13} \\ \mathbf{T} = \dfrac{1}{\sqrt{13}}(-3\mathbf{j} + 2\mathbf{k}) \end{array}}$$

And now

Example 3

Determine the unit tangent vector for the curve

$x = 3t; \quad y = 2t^2; \quad z = t^2 + t$

at the point $(6, 8, 6)$.

On your own. $\mathbf{T} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{T} = \frac{1}{\sqrt{98}}(3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k})}$$

The point $(6, 8, 6)$ corresponds to $t = 2$

$$\mathbf{r} = 3t\mathbf{i} + 2t^2\mathbf{j} + (t^2 + t)\mathbf{k}$$

$$\therefore \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 3\mathbf{i} + 4t\mathbf{j} + (2t + 1)\mathbf{k}$$

At $t = 2$, $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ and $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$

$$\therefore \quad \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right| = (9 + 64 + 25)^{1/2} = \sqrt{98}$$

$$\therefore \quad \mathbf{T} = \frac{\mathrm{d}\mathbf{r}/\mathrm{d}t}{|\mathrm{d}\mathbf{r}/\mathrm{d}t|} = \frac{1}{\sqrt{98}}(3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k})$$

Partial differentiation of vectors

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If a vector $\mathbf{F}$ is a function of two independent variables u and v, then the rules of differentiation follow the usual pattern.
If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then x, y, z will also be functions of u and v.

Then
$$\frac{\partial \mathbf{F}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
$$\frac{\partial \mathbf{F}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$
$$\frac{\partial^2 \mathbf{F}}{\partial u^2} = \frac{\partial^2 x}{\partial u^2}\mathbf{i} + \frac{\partial^2 y}{\partial u^2}\mathbf{j} + \frac{\partial^2 z}{\partial u^2}\mathbf{k}$$
$$\frac{\partial^2 \mathbf{F}}{\partial v^2} = \frac{\partial^2 x}{\partial v^2}\mathbf{i} + \frac{\partial^2 y}{\partial v^2}\mathbf{j} + \frac{\partial^2 z}{\partial v^2}\mathbf{k}$$
$$\frac{\partial^2 \mathbf{F}}{\partial u \partial v} = \frac{\partial^2 x}{\partial u \partial v}\mathbf{i} + \frac{\partial^2 y}{\partial u \partial v}\mathbf{j} + \frac{\partial^2 z}{\partial u \partial v}\mathbf{k}$$

and for small finite changes $\mathrm{d}u$ and $\mathrm{d}v$ in u and v, we have

$$\mathrm{d}\mathbf{F} = \frac{\partial \mathbf{F}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{F}}{\partial v}\mathrm{d}v$$

Example

If $\mathbf{F} = 2uv\mathbf{i} + (u^2 - 2v)\mathbf{j} + (u + v^2)\mathbf{k}$

$$\frac{\partial \mathbf{F}}{\partial u} = \dots\dots\dots\dots; \qquad \frac{\partial \mathbf{F}}{\partial v} = \dots\dots\dots\dots$$
$$\frac{\partial^2 \mathbf{F}}{\partial u^2} = \dots\dots\dots\dots; \qquad \frac{\partial^2 \mathbf{F}}{\partial u \partial v} = \dots\dots\dots\dots$$

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$$\boxed{\begin{array}{ll} \dfrac{\partial \mathbf{F}}{\partial u} = 2v\mathbf{i} + 2u\mathbf{j} + \mathbf{k}; & \dfrac{\partial \mathbf{F}}{\partial v} = 2u\mathbf{i} - 2\mathbf{j} + 2v\mathbf{k} \\ \dfrac{\partial^2 \mathbf{F}}{\partial u^2} = 2\mathbf{j}; & \dfrac{\partial^2 \mathbf{F}}{\partial u \partial v} = 2\mathbf{i} \end{array}}$$

This is straightforward enough.

Integration of vector functions

The process is the reverse of that for differentiation. If a vector $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $\mathbf{F}$, x, y, z are expressed as functions of u, then

$$\int_a^b \mathbf{F}\,\mathrm{d}u = \mathbf{i}\int_a^b x\,\mathrm{d}u + \mathbf{j}\int_a^b y\,\mathrm{d}u + \mathbf{k}\int_a^b z\,\mathrm{d}u.$$

Example 1

If $\mathbf{F} = (3t^2 + 4t)\mathbf{i} + (2t - 5)\mathbf{j} + 4t^3\mathbf{k}$, then

$$\int_1^3 \mathbf{F}\,\mathrm{d}t = \mathbf{i}\int_1^3 (3t^2 + 4t)\,\mathrm{d}t + \mathbf{j}\int_1^3 (2t - 5)\,\mathrm{d}t + \mathbf{k}\int_1^3 4t^3\,\mathrm{d}t = \dots\dots\dots\dots$$

50

$$\boxed{42\mathbf{i} - 2\mathbf{j} + 80\mathbf{k}}$$

Because

$$\int_1^3 \mathbf{F}\,dt = \left[\mathbf{i}(t^3 + 2t^2) + \mathbf{j}(t^2 - 5t) + \mathbf{k}t^4\right]_1^3$$
$$= (45\mathbf{i} - 6\mathbf{j} + 81\mathbf{k}) - (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 42\mathbf{i} - 2\mathbf{j} + 80\mathbf{k}$$

Here is a slightly different one.

Example 2

If $\quad \mathbf{F} = 3u\mathbf{i} + u^2\mathbf{j} + (u+2)\mathbf{k}$

and $\quad \mathbf{V} = 2u\mathbf{i} - 3u\mathbf{j} + (u-2)\mathbf{k}$

evaluate $\displaystyle\int_0^2 (\mathbf{F} \times \mathbf{V})\,du.$

First we must determine $\mathbf{F} \times \mathbf{V}$ in terms of u.

$$\mathbf{F} \times \mathbf{V} = \dots\dots\dots\dots$$

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$$\boxed{\mathbf{F} \times \mathbf{V} = (u^3 + u^2 + 6u)\mathbf{i} - (u^2 - 10u)\mathbf{j} - (2u^3 + 9u^2)\mathbf{k}}$$

Because

$$\mathbf{F} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3u & u^2 & (u+2) \\ 2u & -3u & (u-2) \end{vmatrix}$$

which gives the result above.

Then $\displaystyle\int_0^2 (\mathbf{F} \times \mathbf{V})\,du = \dots\dots\dots\dots$

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$$\boxed{\tfrac{4}{3}\{14\mathbf{i} + 13\mathbf{j} - 24\mathbf{k}\}}$$

Because

$$\int (\mathbf{F} \times \mathbf{V})\,du = \left(\frac{u^4}{4} + \frac{u^3}{3} + 3u^2\right)\mathbf{i} - \left(\frac{u^3}{3} - 5u^2\right)\mathbf{j} - \left(\frac{u^4}{2} + 3u^3\right)\mathbf{k}$$

$$\therefore \int_0^2 (\mathbf{F} \times \mathbf{V})\,du = (4 + \tfrac{8}{3} + 12)\mathbf{i} - (\tfrac{8}{3} - 20)\mathbf{j} - (8 + 24)\mathbf{k}$$
$$= \tfrac{4}{3}\{14\mathbf{i} + 13\mathbf{j} - 24\mathbf{k}\}$$

▶

Example 3

If $\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ where

$$\mathbf{A} = 3t^2\mathbf{i} + (2t-3)\mathbf{j} + 4t\mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} + 4t\mathbf{j} + 3(1-t)\mathbf{k}$$
$$\mathbf{C} = 2t\mathbf{i} - 3t^2\mathbf{j} - 2t\mathbf{k}$$

determine $\int_0^1 \mathbf{F}\mathrm{d}t$.

First we need to find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The simplest way to do this is to use the relationship

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}}$$

So $\quad \mathbf{A} \cdot \mathbf{C} = \ldots\ldots\ldots\ldots$
and $\quad \mathbf{A} \cdot \mathbf{B} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned} &\mathbf{A} \cdot \mathbf{C} = 6t^3 - 6t^3 + 9t^2 - 8t^2 = t^2 \\ &\mathbf{A} \cdot \mathbf{B} = 6t^2 + 8t^2 - 12t + 12t - 12t^2 = 2t^2 \end{aligned}}$$

Then $\quad \mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
$$= t^2\{2\mathbf{i} + 4t\mathbf{j} + 3(1-t)\mathbf{k}\} - 2t^2\{2t\mathbf{i} - 3t^2\mathbf{j} - 2t\mathbf{k}\}$$
$$\therefore \quad \int_0^1 \mathbf{F}\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

Finish off the simplification and complete the integration.

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$$\boxed{\tfrac{1}{60}\{-20\mathbf{i} + 132\mathbf{j} + 75\mathbf{k}\}}$$

Because

$$\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (2t^2 - 4t^3)\mathbf{i} + (4t^3 + 6t^4)\mathbf{j} + (3t^2 + t^3)\mathbf{k}$$

Integration with respect to t then gives the result stated above.

Now let us move on to the next stage of our development

Scalar and vector fields

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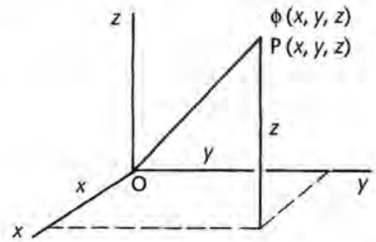
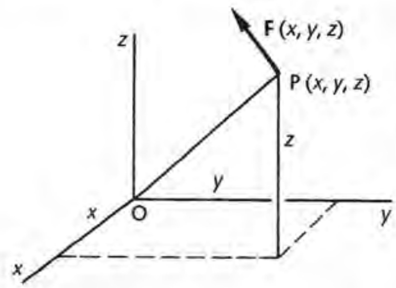


If every point P (x, y, z) of a region R of space has associated with it a scalar quantity $\phi(x, y, z)$, then $\phi(x, y, z)$ is a *scalar function* and a *scalar field* is said to exist in the region R.

Examples of scalar fields are temperature, potential, etc.



Similarly, if every point P (x, y, z) of a region R has associated with it a vector quantity $\mathbf{F}(x, y, z)$, then $\mathbf{F}(x, y, z)$ is a *vector function* and a *vector field* is said to exist in the region R.

Examples of vector fields are force, velocity, acceleration, etc. $\mathbf{F}(x, y, z)$ can be defined in terms of its components parallel to the coordinate axes, OX, OY, OZ.

That is, $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

Note these important definitions:
we shall be making good use of them as we proceed

Grad (gradient of a scalar function)

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If a scalar function $\phi(x, y, z)$ is continuously differentiable with respect to its variables x, y, z, throughout the region, then the *gradient* of ϕ, written *grad* ϕ, is defined as the vector

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \qquad (12)$$

Note that, while ϕ is a scalar function, grad ϕ is a vector function. For example, if ϕ depends upon the position of P and is defined by $\phi = 2x^2yz^3$, then

$$\text{grad}\,\phi = 4xyz^3\mathbf{i} + 2x^2z^3\mathbf{j} + 6x^2yz^2\mathbf{k}$$

▶

Notation

The expression (12) above can be written

$$\text{grad}\,\phi = \left\{\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right\}\phi$$

where $\left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)$ is called a *vector differential operator* and is denoted by the symbol ∇ (pronounced 'del' or sometimes 'nabla')

$$\text{i.e.}\qquad \nabla \equiv \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

Beware! ∇ cannot exist alone: it is an operator and must operate on a stated scalar function $\phi(x, y, z)$.

If $\mathbf{F}$ is a vector function, $\nabla\mathbf{F}$ has no meaning.

So we have:

$$\begin{aligned}\nabla\phi = \text{grad}\,\phi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi \\ &= \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\end{aligned} \qquad (13)$$

Make a note of this definition and then let us see how to use it

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Example 1

If $\phi = x^2yz^3 + xy^2z^2$, determine grad ϕ at the point P (1, 3, 2).

By the definition, $\quad \text{grad}\,\phi = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}.$

All we have to do then is to find the partial derivatives at $x = 1$, $y = 3$, $z = 2$ and insert their values.

$$\therefore\ \nabla\phi = \ldots\ldots\ldots\ldots$$

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$$\boxed{4(21\mathbf{i} + 8\mathbf{j} + 18\mathbf{k})}$$

Because

$$\phi = x^2yz^3 + xy^2z^2 \qquad \therefore\ \frac{\partial\phi}{\partial x} = 2xyz^3 + y^2z^2$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 + 2xyz^2 \qquad \frac{\partial\phi}{\partial z} = 3x^2yz^2 + 2xy^2z$$

Then, at (1, 3, 2)

$$\frac{\partial\phi}{\partial x} = 48 + 36 \qquad \therefore\ \frac{\partial\phi}{\partial x} = 84$$

$$\frac{\partial\phi}{\partial y} = 8 + 24 \qquad \therefore\ \frac{\partial\phi}{\partial y} = 32$$

$$\frac{\partial\phi}{\partial z} = 36 + 36 \qquad \therefore\ \frac{\partial\phi}{\partial z} = 72$$

$$\therefore\ \text{grad}\,\phi = \nabla\phi = 84\mathbf{i} + 32\mathbf{j} + 72\mathbf{k} = 4(21\mathbf{i} + 8\mathbf{j} + 18\mathbf{k})$$

Example 2

If $\mathbf{A} = x^2z\mathbf{i} + xy\mathbf{j} + y^2z\mathbf{k}$

and $\mathbf{B} = yz^2\mathbf{i} + xz\mathbf{j} + x^2z\mathbf{k}$

determine an expression for grad $(\mathbf{A} \cdot \mathbf{B})$.

This we can soon do since we know that $\mathbf{A} \cdot \mathbf{B}$ is a scalar function of x, y and z.

First then, $\mathbf{A} \cdot \mathbf{B} = \dots\dots\dots\dots$

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$$\boxed{\mathbf{A} \cdot \mathbf{B} = x^2yz^3 + x^2yz + x^2y^2z^2}$$

Then $\nabla(\mathbf{A} \cdot \mathbf{B}) = \dots\dots\dots\dots$

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$$\boxed{2xyz(z^2 + 1 + yz)\mathbf{i} + x^2z(z^2 + 1 + 2yz)\mathbf{j} + x^2y(3z^2 + 1 + 2yz)\mathbf{k}}$$

Because

$$\text{if } \phi = \mathbf{A} \cdot \mathbf{B} = (x^2z\mathbf{i} + xy\mathbf{j} + y^2z\mathbf{k}) \cdot (yz^2\mathbf{i} + xz\mathbf{j} + x^2z\mathbf{k})$$

$$= x^2yz^3 + x^2yz + x^2y^2z^2$$

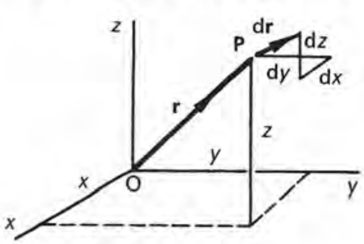
$$\frac{\partial\phi}{\partial x} = 2xyz^3 + 2xyz + 2xy^2z^2 = 2xyz(z^2 + 1 + yz)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 + x^2z + 2x^2yz^2 = x^2z(z^2 + 1 + 2yz)$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 + x^2y + 2x^2y^2z = x^2y(3z^2 + 1 + 2yz)$$

$$\therefore\ \nabla(\mathbf{A} \cdot \mathbf{B}) = 2xyz(z^2 + 1 + yz)\mathbf{i} + x^2z(z^2 + 1 + 2yz)\mathbf{j} + x^2y(3z^2 + 1 + 2yz)\mathbf{k}$$

Now let us obtain another useful relationship.



If $\overline{\text{OP}}$ is a position vector $\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $d\mathbf{r}$ is a small displacement corresponding to changes dx, dy, dz in x, y, z respectively, then

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

If $\phi(x, y, z)$ is a scalar function at P, we know that

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Then grad $\phi \cdot d\mathbf{r} = \dots\dots\dots\dots$

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$$\boxed{\operatorname{grad}\phi\cdot d\mathbf{r}=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz}$$

Because

$$\begin{aligned}\operatorname{grad}\phi\cdot d\mathbf{r}&=\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})\\&=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\\&=\text{the total differential } d\phi \text{ of } \phi\end{aligned}$$

That is

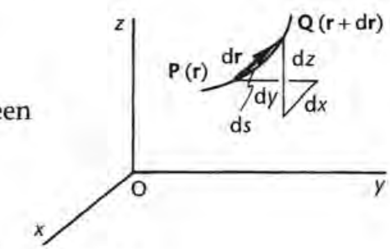
$$d\phi=d\mathbf{r}\cdot\operatorname{grad}\phi \qquad (14)$$

This will certainly be useful, so make a note of it

63 Directional derivatives

We have just established that

$$d\phi=d\mathbf{r}\cdot\operatorname{grad}\phi$$



If ds is the small element of arc between P ($\mathbf{r}$) and Q ($\mathbf{r}+d\mathbf{r}$) then $ds=|d\mathbf{r}|$

$$\frac{d\mathbf{r}}{ds}=\frac{d\mathbf{r}}{|d\mathbf{r}|}$$

and $\dfrac{d\mathbf{r}}{ds}$ is thus a unit vector in the direction of $d\mathbf{r}$.

$$\therefore\ \frac{d\phi}{ds}=\frac{d\mathbf{r}}{ds}\cdot\operatorname{grad}\phi$$

If we denote the unit vector $\dfrac{d\mathbf{r}}{ds}$ by $\hat{\mathbf{a}}$ then the result becomes

$$\frac{d\phi}{ds}=\hat{\mathbf{a}}\cdot\operatorname{grad}\phi$$

$\dfrac{d\phi}{ds}$ is thus the projection of grad ϕ on the unit vector $\hat{\mathbf{a}}$ and is called the *directional derivative* of ϕ in the direction of $\hat{\mathbf{a}}$. It gives the rate of change of ϕ with distance measured in the direction of $\hat{\mathbf{a}}$ and $\dfrac{d\phi}{ds}=\hat{\mathbf{a}}\cdot\operatorname{grad}\phi$ will be a maximum when $\hat{\mathbf{a}}$ and grad ϕ have the same direction, since then

$\hat{\mathbf{a}}\cdot\operatorname{grad}\phi=|\hat{\mathbf{a}}||\operatorname{grad}\phi|\cos\theta$ and θ will be zero.

Thus the direction of grad ϕ gives the direction in which the maximum rate of change of ϕ occurs.

▶

Example 1

Find the directional derivative of the function $\phi = x^2z + 2xy^2 + yz^2$ at the point (1, 2, –1) in the direction of the vector $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

We start off with $\phi = x^2z + 2xy^2 + yz^2$

$$\therefore \quad \nabla\phi = \ldots\ldots\ldots\ldots$$

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$$\boxed{\nabla\phi = (2xz + 2y^2)\mathbf{i} + (4xy + z^2)\mathbf{j} + (x^2 + 2yz)\mathbf{k}}$$

Because

$$\frac{\partial\phi}{\partial x} = 2xz + 2y^2; \qquad \frac{\partial\phi}{\partial y} = 4xy + z^2; \qquad \frac{\partial\phi}{\partial z} = x^2 + 2yz$$

Then, at (1, 2, $-$1)

$$\nabla\phi = (-2+8)\mathbf{i} + (8+1)\mathbf{j} + (1-4)\mathbf{k} = 6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$$

Next we have to find the unit vector $\hat{\mathbf{a}}$ where $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

$$\hat{\mathbf{a}} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\hat{\mathbf{a}} = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})}$$

Because

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \quad \therefore \quad |\mathbf{A}| = \sqrt{4+9+16} = \sqrt{29}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

So we have $\nabla\phi = 6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$ and $\hat{\mathbf{a}} = \dfrac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

$$\therefore \quad \frac{\mathrm{d}\phi}{\mathrm{d}s} = \hat{\mathbf{a}} \cdot \nabla\phi$$
$$= \ldots\ldots\ldots\ldots$$

66

$$\boxed{\frac{d\phi}{ds} = \frac{51}{\sqrt{29}} = 9{\cdot}47}$$

Because

$$\frac{d\phi}{ds} = \hat{\mathbf{a}} \cdot \nabla\phi = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot (6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$$

$$= \frac{1}{\sqrt{29}}(12 + 27 + 12) = \frac{51}{\sqrt{29}} = 9{\cdot}47$$

That is all there is to it.

(a) From the given scalar function ϕ, determine $\nabla\phi$.

(b) Find the unit vector $\hat{\mathbf{a}}$ in the direction of the given vector $\mathbf{A}$.

(c) Then $\dfrac{d\phi}{ds} = \hat{\mathbf{a}} \cdot \nabla\phi$.

Example 2

Find the directional derivative of $\phi = x^2y + y^2z + z^2x$ at the point $(1, -1, 2)$ in the direction of the vector $\mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

Same as before. *Work through it and check the result with the next frame*

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$$\boxed{\frac{d\phi}{ds} = \frac{-23}{3\sqrt{5}} = -3{\cdot}43}$$

Because

$$\phi = x^2y + y^2z + z^2x$$

$$\therefore\ \nabla\phi = (2xy + z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}$$

$$\therefore\ \text{At } (1, -1, 2),\quad \nabla\phi = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$\mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \quad \therefore\ |\mathbf{A}| = \sqrt{16 + 4 + 25} = \sqrt{45} = 3\sqrt{5}$$

$$\therefore\ \hat{\mathbf{a}} = \frac{1}{3\sqrt{5}}(4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$$

$$\therefore\ \frac{d\phi}{ds} = \hat{\mathbf{a}} \cdot \nabla\phi = \frac{1}{3\sqrt{5}}(4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$$

$$= \frac{1}{3\sqrt{5}}(8 - 6 - 25) = \frac{-23}{3\sqrt{5}} = -3{\cdot}43$$

Example 3

Find the direction from the point $(1, 1, 0)$ which gives the greatest rate of increase of the function $\phi = (x + 3y)^2 + (2y - z)^2$.

This appears to be different, but it rests on the fact that the greatest rate of increase of ϕ with respect to distance is in

....

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$$\boxed{\text{the direction of } \nabla\phi}$$

All we need then is to find the vector $\nabla\phi$, which is

....

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$$\boxed{\nabla\phi = 4(2\mathbf{i} + 8\mathbf{j} - \mathbf{k})}$$

Because

$$\phi = (x + 3y)^2 + (2y - z)^2$$

$$\therefore \ \frac{\partial\phi}{\partial x} = 2(x + 3y); \quad \frac{\partial\phi}{\partial y} = 6(x + 3y) + 4(2y - z); \quad \frac{\partial\phi}{\partial z} = -2(2y - z)$$

$$\therefore \ \text{At } (1, 1, 0), \quad \frac{\partial\phi}{\partial x} = 8; \quad \frac{\partial\phi}{\partial y} = 32; \quad \frac{\partial\phi}{\partial z} = -4$$

$$\therefore \ \nabla\phi = 8\mathbf{i} + 32\mathbf{j} - 4\mathbf{k} = 4(2\mathbf{i} + 8\mathbf{j} - \mathbf{k})$$

$\therefore$, greatest rate of increase occurs in direction $2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

So on we go

Unit normal vectors

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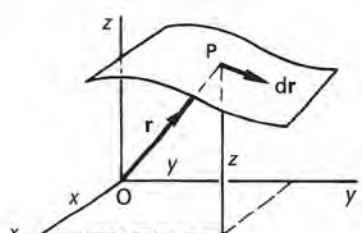
The equation of $\phi(x, y, z) = \text{constant}$ represents a surface in space. For example, $3x - 4y + 2z = 1$ is the equation of a plane and $x^2 + y^2 + z^2 = 4$ represents a sphere centred on the origin and of radius 2.



If $\mathbf{dr}$ is a displacement in this surface, then $d\phi = 0$ since ϕ is constant over the surface.

Therefore our previous relationship $\mathbf{dr} \cdot \text{grad } \phi = d\phi$ becomes

$$\mathbf{dr} \cdot \text{grad } \phi = 0$$

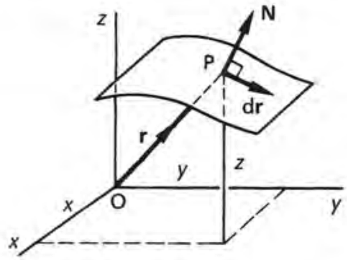
for all such small displacements $\mathbf{dr}$ in the surface.

But $\mathbf{dr} \cdot \text{grad } \phi = |\mathbf{dr}||\text{grad } \phi| \cos\theta = 0$.

$\therefore \ \theta = \frac{\pi}{2} \quad \therefore$ grad ϕ is perpendicular to $\mathbf{dr}$, i.e. grad ϕ is a vector perpendicular to the surface at P, in the direction of maximum rate of change of ϕ. The magnitude of that maximum rate of change is given by $|\text{grad } \phi|$.

▶

The unit vector $\mathbf{N}$ in the direction of grad ϕ is called the *unit normal vector* at P.



$\therefore$ Unit normal vector

$$\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|} \qquad (15)$$

Example 1

Find the unit normal vector to the surface $x^3y + 4xz^2 + xy^2z + 2 = 0$ at the point $(1, 3, -1)$.

Vector normal $= \nabla\phi = \ldots\ldots\ldots\ldots$

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$$\boxed{\nabla\phi = (3x^2y + 4z^2 + y^2z)\mathbf{i} + (x^3 + 2xyz)\mathbf{j} + (8xz + xy^2)\mathbf{k}}$$

Then, at (1, 3, −1), $\quad \nabla\phi = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

and the unit normal at (1, 3, −1) is $\ldots\ldots\ldots\ldots$

72

$$\boxed{\frac{1}{\sqrt{42}}(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})}$$

Because

$$|\nabla\phi| = \sqrt{16 + 25 + 1} = \sqrt{42}$$

and $\quad \mathbf{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{1}{\sqrt{42}}(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})$

One more.

Example 2

Determine the unit normal to the surface

$xyz + x^2y - 5yz - 5 = 0$ at the point (3, 1, 2).

All very straightforward. Complete it.

73

$$\boxed{\text{Unit normal} = \mathbf{N} = \frac{1}{\sqrt{93}}(8\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})}$$

Because

$$\phi = xyz + x^2y - 5yz - 5$$

$$\therefore \ \nabla\phi = (yz + 2xy)\mathbf{i} + (xz + x^2 - 5z)\mathbf{j} + (xy - 5y)\mathbf{k}$$

At (3, 1, 2), $\quad \nabla\phi = 8\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}; \ |\nabla\phi| = \sqrt{64 + 25 + 4} = \sqrt{93}$

$$\therefore \ \text{Unit normal} = \mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{93}}(8\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$$

Collecting our results so far, we have, for $\phi(x, y, z)$ a scalar function

(a) $\operatorname{grad}\phi = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$

(b) $\mathrm{d}\phi = \mathrm{d}\mathbf{r}\cdot\operatorname{grad}\phi$ where $\mathrm{d}\phi = \dfrac{\partial\phi}{\partial x}\mathrm{d}x + \dfrac{\partial\phi}{\partial y}\mathrm{d}y + \dfrac{\partial\phi}{\partial z}\mathrm{d}z$

(c) directional derivative $\dfrac{\mathrm{d}\phi}{\mathrm{d}s} = \hat{\mathbf{a}}\cdot\operatorname{grad}\phi$

(d) unit normal vector $\mathbf{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$.

Copy out this brief summary for future reference. It will help

Grad of sums and products of scalars

74

(a) $$\nabla(A + B) = \mathbf{i}\left\{\frac{\partial}{\partial x}(A + B)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial y}(A + B)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial z}(A + B)\right\}$$

$$= \left\{\frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j} + \frac{\partial A}{\partial z}\mathbf{k}\right\} + \left\{\frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k}\right\}$$

$$\therefore \ \nabla(A + B) = \nabla A + \nabla B$$

(b) $$\nabla(AB) = \mathbf{i}\left\{\frac{\partial}{\partial x}(AB)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial y}(AB)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial z}(AB)\right\}$$

$$= \mathbf{i}\left\{A\frac{\partial B}{\partial x} + B\frac{\partial A}{\partial x}\right\} + \mathbf{j}\left\{A\frac{\partial B}{\partial y} + B\frac{\partial A}{\partial y}\right\} + \mathbf{k}\left\{A\frac{\partial B}{\partial z} + B\frac{\partial A}{\partial z}\right\}$$

$$= \left\{A\frac{\partial B}{\partial x}\mathbf{i} + A\frac{\partial B}{\partial y}\mathbf{j} + A\frac{\partial B}{\partial z}\mathbf{k}\right\} + \left\{B\frac{\partial A}{\partial x}\mathbf{i} + B\frac{\partial A}{\partial y}\mathbf{j} + B\frac{\partial A}{\partial z}\mathbf{k}\right\}$$

$$= A\left\{\frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k}\right\} + B\left\{\frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j} + \frac{\partial A}{\partial z}\mathbf{k}\right\}$$

$$\therefore \ \nabla(AB) = A(\nabla B) + B(\nabla A)$$

Remember that in these results A and B are scalars. The operator ∇ acting on a vector ....

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$$\boxed{\text{has no meaning}}$$

Example

If $A = x^2yz + xz^2$ and $B = xy^2z - z^3$, evaluate $\nabla(AB)$ at the point (2, 1, 3).

We know that $\nabla(AB) = A(\nabla B) + B(\nabla A)$

At (2, 1, 3),

$$\nabla B = \dots\dots\dots\dots; \quad \nabla A = \dots\dots\dots\dots$$

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$$\boxed{\nabla B = 3\mathbf{i} + 12\mathbf{j} - 25\mathbf{k}; \quad \nabla A = 21\mathbf{i} + 12\mathbf{j} + 16\mathbf{k}}$$

$$\nabla B = \frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k} = y^2z\mathbf{i} + 2xyz\mathbf{j} + (xy^2 - 3z^2)\mathbf{k}$$

$$= 3\mathbf{i} + 12\mathbf{j} - 25\mathbf{k} \quad \text{at } (2, 1, 3)$$

$$\nabla A = \frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j} + \frac{\partial A}{\partial z}\mathbf{k} = (2xyz + z^2)\mathbf{i} + x^2z\mathbf{j} + (x^2y + 2xz)\mathbf{k}$$

$$= 21\mathbf{i} + 12\mathbf{j} + 16\mathbf{k} \quad \text{at } (2, 1, 3)$$

Now $\nabla(AB) = A(\nabla B) + B(\nabla A) = \dots\dots\dots\dots$

Finish it

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$$\boxed{\nabla(AB) = 3(-117\mathbf{i} + 36\mathbf{j} - 362\mathbf{k})}$$

Because

$$\nabla(AB) = A(\nabla B) + B(\nabla A)$$

$$A = x^2yz + xz^2 \qquad \therefore \text{ at } (2, 1, 3), \qquad A = 12 + 18 = 30$$

$$B = xy^2z - z^3 \qquad \therefore \text{ at } (2, 1, 3), \qquad B = 6 - 27 = -21$$

$$\therefore \nabla(AB) = 30(3\mathbf{i} + 12\mathbf{j} - 25\mathbf{k}) - 21(21\mathbf{i} + 12\mathbf{j} + 16\mathbf{k})$$

$$= -351\mathbf{i} + 108\mathbf{j} - 1086\mathbf{k}$$

$$= 3(-117\mathbf{i} + 36\mathbf{j} - 362\mathbf{k})$$

So add these to the list of results.

$$\nabla(A + B) = \nabla A + \nabla B$$

$$\nabla(AB) = A(\nabla B) + B(\nabla A)$$

where A and B are scalars.

Now on to the next page

Div (divergence of a vector function)

78

The operator $\nabla \cdot$ (notice the 'dot'; it makes all the difference) can be applied to a vector function $\mathbf{A}(x, y, z)$ to give the *divergence* of $\mathbf{A}$, written in short as *div* $\mathbf{A}$.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\right)$$

$$\therefore \text{ div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

Note that

(a) the grad operator ∇ acts on a scalar and gives a vector

(b) the div operation $\nabla \cdot$ acts on a vector and gives a scalar.

Example 1

If $\mathbf{A} = x^2y\mathbf{i} - xyz\mathbf{j} + yz^2\mathbf{k}$ then

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = 2xy - xz + 2yz}$$

We simply take the appropriate partial derivatives of the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. It could hardly be easier.

Example 2

If $\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$, determine $\nabla \cdot \mathbf{A}$, i.e. div $\mathbf{A}$.

Complete it. $\quad \nabla \cdot \mathbf{A} = \ldots\ldots\ldots\ldots$

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$$\boxed{\nabla \cdot \mathbf{A} = 0}$$

Because

$$\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$$

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \\ &= 4xy - 2(2xy + 3y^2z) + 6y^2z \\ &= 4xy - 4xy - 6y^2z + 6y^2z = 0 \end{aligned}$$

Such a vector $\mathbf{A}$ for which $\nabla \cdot \mathbf{A} = 0$ at all points, i.e. for all values of x, y, z, is called a *solenoidal vector*. It is rather a special case.

▶

Curl (curl of a vector function)

The *curl operator* denoted by $\nabla\times$, acts on a vector and gives another vector as a result.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, then curl $\mathbf{A} = \nabla \times \mathbf{A}$.

$$\text{i.e.}\quad \text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$\therefore\ \nabla\times\mathbf{A} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

Curl $\mathbf{A}$ is thus a vector function. *It is best remembered in its determinant form, so make a note of it.*

If $\nabla\times\mathbf{A} = \mathbf{0}$ then $\mathbf{A}$ is said to be *irrotational*.

Then on for an example

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Example 1

If $\mathbf{A} = (y^4 - x^2z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} - x^2yz\mathbf{k}$, determine curl $\mathbf{A}$ at the point $(1, 3, -2)$.

$$\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^4 - x^2z^2 & x^2 + y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\nabla\times\mathbf{A} = \mathbf{i}\left\{\frac{\partial}{\partial y}(-x^2yz) - \frac{\partial}{\partial z}(x^2 + y^2)\right\} - \mathbf{j}\left\{\frac{\partial}{\partial x}(-x^2yz) - \frac{\partial}{\partial z}(y^4 - x^2z^2)\right\}$$
$$+\mathbf{k}\left\{\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2z^2)\right\}$$

All that now remains is to obtain the partial derivatives and substitute the values of x, y, z.

$$\therefore\ \nabla\times\mathbf{A} = \ldots\ldots\ldots\ldots$$

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$$\boxed{2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}}$$

$$\nabla\times\mathbf{A} = \mathbf{i}\{-x^2z\} - \mathbf{j}\{-2xyz + 2x^2z\} + \mathbf{k}\{2x - 4y^3\}.$$

$$\therefore\ \text{At } (1, 3, -2),\quad \nabla\times\mathbf{A} = \mathbf{i}(2) - \mathbf{j}(12 - 4) + \mathbf{k}(2 - 108)$$
$$= 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$$

▶

Example 2

Determine curl **F** at the point (2, 0, 3) given that

$\mathbf{F} = ze^{2xy}\mathbf{i} + 2xz\cos y\mathbf{j} + (x + 2y)\mathbf{k}$.

In determinant form, curl $\mathbf{F} = \nabla \times \mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ze^{2xy} & 2xz\cos y & x + 2y \end{vmatrix}}$$

Now expand the determinant and substitute the values of x, y and z, finally obtaining curl $\mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = -2(\mathbf{i} + 3\mathbf{k})}$$

Because

$\nabla \times \mathbf{F} = \mathbf{i}\{2 - 2x\cos y\} - \mathbf{j}\{1 - e^{2xy}\} + \mathbf{k}\{2z\cos y - 2xze^{2xy}\}$

$\therefore$ At (2, 0, 3) $\quad \nabla \times \mathbf{F} = \mathbf{i}(2 - 4) - \mathbf{j}(1 - 1) + \mathbf{k}(6 - 12)$

$= -2\mathbf{i} - 6\mathbf{k} = -2(\mathbf{i} + 3\mathbf{k})$

Every one is done in the same way.

Summary of grad, div and curl

(a) *Grad* operator ∇ acts on a *scalar* field to give a *vector* field.

(b) *Div* operator $\nabla\cdot$ acts on a *vector* field to give a *scalar* field.

(c) *Curl* operator $\nabla\times$ acts on a *vector* field to give a *vector* field.

(d) With a *scalar function* $\phi(x, y, z)$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a *vector function* $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

(1) $\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \dfrac{\partial a_x}{\partial x} + \dfrac{\partial a_y}{\partial y} + \dfrac{\partial a_z}{\partial z}$

(2) $\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$

Check through that list, just to make sure. We shall need them all

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By way of revision, here is one further example.

Example 3

If $\phi = x^2y^2 + x^3yz - yz^2$
and $\mathbf{F} = xy^2\mathbf{i} - 2yz\mathbf{j} + xyz\mathbf{k}$
determine for the point P (1, − 1, 2),
(a) $\nabla\phi$, (b) unit normal, (c) $\nabla \cdot \mathbf{F}$, (d) $\nabla \times \mathbf{F}$.

Complete all four parts and then check the results with the next frame

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Here is the working in full. $\phi = x^2y^2 + x^3yz - yz^2$

(a) $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$

$= (2xy^2 + 3x^2yz)\mathbf{i} + (2x^2y + x^3z - z^2)\mathbf{j} + (x^3y - 2yz)\mathbf{k}$

$\therefore$ At (1, −1, 2) $\quad \nabla\phi = -4\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{N} = \dfrac{\nabla\phi}{|\nabla\phi|} \qquad |\nabla\phi| = \sqrt{16+16+9} = \sqrt{41}$

$\therefore\ \mathbf{N} = \dfrac{-1}{\sqrt{41}}(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

(c) $\mathbf{F} = xy^2\mathbf{i} - 2yz\mathbf{j} + xyz\mathbf{k} \qquad \nabla \cdot \mathbf{F} = \dfrac{\partial a_x}{\partial x} + \dfrac{\partial a_y}{\partial y} + \dfrac{\partial a_z}{\partial z}$

$\therefore\ \nabla \cdot \mathbf{F} = y^2 - 2z + xy$

$\therefore$ At (1, − 1, 2) $\quad \nabla \cdot \mathbf{F} = 1 - 4 - 1 = -4 \qquad \therefore\ \nabla \cdot \mathbf{F} = -4$

(d) $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 & -2yz & xyz \end{vmatrix}$$

$\therefore\ \nabla \times \mathbf{F} = \mathbf{i}(xz + 2y) - \mathbf{j}(yz - 0) + \mathbf{k}(0 - 2xy)$

$= (xz + 2y)\mathbf{i} - yz\mathbf{j} - 2xy\mathbf{k}$

$\therefore$ At (1, − 1, 2) $\quad \nabla \times \mathbf{F} = 2\mathbf{j} + 2\mathbf{k} \qquad \therefore\ \nabla \times \mathbf{F} = 2(\mathbf{j} + \mathbf{k})$

Now let us combine some of these operations.

Multiple operations

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We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

Example 1

If $\mathbf{A} = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$

then $\quad \text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \left(\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k})$

$$= 2xy + z^3 + x^3 = \phi \quad \text{say}$$

Then $\quad \text{grad}\,(\text{div}\,\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$

$$= (2y + 3x^2)\mathbf{i} + (2x)\mathbf{j} + (3z^2)\mathbf{k}$$

i.e. $\quad \text{grad div}\,\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) = (2y + 3x^2)\mathbf{i} + 2x\mathbf{j} + 3z^2\mathbf{k}$

Move on for the next example

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Example 2

If $\phi = xyz - 2y^2z + x^2z^2$, determine div grad ϕ at the point (2, 4, 1).

First find grad ϕ and then the div of the result.

At (2, 4, 1), div grad $\phi = \nabla\cdot(\nabla\phi) = \dots\dots\dots$

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$$\boxed{\text{div grad}\,\phi = 6}$$

Because we have $\phi = xyz - 2y^2z + x^2z^2$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$= (yz + 2xz^2)\mathbf{i} + (xz - 4yz)\mathbf{j} + (xy - 2y^2 + 2x^2z)\mathbf{k}$$

$\therefore\ \text{div grad}\,\phi = \nabla\cdot(\nabla\phi) = 2z^2 - 4z + 2x^2$

$\therefore\ $ At (2, 4, 1), $\ \text{div grad}\,\phi = \nabla\cdot(\nabla\phi) = 2 - 4 + 8 = 6$

Example 3

If $\mathbf{F} = x^2yz\mathbf{i} + xyz^2\mathbf{j} + y^2z\mathbf{k}$ determine curl curl $\mathbf{F}$ at the point (2, 1, 1). Determine an expression for curl $\mathbf{F}$ in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

curl curl $\mathbf{F} = \dots\dots\dots$

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$$\boxed{\text{curl curl } \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}}$$

Because

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix}$$

$$= (2yz - 2xyz)\mathbf{i} + x^2y\mathbf{j} + (yz^2 - x^2z)\mathbf{k}$$

$$\text{Then} \quad \text{curl curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2yz - 2xyz & x^2y & yz^2 - x^2z \end{vmatrix}$$

$$= z^2\mathbf{i} - (-2xz - 2y + 2xy)\mathbf{j} + (2xy - 2z + 2xz)\,\mathbf{k}$$

$\therefore$ At (2, 1, 1), $\quad$ curl curl $\mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

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Remember that grad, div and curl are operators and that they must act on a scalar or vector as appropriate. They cannot exist alone and must be followed by a function.

One or two interesting general results appear.

(a) *Curl grad* ϕ where ϕ is a scalar

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\therefore \text{ curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right\} - \mathbf{j}\left\{\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right\} + \mathbf{k}\left\{\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right\}$$

$$= 0$$

$$\therefore \text{ curl grad } \phi = \nabla \times (\nabla \phi) = 0$$

▶

(b) *Div curl* $\mathbf{A}$ where $\mathbf{A}$ is a vector. $\quad \mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$\text{curl }\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\text{Then div curl }\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\nabla \times \mathbf{A})$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_z}{\partial x \partial y} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial y \partial z}$$

$$= 0$$

$$\therefore \text{ div curl }\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = \mathbf{0}$$

(c) *Div grad* ϕ where ϕ is a scalar

$$\text{grad }\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\text{Then div grad }\phi = \nabla \cdot (\nabla \phi)$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \text{ div grad }\phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \nabla^2 \phi, \text{ the Laplacian of } \phi$$

The operator ∇^2 is called the Laplacian.

So these general results are

(a) curl grad $\phi = \nabla \times (\nabla \phi) = 0$

(b) div curl $\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

(c) div grad $\phi = \nabla \cdot (\nabla \phi) = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$.

That brings us to the end of this particular Programme. We have covered quite a lot of new material, so check carefully through the **Revision summary** and **Can You?** checklist that follow: then you can deal with the **Test exercise**. The **Further problems** provide an opportunity for additional practice.

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Revision summary 17

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; $\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$; then we have the following relationships.

1 *Scalar product* (dot product) $\quad \mathbf{A}\cdot\mathbf{B} = AB\cos\theta$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \quad \text{and} \quad \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

If $\mathbf{A}\cdot\mathbf{B} = 0$ and $\mathbf{A}$, $\mathbf{B} \neq \mathbf{0}$ then $\mathbf{A} \perp \mathbf{B}$.

2 *Vector product* (cross product) $\quad \mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{n}$

$\mathbf{n}$ = unit normal vector where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{n}$ form a right-handed set.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ and $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

3 *Unit vectors*

(a) $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$
$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$

(b) $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$
$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}.$

4 *Scalar triple product* $\quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Unchanged by cyclic change of vectors.
Sign reversed by non-cyclic change of vectors.

5 *Coplanar vectors* $\quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = 0.$

6 *Vector triple product* $\quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

and $\quad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}.$

7 *Differentiation of vectors*

If $\mathbf{A}$, a_x, a_y, a_z are functions of u

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u} = \frac{\mathrm{d}a_x}{\mathrm{d}u}\mathbf{i} + \frac{\mathrm{d}a_y}{\mathrm{d}u}\mathbf{j} + \frac{\mathrm{d}a_z}{\mathrm{d}u}\mathbf{k}$$

8 *Unit tangent vector* $\mathbf{T}$

$$\mathbf{T} = \frac{\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}}{\left|\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\right|}$$

▶

9 *Integration of vectors*

$$\int_a^b \mathbf{A}\,\mathrm{d}u = \mathbf{i}\int_a^b a_x\,\mathrm{d}u + \mathbf{j}\int_a^b a_y\,\mathrm{d}u + \mathbf{k}\int_a^b a_z\,\mathrm{d}u$$

10 *Grad* (gradient of a scalar function ϕ)

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\text{'del'} = \text{operator } \nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

(a) *Directional derivative* $\dfrac{\mathrm{d}\phi}{\mathrm{d}s} = \hat{\mathbf{a}} \cdot \text{grad}\,\phi = \hat{\mathbf{a}} \cdot \nabla\phi$ where $\hat{\mathbf{a}}$ is a unit vector in a stated direction. Grad ϕ gives the direction for maximum rate of change of ϕ.

(b) *Unit normal vector* $\mathbf{N}$ to surface $\phi(x, y, z) = \text{constant}$.

$$\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|}$$

11 *Div* (divergence of a vector function $\mathbf{A}$)

$$\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

If $\nabla \cdot \mathbf{A} = 0$ for all points, $\mathbf{A}$ is a solenoidal vector.

12 *Curl* (curl of a vector function $\mathbf{A}$)

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

If $\nabla \times \mathbf{A} = 0$ then $\mathbf{A}$ is an irrotational vector.

13 *Operators*

grad (∇) acts on a *scalar* and gives a *vector*

div ($\nabla\cdot$) acts on a *vector* and gives a *scalar*

curl ($\nabla\times$) acts on a *vector* and gives a *vector*.

14 *Multiple operations*

(a) curl grad $\phi = \nabla \times (\nabla\phi) = 0$

(b) div curl $\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

(c) div grad $\phi = \nabla \cdot (\nabla\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

$= \nabla^2\phi$, the Laplacian of ϕ.

Can You?

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Checklist 17

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:	Frames
• Obtain the scalar and vector product of two vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 4
• Reproduce the relationships between the scalar and vector products of the Cartesian coordinate unit vectors? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	5 to 11
• Obtain the scalar and vector triple products and appreciate their geometric significance? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	12 to 27
• Differentiate a vector field and derive a unit vector tangential to the vector field at a point? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	28 to 48
• Integrate a vector field? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	49 to 55
• Obtain the gradient of a scalar field, the directional derivative and a unit normal to a surface? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	56 to 77
• Obtain the divergence of a vector field and recognise a solenoidal vector field? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	78 to 80
• Obtain the curl of a vector field? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	80 to 86
• Obtain combinations of div, grad and curl acting on scalar and vector fields as appropriate? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	87 to 91

Test exercise 17

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1 Find (a) the scalar product and (b) the vector product of the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$.

2 If $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$; determine

(a) the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

(b) the vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

3 Determine whether the three vectors $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$; $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ are coplanar.

4 If $\mathbf{A} = (u^2 + 5)\mathbf{i} - (u^2 + 3)\mathbf{j} + 2u^3\mathbf{k}$, determine

(a) $\dfrac{d\mathbf{A}}{du}$; (b) $\dfrac{d^2\mathbf{A}}{du^2}$; (c) $\left|\dfrac{d\mathbf{A}}{du}\right|$; all at $u = 2$.

5 Determine the unit tangent vector at the point (2, 4, 3) for the curve with parametric equations

$x = 2u^2$; $y = u + 3$; $z = 4u^2 - u$.

6 If $\mathbf{F} = 2\mathbf{i} + 4u\mathbf{j} + u^2\mathbf{k}$ and $\mathbf{G} = u^2\mathbf{i} - 2u\mathbf{j} + 4\mathbf{k}$, determine

$\displaystyle\int_0^2 (\mathbf{F} \times \mathbf{G})\,du$.

7 Find the directional derivative of the function $\phi = x^2y - 2xz^2 + y^2z$ at the point (1, 3, 2) in the direction of the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

8 Find the unit normal to the surface $\phi = 2x^3z + x^2y^2 + xyz - 4 = 0$ at the point (2, 1, 0).

9 If $\mathbf{A} = x^2y\mathbf{i} + (xy + yz)\mathbf{j} + xz^2\mathbf{k}$; $\mathbf{B} = yz\mathbf{i} - 3xz\mathbf{j} + 2xy\mathbf{k}$; and $\phi = 3x^2y + xyz - 4y^2z^2 - 3$; determine, at the point (1, 2, 1)
(a) $\nabla\phi$; (b) $\nabla \cdot \mathbf{A}$; (c) $\nabla \times \mathbf{B}$; (d) grad div $\mathbf{A}$; (e) curl curl $\mathbf{A}$.

Further problems 17

95

1 If $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$; determine $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

2 If $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

3 If $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$; find

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$; (b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

4 If $\mathbf{F} = x^2\mathbf{i} + (3x + 2)\mathbf{j} + \sin x\mathbf{k}$, find

(a) $\dfrac{d\mathbf{F}}{dx}$; (b) $\dfrac{d^2\mathbf{F}}{dx^2}$; (c) $\left|\dfrac{d\mathbf{F}}{dx}\right|$; (d) $\dfrac{d}{dx}(\mathbf{F} \cdot \mathbf{F})$ at $x = 1$.

▶

5 If $\mathbf{F} = u\mathbf{i} + (1 - u)\mathbf{j} + 3u\mathbf{k}$ and $\mathbf{G} = 2\mathbf{i} - (1 + u)\mathbf{j} - u^2\mathbf{k}$, determine (a) $\dfrac{\mathrm{d}}{\mathrm{d}u}(\mathbf{F} \cdot \mathbf{G})$; (b) $\dfrac{\mathrm{d}}{\mathrm{d}u}(\mathbf{F} \times \mathbf{G})$; (c) $\dfrac{\mathrm{d}}{\mathrm{d}u}(\mathbf{F} + \mathbf{G})$.

6 Find the unit normal to the surface $4x^2y^2 - 3xz^2 - 2y^2z + 4 = 0$ at the point $(2, -1, -2)$.

7 Find the unit normal to the surface $2xy^2 + y^2z + x^2z - 11 = 0$ at the point $(-2, 1, 3)$.

8 Determine the unit vector normal to the surface
$xz^2 + 3xy - 2yz^2 + 1 = 0$ at the point $(1, -2, -1)$.

9 Find the unit normal to the surface $x^2y - 2yz^2 + y^2z = 3$ at the point $(2, -3, 1)$.

10 Determine the directional derivative of $\phi = xe^y + yz^2 + xyz$ at the point $(2, 0, 3)$ in the direction of $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

11 Find the directional derivative of $\phi = (x + 2y + z)^2 - (x - y - z)^2$ at the point $(2, 1, -1)$ in the direction of $\mathbf{A} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

12 Find the scalar triple product of

(a) $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(b) $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

(c) $\mathbf{A} = -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; $\mathbf{C} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$.

13 Find the vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ of the following.

(a) $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$; $\mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(b) $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(c) $\mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

14 If $\mathbf{F} = 4t^3\mathbf{i} - 2t^2\mathbf{j} + 4t\mathbf{k}$, determine when $t = 1$

(a) $\dfrac{\mathrm{d}\mathbf{F}}{\mathrm{d}t}$; (b) $\dfrac{\mathrm{d}^2\mathbf{F}}{\mathrm{d}t^2}$; (c) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F} \cdot \mathbf{F})$.

15 If $\phi = x^2 \sin z + ze^y$ find, at the point $(1, 3, 2)$, the values of

(a) grad ϕ and (b) $|\text{grad } \phi|$.

16 Given that $\phi = xy^2 + yz^2 - x^2$, find the derivative of ϕ with respect to distance at the point $(1, 2, -1)$, measured parallel to the vector $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

17 Find unit vectors normal to the surfaces $x^2 + y^2 - z^2 + 3 = 0$ and $xy - yz + zx - 10 = 0$ at the point $(3, 2, 4)$ and hence find the angle between the two surfaces at that point.

18 If $\mathbf{r} = (t^2 + 3t)\mathbf{i} - 2\sin 3t\mathbf{j} + 3e^{2t}\mathbf{k}$, determine

(a) $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$; (b) $\dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$; (c) the value of $\left|\dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|$ at $t = 0$.

▶

19 (a) Show that curl $(-y\mathbf{i} + x\mathbf{j})$ is a constant vector.

(b) Show that the vector field $(yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})$ has zero divergence and zero curl.

20 If $\mathbf{A} = 2xz^2\mathbf{i} - xz\mathbf{j} + (y + z)\mathbf{k}$, find curl curl $\mathbf{A}$.

21 Determine grad ϕ where $\phi = x^2 \cos(2yz - 0{\cdot}5)$ and obtain its value at the point (1, 3, 1).

22 Determine the value of p such that the three vectors **A**, **B**, **C** are coplanar when $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j} + p\mathbf{k}$; $\mathbf{C} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

23 If $\mathbf{A} = p\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$; $\quad \mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$; $\quad \mathbf{C} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a) find the values of p for which

(1) **A** and **B** are perpendicular to each other

(2) **A**, **B** and **C** are coplanar.

(b) determine a unit vector perpendicular to both **A** and **B** when $p = 2$.

Vector analysis 2

Frames 1 to 87

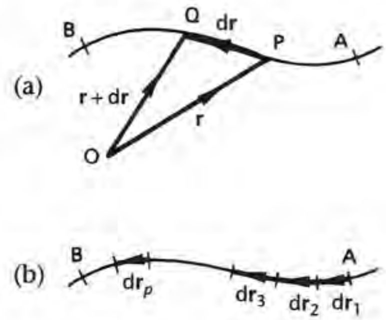
Learning outcomes

When you have completed this Programme you will be able to:

- Evaluate the line integral of a scalar and a vector field in Cartesian coordinates
- Evaluate the volume integral of a vector field
- Evaluate the surface integral of a scalar and a vector field
- Determine whether or not a vector field is a conservative vector field
- Apply Gauss' divergence theorem
- Apply Stokes' theorem
- Determine the direction of unit normal vectors to a surface
- Apply Green's theorem in the plane

We dealt in some detail with line, surface and volume integrals in an earlier Programme, when we approached the subject analytically. In many practical problems, it is more convenient to express these integrals in vector form and the methods often lead to more concise working. 1

Line integrals



Let a point P on the curve c joining A and B be denoted by the position vector $\mathbf{r}$ with respect to a fixed origin O.

If Q is a neighbouring point on the curve with position vector $\mathbf{r} + \mathrm{d}\mathbf{r}$, then $\overline{\mathrm{PQ}} = \mathrm{d}\mathbf{r}$.

The curve c can be divided up into many (n) such small arcs, approximating to $\mathrm{d}\mathbf{r}_1$, $\mathrm{d}\mathbf{r}_2$, $\mathrm{d}\mathbf{r}_3 \ldots$ $\mathrm{d}\mathbf{r}_p \ldots$ so that

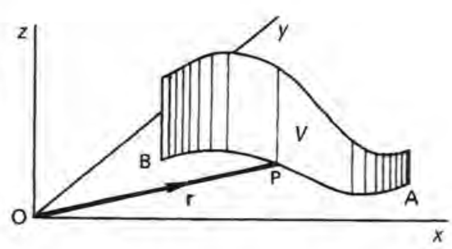
$$\overline{\mathrm{AB}} = \sum_{p=1}^{n} \mathrm{d}\mathbf{r}_p$$

where $\mathrm{d}\mathbf{r}_p$ is a vector representing the element of arc in both magnitude and direction.

Scalar field

If a scalar field V exists for all points on the curve, then $\sum_{p=1}^{n} V\,\mathrm{d}\mathbf{r}_p$ with $\mathrm{d}\mathbf{r} \to 0$, defines the *line integral* of V along the curve c from A to B,

$$\text{i.e. line integral} = \int_{\mathrm{c}} V\,\mathrm{d}\mathbf{r}$$



We can illustrate this integral by erecting a continuous ordinate proportional to V at each point of the curve. $\int_{\mathrm{c}} V\,\mathrm{d}\mathbf{r}$ is then represented by the area of the curved surface between the ends A and B of the curve c.

To evaluate a line integral, the integrand is expressed in terms of x, y, z, with $\mathrm{d}\mathbf{r} = \ldots\ldots\ldots\ldots$

2

$$\boxed{d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz}$$

In practice, x, y and z are often expressed in terms of parametric equations of a fourth variable (say u), i.e. $x = x(u)$; $y = y(u)$; $z = z(u)$. From these, dx, dy and dz can be written in terms of u and the integral evaluated in terms of this parameter u.

The following examples will show the method.

Example 1

If $V = xy^2z$, evaluate $\int_c V\,d\mathbf{r}$ along the curve c having parametric equations $x = 3u$; $y = 2u^2$; $z = u^3$ between A (0, 0, 0) and B (3, 2, 1).

$$V = xy^2z = (3u)(4u^4)(u^3) = 12u^8$$

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz = \dots\dots\dots\dots$$

3

$$\boxed{d\mathbf{r} = \mathbf{i}\,3\,du + \mathbf{j}\,4u\,du + \mathbf{k}\,3u^2\,du}$$

Because

$x = 3u$, $\quad \therefore\ dx = 3\,du$

$y = 2u^2$, $\quad \therefore\ dy = 4u\,du$

$z = u^3$, $\quad \therefore\ dz = 3u^2\,du$

Limits: A (0, 0, 0) corresponds to $u = \dots\dots\dots\dots$

B (3, 2, 1) corresponds to $u = \dots\dots\dots\dots$

4

$$\boxed{A\,(0,0,0) \equiv u = 0 \qquad B\,(3,2,1) \equiv u = 1}$$

$$\therefore \int_c V\,d\mathbf{r} = \int_0^1 12u^8\,(\mathbf{i}\,3\,du + \mathbf{j}\,4u\,du + \mathbf{k}\,3u^2\,du)$$

$$= \dots\dots\dots\dots$$

Finish it off

5

$$\boxed{4\mathbf{i} + \frac{24}{5}\mathbf{j} + \frac{36}{11}\mathbf{k}}$$

Because

$$\int_c V\,d\mathbf{r} = 12\int_0^1 (\mathbf{i}\,3u^8\,du + \mathbf{j}\,4u^9\,du + \mathbf{k}\,3u^{10}\,du)$$

which integrates directly to give the result quoted above.

Now for another example.

6

Example 2

If $V = xy + y^2z$, evaluate $\int_c V\,d\mathbf{r}$ along the curve c defined by $x = t^2$; $y = 2t$; $z = t + 5$ between A (0, 0, 5) and B (4, 4, 7).

As before, expressing V and $d\mathbf{r}$ in terms of the parameter t we have

$V = \ldots\ldots\ldots\ldots \qquad d\mathbf{r} = \ldots\ldots\ldots\ldots$

7

$$\boxed{V = 6t^3 + 20t^2; \quad d\mathbf{r} = \mathbf{i}\,2t\,dt + \mathbf{j}\,2\,dt + \mathbf{k}\,dt}$$

Because

$$V = xy + y^2z = (t^2)(2t) + (4t^2)(t + 5) = 6t^3 + 20t^2.$$

$$\text{Also } \left.\begin{aligned} x &= t^2 & dx &= 2t\,dt \\ y &= 2t & dy &= 2\,dt \\ z &= t + 5 & dz &= dt \end{aligned}\right\} \quad \begin{aligned} \therefore\ d\mathbf{r} &= \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \\ &= \mathbf{i}\,2t\,dt + \mathbf{j}\,2\,dt + \mathbf{k}\,dt \end{aligned}$$

$$\therefore \int_c V\,d\mathbf{r} = \int_c (6t^3 + 20t^2)(\mathbf{i}\,2t + \mathbf{j}\,2 + \mathbf{k})\,dt$$

Limits: A (0, 0, 5) $\equiv t = \ldots\ldots\ldots\ldots$

B (4, 4, 7) $\equiv t = \ldots\ldots\ldots\ldots$

8

$$\boxed{\text{A }(0,0,5) \equiv t = 0; \quad \text{B }(4,4,7) \equiv t = 2}$$

$$\therefore \int_c V\,d\mathbf{r} = \int_0^2 (6t^3 + 20t^2)(\mathbf{i}\,2t + \mathbf{j}\,2 + \mathbf{k})\,dt$$

$= \ldots\ldots\ldots\ldots$ Complete the integration.

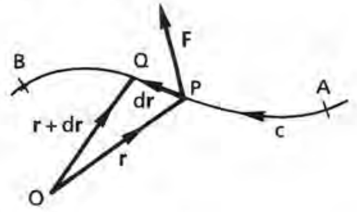
9

$$\boxed{\frac{8}{15}(444\,\mathbf{i} + 290\,\mathbf{j} + 145\,\mathbf{k})}$$

$$\int_c V\,d\mathbf{r} = 2\int_0^2 \{(6t^4 + 20t^3)\mathbf{i} + (6t^3 + 20t^2)\mathbf{j} + (3t^3 + 10t^2)\mathbf{k}\}\,dt$$

The actual integration is simple enough and gives the result shown. All line integrals in scalar fields are done in the same way.

10 Vector field



If a vector field $\mathbf{F}$ exists for all points of the curve c, then for each element of arc we can form the scalar product $\mathbf{F}\cdot d\mathbf{r}$. Summing these products for all elements of arc, we have $\sum_{p=1}^{n} \mathbf{F}\cdot d\mathbf{r}_p$

Then, if $d\mathbf{r}_p \to 0$, the sum becomes the integral $\int_c \mathbf{F}\cdot d\mathbf{r}$,

i.e. the line integral of $\mathbf{F}$ from A to B along the stated curve

$$= \int_c \mathbf{F}\cdot d\mathbf{r}$$

In this case, since $\mathbf{F}\cdot d\mathbf{r}$ is a scalar product, then the line integral is a scalar.

To evaluate the line integral, $\mathbf{F}$ and $d\mathbf{r}$ are expressed in terms of x, y, z and the curve in parametric form. We have

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

and $$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$$

Then $$\mathbf{F}\cdot d\mathbf{r} = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$
$$= F_x\,dx + F_y\,dy + F_z\,dz$$

$$\therefore \int_c \mathbf{F}\cdot d\mathbf{r} = \int_c F_x\,dx + \int_c F_y\,dy + \int_c F_z\,dz$$

Now for an example to show it in operation.

Example 1

If $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} - 2yz\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot d\mathbf{r}$ between A (0, 0, 0) and B (4, 2, 1) along the curve having parametric equations $x = 4t$; $y = 2t^2$; $z = t^3$.

Expressing everything in terms of the parameter t, we have

$$\mathbf{F} = \dots\dots\dots\dots$$

$$dx = \dots\dots\dots\dots; \quad dy = \dots\dots\dots\dots; \quad dz = \dots\dots\dots\dots$$

11

$$\boxed{\begin{array}{c}\mathbf{F} = 32t^4\,\mathbf{i} + 4t^4\,\mathbf{j} - 4t^5\,\mathbf{k}\\ \mathrm{d}x = 4\,\mathrm{d}t; \quad \mathrm{d}y = 4t\,\mathrm{d}t; \quad \mathrm{d}z = 3t^2\,\mathrm{d}t\end{array}}$$

Because

$$\begin{array}{lll} x^2y = (16t^2)(2t^2) = 32t^4 & x = 4t & \therefore\ \mathrm{d}x = 4\,\mathrm{d}t\\ xz = (4t)(t^3) = 4t^4 & y = 2t^2 & \therefore\ \mathrm{d}y = 4t\,\mathrm{d}t\\ 2yz = (4t^2)(t^3) = 4t^5 & z = t^3 & \therefore\ \mathrm{d}z = 3t^2\,\mathrm{d}t \end{array}$$

Then $\displaystyle\int \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int (32t^4\mathbf{i} + 4t^4\mathbf{j} - 4t^5\mathbf{k})\cdot(\mathbf{i}\,4\,\mathrm{d}t + \mathbf{j}\,4t\,\mathrm{d}t + \mathbf{k}\,3t^2\,\mathrm{d}t)$

$$= \int (128t^4 + 16t^5 - 12t^7)\,\mathrm{d}t$$

Limits: A $(0, 0, 0) \equiv t = \ldots\ldots\ldots\ldots$; B $(4, 2, 1) \equiv t = \ldots\ldots\ldots\ldots$

12

$$\boxed{A \equiv t = 0; \quad B \equiv t = 1}$$

$$\therefore \int_c \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^1 (128t^4 + 16t^5 - 12t^7)\,\mathrm{d}t = \ldots\ldots\ldots\ldots$$

13

$$\boxed{\frac{128}{5} + \frac{8}{3} - \frac{3}{2} = \frac{803}{30} = 26{\cdot}77}$$

If the vector field $\mathbf{F}$ is a *force field*, then the line integral $\displaystyle\int_c \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ represents the work done in moving a unit particle along the prescribed curve c from A to B.

Now for another example.

Example 2

If $\mathbf{F} = x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}$, evaluate $\displaystyle\int_c \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ between A (0, 0, 0) and B (1, 2, 3)

(a) along the straight lines c_1 from (0, 0, 0) to (1, 0, 0)
then c_2 from (1, 0, 0) to (1, 2, 0)
and c_3 from (1, 2, 0) to (1, 2, 3)

(b) along the straight line c_4 joining (0, 0, 0) to (1, 2, 3).

As before, we first obtain an expression for $\mathbf{F}\cdot \mathrm{d}\mathbf{r}$ which is

$\ldots\ldots\ldots\ldots$

14

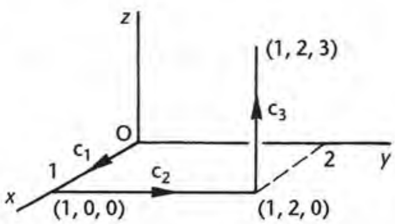
$$\boxed{\mathbf{F}\cdot d\mathbf{r} = x^2y\,dx + 2yz\,dy + 3z^2x\,dz}$$

Because

$$\mathbf{F}\cdot d\mathbf{r} = (x^2y\,\mathbf{i} + 2yz\,\mathbf{j} + 3z^2x\,\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$\therefore\quad \int \mathbf{F}\cdot d\mathbf{r} = \int x^2y\,dx + \int 2yz\,dy + \int 3z^2x\,dz$$

(a) Here the integration is made in three sections, along c_1, c_2 and c_3.



(1) c_1: $y = 0,\ z = 0,\ dy = 0,\ dz = 0$

$$\therefore\quad \int_{c_1} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

(2) c_2: The conditions along c_2 are

....

15

$$\boxed{c_2:\quad x = 1,\quad z = 0,\quad dx = 0,\quad dz = 0}$$

$$\therefore\quad \int_{c_2} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

(3) c_3: $x = 1,\ y = 2,\ dx = 0,\ dy = 0$

$$\therefore\quad \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \dots\dots\dots\dots$$

16

$$\boxed{27}$$

Because

$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + \int_0^3 3z^2\,dz = 27$$

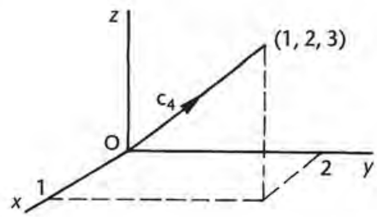
Summing the three partial results

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 27 = 27 \quad \therefore\ \int_{c_1+c_2+c_3} \mathbf{F}\cdot d\mathbf{r} = 27$$

▶

(b) If t is taken as the parameter, the parametric equations of c are

$x = \dots\dots\dots\dots$
$y = \dots\dots\dots\dots$
$z = \dots\dots\dots\dots$



17

$$\boxed{x = t; \quad y = 2t: \quad z = 3t}$$

and the limits of t are $\dots\dots\dots\dots$

18

$$\boxed{t = 0 \quad \text{and} \quad t = 1}$$

As in Example 1, we now express everything in terms of t and complete the integral, finally getting

$$\int_{c_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \dots\dots\dots\dots$$

19

$$\boxed{\int_{c_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{115}{4} = 28{\cdot}75}$$

Because

$$\mathbf{F} = 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}$$
$$\mathrm{d}\mathbf{r} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z = \mathbf{i}\,\mathrm{d}t + \mathbf{j}\,2\,\mathrm{d}t + \mathbf{k}\,3\,\mathrm{d}t$$

$$\begin{aligned} \therefore \quad \int_{c_4} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 (2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\,\mathrm{d}t \\ &= \int_0^1 (2t^3 + 24t^2 + 81t^3)\,\mathrm{d}t = \int_0^1 (83t^3 + 24t^2)\,\mathrm{d}t \\ &= \left[83\frac{t^4}{4} + 8t^3\right]_0^1 = \frac{115}{4} = 28{\cdot}75 \end{aligned}$$

So the value of the line integral depends on the path taken between the two end points A and B

(a) $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ via c_1, c_2 and $c_3 = 27$

(b) $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ via c_4 $= 28{\cdot}75$

We shall refer to this topic later.

One further example on your own. The working is just the same as before.

▶

Example 3

If $\mathbf{F} = x^2y^2\mathbf{i} + y^3z\mathbf{j} + z^2\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot d\mathbf{r}$ along the curve $x = 2u^2$, $y = 3u$, $z = u^3$ between A $(2, -3, -1)$ and B $(2, 3, 1)$. Proceed as before. You will have no difficulty.

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_c \mathbf{F}\cdot d\mathbf{r} = \frac{500}{21} = 23{\cdot}8}$$

Here is the working for you to check.

$$x = 2u^2 \quad y = 3u \quad z = u^3$$

$$x^2y^2 = (4u^4)(9u^2) = 36u^6 \qquad dx = 4u\,du$$

$$y^3z = (27u^3)(u^3) = 27u^6 \qquad dy = 3\,du$$

$$z^2 = u^6 \qquad dz = 3u^2\,du$$

Limits: A $(2, -3, -1)$ corresponds to $u = -1$

B $(2, 3, 1)$ corresponds to $u = 1$

$$\therefore \int_c \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} (x^2y^2\mathbf{i} + y^3z\mathbf{j} + z^2\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= \int_{-1}^{1} (36u^6\mathbf{i} + 27u^6\mathbf{j} + u^6\mathbf{k})\cdot(\mathbf{i}\,4u\,du + \mathbf{j}\,3\,du + \mathbf{k}\,3u^2\,du)$$

$$= \int_{-1}^{1} (144u^7 + 81u^6 + 3u^8)\,du$$

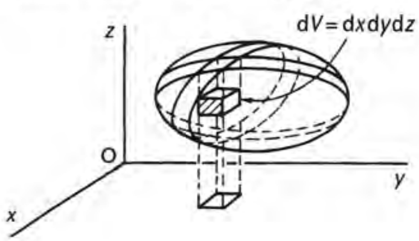
$$= \left[18u^8 + \frac{81u^7}{7} + \frac{u^9}{3}\right]_{-1}^{1} = \frac{500}{21} = 23{\cdot}8$$

Now on to the next section

Volume integrals

21

If V is a closed region bounded by a surface **S** and **F** is a vector field at each point of V and on its boundary surface **S**, then $\int_V \mathbf{F}\,dV$ is the *volume integral* of **F** throughout the region.



$$\int_V \mathbf{F}\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \mathbf{F}\,dz\,dy\,dx$$

Example 1

Evaluate $\int_V \mathbf{F}\,\mathrm{d}V$ where V is the region bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 3$, $z = 0$, $z = 4$, and $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}$.

We start, as in most cases, by sketching the diagram, which is

. . . .

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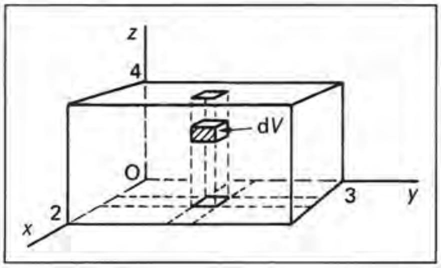

Then $\mathbf{F} = xy\,\mathbf{i} + z\,\mathbf{j} - x^2\,\mathbf{k}$ and $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

$$\therefore \int_V \mathbf{F}\,\mathrm{d}V = \int_0^4\int_0^3\int_0^2 (xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^4\int_0^3 \left[\frac{x^2y}{2}\mathbf{i} + xz\mathbf{j} - \frac{x^3}{3}\mathbf{k}\right]_{x=0}^{x=2} \mathrm{d}y\,\mathrm{d}z$$

$$= \int_0^4\int_0^3 \left(2y\mathbf{i} + 2z\mathbf{j} - \frac{8}{3}\mathbf{k}\right)\mathrm{d}y\,\mathrm{d}z$$

$=$ . . . . Complete the integral.

23

$$\boxed{\int_V \mathbf{F}\,\mathrm{d}V = 4(9\mathbf{i} + 12\mathbf{j} - 8\mathbf{k})}$$

Because

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_0^4 \left[y^2\mathbf{i} + 2yz\mathbf{j} - \frac{8}{3}y\mathbf{k}\right]_{y=0}^{y=3} \mathrm{d}z$$

$$= \int_0^4 (9\mathbf{i} + 6z\mathbf{j} - 8\mathbf{k})\,\mathrm{d}z$$

$$= \left[9z\mathbf{i} + 3z^2\mathbf{j} - 8z\mathbf{k}\right]_0^4$$

$$= 36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}$$

$$= 4(9\mathbf{i} + 12\mathbf{j} - 8\mathbf{k})$$

Now another.

▶

Example 2

Evaluate $\int_V \mathbf{F}\,dV$ where V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 2$, and $\mathbf{F} = 2z\mathbf{i} + y\mathbf{k}$.

To sketch the surface $2x + y + z = 2$, note that

when $z = 0$, $\quad 2x + y = 2 \quad$ i.e. $y = 2 - 2x$
when $y = 0$, $\quad 2x + z = 2 \quad$ i.e. $z = 2 - 2x$
when $x = 0$, $\quad y + z = 2 \quad$ i.e. $z = 2 - y$

Inserting these in the planes $x = 0$, $y = 0$, $z = 0$ will help.

The diagram is therefore

....

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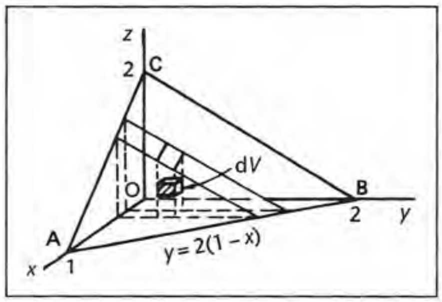

So $2x + y + z = 2$ cuts the axes at A (1, 0, 0); B (0, 2, 0); C (0, 0, 2).

Also $\quad \mathbf{F} = 2z\mathbf{i} + y\mathbf{k}; \quad z = 2 - 2x - y = 2(1 - x) - y$

$$\therefore \int_V \mathbf{F}\,dV = \int_0^1 \int_0^{2(1-x)} \int_0^{2(1-x)-y} (2z\mathbf{i} + y\mathbf{k})\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{2(1-x)} \Big[z^2\mathbf{i} + yz\mathbf{k} \Big]_{z=0}^{z=2(1-x)-y} dy\,dx$$

$$= \int_0^1 \int_0^{2(1-x)} \{[4(1-x)^2 - 4(1-x)y + y^2]\mathbf{i} + [2(1-x)y - y^2]\mathbf{k}\}\,dy\,dx$$

$$= \int_0^1 \left[\left\{ 4(1-x)^2 y - 2(1-x)y^2 + \frac{y^3}{3} \right\}\mathbf{i} + \left\{ (1-x)y^2 - \frac{y^3}{3} \right\}\mathbf{k} \right]_{y=0}^{2(1-x)} dx$$

$= \ldots\ldots\ldots\ldots$

Finish the last stage

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$$\boxed{\int_V \mathbf{F}\,\mathrm{d}V = \frac{1}{3}(2\mathbf{i} + \mathbf{k})}$$

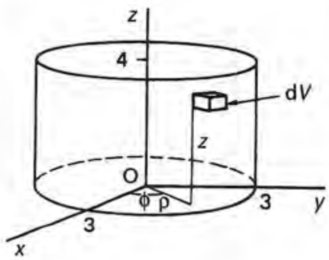
Because

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_0^1 \left\{\frac{8}{3}(1-x)^3\mathbf{i} + \frac{4}{3}(1-x)^3\mathbf{k}\right\}\mathrm{d}x$$
$$= \left[-\frac{2}{3}(1-x)^4\mathbf{i} - \frac{1}{3}(1-x)^4\mathbf{k}\right]_0^1 = \frac{1}{3}(2\mathbf{i} + \mathbf{k})$$

And now one more, slightly different.

Example 3

Evaluate $\int_V \mathbf{F}\,\mathrm{d}V$ where $\mathbf{F} = 2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$ and V is the region bounded by the planes $z = 0$, $z = 4$ and the surface $x^2 + y^2 = 9$.



It will be convenient to use cylindrical polar coordinates (ρ, ϕ, z) so the relevant transformations are

$x = \dots\dots\dots\dots$; $\quad y = \dots\dots\dots\dots$

$z = \dots\dots\dots\dots$; $\quad \mathrm{d}V = \dots\dots\dots\dots$

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$$\boxed{\begin{array}{ll} x = \rho\cos\phi; & y = \rho\sin\phi \\ z = z; & \mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z \end{array}}$$

Then $\int_V \mathbf{F}\,\mathrm{d}V = \iiint_V (2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$

Changing into cylindrical polar coordinates with appropriate change of limits this becomes

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\int_{z=0}^{4}(2\mathbf{i} + 2z\mathbf{j} + \rho\sin\phi\,\mathbf{k})\,\mathrm{d}z\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$
$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3}\left[2z\mathbf{i} + z^2\mathbf{j} + \rho\sin\phi\, z\mathbf{k}\right]_{z=0}^{4}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$
$$= \int_0^{2\pi}\int_0^3 (8\mathbf{i} + 16\mathbf{j} + 4\rho\sin\phi\,\mathbf{k})\,\mathrm{d}\rho\,\mathrm{d}\phi$$
$$= 4\int_0^{2\pi}\int_0^3 (2\rho\mathbf{i} + 4\rho\mathbf{j} + \rho^2\sin\phi\,\mathbf{k})\,\mathrm{d}\rho\,\mathrm{d}\phi$$

Completing the working, we finally get

$$\int_V \mathbf{F}\,\mathrm{d}V = \dots\dots\dots\dots$$

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$$\boxed{72\pi(\mathbf{i}+2\mathbf{j})}$$

Because

$$\begin{aligned}\int_V \mathbf{F}\,\mathrm{d}V &= 4\int_0^{2\pi}\left[\rho^2\mathbf{i}+2\rho^2\mathbf{j}+\frac{\rho^3}{3}\sin\phi\,\mathbf{k}\right]_0^3 \mathrm{d}\phi\\ &= 4\int_0^{2\pi}(9\mathbf{i}+18\mathbf{j}+9\sin\phi\,\mathbf{k})\,\mathrm{d}\phi\\ &= 36\int_0^{2\pi}(\mathbf{i}+2\mathbf{j}+\sin\phi\,\mathbf{k})\,\mathrm{d}\phi\\ &= 36\left[\phi\mathbf{i}+2\phi\mathbf{j}-\cos\phi\,\mathbf{k}\right]_0^{2\pi}\\ &= 36\{(2\pi\mathbf{i}+4\pi\mathbf{j}-\mathbf{k})-(-\mathbf{k})\}\\ &= 72\pi(\mathbf{i}+2\mathbf{j})\end{aligned}$$

You will, of course, remember that in appropriate cases, the use of cylindrical polar coordinates or spherical polar coordinates often simplifies the subsequent calculations. So keep them in mind.

Now let us turn to surface integrals – in the next frame

Surface integrals

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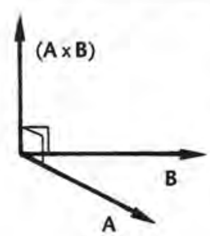
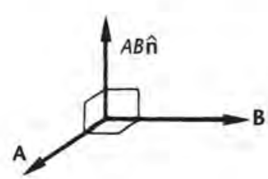


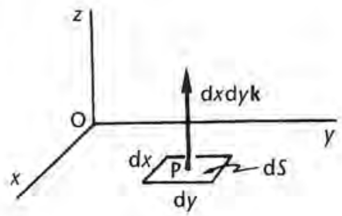
The vector product of two vectors $\mathbf{A}$ and $\mathbf{B}$ has magnitude $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$ at right angles to the plane of $\mathbf{A}$ and $\mathbf{B}$ to form a right-handed set.



If $\theta = \dfrac{\pi}{2}$, then $|\mathbf{A}\times\mathbf{B}| = AB$ in the direction of the normal. Therefore, if $\hat{\mathbf{n}}$ is a unit normal then

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\hat{\mathbf{n}} = AB\,\hat{\mathbf{n}}$$

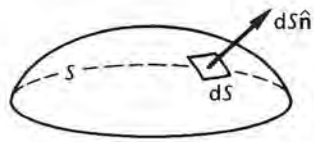
▶



If P (x, y) is a point in the x–y plane, the element of area $dx\,dy$ has a vector area $d\mathbf{S} = (\mathbf{i}\,dx) \times (\mathbf{j}\,dy)$.

i.e. $d\mathbf{S} = dx\,dy(\mathbf{i} \times \mathbf{j}) = dx\,dy\,\mathbf{k}$

i.e. a vector of magnitude $dx\,dy$ acting in the direction of $\mathbf{k}$ and referred to as the *vector area*.



For a general surface S in space, each element of surface dS has a *vector area* dS such that $d\mathbf{S} = dS\,\hat{\mathbf{n}}$.

You will remember we established previously that for a surface S given by the equation $\phi(x, y, z) = \text{constant}$, the unit normal $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$$

Let us see how we can apply these results to the following examples.

Scalar fields

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Example 1

A scalar field $V = xyz$ exists over the curved surface S defined by $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = 3$ in the first octant. Evaluate $\int_S V\,d\mathbf{S}$ over this surface.

We have $V = xyz \quad S\colon \quad x^2 + y^2 - 4 = 0, \quad z = 0 \;\text{ to }\; z = 3$

$$d\mathbf{S} = \hat{\mathbf{n}}\,dS \qquad \text{where } \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$ and

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

Therefore

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\mathbf{i} + y\mathbf{j}}{2} \quad \text{so that } d\mathbf{S} = \hat{\mathbf{n}}\,dS = \frac{x\mathbf{i} + y\mathbf{j}}{2}\,dS$$

$$\therefore \quad \int_S V\,d\mathbf{S} = \int_S V\,\hat{\mathbf{n}}\,dS$$

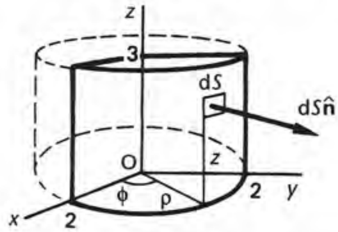
$$= \frac{1}{2}\int_S xyz(x\mathbf{i} + y\mathbf{j})\,dS$$

$$= \frac{1}{2}\int_S (x^2yz\mathbf{i} + xy^2z\mathbf{j})\,dS \tag{1}$$

▶

We have to evaluate this integral over the prescribed surface.

Changing to cylindrical coordinates with $\rho = 2$



$x = \dots\dots\dots\dots$; $\quad y = \dots\dots\dots\dots$

$z = \dots\dots\dots\dots$; $\quad dS = \dots\dots\dots\dots$

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$$\boxed{\begin{array}{ll} x = 2\cos\phi; & y = 2\sin\phi \\ z = z; & dS = 2\,d\phi\,dz \end{array}}$$

$$\begin{aligned} \therefore\ x^2yz &= (4\cos^2\phi)(2\sin\phi)(z) \\ &= 8\cos^2\phi\sin\phi\, z \\ xy^2z &= (2\cos\phi)(4\sin^2\phi)(z) \\ &= 8\cos\phi\sin^2\phi\, z \end{aligned}$$

Then result (1) above becomes

$$\begin{aligned} \int_S V\,d\mathbf{S} &= \frac{1}{2}\int_0^{\pi/2}\int_0^3 (8\cos^2\phi\sin\phi\, z\mathbf{i} + 8\cos\phi\sin^2\phi\, z\mathbf{j})\, 2\,dz\,d\phi \\ &= 4\int_0^{\pi/2}\int_0^3 (\cos^2\phi\sin\phi\,\mathbf{i} + \cos\phi\sin^2\phi\,\mathbf{j})\, 2z\,dz\,d\phi \\ &= 4\int_0^{\pi/2} (\cos^2\phi\sin\phi\,\mathbf{i} + \cos\phi\sin^2\phi\,\mathbf{j})\, 9\,d\phi \end{aligned}$$

and this eventually gives

$$\int_S V\,d\mathbf{S} = \dots\dots\dots\dots$$

$$\boxed{\int_S V\,d\mathbf{S} = 12(\mathbf{i}+\mathbf{j})}$$

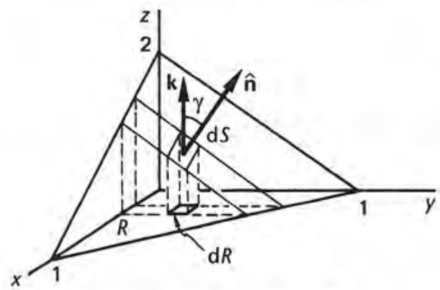
Because

$$\int_S V\,d\mathbf{S} = 36\left[-\frac{\cos^3\phi}{3}\mathbf{i} + \frac{\sin^3\phi}{3}\mathbf{j}\right]_0^{\pi/2} = 12(\mathbf{i}+\mathbf{j})$$

Example 2

A scalar field $V = x + y + z$ exists over the surface S defined by $2x + 2y + z = 2$ bounded by $x = 0,\ y = 0,\ z = 0$ in the first octant.

Evaluate $\int_S V\,d\mathbf{S}$ over this surface.



S: $2x + 2y + z = 2$

$x = 0 \qquad z = 2 - 2y$

$y = 0 \qquad z = 2 - 2x$

$z = 0 \qquad y = 1 - x$

$d\mathbf{S} = \hat{\mathbf{n}}\,dS \qquad$ where $\hat{\mathbf{n}} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

Therefore

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3} \quad \text{so that} \quad d\mathbf{S} = \hat{\mathbf{n}}\,dS = \frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\,dS$$

If we now project dS onto the x–y plane, $dR = dS\cos\gamma$

$$\cos\gamma = \hat{\mathbf{n}}\cdot\mathbf{k} = \frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(\mathbf{k}) = \frac{1}{3}$$

$$\therefore\ dR = \frac{1}{3}dS \qquad \therefore\ dS = 3dR = 3\,dx dy$$

$$\therefore\ \int_S V\,d\mathbf{S} = \int_S V\hat{\mathbf{n}}\,dS = \int_S\int (x+y+z)\frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})3\,dx\,dy$$

But $z = 2 - 2x - 2y$

$$\therefore\ \int_S V\,d\mathbf{S} = \int_{x=0}^{1}\int_{y=0}^{1-x}(2-x-y)(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\,dy\,dx$$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{2}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})}$$

Because

$$\begin{aligned}\int_S V\,\mathrm{d}\mathbf{S} &= \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^{1-x} (2\mathbf{i}+2\mathbf{j}+\mathbf{k})\,\mathrm{d}x \\ &= \left[\frac{3}{2}x - x^2 + \frac{x^3}{6}\right]_0^1 (2\mathbf{i}+2\mathbf{j}+\mathbf{k}) \\ &= \frac{2}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\end{aligned}$$

33 Vector fields

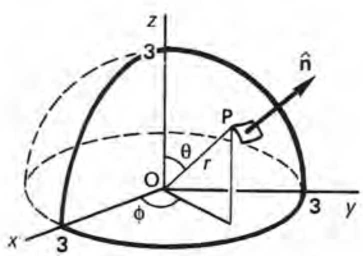
Example 1

A vector field $\mathbf{F} = y\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ exists over a surface S defined by $x^2 + y^2 + z^2 = 9$ bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant. Evaluate $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ over the surface indicated.

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}S \qquad \text{where } \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} \text{ where } \phi = x^2 + y^2 + z^2 - 9 = 0$$

Now $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ and

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{9} = 6$$



$$\therefore\ \hat{\mathbf{n}} = \frac{1}{6}(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) = \frac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\begin{aligned}\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_S (y\mathbf{i} + 2\mathbf{j} + \mathbf{k})\cdot\frac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,\mathrm{d}S \\ &= \frac{1}{3}\int_S (xy + 2y + z)\,\mathrm{d}S\end{aligned}$$

Before integrating over the surface, we convert to spherical polar coordinates.

$x = \ldots\ldots\ldots\ldots$; $\qquad y = \ldots\ldots\ldots\ldots$

$z = \ldots\ldots\ldots\ldots$; $\qquad \mathrm{d}S = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{array}{ll} x = 3\sin\theta\cos\phi; & y = 3\sin\theta\sin\phi \\ z = 3\cos\theta; & dS = 9\sin\theta\,d\theta\,d\phi \end{array}}$$

Limits of θ and ϕ are $\theta = 0$ to $\frac{\pi}{2}$; $\phi = 0$ to $\frac{\pi}{2}$.

$$\therefore \int_S \mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\int_0^{\pi/2}\int_0^{\pi/2}(9\sin^2\theta\sin\phi\cos\phi + 6\sin\theta\sin\phi + 3\cos\theta)\,9\sin\theta\,d\theta\,d\phi$$

$$= 9\int_0^{\pi/2}\int_0^{\pi/2}(3\sin^3\theta\sin\phi\cos\phi + 2\sin^2\theta\sin\phi + \sin\theta\cos\theta)\,d\theta\,d\phi$$

$$= \ldots\ldots\ldots\ldots$$

Complete the integral

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$$\boxed{\int_S \mathbf{F}\cdot d\mathbf{S} = 9\left(1 + \frac{3\pi}{4}\right)}$$

Because

$$\int_S \mathbf{F}\cdot d\mathbf{S} = 9\int_0^{\pi/2}\left(2\sin\phi\cos\phi + \frac{\pi}{2}\sin\phi + \frac{1}{2}\right)d\phi$$

$$= 9\left[\sin^2\phi - \frac{\pi}{2}\cos\phi - \frac{\phi}{2}\right]_0^{\pi/2} = 9\left(1 + \frac{3\pi}{4}\right)$$

Example 2

Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ and S is the surface $x^2 + y^2 = 4$ in the first two octants bounded by the planes $z = 0$, $z = 5$ and $y = 0$.

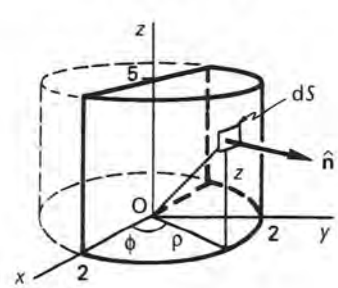
ϕ: $x^2 + y^2 - 4 = 0 \qquad \hat{\mathbf{n}} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\therefore |\nabla\phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$$

$$= 2\sqrt{4} = 4$$

$$\therefore \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{4} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j})$$



$$\therefore \int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_S y^2\,dS}$$

Because

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_S (2y\mathbf{j}+z\mathbf{k})\cdot\frac{1}{2}(x\mathbf{i}+y\mathbf{j})\,dS$$
$$=\frac{1}{2}\int_S (2y^2)\,dS = \int_S y^2\,dS$$

This is clearly a case for using cylindrical polar coordinates.

$x = \dots\dots\dots\dots;\quad y = \dots\dots\dots\dots$

$z = \dots\dots\dots\dots;\quad dS = \dots\dots\dots\dots$

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$$\boxed{\begin{array}{ll} x = 2\cos\phi; & y = 2\sin\phi \\ z = z; & dS = 2\,d\phi\,dz \end{array}}$$

$$\therefore\ \int_S \mathbf{F}\cdot d\mathbf{S} = \int_S y^2\,dS = \int_S\int 4\sin^2\phi\, 2\,d\phi\,dz = 8\int_S\int \sin^2\phi\,d\phi\,dz$$

Limits: $\phi = 0$ to $\phi = \pi$; $z = 0$ to $z = 5$

$$\therefore\ \int_S \mathbf{F}\cdot d\mathbf{S} = \dots\dots\dots\dots$$

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$$\boxed{20\pi}$$

Because

$$\int_S \mathbf{F}\cdot dS = 4\int_{z=0}^{5}\int_{\phi=0}^{\pi}(1-\cos 2\phi)\,d\phi\,dz$$
$$=4\int_0^5\left[\phi-\frac{\sin 2\phi}{2}\right]_0^{\pi}dz$$
$$=4\int_0^5 \pi\,dz = 4\pi\Big[z\Big]_0^5 = 20\pi$$

Example 3

Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}$ is the field $x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}$ and S is the surface $2x + y + 2z = 2$ bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant.

We can sketch the diagram by putting $x = 0$, $y = 0$, $z = 0$ in turn in the equation for S.

When	$x = 0$	$y + 2z = 2$	$z = 1 - \dfrac{y}{2}$
	$y = 0$	$x + z = 1$	$z = 1 - x$
	$z = 0$	$2x + y = 2$	$y = 2 - 2x$

So the diagram is

$\dots\dots\dots\dots$

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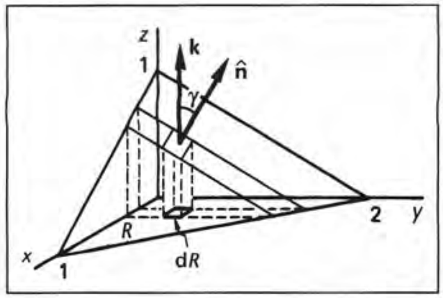

$$\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}; \qquad \phi: \quad 2x + y + 2z - 2 = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \qquad |\nabla\phi| = 3$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

$$= \ldots\ldots\ldots\ldots \text{ (next stage)}$$

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$$\boxed{\frac{1}{3}\int_S (2x^2 - y + 4z)\, dS}$$

Because

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_S (x^2\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}) \cdot \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})\, dS$$

$$= \frac{1}{3}\int_S (2x^2 - y + 4z)\, dS$$

If we now project the element of surface dS onto the x–y plane

$$dR = dS\cos\gamma \qquad \cos\gamma = \hat{\mathbf{n}} \cdot \mathbf{k} \qquad \therefore\ dR = \hat{\mathbf{n}} \cdot \mathbf{k}\, dS \qquad \therefore\ dS = \frac{dx\,dy}{\hat{\mathbf{n}} \cdot \mathbf{k}}$$

$$\therefore\ \hat{\mathbf{n}} \cdot \mathbf{k} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{k}) = \frac{2}{3} \qquad \therefore\ dS = \frac{3}{2}\, dx\, dy$$

Using these new relationships, $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$

$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_R\int\frac{1}{2}(2x^2-y+4z)\,dx\,dy}$$

Because

$$\begin{aligned}\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \frac{1}{3}\int_S(2x^2-y+4z)\,dS\\ &= \frac{1}{3}\int_R\int(2x^2-y+4z)\frac{3}{2}\,dx\,dy\\ &= \frac{1}{2}\int_R\int(2x^2-y+4z)\,dx\,dy\end{aligned}$$

Limits: $y=0$ to $y=2-2x$; $x=0$ to $x=1$

$$\therefore\quad \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_0^1\int_0^{2-2x}(2x^2-y+4z)\,dy\,dx$$

But $2x+y+2z=2 \quad \therefore\ z=\frac{1}{2}(2-2x-y)$

$$\therefore\quad \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \ldots\ldots\ldots\ldots$$

Complete the integration

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$$\boxed{\frac{1}{2}}$$

Here is the rest of the working.

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \frac{1}{2}\int_0^1\int_0^{2-2x}(2x^2-y+4-4x-2y)\,dy\,dx\\ &= \frac{1}{2}\int_0^1\int_0^{2-2x}(2x^2-4x+4-3y)\,dy\,dx\\ &= \frac{1}{2}\int_0^1\left[(2x^2-4x+4)y-\frac{3y^2}{2}\right]_0^{2-2x}dx\\ &= \frac{1}{2}\int_0^1(4x^2-8x+8-4x^3+8x^2-8x-6+12x-6x^2)\,dx\\ &= \frac{1}{2}\int_0^1(6x^2-4x^3-4x+2)\,dx = \int_0^1(3x^2-2x^3-2x+1)\,dx\\ &= \left[x^3-\frac{x^4}{2}-x^2+x\right]_0^1 = \frac{1}{2}\end{aligned}$$

While we are concerned with vector fields, let us move on to a further point of interest.

Conservative vector fields

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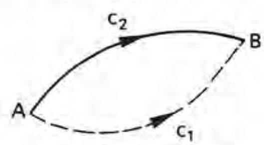
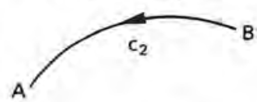
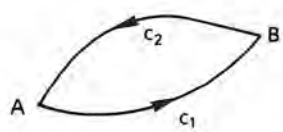
In general, the value of the line integral $\int_c \mathbf{F}\cdot d\mathbf{r}$ between two stated points A and B depends on the particular path of integration followed.

If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field $\mathbf{F}$ is said to be *conservative*.

It follows that, for a closed path in a conservative field, $\oint_c \mathbf{F}\cdot d\mathbf{r} = 0$.







Because, if the field is conservative

$$\int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} = \int_{c_2(AB)} \mathbf{F}\cdot d\mathbf{r}$$

$$\text{But } \int_{c_2(BA)} \mathbf{F}\cdot d\mathbf{r} = -\int_{c_2(AB)} \mathbf{F}\cdot d\mathbf{r}$$

Hence, for the closed path $\mathbf{AB}_{c_1} + \mathbf{BA}_{c_2}$

$$\begin{aligned}\oint \mathbf{F}\cdot d\mathbf{r} &= \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} + \int_{c_2(BA)} \mathbf{F}\cdot d\mathbf{r}\\ &= \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} - \int_{c_2(AB)} \mathbf{F}\cdot d\mathbf{r}\\ &= \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} - \int_{c_1(AB)} \mathbf{F}\cdot d\mathbf{r} = 0\end{aligned}$$

$$\therefore \quad \oint \mathbf{F}\cdot d\mathbf{r} = 0$$

Note that this result holds good only for a closed curve and when the vector field is a conservative field.

Now for an example.

Example

If $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, evaluate the line integral $\int \mathbf{F}\cdot d\mathbf{r}$ between A (0, 0, 0) and B (2, 4, 6)

(a) along the curve c whose parametric equations are $x = u$, $y = u^2$, $z = 3u$

(b) along the three straight lines c_1: (0, 0, 0) to (2, 0, 0); c_2: (2, 0, 0) to (2, 4, 0); c_3: (2, 4, 0) to (2, 4, 6).

Hence determine whether or not $\mathbf{F}$ is a conservative field.

First draw the diagram

. . . .

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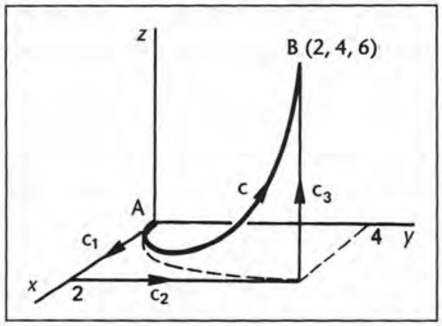

(a) $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

$$x = u; \qquad y = u^2; \qquad z = 3u$$

$$\therefore \ dx = du; \qquad dy = 2u\,du; \qquad dz = 3\,du.$$

$$\mathbf{F}\cdot d\mathbf{r} = (2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k})\cdot(\mathbf{i}\ dx + \mathbf{j}\ dy + \mathbf{k}\ dz)$$

$$= 2xyz\,dx + x^2z\,dy + x^2y\,dz$$

Using the transformations shown above, we can now express $\mathbf{F}\cdot d\mathbf{r}$ in terms of u.

$\mathbf{F}\cdot d\mathbf{r} =$ ....

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$$\boxed{15u^4\,du}$$

Because

$$2xyz\,dx = (2u)(u^2)(3u)\,du = 6u^4\,du$$

$$x^2z\,dy = (u^2)(3u)(2u)\,du = 6u^4\,du$$

$$x^2y\,dz = (u^2)(u^2)3\,du = 3u^4\,du$$

$$\therefore \ \mathbf{F}\cdot d\mathbf{r} = 6u^4\,du + 6u^4\,du + 3u^4\,du = 15u^4\,du$$

The limits of integration in u are

....

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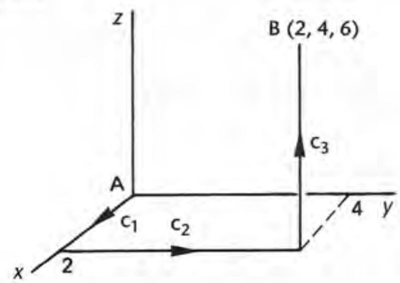
$$\boxed{u = 0 \text{ to } u = 2}$$

$$\therefore \int_c \mathbf{F}\cdot d\mathbf{r} = \int_0^2 15u^4\,du = \left[3u^5\right]_0^2 = 96 \qquad \int_c \mathbf{F}\cdot d\mathbf{r} = 96$$

(b) The diagram for (b) is as shown. We consider each straight line section in turn.



$$\int \mathbf{F}\cdot d\mathbf{r} = \int (2xyz\,dx + x^2z\,dy + x^2y\,dz)$$

c_1: $(0, 0, 0)$ to $(2, 0, 0)$; $y = 0$, $z = 0$, $dy = 0$, $dz = 0$

$$\therefore \int_{c_1} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

In the same way, we evaluate the line integral along c_2 and c_3.

$$\int_{c_2} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots; \qquad \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int_{c_2} \mathbf{F}\cdot d\mathbf{r} = 0; \qquad \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = 96}$$

Because we have $\int \mathbf{F}\cdot d\mathbf{r} = \int (2xyz\,dx + x^2z\,dy + x^2y\,dz)$

c_2: $(2, 0, 0)$ to $(2, 4, 0)$; $x = 2$, $z = 0$, $dx = 0$, $dz = 0$

$$\therefore \int_{c_2} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

$$\int_{c_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

c_3: $(2, 4, 0)$ to $(2, 4, 6)$; $x = 2$, $y = 4$, $dx = 0$, $dy = 0$

$$\therefore \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + \int_0^6 16\,dz = \Big[16z\Big]_0^6 = 96$$

$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = 96$$

Collecting the three results together

$$\int_{c_1+c_2+c_3} \mathbf{F}\cdot d\mathbf{r} = 0 + 0 + 96 \qquad \therefore \int_{c_1+c_2+c_3} \mathbf{F}\cdot d\mathbf{r} = 96$$

▶

In this particular example, the value of the line integral is independent of the two paths we have used joining the same two end points and indicates that **F** may be a conservative field. It follows that

$$\int_c \mathbf{F}\cdot d\mathbf{r} - \int_{c_1+c_2+c_3} \mathbf{F}\cdot d\mathbf{r} = 0 \quad \text{i.e.} \quad \oint \mathbf{F}\cdot d\mathbf{r} = 0$$

So, if **F** is a conservative field, $\oint \mathbf{F}\cdot d\mathbf{r} = 0$

Make a note of this for future use

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Two tests can be applied to establish that a given vector field is conservative.

If **F** is a conservative field

(a) curl $\mathbf{F} = 0$

(b) **F** can be expressed as grad V where V is a scalar field to be determined.

For example, in the work we have just completed, we showed that $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ is a conservative field.

(a) If we determine curl **F** in this case, we have

curl **F** = ...

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$$\boxed{\text{curl } \mathbf{F} = 0}$$

Because

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$$= (x^2 - x^2)\mathbf{i} - (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} = \mathbf{0}$$

$$\therefore \text{ curl } \mathbf{F} = \mathbf{0}$$

(b) We can attempt to express **F** as grad V where V is a scalar in x, y, z.

If $V = f(x, y, z)$

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

and we have $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

$$\therefore \frac{\partial V}{\partial x} = 2xyz \qquad \therefore V = x^2yz + f(y, z)$$

$$\frac{\partial V}{\partial y} = x^2z \qquad \therefore V = \ldots\ldots\ldots\ldots$$

$$\frac{\partial V}{\partial z} = x^2y \qquad \therefore V = \ldots\ldots\ldots\ldots$$

We therefore have to find a scalar function V that satisfies the three requirements. $V = \ldots\ldots\ldots\ldots$

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$$\boxed{V = x^2yz}$$

Because

$$\frac{\partial V}{\partial x} = 2xyz \qquad \therefore\ V = x^2yz + f(y, z)$$

$$\frac{\partial V}{\partial y} = x^2z \qquad \therefore\ V = x^2yz + g(x, z)$$

$$\frac{\partial V}{\partial z} = x^2y \qquad \therefore\ V = x^2yz + h(x, y)$$

These three are satisfied if $f(y, z) = g(z, x) = h(x, y) = 0$

$\therefore\ \mathbf{F} = \text{grad } V$ where $V = x^2yz$

So two tests can be applied to determine whether or not a vector field is conservative. They are

(a) ....

(b) ....

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(a) curl $\mathbf{F} = 0$
(b) $\mathbf{F} =$ grad V

Any one of these conditions can be applied as is convenient.

Now what about these?

Exercise

Determine which of the following vector fields are conservative.

(a) $\mathbf{F} = (x + y)\mathbf{i} + (y - z)\mathbf{j} + (x + y + z)\mathbf{k}$

(b) $\mathbf{F} = (2xz + y)\mathbf{i} + (z + x)\mathbf{j} + (x^2 + y)\mathbf{k}$

(c) $\mathbf{F} = y \sin z\,\mathbf{i} + x \sin z\,\mathbf{j} + (xy \cos z + 2z)\mathbf{k}$

(d) $\mathbf{F} = 2xy\mathbf{i} + (x^2 + 4yz)\mathbf{j} + 2y^2z\mathbf{k}$

(e) $\mathbf{F} = y \cos x \cos z\,\mathbf{i} + \sin x \cos z\,\mathbf{j} - y \sin x \sin z\,\mathbf{k}$.

Complete all five and check your findings with the next frame.

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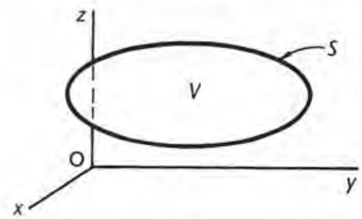
(a) No (b) Yes (c) Yes (d) No (e) Yes

▶

Divergence theorem (Gauss' theorem)



For a closed surface S, enclosing a region V in a vector field $\mathbf{F}$,

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

In general, this means that the volume integral (triple integral) on the left-hand side can be expressed as a surface integral (double integral) on the right-hand side. Let us work through one or two examples.

Example 1

Verify the divergence theorem for the vector field $\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ taken over the region bounded by the planes $z = 0$, $z = 2$, $x = 0$, $x = 1$, $y = 0$, $y = 3$.

Start off, as always, by sketching the relevant diagram, which is

....

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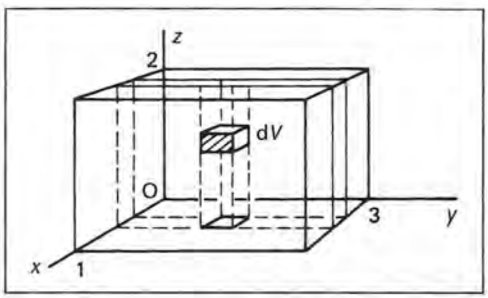

$dV = dx\,dy\,dz$

We have to show that

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

(a) To find $\int_V \operatorname{div} \mathbf{F}\,dV$

$$\operatorname{div} \mathbf{F} = \nabla\cdot\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\,\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(y) = 2x + 0 + 0 = 2x$$

$$\therefore \int_V \operatorname{div} \mathbf{F}\,dV = \int_V 2x\,dV = \iiint_V 2x\,dz\,dy\,dx$$

Inserting the limits and completing the integration

$$\int_V \operatorname{div} \mathbf{F}\,dV = \ldots\ldots\ldots\ldots$$

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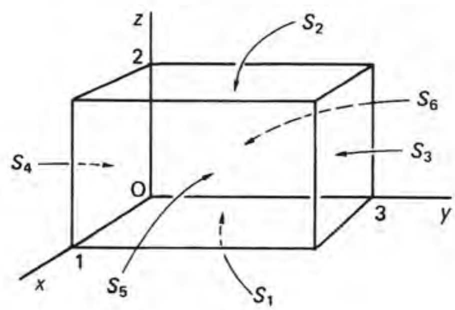
$$\boxed{\int_V \operatorname{div} \mathbf{F}\,\mathrm{d}V = 6}$$

Because

$$\int_V \operatorname{div} \mathbf{F}\,\mathrm{d}V = \int_0^1\int_0^3\int_0^2 2x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\int_0^3\Big[2xz\Big]_0^2\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1\Big[4xy\Big]_0^3\mathrm{d}x = \int_0^1 12x\,\mathrm{d}x = \Big[6x^2\Big]_0^1 = 6$$

Now we have to find $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$

(b) To find $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ i.e. $\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$



The enclosing surface S consists of six separate plane faces denoted as $S_1, S_2, \ldots, S_6$ as shown. We consider each face in turn.

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(1) S_1 (base): $z = 0$; $\hat{\mathbf{n}} = -\mathbf{k}$ (outwards and downwards)

$$\therefore\ \mathbf{F} = x^2\mathbf{i} + y\mathbf{k} \qquad \mathrm{d}S_1 = \mathrm{d}x\,\mathrm{d}y$$

$$\therefore\ \int_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_1}(x^2\mathbf{i} + y\mathbf{k})\cdot(-\mathbf{k})\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1\int_0^3(-y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1\left[-\frac{y^2}{2}\right]_0^3\mathrm{d}x$$
$$= -\frac{9}{2}$$

(2) S_2 (top): $z = 2$; $\hat{\mathbf{n}} = \mathbf{k}$ $\qquad \mathrm{d}S_2 = \mathrm{d}x\,\mathrm{d}y$

$$\therefore\ \int_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{9}{2}}$$

Because

$$\int_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} (x^2\mathbf{i} + 2\mathbf{j} + y\mathbf{k})\cdot(\mathbf{k})\,dy\,dx$$

$$= \int_0^1\int_0^3 y\,dy\,dx = \frac{9}{2}$$

So we go on.

(3) S_3 (right-hand end): $y = 3$; $\quad \hat{\mathbf{n}} = \mathbf{j} \quad dS_3 = dx\,dz$

$$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$$

$$\therefore \int_{S_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_3} (x^2\mathbf{i} + z\mathbf{j} + 3\mathbf{k})\cdot(\mathbf{j})\,dz\,dx$$

$$= \int_0^1\int_0^2 z\,dz\,dx$$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^2 dx = \int_0^1 2\,dx = 2$$

(4) S_4 (left-hand end): $y = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{j} \quad dS_4 = dx\,dz$

$$\therefore \int_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \dots\dots\dots\dots$$

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$$\boxed{-2}$$

Because

$$\int_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_4} (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k})\cdot(-\mathbf{j})\,dz\,dx = \int_0^1\int_0^2 (-z)\,dz\,dx$$

$$= \int_0^1 \left[-\frac{z^2}{2}\right]_0^2 dx = \int_0^1 (-2)\,dx = -2$$

Now for the remaining two sides S_5 and S_6.
Evaluate these in the same manner, obtaining

$$\int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \dots\dots\dots\dots$$

$$\int_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \dots\dots\dots\dots$$

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$$\boxed{\int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 6; \qquad \int_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0}$$

Check:

(5) S_5 (front): $x = 1$; $\hat{\mathbf{n}} = \mathbf{i}$ $\mathrm{d}S_5 = \mathrm{d}y\,\mathrm{d}z$

$$\therefore \int_{S_5} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_5} (\mathbf{i} + z\mathbf{j} + y\mathbf{k})\cdot(\mathbf{i})\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_5} 1\,\mathrm{d}y\,\mathrm{d}z = 6$$

(6) S_6 (back): $x = 0$; $\hat{\mathbf{n}} = -\mathbf{i}$ $\mathrm{d}S_6 = \mathrm{d}y\,\mathrm{d}z$

$$\therefore \int_{S_6} \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \iint_{S_6} (z\mathbf{j} + y\mathbf{k})\cdot(-\mathbf{i})\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_6} 0\,\mathrm{d}y\,\mathrm{d}z = 0$$

Now on to the next frame where we will collect our results together

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For the whole surface S we therefore have

$$\int_S \mathbf{F}\cdot\mathrm{d}S = -\frac{9}{2} + \frac{9}{2} + 2 - 2 + 6 + 0 = 6$$

and from our previous work in section (a) $\int_V \operatorname{div}\mathbf{F}\,\mathrm{d}V = 6$

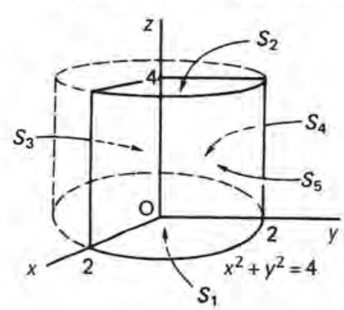
We have therefore verified as required that, in this example

$$\int_V \operatorname{div}\mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F}\cdot\mathrm{d}S$$

We have made rather a meal of this since we have set out the working in detail. In practice, the actual writing can often be considerably simplified. Let us move on to another example.

Example 2

Verify the Gauss divergence theorem for the vector field $\mathbf{F} = x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by the planes $z = 0$, $z = 4$, $x = 0$, $y = 0$ and the surface $x^2 + y^2 = 4$ in the first octant.



Divergence theorem

$$\int_V \operatorname{div}\mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F}\cdot\mathrm{d}S$$

S consists of five surfaces $S_1, S_2, \ldots, S_5$ as shown.

(a) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k})$

$= \ldots\ldots\ldots\ldots$

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$$\boxed{1+2z}$$

$$\therefore \int_V \operatorname{div} \mathbf{F}\,dV = \int_V \nabla \cdot \mathbf{F}\,dV = \iiint_V (1+2z)\,dx\,dy\,dz$$

Changing to cylindrical polar coordinates (ρ, ϕ, z)

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \qquad dV = \rho\,d\rho\,d\phi\,dz$$

Transforming the variables and inserting the appropriate limits, we then have

$$\int_V \operatorname{div} \mathbf{F}\,dV = \ldots\ldots\ldots\ldots$$

Finish it

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$$\boxed{20\pi}$$

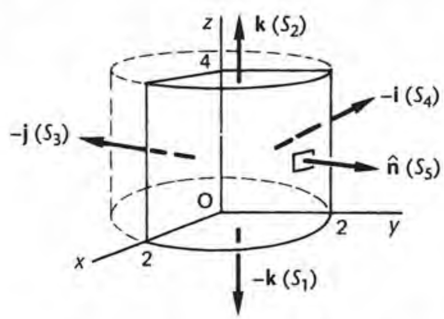
Because

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_0^{\pi/2}\int_0^2\int_0^4 (1+2z)\,dz\,\rho\,d\rho\,d\phi$$

$$= \int_0^{\pi/2}\int_0^2 \left[z+z^2\right]_0^4 \rho\,d\rho\,d\phi = \int_0^{\pi/2}\int_0^2 20\rho\,d\rho\,d\phi$$

$$= \int_0^{\pi/2}\left[10\rho^2\right]_0^2 d\phi = \int_0^{\pi/2} 40\,d\phi = 20\pi \qquad (1)$$

(b) Now we evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ over the closed surface.



The unit normal vector for each surface is shown.

$$\mathbf{F} = x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}$$

(1) S_1: $z = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{k} \qquad \mathbf{F} = x\mathbf{i} + 2\mathbf{j}$

$$\therefore \int_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_{S_1} (x\mathbf{i} + 2\mathbf{j})\cdot(-\mathbf{k})\,dS = 0$$

▶

(2) S_2: $z = 4$; $\quad \hat{\mathbf{n}} = \mathbf{k} \quad \mathbf{F} = x\mathbf{i} + 2\mathbf{j} + 16\mathbf{k}$

$$\therefore \int_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_2} (x\mathbf{i} + 2\mathbf{j} + 16\mathbf{k}) \cdot (\mathbf{k})\, dS = \int_{S_2} 16\, dS$$

$$= 16\left(\frac{\pi 4}{4}\right) = 16\pi$$

In the same way for S_3: $\displaystyle\int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$

and for S_4: $\displaystyle\int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$

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$$\boxed{\int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -16; \qquad \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0}$$

Because we have

(3) S_3: $y = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{j} \quad \mathbf{F} = x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}$

$$\therefore \int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_3} (x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{j})\, dS$$

$$= \int_{S_3} (-2)\, dS = -2(8) = -16$$

(4) S_4: $x = 0$; $\quad \hat{\mathbf{n}} = -\mathbf{i} \quad \mathbf{F} = 2\mathbf{j} + z^2\mathbf{k}$

$$\therefore \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_4} (2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{i})\, dS = 0$$

Finally we have

(5) S_5: $x^2 + y^2 - 4 = 0 \quad \hat{\mathbf{n}} = \dots\dots\dots\dots$

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$$\boxed{\hat{\mathbf{n}} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j})}$$

Because

$$x^2 + y^2 - 4 = 0 \qquad \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\mathbf{i} + y\mathbf{j}}{2}$$

$$\therefore \int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_5} (x\mathbf{i} + 2\mathbf{j} + z^2\mathbf{k}) \cdot \left(\frac{x\mathbf{i} + y\mathbf{j}}{2}\right) dS = \frac{1}{2}\int_{S_5} (x^2 + 2y)\, dS$$

Converting to cylindrical polar coordinates, this gives

$$\int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

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$$\boxed{4\pi + 16}$$

Because we have

$$\int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \frac{1}{2}\int_{S_5} (x^2 + 2y)\, dS$$

also $\quad x = 2\cos\phi; \qquad y = 2\sin\phi$

$\quad\quad\quad z = z; \qquad dS = 2\, d\phi\, dz$

$$\begin{aligned}
\therefore \int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \frac{1}{2}\int_0^4 \int_0^{\pi/2} (4\cos^2\phi + 4\sin\phi)\, 2\, d\phi\, dz \\
&= 2\int_0^4 \int_0^{\pi/2} \{(1 + \cos 2\phi) + 2\sin\phi\}\, d\phi\, dz \\
&= 2\int_0^4 \left[\left(\phi - \frac{\sin 2\phi}{2}\right) - 2\cos\phi\right]_0^{\pi/2} dz \\
&= 2\int_0^4 \left(\frac{\pi}{2} + 2\right) dz = 4\pi + 16
\end{aligned}$$

Therefore, for the total surface S

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0 + 16\pi - 16 + 0 + 4\pi + 16 = 20\pi \qquad (2)$$

$$\therefore \int_V \text{div}\, \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S} = 20\pi$$

Other examples are worked in much the same way. You will remember that, for a closed surface, the normal vectors at all points are drawn in an *outward* direction.

Now we move on to a further important theorem.

Stokes' theorem

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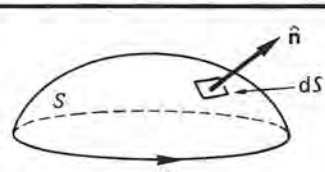


If $\mathbf{F}$ is a vector field existing over an open surface S and around its boundary, closed curve c, then

$$\int_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$$

This means that we can express a surface integral in terms of a line integral round the boundary curve.

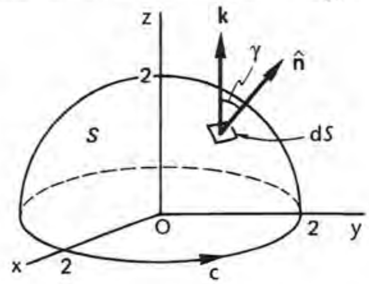
The proof of this theorem is rather lengthy and is to be found in the Appendix. Let us demonstrate its application in the following examples.

▶

Example 1

A hemisphere S is defined by $x^2 + y^2 + z^2 = 4$ $(z \geq 0)$. A vector field $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$ exists over the surface and around its boundary c.

Verify Stokes' theorem, that $\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$.



S: $x^2 + y^2 + z^2 - 4 = 0$
$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$
c is the circle $x^2 + y^2 = 4$.

(a) $$\oint_c \mathbf{F} \cdot d\mathbf{r} = \int_c (2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$
$$= \int_c (2y\,dx - x\,dy + xz\,dz)$$

Converting to polar coordinates

$x = 2\cos\theta$; $\quad y = 2\sin\theta$; $\quad z = 0$

$dx = -2\sin\theta\,d\theta$; $\quad dy = 2\cos\theta\,d\theta$; $\quad$ Limits $\theta = 0$ to 2π

Making the substitutions and completing the integral

$$\oint_c \mathbf{F} \cdot d\mathbf{r} = \dots\dots\dots\dots$$

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$$\boxed{\oint_c \mathbf{F} \cdot d\mathbf{r} = -12\pi}$$

Because

$$\oint_c \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (4\sin\theta[-2\sin\theta\,d\theta] - 2\cos\theta\,2\cos\theta\,d\theta)$$
$$= -4\int_0^{2\pi} (2\sin^2\theta + \cos^2\theta)\,d\theta$$
$$= -4\int_0^{2\pi} (1 + \sin^2\theta)\,d\theta = -2\int_0^{2\pi} (3 - \cos 2\theta)\,d\theta$$
$$= -2\left[3\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = -12\pi \qquad (1)$$

On to the next frame

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(b) Now we determine $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$

$$\int \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS \qquad \mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$$

$$\therefore \ \text{curl}\,\mathbf{F} = \dots\dots\dots\dots$$

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$$\boxed{\text{curl}\,\mathbf{F} = -z\mathbf{j} - 3\mathbf{k}}$$

Because

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y & -x & xz \end{vmatrix} = \mathbf{i}\,(0-0) - \mathbf{j}\,(z-0) + \mathbf{k}\,(-1-2) = -z\mathbf{j} - 3\mathbf{k}$$

Now $\hat{\mathbf{n}} = \dfrac{\nabla S}{|\nabla S|} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2}$

$$\text{Then } \int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_S (-z\mathbf{j} - 3\mathbf{k})\cdot\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{2}\right) dS$$

$$= \frac{1}{2}\int_S (-yz - 3z)\,dS$$

Expressing this in spherical polar coordinates and integrating, we get

$$\int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \dots\dots\dots\dots$$

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$$\boxed{-12\pi}$$

Because

$x = 2\sin\theta\,\cos\phi;\quad y = 2\sin\theta\sin\phi;\quad z = 2\cos\theta;\quad dS = 4\sin\theta\,d\theta\,d\phi$

$$\therefore \int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \frac{1}{2}\int_S\int(-2\sin\theta\sin\phi\,2\cos\theta - 6\cos\theta)4\,\sin\theta\,d\theta\,d\phi$$

$$= -4\int_0^{2\pi}\int_0^{\pi/2}(2\sin^2\theta\cos\theta\sin\phi + 3\sin\theta\,\cos\theta)\,d\theta\,d\phi$$

$$= -4\int_0^{2\pi}\left[\frac{2\sin^3\theta\sin\phi}{3} + \frac{3\sin^2\theta}{2}\right]_0^{\pi/2} d\phi$$

$$= -4\int_0^{2\pi}\left(\frac{2}{3}\sin\phi + \frac{3}{2}\right) d\phi = -12\pi \qquad (2)$$

So we have from our two results (1) and (2)

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Before we proceed with another example, let us clarify a point relating to the direction of unit normal vectors now that we are dealing with surfaces.

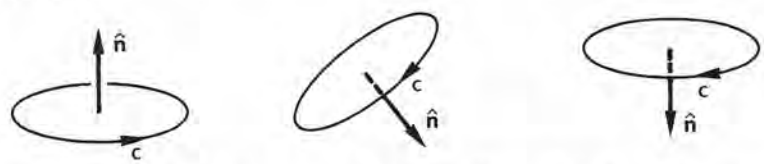
So on to the next frame

Direction of unit normal vectors to a surface S

69

When we were dealing with the divergence theorem, the normal vectors were drawn in a direction outward from the enclosed region.

With an open surface as we now have, there is in fact no inward or outward direction. With any general surface, a normal vector can be drawn in either of two opposite directions. To avoid confusion, a convention must therefore be agreed upon and the established rule is as follows.

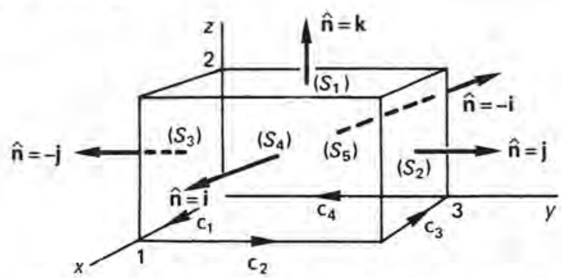


A unit normal $\hat{\mathbf{n}}$ is drawn perpendicular to the surface S at any point in the direction indicated by applying a right-handed screw sense to the direction of integration round the boundary c.

Having noted that point, we can now deal with the next example.

Example 2

A surface consists of five sections formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 3$, $z = 2$ in the first octant. If the vector field $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}$ exists over the surface and around its boundary, verify Stokes' theorem.



If we progress round the boundary along c_1, c_2, c_3, c_4 in an anti-clockwise manner, the normals to the surfaces will be as shown.

We have to verify that $\displaystyle\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$

(a) We will start off by finding $\displaystyle\oint_c \mathbf{F} \cdot d\mathbf{r}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\int \mathbf{F}\cdot d\mathbf{r} = \int (y\,dx + z^2\,dy + xy\,dz)}$$

(1) Along c_1: $y = 0;\quad z = 0;\quad dy = 0;\quad dz = 0$

$$\therefore \int_{c_1} \mathbf{F}\cdot d\mathbf{r} = \int (0+0+0) = 0$$

(2) Along c_2: $x = 1;\quad z = 0;\quad dx = 0;\quad dz = 0$

$$\therefore \int_{c_2} \mathbf{F}\cdot d\mathbf{r} = \int (0+0+0) = 0$$

In the same way

$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \dots\dots\dots\dots \quad \text{and} \quad \int_{c_4} \mathbf{F}\cdot d\mathbf{r} = \dots\dots\dots\dots$$

71

$$\boxed{\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = -3; \quad \int_{c_4} \mathbf{F}\cdot d\mathbf{r} = 0}$$

Because

(3) Along c_3: $y = 3;\quad z = 0;\quad dy = 0;\quad dz = 0$

$$\therefore \int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (3\,dx + 0 + 0) = \Big[3x\Big]_1^0 = -3$$

(4) Along c_4: $x = 0;\quad z = 0;\quad dx = 0;\quad dz = 0$

$$\therefore \int_{c_4} \mathbf{F}\cdot d\mathbf{r} = \int (0+0+0) = 0$$

$$\therefore \oint_c \mathbf{F}\cdot d\mathbf{r} = 0 + 0 - 3 + 0 = -3$$

$$\oint_c \mathbf{F}\cdot d\mathbf{r} = -3 \qquad (1)$$

(b) Now we have to find $\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$.

First we need an expression for curl $\mathbf{F}$.

$$\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + xy\mathbf{k}$$

$$\therefore \operatorname{curl}\mathbf{F} = \dots\dots\dots\dots$$

72

$$\boxed{\text{curl } \mathbf{F} = (x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}}$$

Because

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z^2 & xy \end{vmatrix}$$
$$= \mathbf{i}\,(x - 2z) - \mathbf{j}\,(y - 0) + \mathbf{k}\,(0 - 1) = (x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}$$

Then, for each section, we obtain $\int \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$

(1) S_1 (top): $\quad \hat{\mathbf{n}} = \mathbf{k}$

$$\therefore \int_{S_1} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

73

$$\boxed{-3}$$

Because

$$\int_{S_1} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_1} \{(x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\} \cdot (\mathbf{k})\, dS$$
$$= \int_{S_1} (-1)\, dS = -(\text{area of } S_1) = -3$$

Then, likewise

(2) S_2 (right-hand end): $\quad \hat{\mathbf{n}} = \mathbf{j}$

$$\therefore \int_{S_2} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_2} \{(x - 2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\} \cdot (\mathbf{j})\, dS$$
$$= \int_{S_2} (-y)\, dS$$

But $y = 3$ for this section

$$\therefore \int_{S_2} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{S_2} (-3)\, dS = (-3)(2) = -6$$

(3) S_3 (left-hand end): $\quad \hat{\mathbf{n}} = -\mathbf{j}$

$$\therefore \int_{S_3} \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \dots\dots\dots\dots$$

74

$$\boxed{0}$$

Because

$$\int_{S_3} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{S_3}\{(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\}\cdot(-\mathbf{j})\,\mathrm{d}S$$
$$= \int_{S_3} y\,\mathrm{d}S$$

But $y = 0$ over S_3

$$\therefore \int_{S_3} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0$$

Working in the same way

$$\int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \dots\dots\dots\dots;\quad \int_{S_5} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \dots\dots\dots\dots$$

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$$\boxed{\int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -6;\qquad \int_{S_5} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 12}$$

Because

(4) S_4 (front): $\hat{\mathbf{n}} = \mathbf{i}$

$$\therefore \int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{S_4}\{(x-2z)\mathbf{i} - y\mathbf{j} - \mathbf{k}\}\cdot(\mathbf{i})\,\mathrm{d}S$$
$$= \int_{S_4}(x-2z)\,\mathrm{d}S$$

But $x = 1$ over S_4

$$\therefore \int_{S_4} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^3\int_0^2 (1-2z)\,\mathrm{d}z\,\mathrm{d}y = \int_0^3\Big[z - z^2\Big]_0^2\,\mathrm{d}y$$
$$= \int_0^3 (-2)\,\mathrm{d}y = \Big[-2y\Big]_0^3 = -6$$

(5) S_5 (back): $\hat{\mathbf{n}} = -\mathbf{i}$ with $x = 0$ over S_5

Similar working to that above gives $\int_{S_5} \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 12$

Finally, collecting the five results together gives

$$\int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \dots\dots\dots\dots$$

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$$\boxed{\int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -3-6+0-6+12=-3} \qquad (2)$$

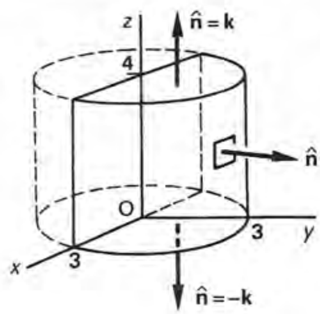
So, referring back to our result for section (a) we see that

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Of course we can, on occasions, make use of Stokes' theorem to lighten the working – as in the next example.

Example 3

A surface S consists of that part of the cylinder $x^2+y^2=9$ between $z=0$ and $z=4$ for $y\geq 0$ and the two semicircles of radius 3 in the planes $z=0$ and $z=4$. If $\mathbf{F}=z\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}$, evaluate $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ over the surface.



The surface S consists of three sections

(a) the curved surface of the cylinder
(b) the top and bottom semicircles.

We could therefore evaluate

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

over each of these separately.

However, we know by Stokes' theorem that

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \ldots\ldots\ldots\ldots$$

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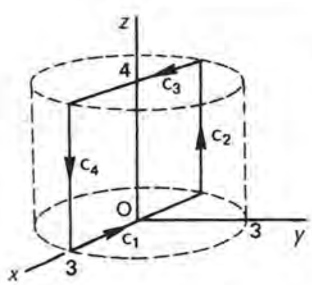
$$\boxed{\oint_c \mathbf{F}\cdot d\mathbf{r} \text{ where c is the boundary of } S}$$



$$\mathbf{F}=z\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}$$

$$\therefore\ \oint_c \mathbf{F}\cdot d\mathbf{r} = \oint_c (z\mathbf{i}+xy\mathbf{j}+xz\mathbf{k})\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)$$

$$= \oint_c (z\,dx+xy\,dy+xz\,dz)$$

Now we can work through this easily enough, taking c_1, c_2, c_3, c_4 in turn, and summing the results, which gives

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots$$

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$\boxed{-24}$

Here is the working in detail. $\oint_c \mathbf{F}\cdot d\mathbf{r} = \oint_c (z\,dx + xy\,dy + xz\,dz)$

(1) c_1: $y = 0$; $z = 0$; $dy = 0$; $dz = 0$

$$\int_{c_1} \mathbf{F}\cdot d\mathbf{r} = \int_{c_1} (0 + 0 + 0) = 0$$

(2) c_2: $x = -3$; $y = 0$; $dx = 0$; $dy = 0$

$$\int_{c_2} \mathbf{F}\cdot d\mathbf{r} = \int_{c_2} (0 + 0 - 3z\,dz) = \left[\frac{-3z^2}{2}\right]_0^4 = -24$$

(3) c_3: $y = 0$; $z = 4$; $dy = 0$; $dz = 0$

$$\int_{c_3} \mathbf{F}\cdot d\mathbf{r} = \int_{c_3} (4\,dx + 0 + 0) = \int_{-3}^{3} 4\,dx = 24$$

(4) c_4: $x = 3$; $y = 0$; $dx = 0$; $dy = 0$

$$\int_{c_4} \mathbf{F}\cdot d\mathbf{r} = \int_{c_4} (0 + 0 + 3z\,dz) = \left[\frac{3z^2}{2}\right]_4^0 = -24$$

Totalling up these four results, we have

$$\oint_c \mathbf{F}\cdot d\mathbf{r} = 0 - 24 + 24 - 24 = -24$$

But $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$ $\quad\therefore\quad \int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = -24$

This working is a good deal easier than calculating $\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ over the three separate surfaces direct.

So, if you have not already done so, make a note of Stokes' theorem:

$$\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Then on to the next section of the work

Green's theorem

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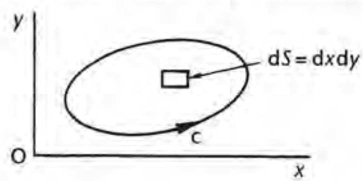
Green's theorem enables an integral over a plane area to be expressed in terms of a line integral round its boundary curve.

We showed in Programme 14 that, if P and Q are two single-valued functions of x and y, continuous over a plane surface S, and c is its boundary curve, then

$$\oint_c (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where the line integral is taken round c in an anticlockwise manner.

In vector terms, this becomes:



S is a two-dimensional space enclosed by a simple closed curve c.

$$dS = dx\,dy$$
$$d\mathbf{S} = \hat{\mathbf{n}}\,dS = \mathbf{k}\,dx\,dy$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ where $P = P(x, y)$ and $Q = Q(x, y)$ then

curl $\mathbf{F}$ = ...

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$$\boxed{\mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$$

Because

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$
$$= \mathbf{i}\left(0 - \frac{\partial Q}{\partial z}\right) - \mathbf{j}\left(0 - \frac{\partial P}{\partial z}\right) + \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

But in the x–y plane, $\dfrac{\partial Q}{\partial z} = \dfrac{\partial P}{\partial z} = 0$. $\therefore$ curl $\mathbf{F} = \mathbf{k}\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)$

So $\int \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ and in the x–y plane, $\hat{\mathbf{n}} = \mathbf{k}$

$$\therefore \int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot(\mathbf{k})\,dS = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\therefore \int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad (1)$$

Now by Stokes' theorem ....

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$$\boxed{\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}}$$

and, in this case, $\displaystyle\oint_c \mathbf{F} \cdot d\mathbf{r} = \oint_c (P\mathbf{i} + Q\mathbf{j}) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$

$$= \oint_c (P\,dx + Q\,dy)$$

$$\therefore \quad \oint_c \mathbf{F} \cdot d\mathbf{r} = \oint_c (P\,dx + Q\,dy) \qquad (2)$$

Therefore from (1) and (2)

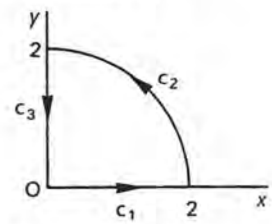
Stokes' theorem $\displaystyle\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$ in two dimensions becomes

Green's theorem $\displaystyle\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_c (P\,dx + Q\,dy)$

Example

Verify Green's theorem for the integral $\displaystyle\oint_c \{(x^2 + y^2)\,dx + (x + 2y)\,dy\}$ taken round the boundary curve c defined by

$$\begin{aligned} y &= 0 & 0 &\le x \le 2 \\ x^2 + y^2 &= 4 & 0 &\le x \le 2 \\ x &= 0 & 0 &\le y \le 2. \end{aligned}$$



Green's theorem: $\displaystyle\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_c (P\,dx + Q\,dy)$

In this case $(x^2 + y^2)\,dx + (x + 2y)\,dy = P\,dx + Q\,dy$

$\therefore\ P = x^2 + y^2$ and $Q = x + 2y$

We now take c_1, c_2, c_3 in turn.

(1) c_1: $y = 0$; $dy = 0$

$$\therefore \quad \int_{c_1} (P\,dx + Q\,dy) = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

(2) c_2: $x^2 + y^2 = 4 \quad \therefore\ y^2 = 4 - x^2 \quad \therefore\ y = (4 - x^2)^{1/2}$

$$x + 2y = x + 2(4 - x^2)^{1/2}$$

$$dy = \frac{1}{2}(4 - x^2)^{-1/2}(-2x)\,dx = \frac{-x}{\sqrt{4 - x^2}}\,dx$$

$$\therefore \quad \int_{c_2} (P\,dx + Q\,dy) = \ldots\ldots\ldots\ldots$$

Make any necessary substitutions and evaluate the line integral for c_2.

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$$\boxed{\pi - 4}$$

Because we have

$$\int_{c_2} (P\,dx + Q\,dy) = \int_{c_2} \left\{4 + (x + 2\sqrt{4 - x^2})\left(\frac{-x}{\sqrt{4 - x^2}}\right)\right\} dx$$
$$= \int_{c_2} \left\{4 - 2x - \frac{x^2}{\sqrt{4 - x^2}}\right\} dx$$

Putting $x = 2\sin\theta$, $\sqrt{4 - x^2} = 2\cos\theta \quad dx = 2\cos\theta\,d\theta$

Limits: $x = 2,\ \theta = \frac{\pi}{2};\quad x = 0,\ \theta = 0.$

$$\therefore \int_{c_2} (P\,dx + Q\,dy) = \int_{\pi/2}^{0} \left\{4 - 4\sin\theta - \frac{4\sin^2\theta}{2\cos\theta}\right\} 2\cos\theta\,d\theta$$
$$= 4\left[2\sin\theta - \sin^2\theta - \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\pi/2}^{0}$$
$$= 4\left[-\left(2 - 1 - \frac{\pi}{4}\right)\right] = \pi - 4$$

Finally

(3) c_3: $\quad x = 0;\quad dx = 0$

$$\therefore \int_{c_3} (P\,dx + Q\,dy) = \int_2^0 2y\,dy = \Big[y^2\Big]_2^0 = -4$$

$\therefore$ Collecting our three partial results

$$\oint_c (P\,dx + Q\,dy) = \frac{8}{3} + \pi - 4 - 4 = \pi - \frac{16}{3} \tag{1}$$

That is one part done. Now we have to evaluate $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$P = x^2 + y^2 \qquad \therefore\ \frac{\partial P}{\partial y} = 2y$$

$$Q = x + 2y \qquad \therefore\ \frac{\partial Q}{\partial x} = 1$$

$$\therefore \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_S (1 - 2y)\,dy\,dx$$

It will be more convenient to work in polar coordinates, so we make the substitutions

$$x = r\cos\theta;\quad y = r\sin\theta;\quad dS = dx\,dy = r\,dr\,d\theta$$

$$\therefore \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^{\pi/2}\int_0^2 (1 - 2r\sin\theta)r\,dr\,d\theta$$
$$= \ldots\ldots\ldots\ldots$$

Complete it

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$$\boxed{\pi - \frac{16}{3}}$$

Here it is:

$$\begin{aligned}\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/2}\int_0^2 (r - 2r^2 \sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{2r^3}{3}\sin\theta\right]_0^2 \mathrm{d}\theta \\ &= \int_0^{\pi/2}\left\{2 - \frac{16}{3}\sin\theta\right\}\mathrm{d}\theta \\ &= \left[2\theta + \frac{16}{3}\cos\theta\right]_0^{\pi/2} = \pi - \frac{16}{3} \qquad (2)\end{aligned}$$

So we have established once again that

$$\oint_c (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

And that brings us to the end of this particular Programme. We have covered a number of important sections, so check carefully down the **Revision summary** and the **Can You?** checklist, and then work through the **Test exercise** that follows. The **Further problems** provide valuable additional practice.

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Revision summary 18

1 *Line integrals*

(a) Scalar field V: $\int_c V\,\mathrm{d}\mathbf{r}$

The curve c is expressed in parametric form.

$\mathrm{d}\mathbf{r} = \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z$

(b) Vector field $\mathbf{F}$: $\int_c \mathbf{F}\cdot\mathrm{d}\mathbf{r}$

$$\begin{aligned}\mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\ \mathrm{d}\mathbf{r} &= \mathbf{i}\,\mathrm{d}x + \mathbf{j}\,\mathrm{d}y + \mathbf{k}\,\mathrm{d}z \\ \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z\end{aligned}$$

2 *Volume integrals*

$\mathbf{F}$ is a vector field; V a closed region with boundary surface S.

$$\int_V \mathbf{F}\,\mathrm{d}V = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \mathbf{F}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

▶

3 *Surface integrals* (surface defined by $\phi(x, y, z) = \text{constant}$)

(a) Scalar field $V(x, y, z)$:

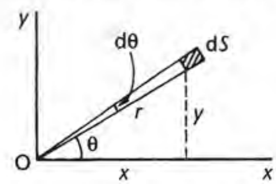
$$\int_S V\,\mathrm{d}\mathbf{S} = \int_S V\hat{\mathbf{n}}\,\mathrm{d}S; \qquad \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

(b) Vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$:

$$\int_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S; \qquad \hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

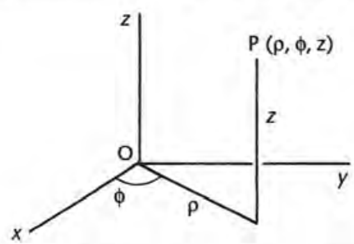
4 *Polar coordinates*

(a) Plane polar coordinates (r, θ)



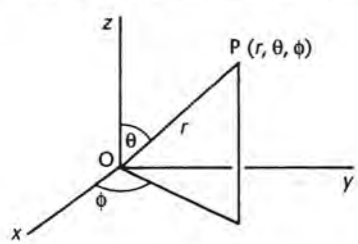
$$x = r\cos\theta; \quad y = r\sin\theta$$
$$\mathrm{d}S = r\,\mathrm{d}r\,\mathrm{d}\theta$$

(b) Cylindrical polar coordinates (ρ, ϕ, z)



$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$
$$\mathrm{d}S = \rho\,\mathrm{d}\phi\,\mathrm{d}z$$
$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$$

(c) Spherical polar coordinates (r, θ, ϕ)



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$\mathrm{d}S = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

5 *Conservative vector fields*

A vector field $\mathbf{F}$ is conservative if

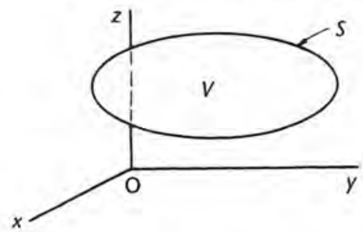
(a) $\oint_c \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$ for all closed curves

(b) $\text{curl}\,\mathbf{F} = 0$

(c) $\mathbf{F} = \text{grad}\,V$ where V is a scalar.

▶

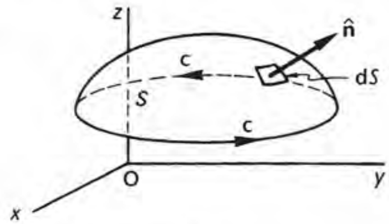
6 *Divergence theorem* (Gauss' theorem)



Closed surface S enclosing a region V in a vector field $\mathbf{F}$.

$$\int_V \operatorname{div} \mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

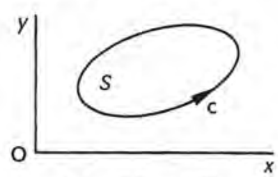
7 *Stokes' theorem*



An open surface S bounded by a simple closed curve c, then

$$\int_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

8 *Green's theorem*



The curve c is a simple closed curve enclosing a plane space S in the x–y plane. P and Q are functions of both x and y.

Then $\displaystyle\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_c (P\,dx + Q\,dy)$.

Can You?

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Checklist 18

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Evaluate the line integral of a scalar and a vector field in Cartesian coordinates? **1** to **20**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate the volume integral of a vector field? **21** to **27**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate the surface integral of a scalar and a vector field? 28 to 42

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine whether or not a vector field is a conservative vector field? 43 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Gauss' divergence theorem? 52 to 63

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Stokes' theorem? 64 to 68

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the direction of unit normal vectors to a surface? 69 to 78

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Green's theorem in the plane? 79 to 83

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 18

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1 If $V = x^3y + 2xy^2 + yz$, evaluate $\int_c V\,d\mathbf{r}$ between A (0, 0, 0) and B (2, 1, -3) along the curve with parametric equations $x = 2t$, $y = t^2$, $z = -3t^3$.

2 If $\mathbf{F} = x^2y^3\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$, evaluate $\int_c \mathbf{F}\cdot d\mathbf{r}$ along the curve $x = 3u^2$, $y = u$, $z = 2u^3$ between A (3, -1, -2) and B (3, 1, 2).

3 Evaluate $\int_V \mathbf{F}\,dV$ where $\mathbf{F} = 3\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$ and V is the region bounded by the planes $z = 0$, $z = 3$ and the surface $x^2 + y^2 = 4$.

4 If V is the scalar field $V = xyz^2$, evaluate $\int_S V\,d\mathbf{S}$ over the surface S defined by $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$ in the first octant.

5 Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$ over the surface S defined by $x^2 + y^2 + z^2 = 4$ for $z \geq 0$ and bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant where $\mathbf{F} = x\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$.

▶

6 Determine which of the following vector fields are conservative.

(a) $\mathbf{F} = (2xy + z)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (x + y^2)\mathbf{k}$

(b) $\mathbf{F} = (yz + 2y)\mathbf{i} + (xz + 2x)\mathbf{j} + (xy + 3)\mathbf{k}$

(c) $\mathbf{F} = (yz^2 + 3)\mathbf{i} + (xz^2 + 2)\mathbf{j} + (2xyz + 4)\mathbf{k}$.

7 By the use of the divergence theorem, determine $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} + 2\mathbf{k}$, taken over the region bounded by the planes $z = 0$, $z = 4$, $x = 0$, $y = 0$ and the surface $x^2 + y^2 = 9$ in the first octant.

8 A surface consists of parts of the planes $x = 0$, $x = 2$, $y = 0$, $y = 2$ and $z = 3 - y$ in the region $z \geq 0$. Apply Stokes' theorem to evaluate $\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$ over the surface where $\mathbf{F} = 2x\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$ where S lies in the $z = 0$ plane.

9 Verify Green's theorem in the plane for the integral

$$\oint_c \left\{ (xy^2 - 2x)\mathrm{d}x + (x + 2xy^2)\mathrm{d}y \right\}$$

where c is the square with vertices at (1, 1), (−1, 1), (−1, −1) and (1, −1).

Further problems 18

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1 If $V = x^2yz$, evaluate $\int_c V\,d\mathbf{r}$ between A (0, 0, 0) and B (6, 2, 4)

(a) along the straight lines c_1: (0, 0, 0) to (6, 0, 0)
c_2: (6, 0, 0) to (6, 2, 0)
c_3: (6, 2, 0) to (6, 2, 4)

(b) along the path c_4 having parametric equations $x = 3t$, $y = t$, $z = 2t$.

2 If $V = xy^2 + yz$, evaluate to one decimal place $\int_c V\,d\mathbf{r}$ along the curve c having parametric equations $x = 2t^2$, $y = 4t$, $z = 3t + 5$ between A (0, 0, 5) and B (8, 8, 11).

3 Evaluate to one decimal place the integral $\int_c (xyz + 4x^2y)\,d\mathbf{r}$ along the curve c with parametric equations $x = 2u$, $y = u^2$, $z = 3u^3$ between A (2, 1, 3) and B (4, 4, 24).

4 If $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + 3xyz\mathbf{k}$, evaluate $\int_c \mathbf{F} \cdot d\mathbf{r}$ between A (0,2,0) and B (3,6,1) where c has the parametric equations $x = 3u$, $y = 4u + 2$, $z = u^2$.

5 $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + yz\mathbf{k}$. Evaluate $\int_c \mathbf{F} \cdot d\mathbf{r}$ between A (2, 1, 2) and B (4, 4, 5) where c is the path with parametric equations $x = 2u$, $y = u^2$, $z = 3u - 1$.

6 A unit particle is moved in an anticlockwise manner round a circle with centre (0, 0, 4) and radius 2 in the plane $z = 4$ in a force field defined as $\mathbf{F} = (xy + z)\mathbf{i} + (2x + y)\mathbf{j} + (x + y + z)\mathbf{k}$. Find the work done.

7 Evaluate $\int_V \mathbf{F}\, dV$ where $\mathbf{F} = \mathbf{i} - y\mathbf{j} + \mathbf{k}$ and V is the region bounded by the plane $z = 0$ and the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$.

8 V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and the surfaces $y = 4 - x^2$ $(z \geq 0)$ and $y = 4 - z^2$ $(y \geq 0)$.

If $\mathbf{F} = 2\mathbf{i} + y^2\mathbf{j} - \mathbf{k}$, evaluate $\int_V \mathbf{F}\, dV$ throughout the region.

9 If $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 2x\mathbf{k}$, evaluate $\int_V \mathbf{F}\, dV$ where V is the region bounded by the planes $y = 0$, $z = 0$, $z = 4 - y$ $(z \geq 0)$ and the surface $x^2 + y^2 = 16$.

10 A scalar field $V = x + y$ exists over a surface S defined by

$$x^2 + y^2 + z^2 = 9,$$

bounded by the planes $x = 0$, $y = 0$, $z = 0$ in the first octant. Evaluate $\int_S V\, d\mathbf{S}$ over the curved surface.

11 A surface S is defined by $y^2 + z = 4$ and is bounded by the planes $x = 0$, $x = 3$, $y = 0$, $z = 0$ in the first octant. Evaluate $\int_S V\, d\mathbf{S}$ over this curved surface where V denotes the scalar field $V = x^2yz$.

12 Evaluate $\int_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$ over the surface S defined by $2x + 2y + z = 2$ and bounded by $x = 0$, $y = 0$, $z = 0$ in the first octant and where $\mathbf{F} = y^2\mathbf{i} + 2yz\mathbf{j} + xy\mathbf{k}$.

13 Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ over the hemisphere defined by $x^2 + y^2 + z^2 = 25$ with $z \geq 0$, where $\mathbf{F} = (x + y)\mathbf{i} - 2z\mathbf{j} + y\mathbf{k}$.

14 A vector field $\mathbf{F} = 2x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ exists over a surface S defined by $x^2 + y^2 + z^2 = 16$, bounded by the planes $z = 0$, $z = 3$, $x = 0$, $y = 0$. Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ over the stated curved surface.

▶

15 Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ is the vector field $x^2\mathbf{i} + 2z\mathbf{j} - y\mathbf{k}$, over the curved surface S defined by $x^2 + y^2 = 25$ and bounded by $z = 0$, $z = 6$, $y \geq 3$.

16 A region V is defined by the quartersphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, $y \geq 0$ and the planes $z = 0$, $y = 0$. A vector field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j} + \mathbf{k}$ exists throughout and on the boundary of the region. Verify the Gauss divergence theorem for the region stated.

17 A surface consists of parts of the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 1$ in the first octant. If $\mathbf{F} = y\mathbf{i} + x^2z\mathbf{j} + xy\mathbf{k}$, verify Stokes' theorem.

18 S is the surface $z = x^2 + y^2$ bounded by the planes $z = 0$ and $z = 4$. Verify Stokes' theorem for a vector field $\mathbf{F} = xy\mathbf{i} + x^3\mathbf{j} + xz\mathbf{k}$.

19 A vector field $\mathbf{F} = xy\mathbf{i} + z^2\mathbf{j} + xyz\mathbf{k}$ exists over the surfaces

$$x^2 + y^2 + z^2 = a^2,\ x = 0 \text{ and } y = 0$$

in the first octant. Verify Stokes' theorem that $\int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$.

20 A surface is defined by $z^2 = 4(x^2 + y^2)$ where $0 \leq z \leq 6$. If a vector field $\mathbf{F} = z\mathbf{i} + xy^2\mathbf{j} + x^2z\mathbf{k}$ exists over the surface and on the boundary circle c, show that $\oint_c \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$.

21 Verify Green's theorem in the plane for the integral

$$\oint_c \{(x - y)\,dx - (y^2 + xy)\,dy\}$$

where c is the circle with unit radius, centred on the origin.

Vector analysis 3

Frames 1 to 40

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the family of curves of constant coordinates for curvilinear coordinates
- Derive unit base vectors and scale factors in orthogonal curvilinear coordinates
- Obtain the element of arc ds and the element of volume dV in orthogonal curvilinear coordinates
- Obtain expressions for the operators grad, div and curl in orthogonal curvilinear coordinates

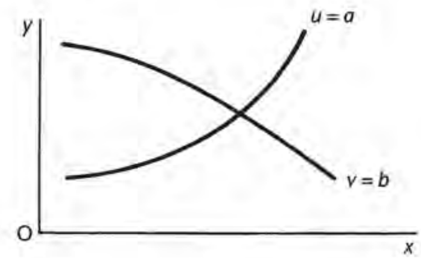
1

This short Programme is an extension of the two previous ones and may not be required for all courses. It can well be bypassed without adversely affecting the rest of the work.

Curvilinear coordinates

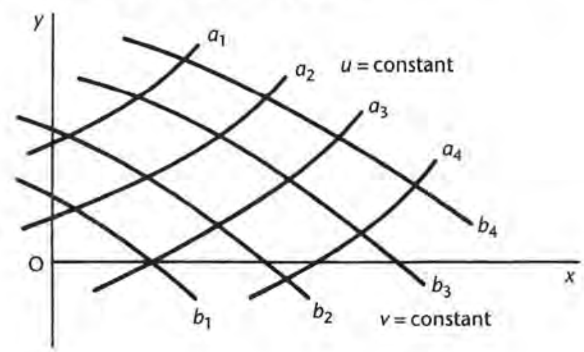
Let us consider two variables u and v, each of which is a function of x and y

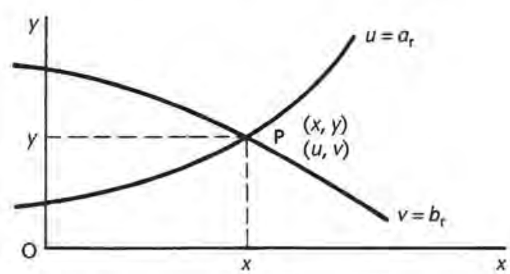
i.e. $u = f(x, y)$
$v = g(x, y)$



If u and v are each assigned a constant value a and b, the equations will, in general, define two intersecting curves.

If u and v are each given several such values, the equations define a network of curves covering the x–y plane.





A pair of curves $u = a_r$ and $v = b_r$ pass through each point in the plane. Hence, any point in the plane can be expressed in *rectangular coordinates* (x, y) or in *curvilinear coordinates* (u, v).

Let us see how this works out in an example, so move on

2

Example 1

Let us consider the case where $u = xy$ and $v = x^2 - y$.

(a) With $u = xy$, if we put $u = 4$, then $y = \dfrac{4}{x}$ and we can plot y against x to obtain the relevant curve.

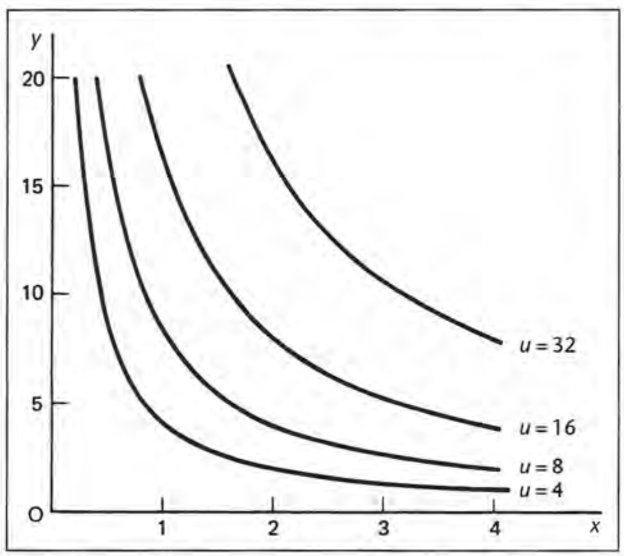
Similarly, putting $u = 8, 16, 32, \ldots$ we can build up a family of curves, all of the pattern $u = xy$.

x		0·5	1·0	2·0	3·0	4·0
y	$u = 4$	8	4	2	1·33	1·0
	$u = 8$	16	8	4	2·67	2
	$u = 16$	32	16	8	5·33	4
	$u = 32$	64	32	16	10·67	8

If we plot these on graph paper between $x = 0$ and $x = 4$ with a range of y from $y = 0$ to $y = 20$, we obtain

. . . .

3

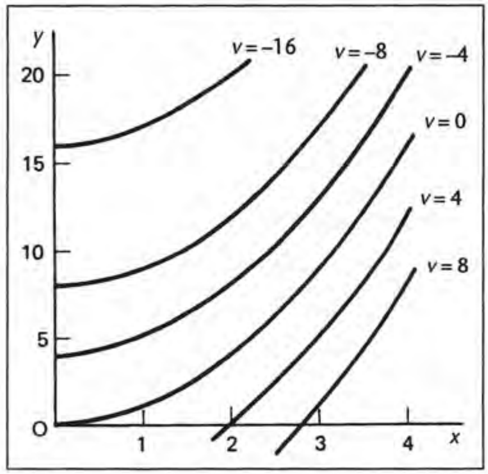


Note that each graph is labelled with its individual u-value.

(b) With $v = x^2 - y$, we proceed in just the same way. We rewrite the equation as $y = x^2 - v$; assign values such as 8, 4, 0, −4, −8, −12, −16, ... to v; and draw the relevant curve in each case. If we do that for $x = 0$ to $x = 4$ and limit the y-values to the range $y = 0$ to $y = 20$, we obtain the family of curves

. . . .

4



The table of function values is as follows.

x		0	1	2	3	4
y	$v = 8$	−8	−7	−4	1	8
	$v = 4$	−4	−3	0	5	12
	$v = 0$	0	1	4	9	16
	$v = -4$	4	5	8	13	20
	$v = -8$	8	9	12	17	24
	$v = -12$	12	13	16	21	28
	$v = -16$	16	17	20	25	32

Note again that we label each graph with its own v-value.

This again is a family of curves with the common pattern $v = x^2 - y$, the members being distinguished from each other by the value assigned to v in each case.

Now we draw both sets of curves on a common set of x–y axes, taking

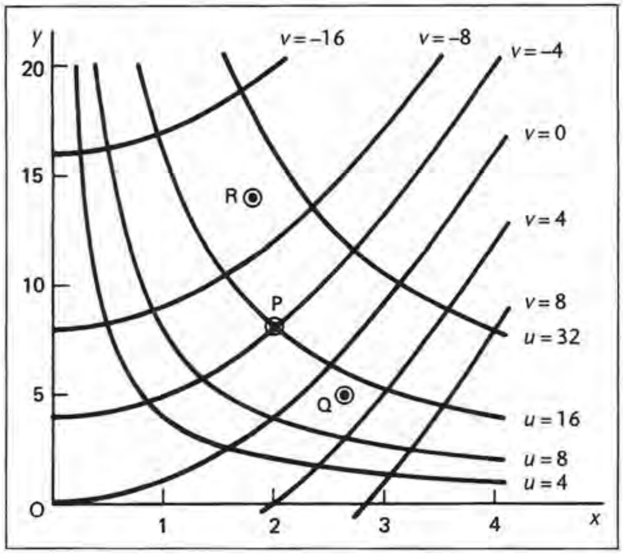
the range of x from $x = 0$ to $x = 4$

and the range of y from $y = 0$ to $y = 20$.

It is worthwhile taking a little time over it – and good practice!

When you have the complete picture, move on to the next frame

5



The position of any point in the plane can now be stated in two ways. For example, the point P has Cartesian rectangular coordinates $x = 2$, $y = 8$. It can also be stated in curvilinear coordinates $u = 16$, $v = -4$, for it is at the point of intersection of the two curves corresponding to $u = 16$ and $v = -4$.

Likewise, for the point Q, the position in rectangular coordinates is $x = 2{\cdot}65$, $y = 5{\cdot}0$ and for its position in curvilinear coordinates we must estimate it within the network. Approximate values are $u = 13$, $v = 2$.

Similarly, the curvilinear coordinates of R $(x = 1{\cdot}8, y = 14)$ are approximately

$$u = \dots\dots\dots\dots; \quad v = \dots\dots\dots\dots$$

6

$$\boxed{u = 26; \quad v = -11}$$

Their actual values are in fact $u = 25{\cdot}2$ and $v = -10{\cdot}76$.

Now let us deal with another example.

7

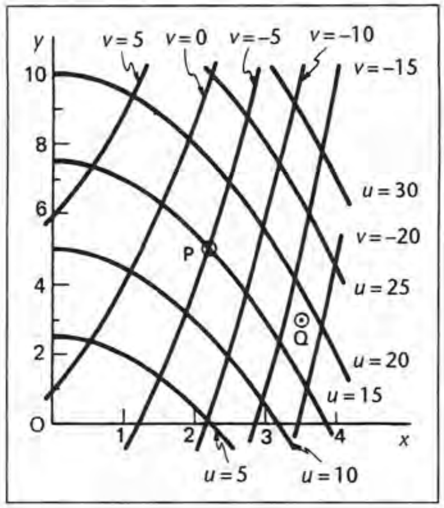
Example 2

If $u = x^2 + 2y$ and $v = y - (x+1)^2$, these can be rewritten as $y = \frac{1}{2}(u - x^2)$ and $y = v + (x+1)^2$. We can now plot the family of curves, say between $x = 0$ and $x = 4$, with $u = 5(5)30$ and $v = -20(5)5$, i.e. values of u from 5 to 30 at intervals of 5 units and values of v from -20 to 5 at intervals of 5 units.

The resulting network is easily obtained and appears as

....

8



For P, the rectangular coordinates are $(x = 2{\cdot}18, y = 5{\cdot}1)$ and the curvilinear coordinates are $(u = 15, v = -5)$.
For Q, the rectangular coordinates are ....
and the curvilinear coordinates are ....

9

Q: $(x = 3{\cdot}5, y = 3{\cdot}0)$; $(u = 18{\cdot}5, v = -17)$

Orthogonal curvilinear coordinates

If the coordinate curves for u and v forming the network cross at right angles, the system of coordinates is said to be *orthogonal*. The test for orthogonality is given by the dot product of the vectors formed from the partial derivatives. This is, if

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \text{ then } u \text{ and } v \text{ are orthogonal.}$$

Example 3

Given the curvilinear coordinates u and v where $u = xy$ and $v = x^2 - y^2$ then

u and v form a coordinate system that is ....

10

orthogonal

Because

$$u = xy \text{ so } \frac{\partial u}{\partial x} = y \text{ and } \frac{\partial u}{\partial y} = x,\ v = x^2 - y^2 \text{ so } \frac{\partial v}{\partial x} = 2x \text{ and } \frac{\partial v}{\partial y} = -2y.$$

Then $\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y} = 2xy - 2xy = 0$ and so u and v form a coordinate system that is orthogonal.

Example 4

Given the curvilinear coordinates u and v where $u = x^2 + 2y$ and $v = y - (x+1)^2$ then

u and v form a coordinate system that is ....

11

not orthogonal

Because

$$u = x^2 + 2y \text{ so } \frac{\partial u}{\partial x} = 2x \text{ and } \frac{\partial u}{\partial y} = 2,\ v = y - (x+1)^2 \text{ so } \frac{\partial v}{\partial x} = -2(x+1)$$

and $\dfrac{\partial v}{\partial y} = 1$.

Then

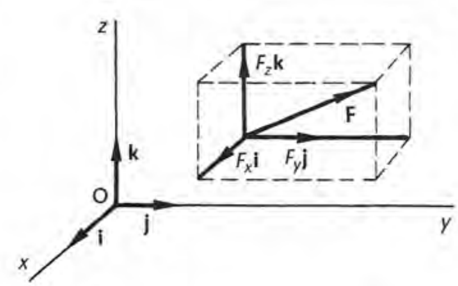
$\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y} = -4x(x+1) + 2 \neq 0$ and so u and v form a coordinate system that is not orthogonal.

Let us extend these ideas to three dimensions. Move on

Orthogonal coordinate systems in space

12

Any vector $\mathbf{F}$ can be expressed in terms of its components in three mutually perpendicular directions, which have normally been the directions of the coordinate axes, i.e.

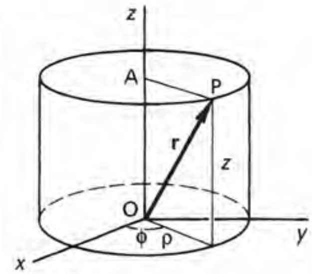


$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors parallel to the x, y, z axes respectively.

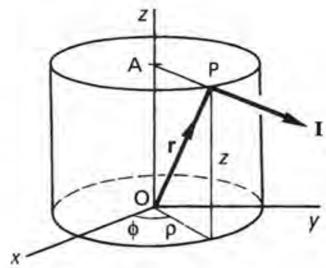
Situations can arise, however, where the directions of the unit vectors do not remain fixed, but vary from point to point in space according to prescribed conditions. Examples of this occur in cylindrical and spherical polar coordinates, with which we are already familiar.

1 *Cylindrical polar coordinates* (ρ, ϕ, z)
Let P be a point with cylindrical coordinates (ρ, ϕ, z) as shown. The position of P is a function of the three variables ρ, ϕ, z



(a) If ϕ and z remain constant and ρ varies, then P will move out along AP by an amount $\dfrac{\partial \mathbf{r}}{\partial \rho}$ and the unit vector **I** in this direction will be given by

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \rho}\right|$$

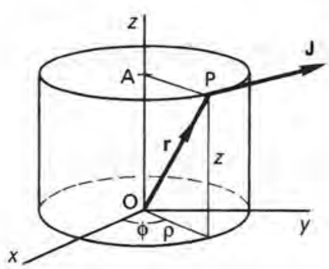


(b) If, instead, ρ and z remain constant and ϕ varies, P will move

....

13

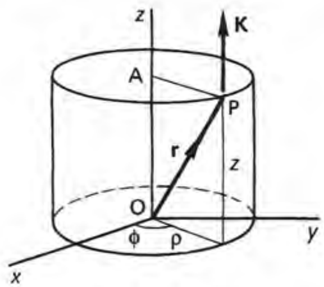
round the circle with AP as radius



$\dfrac{\partial \mathbf{r}}{\partial \phi}$ is therefore a vector along the tangent to the circle at P and the unit vector **J** at P will be given by

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

▶



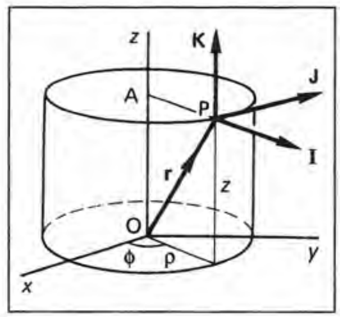
(c) Finally, if ρ and ϕ remain constant and z increases, the vector $\dfrac{\partial \mathbf{r}}{\partial z}$ will be parallel to the z-axis and the unit vector $\mathbf{K}$ in this direction will be given by

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z} \Big/ \left|\frac{\partial \mathbf{r}}{\partial z}\right|$$

Putting our three unit vectors on to one diagram, we have

....

14



Note that $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are mutually perpendicular and form a right-handed set. But note also that, unlike the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the Cartesian system, the unit vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, or *base vectors* as they are called, are not fixed in directions, but change as the position of P changes.

So we have, for cylindrical polar coordinates

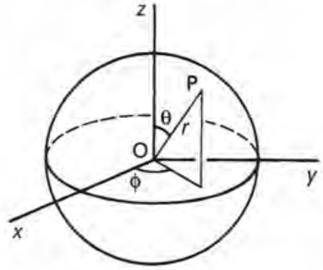
$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \rho}\right|$$
$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$
$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z} \Big/ \left|\frac{\partial \mathbf{r}}{\partial z}\right|$$

If $\mathbf{F}$ is a vector associated with P, then $\mathbf{F(r)} = F_\rho \mathbf{I} + F_\phi \mathbf{J} + F_z \mathbf{K}$ where F_ρ, F_ϕ, F_z are the components of $\mathbf{F}$ in the directions of the unit base vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$.

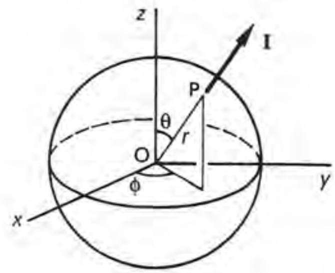
Now let us attend to spherical coordinates in the same way.

15

2 *Spherical polar coordinates* $(r,\ \theta,\ \phi)$

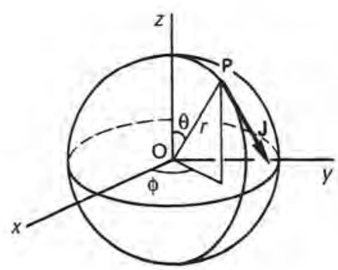


P is a function of the three variables r, θ, ϕ.



(a) If θ and ϕ remain constant and r increases, P moves outwards in the direction OP. $\dfrac{\partial \mathbf{r}}{\partial r}$ is thus a vector normal to the surface of the sphere at P and the unit vector $\mathbf{I}$ in that direction is therefore

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial r} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial r}\right|$$



(b) If r and ϕ remain constant and θ increases, P will move along the 'meridian' through P, i.e. $\dfrac{\partial \mathbf{r}}{\partial \theta}$ is a tangent vector to this circle at P and the unit vector $\mathbf{J}$ is given by

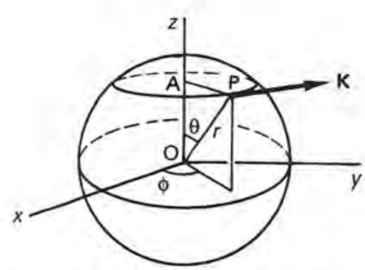
$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \theta} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial \theta}\right|$$

(c) If r and θ remain constant and ϕ increases, P will move

....

16

along the circle through P perpendicular to the z-axis

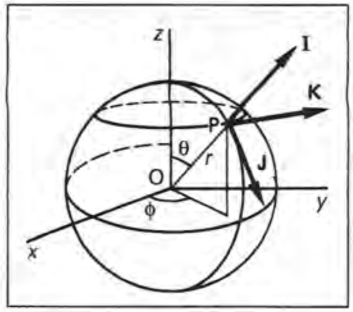


$\dfrac{\partial \mathbf{r}}{\partial \phi}$ is therefore a tangent vector at P and the unit vector $\mathbf{K}$ in this direction is given by

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \phi} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

So, putting the three results on one diagram, we have ....

17



Once again, the three unit vectors at P (base vectors) are mutually perpendicular and form a right-handed set. Their directions in space, however, change as the position of P changes.

A vector $\mathbf{F}$ associated with P can therefore be expressed as $\mathbf{F} = F_r\mathbf{I} + F_\theta\mathbf{J} + F_\phi\mathbf{K}$ where F_r, F_θ, F_ϕ are the components of $\mathbf{F}$ in the directions of the base vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$.

Both cylindrical and spherical polar coordinate systems are

. . . .

18

orthogonal

Scale factors

Collecting the recent results together, we have:

1 For cylindrical polar coordinates, the unit base vectors are

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial\rho}\Big/\left|\frac{\partial\mathbf{r}}{\partial\rho}\right| = \frac{1}{h_\rho}\frac{\partial\mathbf{r}}{\partial\rho} \qquad \text{where } h_\rho = \left|\frac{\partial\mathbf{r}}{\partial\rho}\right|$$

$$\mathbf{J} = \frac{\partial\mathbf{r}}{\partial\phi}\Big/\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{1}{h_\phi}\frac{\partial\mathbf{r}}{\partial\phi} \qquad \text{where } h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right|$$

$$\mathbf{K} = \frac{\partial\mathbf{r}}{\partial z}\Big/\left|\frac{\partial\mathbf{r}}{\partial z}\right| = \frac{1}{h_z}\frac{\partial\mathbf{r}}{\partial z} \qquad \text{where } h_z = \left|\frac{\partial\mathbf{r}}{\partial z}\right|$$

2 For spherical polar coordinates, the unit base vectors are

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial r}\Big/\left|\frac{\partial\mathbf{r}}{\partial r}\right| = \frac{1}{h_r}\frac{\partial\mathbf{r}}{\partial r} \qquad \text{where } h_r = \left|\frac{\partial\mathbf{r}}{\partial r}\right|$$

$$\mathbf{J} = \frac{\partial\mathbf{r}}{\partial\theta}\Big/\left|\frac{\partial\mathbf{r}}{\partial\theta}\right| = \frac{1}{h_\theta}\frac{\partial\mathbf{r}}{\partial\theta} \qquad \text{where } h_\theta = \left|\frac{\partial\mathbf{r}}{\partial\theta}\right|$$

$$\mathbf{K} = \frac{\partial\mathbf{r}}{\partial\phi}\Big/\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{1}{h_\phi}\frac{\partial\mathbf{r}}{\partial\phi} \qquad \text{where } h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right|$$

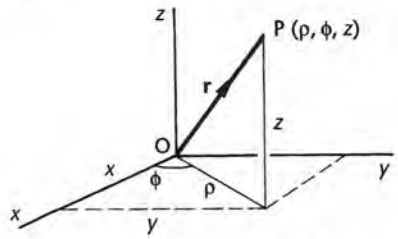
In each case, h is called the *scale factor*.

Move on

19 Scale factors for coordinate systems

1 *Rectangular coordinates* (x, y, z)
With rectangular coordinates, $h_x = h_y = h_z = 1$.

2 *Cylindrical coordinates* (ρ, ϕ, z)



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\therefore\ \mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\,\mathbf{k}$$

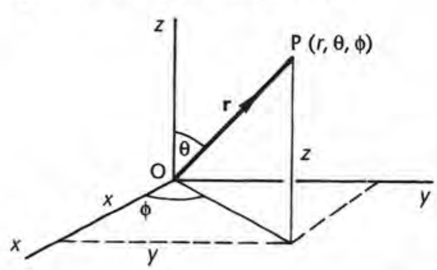
$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial\rho}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\rho}\right| = \frac{1}{h_\rho}\frac{\partial\mathbf{r}}{\partial\rho} \qquad h_\rho = \left|\frac{\partial\mathbf{r}}{\partial\rho}\right| = |\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}|$$
$$= (\cos^2\phi + \sin^2\phi)^{1/2} = 1$$
$$\therefore\ h_\rho = 1$$

$$\mathbf{J} = \frac{\partial\mathbf{r}}{\partial\phi}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{1}{h_\phi}\frac{\partial\mathbf{r}}{\partial\phi} \qquad h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = |-\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}|$$
$$= (\rho^2\sin^2\phi + \rho^2\cos^2\phi)^{1/2} = \rho$$
$$\therefore\ h_\phi = \rho$$

$$\mathbf{K} = \frac{\partial\mathbf{r}}{\partial z}\bigg/\left|\frac{\partial\mathbf{r}}{\partial z}\right| = \frac{1}{h_z}\frac{\partial\mathbf{r}}{\partial z} \qquad h_z = \left|\frac{\partial\mathbf{r}}{\partial z}\right| = |\mathbf{k}| = 1$$
$$\therefore\ h_z = 1$$

$$\therefore\ h_\rho = 1;\ h_\phi = \rho;\ h_z = 1$$

3 *Spherical coordinates* (r, θ, ϕ)



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\therefore\ \mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

Then working as before

$h_r = \dots\dots\dots\dots;\quad h_\theta = \dots\dots\dots\dots;\quad h_\phi = \dots\dots\dots\dots$

$$\boxed{h_r = 1; \quad h_\theta = r; \quad h_\phi = r\sin\theta}$$

Because

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial r}\bigg/\left|\frac{\partial\mathbf{r}}{\partial r}\right| = \frac{1}{h_r}\frac{\partial\mathbf{r}}{\partial r}$$

$$\begin{aligned} h_r = \left|\frac{\partial\mathbf{r}}{\partial r}\right| &= |\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}| \\ &= (\sin^2\theta\cos^2\phi + \sin^2\theta\ \sin^2\phi + \cos^2\theta)^{1/2} \\ &= (\sin^2\theta + \cos^2\theta)^{1/2} = 1 \end{aligned}$$

$$\therefore\ h_r = 1$$

$$\mathbf{J} = \frac{\partial\mathbf{r}}{\partial\theta}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\theta}\right| = \frac{1}{h_\theta}\frac{\partial\mathbf{r}}{\partial\theta}$$

$$\begin{aligned} h_\theta = \left|\frac{\partial\mathbf{r}}{\partial\theta}\right| &= |\,r\ \cos\theta\ \cos\phi\,\mathbf{i} + r\ \cos\theta\ \sin\phi\,\mathbf{j} - r\ \sin\theta\,\mathbf{k}\,| \\ &= (r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta)^{1/2} \\ &= (r^2\cos^2\theta + r^2\sin^2\theta)^{1/2} = r \end{aligned}$$

$$\therefore\ h_\theta = r$$

$$\mathbf{K} = \frac{\partial\mathbf{r}}{\partial\phi}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{1}{h_\phi}\frac{\partial\mathbf{r}}{\partial\phi}$$

$$\begin{aligned} h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right| &= |-r\sin\theta\sin\phi\,\mathbf{i} + r\sin\theta\cos\phi\,\mathbf{j}\,| \\ &= (r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi)^{1/2} \\ &= (r^2\sin^2\theta)^{1/2} = r\sin\theta \end{aligned}$$

$$\therefore\ h_\phi = r\sin\theta$$

$$\therefore\ h_r = 1; \quad h_\theta = r; \quad h_\phi = r\sin\theta$$

So: (a) for cylindrical coordinates

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial\rho}; \quad \mathbf{J} = \frac{1}{\rho}\frac{\partial\mathbf{r}}{\partial\phi}; \quad \mathbf{K} = \frac{\partial\mathbf{r}}{\partial z}$$

(b) for spherical coordinates

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial r}; \quad \mathbf{J} = \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}; \quad \mathbf{K} = \frac{1}{r\sin\theta}\frac{\partial\mathbf{r}}{\partial\phi}$$

General curvilinear coordinate system (u, v, w)

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Any system of coordinates can be treated in like manner to obtain expressions for the appropriate unit vectors **I**, **J**, **K**.

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial u} \Big/ \left|\frac{\partial \mathbf{r}}{\partial u}\right|; \quad \mathbf{J} = \frac{\partial \mathbf{r}}{\partial v} \Big/ \left|\frac{\partial \mathbf{r}}{\partial v}\right|; \quad \mathbf{K} = \frac{\partial \mathbf{r}}{\partial w} \Big/ \left|\frac{\partial \mathbf{r}}{\partial w}\right|$$

These unit vectors are not always at right angles to each other.

If they are mutually perpendicular, the coordinate system is

....

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orthogonal

Unit vectors **I**, **J**, **K** are orthogonal if

$$\mathbf{I} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{I} = 0$$

Exercise

Determine the unit base vectors in the directions of the following vectors and determine whether the vectors are orthogonal.

1 $\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
$2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
$-2\mathbf{i} + \mathbf{j} + \mathbf{k}$

2 $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
$\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$-10\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

3 $4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
$3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
$\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

4 $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
$6\mathbf{i} + \mathbf{j} - \mathbf{k}$

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The results are as follows:

1 $\mathbf{I} = \dfrac{1}{\sqrt{21}}(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$; $\mathbf{J} = \dfrac{1}{\sqrt{14}}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$;

$\mathbf{K} = \dfrac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{I} \cdot \mathbf{J} = 0$; $\mathbf{J} \cdot \mathbf{K} = 0$; $\mathbf{K} \cdot \mathbf{I} = 0$ $\therefore$ orthogonal

2 $\mathbf{I} = \dfrac{1}{\sqrt{17}}(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$; $\mathbf{J} = \dfrac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$;

$\mathbf{K} = \dfrac{1}{\sqrt{153}}(-10\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

$\mathbf{I} \cdot \mathbf{J} = 0$; $\mathbf{J} \cdot \mathbf{K} = 0$; $\mathbf{K} \cdot \mathbf{I} = 0$ $\therefore$ orthogonal

3 $\mathbf{I} = \dfrac{1}{\sqrt{21}}(4\mathbf{i} + 2\mathbf{j} - \mathbf{k})$; $\mathbf{J} = \dfrac{1}{\sqrt{38}}(3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$;

$\mathbf{K} = \dfrac{1}{\sqrt{41}}(\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

$\mathbf{I}\cdot\mathbf{J} = 0$; $\mathbf{J}\cdot\mathbf{K} \neq 0$ $\quad\therefore$ not orthogonal

4 $\mathbf{I} = \dfrac{1}{\sqrt{14}}(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$; $\mathbf{J} = \dfrac{1}{\sqrt{19}}(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$;

$\mathbf{K} = \dfrac{1}{\sqrt{38}}(6\mathbf{i} + \mathbf{j} - \mathbf{k})$

$\mathbf{I}\cdot\mathbf{J} = 0$; $\mathbf{J}\cdot\mathbf{K} = 0$; $\mathbf{K}\cdot\mathbf{I} \neq 0$ $\quad\therefore$ not orthogonal

Transformation equations

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In general coordinates, the transformation equations are of the form

$x = f(u, v, w)$; $y = g(u, v, w)$; $z = h(u, v, w)$

where the functions f, g, h are continuous and single-valued, and whose partial derivatives are continuous.

Then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(u, v, w)\mathbf{i} + g(u, v, w)\mathbf{j} + h(u, v, w)\mathbf{k}$ and coordinate curves can be formed by keeping two of the three variables constant.

$$\text{Now } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \therefore\ \mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{r}}{\partial w}\mathrm{d}w \qquad (1)$$

$\dfrac{\partial \mathbf{r}}{\partial u}$ is a tangent vector to the u-coordinate curve at P

$\dfrac{\partial \mathbf{r}}{\partial v}$ is a tangent vector to the v-coordinate curve at P

$\dfrac{\partial \mathbf{r}}{\partial w}$ is a tangent vector to the w-coordinate curve at P

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial u}\Big/\left|\frac{\partial \mathbf{r}}{\partial u}\right| \quad \therefore\ \frac{\partial \mathbf{r}}{\partial u} = h_u\mathbf{I} \text{ where } h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial v}\Big/\left|\frac{\partial \mathbf{r}}{\partial v}\right| \quad \therefore\ \frac{\partial \mathbf{r}}{\partial v} = h_v\mathbf{J} \text{ where } h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial w}\Big/\left|\frac{\partial \mathbf{r}}{\partial w}\right| \quad \therefore\ \frac{\partial \mathbf{r}}{\partial w} = h_w\mathbf{K} \text{ where } h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|$$

Then (1) above becomes

$$\mathrm{d}\mathbf{r} = h_u\mathrm{d}u\,\mathbf{I} + h_v\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K}$$

where, as before, h_u, h_v, h_w are the scale factors.

Element of arc d*s* and element of volume d*V* in orthogonal curvilinear coordinates

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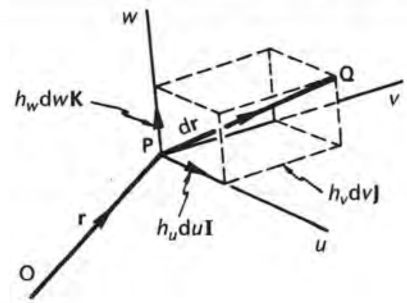

(a) *Element of arc* ds

Element of arc ds from P to Q is given by

$$\mathrm{d}\mathbf{r} = h_u\,\mathrm{d}u\,\mathbf{I} + h_v\,\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K}$$

$$\therefore\ \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = (h_u\mathrm{d}u\,\mathbf{I} + h_v\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K})\cdot(h_u\mathrm{d}u\,\mathbf{I} + h_v\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K})$$

$$\therefore\ \mathrm{d}s^2 = h_u^2\mathrm{d}u^2 + h_v^2\mathrm{d}v^2 + h_w^2\mathrm{d}w^2$$

$$\therefore\ \mathrm{d}s = (h_u^2\mathrm{d}u^2 + h_v^2\mathrm{d}v^2 + h_w^2\mathrm{d}w^2)^{1/2}$$

(b) *Element of volume* dV

$$\mathrm{d}V = (h_u\mathrm{d}u\,\mathbf{I})\cdot(h_v\mathrm{d}v\,\mathbf{J}\times h_w\mathrm{d}w\,\mathbf{K})$$
$$= (h_u\mathrm{d}u\,\mathbf{I})\cdot(h_v\mathrm{d}v\ h_w\mathrm{d}w\,\mathbf{I}) = h_u\mathrm{d}u\ h_v\mathrm{d}v\ h_w\mathrm{d}w$$
$$\therefore\ \mathrm{d}V = h_u\,h_v\,h_w\,\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

Note also that

$$\mathrm{d}V = \left|\frac{\partial\mathbf{r}}{\partial u}\cdot\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\right|\mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w$$
$$= \frac{\partial(x, y, z)}{\partial(u, v, w)}\ \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w$$

where $\dfrac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$ is the Jacobian of the transformation.

Grad, div and curl in orthogonal curvilinear coordinates

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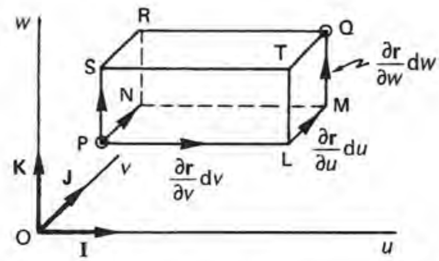
(a) *Grad* V (∇V)



Let a scalar field V exist in space and let $\mathrm{d}V$ be the change in V from P to Q. If the position vector of P is $\mathbf{r}$ then that of Q is $\mathbf{r} + d\mathbf{r}$.

Then $\mathrm{d}V = \dfrac{\partial V}{\partial u}\mathrm{d}u + \dfrac{\partial V}{\partial v}\mathrm{d}v + \dfrac{\partial V}{\partial w}\mathrm{d}w$

Let grad $V = \nabla V = (\nabla V)_u\mathbf{I} + (\nabla V)_v\mathbf{J} + (\nabla V)_w\mathbf{K}$

where $(\nabla V)_{u,v,w}$ are the components of grad V in the u, v, w directions.

Also $\quad \mathrm{d}\mathbf{r} = \dfrac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \dfrac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}v + \dfrac{\partial \mathbf{r}}{\partial w}\,\mathrm{d}w$

But $\quad \dfrac{\partial \mathbf{r}}{\partial u} = \left|\dfrac{\partial \mathbf{r}}{\partial u}\right|\mathbf{I} = h_u\,\mathbf{I}; \qquad \dfrac{\partial \mathbf{r}}{\partial v} = \left|\dfrac{\partial \mathbf{r}}{\partial v}\right|\mathbf{J} = h_v\,\mathbf{J};$

and $\quad \dfrac{\partial \mathbf{r}}{\partial w} = \left|\dfrac{\partial \mathbf{r}}{\partial w}\right|\mathbf{K} = h_w\,\mathbf{K}.$

$\therefore\ \mathrm{d}\mathbf{r} = h_u\,\mathrm{d}u\,\mathbf{I} + h_v\,\mathrm{d}v\,\mathbf{J} + h_w\,\mathrm{d}w\,\mathbf{K}$

We have previously established that $\mathrm{d}V = \text{grad } V \cdot \mathrm{d}\mathbf{r}$

$$\begin{aligned}\therefore\ \mathrm{d}V &= \{(\nabla V)_u\mathbf{I} + (\nabla V)_v\mathbf{J} + (\nabla V)_w\mathbf{K}\} \cdot \\ &\qquad\qquad \{h_u\mathrm{d}u\mathbf{I} + h_v\mathrm{d}v\mathbf{J} + h_w\mathrm{d}w\mathbf{K}\} \\ &= (\nabla V)_u h_u\,\mathrm{d}u + (\nabla V)_v h_v\,\mathrm{d}v + (\nabla V)_w h_w\,\mathrm{d}w\end{aligned}$$

But $\quad \mathrm{d}V = \dfrac{\partial V}{\partial u}\mathrm{d}u + \dfrac{\partial V}{\partial v}\mathrm{d}v + \dfrac{\partial V}{\partial w}\mathrm{d}w$

▶

$\therefore$ Equating coefficients, we then have

$$\frac{\partial V}{\partial u} = (\nabla V)_u h_u \qquad \therefore\ (\nabla V)_u = \frac{1}{h_u}\frac{\partial V}{\partial u}$$

$$\frac{\partial V}{\partial v} = (\nabla V)_v h_v \qquad \therefore\ (\nabla V)_v = \frac{1}{h_v}\frac{\partial V}{\partial v}$$

$$\frac{\partial V}{\partial w} = (\nabla V)_w h_w \qquad \therefore\ (\nabla V)_w = \frac{1}{h_w}\frac{\partial V}{\partial w}$$

$$\therefore\ \text{grad } V = \nabla V = \frac{1}{h_u}\frac{\partial V}{\partial u}\mathbf{I} + \frac{1}{h_v}\frac{\partial V}{\partial v}\mathbf{J} + \frac{1}{h_w}\frac{\partial V}{\partial w}\mathbf{K}$$

$$\text{i.e. grad operator } \nabla = \frac{\mathbf{I}}{h_u}\frac{\partial}{\partial u} + \frac{\mathbf{J}}{h_v}\frac{\partial}{\partial v} + \frac{\mathbf{K}}{h_w}\frac{\partial}{\partial w}$$

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Other results we state without proof.

(b) *Div* $\mathbf{F}$ $(\nabla \cdot \mathbf{F})$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

$$= \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right\}$$

Example 1

Show that the curvilinear expression for div $\mathbf{F}$ agrees with the earlier definition in Cartesian coordinates.

In Cartesian coordinates x, y, z we have $h_x = h_y = h_z = \dots\dots\dots\dots$ so that

$$\text{div } \mathbf{F} = \dots\dots\dots\dots$$

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$$\boxed{\begin{array}{l} h_x = h_y = h_z = 1 \text{ so that} \\ \text{div } \mathbf{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z} \end{array}}$$

(c) *Curl* $\mathbf{F}$ $(\nabla \times \mathbf{F})$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{I} & h_v\mathbf{J} & h_w\mathbf{K} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Example 2

Show that the curvilinear expression for curl $\mathbf{F}$ agrees with the earlier definition in Cartesian coordinates.

In Cartesian coordinates x, y, z we have $h_x = h_y = h_z = \dots\dots\dots\dots$ and $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K} = \dots$, $\dots$, $\dots$ so that

$$\text{curl } \mathbf{F} = \dots\dots\dots\dots$$

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$h_x = h_y = h_z = 1$ and $\mathbf{I}, \mathbf{J}, \mathbf{K} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ so that

$$\text{curl } \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Because in Cartesians

$h_x = h_y = h_z = 1$ and $\mathbf{I}, \mathbf{J}, \mathbf{K} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ so that

$$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{I} & h_v\mathbf{J} & h_w\mathbf{K} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

$$= \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

(d) *Div grad V* $(\nabla^2 V)$

$$\text{div grad } V = \nabla \cdot (\nabla V) = \nabla^2 V$$

$$= \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u} \cdot \frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v} \cdot \frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w} \cdot \frac{\partial V}{\partial w}\right)\right\}$$

Example 3

Show that the curvilinear expression for $\nabla^2 V$ agrees with the earlier definition in Cartesian coordinates.

In Cartesian coordinates x, y, z we have $h_x = h_y = h_z = \dots\dots\dots$ so that

$$\nabla^2 V = \dots\dots\dots\dots$$

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$h_x = h_y = h_z = 1$ so that

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Let's try another example, this time in coordinates other than Cartesians.

Example 4

If $V(u, v, w) = u + v^2 + w^3$ with scale factors $h_u = 2$, $h_v = 1$, $h_w = 1$, find $\nabla^2 V$ at the point (5, 3, 4).

There is very little to it. All we have to do is to determine the various partial derivatives and substitute in the expression above with relevant values.

$$\text{div grad } V = \dots\dots\dots\dots$$

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$$\boxed{26}$$

Because

$$\nabla^2 V = \frac{1}{h_u\, h_v\, h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v\, h_w}{h_u}\cdot\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_u\, h_w}{h_v}\cdot\frac{\partial V}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_u\, h_v}{h_w}\cdot\frac{\partial V}{\partial w}\right)\right\}$$

In this case, $V = u + v^2 + w^3 \;\therefore\; \dfrac{\partial V}{\partial u} = 1;\quad \dfrac{\partial V}{\partial v} = 2v;\quad \dfrac{\partial V}{\partial w} = 3w^2$

Also $h_u = 2,\ h_v = 1,\ h_w = 1$

$$\therefore\ \nabla^2 V = \frac{1}{2}\left\{\frac{\partial}{\partial u}\left(\frac{1}{2}\right)+\frac{\partial}{\partial v}(4v)+\frac{\partial}{\partial w}(6w^2)\right\}$$
$$= \tfrac{1}{2}\{0 + 4 + 12w\}$$

$\therefore$ At $w = 4,\quad \nabla^2 V = 26$

That is all there is to it. Here is another.

Example 5

If $V = (u^2 + v^2)w^3$ with $h_u = 3,\ h_v = 1,\ h_w = 2$, find div grad V at the point $(2,\ -2,\ 1)$.

$$\nabla^2 V = \ldots\ldots\ldots\ldots$$

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$$\boxed{14\tfrac{2}{9}}$$

Because

$$V = (u^2 + v^2)w^3 \quad \therefore\quad \frac{\partial V}{\partial u} = 2uw^3;\ \ \frac{\partial V}{\partial v} = 2vw^3;\ \ \frac{\partial V}{\partial w} = 3(u^2 + v^2)w^2$$

also $h_u = 3,\ \ h_v = 1,\ \ h_w = 2$

$$\therefore\ \nabla^2 V = \frac{1}{6}\left\{\frac{\partial}{\partial u}\left(\frac{2}{3}\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial v}\left(6\frac{\partial V}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{3}{2}\frac{\partial V}{\partial w}\right)\right\}$$
$$= \frac{1}{6}\left\{\frac{\partial}{\partial u}\left(\frac{4}{3}uw^3\right)+\frac{\partial}{\partial v}(12vw^3)+\frac{\partial}{\partial w}\left(\frac{9}{2}(u^2+v^2)w^2\right)\right\}$$

$\therefore$ at $(2,\ -2,\ 1)$

$$\nabla^2 V = \tfrac{1}{6}\{(\tfrac{4}{3}w^3) + (12w^3) + 9(u^2 + v^2)w\}$$
$$= \tfrac{1}{6}\{\tfrac{4}{3} + 12 + 72\} = \frac{256}{18} = 14\tfrac{2}{9}$$

Particular orthogonal systems

We can apply the general results for div, grad and curl to special coordinate systems by inserting the appropriate scale factors – as we shall now see.

▶

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(a) *Cartesian rectangular coordinate system*

If we replace u, v, w by x, y, z and insert values of $h_x = h_y = h_z = 1$, we obtain expressions for grad, div and curl in rectangular coordinates, so that

grad $V = \ldots\ldots\ldots\ldots$; div $\mathbf{F} = \ldots\ldots\ldots\ldots$; curl $\mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned}
\text{grad } V &= \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\\
\text{div } \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\\
\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_x & F_y & F_z \end{vmatrix}\\
\nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}
\end{aligned}}$$

all of which you will surely recognise.

(b) *Cylindrical polar coordinate system*

Here we simply replace u, v, w with ρ, ϕ, z and insert $h_u = h_\rho = 1$, $h_v = h_\phi = \rho$, $h_w = h_z = 1$ giving

grad $V = \ldots\ldots\ldots\ldots$; div $\mathbf{F} = \ldots\ldots\ldots\ldots$;
curl $\mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned}
\text{grad } V &= \frac{\partial V}{\partial \rho}\mathbf{I} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{J} + \frac{\partial V}{\partial z}\mathbf{K}\\
\text{div } \mathbf{F} &= \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right\}\\
\text{curl } \mathbf{F} &= \frac{1}{\rho}\begin{vmatrix} \mathbf{I} & \rho\mathbf{J} & \mathbf{K}\\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z}\\ F_\rho & \rho F_\phi & F_z \end{vmatrix}\\
\nabla^2 V &= \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}
\end{aligned}}$$

(c) *Spherical polar coordinate system*

Replacing u, v, w with r, θ, ϕ with $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$,

grad $V = \ldots\ldots\ldots\ldots$; div $\mathbf{F} = \ldots\ldots\ldots\ldots$;
curl $\mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\text{grad } V = \frac{\partial V}{\partial r}\mathbf{I} + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{J} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, F_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right\}$$

$$\text{curl } \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{I} & r\mathbf{J} & r\sin\theta\,\mathbf{K} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial \phi^2}$$

The results we have compiled are sometimes written in slightly different forms, but they are, of course, equivalent.

That brings us to the end of this Programme which is designed as an introduction to the topic of curvilinear coordinates. It has considerable applications, but these are beyond the scope of this present course of study.

The **Revision summary** follows as usual. Make any further notes as necessary: then you can work through the **Can You?** checklist and the **Test exercise** without difficulty. The Programme ends with the usual **Further problems**.

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Revision summary 19

1 *Curvilinear coordinates in two dimensions*

$u = f(x, y);\quad v = g(x, y)$

2 *Orthogonal coordinate system in space*

(a) *Cartesian rectangular coordinates* (x, y, z)

$\mathbf{F} = F_x\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ Scale factors $h_x = h_y = h_z = 1$

(b) *Cylindrical polar coordinates* (ρ, ϕ, z)

$\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\,\sin\phi\,\mathbf{j} + z\mathbf{k}$

Base unit vectors: Scale factors:

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho}\bigg/\left|\frac{\partial \mathbf{r}}{\partial \rho}\right| \qquad h_\rho = \left|\frac{\partial \mathbf{r}}{\partial \rho}\right| = 1$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi}\bigg/\left|\frac{\partial \mathbf{r}}{\partial \phi}\right| \qquad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \rho$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z}\bigg/\left|\frac{\partial \mathbf{r}}{\partial z}\right| \qquad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1$$

$\mathbf{F} = F_\rho\,\mathbf{I} + F_\phi\,\mathbf{J} + F_z\,\mathbf{K}$

▶

(c) *Spherical polar coordinates* (r, θ, ϕ)

$$\mathbf{r} = r\sin\theta\,\cos\phi\,\mathbf{i} + r\sin\theta\,\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

Base unit vectors:	Scale factors:
$\mathbf{I} = \dfrac{\partial\mathbf{r}}{\partial r}\Big/\left\lvert\dfrac{\partial\mathbf{r}}{\partial r}\right\rvert$	$h_r = \left\lvert\dfrac{\partial\mathbf{r}}{\partial r}\right\rvert = 1$
$\mathbf{J} = \dfrac{\partial\mathbf{r}}{\partial\theta}\Big/\left\lvert\dfrac{\partial\mathbf{r}}{\partial\theta}\right\rvert$	$h_\theta = \left\lvert\dfrac{\partial\mathbf{r}}{\partial\theta}\right\rvert = r$
$\mathbf{K} = \dfrac{\partial\mathbf{r}}{\partial\phi}\Big/\left\lvert\dfrac{\partial\mathbf{r}}{\partial\phi}\right\rvert$	$h_\phi = \left\lvert\dfrac{\partial\mathbf{r}}{\partial\phi}\right\rvert = r\sin\theta$

$$\mathbf{F} = F_r\,\mathbf{I} + F_\theta\,\mathbf{J} + F_\phi\,\mathbf{K}$$

3 *General orthogonal curvilinear coordinates* (u, v, w)
$x = f(u, v, w);\quad y = g(u, v, w);\quad w = h(u, v, w)$

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\frac{\partial\mathbf{r}}{\partial u} = h_u\mathbf{I} \quad \text{where} \quad h_u = \left\lvert\frac{\partial\mathbf{r}}{\partial u}\right\rvert$$

$$\frac{\partial\mathbf{r}}{\partial v} = h_v\mathbf{J} \quad \text{where} \quad h_v = \left\lvert\frac{\partial\mathbf{r}}{\partial v}\right\rvert$$

$$\frac{\partial\mathbf{r}}{\partial w} = h_w\mathbf{K} \quad \text{where} \quad h_w = \left\lvert\frac{\partial\mathbf{r}}{\partial w}\right\rvert$$

Element of arc: $\mathrm{d}s = (h_u^2\,\mathrm{d}u^2 + h_v^2\,\mathrm{d}v^2 + h_w^2\,\mathrm{d}w^2)^{1/2}$

Element of volume: $\mathrm{d}V = h_u h_v h_w\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$

$$= \frac{\partial(x, y, z)}{\partial(u, v, w)}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

4 *Grad, div and curl in orthogonal curvilinear coordinates*

(a) Grad $V = \nabla V = \dfrac{1}{h_u}\dfrac{\partial V}{\partial u}\mathbf{I} + \dfrac{1}{h_v}\dfrac{\partial V}{\partial v}\mathbf{J} + \dfrac{1}{h_w}\dfrac{\partial V}{\partial w}\mathbf{K}$

grad operator $= \nabla = \dfrac{\mathbf{I}}{h_u}\dfrac{\partial}{\partial u} + \dfrac{\mathbf{J}}{h_v}\dfrac{\partial}{\partial v} + \dfrac{\mathbf{K}}{h_w}\dfrac{\partial}{\partial w}$

(b) Div $\mathbf{F} = \dfrac{1}{h_u h_v h_w}\left\{\dfrac{\partial}{\partial u}(h_v h_w F_u) + \dfrac{\partial}{\partial v}(h_w h_u F_v) + \dfrac{\partial}{\partial w}(h_u h_v F_w)\right\}$

(c) Curl $\mathbf{F} = \dfrac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{I} & h_v\mathbf{J} & h_w\mathbf{K} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$

(d) Div grad $V = \nabla \cdot \nabla V = \nabla^2 V$

$$= \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\cdot\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\cdot\frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\cdot\frac{\partial V}{\partial w}\right)\right\}$$

▶

5 *Grad, div and curl in cylindrical and spherical coordinates*

(a) *Cylindrical coordinates* (ρ, ϕ, z)

$$\text{grad } V = \frac{\partial V}{\partial \rho}\mathbf{I} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{J} + \frac{\partial V}{\partial z}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial(\rho F_\rho)}{\partial \rho}\right\} + \frac{1}{\rho}\left\{\frac{\partial F_\phi}{\partial \phi}\right\} + \frac{\partial F_z}{\partial z}$$

$$\text{curl } \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{I} & \rho\mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

(b) *Spherical coordinates* (r, θ, ϕ)

$$\text{grad } V = \frac{\partial V}{\partial r}\mathbf{I} + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{J} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(F_\phi)$$

$$\text{curl } \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{I} & r\mathbf{J} & r\sin\theta\mathbf{K} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Can You?

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Checklist 19

Check this list before and after you try the end of Programme test

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Derive the family of curves of constant coordinates for curvilinear coordinates? **1** to **11**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive unit base vectors and scale factors in orthogonal curvilinear coordinates? **12** to **24**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

▶

- Obtain the element of arc ds and the element of volume dV in orthogonal curvilinear coordinates? **25**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain expressions for the operators grad, div and curl in orthogonal curvilinear coordinates? **26** to **36**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 19

39

1 Determine the unit vectors in the directions of the following three vectors and test whether they form an orthogonal set.

$$3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$-2\mathbf{i} - \mathbf{j} + 4\mathbf{k}.$$

2 If $\mathbf{r} = u \sin 2\theta\, \mathbf{i} + u \cos 2\theta\, \mathbf{j} + v^2\, \mathbf{k}$, determine the scale factors h_u, h_v, h_θ.

3 If P is a point $\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\mathbf{k}$ and a scalar field $V = \rho^2 z \sin 2\phi$ exists in space, using cylindrical polar coordinates (ρ, ϕ, z) determine grad V at the point at which $\rho = 1$, $\phi = \pi/4$, $z = 2$.

4 A vector field $\mathbf{F}$ is given in cylindrical coordinates by

$$\mathbf{F} = \rho\cos\phi\,\mathbf{I} + \rho\sin 2\phi\,\mathbf{J} + z\mathbf{K}$$

Determine (a) div $\mathbf{F}$; (b) curl $\mathbf{F}$.

5 Using spherical coordinates (r, θ, ϕ) determine expressions for (a) an element of arc ds; (b) an element of volume dV.

6 If V is a scalar field such that $V = u^2 v w^3$ and scale factors are $h_u = 1$, $h_v = 2$, $h_w = 4$, determine $\nabla^2 V$ at the point $(2, 3, -1)$.

Further problems 19

40

1 Determine whether the following sets of three vectors are orthogonal.

(a) $4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$
$\mathbf{i} - 11\mathbf{j} + 26\mathbf{k}$ $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

2 If $V(u, v, w) = v^3 w^2 \sin 2u$ with scale factors $h_u = 3$, $h_v = 1$, $h_w = 2$, determine div grad V at the point $(\pi/4, -1, 3)$.

3 A scalar field $V = \dfrac{u^2 e^{2w}}{v}$ exists in space. If the relevant scale factors are $h_u = 2$, $h_v = 3$, $h_w = 1$, determine the value of $\nabla^2 V$ at the point (1, 2, 0).

4 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ in spherical polar coordinates (r, θ, ϕ), prove that, for any vector field $\mathbf{F}$ where

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = F_r\mathbf{I} + F_\theta\mathbf{J} + F_\phi\mathbf{K}$$

then
$$F_x = F_r\sin\theta\cos\phi + F_\theta\cos\theta\cos\phi - F_\phi\sin\phi$$
$$F_y = F_r\sin\theta\sin\phi + F_\theta\cos\theta\sin\phi + F_\phi\cos\phi$$
$$F_z = F_r\cos\theta - F_\theta\sin\theta.$$

5 If V is a scalar field, determine an expression for $\nabla^2 V$

(a) in cylindrical polar coordinates

(b) in spherical polar coordinates.

6 Transformation equations from rectangular coordinates (x, y, z) to parabolic cylindrical coordinates (u, v, w) are

$$x = \frac{u^2 - v^2}{2}; \quad y = uv; \quad z = w$$

V is a scalar field and $\mathbf{F}$ a vector field.

(a) Prove that the (u, v, w) system is orthogonal

(b) Determine the scale factors

(c) Find div $\mathbf{F}$

(d) Obtain an expression for $\nabla^2 V$.

Complex analysis 1

Frames 1 to 70

Learning outcomes

When you have completed this Programme you will be able to:

- Recognise the transformation equation in the form $w = f(z) = u(x, y) + jv(x, y)$
- Illustrate the image of a point in the complex z-plane under a complex mapping onto the w-plane
- Map a straight line in the z-plane onto the w-plane under the transformation $w = f(z)$
- Identify complex mappings that form translations, magnifications, rotations and their combinations
- Deal with the non-linear transformations $w = z^2$, $w = 1/z$, $w = 1/(z - a)$ and $w = (az + b)/(cz + d)$

Prerequisite: Engineering Mathematics (Fifth Edition)

Programmes 1 Complex numbers 1, 2 Complex numbers 2 and 3 Hyperbolic functions

1

The foundations of complex numbers and their application to hyperbolic functions were treated fully in Programmes 1, 2 and 3 of *Engineering Mathematics (Fifth Edition)* and these provide valuable revision should you feel it to be necessary before embarking on the new work.

It will be assumed that you are already familiar with the material covered in those previous Programmes and it would be a wise move to work through the relevant Test exercises to refresh your memory on this all-important part of the course.

Functions of a complex variable

For a function of a single real variable $f(x)$ we can construct the graph of the function by plotting points against two mutually perpendicular Cartesian axes, the x-axis and the $f(x)$-axis. For a function of a single complex variable $w = f(z) = u(x, y) + jv(x, y)$ we have four real variables, x, y, u and v. For example if $z = x + jy$ and $f(z) = z^2$ then

$$\begin{aligned} f(z) &= (x + jy)^2 \\ &= x^2 + 2jxy + (jy)^2 \\ &= x^2 - y^2 + 2jxy \end{aligned}$$

and so

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ \text{and} \quad v(x, y) &= 2xy \end{aligned}$$

We cannot plot the graph of the function $f(z)$ against a single set of axes because to do so we would be required to draw four mutually perpendicular axes which is not possible. Instead, we resort to plotting z-values against x- and y-axes in the complex z-plane and to plotting the corresponding values of $w = f(z)$ against u- and v-axes in the complex w-plane. Accordingly, values of z are plotted on one plane and the corresponding values of $f(z)$ are plotted on another plane. So in our example above for a particular value of z, for example, $z = 4 + j3$

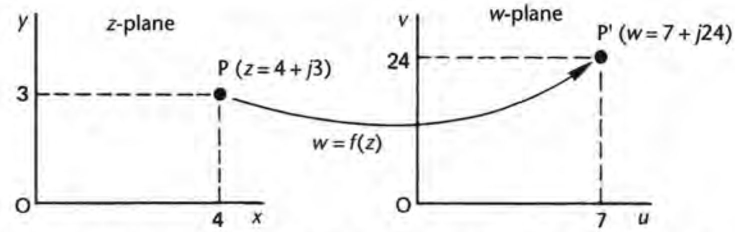
$u = \ldots\ldots\ldots\ldots$

$v = \ldots\ldots\ldots\ldots$

2

$$\boxed{u = 7 \quad v = 24}$$

Because with $z = 4 + j3$, $x = 4$ and $y = 3$. Then $u = 16 - 9 = 7$ and $v = 24$.



Therefore, z (where $z = x + jy$) and w (where $w = u + jv$) are two complex variables related by the equation $w = f(z)$.

Any other point in the z-plane will similarly be transformed into a corresponding point in the w-plane, the resulting position P′ depending on

(a) the initial position of P

(b) the relationship $w = f(z)$, called the *transformation equation* or *transformation function*.

Complex mapping

The transformation of P in the z-plane onto P′ in the w-plane is said to be a *mapping* of P onto P′ under the transformation $w = f(z)$ and P′ is sometimes referred to as the *image* of P.

Example 1

Determine the image of the point P, $z = 3 + j2$, on the w-plane under the transformation $w = 3z + 2 - j$.

$$\begin{aligned} w = u + jv = f(z) &= 3z + 2 - j \\ &= 3(x + jy) + 2 - j \end{aligned}$$

so that, for this example,

$u = \ldots\ldots\ldots\ldots$; $v = \ldots\ldots\ldots\ldots$

3

$$\boxed{u = 3x + 2; \quad v = 3y - 1}$$

Then the point P ($z = 3 + j2$) transforms onto $\ldots\ldots\ldots\ldots$

4

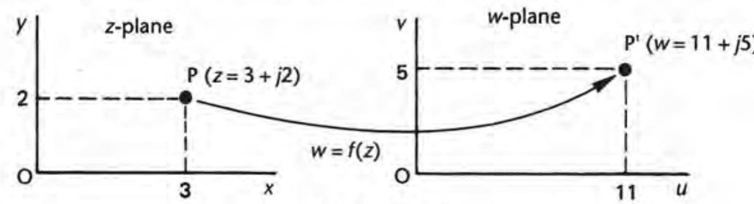
$$\boxed{w = 11 + j5}$$

Because

$z = 3 + j2 \quad \therefore\ x = 3,\ y = 2$

$u = 3x + 2 = 11; \quad v = 3y - 1 = 5; \quad \therefore\ w = 11 + j5$

We can illustrate the transformation thus:



$w = f(z) = 3z + 2 - j$

Here is another.

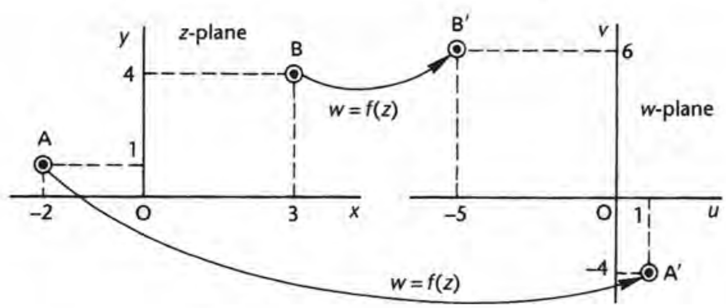
Example 2

Map the points A $(z = -2 + j)$ and B $(z = 3 + j4)$ onto the w-plane under the transformation $w = j2z + 3$ and illustrate the transformation on a diagram.

This is no different from the previous example. Complete the job and check with the next frame.

5

$$\boxed{\text{A}'\ (w = 1 - j4); \quad \text{B}'\ (w = -5 + j6)}$$



Because

$w = f(z) = j2z + 3 = j2(x + jy) + 3 = (3 - 2y) + j2x$

$w = u + jv \quad \therefore\ u = 3 - 2y; \quad v = 2x$

A: $x = -2,\ y = 1 \quad \therefore\ \text{A}': u = 3 - 2 = 1; v = -4 \quad \therefore\ \text{A}': w = 1 - j4$

B: $x = 3,\ y = 4 \quad \therefore\ \text{B}': u = 3 - 8 = -5; v = 6 \quad \therefore\ \text{B}': w = -5 + j6$

There now follows a short practice exercise. Work all four of the items before you check the results. There is no need to illustrate the transformation in each case.

So move on

6

Exercise

Map the following points in the z-plane onto the w-plane under the transformation $w = f(z)$ stated in each case.

1 $z = 4 - j2$ under $w = j3z + j2$
2 $z = -2 - j$ under $w = jz + 3$
3 $z = 3 + j2$ under $w = (1 + j)z - 2$
4 $z = 2 + j$ under $w = z^2$.

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1 $w = 6 + j14$	**2** $w = 4 - j2$
3 $w = -1 + j5$	**4** $w = 3 + j4$

That was easy enough. Now let us extend the ideas.

Mapping of a straight line in the z-plane onto the w-plane under the transformation $w = f(z)$

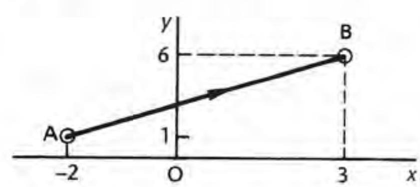
A typical example will show the method.

Example 1

To map the straight line joining A $(-2 + j)$ and B $(3 + j6)$ in the z-plane onto the w-plane when $w = 3 + j2z$.

We first of all map the end points A and B onto the w-plane to obtain A′ and B′ as in the previous cases.

A′: $w =$ ....
B′: $w =$ ....



8

$$\boxed{\text{A}':\ w = 1 - j4;\quad \text{B}':\ w = -9 + j6}$$

Because

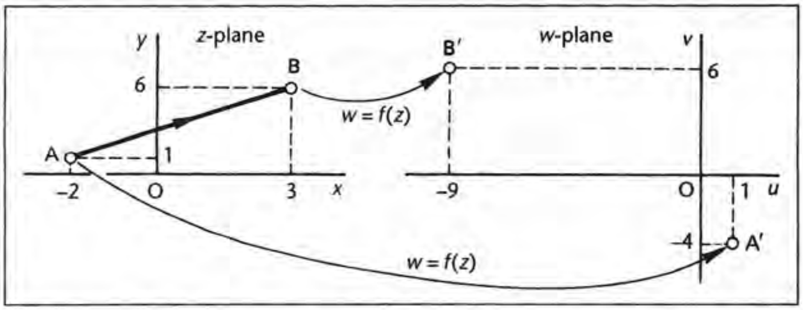
(1) A: $z = -2 + j \qquad w = 3 + j2z$
$\therefore$ A′: $w = 3 + j2(-2 + j) = 3 - j4 - 2 = 1 - j4$

(2) B: $z = 3 + j6$
$\therefore$ B′: $w = 3 + j2(3 + j6) = 3 + j6 - 12 = -9 + j6$

Then, if we illustrate the transformations on a diagram, as before, we get

....

9



As z moves along the line A to B in the z-plane, we cannot assume that its image in the w-plane travels along a straight line from A′ to B′. As yet, we have no evidence of what the path is. We therefore have to find a general point $w = u + jv$ in the w-plane corresponding to a general point $z = x + jy$ in the z-plane.

$$w = u + jv = f(z) = 3 + j2z$$
$$= \dots\dots\dots\dots\dots\dots$$

10

$$\boxed{w = u + jv = (3 - 2y) + j2x}$$

Because

$$w = 3 + j2(x + jy) = 3 + j2x - 2y = (3 - 2y) + j2x$$
$$\therefore\ u = 3 - 2y \quad \text{and} \quad v = 2x$$

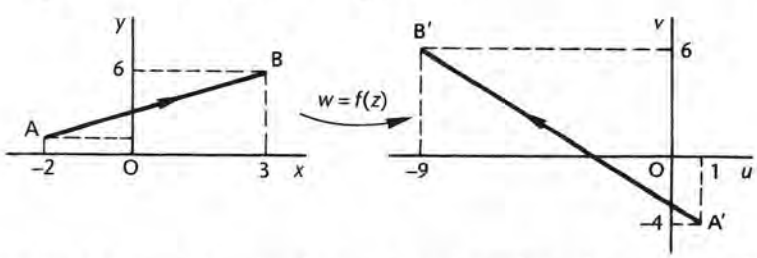
Rearranging these results, we also have $y = \dfrac{3 - u}{2}$; $x = \dfrac{v}{2}$.

Now the Cartesian equation of AB is $y = x + 3$ and substituting from the previous line, we have $\dfrac{3 - u}{2} = \dfrac{v}{2} + 3$ which simplifies to ....

11

$$\boxed{v = -u - 3}$$

which is the equation of a straight line, so, in this case, the path joining A′ and B′ *is* in fact a straight line.



Note that it is useful to attach arrow heads to show the corresponding direction of progression in the transformation.

On to the next

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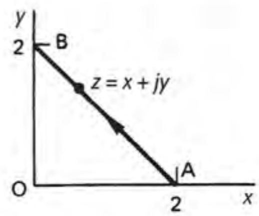
Example 2

If $w = z^2$, find the path traced out by w as z moves along the straight line joining A $(2 + j0)$ and B $(0 + j2)$.



Cartesian equation of AB is

$y = 2 - x$

First we transform the two end points A and B onto A′ and B′ in the w-plane.

A′: ....; B′: ....

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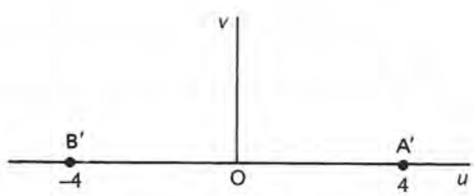
$$\boxed{\text{A}':\ w = 4 + j0;\quad \text{B}':\ w = -4 + j0}$$

Because

$w = z^2$ A: $z = 2$ $\therefore$ A′: $w = 2^2 = 4$

B: $z = j2$ $\therefore$ B′: $w = (j2)^2 = -4$

So we have



Now we have to find the path from A′ to B′.

The Cartesian equation of AB in the z-plane is $y = 2 - x$.

Also $w = z^2 = (x + jy)^2 = (x^2 - y^2) + j2xy$

$\therefore\ u = x^2 - y^2$ and $v = 2xy$

Substituting $y = 2 - x$ in these results we can express u and v in terms of x.

$u =$ ....; $v =$ ....

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$$\boxed{u = 4x - 4; \quad v = 4x - 2x^2}$$

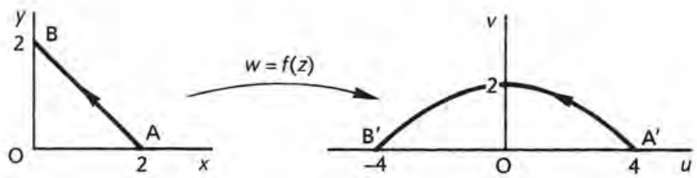
So, from the first of these $\quad x = \dfrac{u+4}{4}$

Substituting in the second $\quad v = 4\left(\dfrac{u+4}{4}\right) - 2\left(\dfrac{u+4}{4}\right)^2$

$$= u + 4 - \frac{1}{8}(u^2 + 8u + 16)$$

$$= -\frac{1}{8}(u^2 - 16)$$

Therefore the path is $v = -\dfrac{1}{8}(u^2 - 16)$ which is a parabola for which at $u = 0$, $v = 2$.

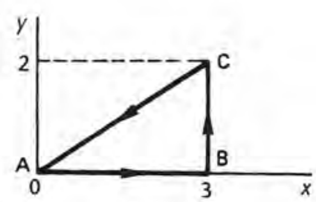


Note that a straight line in the z-plane does not always map onto a straight line in the w-plane. It depends on the particular transformation equation $w = f(z)$.

If the transformation is a *linear equation*, $w = f(z) = az + b$, where a and b may themselves be real or complex, then a straight line in the z-plane maps onto a corresponding straight line in the w-plane.

Example 3

A triangle in the z-plane has vertices at A $(z = 0)$, B $(z = 3)$ and C $(z = 3 + j2)$. Determine the image of this triangle in the w-plane under the transformation equation $w = (2 + j)z$.



$$w = u + jv = f(z) = (2+j)z = (2+j)(x+jy) = (2x - y) + j(2y + x)$$

$$\therefore \; u = 2x - y; \qquad v = 2y + x$$

We now transform each vertex in turn onto the w-plane to determine A′, B′ and C′.

These are A′: ....; B′: ....; C′: ....

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$$A': \ w = 0; \quad B': \ w = 6 + j3; \quad C': \ w = 4 + j7$$

The transformation is linear (of the form $w = az$) so A′B′, B′C′ and C′A′ are straight lines and the transformation can be illustrated in the diagram

. . . .

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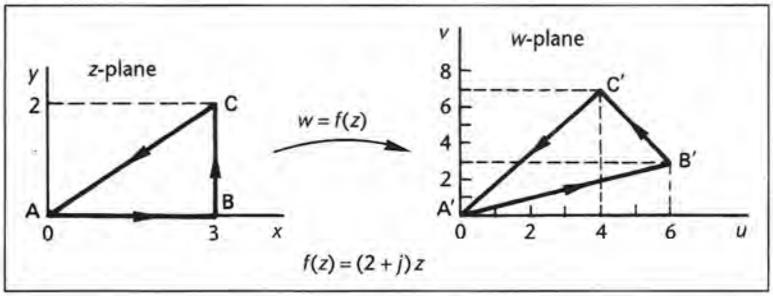


All very straightforward. Let us now take a more detailed look at linear transformations.

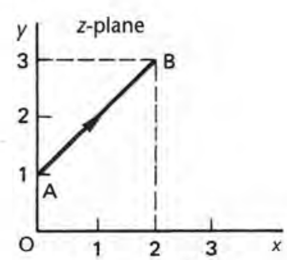
Types of transformation of the form $w = az + b$

where the constants a and b may be real or complex.

1 *Translation*

Let $a = 1$ and $b = 2 - j$ i.e. $w = z + (2 - j)$.

If we apply this to the straight line joining A $(0 + j)$ and B $(2 + j3)$ in the z-plane, then



$$w = x + jy + 2 - j$$
$$= (x + 2) + j(y - 1)$$

so the corresponding end points A′ and B′ in the w-plane are

A′: . . . .; B′: . . . .

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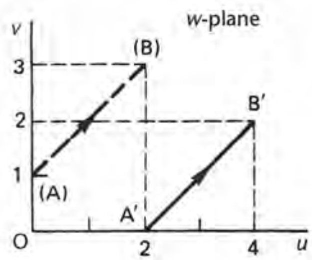
$$\boxed{A': \; w = 2; \quad B': \; w = 4 + j2}$$



The transformed line A′B′ is then as shown. The broken line (A)(B) indicates the position of the original line AB in the *z*-plane.

Note that the whole line AB has moved two units to the right and one unit downwards, while retaining its original magnitude (length) and direction.

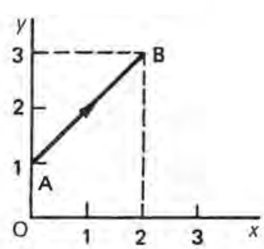
Such a transformation is called a *translation* and occurs whenever the transformation equation is of the form $w = z + b$. The degree of translation is given by the value of b – in this case $(2 - j)$, i.e. 2 units along the positive real axis and 1 unit in the direction of the negative imaginary axis.

On to the next frame

18

2 *Magnification*

Consider now $w = az + b$ where $b = 0$ and a is real, e.g. $w = 2z$.



Applying the transformation to the same line AB as before, we have

$$w = u + jv = 2z = 2(x + jy)$$
$$\therefore \; u = 2x \quad \text{and} \quad v = 2y$$

Transforming the end points A $(0 + j)$ and B $(2 + j3)$ onto A′ and B′ in the *w*-plane, we have

A′: ....; B′: ....

and the *w*-plane diagram becomes

....

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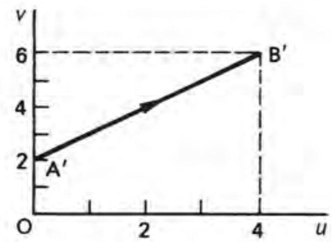
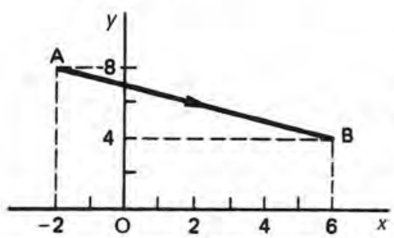
$A': \ w = j2; \quad B': \ w = 4 + j6$



Note that (a) all distances in the z-plane are magnified by a factor 2, and (b) the direction of A′B′ is that of AB unchanged. Any such transformation $w = az$ where a is real, is said to be a *magnification* by the factor a.

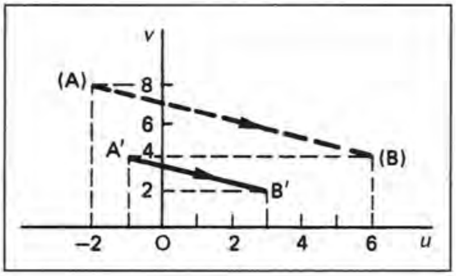


So, if we apply the transformation $w = z/2$ to AB shown here, we can map AB onto A′B′ in the w-plane and obtain

. . . .

Sketch the result

20

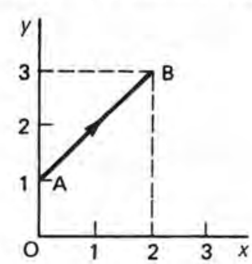


3 *Rotation*

Consider next $w = az + b$ with $b = 0$ and a complex,

e.g. $w = jz$.

$$
\begin{aligned}
w = u + jv &= jz \\
&= j(x + jy) \\
&= -y + jx
\end{aligned}
$$



Transforming the end points as usual, we can sketch the original line AB and the mapping A′B′, which gives . . . .

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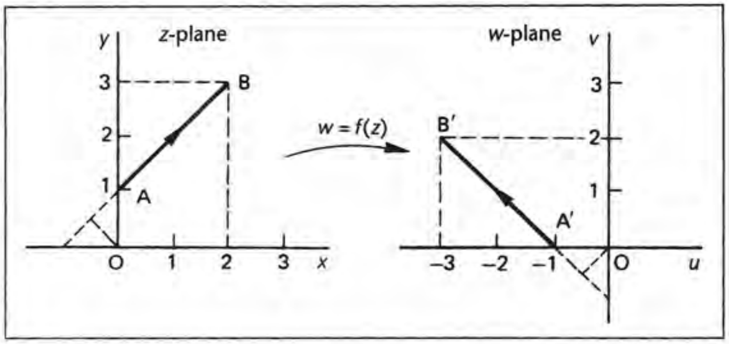

A′ is the point $w = -1 + j0$; B′ is the point $w = -3 + j2$

Note $AB = 2\sqrt{2}$ $A'B' = 2\sqrt{2}$

Slope of $AB = m = 1$ Slope of $A'B' = m_1 = -1$

$mm_1 = 1(-1) = -1$

Therefore in transformation by $w = jz$, AB retains its original length but is rotated about the origin, in this case through 90° in a positive (anticlockwise) direction.

Some degree of rotation always occurs when the transformation equation is of the form $w = az + b$ with a complex.

Move on to the next frame

22

4 *Combined magnification and rotation*

If $w = (a + jb)z$, the effect of transformation is

(a) magnification $|\, a + jb \,| = \sqrt{a^2 + b^2}$

(b) rotation anticlockwise through $\arg(a + jb)$, i.e. $\arctan \dfrac{b}{a}$.

Let us see this with an example.

Example

Map the straight line joining A $(0 + j2)$ and B $(4 + j6)$ in the z-plane onto the w-plane under the transformation $w = (3 + j2)z$.

The working is just as before. Draw the z-plane and w-plane diagrams, which give

....

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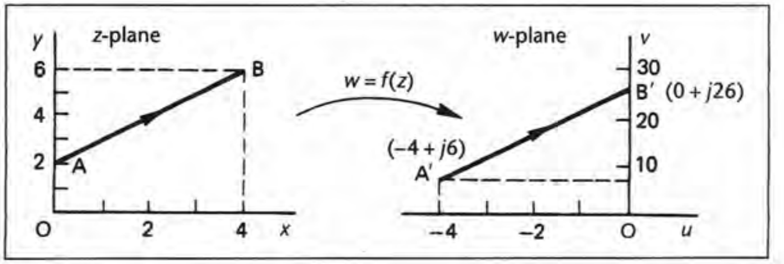

$$w = (3 + j2)z$$

$$\therefore\ u + jv = (3 + j2)(x + jy) = (3x - 2y) + j(2x + 3y)$$

$$\therefore\ u = 3x - 2y \quad \text{and} \quad v = 2x + 3y$$

$$\text{A:}\ \ z = 0 + j2, \ \text{ i.e. } x = 0, \ \ y = 2$$

$$\therefore\ \text{A}'\text{:}\ u = -4, \ \ v = 6 \quad \therefore\ \text{A}'\text{:}\ (-4 + j6)$$

$$\text{B:}\ \ z = 4 + j6, \ \text{ i.e. } x = 4, \ \ y = 6$$

$$\therefore\ \text{B}'\text{:}\ u = 0, \ \ v = 26 \qquad \therefore\ \text{B}'\text{:}\ (0 + j26)$$

By a simple application of Pythagoras, we can now calculate the lengths of AB and A′B′, and then determine the magnification factor (A′B′)/(AB).

AB = ...; A′B′ = ...; magnification = ...

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$$\boxed{\text{AB} = 4\sqrt{2}; \ \ \text{A}'\text{B}' = 4\sqrt{26}; \quad \text{mag} = \sqrt{13}}$$

Because

$$\text{AB} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}$$

$$\text{A}'\text{B}' = \sqrt{16 + 400} = \sqrt{416} = 4\sqrt{26}$$

$$\therefore\ \text{magnification} = \frac{4\sqrt{26}}{4\sqrt{2}} = \sqrt{13}$$

Also $|\, a + jb \,| = |\, 3 + j2 \,| = \sqrt{9 + 4} = \sqrt{13} \quad \therefore\ \text{mag} = |\, a + jb \,|$

Now let us check the rotation.

For AB $\quad \tan\theta_1 = 1 \quad \therefore\ \theta_1 = 45^\circ = 0{\cdot}7854$ radians

For A′B′ $\quad \tan\theta_2 = 5 \quad \therefore\ \theta_2 = 78^\circ\ 41' = 1{\cdot}3733$ radians

$$\therefore\ \text{rotation} = \theta_2 - \theta_1 = 1{\cdot}3733 - 0{\cdot}7854 = 0{\cdot}5879$$

i.e. rotation = 0·5879 radians

Also $\arg(a + jb) = \arg(3 + j2) =$ . . . .

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$$\boxed{0{\cdot}5879 \text{ radians}}$$

Because $\arg(3+j2) = \arctan\frac{2}{3} = 33^\circ\,41' = 0{\cdot}5879$ radians.

So, in transformation $w = (a+jb)z = (3+j2)z$

(a) AB is magnified by $|a+jb|$, i.e. $\sqrt{13}$

(b) AB is rotated anticlockwise through $\arg(a+jb)$, i.e. $\arg(3+j2)$ i.e. $0{\cdot}5879$ radians.

5 *Combined magnification, rotation and translation*

The work we have just done can be extended to include all three effects of transformation.

In general, a transformation equation $w = az + b$, where a and b are each real or complex, results in

magnification $|a|$; rotation $\arg a$; translation b

Therefore, if $w = (3+j)z + 2 - j$

magnification = ...; rotation = ...;
translation = ...

26

mag $= \sqrt{10} = 3{\cdot}162$; rotation $= 18^\circ\,26' = 0{\cdot}3218$ radians;
translation = 2 units to right, 1 unit downwards

Because

(a) magnification $= |3+j| = \sqrt{9+1} = \sqrt{10} = 3{\cdot}162$

(b) rotation $= \arg(3+j) = \arctan\dfrac{1}{3} = 18^\circ\,26' = 0{\cdot}3218$ radians

(c) translation $= 2 - j$, i.e. 2 to the right, 1 downwards.

Let us work through an example in detail.

Example 1

The straight line joining A $(-2-j3)$ and B $(3+j)$ in the z-plane is subjected to the linear transformation equation

$w = (1+j2)z + 3 - j4$

Illustrate the mapping onto the w-plane and state the resulting magnification, rotation and translation involved.

The first part is just like examples we have already done. So,

(a) transform the end points A and B onto A′ and B′ in the w-plane

(b) join A′ and B′ with a straight line, since AB is a straight line and the transformation equation is linear.

That can be done without trouble, the final diagram being

....

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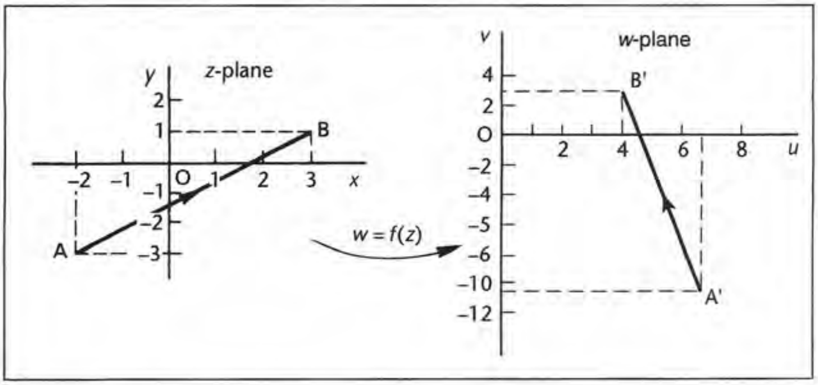

Check the working. $w = (1 + j2)z + 3 - j4$

$$\begin{aligned}
\text{A:}\quad z &= x + jy \\
&= -2 - j3 \\
\text{A}':\quad w &= u + jv = (1 + j2)(-2 - j3) + 3 - j4 \\
&= -2 - j7 + 6 + 3 - j4 \\
&= 7 - j11 \\
\text{B:}\quad z &= x + jy \\
&= 3 + j \\
\text{B}':\quad w &= u + jv = (1 + j2)(3 + j) + 3 - j4 \\
&= 3 + j7 - 2 + 3 - j4 \\
&= 4 + j3
\end{aligned}$$

Now for the second part of the problem, we have to state the magnification, rotation and translation when $w = (1 + j2)z + 3 - j4$. We remember that the 'tailpiece', i.e. $3 - j4$, independent of z, represents the

....

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$\boxed{\text{translation}}$

So, for the moment, we concentrate on $w = (1 + j2)z$, which determines the magnification and rotation. This tells us that

magnification = ...
rotation = ...

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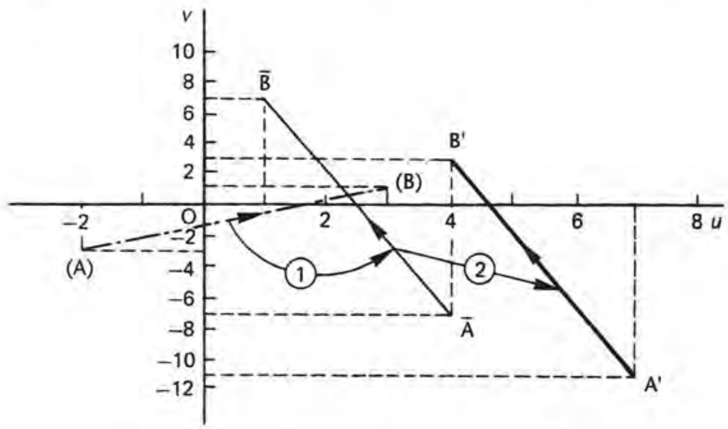
$$\boxed{\begin{array}{l}\text{mag} = |a| = |1 + j2| = \sqrt{1+4} = \sqrt{5} = 2{\cdot}236 \\ \text{rotation} = \arg a = \arctan \frac{2}{1} = 63^\circ\ 26' = 1{\cdot}107 \text{ radians}\end{array}}$$

The translation is given by $(3 - j4)$, i.e. 3 units to the right, 4 units downwards.

We can in fact see the intermediate steps if we deal first with the transformation $w = (1 + j2)z$ and subsequently with the translation $w = 3 - j4$.



Under $w = (1 + j2)z$, A and B map onto $\overline{\text{A}}$ and $\overline{\text{B}}$ where $\overline{\text{A}}$ is $w = 4 - j7$ and $\overline{\text{B}}$ is $w = 1 + j7$.

Then the translation $w = 3 - j4$ moves all points 3 units to the right and 4 units downwards, so that $\overline{\text{A}}$ and $\overline{\text{B}}$ now map onto A′ and B′ where A′ is $w = 7 - j11$ and B′ is $w = 4 + j3$.

Normally, there is no need to analyse the transformation into intermediate steps.

Now for –

Example 2

Map the straight line joining A $(1 + j2)$ and B $(4 + j)$ in the z-plane onto the w-plane using the transformation equation

$$w = (2 - j3)z - 4 + j5$$

and state the magnification, rotation and translation involved.

There are no snags. Complete the working and check with the next frame.

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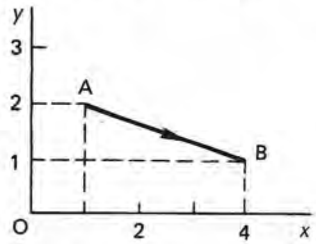
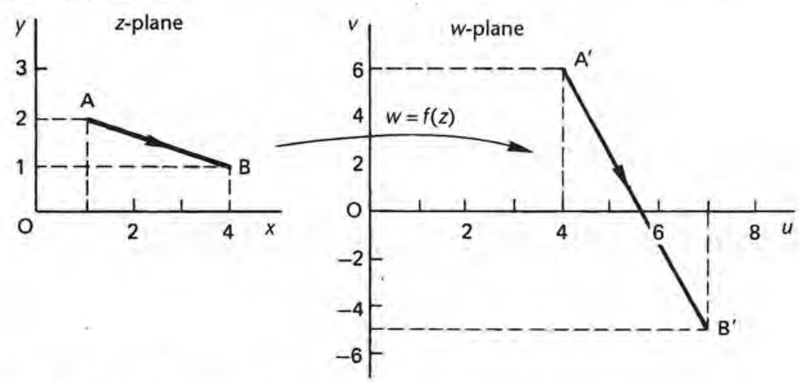
Here is the complete working.



$w = (2 - j3)z - 4 + j5$
A: $z = 1 + j2$
B: $z = 4 + j$

A: $z = 1 + j2$
A′: $w = (2 - j3)(1 + j2) - 4 + j5 = 2 + j + 6 - 4 + j5 = 4 + j6$
B: $z = 4 + j$
B′: $w = (2 - j3)(4 + j) - 4 + j5 = 8 - j10 + 3 - 4 + j5 = 7 - j5$

So we have



Also we have

(a) magnification $= |2 - j3| = \sqrt{4+9} = \sqrt{13} = 3{\cdot}606$
(b) rotation $= \arg(2 - j3) = \arctan(\frac{-3}{2}) = -56°\ 19'$
$= 0{\cdot}9828$ radians clockwise
(c) translation $= -4 + j5$ i.e. 4 units to left, 5 units upwards

All very straightforward. Before we move on, here is a short revision exercise.

Exercise

Calculate (a) the magnification, (b) the rotation, (c) the translation involved in each of the following transformations.

1 $w = (1 - j2)z + 2 - j3$
2 $w = (4 + j3)z - 2 + j5$
3 $w = (2 - j3)z - 1 - j$
4 $w = (j - 4)z + j2 - 3$
5 $w = j2z + 4 - j$
6 $w = (5 + j2)z + j(j3 - 4)$.

Complete all six and then check the results with the next frame.

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Results:

1 $w = (1 - j2)z + 2 - j3$

(a) magnitude $= |1 - j2| = \sqrt{1+4} = \sqrt{5} = 2{\cdot}236$

(b) rotation $= \arg(1 - j2) = \arctan(-2) = -63^\circ\ 26'$

$= 1{\cdot}107$ radians clockwise

(c) translation $= 2 - j3$, i.e. 2 units to right, 3 units downwards.

The others are done in the same way and give the following results.

No.	Magnitude	Rotation (rad)	Translation
2	5	0·6435 ac	2L, 5U
3	3·606	0·9828 c	1L, 1D
4	4·123	0·2450 c	3L, 2U
5	2	1·5708 ac	4R, 1D
6	5·385	0·3805 ac	3L, 4D

Now let us start a new section, so on to the next frame

Non-linear transformations

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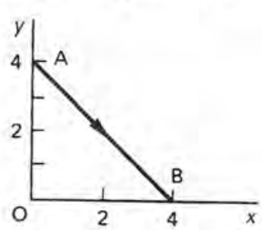
So far, we have concentrated on linear transformations of the form $w = az + b$. We can now proceed to something rather more interesting.

1 *Transformation* $w = z^2$ (refer to Frame 12)

The general principles are those we have used before. An example will show the development.

Example 1

The straight line AB in the z-plane as shown is mapped onto the w-plane by $w = z^2$.



As before, we start by transforming the end points onto A′ and B′ in the w-plane.

A′: $w = \dots\dots\dots\dots$

B′: $w = \dots\dots\dots\dots$

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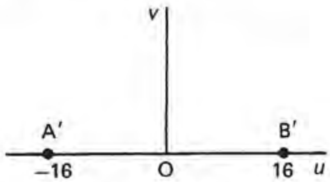
$$\boxed{A': \ w = -16; \quad B': \ w = 16}$$



We cannot however assume that AB maps onto the straight line A′B′, since the transformation is not linear. We therefore have to deal with a general point.

$$w = u + jv = z^2 = (x + jy)^2 = x^2 + j2xy - y^2 = (x^2 - y^2) + j2xy$$

$$\therefore \ u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

The Cartesian equation of AB in the z-plane is $y = 4 - x$. So, substituting in the results of the previous line, we can express u and v in terms of x.

$$u = \dots\dots\dots\dots; \quad v = \dots\dots\dots\dots$$

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$$\boxed{u = 8x - 16; \quad v = 8x - 2x^2}$$

The first gives $x = \dfrac{u + 16}{8}$ and substituting this in the expression for v gives …………

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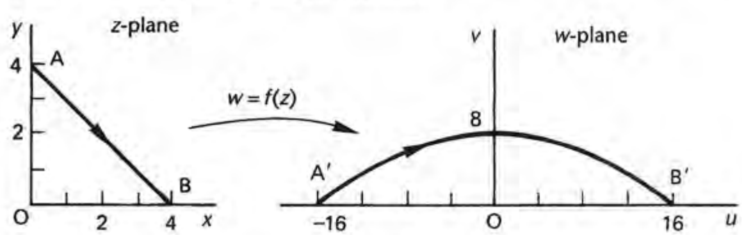
$$\boxed{v = -\tfrac{1}{32}u^2 + 8}$$

Because

$$v = 8\left(\frac{u + 16}{8}\right) - 2\left(\frac{u + 16}{8}\right)^2 = u + 16 - \frac{u^2}{32} - u - 8$$

$$\therefore \ v = -\frac{u^2}{32} + 8$$

which is an 'inverted' parabola, symmetrical about the v-axis, with $v = 8$ at $u = 0$. The mapping is therefore



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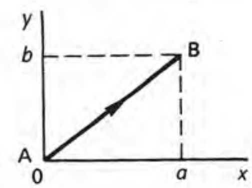
Example 2

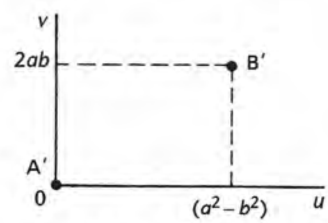
AB is a straight line in the z-plane joining the origin A to the point B $(a + jb)$. Obtain the mapping of AB onto the w-plane under the transformation $w = z^2$.



As always, first map the end points.

A′: $w = 0$

B′: $w = (a + jb)^2 = (a^2 - b^2) + j2ab$



Now to find the path joining A′ and B′, we consider a general point $z = x + jy$.

$$w = u + jv = z^2$$
$$= (x + jy)^2$$
$$= (x^2 - y^2) + j2xy$$
$$\therefore \ u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

The equation of AB is $y = \dfrac{b}{a}x$. We can therefore find u and v in terms of x and hence v in terms of u.

$u =$ ....

$v =$ ....

$v = f(u) =$ ....

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$$u = \left(\frac{a^2 - b^2}{a^2}\right)x^2; \quad v = \left(\frac{2b}{a}\right)x^2; \quad v = \left(\frac{2ab}{a^2 - b^2}\right)u$$

Because

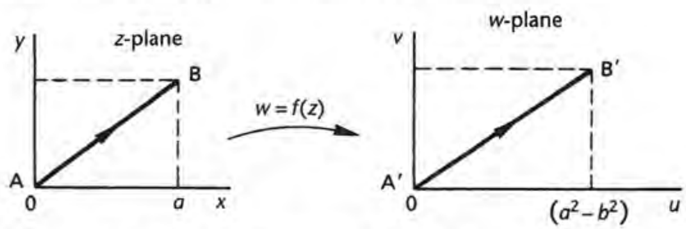
$$u = x^2 - y^2 = x^2 - \left(\frac{b^2}{a^2}\right)x^2 = \left(\frac{a^2 - b^2}{a^2}\right)x^2$$

$$v = 2xy = 2x\left(\frac{b}{a}\right)x = \left(\frac{2b}{a}\right)x^2$$

From the expression for u, $x^2 = \left(\frac{a^2}{a^2 - b^2}\right)u \quad \therefore \; v = \frac{2b}{a}\left(\frac{a^2}{a^2 - b^2}\right)u$

$\therefore \; v = \left(\frac{2ab}{a^2 - b^2}\right)u$ which is of the form $v = ku$.

A′B′ is therefore a straight line through the origin.



Therefore, under the transformation $w = z^2$, a straight line through the origin in the z-plane maps onto a straight line through the origin in the w-plane, whereas a straight line not passing through the origin maps onto a parabola.

This is worth remembering, so make a note of it

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Example 3

A triangle consisting of AB, BC, CA in the z-plane is mapped onto the w-plane by the transformation $w = z^2$.

The transformation is $w = z^2$.

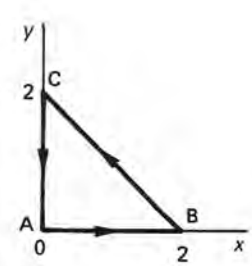
$$\therefore \; w = (x + jy)^2 = (x^2 - y^2) + j2xy$$
$$= u + jv$$
$$\therefore \; u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

First we can map the end points A, B, C onto A′, B′, C′ in the w-plane.

A′: ....

B′: ....

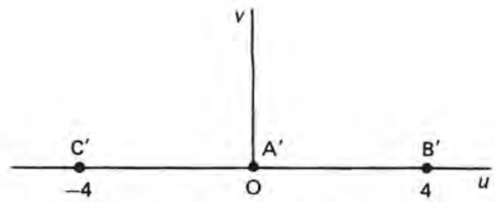
C′: ....



39

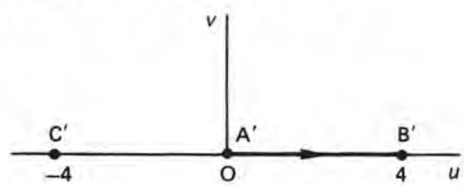
$$A': w = 0; \quad B': w = 4; \quad C': w = -4$$

So we establish



To find the paths joining these three transformed end points, we consider each of the sides of the triangle in turn.

(a) AB: Equation of AB is $y = 0 \quad \therefore \; u = x^2; \quad v = 0$
$\therefore$ Each point in AB maps onto a point between A′ and B′ for which $v = 0$, i.e. part of the u-axis.



(b) BC: Equation of BC is $y = 2 - x$
Substitute in $u = x^2 - y^2$ and $v = 2xy$ and determine v as a function of u.

$u = \ldots\ldots\ldots\ldots$
$v = \ldots\ldots\ldots\ldots$
$v = f(u) = \ldots\ldots\ldots\ldots$

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$$u = 4x - 4; \quad v = 4x - 2x^2; \quad v = 2 - \frac{u^2}{8}$$

Because

$$u = x^2 - y^2 = x^2 - (2 - x)^2 = 4x - 4 \quad \therefore \; x = \frac{u+4}{4}$$
$$v = 2xy = 2x(2 - x) = 4x - 2x^2$$
$$\therefore \; v = 4\left(\frac{u+4}{4}\right) - 2\left(\frac{u+4}{4}\right)^2 = 2 - \frac{u^2}{8}$$

Therefore, the path joining B′ to C′ is an

$\ldots\ldots\ldots\ldots$

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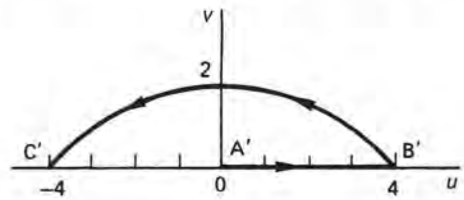
inverted parabola

$v = 2 - \frac{u^2}{8}$ $\therefore$ at $u = 0$, $v = 2$ and the w-plane diagram now becomes



To complete the mapping, we have still to deal with CA. This transforms onto

. . . .

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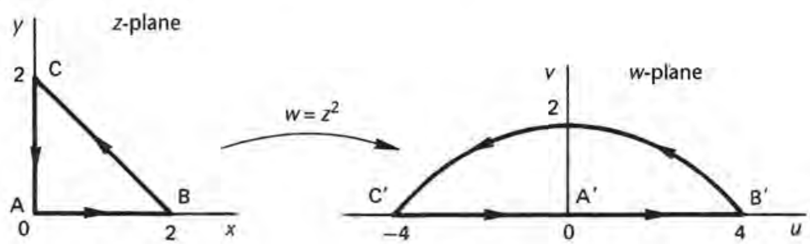
the u-axis between C′ and A′

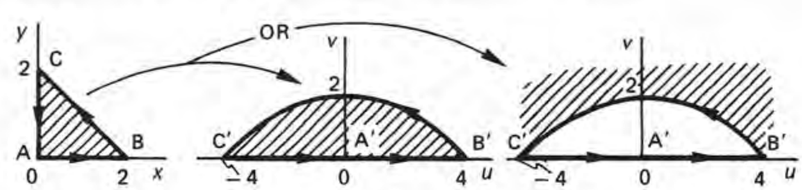
(c) CA: Equation of CA is $x = 0$ $\therefore$ $u = -y^2$, $v = 0$
$\therefore$ Each point between C and A maps onto the negative part of the u-axis between C′ and A′.

So finally we have



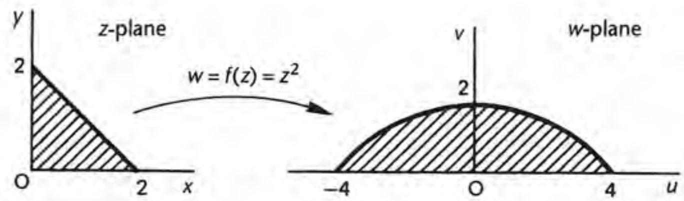
Mapping of regions

In this last example, the three lines AB, BC and CA form the boundary of a triangular region and we have seen how this boundary maps onto the boundary A′B′C′A′ in the w-plane. What we do not know yet is whether the points internal to the triangle map to points internal to the figure in the w-plane or to points external to it.



In the z-plane, the region is on the left-hand side as we proceed round the figure in the direction of the arrows ABCA. The region on the left-hand side as we proceed round the figure A′B′C′A′ in the w-plane determines that the transformed region in this case is, in fact, the internal region.

So



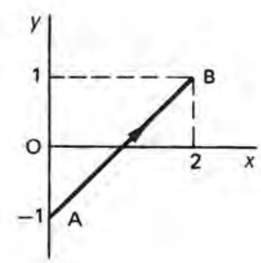
Therefore, every point in the region shaded in the z-plane maps onto a corresponding point in the region shaded in the w-plane.

43

2 *Transformation* $w = \dfrac{1}{z}$ (inversion)

Example 1

A straight line joining A $(-j)$ and B $(2+j)$ in the z-plane is mapped onto the w-plane by the transformation equation $w = \dfrac{1}{z}$.



Proceeding as before

$$w = \frac{1}{z}$$

$$\therefore\ u + jv = \frac{1}{x + jy}$$

$$= \frac{x - jy}{x^2 + y^2}$$

$$\therefore\ u = \frac{x}{x^2 + y^2};\quad v = \frac{-y}{x^2 + y^2}$$

First we map the end points A and B onto the w-plane.

A′: $w = \ldots\ldots\ldots\ldots$

B′: $w = \ldots\ldots\ldots\ldots$

44

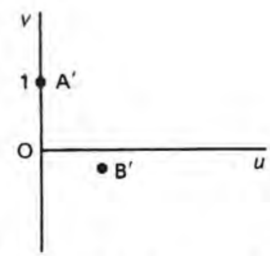
$$\boxed{\text{A}':\ w = j;\quad \text{B}': w = \frac{2}{5} - j\frac{1}{5}}$$

Because

A: $x = 0,\ y = -1$ $\quad\therefore$ A′: $u = 0,\ v = 1$ $\quad\therefore$ A′ is $w = j$

B: $x = 2,\ y = 1$ $\quad\therefore$ B′: $u = \frac{2}{5},\ v = -\frac{1}{5}$ $\quad\therefore$ B′ is $w = \frac{2}{5} - j\frac{1}{5}$

So far then we have



To determine the path A′B′, we can proceed as follows

$$w = \frac{1}{z} \quad \therefore\ z = \frac{1}{w} \quad \text{i.e.} \quad x + jy = \frac{1}{u + jv} = \frac{u - jv}{u^2 + v^2}$$

$$\therefore\ x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

The equation of AB is $y = x - 1$

$$\therefore\ \frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1$$

which simplifies into …………

45

$$\boxed{u^2 + v^2 - u - v = 0}$$

Because

$$\frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1 \quad \therefore\ -v = u - u^2 - v^2$$

$$\therefore\ u^2 + v^2 - u - v = 0$$

We can write this as $(u^2 - u) + (v^2 - v) = 0$ and completing the square in each bracket this becomes

$$\left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2}$$

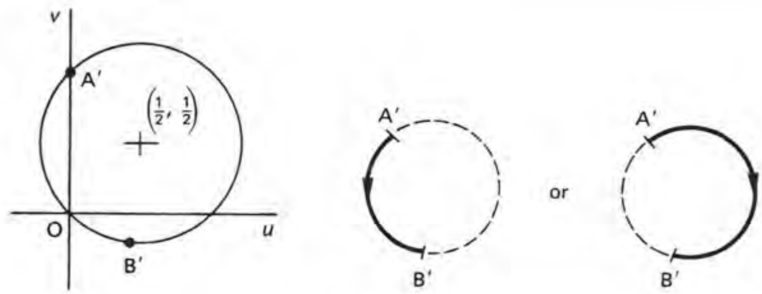
which we recognise as the equation of a …………

46

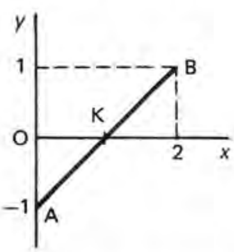
$$\boxed{\text{circle with centre } \left(\frac{1}{2}, \frac{1}{2}\right) \text{ and radius } \frac{1}{\sqrt{2}}}$$

The path joining A′ and B′ is therefore an arc of this circle.

But we still have problems, for it could be the minor arc or the major arc.



To decide which is correct, we take a further convenient point on the original line AB and determine its image on the w-plane.



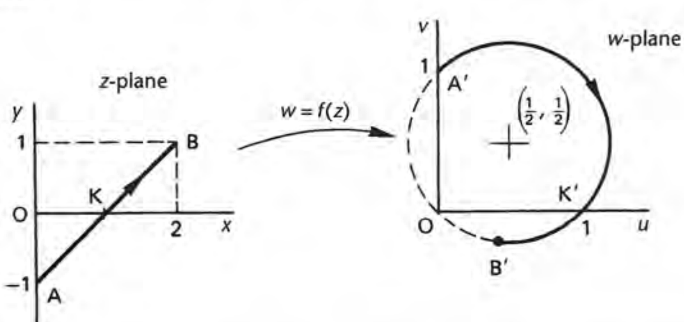
For instance, for K, $x = 1$, $y = 0$

$$\therefore \text{ For K}', \; u = \frac{x}{x^2 + y^2} = 1$$

$$v = \frac{-y}{x^2 + y^2} = 0$$

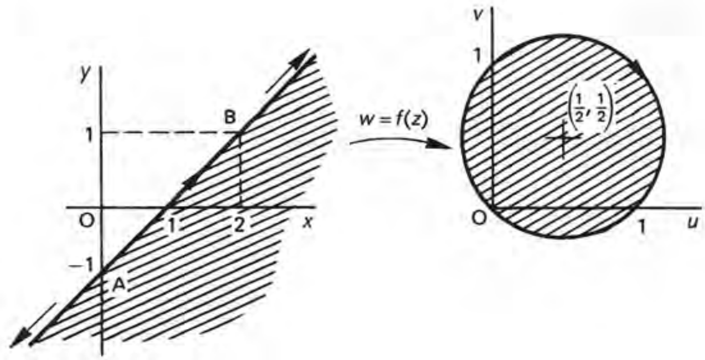
$\therefore$ K′ is the point $w = 1$

The path is, therefore, the major arc A′K′B′ developed in the direction indicated.



47

If we consider the line AB of the previous example extended to infinity in each direction, its image in the w-plane would then be the complete circle.



Furthermore, the line AB cuts the entire z-plane into two regions and

(a) the region on the right-hand side of the line relative to the arrowed direction maps onto the region inside the circle in the w-plane

(b) the region on the left-hand side of the line maps onto

. . . .

48

$\boxed{\text{the region outside the circle in the } w\text{-plane}}$

Let us now consider a general case.

Example 2

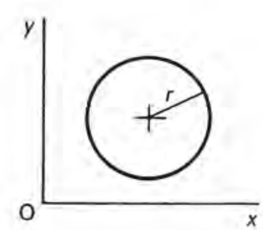
Determine the image in the w-plane of a circle in the z-plane under the inversion transformation $w = \dfrac{1}{z}$.

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with centre $(-g, -f)$

and radius $\sqrt{g^2 + f^2 - c}$.



It is convenient at times to write this as

$$A(x^2 + y^2) + Dx + Ey + F = 0$$

in which case

centre is . . . . and radius is . . . .

49

$$\boxed{\text{centre}\left(-\frac{D}{2A}, -\frac{E}{2A}\right); \quad \text{radius} = \frac{1}{2A}\sqrt{D^2 + E^2 - 4AF}}$$

Because

$$g = \frac{D}{2A}, \quad f = \frac{E}{2A}, \quad c = \frac{F}{A}.$$

As before we have $w = \dfrac{1}{z}$ $\quad \therefore \; z = \dfrac{1}{w}$

$$\therefore \; x + jy = \frac{1}{u + jv} = \frac{u - jv}{u^2 + v^2} \quad \therefore \; x = \frac{u}{u^2 + v^2}; \quad y = \frac{-v}{u^2 + v^2}$$

Then $A(x^2 + y^2) + Dx + Ey + F = 0$

becomes ....

Simplify it as far as possible

50

$$\boxed{A + Du - Ev + F(u^2 + v^2) = 0}$$

Because we have

$$\frac{A(u^2 + v^2)}{(u^2 + v^2)^2} + \frac{Du}{u^2 + v^2} - \frac{Ev}{u^2 + v^2} + F = 0$$

$$\therefore \; A + Du - Ev + F(u^2 + v^2) = 0$$

Changing the order of terms, this can be written

$$F(u^2 + v^2) + Du - Ev + A = 0$$

which is the equation of a circle with

centre ....; radius ....

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$$\boxed{\text{centre}\left(-\frac{D}{2F}, \frac{E}{2F}\right); \quad \text{radius}\,\frac{1}{2F}\sqrt{D^2 + E^2 - 4FA}}$$

Thus any circle in the z-plane transforms, with $w = \dfrac{1}{z}$, onto another circle in the w-plane.

We have already seen previously that, under inversion, a straight line also maps onto a circle. This may be regarded as a special case of the general result, if we accept a straight line as the circumference of a circle of .... radius.

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$\boxed{\text{infinite}}$

Because

$A(x^2 + y^2) + Dx + Ey + F = 0$

If $A = 0$, this becomes $Dx + Ey + F = 0$ i.e. a straight line

and also the centre $\left(-\dfrac{D}{2A}, -\dfrac{E}{2A}\right)$ becomes infinite

and the radius $\dfrac{1}{2A}\sqrt{D^2 + E^2 - 4AF}$ becomes infinite.

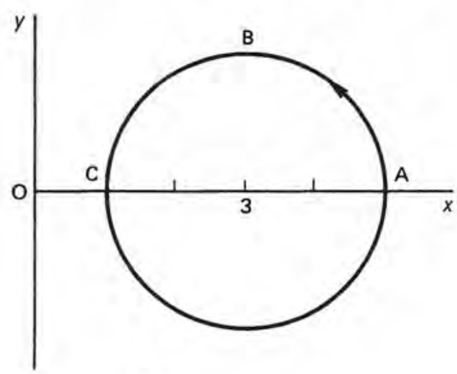
Therefore, combining the results we have obtained, we have this conclusion:

> Under inversion $w = \dfrac{1}{z}$, a circle or a straight line in the z-plane transforms onto a circle or a straight line in the w-plane.

Now for one more example.

Example 3

A circle in the z-plane has its centre at $z = 3$ and a radius of 2 units. Determine its image in the w-plane when transformed by $w = \dfrac{1}{z}$.



Equation of the circle is

$$(x - 3)^2 + y^2 = 4$$
$$x^2 - 6x + 9 + y^2 = 4$$
$$x^2 + y^2 - 6x + 5 = 0.$$

Using $w = \dfrac{1}{z}$, we can obtain x and y in terms of u and v.

$x = \ldots\ldots\ldots\ldots;\quad y = \ldots\ldots\ldots\ldots$

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$$\boxed{x = \frac{u}{u^2+v^2}; \quad y = \frac{-v}{u^2+v^2}}$$

Because $w = \dfrac{1}{z}$,

$$\therefore \; z = \frac{1}{w}$$

$$\therefore \; x + jy = \frac{1}{u+jv}$$

$$= \frac{u - jv}{u^2+v^2}$$

$$\therefore \; x = \frac{u}{u^2+v^2}; \quad y = \frac{-v}{u^2+v^2}$$

Substituting these in the equation of the circle, we get a relationship between u and v, which is

....

54

$$\boxed{5(u^2+v^2) - 6u + 1 = 0}$$

Because the circle is $x^2 + y^2 - 6x + 5 = 0$

$$\therefore \; \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{6u}{u^2+v^2} + 5 = 0$$

$$\frac{1}{u^2+v^2} - \frac{6u}{u^2+v^2} + 5 = 0$$

$$5(u^2+v^2) - 6u + 1 = 0$$

This is of the form $A(u^2+v^2) + Du + Ev + F = 0$
where $A = 5$, $D = -6$, $E = 0$, $F = 1$.

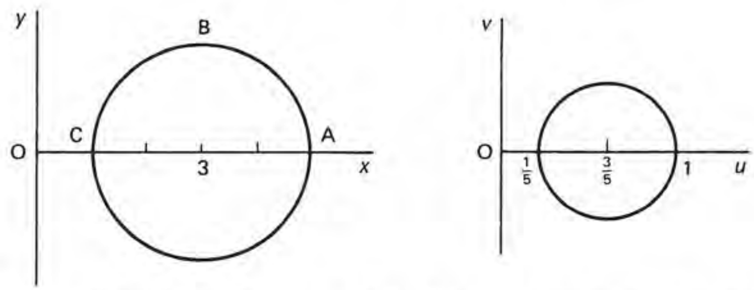
Therefore, the centre is ....
and the radius is ....

55

$$\boxed{\text{centre} = \left(\frac{3}{5},\ 0\right);\quad \text{radius} = \frac{2}{5}}$$

Because the centre is $\left(-\dfrac{D}{2A},\ -\dfrac{E}{2A}\right) = \left(\dfrac{6}{10},\ 0\right)$ i.e. $\left(\dfrac{3}{5},\ 0\right)$

and the radius $= \dfrac{1}{2A}\sqrt{D^2 + E^2 - 4AF} = \dfrac{1}{10}\sqrt{36 + 0 - 20} = \dfrac{2}{5}$.



Taking three sample points A, B, C as shown, we can map these onto the w-plane using $u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$.

A′: ....; B′ : ....; C′: ....

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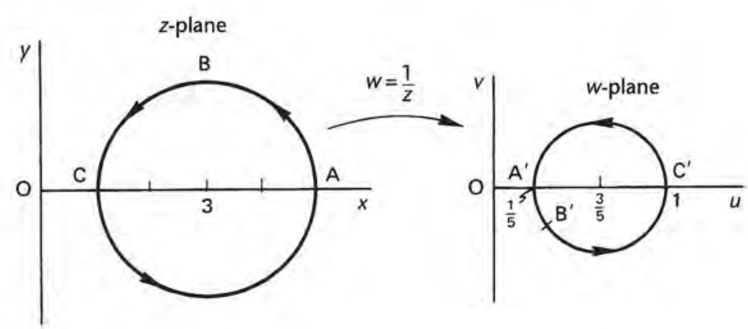
$$\boxed{A'\colon \left(\frac{1}{5}, 0\right);\quad B'\colon \left(\frac{3}{13},\ -\frac{2}{13}\right);\quad C'\colon (1,\ 0)}$$

So we finally have

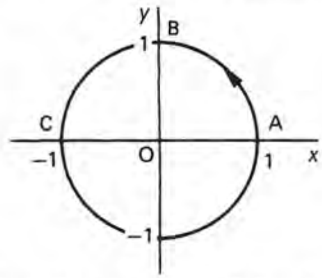


3 *Transformation* $w = \dfrac{1}{z - a}$

An extension of the method we have just applied occurs with transformations of the form $w = \dfrac{1}{z - a}$ where a is real or complex.

Example

A circle $|z| = 1$ in the z-plane is mapped onto the w-plane by $w = \dfrac{1}{z-2}$.



$$w = \frac{1}{z-2} \quad \therefore\ z - 2 = \frac{1}{w}$$
$$x + jy - 2 = \frac{1}{u + jv}$$
$$(x-2) + jy = \frac{u - jv}{u^2 + v^2}$$
$$\therefore\ x = \frac{u}{u^2+v^2} + 2; \quad y = \frac{-v}{u^2+v^2}$$

Cartesian equation of the circle is $x^2 + y^2 = 1$.

We then substitute the expressions for x and y in terms of u and v and obtain the relationship between u and v, which is ....

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$$\boxed{3(u^2+v^2) + 4u + 1 = 0}$$

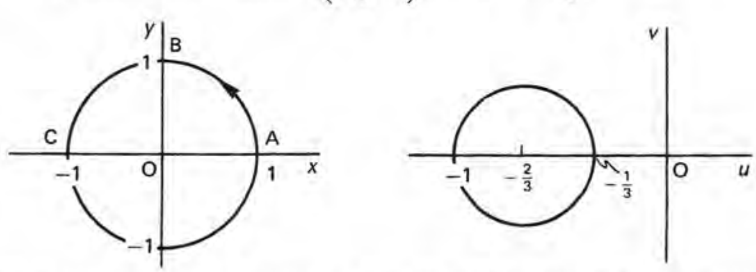
Because we have $\left\{\dfrac{u + 2(u^2+v^2)}{u^2+v^2}\right\}^2 + \left\{\dfrac{-v}{u^2+v^2}\right\}^2 = 1$

$$\{u + 2(u^2+v^2)\}^2 + v^2 = (u^2+v^2)^2$$
$$u^2 + 4u(u^2+v^2) + 4(u^2+v^2)^2 + v^2 = (u^2+v^2)^2$$
$$1 + 4u + 4(u^2+v^2) = u^2 + v^2$$
$$3(u^2+v^2) + 4u + 1 = 0$$

This can be expressed as

$$u^2 + \frac{4}{3}u + v^2 + \frac{1}{3} = 0$$
$$\left(u + \frac{2}{3}\right)^2 + v^2 = \left(\frac{1}{3}\right)^2$$

which is a circle with centre $\left(-\dfrac{2}{3},\ 0\right)$ and radius $\dfrac{1}{3}$.



To determine the direction of development relative to the arrowed direction in the z-plane, we consider the mapping of three sample points A, B, C as shown onto the w-plane, giving A′, B′, C′.

A′: ....; B′: ....; C′: ....

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$$\boxed{\text{A}'\text{:}\quad w = (-1, 0);\quad \text{B}'\text{:}\quad w = \left(-\frac{2}{5}, -\frac{1}{5}\right);\quad \text{C}'\text{:}\quad w = \left(\frac{1}{3}, 0\right)}$$

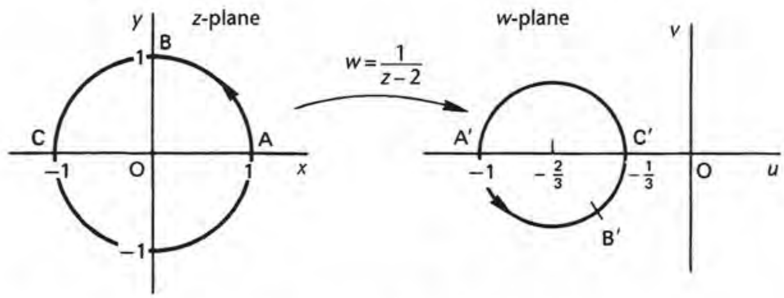
Because

$$\text{A: } z = 1 \qquad \therefore\ w = \frac{1}{z-2} = -1 \qquad \therefore\ \text{A}' = (-1, 0)$$

$$\text{B: } z = j \qquad \therefore\ w = \frac{1}{j-2} = \frac{j+2}{-5} \qquad \therefore\ \text{B}' = \left(-\frac{2}{5}, -\frac{1}{5}\right)$$

$$\text{C: } z = -1 \qquad \therefore\ w = -\frac{1}{3} \qquad \therefore\ \text{C}' = \left(-\frac{1}{3}, 0\right)$$

Whereupon we have



We now have one further transformation which is important, so move on to the next frame for a fresh start

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4 *Bilinear transformation* $w = \dfrac{az+b}{cz+d}$

Transformation of the form $w = \dfrac{az+b}{cz+d}$ where a, b, c, d are, in general, complex.

Note that

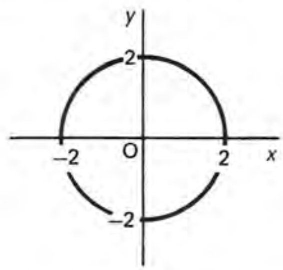
(a) if $cz + d = 1$, $w = az + b$, i.e. the general linear transformation

(b) if $az + b = 1$, $w = \dfrac{1}{cz+d}$, i.e. the form of inversion just considered.

▶

Example

Determine the image in the w-plane of the circle $|z| = 2$ in the z-plane under the transformation $w = \dfrac{z+j}{z-j}$ and show the region in the w-plane onto which the region within the circle is mapped.



We begin in very much the same way as before by expressing u and v in terms of x and y.

u ....; $v =$ ....

60

$$\boxed{u = \frac{x^2+y^2-1}{x^2+y^2-2y+1}; \quad v = \frac{2x}{x^2+y^2-2y+1}}$$

Because

$$w = u + jv = \frac{z+j}{z-j} = \frac{x+j(y+1)}{x+j(y-1)}$$
$$= \frac{\{x+j(y+1)\}\{x-j(y-1)\}}{\{x+j(y-1)\}\{x-j(y-1)\}}$$
$$= \frac{x^2+jx(y+1-y+1)+y^2-1}{x^2+(y-1)^2}$$
$$= \frac{x^2+y^2-1+j2x}{x^2+y^2-2y+1}$$
$$\therefore\ u = \frac{x^2+y^2-1}{x^2+y^2-2y+1} \quad \text{and} \quad v = \frac{2x}{x^2+y^2-2y+1}$$

But the equation of the circle is $x^2 + y^2 = 4$, so these expressions simplify to

$u =$ .... and $v =$ ....

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$$\boxed{u = \frac{3}{5-2y}; \quad v = \frac{2x}{5-2y}}$$

From these, we can form expressions for x and y in terms of u and v.

$x =$ ....; $y =$ ....

62

$$\boxed{x = \frac{3v}{2u}; \quad y = \frac{5u-3}{2u}}$$

Because, from the first, $y = \dfrac{5u-3}{2u}$ and substituting in the second gives $x = \dfrac{3v}{2u}$.

But $x^2 + y^2 = 4 \quad \therefore \quad \dfrac{9v^2}{4u^2} + \dfrac{(5u-3)^2}{4u^2} = 4$

which can be simplified to ....

63

$$\boxed{9(u^2 + v^2) - 30u + 9 = 0}$$

Because

$$9v^2 + 25u^2 - 30u + 9 = 16u^2 \quad \therefore \quad 9(u^2 + v^2) - 30u + 9 = 0.$$

Dividing through by 9, we can now rearrange this to

$$\left(u^2 - \frac{30}{9}u\right) + v^2 + 1 = 0$$

i.e. $$\left(u - \frac{5}{3}\right)^2 + v^2 + 1 - \frac{25}{9} = 0$$

$$\left(u - \frac{5}{3}\right)^2 + v^2 = \left(\frac{4}{3}\right)^2$$

which, you will recognise, is a circle in the w-plane with

centre .... and radius ....

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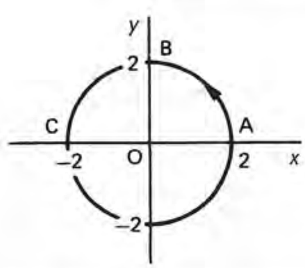
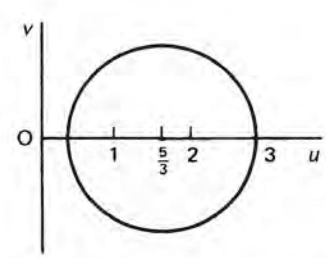
$$\boxed{\text{centre} = \left(\frac{5}{3}, 0\right); \quad \text{radius} = \frac{4}{3}}$$





To find the direction of development, we map three sample points A, B, C onto A′, B′, C′ as usual.

A′: ....; B′: ....; C′: ....

65

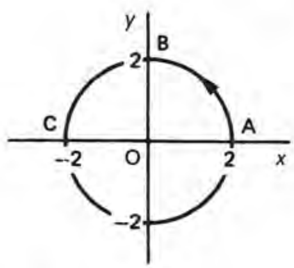
$$\boxed{A'\colon w=\frac{3}{5}+j\frac{4}{5};\quad B'\colon w=3;\quad C'\colon w=\frac{3}{5}-j\frac{4}{5}}$$

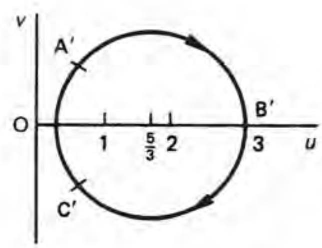
Because

$$\text{A: } z=2 \qquad \therefore\ w=\frac{2+j}{2-j}=\frac{(2+j)^2}{5}=\frac{4+j4-1}{5}=\tfrac{3}{5}+j\tfrac{4}{5}\quad \text{i.e. A}'$$

$$\text{B: } z=j2 \qquad \therefore\ w=\frac{j2+j}{j2-j}=\frac{j3}{j}=3\quad \therefore\ w=3\quad \text{i.e. B}'$$

$$\text{C: } z=-2 \qquad \therefore\ w=\frac{-2+j}{-2-j}=\frac{2-j}{2+j}=\frac{(2-j)^2}{5}=\tfrac{3}{5}-j\tfrac{4}{5}\quad \text{i.e. C}'$$





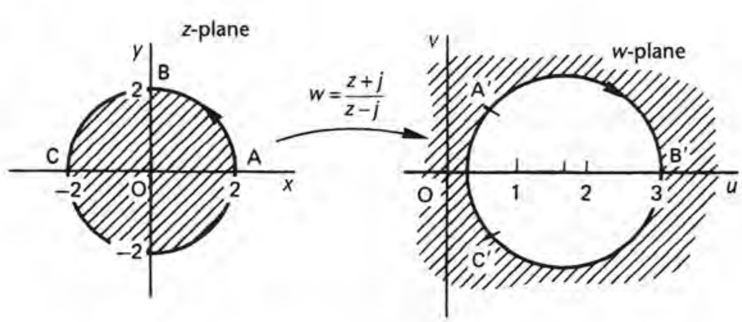
So an anticlockwise progression in the z-plane becomes a clockwise progression in the w-plane with this particular example.

Now we can complete the problem, for the region inside the circle in the z-plane maps onto .... in the w-plane.

66

the region outside the circle

Because the enclosed region in the z-plane is on the left-hand side of the direction of progression. The region on the left-hand side of the direction of progression in the w-plane is thus the region outside the transformed circle.



▶

And that brings us successfully to the end of this Programme. We shall pursue the topic further in the succeeding Programe. Meanwhile, all that remains is to check down the **Revision summary** and the **Can You?** checklist before working through the **Test exercise**. All very straightforward. The **Further problems** will give you valuable additional practice.

Revision summary 20

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1 *Transformation equation*

$z = x + jy \qquad w = u + jv$

The transformation equation is the relationship between z and w, i.e. $w = f(z)$.

2 *Linear transformation* $w = az + b$ where a and b are real or complex. A straight line in the z-plane maps onto a corresponding straight line in the w-plane.

3 *Types of transformation* $w = az + b$

(a) *magnification* – given by $|a|$

(b) *rotation* – given by arg a

(c) *translation* – given by b.

4 *Non-linear transformation*

(a) $w = z^2$

A straight line through the origin maps onto a corresponding straight line through the origin in the w-plane. A straight line not passing through the origin maps onto a parabola.

(b) $w = \dfrac{1}{z}$ (inversion)

A straight line or a circle maps onto a straight line or a circle in the w-plane.

A straight line may be regarded as a circle of infinite radius.

(c) $w = \dfrac{az + b}{cz + d}$ (bilinear transformation) – with a, b, c, d real or complex.

5 *Mapping of a region* depends on the direction of development. Right-hand regions map onto right-hand regions: left-hand regions onto left-hand regions.

Can You?

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Checklist 20

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: — Frames

- Recognise the transformation equation in the form $w = f(z) = u(x, y) + jv(x, y)$? — Frames 1 and 2

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Illustrate the image of a point in the complex z-plane under a complex mapping onto the w-plane? — Frames 2 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Map a straight line in the z-plane onto the w-plane under the transformation $w = f(z)$? — Frames 7 to 16

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Identify complex mappings that form translations, magnifications, rotations and their combinations? — Frames 16 to 31

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Deal with the non-linear transformations $w = z^2$, $w = 1/z$, $w = 1/(z - a)$ and $w = (az + b)/(cz + d)$? — Frames 32 to 66

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 20

69

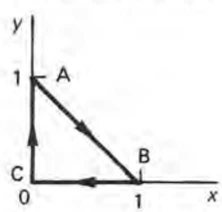
1 Map the following points in the z-plane onto the w-plane under the transformation $w = f(z)$.

(a) $z = 3 + j2$; $w = 2z - j6$ (c) $z = j(1 - j)$; $w = (2 + j)z - 1$

(b) $z = -2 + j$; $w = 4 + jz$ (d) $z = j - 2$; $w = (1 - j)(z + 3)$.

2 Map the straight line joining A $(2 - j)$ and B $(4 - j3)$ in the z-plane onto the w-plane using the transformation $w = (1 + j2)z + 1 - j3$. State the magnification, rotation and translation involved.

3 A triangle ABC in the z-plane as shown is mapped onto the w-plane under the transformation $w = z^2$.



Determine the image in the w-plane and indicate the mapping of the interior triangular region ABC.

4 Map the straight line joining A ($z = j$) and B ($z = 3 + j4$) in the z-plane onto the w-plane under the inversion transformation $w = \dfrac{1}{z}$.
Sketch the image of AB in the w-plane.

5 The unit circle $|z| = 1$ in the z-plane is mapped onto the w-plane by $w = \dfrac{1}{z - j2}$. Determine (a) the position of the centre and (b) the radius of the circle obtained.

6 The circle $|z| = 2$ is mapped onto the w-plane by the transformation $w = \dfrac{z + j2}{z + j}$. Determine the centre and radius of the resulting circle in the w-plane.

Further problems 20

1 A triangle ABC in the z-plane with vertices A $(-1 - j)$, B $(2 + j2)$, C $(-1 + j2)$ is mapped onto the w-plane under the transformation $w = (1 - j)z + (1 + j2)$. Determine the image A′B′C′ of ABC in the w-plane. **70**

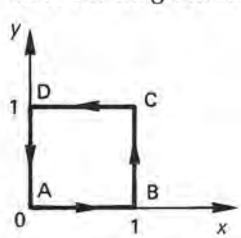
2 The straight line joining A $(1 + j2)$ and B $(4 - j3)$ in the z-plane is mapped onto the w-plane by the transformation equation $w = (2 + j5)z$. Determine (a) the images of A and B, (b) the magnification, rotation and translation involved.

3 Map the straight line joining A $(-2 + j3)$ and B $(1 + j2)$ in the z-plane onto the w-plane using the transformation equation

$$w = (-3 + j)z + 2 + j4.$$

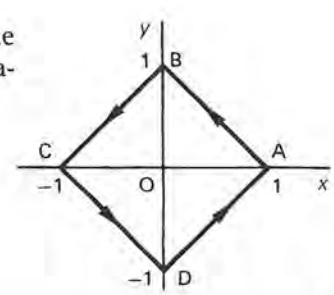
State the magnification, rotation and translation occurring in the process.

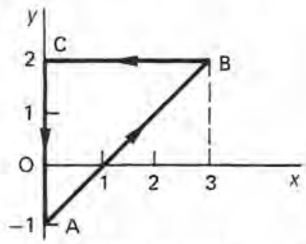
4



Transform the square ABCD in the z-plane onto the w-plane under the transformation $w = z^2$.

5 Map the square ABCD in the z-plane onto the w-plane using the transformation $w = 2z^2 + 2$.



6 

The triangle ABC in the z-plane is mapped onto the w-plane by the transformation $w = j2z^2 + 1$. Determine the image of ABC in the w-plane.

7 A circle in the z-plane has its centre at the point $(-\frac{3}{4} - j)$ and radius $\frac{7}{4}$. Show that its Cartesian equation can be expressed as

$$2(x^2 + y^2) + 3x + 4y - 3 = 0$$

Determine the image of the circle in the w-plane under the inversion transformation $w = \dfrac{1}{z}$.

8 The transformation $w = \dfrac{1}{z - 1}$ is applied to the circle $|z| = 2$ in the z-plane. Determine

(a) the image of the circle in the w-plane

(b) the region in the w-plane onto which the region enclosed within the circle in the z-plane is mapped.

9 The circle $|z| = 4$ is described in the z-plane in an anticlockwise manner. Obtain its image in the w-plane under the transformation $w = \dfrac{z + 1}{z - 2}$ and state the direction of development.

10 The bilinear transformation $w = \dfrac{z - j}{z + j2}$ is applied to the circle $|z| = 3$ in the z-plane. Determine the equation of the image in the w-plane and state its centre and radius.

11 The unit circle $|z| = 1$ in the z-plane is mapped onto the w-plane under the transformation $w = \dfrac{z - 1}{z - 3}$. Determine the equation of its image and the region onto which the region within the circle is mapped.

12 Obtain the image of the unit circle $|z| = 1$ in the z-plane under the transformation $w = \dfrac{z + j3}{z - j2}$.

13 The circle $|z| = 2$ is mapped onto the w-plane by the transformation $w = \dfrac{z + j}{2z - j}$. Determine

(a) the image of the circle in the w-plane

(b) the mapping of the region enclosed by $|z| = 2$.

14 Show that the transformation equation $w = \dfrac{z - a}{z - b}$ where $z = x + jy$, $a = 1 + j4$ and $b = 2 + j3$, transforms the circle $(x - 3)^2 + (y - 5)^2 = 5$ into a straight line through the origin in the w-plane.

Complex analysis 2

Frames 1 to 79

Learning outcomes

When you have completed this Programme you will be able to:

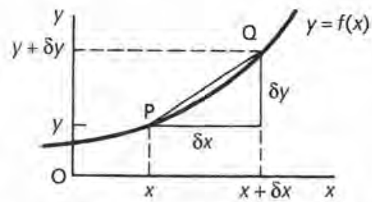
- Appreciate when the derivative of a function of a complex variable exists
- Understand the notions of regular functions and singularities and be able to obtain the derivative of a regular function from first principles
- Derive the Cauchy–Riemann equations and apply them to find the derivative of a regular function
- Understand the notion of an harmonic function and derive a conjugate function
- Evaluate line and contour integrals in the complex plane
- Derive and apply Cauchy's theorem
- Apply Cauchy's theorem to contours around regions that contain singularities
- Define the essential characteristics of and conditions for a conformal mapping
- Locate critical points of a function of a complex variable
- Determine the image in the w-plane of a figure in the z-plane under a conformal transformation $w = f(z)$
- Describe and apply the Schwarz–Christoffel transformation

1

In the previous Programme we introduced the ideas of mapping from one complex plane to another and considered some of the more common transformation functions. Now we pursue our consideration of the complex variable a little further.

Differentiation of a complex function

In differentiation of a function of a single real variable, $y = f(x)$, the derivative of y with respect to x can be defined as the limiting value of $\frac{(y+\delta y)-y}{\delta x}$ as δx tends to zero.



$$y = f(x) \quad \delta y = f(x+\delta x) - f(x)$$

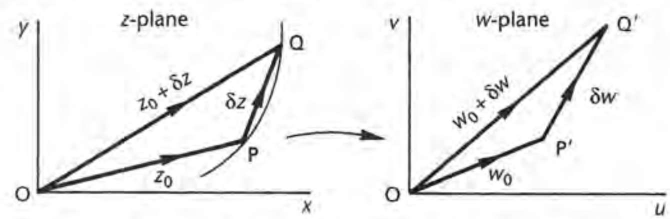
$$\text{i.e. } \frac{\mathrm{d}y}{\mathrm{d}x} = \underset{\delta x \to 0}{Lim}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$$

In considering the differentiation of a function of a complex variable, $w = f(z)$, the derivative of w with respect to z can similarly be defined as the limiting value of .... as δz tends to zero.

2

$$\boxed{\frac{(w+\delta w)-w}{\delta z} \quad \text{i.e.} \quad \frac{f(z+\delta z)-f(z)}{\delta z}}$$

Now, of course, we are dealing in vectors.



If P and Q in the z-plane map onto P′ and Q′ in the w-plane, then

$$\mathrm{P'Q'} = \delta w = (w_0 + \delta w) - w_0 = f(z_0 + \delta z) - f(z_0)$$

Therefore, the derivative of w at P′ $(z = z_0)$ is the limiting value of $\frac{\delta w}{\delta z}$ as

$$\delta z \to 0, \text{ i.e. } \left[\frac{\mathrm{d}w}{\mathrm{d}z}\right]_{z_0} = \underset{\delta z \to 0}{Lim}\left\{\frac{f(z_0+\delta z) - f(z_0)}{\delta z}\right\} = \underset{\mathrm{Q}\to\mathrm{P}}{Lim}\left(\frac{\mathrm{P'Q'}}{\mathrm{PQ}}\right)$$

If this limiting value exists – which is not always the case as we shall see – the function $f(z)$ is said to be *differentiable at* P.

Also, if $w = f(z)$ and $f'(z)$ has a limit for all points z_0 within a given region for which $w = f(z)$ is defined, then $f(z)$ is said to be differentiable in that region. From this, it follows that the limit exists whatever the path of approach from Q $(z = z_0 + \delta z)$ to P $(z = z_0)$.

Regular function

A function $w = f(z)$ is said to be *regular* (or analytic) at a point $z = z_0$, if it is defined and single-valued, and has a derivative at every point at and around z_0. Points in a region where $f(z)$ ceases to be regular are called *singular points*, or *singularities*.

A function of a complex variable that is analytic over the entire finite complex plane is called an *entire* function. Examples of entire functions are polynomials, e^z, $\sin z$ and $\cos z$.

We have introduced quite a few new definitions, so let us pause here while you make a note of them. We shall be meeting the various terms quite often.

3

In those cases where a derivative exists, the usual rules of differentiation apply. For example, the derivative of $w = z^2$ can be found from first principles in the normal way.

$$w = z^2 \quad \therefore \ w + \delta w = (z + \delta z)^2 = z^2 + 2z\delta z + \delta z^2$$

$$\therefore \ \delta w = 2z\delta z + \delta z^2 \quad \therefore \ \frac{\delta w}{\delta z} = 2z + \delta z$$

$\therefore \ \dfrac{dw}{dz} = \underset{\delta z \to 0}{Lim}\,(2z + \delta z) = 2z$ and does not depend on the path along which δz tends to zero.

That was elementary. Here is a rather different one.

Example

To find the derivative of $w = z\bar{z}$ where $z = x + jy$ and $\bar{z} = x - jy$.

We have $w = z\bar{z} \quad \therefore \ w + \delta w = (z + \delta z)(\bar{z} + \delta\bar{z})$ from which

$$\frac{\delta w}{\delta z} = \dots\dots\dots\dots$$

4

$$\boxed{\frac{\delta w}{\delta z} = \bar{z} + z\frac{\delta\bar{z}}{\delta z} + \delta\bar{z}}$$

Because

$$w + \delta w = (z + \delta z)(\bar{z} + \delta\bar{z}) = z\bar{z} + \bar{z}\delta z + z\delta\bar{z} + \delta z\delta\bar{z}$$

$$\therefore \ \delta w = \bar{z}\delta z + z\delta\bar{z} + \delta z\delta\bar{z} \quad \therefore \ \frac{\delta w}{\delta z} = \bar{z} + z\frac{\delta\bar{z}}{\delta z} + \delta\bar{z}$$

Now since $z = x + jy$ and $\bar{z} = x - jy$, we can express $\dfrac{\delta w}{\delta z}$ in terms of x and y. $\quad \dfrac{\delta w}{\delta z} = \dots\dots\dots\dots$

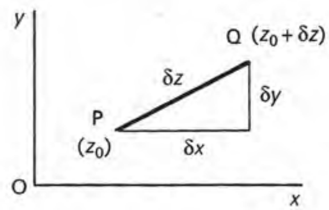
5

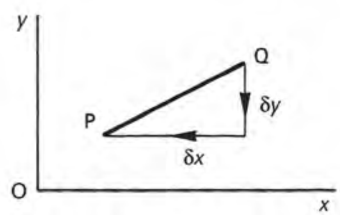
$$\boxed{\frac{\delta w}{\delta z} = (x - jy) + (x + jy)\left\{\frac{\delta x - j\delta y}{\delta x + j\delta y}\right\} + \delta x - j\delta y}$$

Because

$$\left.\begin{aligned} z &= x + jy \quad \therefore\ \delta z = \delta x + j\delta y \\ \bar{z} &= x - jy \quad \therefore\ \delta\bar{z} = \delta x - j\delta y \end{aligned}\right\} \quad \therefore\ \frac{\delta\bar{z}}{\delta z} = \frac{\delta x - j\delta y}{\delta x + j\delta y}$$

Then $\dfrac{\delta w}{\delta z} = \bar{z} + z\dfrac{\delta\bar{z}}{\delta z} + \delta\bar{z}$ gives the expression quoted above.





The next step is to reduce δz to zero. But δz consists of $\delta x + j\delta y$ and so reducing δz to zero can be done in one of two ways.

(1) First let $\delta y \to 0$ and afterwards let $\delta x \to 0$.

If $\delta y \to 0$, $\quad \dfrac{\delta w}{\delta z} = x - jy + (x + jy)\dfrac{\delta\bar{x}}{\delta x} + \delta\bar{x}$

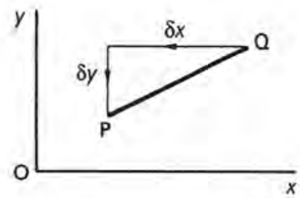
Then $\dfrac{\mathrm{d}w}{\mathrm{d}z} = \underset{\delta x \to 0}{Lim}\{x - jy + x + jy + \delta x\}$

$= \dots\dots\dots\dots$

6

$$\boxed{\frac{dw}{dz} = 2x}$$

On the other hand, we could have reduced δz to zero in the second way.



(2) First let $\delta x \to 0$ and afterwards let $\delta y \to 0$.

We have $\quad \dfrac{\delta w}{\delta z} = x - jy + (x + jy)\left\{\dfrac{\delta x - j\delta y}{\delta x + j\delta y}\right\} + \delta x - j\delta y$

If $\delta x \to 0 \quad \dfrac{\delta w}{\delta z} = x - jy + (x + jy)(-1) - j\delta y = -j2y - j\delta y$

Then $\quad \dfrac{dw}{dz} = \underset{\delta y \to 0}{Lim}\{-j2j - j\delta y\} = -j2y$

So, in the first case, $\dfrac{dw}{dz} = 2x$ and in the second case $\dfrac{dw}{dz} = -j2y$.

These two results are clearly not the same for all values of x and y – with one exception, i.e.

. . . .

7

$$\boxed{\text{when } x = y = 0}$$

Therefore $w = z\bar{z}$ is a function that has no specific derivative, except at $z = 0$ – and there are others. It would be convenient, therefore, to have some form of test to see whether a particular function $w = f(z)$ has a derivative $f'(z)$ at $z = z_0$. This useful tool is provided by the Cauchy–Riemann equations.

Cauchy–Riemann equations

The development is very much along the same lines as in the previous example. If $w = f(z) = u + jv$, we have to establish conditions for $w = f(z)$ to have a derivative at a given point $z = z_0$.

$$w = u + jv \quad \therefore\ \delta w = \delta u + j\delta v; \qquad z = x + jy \quad \therefore\ \delta z = \delta x + j\delta y$$

$$\text{Then} \quad f'(z) = \frac{dw}{dz} = \underset{\delta z \to 0}{Lim}\left\{\frac{\delta u + j\delta v}{\delta z}\right\} = \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{Lim}\left\{\frac{\delta u + j\delta v}{\delta x + j\delta y}\right\} \qquad (1)$$

(a) Let $\delta x \to 0$, followed by $\delta y \to 0$

Then from (1) above, $\quad f'(z) = \dfrac{dw}{dz} = \ldots\ldots\ldots\ldots$

8

$$\boxed{\frac{dw}{dz} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}}$$

Because

$$f'(z) = \underset{\delta y \to 0}{Lim}\left\{\frac{\delta u + j\delta v}{j\delta y}\right\} = \underset{\delta y \to 0}{Lim}\left\{\frac{\delta v}{\delta y} - j\frac{\delta u}{\delta y}\right\} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} \qquad (2)$$

We use the 'partial' notation since u and v are functions of both x and y.

Or (b) Let $\delta y \to 0$, followed by $\delta x \to 0$.

This gives ....

9

$$\boxed{\frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}}$$

Because

$$f'(z) = \underset{\delta x \to 0}{Lim}\left\{\frac{\delta u + j\delta v}{\delta x}\right\} = \underset{\delta x \to 0}{Lim}\left\{\frac{\delta u}{\delta x} + j\frac{\delta v}{\delta x}\right\} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \qquad (3)$$

If the results (2) and (3) are to have the same value for $f'(z)$ irrespective of the path chosen for δz to tend to zero, then

....

10

$$\boxed{\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y}}$$

Equating real and imaginary parts, this gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the *Cauchy–Riemann equations*.

So, to sum up:

A necessary condition for $w = f(z) = u + jv$ to be regular at $z = z_0$ is that u, v and their partial derivatives are continuous and that in the neighbourhood of $z = z_0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Make a note of this important result – then move on to the next frame

11

We said earlier that where a function fails to be regular, a *singular point*, or *singularity* occurs, for example where $w = f(z)$ is not continuous or where the Cauchy–Riemann test fails.

Exercise

Determine where each of the following functions fails to be regular, i.e. where singularities occur.

1 $w = z^2 - 4$

2 $w = \dfrac{z}{z-2}$

3 $w = \dfrac{z+5}{z+1}$

4 $w = \dfrac{1}{(z-2)(z-3)}$

5 $w = z\bar{z}$

6 $w = \dfrac{x + jy}{x^2 + y^2}$

Finish all six: then check with the next frame

12

Conclusions:

1 Putting $z = x + jy$, the Cauchy–Riemann conditions are satisfied everywhere. Therefore, no singularity in $w = z^2 - 4$.

2 The function becomes discontinuous at $z = 2$. Singularity at $z = 2$.

3 The function is discontinuous at $z = -1$. Singularity at $z = -1$.

4 Singularities at $z = 2$ and $z = 3$.

5 We have already seen that $w = z\bar{z}$ has no derivative for all values of z apart from $z = 0$. All points on $w = z\bar{z}$ are singularities.

6 Singularity occurs where $x^2 + y^2 = 0$, i.e. $x = 0$ and $y = 0$ $\therefore$ $z = 0$. At all other points the Cauchy–Riemann equations do not hold.

Harmonic functions

13

If a function of two real variables $f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 0$$

then we say that $f(x, y)$ is an *harmonic* function. It is relatively straightforward to demonstrate that the real and imaginary parts of an analytic function are both harmonic.

Let $f(z) = u(x, y) + jv(x, y)$ be an analytic function in some region of the z-plane. Because $f(z)$ is analytic the Cauchy–Riemann equations hold true. That is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Differentiating the first with respect to x and the second with respect to y shows us that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}$$

$$\text{and so } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

By a similar reasoning

$$\frac{\partial^2 \ldots}{\partial x^2} + \frac{\partial^2 \ldots}{\partial y^2} = 0$$

14

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Because

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial y} \text{ and } \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\text{and so } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

The functions $u(x, y)$ and $v(x, y)$ are called *conjugate* functions. In addition, the curves $u = \text{constant}$, $v = \text{constant}$ are orthogonal.

Example 1

Show that the real and imaginary parts of the function defined by $f(z) = z^2$ are harmonic.

$$\begin{aligned} f(z) &= z^2 \\ &= (x + jy)^2 \\ &= (x^2 - y^2) + 2jxy \end{aligned}$$

and so $u = x^2 - y^2$ and $v = 2xy$ and therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \ldots\ldots\ldots\ldots \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \ldots\ldots\ldots\ldots$$

15

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ and } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0}$$

Because

$$\frac{\partial u}{\partial x}=2x \text{ so } \frac{\partial^2 u}{\partial x^2}=2 \text{ and } \frac{\partial u}{\partial y}=-2y \text{ so } \frac{\partial^2 u}{\partial y^2}=-2$$

$$\text{therefore } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

and

$$\frac{\partial v}{\partial x}=2y \text{ so } \frac{\partial^2 v}{\partial x^2}=0 \text{ and } \frac{\partial v}{\partial y}=2x \text{ so } \frac{\partial^2 v}{\partial y^2}=0$$

$$\text{therefore } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Example 2

Show that $u(x, y)=x^3y-y^3x$ is an harmonic function and find the function $v(x, y)$ that ensures that $f(z)=u(x,u)+jv(x, y)$ is analytic. That is, find the function $v(x, y)$ that is conjugate to $u(x, y)$.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}= \dots\dots\dots\dots$$

16

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

Because

$$\frac{\partial u}{\partial x}=3x^2y-y^3 \text{ so } \frac{\partial^2 u}{\partial x^2}=6xy \text{ and } \frac{\partial u}{\partial y}=x^3-3y^2x \text{ so } \frac{\partial^2 u}{\partial y^2}=-6xy$$

$$\text{therefore } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

This means that $u(x, y)=x^3y-y^3x$ is harmonic.

Now, if $f(z)=u(x,u)+jv(x, y)$ is analytic then $u(x, y)$ and $v(x, y)$ satisfy the .... equations.

17

$$\boxed{\text{Cauchy–Riemann}}$$

That is

$$\frac{\partial u}{\partial x} = 3x^2y - y^3 = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = x^3 - 3y^2x = -\frac{\partial v}{\partial x}$$

Integrating $\frac{\partial v}{\partial y} = 3x^2y - y^3$ with respect to y gives

$$v(x, y) = \ldots\ldots\ldots\ldots$$

18

$$\boxed{v(x, y) = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + a(x)}$$

Because

$\frac{\partial v}{\partial y} = 3x^2y - y^3$ and so x is treated as a constant and the integral of y^n is $y^{n+1}/(n+1)$.

Did you miss the constant term in the form of $a(x)$? *Because x is treated as a constant, the integration determines y up to an expression involving x.* Differentiate the result with respect to y and you will reclaim the original form for $\frac{\partial v}{\partial y}$.

Now, differentiating this expression with respect to x gives

$$\frac{\partial v}{\partial x} = \ldots\ldots\ldots\ldots$$

19

$$\boxed{\frac{\partial v}{\partial x} = 3xy^2 + a'(x)}$$

Because

$v(x, y) = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + a(x)$ and so $\frac{\partial v}{\partial x} = 3xy^2 + a'(x)$ and this is equal to $-\frac{\partial u}{\partial y}$. Now $-\frac{\partial u}{\partial y} = -x^3 + 3y^2x$ and so

$a'(x) = \ldots\ldots\ldots\ldots$ giving $a(x) = \ldots\ldots\ldots\ldots$

Therefore $v(x, y) = \ldots\ldots\ldots\ldots$

20

$$\boxed{\begin{array}{l} a'(x) = -x^3 \text{ giving } a(x) = -\dfrac{x^4}{4} + C. \\ \text{Therefore } v(x, y) = \dfrac{3x^2y^2}{2} - \dfrac{y^4}{4} - \dfrac{x^4}{4} + C \end{array}}$$

Because

Comparing $\dfrac{\partial v}{\partial x} = 3xy^2 + a'(x)$ and $-\dfrac{\partial u}{\partial y} = -x^3 + 3y^2x$

where $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ then it is seen that $a'(x) = -x^3$.

Therefore $a(x) = -\dfrac{x^4}{4} + C$ giving $v(x, y) = \dfrac{3x^2y^2}{2} - \dfrac{y^4}{4} - \dfrac{x^4}{4} + C$

Try one for yourself.

Example 3

Given $u(x, y) = e^{-x}\cos y$, show that $u(x, y)$ is an harmonic function and find the function $v(x, y)$ that ensures that $f(z) = u(x, y) + jv(x, y)$ is analytic. That is, find the function $v(x, y)$ that is conjugate to $u(x, y)$.

$$\frac{\partial^2 \ldots}{\partial x^2} + \frac{\partial^2 \ldots}{\partial y^2} = \ldots\ldots\ldots\ldots$$

21

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Because

$u = e^{-x}\cos y$ so $\dfrac{\partial u}{\partial x} = -e^{-x}\cos y$ and $\dfrac{\partial^2 u}{\partial x^2} = e^{-x}\cos y$.

Also $\dfrac{\partial u}{\partial y} = -e^{-x}\sin y$ so $\dfrac{\partial^2 u}{\partial y^2} = -e^{-x}\cos y$. Therefore $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, that is $u(x, y)$ is harmonic. The conjugate function $v(x, y)$ is then

$$v(x, y) = \ldots\ldots\ldots\ldots$$

22

$$\boxed{v = -e^{-x}\sin y + C}$$

Because

By the Cauchy–Riemann equation $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -e^{-x}\cos y$. Integrating with respect to y gives $v = -e^{-x}\sin y + a(x)$. Differentiating this with respect to x gives $\dfrac{\partial v}{\partial x} = e^{-x}\sin y + a'(x)$.

Now, by the other Cauchy–Riemann equation $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = e^{-x}\sin y$, so that $a'(x) = 0$ giving $a(x) = C$. Therefore, $v = -e^{-x}\sin y + C$.

Now we shall look at complex integration. Move to the next frame

Complex integration

23

At the beginning of this Programme, we defined differentiation with respect to z in the case of a complex function, since z is a function of two independent variables x and y, i.e. $z = x + jy$. Complex integration is approached in the same way.

$z = x + jy$ and $w = f(z) = u + jv$ where u and v are also functions of x and y.

Also $\mathrm{d}z = \mathrm{d}x + j\,\mathrm{d}y$ and $\mathrm{d}w = \mathrm{d}u + j\,\mathrm{d}v$

$$\therefore \int w\,\mathrm{d}z = \int f(z)\,\mathrm{d}z = \int (u + jv)(\mathrm{d}x + j\,\mathrm{d}y)$$

$$= \int \{(u\,\mathrm{d}x - v\,\mathrm{d}y) + j(v\,\mathrm{d}x + u\,\mathrm{d}y)\}$$

$$\therefore \int f(z)\,\mathrm{d}z = \int (u\,\mathrm{d}x - v\,\mathrm{d}y) + j\int (v\,\mathrm{d}x + u\,\mathrm{d}y)$$

That is, the integral reduces to two real-variable integrals

$$\int (u\,\mathrm{d}x - v\,\mathrm{d}y) \quad \text{and} \quad \int (v\,\mathrm{d}x + u\,\mathrm{d}y)$$

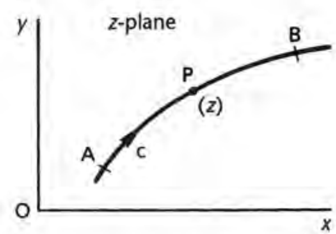
Note that each of these two integrals is of the general form $\int (P\,\mathrm{d}x + Q\,\mathrm{d}y)$ which we met before during our work on *line integrals* and, in the complex plane, this rather neatly leads us into *contour integration.*

Let us make a fresh start

24

Contour integration – line integrals in the *z*-plane

If z moves along the curve c in the z-plane and at each position z has associated with it a function of z, i.e. $f(z)$, then summing up $f(z)$ for all such points between A and B means that we are evaluating a line integral in the z-plane between A ($z = z_1$) and B ($z = z_2$) along the curve c,



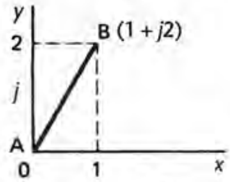
i.e. we are evaluating $\int_c f(z)\,\mathrm{d}z$ where c is the particular path joining A to B.

The evaluation of line integrals in the complex plane is known as *contour integration*. Let us see how it works in practice.

25

Example

Evaluate the integral $\int_c f(z)\,dz$ where $f(z) = (z - j)^2$ and c is the straight line joining A $(z = 0)$ to B $(z = 1 + j2)$.



$$z = x + jy; \quad dz = dx + j\,dy$$

$$f(z) = (z - j)^2 = \{x + j(y - 1)\}^2 = x^2 - (y - 1)^2 + j\,2x(y - 1)$$

$$\therefore\ I = \int\{(x^2 - y^2 + 2y - 1) + j(2xy - 2x)\}\{dx + j\,dy\}$$

$$= \int\{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$

$$+ j\int\{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\}$$

Now the equation of AB is $y = 2x$. $\therefore$ $dy = 2\,dx$ and substituting these in the expression for I, between the limits $x = 0$ and $x = 1$, gives

$I = \dots\dots\dots\dots$ Finish it.

26

$$\boxed{I = \frac{1}{3}(-2 + j)}$$

Because

$$I = \int_0^1\{(x^2 - 4x^2 + 4x - 1)\,dx - (4x^2 - 2x)2\,dx\}$$

$$+ j\int_0^1\{(4x^2 - 2x)\,dx + (2x^2 - 8x^2 + 8x - 2)\,dx\}$$

$$= \int_0^1(-11x^2 + 8x - 1)\,dx + j\int_0^1(-2x^2 + 6x - 2)\,dx$$

and this, by elementary integration, gives $I = \frac{1}{3}(-2 + j)$.

Now you will remember that, in general, the value of a line integral depends on the path of integration between the end points, but that the line integral $\int(P\,dx + Q\,dy)$ is independent of the path of integration in a simply connected region if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout the region.

In our example

$$I = \int\{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$

$$+ j\int\{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\} \equiv I_1 + jI_2$$

If we apply the test to I_1, we get $\dots\dots\dots\dots$

27

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Because

$$\text{for } I_1 = \int\{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dx\} \equiv \int(P\,dx + Q\,dy)$$

$$\left.\begin{aligned} \therefore\ P &= x^2 - y^2 + 2y - 1 \ \therefore\ \frac{\partial P}{\partial y} = -2y + 2 \\ Q &= -2xy + 2x \qquad\quad \therefore\ \frac{\partial Q}{\partial x} = -2y + 2 \end{aligned}\right\}\ \therefore\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

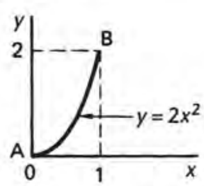
Similarly

$$\text{for } I_2 = \int\{(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)dy\} \equiv \int(Pdx + Qdy)$$

$$\left.\begin{aligned} \therefore\ P &= 2xy - 2x \qquad\quad \therefore\ \frac{\partial P}{\partial y} = 2x \\ Q &= x^2 - y^2 + 2y - 1 \ \therefore\ \frac{\partial Q}{\partial x} = 2x \end{aligned}\right\}\ \therefore\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Therefore, in this example, the value of the line integral is independent of the path of integration.

Just to satisfy our conscience, determine the value of the line integral between the same two end points, but along the parabola $y = 2x^2$.



$$f(z) = (z - j)^2$$

$$y = 2x^2 \quad \therefore\ dy = 4x\,dx$$

As before we have

$$I = \int\{(x^2 - y^2 + 2y - 1)\,dx - (2xy - 2x)\,dy\}$$

$$+ j\int(2xy - 2x)\,dx + (x^2 - y^2 + 2y - 1)\,dy\}$$

Substituting $y = 2x^2$ and $dy = 4x\,dx$, the evaluation gives

$$I = \ldots\ldots\ldots\ldots$$

28

$$\boxed{I = \frac{1}{3}(-2+j)}$$

We have

$$I = \int_0^1 \{(x^2 - 4x^4 + 4x^2 - 1)\,dx - (4x^3 - 2x)4x\,dx\}$$
$$+ j\int_0^1 \{(4x^3 - 2x)\,dx + (x^2 - 4x^4 + 4x^2 - 1)4x\,dx\}$$
$$= \int_0^1 (-20x^4 + 13x^2 - 1)\,dx + j\int_0^1 (-16x^5 + 24x^3 - 6x)\,dx$$

The rest is easy enough, giving $I = \frac{1}{3}(-2+j)$ which is, of course, the same result as before. Note that all results in Frames 25–28 can be obtained very easily by integrating the function of z with respect to z.

For example, the integral $\int_c f(z)\,dz$ where $f(z) = (z-j)^2$ and c is the straight line joining A $(z=0)$ to B $(z = 1 + j2)$ can be evaluated as

$$\int_c f(z)\,dz = \int_{z=0}^{1+j2} (z-j)^2\,dz$$
$$= \left[\frac{(z-j)^3}{3}\right]_0^{1+j2}$$
$$= \left(\frac{(1+j2-j)^3}{3} - \frac{(-j)^3}{3}\right)$$
$$= \frac{1}{3}(-2+j)$$

Now on to the next frame

Cauchy's theorem

29

We have already seen that if $w = f(z)$ where, as usual, $w = u + jv$ and $z = x + jy$, then $dz = dx + j dy$ and

$$\int f(z)\,dz = \int (u + jv)(dx + j\,dy)$$
$$= \int (u\,dx - v\,dy) + j\int (v\,dx + u\,dy)$$

If c is a closed curve as the path of integration, then

$$\oint_c f(z)\,dz = \oint_c (u\,dx - u\,dy) + j\oint_c (v\,dx + u\,dy)$$

▶

Applying Green's theorem to each of the two integrals on the right-hand side in turn, we have

(a) $\oint_c (u\,dx - v\,dy) = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

where S is the region enclosed by the curve c.

Also, if $f(z)$ is regular at every point within and on c, then the Cauchy–Riemann equations give

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and therefore } -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\therefore \quad \oint_c (u\,dx - v\,dy) = 0 \qquad (1)$$

(b) Similarly, with the second integral, we have

....

30

$$\boxed{\oint_c (v\,dx + u\,dy) = 0}$$

Because

$$\oint_c (v\,dx + u\,dy) = \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Again, if $f(z)$ is regular at every point within and on c, then the Cauchy–Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and therefore } \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\therefore \quad \oint_c (v\,dx + u\,dy) = 0 \qquad (2)$$

Combining the two results (1) and (2) we have the following result.

If $f(z)$ is regular at every point within and on a closed curve c, then

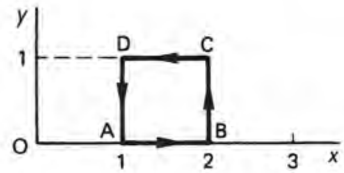
$$\oint_c f(z)\,dz = 0$$

This is Cauchy's theorem. Make a note of the result; then we can see an example

31

Example 1

Verify Cauchy's theorem by evaluating the integral $\oint_c f(z)\,dz$ where $f(z) = z^2$ around the square formed by joining the points $z = 1$, $z = 2$, $z = 2 + j$, $z = 1 + j$.



$$z = x + jy$$
$$z^2 = x^2 - y^2 + j\,2xy$$
$$dz = dx + j\,dy$$

$$\oint_c f(z)\,dz = \oint_c z^2\,dz = \oint_c \{x^2 - y^2 + j\,2xy\}\{dx + j\,dy\}$$
$$= \oint_c \{(x^2 - y^2)\,dx - 2xy\,dy\} + j\oint_c \{2xy\,dx + (x^2 - y^2)\,dy\}$$

We now take each of the sides in turn.

(a) AB: $\quad y = 0 \quad \therefore\ dy = 0$

$$\therefore\ \int_{AB} f(z)\,dz = \int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(b) BC: $\quad x = 2 \quad \therefore\ dx = 0$

$$\therefore\ \int_{BC} f(z)\,dz = \int_0^1 (-4y\,dy) + j\int_0^1 (4 - y^2)\,dy$$
$$= \left[-2y^2\right]_0^1 + j\left[4y - \frac{y^3}{3}\right]_0^1$$
$$= -2 + j\left(4 - \frac{1}{3}\right) = -2 + j\frac{11}{3}$$

Continuing in the same way, the results for the remaining two sides are

.... and ....

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$$\boxed{\text{CD: } -\frac{4}{3} - j3; \quad \text{DA: } 1 - j\frac{2}{3}}$$

Because

(c) CD: $\quad y = 1 \quad \therefore\ dy = 0$

$$\therefore\ \int_{CD} f(z)\,dz = \int_2^1 (x^2 - 1)\,dx + j\int_2^1 2x\,dx$$
$$= \left[\frac{x^3}{3} - x\right]_2^1 + j\left[x^2\right]_2^1 = -\tfrac{4}{3} - j\,3$$

▶

(d) DA: $x = 1 \quad \therefore \; dx = 0$

$$\therefore \int_{DA} f(z)\,dz = \int_1^0 (-2y\,dy) + j\int_1^0 (1 - y^2)\,dy$$
$$= \left[-y^2\right]_1^0 + j\left[y - \frac{y^3}{3}\right]_1^0 = 1 - j\tfrac{2}{3}$$

So, collecting the four results, $\oint_c f(z)\,dz = \ldots\ldots\ldots\ldots$

33

$$\boxed{\oint_c f(z)\,dz = 0}$$

Because

$$\oint_c f(z)\,dz = \frac{7}{3} + \left(-2 + j\frac{11}{3}\right) + \left(-\frac{4}{3} - j3\right) + \left(1 - j\frac{2}{3}\right) = 0$$

Example 2

A region in the z-plane has a boundary c consisting of

(a) OA joining $z = 0$ to $z = 2$

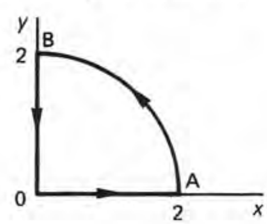
(b) AB a quadrant of the circle $|z| = 2$ from $z = 2$ to $z = j2$

(c) BO joining $z = j2$ to $z = 0$.

Verify Cauchy's theorem by evaluating the integral $\int_c (z^2 + 1)\,dz$

(1) along the arc from A to B

(2) along BO and OA.



$$f(z) = z^2 + 1 = (x + jy)^2 + 1$$
$$= (x^2 - y^2 + 1) + j\,2xy$$
$$z = x + jy \quad \therefore \; dz = dx + j\,dy$$

So the general expression for $\int f(z)\,dz = \ldots\ldots\ldots\ldots$

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$$\int\{(x^2-y^2+1)+j\,2xy\}\{\mathrm{d}x+j\,\mathrm{d}y\}$$
$$=\int\{(x^2-y^2+1)\,\mathrm{d}x-2xy\,\mathrm{d}y\}+j\int\{2xy\,\mathrm{d}x+(x^2-y^2+1)\,\mathrm{d}y\}$$

(1) Arc AB: $x^2+y^2=4 \quad \therefore\ y^2=4-x^2 \quad \therefore\ y=\sqrt{4-x^2}$

$$\mathrm{d}y=\frac{1}{2}(4-x^2)^{-1/2}(-2x)\,\mathrm{d}x \qquad \therefore\ \mathrm{d}y=\frac{-x}{\sqrt{4-x^2}}\mathrm{d}x$$

$$\therefore\ \int_{AB}f(z)\,\mathrm{d}z$$
$$=\int_2^0\left\{(x^2-4+x^2+1)\,\mathrm{d}x-2x\sqrt{4-x^2}\left(\frac{(-x)}{\sqrt{4-x^2}}\right)\mathrm{d}x\right\}$$
$$+\,j\int_2^0\left\{2x\sqrt{4-x^2}\,\mathrm{d}x+(x^2-4+x^2+1)\left(\frac{(1-x)}{\sqrt{4-x^2}}\right)\mathrm{d}x\right\}$$
$$=\int_2^0(4x^2-3)\,\mathrm{d}x+j\int_2^0\frac{11x-4x^3}{\sqrt{4-x^2}}\,\mathrm{d}x=-\frac{14}{3}+jI_1$$

Now we must attend to $I_1=\displaystyle\int_2^0\frac{11x-4x^3}{\sqrt{4-x^2}}\,\mathrm{d}x$.

Substituting $x=2\sin\theta$ and $\mathrm{d}x=2\cos\theta\,\mathrm{d}\theta$ with appropriate limits we have

....

35

$$\boxed{I_1=-\frac{2}{3}}$$

Because

$$I_1=\int_{\pi/2}^0\left(\frac{22\sin\theta-32\sin^3\theta}{2\cos\theta}\right)2\cos\theta\,\mathrm{d}\theta$$
$$=\int_0^{\pi/2}(32\sin^3\theta-22\sin\theta)\,\mathrm{d}\theta$$
$$=32\frac{2}{(3)(1)}+\Big[22\cos\theta\Big]_0^{\pi/2}=\frac{64}{3}-22=-\frac{2}{3}$$
$$\therefore\ \int_{AB}f(z)\,\mathrm{d}z=-4\frac{2}{3}-j\frac{2}{3}=-\frac{2}{3}(7+j)$$

(2) Along BO and OA. Complete this section on your own in the same way.

$$\int_{BO}f(z)\,\mathrm{d}z=\ldots\ldots\ldots\ldots;\quad \int_{OA}f(z)\,\mathrm{d}z=\ldots\ldots\ldots\ldots$$

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$$\int_{BO} f(z)\,dz = j\frac{2}{3}; \quad \int_{OA} f(z)\,dz = 4\frac{2}{3}$$

Because we have

BO: $x = 0 \quad \therefore \; dx = 0$

$$\therefore \; \int_{BO} f(z)\,dz = j\int_{2}^{1}(1 - y^2)\,dy = j\left[y - \frac{y^3}{3}\right]_2^0 = j\frac{2}{3}$$

OA: $y = 0 \quad \therefore \; dy = 0$

$$\therefore \; \int_{OA} f(z)\,dz = \int_0^2 (x^2 + 1)\,dx = \left[\frac{x^3}{3} + x\right]_0^2 = 4\frac{2}{3}$$

Collecting the results together, therefore

$$\int_{AB} f(z)\,dz = -\frac{14}{3} - j\frac{2}{3}$$

$$\int_{BO+OA} f(z)\,dz = j\frac{2}{3} + 4\frac{2}{3} = \frac{14}{3} + j\frac{2}{3}$$

$$\therefore \; \oint_c f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BO+OA} f(z)\,dz = 0$$

which, once again, verifies Cauchy's theorem.

Just by way of revision, Cauchy's theorem actually states that

....

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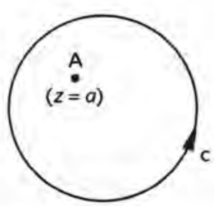
If $f(z)$ is *regular* at every point within and on a closed curve c, then $\oint_c f(z)\,dz = 0$

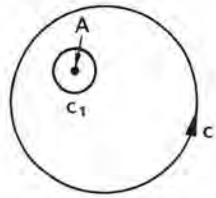
In our examples so far, $f(z)$ has been regular and no problems have arisen. Let us now consider a case where one or more singularities occur within the region enclosed by the curve c.

Deformation of contours at singularities

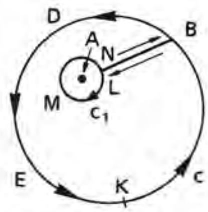


If c is the boundary curve (or *contour*) of a region and $f(z)$ is regular for all points within and on the contour, then the evaluation of $\oint_c f(z)\,dz$ around the contour is straightforward.

However, if $f(z) = \frac{1}{z - a}$, where a is a complex constant, and point A corresponds to $z = a$, then at A, $f(z)$ ceases to be regular and a singularity occurs at that point.



We can isolate A in a very small region within a contour c_1 and then $f(z)$ will be regular at all points within the region c and outside c_1. But the original region is now no longer simply connected (it now has a 'hole' in it) and this was one of our initial conditions.



However, all is not lost! We select a suitable point B on the contour c and join it to the inner contour c_1. If we now consider the integration $\int f(z)\,dz$ starting from a point K and proceeding anticlockwise, the path of integration can be taken as K B L M N B D E K.

Therefore

$$\int f(z)\,dz = I = I_{KB} + I_{BL} + I_{LMN} + I_{NB} + I_{BDEK} = \ldots\ldots\ldots\ldots$$

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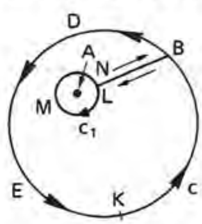
$\boxed{0}$

The function $f(z)$ is now regular at all points within and on the deformed contour. Remember that the inner contour c_1 can be made as small as we wish.

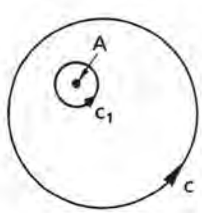


Note that $I_{NB} = -I_{BL}$, being in opposite directions, and these therefore cancel out.

The previous result then becomes

$$I_{KB} + I_{LMN} + I_{BDEK} = 0 \qquad \text{i.e.} \qquad I_{KB} + I_{BDEK} + I_{LMN} = 0$$

But $I_{KB} + I_{BDEK} = \oint_c f(z)\,dz$ and $I_{LMN} = \oint_{c_1} f(z)\,dz$ (clockwise)



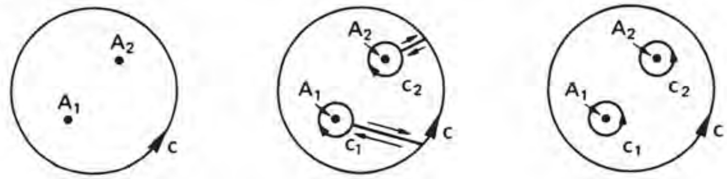
$$\therefore \quad \oint_c f(z)\,dz + \oint_{c_1} f(z)\,dz = 0$$

$$\therefore \quad \oint_c f(z)\,dz - \oint_{c_1} f(z)\,dz = 0$$

$$\therefore \quad \oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz$$

▶

The process can, of course, be extended to cases with more than one such singularity.



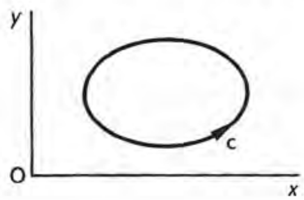
The corresponding result then becomes

$$\oint_c f(z)\,\mathrm{d}z = \oint_{c_1} f(z)\,\mathrm{d}z + \oint_{c_2} f(z)\,\mathrm{d}z \ldots \text{etc.}$$

Now let us apply these ideas to an example.

Example 1

Consider the integral $\oint_c f(z)\,\mathrm{d}z$ where $f(z) = \dfrac{1}{z}$, evaluated round a closed contour in the z-plane.



We first check the function $f(z) = \dfrac{1}{z}$ for singularities and find at once that

. . . .

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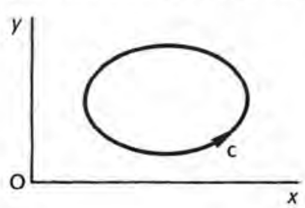
At $z = 0$, $f(z) = \dfrac{1}{z}$ ceases to be regular and a singularity occurs at that point

The actual position of the closed contour is not specified in the problem, so there are two possibilities: either the contour does enclose the origin, or it does not.

Let us consider them in turn.

(a) The contour does not enclose the origin.



No difficulty arises here and by Cauchy's theorem

. . . .

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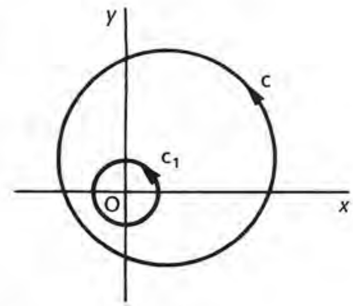
$$\boxed{\oint_c f(z)\,dz = 0}$$

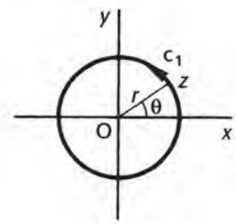
(b) If the contour *does* enclose the origin, the singularity must be taken into account. Then

$$\oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz = \oint_{c_1} \frac{1}{z}dz$$

and we attend to evaluating $\oint_{c_1} \frac{1}{z}dz$ where c_1 is a small circle of radius r entirely within the region bounded by c.



If we take an enlarged view of the small circle c_1, we have $z = x + jy$ which can be expressed in polar form .... and in exponential form ....



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$$\boxed{\begin{aligned} z &= r\,(\cos\theta + j\sin\theta) \\ z &= re^{j\theta} \end{aligned}}$$

Using $z = re^{j\theta}$ then $dz = jre^{j\theta}d\theta$ and $\oint_{c_1} \frac{1}{z}dz = $ ....

Complete it

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$$\boxed{j2\pi}$$

Because

$$\oint_{c_1} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{j\theta}}\{jre^{j\theta}\}d\theta = \int_0^{2\pi} j\,d\theta = j2\pi$$

$$\therefore \oint_c \frac{1}{z}dz = \oint_{c_1} \frac{1}{z}dz = j2\pi$$

So we have:

(a) $\oint_c \frac{1}{z}dz = 0$ if the contour c does not enclose the origin

(b) $\oint_c \frac{1}{z}dz = j2\pi$ if the contour c does enclose the origin.

These two constitute an important result, so note them well

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Example 2

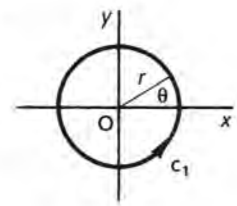
Consider the integral $\oint_c f(z)\,dz$ where $f(z)=\dfrac{1}{z^n}$ $(n=2,3,4,\ldots)$.

Again, a singularity clearly occurs at $z=0$ and again also we have two possible cases.

(a) If the contour c does not enclose the origin, then by Cauchy's theorem $\oint_c f(z)\,dz=0$.

(b) If the contour c does enclose the origin, then we proceed very much as before.

Using $z=re^{j\theta}$, $dz=jre^{j\theta}d\theta$ and $z^n=r^n e^{jn\theta}$



Then
$$\oint_c f(z)\,dz=\oint_{c_1} f(z)\,dz$$
$$=\int_0^{2\pi}\frac{1}{r^n e^{jn\theta}}\{jre^{j\theta}\}\,d\theta$$
$$=\frac{j}{r^{n-1}}\int_0^{2\pi}e^{-j(n-1)\theta}d\theta$$
$$=\frac{-1}{(n-1)r^{n-1}}\left[e^{-j(n-1)\theta}\right]_0^{2\pi}$$
$$=\ldots\ldots\ldots\ldots$$

Finish it off

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$$\boxed{0}$$

Because
$$\oint_c \frac{1}{z^n}dz=\frac{-1}{(n-1)r^{n-1}}\{e^{-j(n-1)2\pi}-1\}$$
$$=\frac{-1}{(n-1)r^{n-1}}\{\cos(n-1)2\pi-j\sin(n-1)2\pi-1\}$$
$$=0\quad\text{since}\quad\left.\begin{array}{r}\cos(n-1)2\pi=1\\ \sin(n-1)2\pi=0\end{array}\right\}\ n=2,3,4,\ldots$$

So $\oint_c \dfrac{1}{z^n}dz=0$ for all positive integer values of n other than $n=1$, where c is any closed contour.

The particular case when $n=1$ we have seen in Example 1.

Now we can easily cope with this next example.

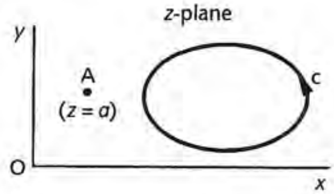
▶

Example 3

Consider $\oint_c f(z)\,dz$ where $f(z) = \dfrac{1}{(z-a)^n}$ for $n = 1, 2, 3, \ldots$

This is a simple extension of the previous piece of work. Here we see that a singularity occurs at $z = a$ and yet again we have two cases to consider.

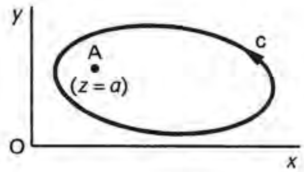
(a)



If the contour c does not enclose $z = a$, then by Cauchy's theorem

$$\oint_c f(z)\,dz = 0$$

(b)



If c encloses A $(z = a)$ we consider separately the cases when

(1) $n = 1$ and (2) $n > 1$.

(1) If $n = 1$, $\displaystyle\oint_c f(z)\,dz = \oint_c \frac{1}{z-a}\,dz$

Putting $z - a = w \quad \therefore \; dz = dw \quad \therefore \; \displaystyle\oint_c \frac{1}{z-a}\,dz = \oint_c \frac{1}{w}\,dw$

and this we have already established has a value ....

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$\boxed{j2\pi}$

(2) If $n > 1$, $\displaystyle\oint_c f(z)\,dz = \oint_c \frac{1}{(z-a)^n}\,dz = \oint_c \frac{1}{w^n}\,dw = 0$ for $n \neq 1$.

So collecting our results together, we have the following.

For $\oint_c f(z)\,dz$, where $f(z) = \dfrac{1}{(z-a)^n}$, $n = 1, 2, 3, \ldots$ and c is a closed contour

$$\begin{aligned} \oint_c \frac{1}{(z-a)^n}\,dz &= 0 && n \neq 1 \\ &= 0 && n = 1 \text{ and c does not enclose } z = a \\ &= j2\pi && n = 1 \text{ and c does enclose } z = a. \end{aligned}$$

You will notice that this is a more general result and includes the results obtained from Examples 1 and 2. *Make a note of it, therefore: it is quite important.*

Then on to Example 4

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Example 4

Finally, we can go one stage further and consider the contour integral of functions such as $f(z) = \dfrac{z-j-4}{(z+j)(z-2)}$.

First we express $f(z)$ in partial fractions

$$\frac{z-j-4}{(z+j)(z-2)} = \frac{A}{z+j} + \frac{B}{z-2}$$

One quick way of finding A and B is by the 'cover up' method.

(a) *To find A*, temporarily cover up the denominator $(z+j)$ in the partial fraction $\dfrac{A}{[z+j]}$ and in the function $\dfrac{z-j-4}{[z+j](z-2)}$ and substitute $z+j=0$, i.e. $z=-j$ in the remainder of the function.

$$A = \frac{-j-j-4}{-j-2} = \frac{4+j2}{2+j} = 2 \quad \therefore \; A = 2$$

(b) *To find B*, cover up the denominator $(z-2)$ in the partial fraction $\dfrac{B}{[z-2]}$ and in the function $\dfrac{z-j-4}{(z+j)[z-2]}$ and substitute $z-2=0$, i.e. $z=2$ in the remainder of the function.

$B = \ldots\ldots\ldots\ldots$

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$$\boxed{B=-1}$$

Because

$$\begin{aligned} B &= \frac{2-j-4}{2+j} \\ &= \frac{-2-j}{2+j} \\ &= -1 \end{aligned}$$

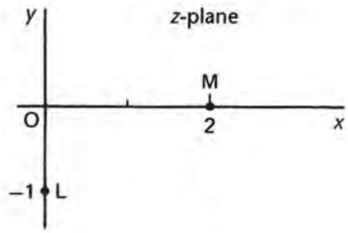
Therefore the function $f(z)$ becomes

$$f(z) = \frac{z-j-4}{(z+j)(z-2)} \equiv \frac{2}{z+j} - \frac{1}{z-2}$$

Now we can see that there are singularities at $\ldots\ldots\ldots\ldots$

$$\boxed{z = -j \quad \text{and} \quad z = 2}$$

Denote the singularities by L and M.

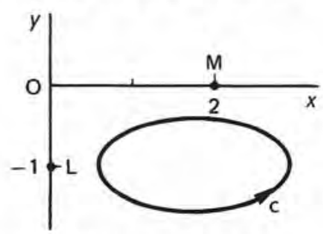


$$\therefore \oint_c \frac{z - j - 4}{(z + j)(z - 2)} \mathrm{d}z = \oint_c \left\{ \frac{2}{z + j} - \frac{1}{z - 2} \right\} \mathrm{d}z$$

$$= \oint_c \left\{ 2\left(\frac{1}{z + j} \right) - \frac{1}{z - 2} \right\} \mathrm{d}z$$

So we now have *four* cases to consider, depending on whether L, M, neither, or both, are enclosed within the contour c.

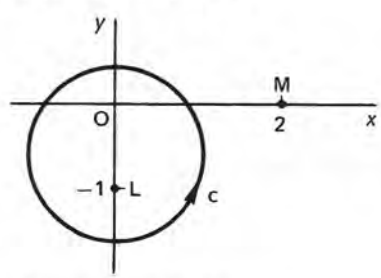
(a) *Neither L nor M enclosed*



Then, once again, by Cauchy's theorem

$$\oint_c f(z)\,\mathrm{d}z = 0$$

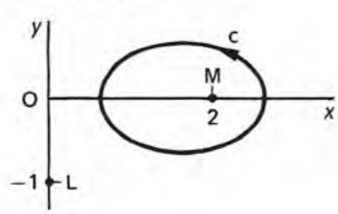
(b) *L enclosed but not M*



Then, in this case

$$\oint_c f(z)\,\mathrm{d}z = 2(j\,2\pi) - 0 = j\,4\pi$$

(c) *M enclosed but not L*

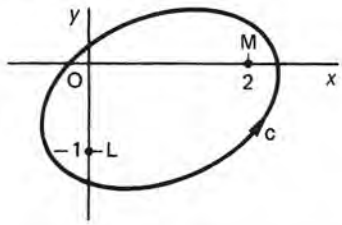


Here

$$\oint_c f(z)\,\mathrm{d}z = 0 - (j\,2\pi) = -j\,2\pi$$

▶

(d) *Both L and M enclosed*



In this case

$$\oint_c f(z)\,dz = \ldots\ldots\ldots\ldots$$

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$$\boxed{j2\pi}$$

Because, when both L and M are enclosed

$$\oint_c f(z)\,dz = \oint_c \left\{2\left(\frac{1}{z+j}\right) - \frac{1}{z-2}\right\}dz$$
$$= 2(j2\pi) - j2\pi$$
$$= j2\pi$$

The key is provided by the results we established earlier.

$$\oint_c \frac{1}{(z-a)^n}\,dz \begin{aligned} &= \ldots\ldots\ldots\ldots \text{ if } \ldots\ldots\ldots\ldots \\ &= \ldots\ldots\ldots\ldots \text{ if } \ldots\ldots\ldots\ldots \\ &= \ldots\ldots\ldots\ldots \text{ if } \ldots\ldots\ldots\ldots \end{aligned}$$

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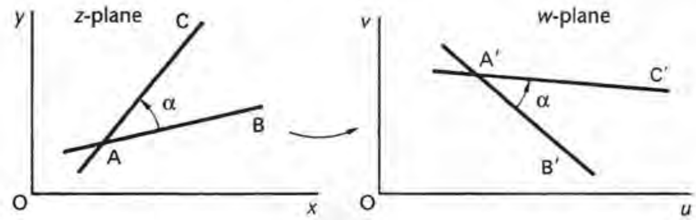
$$\boxed{\begin{aligned} \oint_c \frac{1}{(z-a)^n}\,dz &= 0 && \text{if } n \neq 1 \\ &= 0 && \text{if } n = 1 \text{ and c does not enclose } z = a \\ &= j2\pi && \text{if } n = 1 \text{ and c does enclose } z = a. \end{aligned}}$$

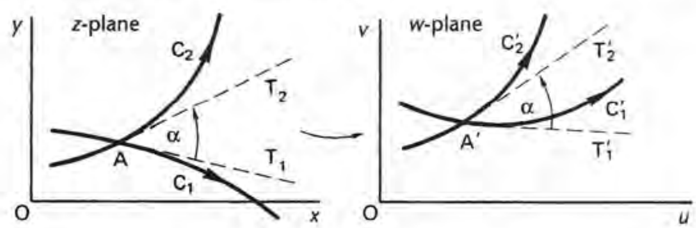
Now for something somewhat different.

Conformal transformation (conformal mapping)

A mapping from the z-plane onto the w-plane is said to be *conformal* if the angles between lines in the z-plane are preserved both in magnitude and in sense of rotation when transformed onto the corresponding lines in the w-plane.



The angle between two intersecting curves in the z-plane is defined by the angle α $(0 \leq \alpha \leq \pi)$ between their two tangents at the point of intersection, and this is preserved.



The essential characteristic of a conformal mapping is that

. . . .

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angles are preserved both in magnitude and in sense of rotation

Conditions for conformal transformation

The conditions necessary in order that a transformation shall be conformal are as follows.

1 The transformation function $w = f(z)$ must be a regular function of z. That is, it must be defined and single-valued, have a continuous derivative at every point in the region and satisfy the Cauchy–Riemann equations.

2 The derivative $\dfrac{dw}{dz}$ must not be zero, i.e. $f'(z) \neq 0$ at a point of intersection.

▶

Critical points

A point at which $f'(z) = 0$ is called a *critical point* and, at such a point, the transformation is not conformal.

So, if $w = f(z)$ is a regular function, then, except for points at which $f'(z) = 0$, the transformation function will preserve both the magnitude of the angle and its sense of rotation.

Now for a short exercise by way of practice.

Exercise

Determine critical points (if any) which occur in the following transformations $w = f(z)$.

1 $f(z) = (z-1)^2$	**5** $f(z) = (2z+3)^3$
2 $f(z) = e^z$	**6** $f(z) = z^3 + 6z + 9$
3 $f(z) = \dfrac{1}{z^2}$	**7** $f(z) = \dfrac{z-j}{z+j}$
4 $f(z) = z + \dfrac{1}{z}$	**8** $f(z) = (z+3)(z-j)$.

Finish the whole set before checking with the results in the next frame.

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1 $z = 1$	**5** $z = -\frac{3}{2}$
2 none	**6** $z = \pm j\sqrt{2}$
3 none	**7** none
4 $z = \pm 1$	**8** $z = \frac{1}{2}(j-3)$

All that is required is to differentiate each function and to find for which values of z, $f'(z) = 0$.

Now one or two simple examples on conformal mapping.

Example 1

Linear transformation $w = az + b$, $a \neq 0$, a and b complex.

(1) Cauchy–Riemann conditions satisfied.

(2) $f'(z) = a$ i.e. not zero $\therefore$ no critical points.

Therefore, the transformation $w = az + b$ provides conformal mapping throughout the entire z-plane.

Example 2

Non-linear transformation $w = z^2$.

First check for singularities and critical points. These, if any, occur at

. . . .

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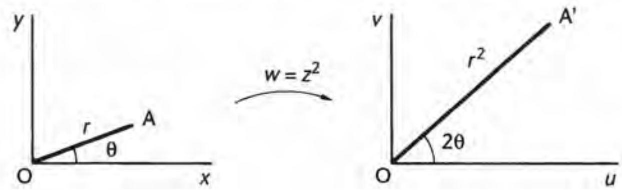
$\boxed{\text{no singularities; critical point at } z = 0}$

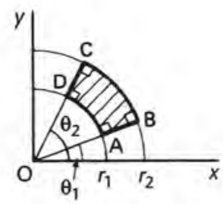
Because

$$f'(z) = 2z \quad \therefore \; f'(z) = 0 \text{ at } z = 0.$$

Therefore, the transformation is not conformal at the origin.

If we choose to express z in exponential form $z = x + jy = re^{j\theta}$, then $w = z^2 = r^2 e^{j2\theta}$, i.e. r is squared and the angle doubled.





So ABCD, a section of an annulus of inner and outer radii r_1 and r_2 respectively, will be mapped onto

. . . .

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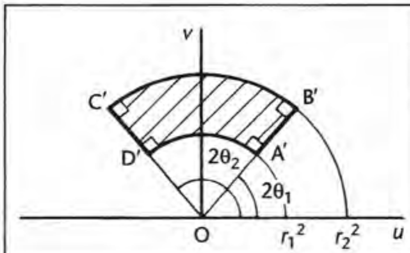

The angles at the origin are doubled, but notice that the right angles at A, B, C, D are preserved at A′, B′, C′, D′, i.e. the transformation there is conformal.

Example 3

Consider the mapping of the circle $|z| = 1$ under the transformation $w = z + \dfrac{4}{z}$ onto the w-plane.

First, as always, check for singularities and critical points. We find

. . . .

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$$\boxed{\text{singularity at } z = 0; \text{ critical points at } z = \pm 2}$$

A singularity occurs at $z = 0$, i.e. $f'(z)$ does not exist at $z = 0$. Also $f(z) = z + \dfrac{4}{z}$ $\therefore$ $f'(z) = 1 - \dfrac{4}{z^2}$ $\therefore$ $f'(z) = 0$ at $z = \pm 2$.
Therefore the transformation is not conformal at $z = 0$ and at $z = \pm 2$.

In fact, if we carry out the transformation $w = z + \dfrac{4}{z}$ on the unit circle $|z| = 1$, we get ....

Complete it: it is good revision

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$$\boxed{\text{the ellipse } \frac{u^2}{5^2} + \frac{v^2}{3^2} = 1}$$

Because

$$w = u + jv = z + \frac{4}{z}$$

$$= x + jy + \frac{4}{x + jy}$$

$$= x + jy + \frac{4(x - jy)}{x^2 + y^2}$$

$$\therefore \; u = x + \frac{4x}{x^2 + y^2}; \qquad v = y - \frac{4y}{x^2 + y^2}$$

$$|z| = 1 \quad \therefore \; x^2 + y^2 = 1 \; \therefore \; u = x(1 + 4) = 5x; \quad v = y(1 - 4) = -3y$$

$$\therefore \; x = \frac{u}{5} \text{ and } y = -\frac{v}{3}$$

Then $x^2 + y^2 = 1$ gives $\dfrac{u^2}{5^2} + \dfrac{v^2}{3^2} = 1$

The image of the unit circle is therefore an ellipse with centre at the origin; semi major axis 5; semi minor axis 3.

Now let us move on to a new section

Schwarz–Christoffel transformation

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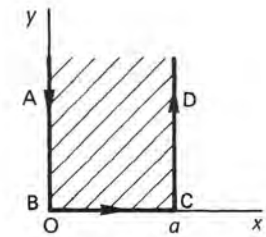
Example 1

Consider a semi-infinite strip on BC as base, the arrows at A and D indicating that the ordinate boundaries extend to infinity in the positive y-direction and that progression round the boundary is to be taken in the direction indicated.



Let us apply the transformation $w = -\cos\dfrac{\pi z}{a}$ to the shaded region.

$$\text{Then } w = u + jv = -\cos\frac{\pi z}{a}$$
$$= -\cos\frac{\pi(x+jy)}{a}$$
$$= -\left\{\cos\frac{\pi x}{a}\cos\frac{j\pi y}{a} - \sin\frac{\pi x}{a}\sin\frac{j\pi y}{a}\right\}$$

Now $\cos j\theta = \cosh\theta$ and $\sin j\theta = j\sinh\theta$.

$$\therefore\ w = u + jv$$
$$= -\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a} + j\sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}$$
$$\therefore\ u = -\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a}; \qquad v = \sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}$$

So B and C map onto B′ and C′ where

B′ = ...; C′ = ...

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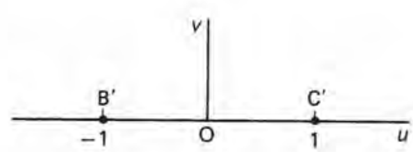
$$\boxed{\text{B}'\text{: } u = -1,\ v = 0; \quad \text{C}'\text{: } u = 1,\ v = 0}$$

Because

(1) at B, $x = 0$, $y = 0$ $\therefore\ u = -(1)(1) = -1$; $v = (0)(0) = 0$

and (2) at C, $x = a$, $y = 0$ $\therefore\ u = -(-1)(1) = 1$; $v = (0)(0) = 0$

So we have



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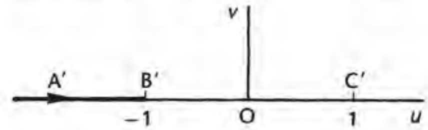
Now we map AB, BC, CD onto the w-plane giving A′B′, B′C′, C′D′.

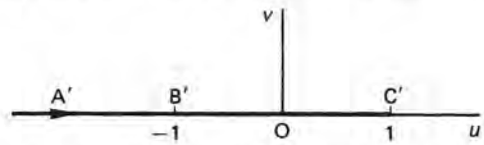
(a) AB: $x = 0$ $\therefore$ A′B′: $u = -\cosh\dfrac{\pi y}{a}$; $v = 0$

$\therefore$ As y decreases from ∞ to 0, u increases from $-\infty$ to -1.



(b) BC: $y = 0$ $\therefore$ B′C′: $u = -\cos\dfrac{\pi x}{a}$; $v = 0$

$\therefore$ As x increases from 0 to a, u increases from -1 to 1.



(c) CD: In the same way you can map CD and C′D′ in the w-plane and the mapping then becomes ....

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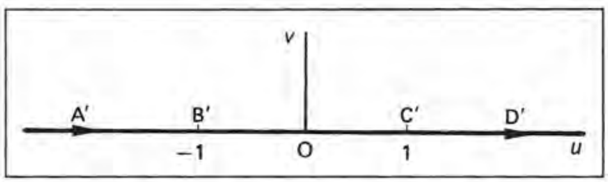
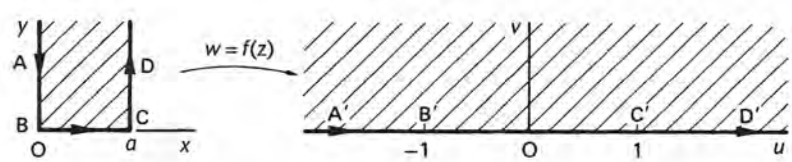


Because

CD: $x = a$ $\therefore$ C′D′: $u = \cosh\dfrac{\pi y}{a}$; $v = 0$.

Therefore, as y increases from 0 to ∞, u increases from 1 to ∞.

Notice the direction of the arrows. These correspond to the directed travel round the boundary shown in the z-plane.

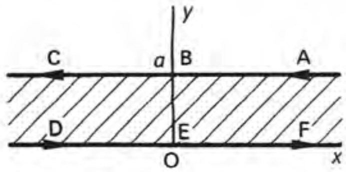


The shaded region in the z-plane is on the left-hand side of the boundary as traversed. This maps onto the left-hand side of the image on the w-plane, i.e. the entire upper half of the plane.

Note that $\dfrac{dw}{dz} = \dfrac{\pi}{a}\sin\dfrac{\pi z}{a}$ $\therefore$ at B ($z = 0$) and C ($z = a$), $\dfrac{dw}{dz} = 0$. Therefore, the conformal property does not hold at these points. The internal angle at B and at C is $\dfrac{\pi}{2}$, while at B′ and C′ it is π.

Example 2

Consider an infinite strip in the z-plane bounded by the real axis and $z = ja$



Note the arrows. The boundary comes from $+\infty$ (A) and continues to $-\infty$ (C); then returns from $-\infty$ (D) to $+\infty$ (F).

The strip can be considered as a closed figure with the left- and right-hand vertices at infinity.

We now map the infinite strip onto the w-plane by the transformation $w = e^{\pi z/a}$.

$$\therefore \; w = u + jv = e^{\pi z/a}, \text{ from which}$$
$$u = \ldots\ldots\ldots\ldots; \; v = \ldots\ldots\ldots\ldots$$

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$$\boxed{u = e^{\pi x/a}\cos\frac{\pi y}{a}; \quad v = e^{\pi x/a}\sin\frac{\pi y}{a}}$$

Because

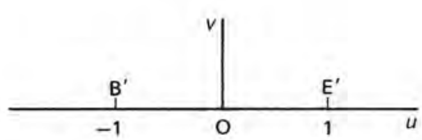
$$\begin{aligned} u + jv &= e^{\pi z/a} \\ &= e^{\pi(x+jy)/a} \\ &= e^{\pi x/a}e^{j\pi y/a} \\ &= e^{\pi x/a}\left(\cos\frac{\pi y}{a} + j\sin\frac{\pi y}{a}\right) \end{aligned}$$
$$\therefore \; u = e^{\pi x/a}\cos\frac{\pi y}{a}; \qquad v = e^{\pi x/a}\sin\frac{\pi y}{a}$$

Now we map points B and E onto B′ and E′.

(1) B: $x = 0, \; y = a \quad \therefore$ B′: $u = -1, \; v = 0$

(2) E: $x = 0, \; y = 0 \quad \therefore$ E′: $u = 1, \; v = 0$

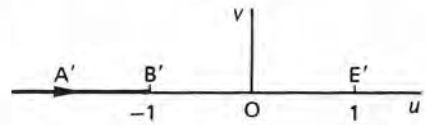
i.e.



Now we map the lines AB, BC, DE, EF onto the w-plane.

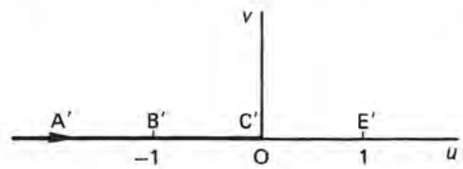
(a) AB: $y = a \quad \therefore \quad u = -e^{\pi x/a}, \quad v = 0$

$\therefore$ As x decreases from $+\infty$ to 0, u increases from $-\infty$ to -1.



(b) BC: $y = a \quad \therefore \quad u = -e^{\pi x/a}, \quad v = 0$ (as for AB)

$\therefore$ As x decreases from 0 to $-\infty$, u increases from -1 to 0.



(c) Now there is DE which maps onto

. . . .

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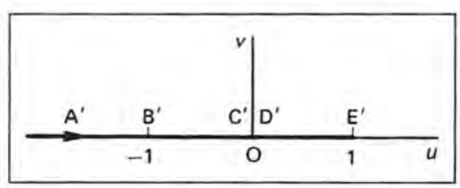


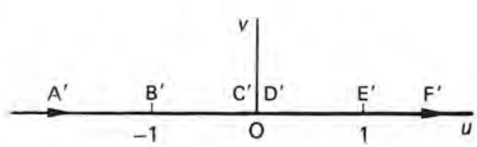
Because

(c) DE: $y = 0 \quad \therefore \quad u = e^{\pi x/a}, \quad v = 0$

$\therefore$ As x increases from $-\infty$ to 0, u increases from 0 to 1.

(d) EF: $y = 0 \quad \therefore \quad u = e^{\pi x/a}, \quad v = 0$ (as for DE)

$\therefore$ As x increases from 0 to $+\infty$, u increases from 1 to $+\infty$.



Notice that C and D map to the same point, namely $u = v = 0$.

Finally, what about the shaded region in the z-plane? This maps onto

. . . .

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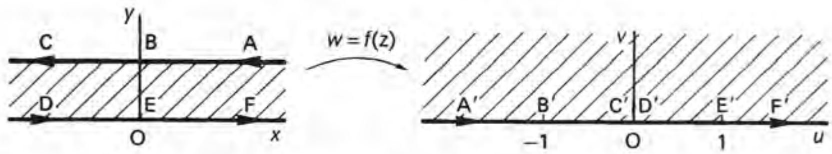
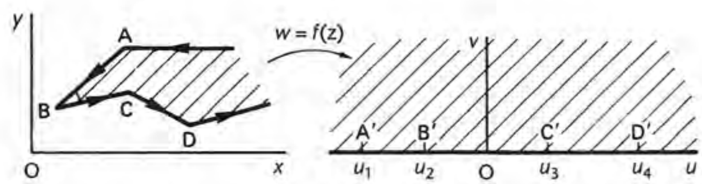
$\boxed{\text{the upper half of the } w\text{-plane}}$

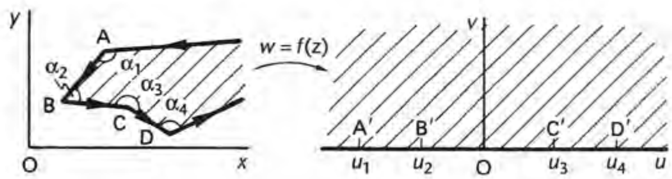
because it is on the left-hand side of the directed boundary in the z-plane.



The previous two examples have been simple cases of the application of the Schwarz–Christoffel transformation under which any polygon in the z-plane can be made to map onto the entire *upper half* of the w-plane and the boundary of the polygon onto the *real axis* of the w-plane.



The process depends, of course, on the right choice of transformation function for any particular polygon, which can be defined by its vertices and the internal angle at each vertex.



The Schwarz–Christoffel transformation function is given by

$$\frac{dz}{dw} = A(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}(w-u_3)^{\alpha_3/\pi-1}\dots$$

$$\therefore\ z = A\int(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}\dots(w-u_n)^{\alpha_n/\pi-1}dw + B$$

where A and B are complex constants, determined by the physical properties of the polygon.

This is not as bad as it looks!

Make a careful note of it: then we will apply it

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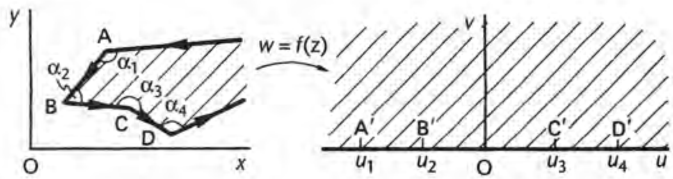
Here it is again.



$$\frac{dz}{dw} = A(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}(w-u_3)^{\alpha_3/\pi-1}\ldots$$

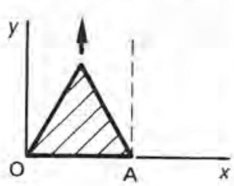
$$\therefore\ z = A\int(w-u_1)^{\alpha_1/\pi-1}(w-u_2)^{\alpha_2/\pi-1}\ldots(w-u_n)^{\alpha_n/\pi-1}dw + B$$

where A and B are complex constants.

Three other points also have to be noted.

1 Any three points u_1, u_2, u_3 on the u-axis can be selected as required.
2 It is convenient to choose one such point, u_n, at infinity, in which case the relevant factor in the integral above does not occur.
3 Infinite open polygons are regarded as limiting cases of closed polygons where one (or more) vertex is taken to infinity.

Open polygons



We have already introduced these in Examples 1 and 2 of this section.

In Example 1, the semi-infinite strip is a case of a triangle with one vertex that is

....

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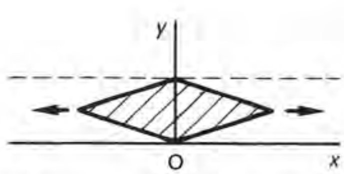
taken to infinity in the y-direction



In Example 2, the infinite strip is a case of a double triangle, or quadrilateral, with two vertices taken to infinity.

An open polygon with n sides with one vertex at infinity will have $(n-1)$ internal angles.

An open polygon with n sides with two vertices at infinity will have $(n-2)$ internal angles.

Now for an example to see how all this works.

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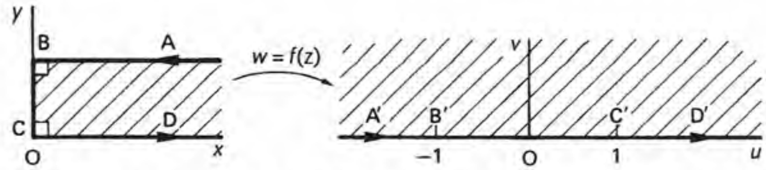
Example 3

To determine the transformation that will map the semi-infinite strip ABCD onto the w-plane so that the images of B and C occur at $u=-1$ and $u=1$, respectively, and the shaded region maps onto the upper half of the w-plane.



In this case, B′ is $u_1=-1$ and C′ is $u_2=1$.

The corresponding internal angles are:

at B $(z=ja)$, $\alpha_1=\dfrac{\pi}{2}$ and at C $(z=0)$, $\alpha_2=\dfrac{\pi}{2}$.

So we have

$$\frac{dz}{dw}=A(w+1)^{(\pi/2)/\pi-1}(w-1)^{(\pi/2)/\pi-1}$$ where A is a complex constant

$$=A(w+1)^{-1/2}(w-1)^{-1/2}$$

$$=A(w^2-1)^{-1/2}$$

$$=K(1-w^2)^{-1/2}=\frac{K}{\sqrt{1-w^2}}$$

$$\therefore\ z=\int\frac{K}{\sqrt{1-w^2}}\,dw=\dots\dots\dots\dots$$

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$$\boxed{z=K\arcsin w+\overline{B}}$$

$$\therefore\ \arcsin w=\frac{z-\overline{B}}{K}\qquad\therefore\ w=\sin\frac{z-\overline{B}}{K}$$

Now we have to find $\overline{B}$ and K.

(a) We require B $(z=ja)$ to map onto B′ $(w=-1)$

$$\therefore\ -1=\sin\frac{ja-\overline{B}}{K}$$

$$\therefore\ \frac{ja-\overline{B}}{K}=-\frac{\pi}{2}\qquad\therefore\ 2ja-2\overline{B}=-K\pi\qquad(1)$$

(b) We also require C $(z=0)$ to map onto C′ $(w=1)$ $\therefore\ 1=\sin\dfrac{0-\overline{B}}{K}$

$$\therefore\ -\frac{\overline{B}}{K}=\frac{\pi}{2}\qquad\therefore\ -2\overline{B}=K\pi\qquad(2)$$

Then, from (1) and (2), $\overline{B}=\dots\dots\dots\dots$; $K=\dots\dots\dots\dots$;

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$$\boxed{\bar{B}=\frac{ja}{2};\quad K=-\frac{ja}{\pi}}$$

$$\therefore\ w=\sin\left\{\frac{z-(ja)/2}{-ja/\pi}\right\}=\sin\left\{jz\frac{\pi}{a}+\frac{\pi}{2}\right\}=\cos\frac{jz\pi}{a}$$

But $\cos j\theta=\cosh\theta\quad\therefore\ w=\cosh\ \dfrac{\pi z}{a}$

To verify that this is the required transformation, let us apply it to the figure given in the z-plane.

We will do that in the next frame

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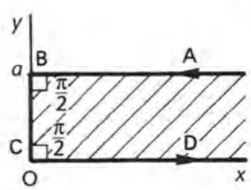
We have

$$w=u+jv=\cosh\frac{\pi z}{a}=\cosh\frac{(x+jy)\pi}{a}$$
$$\therefore\ u+jv=\cosh\frac{x\pi}{a}\cosh\frac{jy\pi}{a}+\sinh\frac{x\pi}{a}\sinh\frac{jy\pi}{a}$$

But $\cosh j\theta=\cosh\theta$ and $\sinh j\theta=j\sin\theta$

$$\therefore\ u+jv=\cosh\frac{x\pi}{a}\cos\frac{y\pi}{a}+j\sinh\frac{x\pi}{a}\sin\frac{y\pi}{a}$$
$$\therefore\ u=\cosh\frac{x\pi}{a}\cos\frac{y\pi}{a};\quad v=\sinh\frac{x\pi}{a}\sin\frac{y\pi}{a}$$



First map the points B and C onto B′ and C′ in the w-plane.

B′: ....; C′: ....

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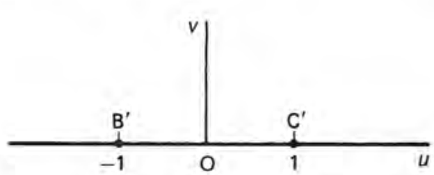
$$\boxed{\text{B}':\ u=-1, v=0;\quad \text{C}':\ u=1, v=0}$$

Because

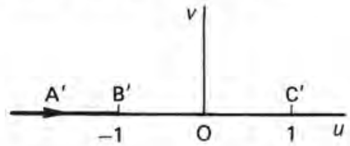
B: $x=0,\ y=a\quad\therefore$ B′: $u=\cos\pi=-1,\ v=0\quad\therefore$ B′: $u=-1,\ v=0$
C: $x=0,\ y=0\quad\therefore$ C′: $u=1,\ v=0\quad\therefore$ C′: $u=1,\ v=0.$



Now we map AB, BC, CD in turn.

(a) AB: $y = a \quad \therefore \ u = -\cosh\dfrac{x\pi}{a}, \quad v = 0$

$\therefore$ As x decreases from ∞ to 0, u increases from $-\infty$ to -1.



(b) BC: (c) CD: } Complete the working and show the mapped region which is …………

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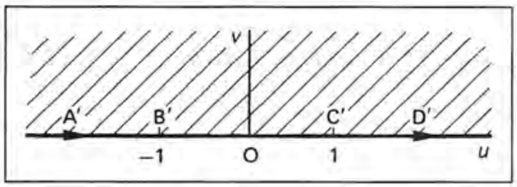

Because we have

(b) BC: $x = 0 \quad \therefore \ u = \cos\dfrac{y\pi}{a}, \quad v = 0$

$\therefore$ As y decreases from a to 0, u increases from -1 to 1.

CD: $y = 0 \quad \therefore \ u = \cosh\dfrac{x\pi}{a}, \quad v = 0$

$\therefore$ As x increases from 0 to ∞, u increases from 1 to ∞.

In each plane, the shaded region is on the left-hand side of the boundary.

We will now finish with one further example.

So move on

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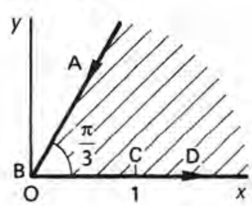
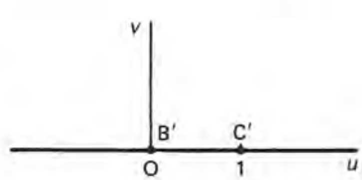
Example 4





Determine the transformation function $w = f(z)$ that maps the infinite sector in the z-plane onto the upper half of the w-plane with points B and C mapping onto B′ and C′ as shown.

The transformation function $w = f(z)$ is given by …………

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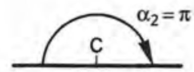
$$\frac{dz}{dw} = A(w - u_1)^{\alpha_1/\pi - 1}(w - u_2)^{\alpha_2/\pi - 1} \dots (w - u_n)^{\alpha_n/\pi - 1}$$

At B, $\alpha_1 = \frac{\pi}{3}$. At C, $\alpha_2 = \pi$.



With that reminder, you can now work through on your own, just as we did before, finally obtaining

$w = \dots\dots\dots\dots$

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$$\boxed{w = z^3}$$

Check with the working.

$$\frac{dz}{dw} = A(w - 0)^{(\pi/3)/\pi - 1}(w - 1)^{\pi/\pi - 1}$$
$$= Aw^{-2/3}(w - 1)^0$$
$$= Aw^{-2/3}$$
$$\therefore\ z = 3Aw^{1/3} + \overline{B}$$
$$= Kw^{1/3} + \overline{B}$$
$$\therefore\ w = \left(\frac{z - \overline{B}}{K}\right)^3$$

To find $\overline{B}$ and K

(a) At B: $z = 0$ At B′: $w = 0$ $\therefore\ 0 = \left(\frac{-\overline{B}}{K}\right)^3$ $\therefore\ \overline{B} = 0$ $\therefore\ w = \left(\frac{z}{K}\right)^3$

(b) At C: $z = 1$ At C′: $w = 1$ $\therefore\ 1 = \left(\frac{1}{K}\right)^3$ $\therefore\ K = 1$ $\therefore\ w = z^3$

$\therefore$ the transformation function is $w = z^3$

Finally, as a check – and a little more valuable practice – apply the function $w = z^3$ to the region shaded in the z-plane.

$$w = u + jv = (x + jy)^3 = x^3 + 3x^2(jy) + 3x(jy)^2 + (jy)^3$$

$$\therefore\ u = \dots\dots\dots\dots;\quad v = \dots\dots\dots\dots$$

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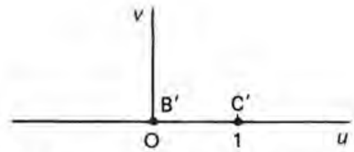
$$\boxed{u = x^3 - 3xy^2; \quad v = 3x^2y - y^3}$$

At B: $x = 0,\ y = 0 \quad \therefore\ u = 0,\ v = 0 \quad \therefore\ \text{B}'\!:\ u = 0,\ v = 0$
At C: $x = 1,\ y = 0 \quad \therefore\ u = 1,\ v = 0 \quad \therefore\ \text{C}'\!:\ u = 1,\ v = 0$



Now we map AB, BC, CD onto A′B′, B′C′, C′D′.

AB: $y = \sqrt{3}x \quad \therefore\ u = x^3 - 9x^3 = -8x^3, \quad v = 0$

$\therefore$ As x decreases from ∞ to 0, u increases from $-\infty$ to 0.

You can now deal with BC and CD in the same way and finally show the transformed region.

So we get …………

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Here is the remaining working.

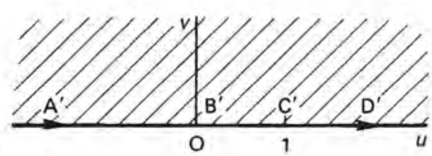
BC: $y = 0 \ \therefore\ u = x^3,\ v = 0$

$\therefore$ As x increases from 0 to 1, u increases from 0 to 1.

CD: $y = 0 \ \therefore\ u = x^3,\ v = 0$

$\therefore$ As x increases from 1 to ∞, u increases from 1 to ∞.

So we have



The shaded region is to the left of the directed boundary in the z-plane. This therefore maps onto the region to the left of the directed real axis in the w-plane, i.e. the upper half of the plane.

We have just touched on the fringe of the work on Schwarz–Christoffel transformation. The whole topic of mapping between planes has applications in fluid mechanics, heat conduction, electromagnetic theory, etc. and it is at times convenient to solve a problem relating to the z-plane by transforming to the upper half of the w-plane and later to transform back to the z-plane. The transformation function can be operated in either direction.

And that is it. The **Revision summary** follows and the **Can You?** checklist. Then on to the **Test exercise** and the **Further problems** for additional practice.

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Revision summary 21

1 *Differentiation of a complex function*

$$w = f(z) \qquad \frac{\mathrm{d}w}{\mathrm{d}z} = f'(z) = \underset{\delta z \to 0}{Lim}\left\{\frac{f(z_0 + \delta z) - f(z_0)}{\delta z}\right\}$$

2 *Regular (or analytic) function*
$w = f(z)$ is *regular* at z_0 if it is defined, single-valued and has a derivative at every point at and around $z = z_0$.

3 *Singularities* or singular points – points at which $f(z)$ ceases to be regular.

4 *Cauchy–Riemann equations* test whether $w = f(z)$ has a derivative $f'(z)$ at $z = z_0$. $\quad w = u + jv = f(z)$ where $z = x + jy$.

Then $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

5 If a function of two real variables $f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 0$$

then $f(x, y)$ is an harmonic function. The real and imaginary parts of an analytic function are both harmonic and form a conjugate pair of functions.

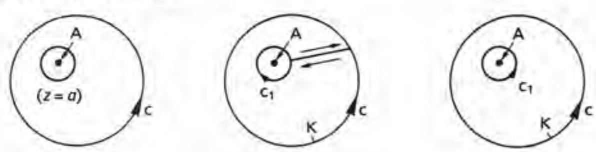
6 *Complex integration*

$$\int w\mathrm{d}z = \int f(z)\mathrm{d}z = \int (u\mathrm{d}x - v\mathrm{d}y) + j\int (v\mathrm{d}x + u\mathrm{d}y)$$

7 *Contour integration* – evaluation of line integrals in the z-plane.

8 *Cauchy's theorem* If $f(z)$ is regular at every point within and on closed curve c, then $\oint_c f(z)\,\mathrm{d}z = 0$.

9 *Deformation of contours*



(a) (b) (c)

(a) Singularity at A
(b) Restored to a closed curve
(c) $\oint_c f(z)\,\mathrm{d}z = \oint_{c_1} f(z)\,\mathrm{d}z.$

For $\oint_c f(z)\,\mathrm{d}z$ where $f(z) = \dfrac{1}{(z-a)^n} \quad n = 1, 2, 3, \ldots$

$$\begin{aligned}\oint_c \frac{1}{(z-a)^n}\mathrm{d}z &= 0 && \text{if } n \neq 1\\ &= 0 && \text{if } n = 1 \text{ and c does not enclose } z = a\\ &= j2\pi && \text{if } n = 1 \text{ and c does enclose } z = a.\end{aligned}$$

10 *Conformal transformation* – mapping in which angles are preserved in size and sense of rotation.

Conditions

1 $w = f(z)$ must be a regular function of z.

2 $f'(z)$, i.e. $\dfrac{dw}{dz}$, $\neq 0$ at the point of intersection.

If $f'(z) = 0$ at $z = z_0$, then z_0 is a *critical point*.

11 *Schwarz–Christoffel transformation* maps any polygon in the z-plane onto the entire *upper half* of the w-plane and the boundary of the polygon onto the *real axis* of the w-plane.

$$\frac{dz}{dw} = A(w - u_1)^{\alpha_1/\pi - 1}(w - u_2)^{\alpha_2/\pi - 1} \ldots (w - u_n)^{\alpha_n/\pi - 1}$$

1 Any three points u_1, u_2, u_3 can be selected on the u-axis.

2 One such point can be chosen at infinity.

3 Infinite open polygons are regarded as limiting cases of closed polygons.

Can You?

Checklist 21

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Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Appreciate when the derivative of a function of a complex variable exists? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the notions of regular functions and singularities and be able to obtain the derivative of a regular function from first principles? 3 to 6

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive the Cauchy–Riemann equations and apply them to find the derivative of a regular function? 7 to 12

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Understand the notion of an harmonic function and derive a conjugate function? 13 to 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate line and contour integrals in the complex plane? 23 to 28

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive and apply Cauchy's theorem? 29 to 36

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply Cauchy's theorem to contours around regions that contain singularities? 37 to 49

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Define the essential characteristics of and conditions for a conformal mapping? 50 and 51

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Locate critical points of a function of a complex variable? 51 and 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the image in the w-plane of a figure in the z-plane under a conformal transformation $w = f(z)$? 52 to 56

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Describe and apply the Schwarz–Christoffel transformation? 57 to 75

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Text exercise 21

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1 Determine where each of the following functions fails to be regular.

(a) $w = z^3 + 4$

(b) $w = \dfrac{z}{z+5}$

(c) $w = e^{2z+4}$

(d) $w = \dfrac{z-2}{(z-4)(z+1)}$

(e) $w = \dfrac{x - jy}{x^2 + y^2}$.

2 Demonstrate that each of the following is harmonic and obtain the conjugate function.

(a) $u(x, y) = \sinh x \cos y$

(b) $u(x, y) = 4y(1 + 3x)$.

3 Verify Cauchy's theorem by evaluating $\oint_c f(z)\,dz$ where $f(z) = z^2$ round the rectangle formed by joining the points $z = 2 + j$, $z = 2 + j4$, $z = j4$, $z = j$.

4 Evaluate the integral $\oint_c f(z)dz$ where $f(z) = \dfrac{3z - 6 - j}{(z - j)(z - 3)}$ round the contour $|z| = 2$.

▶

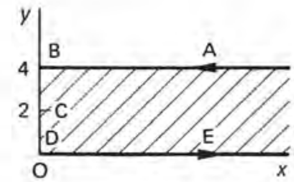
5 Determine critical points, if any, at which the following transformation functions $w = f(z)$ fail to be conformal.

(a) $w = z^4$ (d) $w = z + \dfrac{2}{z}$

(b) $w = z^3 - 3z$ (e) $w = e^{(z^2)}$

(c) $w = e^{1-z}$ (f) $w = \dfrac{z+j}{z-j}$.

6 Determine the Schwarz–Christoffel transformation function $w = f(z)$ that will map the semi-infinite strip shaded in the z-plane onto the upper half of the w-plane, so that the image of B is B′ ($w = -1$) and that of C is C′ ($w = 0$). Obtain the image of the point D.



Further problems 21

79

1 Verify Cauchy's theorem for the closed path c consisting of three straight lines joining A $(1 + j)$, B $(3 + j3)$, C $(-1 + j3)$ where $f(z) = z - 1 + j$.

2 If $z = 2 + jy$ is mapped onto the w-plane under the transformation $w = f(z) = \dfrac{1}{z}$, show that the locus of w is a circle with centre $w = 0{\cdot}25$ and radius $0{\cdot}25$.

3 Determine the image in the w-plane of the circle $|z - 2| = 1$ in the z-plane under the transformation $w = (1 - j)z + 3$.

4 The unit circle $|z| = 1$ in the z-plane is generated in an anticlockwise manner from the point A ($z = 1$) and is transformed onto the w-plane by $w = \dfrac{z}{z-2}$. Determine the locus of w and the direction in which it is generated.

5 Find the conjugate function of each of the following.

(a) $u(x, y) = x^2 - 2x - y^2$

(b) $u(x, y) = x^3 - 3xy^2 - x^2 + y^2 + x$

(c) $u(x, y) = 2y(x - 1)$

(d) $u(x, y) = e^{x^2 - y^2} \cos 2xy$.

6 Evaluate $\oint_c f(z)\,dz$ where $f(z) = \dfrac{5z - 2 - j3}{(z-j)(z-1)}$ around the closed contour c for the two cases when

(a) c is the path $|z| = 2$

(b) c is the path $|z - 1| = 1$.

7 If $f(z) = \dfrac{5z + j}{(z-j)(z+j2)}$, evaluate $\oint_c f(z)dz$ along the contours

(a) $|z - 1| = 1$; (b) $|z| = \dfrac{3}{2}$; (c) $|z| = 3$.

8 If $z = x + jy$ and $w = f(z)$, show that, if $\dfrac{j(w+z)}{w-z}$ is entirely real, then $|w| = |z|$.

9 Evaluate $\oint_c f(z)\,dz$, where $f(z) = \dfrac{3z - j5}{(z + 1 - j2)(z - 2 - j)}$, around the perimeter of the rectangle formed by the lines $z = 1$, $z = j3$, $z = -2$, $z = -j$.

10 If $f(z) = \dfrac{8z^2 - 2}{z(z-1)(z+1)}$, evaluate $\oint_c f(z)\,dz$ along the contour c where c is the triangle joining the points $z = 2$, $z = j$, $z = -1 - j$.

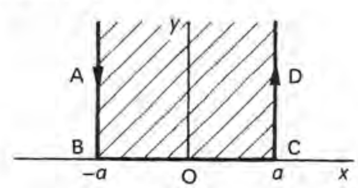
11 (a) For the transformation $w = z + \dfrac{1}{z}$, state (1) singularities, (2) critical points.

(b) Apply $w = z + \dfrac{1}{z}$ to map the circle $|z| = 2$ onto the w-plane.

12 Find the images in the w-plane of (a) the line $y = 0$ and (b) the line $y = x$ that result from the mapping $w = \dfrac{z-j}{z+j}$. Show that the curves intersect at the points $(\pm 1, 0)$ in the w-plane and determine the angle at which they intersect.

13 Use the transformation $w = \dfrac{j(1+z)}{1-z}$ to map the unit circle $|z| = 1$ in the z-plane onto the w-plane. Determine also the image in the w-plane of the region bounded by $|z| = 1$ and inside the circle.

14 Determine the transformation that will map the semi-infinite strip shown, onto the upper half of the w-plane, where the image of B is B′ ($w = -1$) and that of C is C′ ($w = 1$).



Complex analysis 3

Frames 1 to 38

Learning outcomes

When you have completed this Programme you will be able to:

- Expand a function of a complex variable about the origin in a Maclaurin series
- Determine the circle and radius of convergence of a Maclaurin series expansion
- Recognise singular points in the form of poles of order n, removable and essential singularities
- Expand a function of a complex variable about a point in the complex plane in a Taylor series, transforming the coordinates with a shift of origin
- Expand a function of a complex variable about a singular point in a Laurent series
- Recognise the principal and analytic parts of the Laurent series and link the form of the principal part to the type of singularity
- Recognise the residue of a Laurent series and state the Residue theorem
- Calculate the residues at the poles of an expression without resort to deriving the Laurent series
- Evaluate certain types of real integrals using the Residue theorem

Maclaurin series

1

You will recall that the Maclaurin series expansion of the function of a real variable x with output $f(x)$ is given as

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots + x^n\frac{f^{(n)}(0)}{n!} + \cdots$$

This is an infinite series expansion of $f(x)$ about the point $x = 0$. Because the series on the right-hand side of this equation contains an infinite number of terms, the right-hand side may only converge for a restricted set of values of x. Consequently, this expansion is only valid for that restricted set of values. For example, the expression $f(x) = (1-x)^{-1}$ has the Maclaurin series expansion ....

2

$$\boxed{f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots}$$

Because

$$\begin{aligned}
f(x) &= (1-x)^{-1} \text{ and so } f(0) = (1-0)^{-1} = 1\\
f'(x) &= (1-x)^{-2} \text{ and so } f'(0) = (1-0)^{-2} = 1\\
f''(x) &= 2(1-x)^{-3} \text{ and so } f''(0) = 2(1-0)^{-3} = 2\\
f'''(x) &= 3!(1-x)^{-4} \text{ and so } f'''(0) = 3!(1-0)^{-4} = 3!\\
&\vdots \qquad\qquad\qquad\qquad \vdots\\
f^{(n)}(x) &= n!(1-x)^{-(n+1)} \text{ and so } f^{(n)}(0) = n!(1-0)^{-(n+1)} = n!
\end{aligned}$$

Therefore, substituting into the Maclaurin series expansion, we find

$$\begin{aligned}
f(x) &= f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + \cdots + x^n\frac{f^{(n)}(0)}{n!} + \cdots\\
&= 1 + x \times 1 + x^2 \times \frac{2!}{2!} + x^3 \times \frac{3!}{3!} + \cdots + x^n \times \frac{n!}{n!} + \cdots\\
&= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots
\end{aligned}$$

This same result could also be derived by using the binomial theorem or even by performing the long division of 1 by $1 - x$. However, performing the algorithmic procedure is one thing, but knowing that the result of the procedure is valid is another. To determine the validity of the expansion we resort to convergence tests, and in this case we use the ratio test. To refresh your memory, the ratio test for the infinite series

$$f(x) = a_0(x) + a_1(x) + a_2(x) + a_3(x) + \cdots + a_n(x) + \cdots$$

is that given

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = L \text{ then if } \begin{array}{l} L < 1 \text{ the series converges}\\ L > 1 \text{ the series diverges}\\ L = 1 \text{ the test fails and an alternative}\\ \qquad\quad \text{convergence test is required.}\end{array}$$

Applying the ratio test to the Maclaurin series expansion

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

tells us that

The series converges for ....
The series diverges for ....
The test fails for ....

3

The series converges for $-1 < x < 1$
The series diverges for $x < -1$ or $x > 1$
The test fails for $x = \pm 1$

Because

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \underset{n\to\infty}{Lim}\left|\frac{x^{n+1}}{x^n}\right| = \underset{n\to\infty}{Lim}|x| = |x|, \text{ so}$$

if $|x| < 1$, that is $-1 < x < 1$, the series converges and so the expansion is valid
$|x| > 1$, that is $x < -1$ or $x > 1$, the series diverges and so the expansion is invalid
$|x| = 1$, that is $x = \pm 1$, the ratio test fails to give a conclusion.

By inspection, when $x = 1$ the series clearly diverges and when $x = -1$ the sum of terms alternates between 1 and 0 as each successive term is added. Clearly the series does not converge and so, therefore, it must diverge when $x = -1$.

Everything that has been said about the Maclaurin series expansion of an expression involving a real variable x can equally be said about an expression involving a complex variable z. That is, if $f(z)$ is a function in the complex variable z, analytic at $z = 0$, then the Maclaurin series expansion is

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + z^3\frac{f'''(0)}{3!} + \cdots$$

So, the Maclaurin series expansion of $f(z) = \sin z$ is ....

4

$$\boxed{f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots}$$

Because

$f(z) = \sin z$ and so $f(0) = \sin 0 = 0$

$f'(z) = \cos z$ and so $f'(0) = \cos 0 = 1$

$f''(z) = -\sin z$ and so $f''(0) = -\sin 0 = 0$

$f'''(z) = -\cos z$ and so $f'''(0) = -\cos 0 = -1$

$\vdots \qquad\qquad \vdots$

Therefore

$$\begin{aligned} f(z) &= f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + z^3\frac{f'''(0)}{3!} + \cdots \\ &= 0 + z \times 1 + z^2 \times \frac{0}{2!} + z^3 \times \frac{(-1)}{3!} + \cdots \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

Furthermore, applying the ratio test tells us that this series expansion is valid for ....

5

all finite values of z

Because

$$\begin{aligned} \underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| &= \underset{n\to\infty}{Lim}\left|\frac{(-1)^{n+1}z^{2(n+1)+1}/[2(n+1)+1]!}{(-1)^n z^{2n+1}/[2n+1]!}\right| \\ &= \underset{n\to\infty}{Lim}\left|\frac{z^2}{(2n+3)(2n+2)}\right| = 0 < 1 \end{aligned}$$

So the expansion is valid for all finite values of z.

Try this one. The Maclaurin series expansion of $f(z) = \ln(1+z)$ is

$\ln(1+z) = \ldots\ldots\ldots\ldots$

6

$$\boxed{\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + \frac{(-1)^{n+1}z^n}{n} + \cdots \quad n = 1, 2, \ldots}$$

Because

$f(z) = \ln(1+z)$ and so $f(0) = (1+0) = 0$

$f'(z) = (1+z)^{-1}$ and so $f'(0) = (1+0)^{-1} = 1$

$f''(z) = -(1+z)^{-2}$ and so $f''(0) = -(1+0)^{-2} = -1$

$f'''(z) = 2(1+z)^{-3}$ and so $f'''(0) = 2(1+0)^{-3} = 2$

$f^{(iv)}(z) = -3!(1+z)^{-4}$ and so $f^{(iv)}(0) = -3!(1+0)^{-4} = -3!$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(z) = (-1)^{n+1}n!(1+z)^{-n}$ and so $f^{(n)}(0) = (-1)^{n+1}n!(1+0)^{-n}$

$= (-1)^{n+1}n!$

Therefore

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + \frac{(-1)^{n+1}z^n}{n} + \cdots$$

This series is valid for

....

7

$$\boxed{|z| < 1}$$

Because

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| = \underset{n\to\infty}{Lim}\left|\frac{(-1)^{n+2}z^{n+1}/[n+1]}{(-1)^{n+1}z^n/[n]}\right| = \underset{n\to\infty}{Lim}\left|\frac{nz}{n+1}\right| = |z|$$

So if $|z| < 1$ the series converges and so the expansion is valid

$|z| > 1$ the series diverges and so the expansion is invalid

$|z| = 1$ the ratio test fails

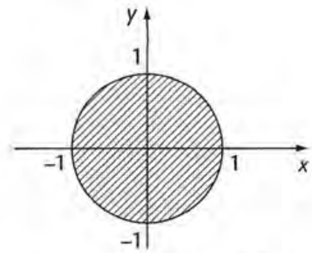
We shall look at the case $|z| = 1$ a little later.

Move to the next frame

Radius of convergence

8

We have just seen that the Maclaurin expansion of $\ln(1+z)$ is valid for $|z|<1$. This inequality defines the interior of a circle of radius 1 centred on the origin, namely $z = 1e^{j\theta}$.



This means that the expansion is valid for all z-values lying within this circle. The radius of the circle within which a series expansion is valid is called the *radius of convergence* of the series and the circle is called the *circle of convergence*.

Example

To find the infinite series expansion and radius of convergence of the expression $f(z) = \dfrac{z}{(1-3z)^2}$, we progress in stages, noting that $\dfrac{z}{(1-3z)^2} = z(1-3z)^{-2}$. We expand $(1-3z)^{-2}$ first.

By the binomial theorem, the expansion of $(1-3z)^{-2}$ is

$$(1-3z)^{-2} = \dots\dots\dots\dots$$

9

$$\boxed{(1-3z)^{-2} = 1 + 6z + 27z^2 + 108z^3 + 405z^4 + \cdots}$$

Because

$$\begin{aligned}(1-3z)^{-2} &= \left(1 + (-2)\times(-3z) + \frac{(-2)(-3)\times(-3z)^2}{2!}\right.\\ &\qquad\left. + \frac{(-2)(-3)(-4)\times(-3z)^3}{3!} + \dots\right)\\ &= \left(1 + 6z + 3(-3z)^2 - 4(-3z)^3 + 5(-3z)^4 + \dots\right.\\ &\qquad\left. + (-1)^n(n+1)(-3)^n z^n + \dots\right)\\ &= 1 + 6z + 27z^2 + 108z^3 + 405z^4 + \dots + (n+1)3^n z^n + \dots\end{aligned}$$

and so

$$z(1-3z)^{-2} = z + 6z^2 + 27z^3 + 108z^4 + 405z^5 + \dots + (n+1)z^n z^{n+1} + \dots$$

The radius of convergence is then

$$\dots\dots\dots\dots$$

10

$\boxed{1/3}$

Because

The general term of the expansion is $a_n(z) = (n+1)3^n z^{n+1}$ and so the ratio test tells us that

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| = \underset{n\to\infty}{Lim}\left|\frac{(n+2)3^{n+1}z^{n+2}}{(n+1)3^n z^{n+1}}\right| = \underset{n\to\infty}{Lim}\left|\frac{3(n+2)z}{(n+1)}\right| = |3z|$$

So, if $|3z| < 1$, that is $|z| < 1/3$, then the series converges and the expansion is valid. The radius of convergence is therefore $1/3$.

Move to the next frame

Singular points

11

Any point at which $f(z)$ fails to be analytic, that is where the derivative does not exist, is called a *singular point* (also called a singularity). For example

$$f(z) = \frac{1}{z-1}$$

is analytic everywhere in the finite complex plane except at the point $z = 1$ where not only is the derivative $f'(z)$ not defined but neither is $f(z)$. Accordingly, the point $z = 1$ is a singular point. There are different types of singular points, for now we shall look at just two of them.

Poles

If $f(z)$ has a singular point at z_0 and for some natural number n, $\underset{z\to z_0}{Lim}\{(z-z_0)^n f(z)\} = L \neq 0$ then the singular point is called a *pole of order n*. For example

$$f(z) = \frac{2z}{(z+4)^2}$$

has a singular point at $z = -4$ and because

$$\underset{z\to -4}{Lim}\left\{(z+4)^2 f(z)\right\} = \underset{z\to -4}{Lim}\{2z\} = -8 \neq 0$$

the singularity is a *pole of order* 2 (also called a *double pole*).

▶

Removable singularities

If $f(z)$ has a singular point at z_0 and $\underset{z \to z_0}{Lim}\{f(z)\}$ exists then the singular point is called a *removable singularity*. For example

$$f(z) = \frac{\sin z}{z}$$

has a singular point at $z = 0$. However, $\underset{z \to 0}{Lim}\left\{\frac{\sin z}{z}\right\} = 1$ and so the singularity at $z = 0$ is a removable singularity. We can see this from the Maclaurin series expansion of $f(z)$ where

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

While we cannot substitute $z = 0$ into $f(z) = \frac{\sin z}{z}$, we can define $f(0) = 1$ in complete consistency with the series expansion. In this sense the singularity at $z = 0$ is removable by virtue of the fact that we can assign a value to $f(z)$ at the singularity which is consistent with the series expansion.

Move to the next frame

Circle of convergence

12

When an expression is expanded in a Maclaurin series, the circle of convergence is always centred on the origin and the radius of convergence is determined by the location of the first singular point met as $|z|$ increases from $|z| = 0$. For example, the Maclaurin series expansion of $f(z) = \ln(1 + z)$ is

$$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + \frac{(-1)^{n+1}z^n}{n} + \cdots$$

which is valid inside the circle of convergence $|z| = 1$. The first singular point met by this function as $|z|$ increases from zero is at $z = -1$, for at that point $\ln(1 + z)$ is not defined and the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots$$

diverges – it is the negative of the harmonic series. Hence the radius of convergence is 1. When $z = 1$, substitution into the series expansion gives

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

▶

The right-hand side is the alternating harmonic series which we know converges by the *alternating sign test* which states that if the magnitude of the terms decreases and the signs alternate then the series converges. Now we know that it converges to $\ln 2$. Notice that the circle of convergence is identified by the location of the *first* singularity as $|z|$ increases from $|z| = 0$. This does not mean that the function is singular at all points on the circle of convergence.

There are times when it is desirable to have a series expansion of an expression that is singular at the origin. Because the Maclaurin expansion requires the function to be analytic everywhere within the circle of convergence which is centred on the origin, we cannot use that method. Fortunately, we do have a method of expanding a function about *any point* in the complex plane – this is Taylor's expansion.

Move to the next frame

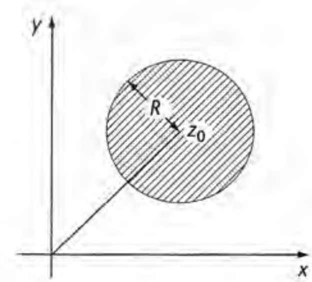
Taylor's series

13

Provided $f(z)$ is analytic inside and on a simple closed curve c, the Taylor series expansion of $f(z)$ about the point z_0 which is interior to c is given as

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2 f''(z_0)}{2!} + \cdots + \frac{(z - z_0)^n f^{(n)}(z_0)}{n!} + \cdots$$

where here, the point z_0 is the centre of the circle of convergence. The circle of convergence is given as $|z - z_0| = R$. That is $z - z_o = Re^{j\theta}$ or $z = z_0 + Re^{j\theta}$ where R is the radius of convergence.



Notice that Maclaurin's series is a special case of Taylor's series where $z_0 = 0$.

Example

Expand $f(z) = \dfrac{1}{z+1}$ in a Taylor series about the point $z = 1$ and find the values of z for which the expansion is valid.

The simplest way of doing this is to perform a coordinate transformation that moves the origin of the new coordinate to the point $z = 1$ and then derive the series about the new origin. To do this we define a new complex variable $u = z - 1$ so that $z = u + 1$ and so

$$\frac{1}{z+1} \text{ becomes } \frac{1}{u+2} = (2+u)^{-1} = \frac{1}{2}\left(1 + \frac{u}{2}\right)^{-1}.$$

The expansion of this expression can now be derived using either Maclaurin or, as here, the binomial theorem to obtain

$$\frac{1}{u+2}=\frac{1}{2}\left(1+(-1)\frac{u}{2}+\frac{(-1)(-2)}{2!}\left(\frac{u}{2}\right)^2+\cdots\right)$$
$$=\frac{1}{2}-\frac{u}{4}+\frac{u^2}{8}-\frac{u^3}{16}+\cdots$$

Transforming back to the original variable z gives

$$\frac{1}{z+1}=\frac{1}{2}-\frac{z-1}{4}+\frac{(z-1)^2}{8}-\frac{(z-1)^3}{16}+\cdots$$

The circle of convergence is given by $\left|\frac{u}{2}\right|=1$, that is $\left|\frac{z-1}{2}\right|=1$ or $|z-1|=2$. Consequently, this series expansion is valid provided z is inside the circle defined by

$$z-1=2e^{j\theta} \text{ that is } z=1+2e^{j\theta}$$

By the same reasoning, the Taylor series expansion of $f(z)=\cos z$ about the point $z=\pi/3$ is

. . . .

14

$$\boxed{\frac{1}{2}\left(1-\sqrt{3}(z-\pi/3)-\frac{(z-\pi/3)^2}{2!}+\sqrt{3}\frac{(z-\pi/3)^3}{3!}+\frac{(z-\pi/3)^4}{4!}-\cdots\right)}$$

Because

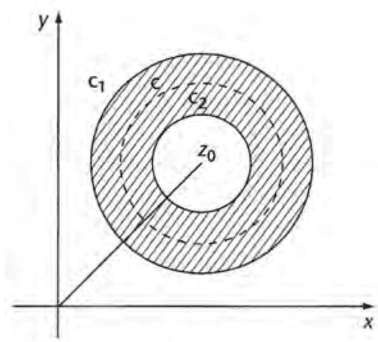
If $u=z-\pi/3$ then

$$\cos z=\cos(u+\pi/3)$$
$$=\cos u\cos\pi/3-\sin u\sin\pi/3$$
$$=\frac{1}{2}\left(\cos u-\sqrt{3}\sin u\right)$$
$$=\frac{1}{2}\left(\left[1-\frac{u^2}{2!}+\frac{u^4}{4!}-\cdots\right]-\sqrt{3}\left[u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right]\right)$$
$$=\frac{1}{2}\left(1-\frac{u^2}{2!}+\frac{u^4}{4!}-\cdots-\sqrt{3}u+\sqrt{3}\frac{u^3}{3!}-\sqrt{3}\frac{u^5}{5!}-\cdots\right)$$
$$=\frac{1}{2}\left(1-\sqrt{3}u-\frac{u^2}{2!}+\sqrt{3}\frac{u^3}{3!}+\frac{u^4}{4!}-\sqrt{3}\frac{u^5}{5!}-\cdots\right)$$
$$=\frac{1}{2}\left(1-\sqrt{3}(z-\pi/3)-\frac{(z-\pi/3)^2}{2!}+\sqrt{3}\frac{(z-\pi/3)^3}{3!}+\frac{(z-\pi/3)^4}{4!}-\cdots\right) \text{ for } z<\infty$$

Laurent's series

15

Sometimes a valid series expansion of a function is required within a specific region of the complex plane that contains a singular point. In this case we cannot avoid the singular point as we did with Taylor's series by expanding about an alternative non-singular point, because then we move away from part of the specified region. To accommodate this case we can use the *Laurent series expansion* which provides a series expansion valid within an annular region *centred on the singular point*.



Let $f(z)$ be singular at $z = z_0$ and let c_1 and c_2 be two concentric circles centred on z_0. Then if $f(z)$ is analytic in the annular region between c_1 and c_2 and if c is any concentric circle lying within the annular region between c_1 and c_2 we can expand $f(z)$ as a Laurent series in the form

$$\begin{aligned} f(z) &= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \\ &= \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \end{aligned}$$

where $a_n = \dfrac{1}{2\pi j}\displaystyle\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

Example

Expand $\dfrac{e^{3z}}{(z-2)^4}$ in a Laurent series about the point $z = 2$ and determine the nature of the singularity at $z = 2$.

$f(z) = \dfrac{e^{3z}}{(z-2)^4}$ and $f'(z) = \dfrac{e^{3z}(3z-10)}{(z-2)^5}$ so $f(z)$ is analytic everywhere except at $z = 2$. The first thing we must do is to transform the coordinate system by shifting the origin to the point $z = 2$ by defining $u = z - 2$ so that $z = u + 2$. Then

$$\frac{e^{3z}}{(z-2)^4} = \frac{e^{3(u+2)}}{u^4} = e^6\frac{e^{3u}}{u^4}.$$

▶

Now we can expand using the Maclaurin series expansion

$$= \frac{e^6}{u^4}\left\{1 + 3u + \frac{(3u)^2}{2!} + \frac{(3u)^3}{3!} + \frac{(3u)^4}{4!} + \frac{(3u)^5}{5!} + \cdots\right\}$$

$$= e^6\left\{\frac{1}{u^4} + \frac{3u}{u^4} + \frac{(3u)^2}{2!u^4} + \frac{(3u)^3}{3!u^4} + \frac{(3u)^4}{4!u^4} + \frac{(3u)^5}{5!u^4} + \cdots\right\}$$

$$= e^6\left\{\frac{1}{u^4} + \frac{3}{u^3} + \frac{9}{2u^2} + \frac{27}{6u} + \frac{81}{24} + \frac{243u}{120} + \cdots\right\}$$

$$= e^6\left\{\frac{1}{(z-2)^4} + \frac{3}{(z-2)^3} + \frac{9}{2(z-2)^2} + \frac{9}{2(z-2)} + \frac{27}{8} + \frac{81(z-2)}{40} + \cdots\right\}$$

This series converges for all finite z except $z = 2$ at which point there is a pole of order 4.

The part of the Laurent series that contains negative powers of the variable is called the *principal part* of the series and the remaining terms constitute what is called the *analytic part* of the series. *If, in the principal part the highest power of* $1/z$ *is* n*, then the function possesses a pole of order* n*; and if the principal part contains an infinite number of terms, the function possesses an* **essential singularity**.

Now you try one

16

The Laurent series expansion of $z^2 \cos\frac{1}{z}$ about the point $z = 0$

is .... valid for ....

at which point there is ....

17

$$z^2 - \frac{1}{2!} + \frac{1}{4!z^2} - \frac{1}{6!z^4} + \cdots \text{ valid for all } z \neq 0$$
at which point there is an essential singularity

Because

$f(z) = z^2 \cos\frac{1}{z}$ and $f'(z) = 2z\cos\frac{1}{z} + \sin\frac{1}{z}$ and so $f(z)$ is analytic everywhere except at $z = 0$. Expanding about $z = 0$ gives

$$z^2 \cos\frac{1}{z} = z^2\left(1 - \frac{(1/z)^2}{2!} + \frac{(1/z)^4}{4!} - \frac{(1/z)^6}{6!} + \cdots\right)$$
$$= z^2 - \frac{1}{2!} + \frac{1}{4!z^2} - \frac{1}{6!z^4} + \cdots$$

valid for all $z \neq 0$, at which point there is an essential singularity because there is an infinity of terms in the principal part of the series.

Try another. The Laurent series expansion of $\dfrac{z}{(z+2)(z+4)}$ valid for

$2 < |z| < 4$ is ....

$$\boxed{\cdots+\frac{8}{z^4}-\frac{4}{z^3}+\frac{2}{z^2}-\frac{1}{z}+\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots}$$

Because

$$\frac{z}{(z+2)(z+4)}=\frac{2}{z+4}-\frac{1}{z+2} \qquad \text{(separating into partial fractions)}$$

If $|z|>2$ then we can write $\dfrac{1}{z+2}=\dfrac{1}{z(1+2/z)}=\dfrac{(1+2/z)^{-1}}{z}$

and because $|z|>2$, that is, $|2/z|<1$, we can now use the binomial theorem

$$\frac{1}{z+2}=\frac{1}{z(1+2/z)}=\frac{1}{z}\left\{1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right\}=\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots$$

and if $|z|<4$ then

$$\begin{aligned}\frac{2}{z+4}=\frac{1}{2(1+z/4)}&=\frac{1}{2}\left\{1-\frac{z}{4}+\frac{z^2}{16}-\frac{z^3}{64}+\cdots\right\}\\&=\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\end{aligned}$$

Note the expansion of $(1+z/4)^{-1}$ which is valid for $|z/4|<1$, that is $|z|<4$.

The first expansion for $|z|>2$ is still valid for $|z|<4$ since $4>2$ and the second expansion for $|z|<4$ is still valid for $|z|>2$ since $2<4$. Consequently, if $2<|z|<4$, then, by subracting the first series from the second

$$\begin{aligned}\frac{z}{(z+2)(z+4)}&=\frac{2}{z+4}-\frac{1}{z+2}\\&=\left\{\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\right\}-\left\{\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right\}\\&=\cdots+\frac{8}{z^4}-\frac{4}{z^3}+\frac{2}{z^2}-\frac{1}{z}+\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\end{aligned}$$

Take care here! You may be tempted to think that this displays an essential singularity at $z=0$. This is not the case because the expansion is only valid inside the annular region $2<|z|<4$ centred on the origin. Consequently, the point $z=0$ is outside this region and the series expansion is invalid at that point.

The series expansion of the same function valid for $|z|<2$ is

. . . .

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$$\boxed{\frac{z}{8}-\frac{3z^2}{32}+\frac{7z^3}{128}+\cdots}$$

Because

If $|z|<2$ then $\dfrac{1}{z+2}=\dfrac{1}{2(1+z/2)}=\dfrac{1}{2}\left\{1-\dfrac{z}{2}+\dfrac{z^2}{4}-\dfrac{z^3}{8}+\cdots\right\}$

$$=\frac{1}{2}-\frac{z}{4}+\frac{z^2}{8}-\frac{z^3}{16}+\cdots$$

We have already seen that if $|z|<4$ then

$$\frac{2}{z+4}=\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots$$

This is still valid for $|z|<2$ since $2<4$. Conseqently, if $|z|<2$, then, by subtracting the first series from the second

$$\begin{aligned}\frac{z}{(z+2)(z+4)}&=\frac{2}{z+4}-\frac{1}{z+2}\\&=\left\{\frac{1}{2}-\frac{z}{8}+\frac{z^2}{32}-\frac{z^3}{128}+\cdots\right\}-\left\{\frac{1}{2}-\frac{z}{4}+\frac{z^2}{8}-\frac{z^3}{16}+\cdots\right\}\\&=\frac{z}{8}-\frac{3z^2}{32}+\frac{7z^3}{128}-\cdots\end{aligned}$$

Notice that for different regions of convergence we obtain different series expansions. Furthermore, each series expansion is unique within its own particular radius of convergence.

Try one more just to make sure that you can derive these expansions.

The Laurent series of $\dfrac{1-\cos(z-6)}{(z-6)^2}$ about the point $z=6$ is .... valid for .... at which point there is ....

20

$$\boxed{\begin{array}{l}\dfrac{1}{2!}-\dfrac{(z-6)^2}{4!}+\dfrac{(z-6)^4}{6!}-\cdots \text{ valid for all } z\neq 6\\ \text{at which point there is a removable singularity}\end{array}}$$

Because

If we let $u=z-6$ then

$$\begin{aligned}\frac{1-\cos(z-6)}{(z-6)^2}&=\frac{1-\cos u}{u^2}\\&=\frac{1}{u^2}\left\{1-\left(1-\frac{u^2}{2!}+\frac{u^4}{4!}-\frac{u^6}{6!}+\cdots\right)\right\}\\&=\frac{1}{2!}-\frac{u^2}{4!}+\frac{u^4}{6!}-\cdots\\&=\frac{1}{2!}-\frac{(z-6)^2}{4!}+\frac{(z-6)^4}{6!}-\cdots\end{aligned}$$

This is valid for all finite values of $z \neq 6$ at which point there is a removable singularity which can be removed by defining $\dfrac{1-\cos(z-6)}{(z-6)^2}$ at $z=6$ as $\dfrac{1}{2!}$. Notice that here the principal part has no terms, so that the Laurent series is identical to the Taylor series.

Next frame

Residues

21

In the Laurent series

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

the coefficient a_{-1} is referred to as the *residue* of $f(z)$ for reasons that will soon become apparent. Recall the integral in Frame 45 of Programme 21 which states that if the simple closed contour c has z_0 as an interior point, then

$$\oint_c \frac{dz}{(z-z_0)^n} = 2\pi j\delta_{n1}$$

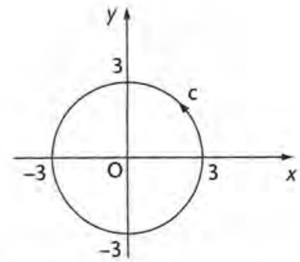
where the Kronecker delta $\delta_{n1} = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases}$. Applying this fact to the Laurent series of $f(z)$ yields

$$\begin{aligned}\oint_c f(z)\,dz &= \oint_c \left[\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0)\right.\\ &\qquad\qquad \left. + a_2(z-z_0)^2 + \cdots\right] dz\\ &= \cdots + \oint_c \frac{a_{-2}\,dz}{(z-z_0)^2} + \oint_c \frac{a_{-1}\,dz}{(z-z_0)} + \oint_c a_0\,dz\\ &\qquad + \oint_c a_1(z-z_0)\,dz + \oint_c a_2(z-z_0)^2 dz + \cdots\\ &= \cdots + 0 + 2\pi j a_{-1} + 0 + 0 + 0 + \cdots\\ &= 2\pi j a_{-1}\end{aligned}$$

That is, provided $f(z)$ is analytic at all points inside and on the simple closed contour c, apart from the single isolated singularity at z_0 which is interior to c, then

$$\oint_c f(z)\,dz = 2\pi j a_{-1}$$

Hence the name *residue* for a_{-1} because it is all that remains when the Laurent series is integrated term by term. This statement is called the **Residue theorem** and it has many far reaching consequences – we shall see some of these later. For now, just try an example.



If c is a circle, centred on the origin and of radius 3, then

$$\oint_c \frac{z\,dz}{(z+2)(z+4)} = \ldots\ldots\ldots\ldots$$

22

$$\boxed{\oint_c \frac{z\,dz}{(z+2)(z+4)} = -2\pi j}$$

Because

The circle $|z| = 3$ lies within the annular region $2 < |z| < 4$ and we have already found the Laurent series for the integrand valid for $2 < |z| < 4$ in Frame 18, namely

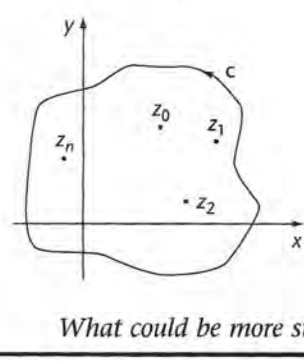
$$\frac{z}{(z+2)(z+4)} = \frac{2}{z+4} - \frac{1}{z+2}$$
$$= \cdots + \frac{8}{z^4} - \frac{4}{z^3} + \frac{2}{z^2} - \frac{1}{z} + \frac{1}{2} - \frac{z}{8} + \frac{z^2}{32} - \frac{z^3}{128} + \cdots$$

Here the residue is $a_{-1} = -1$ and so $\oint_c \dfrac{z\,dz}{(z+2)(z+4)} = 2\pi j(-1) = -2\pi j$

where c lies entirely within the region of convergence.

The Residue theorem extends to the case where the contour contains a finite number of singularities. If $f(z)$ is analytic inside and on the simple closed contour c except at the finite number of points z_0, z_1, $z_2, \ldots$, each with a Laurent series expansion and each with corresponding residues $\overset{(0)}{a}_{-1}$, $\overset{(1)}{a}_{-1}$, $\overset{(2)}{a}_{-1}, \ldots$ then

$$\oint_c f(z)\,dz = 2\pi j\left\{\overset{(0)}{a}_{-1} + \overset{(1)}{a}_{-1} + \overset{(2)}{a}_{-1}\right\} = 2\pi j\{\text{sum of residues inside c}\}$$



What could be more straightforward? Next frame

Calculating residues

23

When evaluating these integrals the major part of the exercise is to find the residues, and it would be very tedious if we had to find a Laurent series for each and every singularity. Fortunately there is a simpler method for poles. If $f(z)$ is analytic inside and on the simple closed contour c except at the interior point z_0 at which there is a pole of order n, then

$$a_{-1} = \underset{z\to z_0}{Lim}\left[\frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left((z-z_0)^n f(z)\right)\right]$$

Example

Find the residues at all the poles of $f(z) = \dfrac{3z}{(z+2)^2(z^2-1)}$.

$f(z)$ has a pole of order 2 (a double pole) at $z = -2$ and two poles of order 1 (simple poles) at $z = \pm 1$.

At $z = -2$ the residue is $a_{-1} = \underset{z\to -2}{Lim}\left[\dfrac{1}{(2-1)!}\dfrac{\mathrm{d}^{2-1}}{\mathrm{d}z^{2-1}}\left((z+2)^2 f(z)\right)\right]$

$$= \underset{z\to -2}{Lim}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{3z}{z^2-1}\right)\right]$$

$$= \underset{z\to -2}{Lim}\left[\frac{3(z^2-1)-6z^2}{(z^2-1)^2}\right]$$

$$= \frac{3(4-1)-24}{(4-1)^2} = -\frac{5}{3}$$

At $z = 1$ the residue is ....

24

$$\boxed{\frac{1}{6}}$$

Because

At $z = 1$ the residue is $a_{-1} = \underset{z\to 1}{Lim}\left[\dfrac{1}{(1-1)!}\dfrac{\mathrm{d}^{1-1}}{\mathrm{d}z^{1-1}}\left((z-1)f(z)\right)\right]$

$$= \underset{z\to 1}{Lim}\left[\frac{\mathrm{d}^0}{\mathrm{d}z^0}\left(\frac{3z}{(z+2)^2(z+1)}\right)\right]$$

The zeroth derivative of an expression is the expression itself

$$= \underset{z\to 1}{Lim}\left[\frac{3z}{(z+2)^2(z+1)}\right]$$

$$= \frac{3}{(3)^2(2)} = \frac{1}{6}$$

At $z = -1$ the residue is ....

25

$$\boxed{\frac{3}{2}}$$

Because

$$\text{At } z=-1 \text{ the residue is } a_{-1} = \underset{z\to -1}{Lim}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}((z+1)f(z))\right]$$

$$= \underset{z\to -1}{Lim}\left[\frac{d^0}{dz^0}\left(\frac{3z}{(z+2)^2(z-1)}\right)\right]$$

$$= \underset{z\to -1}{Lim}\left[\frac{3z}{(z+2)^2(z-1)}\right]$$

$$= \frac{-3}{(1)^2(-2)}$$

$$= \frac{3}{2}$$

Move to the next frame

Integrals of real functions

26

The Residue theorem can be very usefully employed to evaluate integrals of real functions that cannot be evaluated using the real calculus. Even when an integral is susceptible to evaluation by the real calculus, the use of the residue calculus can often save a great amount of effort. We shall look at three types of real integral and in each case we shall proceed by example.

Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Example

Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{4\cos\theta - 5}\, d\theta.$

To evaluate this integral we make use of the exponential representation of a complex number of unit length, namely $z = e^{j\theta}$,and the exponential form of the trigonometric functions

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} = \frac{z+z^{-1}}{2} \text{ and } \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j} = \frac{z-z^{-1}}{2j},$$

and finally $dz = je^{j\theta}\, d\theta = jz\, d\theta$ so that $d\theta = dz/jz$

▶

Using these relations we can transform the real integral from 0 to 2π into a contour integral in the complex plane where the contour c is the *unit circle centred on the origin*. That is

$$\begin{aligned}\int_0^{2\pi}\frac{1}{4\cos\theta-5}\,d\theta &= \oint_c \frac{1}{4\dfrac{z+z^{-1}}{2}-5}\times\frac{dz}{jz}\\ &= -j\oint_c \frac{1}{2z^2-5z+2}\,dz\\ &= -j\oint_c \frac{1}{(2z-1)(z-2)}\,dz\end{aligned}$$

The complex integrand has two simple poles, one at $z=\dfrac{1}{2}$ which is inside the contour c and another at $z=2$ which is outside the contour c. Using the Residue theorem

$$-j\oint_c \frac{1}{(2z-1)(z-2)}\,dz = -j\times 2\pi j\times\{\text{residue at } z=1/2\}$$

The residue at $z=1/2$ is

$$\begin{aligned}\underset{z\to 1/2}{Lim}\left\{(z-1/2)\frac{1}{(2z-1)(z-2)}\right\} &= \underset{z\to 1/2}{Lim}\left\{\frac{1}{2(z-2)}\right\}\\ &= -\frac{1}{3}\end{aligned}$$

so that

$$\begin{aligned}\int_0^{2\pi}\frac{1}{4\cos\theta-5}\,d\theta &= -j\oint_c \frac{1}{(2z-1)(z-2)}\,dz\\ &= -j\times 2\pi j\times\{\text{residue at } z=1/2\}\\ &= -2\pi/3\end{aligned}$$

Now you try one

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \ldots\ldots\ldots\ldots$$

27

$$\boxed{\frac{2\pi}{\sqrt{3}}}$$

Because

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_c \frac{dz/jz}{2+\dfrac{z+z^{-1}}{2}}$$ where c is the unit circle centred on the origin.

$$= -j\oint_c \frac{2\,dz}{z^2+4z+1}$$

$$= -j\oint_c \frac{2\,dz}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

The integrand has two simple poles, one at $z=-2+\sqrt{3}$ which is inside c and another at $z=-2-\sqrt{3}$ which is outside c. Therefore

$$-j\oint_c \frac{2\,dz}{(z+2-\sqrt{3})(z+2+\sqrt{3})} = -j\times 2\pi j\times\left\{\text{residue at } z=-2+\sqrt{3}\right\}$$

The residue is

$$\underset{z\to -2+\sqrt{3}}{Lim}\left\{\left(z+2-\sqrt{3}\right)\frac{2}{(z+2-\sqrt{3})(z+2+\sqrt{3})}\right\}$$

$$= \underset{z\to -2+\sqrt{3}}{Lim}\left\{\frac{2}{(z+2+\sqrt{3})}\right\} = \frac{1}{\sqrt{3}} \text{ and so}$$

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = -j\oint_c \frac{2\,dz}{(z+2-\sqrt{3})(z+2+\sqrt{3})} = -j\times 2\pi j\times\frac{1}{\sqrt{3}}$$

$$= 2\pi\times\frac{1}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

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Integrals of the form $\int_{-\infty}^{\infty} F(x)\,dx$

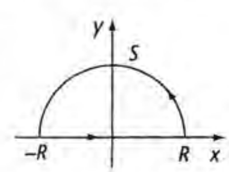
Example

Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^4}\,dx$.

To evaluate this integral we must consider the integral $\displaystyle\oint_c \frac{1}{1+z^4}\,dz$ where c is the contour shown in the figure, so that

$$\oint_c \frac{1}{1+z^4}\,dz = \int_s \frac{dz}{1+z^4} + \int_{-R}^{R}\frac{dx}{1+x^4} = 2\pi j\{\text{sum of residues inside c}\}$$

Notice that along the real axis between $-R$ and R, $z=x$. Provided $R>1$ we can evaluate this integral using the Residue theorem. That is



$$\oint_c \frac{1}{1+z^4}\,dz = 2\pi j\times\{\text{sum of residues inside c}\}$$

29

$$\boxed{\frac{\pi}{\sqrt{2}}}$$

Because

The integrand $\dfrac{1}{1+z^4}$ possesses four simple poles at $z = e^{\pi j/4}$, $e^{3\pi j/4}$, $e^{5\pi j/4}$, $e^{7\pi j/4}$ of which only the first two are inside c.

The residue at $z = e^{\pi j/4}$ is $\underset{z \to e^{\pi j/4}}{Lim}\left\{\left(z - e^{\pi j/4}\right) \times \dfrac{1}{1+z^4}\right\}$

$$= \underset{z \to e^{\pi j/4}}{Lim}\left\{\frac{1}{4z^3}\right\} \text{ by L'Hôpital's rule}$$

$$= \frac{e^{-3\pi j/4}}{4}$$

The residue at $z = e^{3\pi j/4}$ is $\underset{z \to e^{3\pi j/4}}{Lim}\left\{\left(z - e^{3\pi j/4}\right) \times \dfrac{1}{1+z^4}\right\}$

$$= \underset{z \to e^{3\pi j/4}}{Lim}\left\{\frac{1}{4z^3}\right\} \text{ by L'Hôpital's rule}$$

$$= \frac{e^{-9\pi j/4}}{4}$$

$$= \frac{e^{-\pi j/4}}{4}$$

Therefore

$$\oint_c \frac{1}{1+z^4}\,dz = 2\pi j \times \left\{\frac{1}{4}\left(e^{-3\pi j/4} + e^{-\pi j/4}\right)\right\}$$

Now $e^{-3\pi j/4} = \cos\dfrac{3\pi}{4} - j\sin\dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}} - \dfrac{j}{\sqrt{2}}$ and

$e^{-\pi j/4} = \cos\dfrac{\pi}{4} - j\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} - \dfrac{j}{\sqrt{2}}$ and so

$$\oint_c \frac{1}{1+z^4}\,dz = 2\pi j \times \left\{\frac{1}{4}\left(\frac{-2j}{\sqrt{2}}\right)\right\} = \frac{\pi}{\sqrt{2}}$$

We now look at the components of this integral in the next frame

30

We now recognise that

$$\oint_c \frac{1}{1+z^4}\,dz = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_S \frac{1}{1+z^4}\,dz$$

because $z = x$ along the real line.

Now we let R increase indefinitely and take limits, so that

$$\underset{R\to\infty}{Lim}\oint_c \frac{1}{1+z^4}\,dz = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx + \underset{R\to\infty}{Lim}\int_S \frac{1}{1+z^4}\,dz = \frac{\pi}{\sqrt{2}}$$

because the value of the contour integral is independent of the value of R. We shall now proceed to show that $\underset{R\to\infty}{Lim}\displaystyle\int_S \frac{1}{1+z^4}\,dz = 0$. ▶

Writing $z = Re^{j\theta}$ so that, on S, $dz = Re^{j\theta}\,d\theta$, the limit of the integral becomes

$$\underset{R\to\infty}{Lim}\int_S \frac{Re^{j\theta}}{1+R^4e^{j4\theta}}\,d\theta = 0$$

Notice that the requirement that ensures that the integral along the semicircle vanishes in the limit is equivalent to the requirement that the degree of the denominator be at least two degrees higher than the numerator.

Now you try one.

$$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2} = \ldots\ldots\ldots\ldots$$

31

$$\boxed{\frac{\pi}{2}}$$

Because

Consider the integral $\oint_c \frac{z^2\,dz}{(z^2+1)^2}$ where the contour c is the same semicircular contour as in the previous example. Here the integrand has two double poles at $z = j$ and $z = -j$ but only the pole at $z = j$ is inside the contour. The residue at $z = j$ is

$$\underset{z\to j}{Lim}\left\{\frac{d}{dz}(z-j)^2\frac{z^2}{(z-j)^2(z+j)^2}\right\} = \underset{z\to j}{Lim}\left\{\frac{2z(z+j)^2 - z^2 2(z+j)}{(z+j)^4}\right\}$$
$$= -\frac{j}{4}$$

Therefore

$$\oint_c \frac{z^2\,dz}{(z^2+1)^2} = 2\pi j\left(-\frac{j}{4}\right) = \frac{\pi}{2}$$

Taking limits

$$\underset{R\to\infty}{Lim}\oint_c \frac{z^2\,dz}{(z^2+1)^2} = \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2} + \underset{R\to\infty}{Lim}\int_S \frac{z^2\,dz}{(z^2+1)^2} = \frac{\pi}{2}$$

Where, in the second integral on the right-hand side, the degree of the denominator is two higher than the degree of the numerator, and so

$$\underset{R\to\infty}{Lim}\int_S \frac{z^2\,dz}{(z^2+1)^2} = 0, \text{ therefore } \int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2} = \frac{\pi}{2}$$

32

Integrals of the form $\int_{-\infty}^{\infty} F(x)\begin{cases}\sin x\\ \cos x\end{cases} dx$

These integrals are often referred to as Fourier integrals because of their appearances within Fourier analysis.

Example

Evaluate $\int_{-\infty}^{\infty} \frac{\cos kx}{a^2+x^2}\,dx$ where $a > 0$ and $k > 0$.

To evaluate this integral we consider the contour integral $\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz$ where c is the semicircular contour of the previous problems and whose integrand possesses two simple poles at $z = aj$ and $z = -aj$ of which only the first is inside the contour. Consequently

$$\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz = 2\pi j\{\text{residue at } z = aj\} = \ldots\ldots\ldots\ldots$$

33

$$\boxed{\frac{\pi e^{-ka}}{a}}$$

Because

The residue at $z = aj$ is

$$\underset{z\to aj}{Lim}\left\{(z-aj)\frac{e^{jkz}}{a^2+z^2}\right\} = \underset{z\to aj}{Lim}\left\{\frac{e^{jkz}}{z+aj}\right\} = \frac{e^{jk(aj)}}{2aj} = -\frac{je^{-ka}}{2a} \text{ and so}$$

$$\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz = 2\pi j\left\{-\frac{je^{-ka}}{2a}\right\} = \frac{\pi e^{-ka}}{a}$$

Taking limits as $R \to \infty$

$$\underset{R\to\infty}{Lim}\oint_c \frac{e^{jkz}}{a^2+z^2}\,dz = \int_{-\infty}^{\infty} \frac{e^{jkz}}{a^2+z^2}\,dz + \underset{R\to\infty}{Lim}\int_S \frac{e^{jkz}}{a^2+z^2}\,dz = \frac{\pi e^{-ka}}{a}$$

In the second integral on the right-hand side, the degree of the denominator is two higher than the degree of the numerator, and so

$\underset{R\to\infty}{Lim}\int_S \frac{e^{jkz}}{a^2+z^2}\,dz = 0$, therefore $\int_{-\infty}^{\infty} \frac{e^{jkx}}{a^2+x^2}\,dx = \frac{\pi e^{-ka}}{a}$. That is

$$\int_{-\infty}^{\infty} \frac{\cos kx + j\sin kx}{a^2+x^2}\,dx = \frac{\pi e^{-ka}}{a} = 2\pi j\{\text{residue at } z = aj\}.$$

Consequently

$$\int_{-\infty}^{\infty} \frac{\cos kx}{a^2+x^2}\,dx = \frac{\pi e^{-ka}}{a} = -2\pi\,\text{Im}\{\text{residue at } z = aj\} \text{ and}$$

$$\int_{-\infty}^{\infty} \frac{\sin kx}{a^2+x^2}\,dx = 0 = 2\pi\,\text{Re}\{\text{residue at } z = aj\}$$

▶

Notice that e^{jkz} is easier to use than $\cos kx = \left(e^{jkx} + e^{-jkx}\right)/2$, and it also gives the solution to the related integral with $\cos kx$ replaced with $\sin kx$.

Finally, to finish off the Programme, here is one for you to try.

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{x^2 + x + 1}\, dx = \ldots\ldots\ldots\ldots$$

34

$$\boxed{0}$$

Because

Consider $\oint_c \frac{e^{j\pi z}}{z^2 + z + 1}\, dz$ where c is the semicircular contour of the previous problem. The integrand is singular at the simple poles $z = (-1 \pm j\sqrt{3})/2$ where only $z = (-1 + j\sqrt{3})/2$ is inside the contour. The residue at $z = (-1 + j\sqrt{3})/2$ is then

$$\underset{z \to (-1+j\sqrt{3})/2}{Lim} \left\{ \left(z - \left[-1 + j\sqrt{3}\right]/2\right) \frac{e^{j\pi z}}{z^2 + z + 1} \right\}$$

$$= \underset{z \to (-1+j\sqrt{3})/2}{Lim} \left\{ \frac{e^{j\pi z}}{z - \left[-1 - j\sqrt{3}\right]/2} \right\}$$

$$= \frac{e^{j\pi(-1+j\sqrt{3})/2}}{j\sqrt{3}}$$

$$= \frac{e^{-j\pi/2} e^{-\sqrt{3}\pi/2}}{j\sqrt{3}}$$

$$= -\frac{e^{-\sqrt{3}\pi/2}}{\sqrt{3}} \qquad \text{since } e^{-j\pi/2} = -j$$

Therefore

$$\oint_c \frac{e^{j\pi z}}{z^2 + z + 1}\, dz = 2\pi j \left\{ \frac{e^{-\sqrt{3}\pi/2}}{\sqrt{3}} \right\} = -j \frac{2\pi e^{-\sqrt{3}\pi/2}}{\sqrt{3}}$$

that is

$$\oint_c \frac{e^{j\pi z}}{z^2 + z + 1}\, dz = \oint_c \frac{\cos \pi z + j \sin \pi z}{z^2 + z + 1}\, dz = -j \frac{2\pi e^{-\sqrt{3}\pi/2}}{\sqrt{3}}$$

and so

$$\oint_c \frac{\cos \pi z}{z^2 + z + 1}\, dz = 0 \text{ and } \oint_c \frac{\sin \pi z}{z^2 + z + 1}\, dz = -\frac{2\pi e^{-\sqrt{3}\pi/2}}{\sqrt{3}}$$

Note that, again, the contribution from the contour integral along the semicircle is zero.

The **Revision summary** now follows. Check it through in conjunction with the **Can You?** checklist before goint on to the **Test exercise**. The **Further problems** provide additional practice.

Revision summary 22

1 *Maclaurin series*
The Maclaurin series expansion of a function of a complex variable z is

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + z^3\frac{f'''(0)}{3!} + \cdots$$

2 *Ratio test for convergence*
The ratio test for convergence of a series of terms of a complex variable

$$f(z) = a_0(z) + a_1(z) + a_2(z) + a_3(z) + \cdots + a_n(z) + \cdots$$

is that given

$$\underset{n\to\infty}{Lim}\left|\frac{a_{n+1}(z)}{a_n(z)}\right| = L$$

then if $L < 1$ the series converges and so the expansion is valid
$L > 1$ the series diverges and so the expansion is invalid
$L = 1$ the ratio test fails to give a conclusion.

3 *Radius and circle of convergence*
The radius of the circle within which a series expansion is valid is called the *radius of convergence* of the series and the circle is called the *circle of convergence*. The radius of convergence can be found using the ratio test for convergence.

4 *Singular points*
Any point at which $f(z)$ fails to be analytic, that is where the derivative does not exist, is called a *singular point*.

Poles
If $f(z)$ has a singular point at z_0 and for some natural number n

$$\underset{z\to z_0}{Lim}\left\{(z - z_0)^n f(z)\right\} = L \neq 0$$

then the singular point (also called a singularity) is called *a pole of order n*.

Removable singularity
If $f(z)$ has a singular point at z_0 but $\underset{z\to z_0}{Lim}\{f(z)\}$ exists then the singular point is called a *removable singularity*.

5 *Circle of convergence*
When an expression is expanded in a Maclaurin series, the *circle of convergence* is always centred on the origin and the *radius of convergence* is determined by the location of the first singular point met as z moves out from the origin.

▶

6 *Taylor's series*

Provided $f(z)$ is analytic inside and on a simple closed curve c, the Taylor series expansion of $f(z)$ about a point z_0 which is interior to c is given as

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2 f''(z_0)}{2!} + \cdots + \frac{(z - z_0)^n f^{(n)}(z_0)}{n!} + \cdots$$

where, here, the expansion is about the point z_0 which is the centre of the circle of convergence. The circle of convergence is given as $|z - z_0| = R$ where R is the radius of convergence. Maclaurin's series is a special case of Taylor's series where $z_0 = 0$.

7 *Laurent's series*

The *Laurent series expansion* provides a series expansion valid within an annular region *centred on the singular point.*

Let $f(z)$ be singular at $z = z_0$ and let c_1 and c_2 be two concentric circles centred on z_0. Then if $f(z)$ is analytic in the annular region between c_1 and c_2 and c is any concentric circle lying within the annular region between c_1 and c_2 we can expand $f(z)$ as a Laurent series in the form

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$
$$= \sum_{n \to -\infty}^{\infty} a_n(z - z_0)^n \text{ where } a_n = \frac{1}{2\pi j}\oint_c \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

8 *Residues*

In the Laurent series

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

the coefficient a_{-1} is referred to as the *residue* of $f(z)$.

Residue theorem

Provided $f(z)$ is analytic at all points inside and on the simple closed contour c, apart from the single isolated singularity at z_0 which is interior to c, then

$$\oint_c f(z)\,dz = 2\pi j a_{-1}$$

9 The Residue theorem extends to the case where the contour contains a finite number of singularities. If $f(z)$ is analytic inside and on the simple closed contour c except at the finite number of points z_0, z_1, $z_2, \cdots$ each with a Laurent series expansion and each with corresponding residues $\overset{(0)}{a}_{-1}, \overset{(1)}{a}_{-1}, \overset{(2)}{a}_{-1}, \cdots$ then

$$\oint_c f(z)\,dz = 2\pi j\left\{\overset{(0)}{a}_{-1} + \overset{(1)}{a}_{-1} + \overset{(2)}{a}_{-1} + \cdots\right\}$$

▶

10 *Calculating residues*

$$a_{-1} = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right) \right]$$

11 *Real integrals*

The Residue theorem can be very usefully employed to evaluate integrals of real functions.

Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Use $z = e^{j\theta}$ and the exponential form of the trigonometric functions $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2} = \dfrac{z + z^{-1}}{2}$, $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} = \dfrac{z - z^{-1}}{2j}$ and $dz = je^{j\theta}\, d\theta = jz\, d\theta$ so that $d\theta = dz/jz$. Convert the integral into a contour integral around the unit circle centred on the origin and use the Residue theorem.

Integrals of the form $\int_{-\infty}^{\infty} F(x)\, dx$ and $\int_{-\infty}^{\infty} F(x) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx$

Consider integrals of the form $\oint_c F(z)\, dz$ and $\oint_c F(z) e^{jz}\, dz$ respectively, where the contour c is a semicircle with the diameter lying along the real axis. The principle is that the integral can be evaluated by the Residue theorem and then the contour can be expanded to cover the required extent of the real axis, the integration along the semicircle giving a zero contribution.

☑ Can You?

Checklist 22

36

Check this list before and after you try the end of Programme test

On a scale of 1 to 5 how confident are you that you can:	Frames
• Expand a function of a complex variable about the origin in a Maclaurin series? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	1 to 7
• Determine the circle and radius of convergence of a Maclaurin series expansion? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	8 to 10
• Recognise singular points in the form of poles of order n, removable and essential singularities? *Yes* ☐ ☐ ☐ ☐ ☐ *No*	11

▶

- Expand a function of a complex variable about a point in the complex plane in a Taylor series, transforming the coordinates with a shift of origin? 12 to 14

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Expand a function of a complex variable about a singular point in a Laurent series? 15

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the principal and analytic parts of the Laurent series and link the form of the principal part to the type of singularity? 16 to 20

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the residue of a Laurent series and state the Residue theorem? 21 and 22

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Calculate the residues at the poles of an expression without resort to deriving the Laurent series? 23 to 25

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Evaluate certain types of real integrals using the Residue theorem? 26 to 34

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 22

37

1 Expand each of the following in a Maclaurin series and determine the radius and the circle of convergence in each case.

(a) $f(z) = e^z$

(b) $f(z) = \ln(1 + 4z)$.

2 Determine the location and nature of the singular points in each of the following.

(a) $f(z) = \dfrac{3z}{(z+1)^5}$

(b) $f(z) = z^{10}e^{1/z}$

(c) $f(z) = z\sin(1/z)$

(d) $f(z) = \dfrac{1-\cos z}{z^2}$

3 Expand $f(z) = \sin z$ in a Taylor series about the point $z = \pi/4$ and determine the radius of convergence.

▶

4 Expand each of the following in a Laurent series. In (a) and (c) determine the nature of the singularity from the principal part of the series.

(a) $f(z) = (5 - z)\cos\dfrac{1}{z+3}$ about the point $z = -3$

(b) $f(z) = \dfrac{2z}{(z+1)(z+3)}$ valid for $1 < |z| < 3$

(c) $f(z) = \dfrac{1}{z^3(z-2)^2}$ about the point $z = 2$.

5 Calculate the residues at each of the singularities of
$$f(z) = \frac{3z-1}{z^2(z+1)^2(z-1)}.$$

6 Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5\cos\theta - 13}$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{x^4 + 2x^2 + 1}\,dx$

Further problems 22

38

1 For each of the following find the Maclaurin series expansion and determine the radius of convergence.

(a) $\sinh z$

(b) $\tan z$

(c) $\ln\left(\dfrac{1+z}{1-z}\right)$

(d) a^z, where $a > 0$

(e) $\dfrac{15z^2}{(5-3z)^3}$.

2 By using the appropriate Maclaurin series expansions, show that

(a) $(\cos z)' = -\sin z$

(b) $\cos z = \dfrac{e^z + e^{-z}}{2}$

(c) $(e^z)' = e^z$.

▶

3 Given the series expansion for $(1+z)^{-1}$

(a) show by integration that this is compatible with the series expansion for $\ln(1+z)$

(b) by differentiation find $\sum_{n=1}^{\infty}(-1)^n nz^n$ and $\sum_{n=1}^{\infty}(-1)^n n^2 z^n$.

4 Use the ratio test to test each of the following for convergence.

(a) $\sum_{n=0}^{\infty}\dfrac{(2n)!}{(n!)^2}z^n$

(b) $\sum_{n=0}^{\infty}\dfrac{z^n}{1-3n}$

(c) $\sum_{n=0}^{\infty}\dfrac{n^2z^n}{1-3n}$

(d) $\sum_{n=0}^{\infty}\dfrac{(\cos n\pi)z^n}{2n-1}$

(e) $\sum_{n=1}^{\infty}\dfrac{(-1)^n z^n}{(n+1)!}$

5 Find the Taylor series about the point indicated of each of the following.

(a) e^z about the point $z=2$

(b) $\cos z$ about the point $z=\pi/6$

(c) $(z-3)\sin(z+3)$ about the point $z=3$

(d) $(2z-5)^{-1}$ about the point $z=1/3$

(e) $(2z-5)^{-1}$ about the point $z=3$.

6 Find the series expansion of $z\ln z$ valid for $|z-1|<1$.

7 Find the circle of convergence of each of the following when expanded in a Taylor series about the point indicated.

(a) $e^{-z}\cos(z-2)$ about the point $z=1$

(b) $\dfrac{z^3}{(z^2+6)}$ about the point $z=0$

(c) $\dfrac{z-2}{(z-6)(z-4)}$ about the point $z=5$

(d) $\dfrac{z^2}{(e^z+1)}$ about the point $z=0$.

8 Locate and classify all of the singularities of each of the following.

(a) $\dfrac{(z-1)^3}{z^2(z^2-1)^2}$

(b) $z^{-2}e^{-1/z}$.

▶

9 Find the Laurent series about the point indicated of each of the following.

(a) $\dfrac{1}{z}\sin\left(\dfrac{1}{z}\right)$ about the point $z = 0$

(b) $\dfrac{1}{2z-3}$ about the point $z = 3/2$

(c) $\dfrac{z}{(z-2)(z-3)}$ about the point $z = 3$.

10 Find the Laurent series of $\dfrac{z-1}{(z+2)(z+5)}$ that is valid for

(a) $2 < |z| < 5$

(b) $|z| > 5$

(c) $|z| < 2$.

11 Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{\alpha+\beta\sin\theta}$ for $\alpha > |\beta|$

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\alpha^2-2\alpha\cos\theta}$ where $0 < \alpha < 1$

(d) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{5-4\cos\theta}$

(e) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+13}$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4+6x^2+13}$

(h) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+4)^2}$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2+x+1}{x^4+x^2+1}dx$

(j) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$

(k) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\sin\pi x\, dx}{x^4+6x^2+13}$

(l) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin\pi x}{x^4+x^2+1}dx.$

Optimization and linear programming

Frames 1 to 83

Learning outcomes

When you have completed this Programme you will be able to:

- Describe an optimization problem in terms of the objective function and a set of constraints
- Algebraically manipulate and graphically describe inequalities
- Solve a linear programming problem in two real variables
- Use the simplex method to describe a linear programming problem in two real variables as a problem in two real variables with two slack variables
- Set up the simplex tableau and compute the simplex
- Use the simplex method to solve a linear programming problem in three real variables with three slack variables
- Introduce artificial variables into the solution method as and when the need arises
- Solve minimisation problems using the simplex method
- Construct the algebraic form of the objective function and the constraints for a problem stated in words

Optimization

1

An *optimization problem* is one requiring the determination of the *optimal (maximum or minimum) value* of a given function, called the *objective function*, subject to a set of stated restrictions, or *constraints*, placed on the variables concerned.

In practice, for example, we may need to maximise an objective function representing units of output in a manufacturing situation, subject to constraints reflecting the availability of labour, machine time, stocks of raw materials, transport conditions, etc.

Linear programming (or linear optimization)

Linear programming is a method of solving an optimization problem when the objective function is a *linear function* and the constraints are *linear equations* or *linear inequalities*.

In this Programme, we shall restrict our considerations to problems of this type that form an important introduction to the much wider study of operational research.

Let us consider a simple example, so move on to the next frame

2

A simple linear programming problem may look like this:

$$\begin{array}{lll} \text{Maximise} & P = x + 2y & (\textit{objective function}) \\ \text{subject to} & \left.\begin{array}{l} y \leq 3 \\ x + y \leq 5 \\ x - 2y \leq 2 \\ x \geq 0;\ y \geq 0 \end{array}\right\} & (\textit{constraints}) \end{array}$$

The last two constraints, i.e. $x \geq 0$ and $y \geq 0$, apply to all linear programming (LP) problems and indicate that the problem variables, x and y, are restricted to non-negative values: they may have zero or positive values, but NOT negative values. These two constraints are often combined and written $x, y \geq 0$ – or omitted altogether since they are taken for granted in all LP problems.

Before we proceed, we will take a brief look at linear inequalities in general.

On, then, to Frame 3

3

Linear inequalities

In most respects, *linear inequalities* can be manipulated in the same manner as can equations.

(a) Both sides may be increased or decreased by a common term, e.g.

$$2x \leq y + 4 \quad \therefore \; 2x - y \leq 4$$

(b) Both sides may be multiplied or divided by a positive factor, e.g.

$$4x + 6y \geq 12 \quad \therefore \; 2x + 3y \geq 6$$

But NOTE this:

(c) If both sides are multiplied or divided by a negative factor, e.g. (-1), then the inequality sign must be reversed, i.e. $\geq$ becomes $\leq$ and vice versa.

Here, then, is a short exercise.

Exercise

Simplify the following inequalities so that each right-hand side consists of a positive constant term only.

(a) $3x - 5 \leq 4y$ (b) $2(x + 2y) \leq -8$

(c) $4x - 6y \leq -10$ (d) $2x + 3 \geq -(y + 4)$

(e) $-(x - 3y + 5) \geq 2x + 4y - 6$

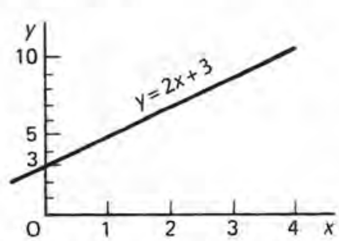
Check the results in the next frame

4

(a) $3x - 4y \leq 5$	(b) $-x - 2y \geq 4$
(c) $-2x + 3y \geq 5$	(d) $-2x - y \leq 7$
(e) $3x + y \leq 1$	

Graphical representation of linear inequalities

Consider the inequality $y - 2x \leq 3$. We can add $2x$ to each side, so that $y \leq 2x + 3$.



The equation $y = 2x + 3$ can be represented by a straight line dividing the x–y plane into two parts.

For all points on the line,

$y = 2x + 3$.

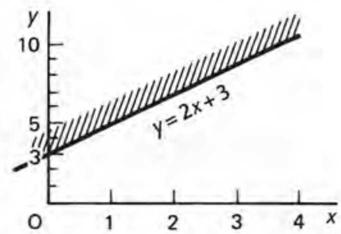
For all points below the line,

....

5

$$\boxed{y < 2x + 3}$$

$\therefore\ y \le 2x + 3$ indicates all points on or below the straight line, but excludes all points above it. We can indicate this exclusion zone by shading the upper side of the line.



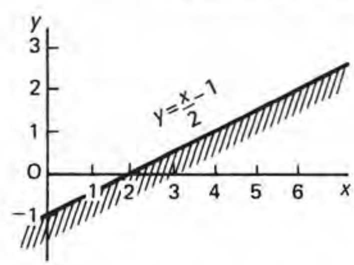
Arguing in much the same way, $x - 2y \le 2$ can be rewritten as $y \ge \dfrac{x}{2} - 1$ and we can draw the line $y = \dfrac{x}{2} - 1$ and shade in the exclusion zone

. . . .

6

$$\boxed{\text{below the line}}$$

i.e.



Example 1

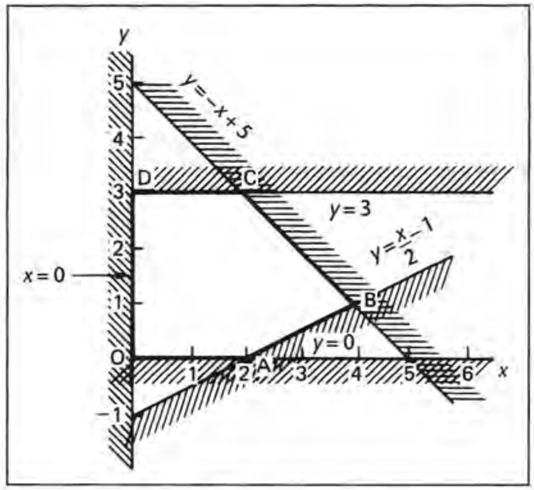
The problem we quoted earlier in Frame 2 was

$$\begin{array}{lll} \text{Maximise} & P = x + 2y & (\textit{objective function}) \\ \text{subject to} & \left.\begin{array}{l} y \le 3 \\ x + y \le 5 \\ x - 2y \le 2 \\ x \ge 0;\ y \ge 0 \end{array}\right\} & (\textit{constraints}) \end{array}$$

Now, on a common pair of x and y axes, we can represent the five constraints and shade in the exclusion zone for each. We then have the composite diagram

. . . .

7

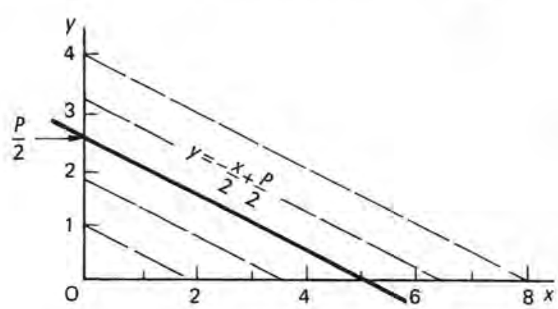


The coordinates of all points on the boundary of the polygon OABCD, or within the figure so formed, satisfy the system of constraints. The set of variables for each such point is called a *feasible point* or *feasible solution* and the figure OABCD is the *feasible domain* or *feasible polygon*.

Note these definitions.

8

Our problem now is to find the particular point within this domain that makes the objective function $P = x + 2y$ a maximum. The equation can be rewritten as $y = -\dfrac{x}{2} + \dfrac{P}{2}$ and this represents a set of parallel lines with different values of the intercept $\dfrac{P}{2}$.



If we draw one sample line of this set to cross the feasible polygon we have just obtained, we get, using $P = 3$

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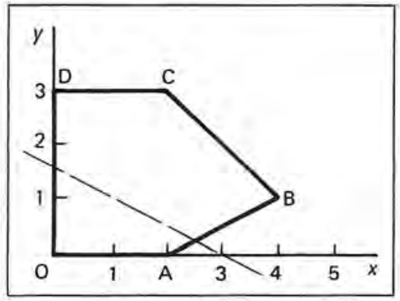
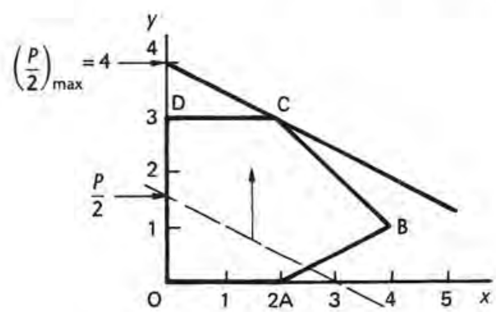


We can increase the value of P and hence raise the objective line up the page until it passes through the extreme point C. Any further increase in the value of P would take the line outside the feasible polygon and hence fail to conform to the given set of constraints.



In this example, then, point C gives the optimal solution.

From the graph it can be seen that the line with maximal P passes through the point of intersection of the two lines $y = 3$ and $y = -x + 5$. This means that $y = 3$, $x = 2$ and so $P_{max} = x + 2y = 8$.

A graphical method of solution is clearly limited to linear programming problems involving two variables only. However, it is a useful introduction to other techniques, so let us deal with another example.

Example 2

$$\begin{aligned} \text{Maximise} \quad & P = x + 4y \\ \text{subject to} \quad & -x + 2y \leq 6 \\ & 5x + 4y \leq 40 \\ & x, y \geq 0 \end{aligned}$$

First of all, plot the appropriate straight line graphs to obtain the feasible polygon. This gives

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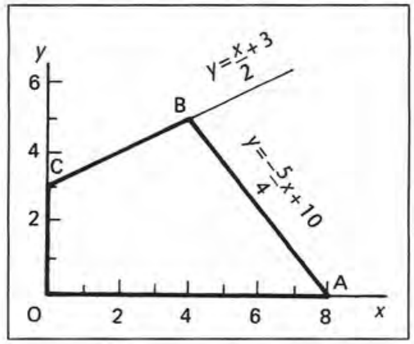
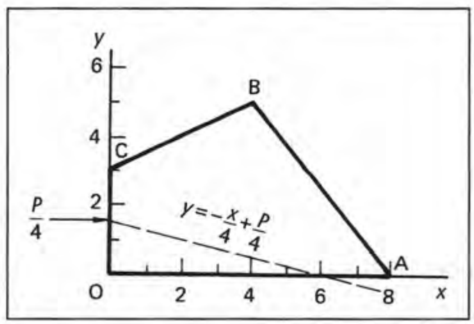
10



The objective function $P = x + 4y$ can be expressed in the form $y = -\dfrac{x}{4} + \dfrac{P}{4}$ and its graph added to the feasible polygon, as before. We then obtain

....

11



The line $y = -\dfrac{x}{4} + \dfrac{P}{4}$ is then raised to give the optimal solution, which is

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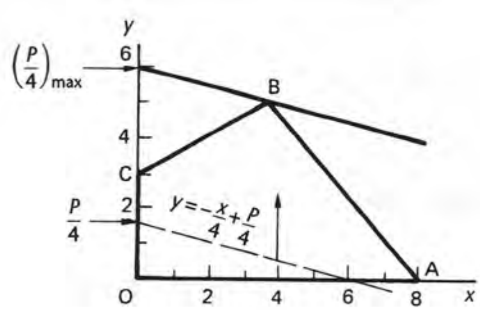
12

$$\boxed{P_{\max} = 24 \text{ with } x = 4, y = 5}$$



▶

From the graph it can be seen that the line with maximal P passes through the point of intersection of the two lines $y = \dfrac{x}{2} + 3$ and $y = -\dfrac{5x}{4} + 10$. That is $x = 4$, $y = 5$ and so $P_{\text{max}} = x + 4y = 24$.

As easy as that.

Now this one.

Example 3

Minimise $\quad P = -4x + 6y$

subject to $\quad -x + 6y \geq 24$

$2x - y \leq 7$

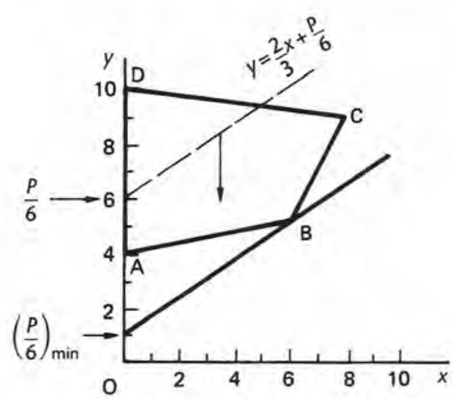
$x + 8y \leq 80$

$x, y \geq 0$

It is very much as before. Complete it on your own.

13

$$\boxed{P_{\text{min}} = 6 \quad \text{with} \quad x = 6,\ y = 5}$$



To obtain the minimum optimal value of P, the graph of the objective function is, of course, lowered to the appropriate extreme point.

In practice, linear programming problems usually contain many more variables than the two we have so far considered and a computational method is then required. One such technique is the *simplex method* and the remainder of this Programme will be devoted to the steps necessary to put it into practice.

So move on to Frame 14

The simplex method

14

The first step in the *simplex method* is to ensure that each constraint is written with a *positive* right-hand side constant term. Then we express all inequalities as equations by the introduction of *slack variables*.

For example, $-x+2y \le 6$ can be written $-x+2y+w_1 \quad = 6$
and $5x+4y \le 40$ can be written $5x+4y \quad +w_2 = 40$

where w_1 and w_2 are positive (or zero) variables with unit coefficients, required to make up the left-hand side to the value of the right-hand side constant term. The new variables, w_1 and w_2, are called *slack variables*.

Let us look again at the problem we solved earlier.

Example 1

Maximise $P = x + 4y$
subject to $-x+2y \le 6$
$5x+4y \le 40$ (as always, $x, y \ge 0$)

The constraints now become
$$-x+2y+w_1 \quad = 6$$
$$5x+4y \quad + w_2 = 40$$
and the objective function
$$P - x - 4y \quad = 0$$

From this, we can now begin to form the simplex tableau (or table).

So make a note of the above information – and then move on

15

Setting up the simplex tableau

(a) *Framework* First construct a framework with the headings shown.

x y w_1 w_2	b	*check*

Next, we enter, in the framework, the coefficients of the problem variables and of the slack variables in the constraints, together with the right-hand side constants in the column headed b. (Ignore the *check* column for the time being.)

So we have

. . . .

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Problem variables		Slack variables		Const.	
x	y	w_1	w_2	b	*check*
-1	2	1	0	6	
5	4	0	1	40	
body		unity matrix			

(b) *Check column* The right-hand side column is included to provide a check on the numerical calculations as we develop the simplex, so, for each row, total up the entries in that row, including the constant column, and enter the sum in the check column.

Do that

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Basis	x	y	w_1	w_2	b	*check*
w_1	-1	2	1	0	6	8
w_2	5	4	0	1	40	50

(c) *Starting basic solution* The two constraints now contain four variables, but if we start by letting x and y each be zero, then we have the temporary solutions, $w_1 = 6$ and $w_2 = 40$, and we indicate these variables in the extra left-hand side column, as shown.

Note (1) The columns with the slack variables form a unity matrix.
(2) There are now four variables, x, y, w_1, w_2 ($n = 4$).
(3) There are two constraints ($m = 2$).
(4) We put $(n - m)$ variables, i.e. two variables (x and y), equal to zero as a start.

Finally, we have to deal with the objective function, so move on to the next frame

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(d) *The objective function* The objective function, $P = x + 4y$, is written $P - x - 4y = 0$ and the coefficients of this form the bottom row, or *index row*, of the tableau, thus

Basis	x	y	w_1	w_2	b	*check*
w_1	-1	2	1	0	6	8
w_2	5	4	0	1	40	50
P	-1	-4	0	0	0	-5

Complete your tableau, if you have not already done so, and then we will see how the computation is carried out.

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Computation of the simplex

1 First we select the column containing the most negative entry in the index row: in this case −4. This is called the *key column* and we enclose it as shown.

Basis	x	y	w_1	w_2	b	*check*
w_1	−1	2	1	0	6	8
w_2	5	4	0	1	40	50
P	−1	−4	0	0	0	−5

key column

2 In each row, we now divide the value in the b column by the positive entry in the key column: the smaller ratio determines the *key row*.

$$\left.\begin{aligned}\text{For row 1 } (w_1),\quad r &= \frac{6}{2} = 3\\ \text{row 2 } (w_2),\quad r &= \frac{40}{4} = 10\end{aligned}\right\}\quad \begin{aligned}&\therefore \text{ row 1 is the key row.}\\ &\text{Enclose it as shown below.}\end{aligned}$$

pivot

Basis	x	y	w_1	w_2	b	*check*	
w_1	−1	2	1	0	6	8	← key row
w_2	5	4	0	1	40	50	
P	−1	−4	0	0	0	−5	

The number at the intersection of the key column and key row is the *key number* or *pivot*: in this case the number 2.

3 We now divide all entries in the key row by the pivot to reduce the pivot to *unity* – which we then circle. The new version of the key row is sometimes called the *main row*. The rest of the tableau remains unchanged, so we then get ....

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unit pivot

Basis	x	y	w_1	w_2	b	*check*	
w_1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4	← main row
w_2	5	4	0	1	40	50	
P	−1	−4	0	0	0	−5	← index

key column

So far, so good. Now we deal with the actual calculations.

Next frame

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4 Using the main row, we now operate on the remaining rows of the tableau, including the index row, to reduce the other entries in the key column to zero. Note that the main row remains unaltered. The new value in any position in the other rows, including the b column and the check column, can be calculated as follows:

New number = old number – the product of the corresponding entries in the main row and key column

1	A	main row
B	K	

K is replaced by $K - (A \times B)$

For example, in the second row (w_2):

5 is replaced by $5 - (-\frac{1}{2})(4) = 5 + 2 = 7$

4 is replaced by $4 - (1)(4) \quad = 4 - 4 = 0$

0 is replaced by $0 - (\frac{1}{2})(4) \quad = 0 - 2 = -2$

1 is replaced by $1 - (0)(4) \quad = 1 - 0 = 1$

40 is replaced by $40 - (3)(4) \quad = 40 - 12 = 28$

50 is replaced by $50 - (4)(4) \quad = 50 - 16 = 34$

and, in the third row (P):

-1 is replaced by $-1 - (-\frac{1}{2})(-4) = -1 - 2 = -3.$

Completing the operations for rows (w_2) and (P), we have

....

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Basis	x	y	w_1	w_2	b	*check*
w_1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	7	0	-2	1	28	34
P	-3	0	2	0	12	11

Now confirm that the new values in the check column are, indeed, the sums of the entries in the corresponding rows. If not, there is a mistake somewhere in the working to be corrected before we proceed.

If all is well, move on to the next frame

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5 *Change of basic variables* In its final form, the key column consists of a single 1 and the remaining entries zero. This is in the column headed y which indicates that the basic variable w_1 in the main row can be replaced by y.

Basis	x	y	w_1	w_2	b	check
$y\ \cancel{w_1}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	7	0	-2	1	28	34
P	-3	0	2	0	12	11

Note that there are two columns containing a single 1 and the rest 0. These are headed y and w_2 which are now also the basic variables in the left-hand side column. Reading the values in the b column therefore gives a basic solution $y = 3$ and $w_2 = 28$, and at this stage $P = 12$. Any variable not listed in the basis column is zero. One basic solution at this stage is therefore $x = 0$, $y = 3$, $w_2 = 28$. However, we are not finished.

The index row (P) still contains another negative entry, so we have to repeat the simplex process using the same steps as before.

Basis	x	y	w_1	w_2	b	check	
y	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4	
w_2	7	0	-2	1	28	34	← key row
P	-3	0	2	0	12	11	

key column

Now divide the key row by the key number (7) to reduce the pivot to a unit pivot. This gives

. . . .

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Basis	x	y	w_1	w_2	b	check	
y	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4	
w_2	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$	← main row
P	-3	0	2	0	12	11	

Using the main row, operate on the remaining rows (including the index row) to reduce the other entries in the key column to zero. Complete that stage and we have

. . . .

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Basis	x	y	w_1	w_2	b	check
y	0	1	$\frac{5}{14}$	$\frac{1}{14}$	5	$\frac{45}{7}$
w_2	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	0	0	$\frac{8}{7}$	$\frac{3}{7}$	24	$\frac{179}{7}$

Again, at this stage, check your working by totalling up the entries in each row and satisfy yourself that the sum agrees with the value in the check column.

Note that w_2 in the basis column can now be replaced by x which was the heading of the column containing the last unit pivot.

So finally, we have

Basis	x	y	w_1	w_2	b	check
y	0	1	$\frac{5}{14}$	$\frac{1}{14}$	5	$\frac{45}{7}$
x	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	0	0	$\frac{8}{7}$	$\frac{3}{7}$	24	$\frac{179}{7}$

A new basic solution now emerges as $x = 4$, $y = 5$.

Furthermore, since there is no further negative entry in the index row, this is also the optimal solution and the optimal value of P is given in the b column, i.e. $P_{\max} = 24$.

For interest, you may wish to compare this result with that obtained in Frame 12.

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We have been through the simplex operation in some detail by way of explanation. Many problems involve more than just two variables, but the method of computation is basically the same, being an iterative process which is repeated until the index row contains no negative entry, at which point the optimal value of the objective function has been attained.

The problem we have just solved would normally look like this:

$$\begin{aligned} &\text{Maximise} && P = x + 4y \\ &\text{subject to} && -x + 2y \leq 6 \\ &&& 5x + 4y \leq 40 \quad (x,\ y \geq 0) \end{aligned}$$

Entering slack variables, etc., this is written

$$\begin{aligned} -x + 2y + w_1 \qquad &= 6 \\ 5x + 4y \qquad + w_2 &= 40 \\ P - x - 4y \qquad &= 0 \end{aligned}$$

The complete tableau is given in the next frame

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Basis	x	y	w_1	w_2	b	*check*
w_1	-1	2	1	0	6	8
w_2	5	4	0	1	40	50
P	-1	-4	0	0	0	-5
$y\ \cancel{w_1}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	5	4	0	1	40	50
P	-1	-4	0	0	0	-5
$y\ \cancel{w_1}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	7	0	-2	1	28	34
P	-3	0	2	0	12	11
y	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	3	4
w_2	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	-3	0	2	0	12	11
y	0	1	$\frac{5}{14}$	$\frac{1}{14}$	5	$\frac{45}{7}$
$x\ \cancel{w_2}$	1	0	$-\frac{2}{7}$	$\frac{1}{7}$	4	$\frac{34}{7}$
P	0	0	$\frac{8}{7}$	$\frac{3}{7}$	24	$\frac{179}{7}$

$$P_{\max} = 24 \text{ with } x = 4,\ y = 5$$

Now for another example – so move on to Frame 28

28

Here is one for you to do on your own. The method is just the same as before so you will have no difficulty.

Example 2

$$\begin{aligned}
\text{Maximise}\quad & P = 4x + 3y \\
\text{subject to}\quad & -x + \ \ y \le 4 \\
& \ \ x + 2y \le 14 \\
& 2x + \ \ y \le 16 \quad (x, y \ge 0)
\end{aligned}$$

We have three inequalities this time, so we shall need to introduce three slack variables. Converting the inequalities into equations, we obtain

....

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$$
\begin{array}{|l|}
\hline
-x + \ \ y + w_1 \qquad\qquad\quad = 4 \\
\ \ x + 2y \qquad + w_2 \qquad = 14 \\
2x + \ \ y \qquad\qquad\quad + w_3 = 16 \\
\hline
\end{array}
$$

Then we set out the simplex framework with appropriate headings, i.e.

....

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Basis	x y w_1 w_2 w_3	b	check

Remembering that the index row uses $P - 4x - 3y = 0$, we can set out the first tableau. Choosing x and y, as usual, to be zero for a start, we have

....

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Basis	x	y	w_1	w_2	w_3	b	check
w_1	−1	1	1	0	0	4	5
w_2	1	2	0	1	0	14	18
w_3	2	1	0	0	1	16	20
P	−4	−3	0	0	0	0	−7

Carry on now and complete the working on this first tableau.

Check with the next frame

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Here is the working so far.

Basis	x	y	w_1	w_2	w_3	b	check
w_1	-1	1	1	0	0	4	5
w_2	1	2	0	1	0	14	18
w_3	2	1	0	0	1	16	20
P	-4	-3	0	0	0	0	-7
w_1	-1	1	1	0	0	4	5
w_2	1	2	0	1	0	14	18
w_3	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	-4	-3	0	0	0	0	-7
w_1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	15
w_2	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	6	8
x $\cancel{w_3}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	0	-1	0	0	2	32	33

(a) The basic variable (w_3) of the unit pivot can now be replaced by the variable at the heading of the unit pivot (x).

(b) We see there is still a negative value in the index row, so we repeat the process for this second tableau.

Now you can finish it off

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Check to see if you agree.

Basis	x	y	w_1	w_2	w_3	b	*check*
w_1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	15
w_2	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	6	8
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	0	-1	0	0	2	32	33
w_1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	12	15
w_2	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	4	$\frac{16}{3}$
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	8	10
P	0	-1	0	0	2	32	33
w_1	0	0	1	-1	1	6	7
y ~~w_2~~	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	4	$\frac{16}{3}$
x	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	6	$\frac{22}{3}$
P	0	0	0	$\frac{2}{3}$	$\frac{5}{3}$	36	$\frac{115}{3}$

The basic variable (w_2) can now be replaced by y, being the heading of the unit pivot column.

There is no further negative entry in the index row: therefore, the optimal value of P has been attained.

$\therefore\ P_{\max} = 36 \quad \text{with} \quad x = 6,\ y = 4.$

Note: We also see that $w_1 = 6$, since the unity matrix has headings x, y, w_1. The full result, therefore, is

$$P_{\max} = 36 \quad \text{with} \quad x = 6,\ y = 4,\ w_1 = 6,\ w_2 = 0,\ w_3 = 0$$

though we do not normally require this extra information.

The meaning of $w_1 = 0$ and $w_2 = 0$ is that the second and third constraints become

$x + 2y = 14$ and $2x + y = 16$ respectively rather than
$x + 2y \leq 14$ and $2x + y \leq 16$.

The meaning of $w_1 = 6$ gives the first constraint as $-x + y < 14$ rather than $-x + y \leq 14$.

Now we will extend the simplex method to an example involving three problem variables.

Next frame

34 Simplex with three problem variables

$$\begin{aligned}\text{Maximise}\quad & P = p_1x + p_2y + p_3z\\ \text{subject to}\quad & a_{11}x + a_{12}y + a_{13}z \le b_1\\ & a_{21}x + a_{22}y + a_{23}z \le b_2\\ & a_{31}x + a_{32}y + a_{33}z \le b_3\\ & x, y, z \ge 0\end{aligned}$$

Introducing slack variables we have

$$\begin{aligned}a_{11}x + a_{12}y + a_{13}z + w_1 \qquad\qquad &= b_1\\ a_{21}x + a_{22}y + a_{23}z \qquad + w_2 \qquad &= b_2\\ a_{31}x + a_{32}y + a_{33}z \qquad\qquad + w_3 &= b_3\\ P - p_1x - p_2y - p_3z \qquad\qquad\qquad &= 0\end{aligned}$$

If there is now a total of n variables and m constraints, then at least $(n-m)$ variables are equated to zero. The remainder form the basic variable column entries. Equating x, y, z to zero, then, the basic variables are w_1, w_2, w_3.

Basis	x	y	z	w_1	w_2	w_3	b	*check*
w_1	a_{11}	a_{12}	a_{13}	1	0	0	b_1	
w_2	a_{21}	a_{22}	a_{23}	0	1	0	b_2	
w_3	a_{31}	a_{32}	a_{33}	0	0	1	b_3	
P	$-p_1$	$-p_2$	$-p_3$	0	0	0	0	

The variables in the basis column are the variables heading the unity matrix. The method is exactly as before.

(a) Select the most negative entry in the index row to determine the *key column*.
(b) Divide the entries in the constant column (b) by the corresponding positive entries in the key column. The smallest positive ratio determines the *key row*.
(c) The entry at the intersection of the key column and the key row is the *key number* or *pivot*.
(d) Divide each entry in the key row by the pivot to reduce the key number to a *unit pivot*. The revised key row is now called the *main row*.
(e) Use the main row to operate on the remaining rows to reduce all other entries in the key column to zero.
(f) Repeat steps (a) to (e) until no negative entry remains in the index row.

Now for an example

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Example 1

Maximise $P = 2x + 6y + 4z$

subject to
$$\begin{aligned} 2x + 5y + 2z &\le 38 \\ 4x + 2y + 3z &\le 57 \\ x + 3y + 5z &\le 57 \\ x, y, z &\ge 0 \end{aligned}$$

Rewriting the inequalities as equations gives

....

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$$\boxed{\begin{array}{lllll} 2x + 5y + 2z + w_1 & & & = 38 \\ 4x + 2y + 3z & + w_2 & & = 57 \\ \;x + 3y + 5z & & + w_3 & = 57 \end{array}}$$

We also have $P - 2x - 6y - 4z = 0$, so we can now set up the simplex tableau ready for solution. That is ....

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Basis	x	y	z	w_1	w_2	w_3	b	*check*
w_1	2	5	2	1	0	0	38	48
w_2	4	2	3	0	1	0	57	67
w_3	1	3	5	0	0	1	57	67
P	−2	−6	−4	0	0	0	0	− 12

Now we just apply the normal simplex routine until there is no negative entry in the index row.

Remember:

(1) to replace the basic variables as the problem variables become available at each stage, and
(2) any variable not appearing in the basis column has zero value.

Now you can work the solution right through and then check the result with the next frame. Take your time: there are no snags.

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Basis	x	y	z	w_1	w_2	w_3	b	check
w_1	2	5	2	1	0	0	38	48
w_2	4	2	3	0	1	0	57	67
w_3	1	3	5	0	0	1	57	67
P	-2	-6	-4	0	0	0	0	-12
w_1	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{38}{5}$	$\frac{48}{5}$
w_2	4	2	3	0	1	0	57	67
w_3	1	3	5	0	0	1	57	67
P	-2	-6	-4	0	0	0	0	-12
y $\cancel{w_1}$	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{38}{5}$	$\frac{48}{5}$
w_2	$\frac{16}{5}$	0	$\frac{11}{5}$	$-\frac{2}{5}$	1	0	$\frac{209}{5}$	$\frac{239}{5}$
w_3	$-\frac{1}{5}$	0	$\frac{19}{5}$	$-\frac{3}{5}$	0	1	$\frac{171}{5}$	$\frac{191}{5}$
P	$\frac{2}{5}$	0	$-\frac{8}{5}$	$\frac{6}{5}$	0	0	$\frac{228}{5}$	$\frac{228}{5}$
y	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{38}{5}$	$\frac{48}{5}$
w_2	$\frac{16}{5}$	0	$\frac{11}{5}$	$-\frac{2}{5}$	1	0	$\frac{209}{5}$	$\frac{239}{5}$
w_3	$-\frac{1}{19}$	0	1	$-\frac{3}{19}$	0	$\frac{5}{19}$	9	$\frac{191}{19}$
P	$\frac{2}{5}$	0	$-\frac{8}{5}$	$\frac{6}{5}$	0	0	$\frac{228}{5}$	$\frac{228}{5}$
y	$\frac{8}{19}$	1	0	$\frac{5}{19}$	0	$-\frac{2}{19}$	4	$\frac{106}{19}$
w_2	$\frac{63}{19}$	0	0	$-\frac{1}{19}$	1	$-\frac{11}{19}$	22	$\frac{448}{19}$
z $\cancel{w_3}$	$-\frac{1}{19}$	0	1	$-\frac{3}{19}$	0	$\frac{5}{19}$	9	$\frac{191}{19}$
P	$\frac{16}{19}$	0	0	$\frac{18}{19}$	0	$\frac{8}{19}$	60	$\frac{1172}{19}$

$$\therefore\ P_{\max} = 60 \text{ with } x = 0,\ y = 4,\ z = 9.$$

Do you agree?

If so, on to the next frame

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Example 2

Maximise $P = 3x + 4y + 5z$

subject to

$$\begin{aligned} 2x + 4y + 3z &\leq 80 \\ 4x + 2y + z &\leq 48 \\ x + y + 2z &\leq 40 \\ x, y, z &\geq 0 \end{aligned}$$

It is much the same as before. Work through it carefully and then check with the next frame.

40

$$
\begin{aligned}
2x + 4y + 3z + w_1 \quad\quad\quad\quad &= 80 \\
4x + 2y + \ \ z \quad\ + w_2 \quad\quad &= 48 \\
x + \ \ y + 2z \quad\quad\quad + w_3 &= 40 \\
P - 3x - 4y - 5z \quad\quad\quad\quad &= 0
\end{aligned}
$$

Basis	x	y	z	w_1	w_2	w_3	b	*check*
w_1	2	4	3	1	0	0	80	90
w_2	4	2	1	0	1	0	48	56
w_3	1	1	2	0	0	1	40	45
P	-3	-4	-5	0	0	0	0	-12
w_1	2	4	3	1	0	0	80	90
w_2	4	2	1	0	1	0	48	56
w_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	20	$\frac{45}{2}$
P	-3	-4	-5	0	0	0	0	-12
w_1	$\frac{1}{2}$	$\frac{5}{2}$	0	1	0	$-\frac{3}{2}$	20	$\frac{45}{2}$
w_2	$\frac{7}{2}$	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	28	$\frac{67}{2}$
z $\cancel{w_3}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	20	$\frac{45}{2}$
P	$-\frac{1}{2}$	$-\frac{3}{2}$	0	0	0	$\frac{5}{2}$	100	$\frac{201}{2}$
w_1	$\frac{1}{5}$	1	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	8	9
w_2	$\frac{7}{2}$	$\frac{3}{2}$	0	0	1	$-\frac{1}{2}$	28	$\frac{67}{2}$
z	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	20	$\frac{45}{2}$
P	$-\frac{1}{2}$	$-\frac{3}{2}$	0	0	0	$\frac{5}{2}$	100	$\frac{201}{2}$
y $\cancel{w_1}$	$\frac{1}{5}$	1	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	8	9
w_2	$\frac{16}{5}$	0	0	$-\frac{3}{5}$	1	$\frac{2}{5}$	16	20
z	$\frac{2}{5}$	0	1	$-\frac{1}{5}$	0	$\frac{4}{5}$	16	18
P	$-\frac{1}{5}$	0	0	$\frac{3}{5}$	0	$\frac{8}{5}$	112	114
y	$\frac{1}{5}$	1	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	8	9
w_2	1	0	0	$-\frac{3}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	5	$\frac{25}{4}$
z	$\frac{2}{5}$	0	1	$-\frac{1}{5}$	0	$\frac{4}{5}$	16	18
P	$-\frac{1}{5}$	0	0	$\frac{3}{5}$	0	$\frac{8}{5}$	112	114
y	0	1	0	$\frac{7}{16}$	$-\frac{1}{16}$	$-\frac{5}{8}$	7	$\frac{31}{4}$
x $\cancel{w_2}$	1	0	0	$-\frac{3}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	5	$\frac{25}{4}$
z	0	0	1	$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{4}$	14	$\frac{31}{2}$
P	0	0	0	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{13}{8}$	113	$\frac{461}{4}$

$$\therefore\ P_{\max} = 113 \quad \text{with } x = 5,\ y = 7,\ z = 14.$$

Now let us meet a further complication. *Next frame*

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Artificial variables

So far, our approach to each problem has been the same.

(a) We first of all convert the 'less than' inequalities into equations by the inclusion of slack variables.
(b) If there are now n variables and m constraints, then at least $(n - m)$ variables are equated to zero – usually x, y, z, etc. – so that the initial basic solution is given by the slack variables, the coefficients of which form the unity matrix in the tableau.
(c) We then proceed by the simplex method to convert the basic solution to one containing the problem variables, the tableau entries for which now form a new unity matrix.
(d) The method is repeated as necessary. When no negative entry remains in the index row, the value of P denoted in the constant column is the optimal value of the objective function.

Now let us look at this example.

Example 1

Maximise $\quad P = 7x + 4y$

subject to
$$2x + y \leq 150$$
$$4x + 3y \leq 350$$
$$x + y \geq 80 \quad (x, y \geq 0)$$

Converting the inequalities to equations, we have

....

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$$\boxed{\begin{array}{l} 2x + y + w_1 = 150 \\ 4x + 3y + w_2 = 350 \\ x + y - w_3 = 80 \end{array}}$$

Also, of course, $P - 7x - 4y = 0$.

NOTE that since the third constraint is a 'greater than' statement, we must subtract the slack positive variable (w_3) to form the equation.

Alternatively, we could have written the inequality as $-x - y \leq -80$ so that $-x - y + w_3 = -80$ and so $x + y - w_3 = 80$.

Forming the first tableau, in the usual manner, we obtain

....

43

Basis	x	y	w_1	w_2	w_3	b	*check*
w_1	2	1	1	0	0	150	154
w_2	4	3	0	1	0	350	358
w_3	1	1	0	0	−1	80	81
P	−7	−4	0	0	0	0	−11

Now we are stuck, for we do not have a unity matrix to start off with. The entry in the w_3 column is –1 and not the necessary +1, and no amount of manipulation will help since the entries in the constant column (b) are, by definition, positive.

So how can we find a starting technique?

Let us restate the problem.

44

Maximise $P = 7x + 4y$

subject to
$$\begin{aligned} 2x + y + w_1 &= 150 \\ 4x + 3y + w_2 &= 350 \\ x + y - w_3 &= 80 \end{aligned}$$

The trouble comes in the third constraint by virtue of the negative sign of the slack variable. To save the situation, we introduce a new small positive variable (w_4) so that w_1, w_2 and w_4 will give rise to a unity matrix and the simplex computation can then proceed. Of course, w_4 is ficticious, is extremely small and cannot appear in the final basic solution. To establish this, we include in the objective function a new term $-Mw_4$, where M is an extremely large positive value which will ensure that w_4 will ultimately vanish. So we now write

$$P = 7x + 4y - Mw_4$$

The new variable, w_4, is called an *artificial variable*: it is introduced solely so that the simplex procedure can be carried out; and it must not appear in the final basic solution listed in the basis column.

The third constraint above now becomes

$$x + y \qquad - w_3 + w_4 = 80$$

Next frame

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We now have

$$\begin{aligned} 2x + y + w_1 \qquad\qquad\qquad &= 150 \\ 4x + 3y \qquad + w_2 \qquad\qquad &= 350 \\ x + y \qquad\qquad - w_3 + w_4 &= 80 \\ P - 7x - 4y \qquad\qquad + Mw_4 &= 0 \end{aligned}$$

Forming the tableau in the usual way:

Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	2	1	1	0	0	0	150	154
w_2	4	3	0	1	0	0	350	358
w_4	1	1	0	0	−1	1	80	82
P	−7	−4	0	0	0	M	0	$M - 11$

Note that:

(1) The columns headed w_1, w_2, w_4 now form the unity matrix.
(2) There are now 6 variables and 3 constraints, i.e. $n = 6$ and $m = 3$. At least $(n - m)$, i.e. $6 - 3 = 3$, variables are put equal to zero. We start off with x, y, w_3 as zero and w_1, w_2, w_4 form the first basic solution with the values given in the b column.

We now proceed in the normal way. Solve the first tableau and check the results so far in the following frame.

46

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
w_1	2	1	1	0	0	0	150	154
w_2	4	3	0	1	0	0	350	358
w_4	1	1	0	0	-1	1	80	82
P	-7	-4	0	0	0	M	0	$M-11$
w_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	4	3	0	1	0	0	350	358
w_4	1	1	0	0	-1	1	80	82
P	-7	-4	0	0	0	M	0	$M-11$
$x\ \cancel{w_1}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	0	1	-2	1	0	0	50	50
w_4	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1	5	5
P	0	$-\frac{1}{2}$	$\frac{7}{2}$	0	0	M	525	$M+528$

The basic variable w_1 can be replaced by x and we continue as before to remove the further negative entry in the index row. *Do that*

47

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	0	1	-2	1	0	0	50	50
w_4	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1	5	5
P	0	$-\frac{1}{2}$	$\frac{7}{2}$	0	0	M	525	$M+528$
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	77
w_2	0	1	-2	1	0	0	50	50
w_4	0	1	-1	0	-2	2	10	10
P	0	$-\frac{1}{2}$	$\frac{7}{2}$	0	0	M	525	$M+528$
x	1	0	1	0	1	-1	70	72
w_2	0	0	-1	1	2	-2	40	40
$y\ \cancel{w_4}$	0	1	-1	0	-2	2	10	10
P	0	0	3	0	-1	$M+1$	530	$M+533$

The basic variable w_4 is now replaced by y (the column of the last unit pivot). We now have another negative entry in the index row, so we have to perform the simplex calculation yet again.

For the next round, we get ....

48

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
x	1	0	1	0	1	-1	70	72
w_2	0	0	-1	1	2	-2	40	40
y	0	1	-1	0	-2	2	10	10
P	0	0	3	0	-1	$M+1$	530	$M+533$
x	1	0	1	0	1	-1	70	72
w_2	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	20	20
y	0	1	-1	0	-2	2	10	10
P	0	0	3	0	-1	$M+1$	530	$M+533$
x	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	50	52
w_3 $\cancel{w_2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	20	20
y	0	1	-2	1	0	0	50	50
P	0	0	$\frac{5}{2}$	$\frac{1}{2}$	0	M	550	$M+553$

No further negative entry remains in the index row. The optimal solution has been found, i.e.

$P_{\text{max}} = 550$ with $x = 50$, $y = 50$.

In addition, we see that $w_3 = 20$ while $w_1 = w_2 = w_4 = 0$ since they do not occur in the basic variable column.

Notice, also, that w_4, the artificial variable, does not figure in the optimal solution – as indeed it must not.

Next frame

49

Here is one for you to do in the same way.

Example 2

$$\begin{aligned}
\text{Maximise} \quad & P = 2x + 5y \\
\text{subject to} \quad & x + 4y \le 60 \\
& 3x + 2y \le 40 \\
& x + y \ge 12 \quad (x, y \ge 0)
\end{aligned}$$

Work right through it, just as before. The result is

. . . .

50

$$\boxed{P_{\max} = 78 \quad \text{with } x = 4,\ y = 14}$$

Because

$$\begin{aligned} x + 4y + w_1 \qquad\qquad\qquad &= 60 \\ 3x + 2y \qquad + w_2 \qquad\qquad &= 40 \\ x + y \qquad\qquad - w_3 + w_4 &= 12 \\ P - 2x - 5y \qquad\qquad + Mw_4 &= 0 \end{aligned}$$

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
w_1	1	4	1	0	0	0	60	66
w_2	3	2	0	1	0	0	40	46
w_4	1	1	0	0	-1	1	12	14
P	-2	-5	0	0	0	M	0	$M-7$
w_1	-3	0	1	0	4	-4	12	10
w_2	1	0	0	1	2	-2	16	18
$y\ \cancel{w_4}$	1	1	0	0	-1	1	12	14
P	3	0	0	0	-5	$M+5$	60	$M+63$
w_1	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	1	-1	3	$\frac{5}{2}$
w_2	1	0	0	1	2	-2	16	18
y	1	1	0	0	-1	1	12	14
P	3	0	0	0	-5	$M+5$	60	$M+63$
$w_3\ \cancel{w_1}$	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	1	-1	3	$\frac{5}{2}$
w_2	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	0	0	10	13
y	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	0	15	$\frac{33}{2}$
P	$-\frac{3}{4}$	0	$\frac{5}{4}$	0	0	M	75	$M+\frac{151}{2}$
w_3	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	1	-1	3	$\frac{5}{2}$
w_2	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	0	4	$\frac{26}{2}$
y	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	0	15	$\frac{33}{2}$
P	$-\frac{3}{4}$	0	$\frac{5}{4}$	0	0	M	75	$M+\frac{151}{2}$
w_3	0	0	$\frac{1}{10}$	$\frac{3}{10}$	1	-1	6	$\frac{32}{5}$
$x\ \cancel{w_2}$	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	0	4	$\frac{26}{5}$
y	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	0	0	14	$\frac{76}{5}$
P	0	0	$\frac{11}{10}$	$\frac{3}{10}$	0	M	78	$M+\frac{397}{5}$

$\therefore\ P_{\max} = 78$ with $x = 4,\ y = 14$.

On to the next frame

51

We are not always as lucky as we were in the previous two examples and other steps sometimes have to be taken to remove the artificial variable. Consider the following case.

Example 3

$$\begin{array}{ll} \text{Maximise} & P = 8x + 4y \\ \text{subject to} & 2x + 3y \leq 120 \\ & x + \ \ y \leq \ \ 45 \\ & -3x + 5y \geq \ \ 25 \quad (x, y \geq 0) \end{array}$$

Inserting the slack variables and artificial variable as required, we have

....

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$$\boxed{\begin{array}{rl} 2x + 3y + w_1 \qquad\qquad\qquad & = 120 \\ x + \ \ y \qquad + w_2 \qquad\qquad & = \ \ 45 \\ -3x + 5y \qquad\qquad - w_3 + \ \ w_4 & = \ \ 25 \\ P - 8x - 4y \qquad\qquad\qquad + Mw_4 & = \ \ \ \ 0 \end{array}}$$

That is very much as before, so work through it and check with the next frame.

53

Here it is.

Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	2	3	1	0	0	0	120	126
w_2	1	1	0	1	0	0	45	48
w_4	−3	5	0	0	−1	1	25	27
P	−8	−4	0	0	0	M	0	$M - 12$
w_1	0	1	1	−2	0	0	30	30
x ~~w_2~~	1	1	0	1	0	0	45	48
* → w_4	0	8	0	3	−1	1	160	171
P	0	4	0	8	0	M	360	$M + 372$

There is no further negative entry in the index row, so it looks as though the optimal solution has been attained. However, the artificial variable w_4 still remains in the basic variable column at * and thus must be removed. Therefore, we take the entry at the junction of the y column and the w_4 row as the pivot and proceed to eliminate w_4 by simplifying the tableau a stage further.

If we do that, we get ....

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Basis	x	y	w_1	w_2	w_3	w_4	b	check
w_1	0	1	1	-2	0	0	30	30
x	1	1	0	1	0	0	45	48
* → w_4	0	$\boxed{8}$	0	3	-1	1	160	171
P	0	4	0	8	0	M	360	$M+372$
w_1	0	1	1	-2	0	0	30	30
x	1	1	0	1	0	0	45	48
* → w_4	0	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	20	$\frac{171}{8}$
P	0	4	0	8	0	M	360	$M+372$
w_1	0	0	1	$-\frac{19}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	10	$\frac{69}{8}$
x	1	0	0	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	25	$\frac{213}{8}$
$y\ \cancel{w_4}$	0	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	50	$\frac{171}{8}$
P	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	$M-\frac{1}{2}$	280	$M+\frac{573}{2}$

The artificial variable, w_4, is now replaced by y in the basic variable column and the optimal solution has been reached.

$\therefore\ P_{\max} = 280 \quad \text{with} \quad x = 25,\ y = 20.$

Next frame

55

Now here is one for you to deal with.

Example 4

$$\begin{aligned}
\text{Maximise}\quad & P = 10x + 2y \\
\text{subject to}\quad & -x + 2y \le 60 \\
& 5x + 4y \le 260 \\
& -x + 8y \ge 80 \quad (x,\ y \ge 0)
\end{aligned}$$

Work through it as before and see if you agree with the solution in the next frame.

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$$\begin{array}{rcrcrcrcrcl}
-x+2y+w_1 & & & & & & & = & 60 \\
5x+4y & & +w_2 & & & & & = & 260 \\
-x+8y & & & & -w_3 & + & w_4 & = & 80 \\
P-10x-2y & & & & & & +Mw_4 & = & 0
\end{array}$$

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
w_1	-1	2	1	0	0	0	60	62
w_2	5	4	0	1	0	0	260	270
w_4	-1	8	0	0	-1	1	80	87
P	-10	-2	0	0	0	M	0	$M-12$
w_1	-1	2	1	0	0	0	60	62
w_2	1	$\frac{4}{5}$	0	$\frac{1}{5}$	0	0	52	54
w_4	-1	8	0	0	-1	1	80	87
P	-10	-2	0	0	0	M	0	$M-12$
w_1	0	$\frac{14}{5}$	1	$\frac{1}{5}$	0	0	112	116
x $\cancel{w_2}$	1	$\frac{4}{5}$	0	$\frac{1}{5}$	0	0	52	54
* → w_4	0	$\boxed{\frac{44}{5}}$	0	$\frac{1}{5}$	-1	1	132	141
P	0	6	0	2	0	M	520	$M+528$
w_1	0	$\frac{14}{5}$	1	$\frac{1}{5}$	0	0	112	116
x	1	$\frac{4}{5}$	0	$\frac{1}{5}$	0	0	52	54
* → w_4	0	1	0	$\frac{1}{44}$	$-\frac{5}{44}$	$\frac{5}{44}$	15	$\frac{705}{44}$
P	0	6	0	2	0	M	520	$M+528$
w_1	0	0	1	$\frac{3}{22}$	$\frac{7}{22}$	$-\frac{7}{22}$	70	$\frac{1565}{22}$
x	1	0	0	$\frac{2}{11}$	$\frac{1}{11}$	$-\frac{1}{11}$	40	$\frac{453}{11}$
y $\cancel{w_4}$	0	1	0	$\frac{1}{44}$	$-\frac{5}{44}$	$\frac{5}{44}$	15	$\frac{705}{44}$
P	0	0	0	$\frac{41}{22}$	$\frac{15}{22}$	$M-\frac{15}{22}$	430	$M+\frac{9501}{22}$

$$\therefore\ P_{\max} = 430 \quad \text{with} \quad x = 40,\ y = 15.$$

Now for another example

Example 5

This one is slightly different, so take note.

Maximise $P = 11x + 15y$

subject to
$$\begin{aligned} 3x + 5y &\le 130 \\ -4x + 5y &\ge 25 \\ x + 5y &\ge 75 \quad (x, y \ge 0) \end{aligned}$$

In this problem, notice that there are two 'greater than' inequalities so that there will be two slack variables to be subtracted and two artificial variables to be incorporated. In the objective function, we can use the same factor, M, for both artificial variables, since neither of those two variables will appear in the final optimal solution. So, we have:

$$\begin{aligned} 3x + 5y + w_1 &= 130 \\ -4x + 5y - w_2 + w_4 &= 25 \\ x + 5y - w_3 + w_5 &= 75 \\ P - 11x - 15y + Mw_4 + Mw_5 &= 0 \end{aligned}$$

w_1, w_4, w_5 now form the unity matrix from which to start. The method is just the same as in previous examples, so finish it off.

58

Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	check
w_1	3	5	1	0	0	0	0	130	139
w_4	-4	5	0	-1	0	1	0	25	26
w_5	1	5	0	0	-1	0	1	75	81
P	-11	-15	0	0	0	M	M	0	$2M-26$
w_1	3	5	1	0	0	0	0	130	139
w_4	$-\frac{4}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	5	$\frac{26}{5}$
w_5	1	5	0	0	-1	0	1	75	81
P	-11	-15	0	0	0	M	M	0	$2M-26$
w_1	7	0	1	1	0	-1	0	105	113
$y\ \cancel{w_4}$	$-\frac{4}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	5	$\frac{26}{5}$
w_5	5	0	0	1	-1	-1	1	50	55
P	-23	0	0	-3	0	$M+3$	M	75	$2M+52$
w_1	7	0	1	1	0	-1	0	105	113
y	$-\frac{4}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	5	$\frac{26}{5}$
w_5	1	0	0	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	10	11
P	-23	0	0	-3	0	$M+3$	M	75	$2M+52$
w_1	0	0	1	$-\frac{2}{5}$	$\frac{7}{5}$	$\frac{2}{5}$	$-\frac{7}{5}$	35	36
y	0	1	0	$-\frac{1}{25}$	$-\frac{4}{25}$	$\frac{1}{25}$	$\frac{4}{25}$	13	14
$x\ \cancel{w_5}$	1	0	0	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	10	11
P	0	0	0	$\frac{8}{5}$	$-\frac{23}{5}$	$M-\frac{8}{5}$	$M+\frac{23}{5}$	305	$2M+305$
w_1	0	0	$\frac{5}{7}$	$-\frac{2}{7}$	1	$\frac{2}{7}$	-1	25	$\frac{180}{7}$
y	0	1	0	$-\frac{1}{25}$	$-\frac{4}{25}$	$\frac{1}{25}$	$\frac{4}{25}$	13	14
x	1	0	0	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	10	11
P	0	0	0	$\frac{8}{5}$	$-\frac{23}{5}$	$M-\frac{8}{5}$	$M+\frac{23}{5}$	305	$2M+305$
$w_3\ \cancel{w_1}$	0	0	$\frac{5}{7}$	$-\frac{2}{7}$	1	$\frac{2}{7}$	-1	25	$\frac{180}{7}$
y	0	1	$\frac{4}{35}$	$-\frac{3}{35}$	0	$\frac{3}{35}$	0	17	$\frac{634}{35}$
x	1	0	$\frac{1}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	0	15	$\frac{113}{7}$
P	0	0	$\frac{23}{7}$	$\frac{2}{7}$	0	$M-\frac{2}{7}$	M	420	$2M+\frac{2963}{7}$

So there it is. $P_{\max} = 420$ with $x = 15,\ y = 17$.

Incidentally, also, $w_3 = 25$ and $w_1 = w_2 = w_4 = w_5 = 0$.

59

Our examples on the use of artificial variables have so far concerned only two problem variables, x and y. The method, however, is exactly the same when more problem variables are involved, though, naturally, the solution then becomes somewhat longer.

Here is one for you to work through: it brings in most of what we have covered and provides excellent revision. The result is given in the next frame.

Example 6

$$\begin{aligned} \text{Maximise} \quad & P = 24x + 21y + 30z \\ \text{subject to} \quad & 12x + 4y + 8z \leq 240 \\ & 8x + 3y + 3z \leq 140 \\ & 6x + 2y + 3z \geq 110 \quad (x, y, z \geq 0) \end{aligned}$$

60

$$\boxed{P_{\max} = 750 \text{ with } x = 10, y = 10, z = 10}$$

The simplex technique is designed to maximise a given objective function in the light of stated constraints. However, a problem requiring the minimisation of an objective function (denoting costs, machine idling time, etc.) can easily be converted for solution by the same method.

For this, move on to the next frame

Minimisation

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If P denotes the objective function to be minimised, we write Q as the negative of this function. $Q_{\max}$ is then determined by the usual simplex method and, finally, the negative value of $Q_{\max}$ is the value of the required $P_{\min}$.

i.e. Write $Q = -P$. Determine $Q_{\max}$ in the normal way.

Then $\quad P_{\min} = -(Q_{\max})$.

Example 1

$$\begin{aligned} \text{Minimise} \quad & P = -3x + 4y \\ \text{subject to} \quad & x + 3y \leq 54 \\ & 3x + \ y \leq 34 \\ & -x + 2y \geq 12 \quad (x, y \geq 0) \end{aligned}$$

First write $Q = -P$, i.e. $Q = 3x - 4y$, and maximise Q.

Inserting the usual slack variables and artificial variable as needed, we have . . . .

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$$\begin{array}{lr} x + 3y + w_1 & = 54 \\ 3x + \ \ y \qquad + w_2 & = 34 \\ -x + 2y \qquad\qquad - w_3 + \ \ w_4 & = 12 \\ Q - 3x + 4y \qquad\qquad\qquad + Mw_4 & = \ \ 0 \end{array}$$

Now we just carry out the usual simplex routine to evaluate Q_{max} and hence P_{min}, since $P_{min} = -(Q_{max})$. $P_{min} = \ldots\ldots\ldots\ldots$

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$$\boxed{P_{min} = 16 \quad \text{with} \quad x = 8,\ y = 10}$$

Because $Q_{max} = -16$ and hence $P_{min} = -(Q_{max}) = 16$.

The full working is available in the next frame, should you need to refer to it.

If not, move straight on to Frame 65

64

Basis	x	y	w_1	w_2	w_3	w_4	b	*check*
w_1	$\boxed{1}$	3	1	0	0	0	54	59
w_2	$\boxed{3}$	1	0	1	0	0	34	39
w_4	$\boxed{-1}$	2	0	0	-1	1	12	13
Q	-3	4	0	0	0	M	0	$M+1$
w_1	1	3	1	0	0	0	54	59
w_2	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{34}{3}$	13
w_4	-1	2	0	0	-1	1	12	13
Q	-3	4	0	0	0	M	0	$M+1$
w_1	0	$\frac{8}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{128}{3}$	46
x ~~w_2~~	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{34}{2}$	13
* → w_4	0	$\boxed{\frac{7}{3}}$	0	$\frac{1}{3}$	-1	1	$\frac{70}{3}$	26
Q	0	5	0	1	0	M	34	$M+40$
w_1	0	$\frac{8}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{128}{3}$	46
x	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	$\frac{34}{3}$	13
w_4	0	1	0	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	10	$\frac{78}{7}$
Q	0	5	0	1	0	M	34	$M+40$
w_1	0	0	1	$-\frac{5}{7}$	$\frac{8}{7}$	$-\frac{8}{7}$	16	$\frac{114}{7}$
x	1	0	0	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	8	$\frac{65}{7}$
y ~~w_4~~	0	1	0	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	10	$\frac{78}{7}$
Q	0	0	0	$\frac{2}{7}$	$\frac{15}{7}$	$M-\frac{15}{7}$	-16	$M-\frac{110}{7}$

$Q_{max} = -16 \quad \therefore\ P_{min} = 16$ with $x = 8,\ y = 10$.

65

Example 2

$$\begin{aligned} \text{Minimise} \quad & P = -2x + 8y \\ \text{subject to} \quad & 3x + 4y \leq 80 \\ & -3x + 4y \geq 8 \\ & x + 4y \geq 40 \quad (x, y \geq 0) \end{aligned}$$

Note that we have two constraints that are 'greater than' inequalities, so, beside the slack variables, we shall need two artificial variables.

The three constraints in their new form therefore become

....

66

$$\boxed{\begin{aligned} 3x + 4y + w_1 \qquad\qquad\qquad\qquad &= 80 \\ -3x + 4y \qquad - w_2 \qquad + w_4 \qquad &= 8 \\ x + 4y \qquad\qquad - w_3 \qquad + w_5 &= 40 \end{aligned}}$$

and, in the subsequent manipulation, we must see that w_4 and w_5 disappear from the basic solution before the optimal solution is obtained.

The objective function P is now replaced by Q $(= -P)$ and the new form of Q is written as

....

67

$$\boxed{Q - 2x + 8y + Mw_4 + Mw_5 = 0}$$

because

$$P = -2x + 8y \quad \therefore \; Q = -P = 2x - 8y$$

and with the artificial variables $Q = 2x - 8y \; - Mw_4 - Mw_5$.

$$\therefore \; Q - 2x + 8y + Mw_4 + Mw_5 = 0$$

In this example, w_1, w_4, w_5 will form the unity matrix, so work through the solution in the usual way. Simplify the initial tableau and then refer to the next frame.

68

Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	check
w_1	3	4	1	0	0	0	0	80	$\boxed{88}$
w_4	-3	4	0	-1	0	1	0	8	9
w_5	1	4	0	0	-1	0	1	40	45
Q	-2	8	0	0	0	M	M	0	$2M+6$
w_1	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
w_4	-3	4	0	-1	0	1	0	8	9
w_5	1	4	0	0	-1	0	1	40	45
Q	-2	8	0	0	0	M	M	0	$2M+6$
x $\cancel{w_1}$	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
* → w_4	0	8	1	-1	0	1	0	88	97
* → w_5	0	$\frac{8}{3}$	$-\frac{1}{3}$	0	-1	0	1	$\frac{40}{3}$	$\frac{47}{3}$
Q	0	$\frac{32}{3}$	$\frac{2}{3}$	0	0	M	M	$\frac{160}{3}$	$2M+\frac{194}{3}$

At this stage, there is no further negative entry in the index row, but we still must get rid of w_4 and w_5 from the basic variable column. Let us start by dealing with w_5.

We will take the entry $\frac{8}{3}$ at the intersection of the w_5 row and the y column as the next pivot and launch forth on the next stage. Complete the second stage and then again refer to the next frame.

69

Here is the working of stage 2.

Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	check
x	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
w_4	0	8	1	-1	0	1	0	88	97
w_5	0	$\boxed{\frac{8}{3}}$	$-\frac{1}{3}$	0	-1	0	1	$\frac{40}{3}$	$\frac{47}{3}$
Q	0	$\frac{32}{3}$	$\frac{2}{3}$	0	0	M	M	$\frac{160}{3}$	$2M+\frac{194}{3}$
x	1	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	0	$\frac{80}{3}$	$\frac{88}{3}$
w_4	0	8	1	-1	0	1	0	88	97
w_5	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	$\frac{32}{3}$	$\frac{2}{3}$	0	0	M	M	$\frac{160}{3}$	$2M+\frac{194}{3}$
x	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	20	$\frac{43}{2}$
* → w_4	0	0	2	-1	3	1	-3	48	50
y $\cancel{w_5}$	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	0	2	0	4	M	$M-4$	0	$2M+2$

▶

At this point, w_5 is replaced by y in the basic variable column.

Now we deal with w_4 by taking the entry 2 at the junction of the w_4 row and the w_1 column as the next pivot. That should do the trick, so finish off the solution and check with the next frame.

70

$$\boxed{P_{\text{min}} = 48 \quad \text{with} \quad x = 8, y = 8}$$

Basis	x	y	w_1	w_2	w_3	w_4	w_5	b	*check*
x	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	20	$\frac{43}{2}$
$* \rightarrow w_4$	0	0	$\boxed{2}$	-1	3	1	-3	48	50
y	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	0	2	0	4	M	$M-4$	0	$2M+2$
x	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	20	$\frac{43}{2}$
w_4	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	24	25
y	0	1	$-\frac{1}{8}$	0	$-\frac{3}{8}$	0	$\frac{3}{8}$	5	$\frac{47}{8}$
Q	0	0	2	0	4	M	$M-4$	0	$2M+2$
x	1	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	8	9
w_1 ~~w_4~~	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	24	25
y	0	1	0	$-\frac{1}{16}$	$-\frac{3}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	8	9
Q	0	0	0	1	1	$M-1$	$M-1$	-48	$2M-48$

w_4 in the basic variable column is now replaced by w_1, so the conditions are satisfied at last. From the final tableau, we have

$$Q_{\text{max}} = -48 \qquad \text{But } P_{\text{min}} = -(Q_{\text{max}}) = 48$$
$$\therefore\ P_{\text{min}} = 48 \quad \text{with} \quad x = 8,\ y = 8.$$

By this means, then, we can solve minimisation problems by the simplex method and so widen the scope of this valuable technique.

Applications

71

So far we have seen how to solve a typical linear programming problem by the simplex method, when the data are presented as a linear objective function and a number of linear constraints in the form of equations or inequalities. A practical problem, however, must first be interpreted into algebraic form and we conclude the Programme with a brief reference to this initial requirement. Let us consider the following example.

▶

Example 1

A firm manufactures two types of couplings, A and B, each of which requires processing time on lathes, grinders and polishers. The machine times needed for each type of coupling are given in the table.

Coupling type	*Time required (hours)*		
	Lathe	*Grinder*	*Polisher*
A	2	8	5
B	5	5	2

The total machine time available is 250 hours on lathes, 310 hours on grinders and 160 hours on polishers. The net profit per coupling of type A is £9 and of type B £10.

Determine

(a) the number of each type to be produced to maximise profit

(b) the maximum profit.

If we let $x =$ the number of type A units to be produced

$y =$ the number of type B units to be produced

the objective function to be maximised can be expressed as ....

72

$$\boxed{P = 9x + 10y}$$

Now we have to sort out the constraints from the given data.

Total time available on lathes = 250 hours

$\therefore\ 2x + 5y \leq 250$ (lathes)

Similar statements for the grinders and polishers are ....

73

$$\boxed{\begin{array}{l} 8x + 5y \leq 310 \text{ (grinders)} \\ 5x + 2y \leq 160 \text{ (polishers)} \end{array}}$$

The problem now can be expressed as

$$\begin{array}{ll} \text{Maximise} & P = 9x + 10y \\ \text{subject to} & 2x + 5y \leq 250 \\ & 8x + 5y \leq 310 \\ & 5x + 2y \leq 160 \quad (x, y \geq 0) \end{array}$$

Then we go through the usual process. Inserting slack variables to convert the inequalities into equations, we have ....

74

$$\boxed{\begin{aligned} 2x + 5y + w_1 \quad\quad\quad &= 250 \\ 8x + 5y \quad + w_2 \quad &= 310 \\ 5x + 2y \quad\quad + w_3 &= 160 \\ P - 9x - 10y \quad\quad\quad &= 0 \end{aligned}}$$

and the solution then develops in the usual way. Work through it carefully – it is all good practice – and see if you agree with the result given in the next frame.

The result is ....

75

$$\boxed{P_{\max} = 550 \quad \text{with} \quad x = 10, y = 46}$$

The maximum profit of £550 occurs with a manufacturing schedule of

10 couplings of type A

and 46 couplings of type B.

Now for another, so move on.

76

Example 2

A firm produces three types of pumps, A, B, C, each of which requires the four processes of turning, drilling, assembling and testing.

Pump type	*Process time (hours) per pump*				*Profit per pump £*
	Turning	*Drilling*	*Assembling*	*Testing*	
A	2	1	3	4	84
B	1	1	4	3	72
C	2	1	2	2	52
Total available time (h/week)	98	60	145	160	

From the information given in the table, determine

(a) the weekly output of each type of pump to maximise profit

(b) the maximum profit.

So, if we let $x =$ the number of pumps, type A

$y =$ the number of pumps, type B

$z =$ the number of pumps, type C

we can interpret the problem into its algebraic form, which is

....

77

$$\begin{array}{ll} \text{Maximise} & P = 84x + 72y + 52z \\ \text{subject to} & 2x + y + 2z \le 98 \\ & x + y + z \le 60 \\ & 3x + 4y + 2z \le 145 \\ & 4x + 3y + 2z \le 160 \qquad (x, y, z \ge 0) \end{array}$$

Inserting the slack variables and expressing the problem as equations, we have

....

78

$$\begin{array}{rl} 2x + y + 2z + w_1 & = 98 \\ x + y + z + w_2 & = 60 \\ 3x + 4y + 2z + w_3 & = 145 \\ 4x + 3y + 2z + w_4 & = 160 \\ P - 84x - 72y - 52z & = 0 \end{array}$$

Now you can proceed to set up the simplex tableau and solve the problem on your own in the usual manner. It is very similar to the other examples you have worked earlier in the Programme.

The result you no doubt get is

....

79

$$\boxed{P_{\max} = 3652 \quad \text{with} \quad x = 23, y = 8, z = 22}$$

i.e. by producing 23 pumps, type A
8 pumps, type B
22 pumps, type C

the maximum profit of £ 3652 is attained.

Care with the calculations and constant use of the check column provide the key to avoiding errors in the working.

That completes the Programme. Check down the **Revision summary** that comes next, in conjunction with the **Can You?** checklist, before working through the **Test exercise** that follows thereafter. As usual, a set of **Further problems** provides further necessary practice in these useful techniques.

Revision summary 23

80

1 *Optimization* – determination of an optimal value (maximum or minimum) of an objective function subject to a set of constraints.

2 *Linear programming (linear optimization)* – optimization where the objective function is a linear function and the constraints are linear equations or linear inequalities.

3 *Inequalities* – multiplying or dividing both sides by a negative factor $(-k)$ reverses the inequality, i.e. $\geq$ becomes $\leq$ and $\leq$ becomes $\geq$.

4 *Problem variables* $(x, y, z$, etc.) are always non-negative.

5 *Feasible solution* – a set of variables that satisfies all the given constraints.

6 *Optimal solution* – a feasible solution for which the objective function becomes a maximum (or minimum) within the constraints.

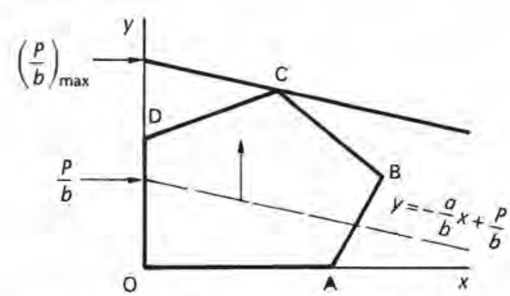
7 *Basic feasible solution* – a feasible solution for which at least $(n - m)$ of the total variables are zero, where
$n =$ total number of variables in the constraints
$m =$ number of constraints.

8 *Basis* – collection of the m variables which are not put equal to zero.

9 *Basic solution* – solution obtained by equating $(n - m)$ variables to zero and solving for the remaining m variables.

10 *Graphical solution*

(a) Constraints – graphs of constraints form the feasible polygon or feasible domain.



Feasible point or feasible solution – coordinates of all points within the feasible polygon or on its boundary (OABCD).

(b) Objective function $P = ax + by$ $\therefore$ $y = -\frac{a}{b}x + \frac{P}{b}$ represented by a set of parallel lines, slope $-\frac{a}{b}$, intercept $\frac{P}{b}$. Line through the extreme point C gives P_{max}, the optimal value of P.

▶

11 *Slack variable* – non-negative variable added to, or subtracted from, a linear inequality to form a linear equation.

12 *Simplex method of solution* – computation.

Refer back to Frame 34.

Where necessary, we multiply an inequality by (–1), with consequent reversal of inequality sign, to ensure that the right-hand side constant term $b_i \geq 0$.

13 *Artificial variable* – to convert a 'greater than' inequality to an equation, the slack variable required must be subtracted. To complete the unity matrix in the tableau, a further artificial variable w_i is included to allow the simplex procedure to continue. Such artificial variables must be eliminated before the optimal solution is finally attained.

The objective function $P = ax + by$ becomes $P = ax + by - Mw_i$.

14 *Minimisation* – If P is the objective function to be minimised

(a) write $Q = -P$

(b) maximise Q by the usual simplex method

(c) then $Q_{\max} = (-P)_{\max} = -(P_{\min})$

i.e. $P_{\min} = -(Q_{\max})$.

Can You?

81 Checklist 23

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Describe an optimization problem in terms of the objective function and a set of constraints? 1 and 2

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Algebraically manipulate and graphically describe inequalities? 3 to 6

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve a linear programming problem in two real variables? 6 to 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the simplex method to describe a linear programming problem in two real variables as a problem in two real variables with two slack variables? 14

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Set up the simplex tableau and compute the simplex? 15 to 32

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Use the simplex method to solve a linear programming problem in three real variables with three slack variables? 33 to 40

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Introduce artificial variables into the solution method as and when the need arises? 41 to 60

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Solve minimisation problems using the simplex method? 61 to 70

  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Construct the algebraic form of the objective function and the constraints for a problem stated in words? 71 to 79

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 23

82

1 Using a *graphical method*, maximise $P = x + 2y$ subject to the constraints

$$\begin{aligned} -3x + 4y &\le 8 \\ x + 4y &\le 16 \\ 3x + 2y &\le 18 \\ x, y &\ge 0. \end{aligned}$$

Note: Use the *simplex method* to solve Exercises **2** to **6**. In each case, all variables are non-negative.

2 Maximise $P = -3x + 4y$ subject to

$$\begin{aligned} 3x - 2y &\le 15 \\ x + y &\le 10 \\ -x + 4y &\le 15 \\ -2x + y &\le 2. \end{aligned}$$

3 Maximise $P = 8x + 12y + 10z$ subject to

$$\begin{aligned} 4x + 3y + 2z &\le 64 \\ 2x + y + 4z &\le 48 \\ x + 2y + z &\le 24. \end{aligned}$$

4 Maximise $P = 44x + 20y$ subject to

$$\begin{aligned} 12x + 6y &\le 84 \\ 3x + 2y &\ge 24. \end{aligned}$$

5 Minimise $P = 3y - 4x$ subject to

$$\begin{aligned} x + 4y &\le 60 \\ 2x + y &\le 22 \\ -x + y &\ge 7. \end{aligned}$$

▶

6 A firm makes two types of containers, A and B, each of which requires cutting, assembly and finishing. The maximum available machine capacity in hours per week for each process is: cutting 50, assembly 84, finishing 72.

The process times for one unit of each type are as follows:

Process	*Time in hours* A	B
Cutting	2	5
Assembly	4	8
Finishing	4	5

If the profit margin is £600 per unit A and £1000 per unit B, determine

(a) the optimum weekly output of containers

(b) the maximum profit.

Further problems 23

83 *All variables in the following problems are non-negative.*

Graphical Solution

1 Maximise $P = -x + 8y$
subject to
$$\begin{aligned} -3x + 4y &\le 10 \\ -x + 4y &\le 14 \\ 3x + 2y &\le 21 \\ 3x + y &\le 18. \end{aligned}$$

2 Maximise $P = -4x + 8y$
subject to
$$\begin{aligned} x + 3y &\le 57 \\ 7x + 4y &\le 110 \\ -x + 5y &\le 40. \end{aligned}$$

3 Maximise $P = 5x + 4y$
subject to
$$\begin{aligned} x - 2y &\le 2 \\ 3x - 4y &\le 8 \\ 5x + 6y &\le 45 \\ x + 3y &\le 18. \end{aligned}$$

Simplex Solution

4 Maximise $P = 2x + y$
subject to
$$\begin{aligned} x + 4y &\le 24 \\ x + y &\le 9 \\ x - y &\le 3 \\ x - 2y &\le 2. \end{aligned}$$

5 Maximise $P = -3x + 4y$
subject to
$$\begin{aligned} 3x - 4y &\le 12 \\ 5x + 4y &\le 36 \\ -x + 3y &\le 8 \\ -3x + y &\le 0. \end{aligned}$$

6 Maximise $P = x + 2y$
subject to
$$\begin{aligned} -2x + y &\le 1 \\ -x + y &\le 2 \\ x + y &\le 6 \\ 2x - 3y &\le 2. \end{aligned}$$

7 Maximise $P = 4y - 3x$
subject to
$$\begin{aligned} x - 2y &\le 0 \\ x - y &\le 2 \\ x + 2y &\le 14 \\ -x + 2y &\le 6 \\ -3x + 2y &\le 2. \end{aligned}$$

8 Maximise $P = 3x + 4y + 5z$
subject to
$$\begin{aligned} 5x + 4y + 8z &\le 40 \\ 3x + 2y + 12z &\le 30 \\ y &\le 8. \end{aligned}$$

9 Maximise $P = 3x + 4y + 3z$
subject to
$$\begin{aligned} 2x + 3y + 4z &\le 58 \\ 4x + 2y + 3z &\le 51 \\ 3x + 4y + 2z &\le 62. \end{aligned}$$

10 Maximise $P = 4x + 3y + 3z$
subject to
$$\begin{aligned} 4x + y + 2z &\le 40 \\ x + 4y + z &\le 50 \\ 2x + 3y + 4z &\le 60. \end{aligned}$$

11 Maximise $P = 5{\cdot}3x + 3{\cdot}6y + 2{\cdot}0z$
subject to
$$\begin{aligned} 2{\cdot}1x + 4{\cdot}3y + 1{\cdot}5z &\le 70 \\ 3{\cdot}2x + 1{\cdot}4y + 2{\cdot}2z &\le 60 \\ 1{\cdot}6x + 6{\cdot}2y + 3{\cdot}1z &\le 100. \end{aligned}$$

Artificial Variables

12 Maximise $P = 8x + 5y$
subject to
$$\begin{aligned} 2x + y &\le 80 \\ x + 3y &\le 90 \\ x + y &\ge 30. \end{aligned}$$

13 Maximise $P = 12x + 8y$
subject to
$$\begin{aligned} x + 2y &\le 20 \\ 4x - y &\le 8 \\ -x + y &\ge 1. \end{aligned}$$

14 Maximise $P = 3x + 4y$
subject to
$$\begin{aligned} x + 4y &\le 76 \\ -5x + 8y &\ge 40 \\ -x + 4y &\ge 32. \end{aligned}$$

15 Minimise $P = 4x + 5y$
subject to
$$\begin{aligned} x + 2y &\le 63 \\ 3x + y &\le 70 \\ 2x + y &\ge 42 \\ x + 4y &\ge 84. \end{aligned}$$

16 Maximise $P = 65x - 23y$
subject to
$$\begin{aligned} 5x - y &\le 30 \\ 10x + 4y &\ge 150. \end{aligned}$$

17 Maximise $P = 24x - 8y$
subject to
$$\begin{aligned} x + 3y &\le 360 \\ 2x + y &\le 850 \\ -5x + 25y &\ge 320. \end{aligned}$$

18 Maximise $P = 4x + 2y$
subject to
$$\begin{aligned} x + 2y &\le 60 \\ 3x + 2y &\le 80 \\ -3x + 10y &\ge 40. \end{aligned}$$

19 Maximise $P = 18x + 40y + 24z$
subject to
$$\begin{aligned} 5x + 2y + 4z &\le 63 \\ 2x + 4y + 2z &\le 42 \\ 2x + 3y + z &\ge 35. \end{aligned}$$

20 Maximise $P = 60x + 45y + 25z$
subject to
$$\begin{aligned} 4x + 8y + 2z &\le 160 \\ 6x + 3y + 4z &\le 168 \\ 4x + 3y + 3z &\ge 128. \end{aligned}$$

21 Maximise $P = 12x + 8y - 10z$
subject to
$$\begin{aligned} 4x + 2y - 3z &\le 210 \\ 6x + 8y + z &\le 630 \\ 2x - y + 4z &\ge 210 \\ x + y + z &\le 180. \end{aligned}$$

Minimisation

22 Minimise $P = -4x + 3y$
subject to
$$\begin{aligned} x + 4y &\le 20 \\ 2x + y &\le 12 \\ x - y &\le 3. \end{aligned}$$

23 Minimise $P = -5x + 8y$
subject to
$$\begin{aligned} x + 2y &\le 40 \\ 3x + 2y &\le 48 \\ -x + 4y &\ge 40. \end{aligned}$$

24 Minimise $P = -4x + 8y$
subject to
$$\begin{aligned} -5x + 4y &\le 32 \\ 7x + 4y &\le 80 \\ -x + 8y &\ge 40. \end{aligned}$$

25 Minimise $P = 2x + 8y$
subject to
$$\begin{aligned} -x + 2y &\le 24 \\ 7x + 6y &\le 132 \\ -x + 2y &\ge 4 \\ x + 2y &\ge 12. \end{aligned}$$

26 Minimise $P = 4x - 8y + 5z$
subject to
$$\begin{aligned} 2x + 3y + z &\le 70 \\ x + 2y + 2z &\le 60 \\ 3x + 4y + z &\le 84 \\ x + y + z &\ge 33. \end{aligned}$$

27 Minimise $P = 6x - 5y - 3z$
subject to
$$\begin{aligned} 5x + 8y + 4z &\le 220 \\ 2x + y + 6z &\le 154 \\ 4x + 2y + z &\ge 77 \\ x + y + 2z &\ge 55. \end{aligned}$$

▶

Applications

28 A firm manufacturing two types of switching module, A and B, is under contract to produce a daily output of at least 35 modules in all. Assembly and testing times for each type of module are as follows:

Module type	*Processing time (hours)*	
	Assembly	*Testing*
A	1·0	2·0
B	2·0	1·0

Available staff resources provide a daily maximum of 80 hours for assembly and 55 hours for testing.
The profit on the sale of each A-module is £40 and of each B-module £50. Determine

(a) the daily production schedule for maximum profit.

(b) the maximum daily profit.

29 Three different types of coupling units are produced by a firm. The times required for machining, polishing and assembling a unit of each type are included in the information given in the following table.

Type of unit	*Process time (hours) per unit*			*Profit (£) per unit*
	Machining	*Polishing*	*Assembling*	
A	4	1	2	110
B	2	3	1	100
C	3	2	4	120
Available time (h/week)	320	250	280	

The firm is required to supply a total of at least 100 units of mixed types each week. Determine

(a) the weekly output of each type to maximise profit

(b) the maximum weekly profit.

30 A firm makes three types of wooden cabinets, A, B, C, with profit margins of £35, £30, £24 per unit respectively.

	Time in hours per cabinet		
Process	A	B	C
Preparation	2	5	4
Assembly	2	3	2
Finishing	5	4	3

The manufacturer has 25 men available for preparation, 20 men for assembly and 30 men for polishing, and all staff work a 40 hour week. To remain competitive, at least 300 cabinets in all must be produced each week. Determine

(a) the number of each model to be manufactured each week in order to maximise the profit

(b) the maximum weekly profit.

Appendix

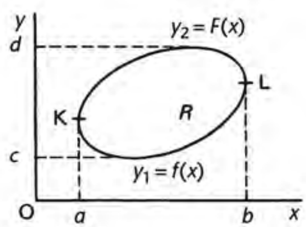
1 Green's theorem

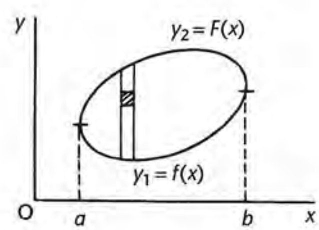
If P and Q are two functions in x and y, finite and continuous inside a region R and on its boundary c in the x–y plane, with continuous first partial derivatives, then Green's theorem states that

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \mathrm{d}x\,\mathrm{d}y = -\oint_c \{P\mathrm{d}x + Q\mathrm{d}y\}$$

Proof of Green's theorem

Let the lower boundary of the region be the curve $y_1 = f(x)$ and the upper boundary the curve $y_2 = F(x)$.

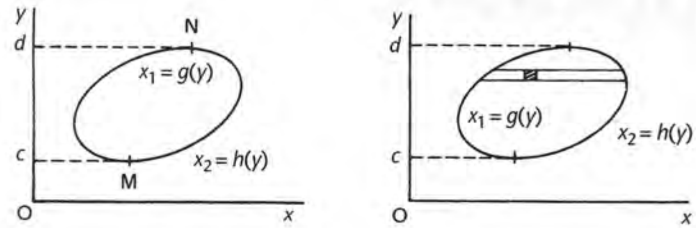




Using vertical strips, we then have

$$\begin{aligned}\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b\int_{y_1}^{y_2} \frac{\partial P}{\partial y}\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \Big[P\Big]_{y_1=f(x)}^{y_2=F(x)}\,\mathrm{d}x \\ &= \int_a^b \{P(x,y_2) - P(x,y_1)\}\,\mathrm{d}x \\ &= -\int_a^b P(x,y_1)\,\mathrm{d}x - \int_b^a P(x,y_2)\,\mathrm{d}x \\ &= -\left\{\int_a^b P(x,y_1)\,\mathrm{d}x + \int_b^a P(x,y_2)\,\mathrm{d}x\right\} \\ &= -\oint P(x,y)\,\mathrm{d}x \qquad (1)\end{aligned}$$

Similarly, using horizontal strips, we have



$$\iint_R \frac{\partial Q}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_{x_1}^{x_2} \frac{\partial Q}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_c^d \Big[Q \Big]_{x_1=g(y)}^{x_2=h(y)} \mathrm{d}y$$

where $x_1 = g(y)$ and $x_2 = h(y)$ are the left-hand and right-hand portions of the boundary curve c.

$$\therefore \quad \iint_R \frac{\partial Q}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_c^d Q(x_2, y)\,\mathrm{d}y - \int_c^d Q(x_1, y)\,\mathrm{d}y$$

$$= \int_c^d Q(x_2, y)\,\mathrm{d}y + \int_d^c Q(x_1, y)\,\mathrm{d}y$$

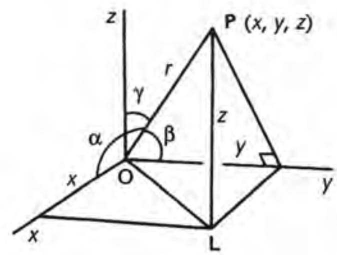
$$= \oint_{\mathrm{c}} Q(x, y)\,\mathrm{d}y \qquad (2)$$

$$\therefore \quad \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x\,\mathrm{d}y = -\oint_{\mathrm{c}} P(x, y)\,\mathrm{d}x - \oint_{\mathrm{c}} Q(x, y)\,\mathrm{d}y$$

$$= -\oint_{\mathrm{c}} \{P\,\mathrm{d}x - Q\,\mathrm{d}y\}$$

2 Proof that $\sec\gamma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$

Let α, β, γ be the angles that OP makes with the x, y and z axes respectively.



Then $x = r\cos\alpha$; $y = r\cos\beta$; $z = r\cos\gamma$

Also $x^2 + y^2 + z^2 = r^2$

If $r = 1$ unit, then $x^2 + y^2 + z^2 = 1 \quad \therefore\ z^2 = 1 - x^2 - y^2$

$$\therefore\ z = (1 - x^2 - y^2)^{1/2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y)$$

$$= \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

$$\therefore\ 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2}$$

$$= \frac{1 - x^2 - y^2 + x^2 + y^2}{1 - x^2 - y^2}$$

$$= \frac{1}{1 - x^2 - y^2} = \frac{1}{z^2}$$

But, with $r = 1$, $z = \cos\gamma \qquad \therefore\ \frac{1}{z^2} = \sec^2\gamma$

$$\therefore\ \sec\gamma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

3 Vector triple products

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$

(b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\,\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\,\mathbf{A}$

Let $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k};\quad \mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k};$

$\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$

Then

$$\mathbf{B} \times \mathbf{C} = (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \times (c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}$$

Then

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} & \begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} & \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \end{vmatrix}$$

$$= \mathbf{i}\left\{ a_y\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} - a_z\begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} \right\} - \mathbf{j}\left\{ a_x\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} - a_z\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} \right\}$$

$$+ \mathbf{k}\left\{ a_x\begin{vmatrix} b_z & b_x \\ c_z & c_x \end{vmatrix} - a_y\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} \right\}$$

$$= \mathbf{i}\,\{a_y(b_xc_y - b_yc_x) - a_z(b_zc_x - b_xc_z)\}$$
$$+ \mathbf{j}\,\{a_z(b_yc_z - c_yb_z) - a_x(b_xc_y - b_yc_x)\}$$
$$+ \mathbf{k}\,\{a_x(b_zc_x - b_xc_z) - a_y(b_yc_z - b_zc_y)\}$$

$$= \mathbf{i}\,\{b_xa_xc_x + b_xa_yc_y + b_xa_zc_z - c_xa_xb_x - c_xa_yb_y - c_xa_zb_z\}$$
$$+ \mathbf{j}\,\{b_ya_xc_x + b_ya_yc_y + b_ya_zc_z - c_ya_xb_x - c_ya_yb_y - c_ya_zb_z\}$$
$$+ \mathbf{k}\,\{b_za_xc_x + b_za_yc_y + b_za_zc_z - c_za_xb_x - c_za_yb_y - c_za_zb_z\}$$

$$= \mathbf{i}\,\{b_x(a_xc_x + a_yc_y + a_zc_z) - c_x(a_xb_x + a_yb_y + a_zb_z)\}$$
$$+ \mathbf{j}\,\{b_y(a_xc_x + a_yc_y + a_zc_z) - c_y(a_xb_x + a_yb_y + a_zb_z)\}$$
$$+ \mathbf{k}\,\{b_z(a_xc_x + a_yc_y + a_zc_z) - c_z(a_xb_x + a_yb_y + a_zb_z)\}$$

Now $\quad \mathbf{A}\cdot\mathbf{C} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k})$

$\qquad = a_xc_x + a_yc_y + a_zc_z$

and similarly $\quad \mathbf{A}\cdot\mathbf{B} = a_xb_x + a_yb_y + a_zb_z$

$$\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{i}\{b_x(\mathbf{A}\cdot\mathbf{C}) - c_x(\mathbf{A}\cdot\mathbf{B})\} + \mathbf{j}\{b_y(\mathbf{A}\cdot\mathbf{C}) - c_y(\mathbf{A}\cdot\mathbf{B})\} + \mathbf{k}\{b_z(\mathbf{A}\cdot\mathbf{C}) - c_z(\mathbf{A}\cdot\mathbf{B})\}.$$

$$\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\{\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z\} - (\mathbf{A}\cdot\mathbf{B})\{\mathbf{i}c_x + \mathbf{j}c_y + \mathbf{k}c_z\}$$

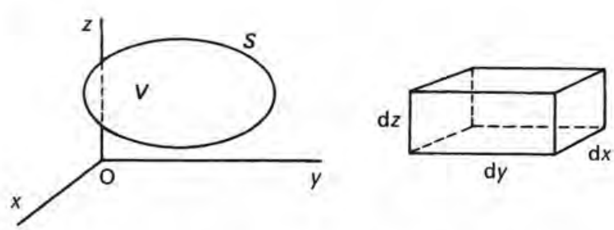
$$\therefore\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

In the same way, it can be established that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}$$

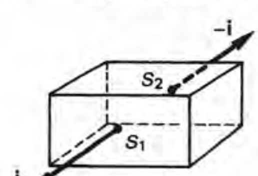
4 Divergence theorem (Gauss' theorem)

To prove that $\int_V \operatorname{div}\mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ for the region V bounded by the surface S.



Consider an element of volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and let the components of $\mathbf{F}$ in the x, y and z directions be denoted by $F_x\mathbf{i}$, $F_y\mathbf{j}$ and $F_z\mathbf{k}$ respectively at any point P. We then determine $\int \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ over the element $\mathrm{d}V$ and finally sum the results for all such elements throughout the region.

(a) S_1: $\quad \mathrm{d}S_1 = \mathrm{d}y\,\mathrm{d}z$; $\quad \mathbf{n} = \mathbf{i}$



$$(\mathbf{F}\cdot\mathrm{d}\mathbf{S})_1 = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{n})\,\mathrm{d}S_1 = (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(\mathbf{i})\,\mathrm{d}S_1 = F_x\,\mathrm{d}S_1$$

(b) S_2 : $\quad dS_2 = dy\,dz; \qquad \mathbf{n} = -\mathbf{i}$

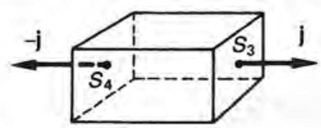
$$\begin{aligned}\therefore\ (\mathbf{F}\cdot d\mathbf{S})_2 &= (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})\cdot(-\mathbf{i})\,dS_2\\ &= -F_x\,dS_2\end{aligned}$$

Combining these two results, we have

$$\begin{aligned}(\mathbf{F}\cdot d\mathbf{S})_1 + (\mathbf{F}\cdot d\mathbf{S})_2 &= (F_x\,dS)_1 - (F_x\,dS)_2\\ &= \frac{\partial}{\partial x}(F_x\,dS)\,dx\end{aligned}$$

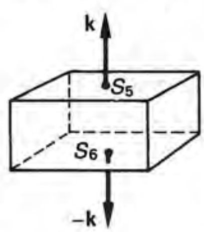
$$\therefore \int_{S_1+S_2} \mathbf{F}\cdot d\mathbf{S} = \frac{\partial F_x}{\partial x}\,dS\,dx = \left(\frac{\partial F_x}{\partial x}\right) dx\,dy\,dz \tag{1}$$

Similarly, for S_3 and S_4 we have



$$\int_{S_3+S_4} \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_y}{\partial y}\right) dx\,dy\,dz \tag{2}$$

and for S_5 and S_6



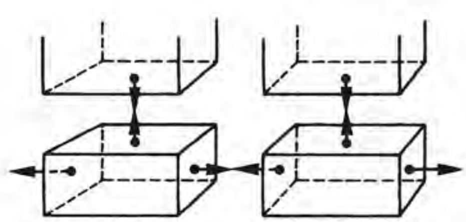
$$\int_{S_5+S_6} \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_z}{\partial z}\right) dx\,dy\,dz \tag{3}$$

These three results together cover the total surface of the element dV.

$$\int_{S_1\ldots S_6} \mathbf{F}\cdot d\mathbf{S} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz = \text{div}\,\mathbf{F}\,dV$$

Finally, summing the results for all such elements throughout the region with $dV \to 0$ and $d\mathbf{S} \to 0$, we obtain

$$\int_V \text{div}\,\mathbf{F}\,dV = \sum \int \mathbf{F}\cdot d\mathbf{S} \quad \text{with } d\mathbf{S} \to 0$$

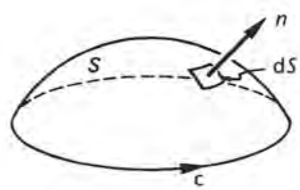


On the common boundaries between adjacent elements, the values of $\int \mathbf{F}\cdot d\mathbf{S}$ cancel out. On the boundary surface, however, there are no such adjacent faces and the integral $\oint_S \mathbf{F}\cdot d\mathbf{S}$ remains.

$$\therefore \int_V \operatorname{div}\mathbf{F}\,dV = \int_S \mathbf{F}\cdot d\mathbf{S}$$

5 Stokes' theorem

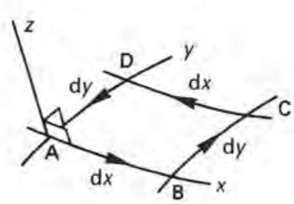
If $\mathbf{F}$ is a single-valued vector field, continuous and differentiable over an open surface S and on the boundary c of the surface, then



$$\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_c \mathbf{F}\cdot d\mathbf{r}$$

Proof of Stokes' theorem

Consider the surface S divided into small rectangular elements and let ABCD be one such element. If axes of reference x and y be arranged to coincide with AB and AD respectively as shown, a third axis z will then be normal to the surface at A.



If AB $= dx$, then $d\mathbf{x} = \mathbf{i}\,dx$ and

if AD $= dy$, then $d\mathbf{y} = \mathbf{j}\,dy$.

Let $\mathbf{F}_a$ denote the vector field at A; $\mathbf{F}_b$ that at B; $\mathbf{F}_c$ that at C; and $\mathbf{F}_d$ that at D. Now consider each side in turn.

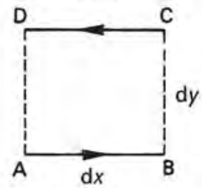
AB: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_a\cdot d\mathbf{x} = \{F_{ax}\mathbf{i} + F_{ay}\mathbf{j} + F_{az}\mathbf{k}\}\cdot\{\mathbf{i}\,dx\} = F_{ax}\,dx$

BC: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_b\cdot d\mathbf{y} = \{F_{bx}\mathbf{i} + F_{by}\mathbf{j} + F_{bz}\mathbf{k}\}\cdot\{\mathbf{j}\,dy\} = F_{by}\,dy$

CD: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_c\cdot d\mathbf{x} = \{F_{cx}\mathbf{i} + F_{cy}\mathbf{j} + F_{cz}\mathbf{k}\}\cdot\{-\mathbf{i}\,dx\} = -F_{cx}\,dx$

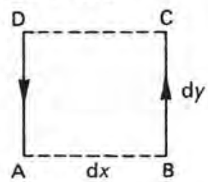
DA: $\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}_d\cdot d\mathbf{y} = \{F_{dx}\mathbf{i} + F_{dy}\mathbf{j} + F_{dz}\mathbf{k}\}\cdot\{-\mathbf{j}\,dy\} = -F_{dy}\,dy$

(a) AB + CD:



$$\int \mathbf{F}\cdot d\mathbf{r} = F_{ax}\,dx - F_{cx}\,dx$$
$$= -(F_{cx} - F_{ax})\,dx$$
$$= -\,\delta F_x\,dx$$
$$= -\frac{\partial F_x}{\partial y}\,dy\,dx$$
$$\therefore \quad \int_{(AB+CD)} \mathbf{F}\cdot d\mathbf{r} = -\frac{\partial F_x}{\partial y}\,dx\,dy \qquad (1)$$

(b) BC + DA:



$$\int \mathbf{F}\cdot d\mathbf{r} = F_{by}\,dy - F_{dy}\,dy$$
$$= (F_{by} - F_{dy})\,dy$$
$$= \delta F_y\,dy$$
$$= \frac{\partial F_y}{\partial x}\,dx\,dy$$
$$\therefore \quad \int_{(BC+DA)} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial F_y}{\partial x}\,dx\,dy \qquad (2)$$

Adding these two results together for the complete rectangle, we have

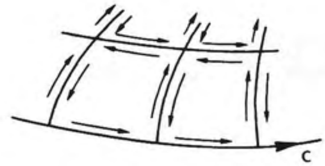
$$\int_{(ABCD)} \mathbf{F}\cdot d\mathbf{r} = \left\{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right\} dx\,dy \qquad (3)$$

$$\text{Now curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$
$$= \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$
$$\therefore \quad \left\{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right\} = (\text{curl } \mathbf{F})\cdot(\mathbf{k}) \qquad (4)$$

From (3) $\displaystyle\int_{ABCD} \mathbf{F}\cdot d\mathbf{r} = \text{curl } \mathbf{F}\cdot\mathbf{k}\,dx\,dy = \text{curl } \mathbf{F}\cdot d\mathbf{S}$

Summing for all such elements over the surface

$$\int_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = \mathop{Lim}_{d\mathbf{r}\to 0} \sum \left\{ \int_{ABCD} \mathbf{F}\cdot d\mathbf{r} \right\} \qquad (5)$$



$\int \mathbf{F} \cdot d\mathbf{r}$ on boundary lines between adjacent rectangular elements will cancel out, except on the boundary curve c of the surface S. The integral then becomes $\oint_c \mathbf{F} \cdot d\mathbf{r}$.

$$\therefore \int_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \oint_c \mathbf{F} \cdot d\mathbf{r}$$

Answers

Test exercise 1 (page 42)

1 $x = -1 - j\sqrt{3}$; $x^2 + 2x + 4 = 0$ **2** $x = -4, 6, 3/2$ **3** $x = -1{\cdot}\dot{6} = -5/3$
4 $x \approx 1{\cdot}710$ **5** $x \approx 0{\cdot}454304$ **6** $x \approx 1{\cdot}317672$ **7** (a) 39·375
(b) 103·392 (c) 481·528 **8** −12·8

Further problems 1 (page 43)

1 $\dfrac{-1 + j\sqrt{3}}{2}$, $\dfrac{-1-j}{\sqrt{2}}$, $x^4 + (1+\sqrt{2})x^3 + (2+\sqrt{2})x^2 + (1+\sqrt{2})x + 1 = 0$
2 $x = 1, 6, -2$ **3** $p = -5, q = -1$ **4** $p = 4, q = 9$ **5** $x = 2, 3, -3$
6 $x = 1, -3, 9$ **7** $y^3 - 5y^2 + 17y - 13 = 0$ **8** $y^3 - 13y^2 + 52y - 60 = 0$
9 $x = \frac{1}{2}, \frac{3}{2}, -1$ **10** $x = -2, 4, 8$ **11** $2y^3 - 15y^2 + 25y = 0$ **13** 0·8934
14 $x = 2{\cdot}732, -0{\cdot}732, -2{\cdot}000$ **15** $y^3 - 3y + 2 = 0$; $x = -4, -1, -1$
16 $x = 1{\cdot}646$ **17** (a) −0·6736 (b) 0·3717
18 (a) −2·3301, 0·2016, 2·1284 (b) $1, -0{\cdot}50 \pm j1{\cdot}66$
(c) −2·115, 0·254, 1·861 **19** (a) $-4{\cdot}104, -0{\cdot}9481 \pm j0{\cdot}5652$
(b) 0·5, −1·5, −1·5 (c) $0{\cdot}25, 1 \pm j3$ **20** (a) −2·456 (b) 1·765
(c) 0·739 (d) 1·812 (e) 1·8175 (f) 0·5170 (g) 0·8449 (h) 0·8806
21 (a) 32·872 (b) 204·328 (c) 381·429 **22** (a) −1·375 and 81·104
(b) 136·971 and −363·429 **23** (a) −6·048 (b) 461·496
24 (a) 133 and −9·048 (b) 0·136 and −65·433 (c) −199·112 and −867
25 0·02768 **26** −1·0670 **27** (a) −2·54846 (b) −2·41734 (c) −1·87134

Test exercise 2 (page 90)

1 (a) $\dfrac{32 - 2s}{s^2 - 16}$ (b) $\dfrac{s+4}{s^2+16}$ (c) $\dfrac{1}{s^4}\{4s^3 - s^2 + 4s + 6\}$ (d) $\dfrac{s+2}{s^2 + 4s + 29}$
(e) $\dfrac{6s}{(s^2+9)^2}$ (f) $\ln\left\{\dfrac{s+2}{s+1}\right\}$ **2** (a) $2e^{3t} - e^{4t}$
(b) $2\cos\sqrt{2}t + \dfrac{5}{\sqrt{2}}\sin\sqrt{2}t - e^t$ (c) $e^t(3t+2) - e^{3t}$
(d) $\frac{1}{8}\{e^t(17\cos 2t + 9\sin 2t) - e^{3t}\}$ **3** (a) $x = e^{-2t} + e^{-3t}$
(b) $x = \frac{1}{12}\{13e^{2t} - \cos 2t - \sin 2t\}$ (c) $x = \frac{1}{6} - \frac{5}{3}e^{3t} + \frac{5}{2}e^{4t}$
(d) $x = e^t\left(1 - t + \dfrac{t^3}{6}\right)$
4 $x = \frac{1}{2}\{9\cos t - 7\sin t - e^{-3t}\}$ $y = 3\sin t - 2\cos t + e^{-2t}$

Further problems 2 (page 91)

1 (a) $\dfrac{s-4}{s^2 - 8s + 20}$ (b) $\dfrac{4s}{(s^2+4)^2}$ (c) $\dfrac{6}{s^4} + \dfrac{8}{s^3} + \dfrac{5}{s}$ (d) $\dfrac{4s^2 - 24s + 38}{(s-3)^3}$
(e) $\dfrac{2s^3 - 6s}{(s^2+1)^3}$ (f) $\ln\sqrt{\dfrac{s+2}{s-2}}$ **2** (a) $e^{2t} + e^{4t}$ (b) $3e^{4t} + 2$
(c) $e^{2t}\left\{\dfrac{3t^2}{2} + 2t + 1\right\}$ (d) $e^{-t}\{2\cos t - 5\sin t\} - 2e^{2t}$

(e) $\frac{1}{3}(\cos t - \cos 2t)$ (f) $e^{-2t}\{\cos 4t - \frac{7}{4}\sin 4t\}$ **3** $x = 4e^{4t} - 2$

4 $x = \frac{35}{78}e^{4t/3} - \frac{3}{26}\{\cos 2t + \frac{2}{3}\sin 2t\}$ **5** $x = e^t(2t+1) + 2t + 4 + \cos t$

6 $x = \frac{3}{2}e^{4t} - e^{3t} - \frac{1}{2}e^{2t}$ **7** $x = \frac{4}{5}\cos 3t + \sin 3t + \frac{1}{5}\cos 2t$

8 $x = \frac{1}{5}\{e^{2t} - e^t(\cos 2t - 2\sin 2t)\}$ **9** $x = \frac{1}{8}\{2t^2 - 4t + 3 + e^{-2t}(4t^2 + 6t + 1)\}$

10 $x = \frac{2}{5}\{2(e^{-4t} - 1)\cos 4t + (e^{-4t} + 1)\sin 4t\}$ **11** $x = (2t+1)\cos 5t + t\sin 5t$

12 $x = \frac{1}{13}\{2e^{2t} + 3e^{-2t} - 5(\cos 3t - \sin 3t)\}$

$y = \frac{1}{13}\{5(\cos 3t + \sin 3t) - 3e^{2t} - 2e^{-2t}\}$

13 $x = \frac{1}{6}\{7e^{-6t} + 5\}$ $y = \frac{1}{3}\{7e^{-6t} + 5\}$ **14** $x = 10e^{-4t} + 2$ $y = 5e^{-4t} + 3$

15 $x = e^{-2t} - e^t + 2t$ $y = 3e^t + \frac{1}{2}e^{-2t} + t - \frac{7}{2}$ **16** $x = 5e^t + 3e^{-t}$ $y = 4e^t - e^{-t}$

17 $x = 4\cos t - 2\sin t - \frac{1}{3}\{8e^{-t} + e^{2t}\}$ $y = 6\cos t + 2\sin t - \frac{4}{3}\{2e^{-t} + e^{2t}\}$

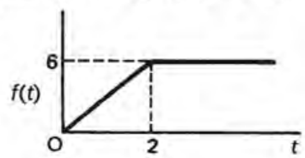
18 $x = \frac{5}{3}\{\cos 2t + \sin 2t - \cosh\sqrt{2}t - \sqrt{2}\sinh\sqrt{2}t\}$

19 $y = \frac{1}{5}\{3\sin 2t - 4\cos 2t + \frac{4}{3}\sin 3t + \frac{48}{7}\cos 3t\} - \frac{4}{7}\cos 4t$

20 $x = \cos\left(t\sqrt{\frac{3}{10}}\right) + \frac{3}{4}\cos(t\sqrt{6})$ $y = \frac{5}{4}\cos\left(t\sqrt{\frac{3}{10}}\right) - \frac{1}{4}\cos(t\sqrt{6})$

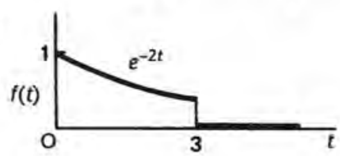
Test exercise 3 (page 109)

1 (a)



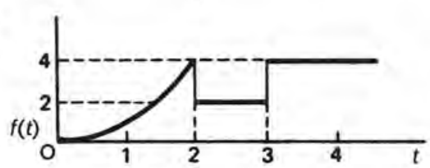
$$F(s) = \frac{3}{s^2}\{1 - e^{-2s}\}$$

(b)



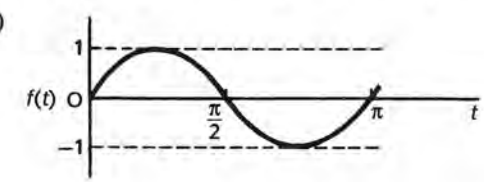
$$F(s) = \frac{1}{s+2}\{1 - e^{-6}e^{-3s}\}$$

(c)



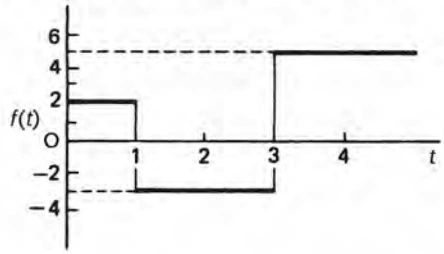
$$F(s) = \frac{2}{s^3} - 2e^{-2s}\left\{\frac{1}{s^3} + \frac{2}{s^2} - \frac{1}{s}\right\} + \frac{2}{s}e^{-3s}$$

(d)

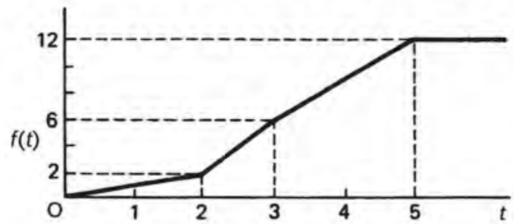


$$F(s) = \frac{2}{s^2+4}\{1 - e^{-\pi s}\}$$

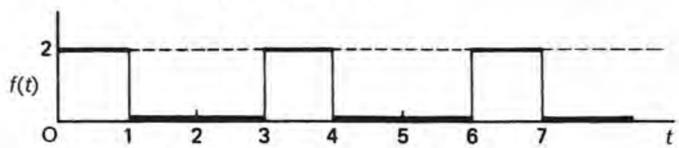
2 $f(t) = 2 \cdot u(t) - 5 \cdot u(t-1) + 8 \cdot u(t-3)$



3 $f(t) = t \cdot u(t) + 3(t-2) \cdot u(t-2) - (t-3) \cdot u(t-3) - 3(t-5) \cdot u(t-5)$



4 $f(t) = 2 \cdot u(t) - 2 \cdot u(t-1) + 2 \cdot u(t-3) - 2 \cdot u(t-4)$
$\quad + 2 \cdot u(t-6) - 2 \cdot u(t-7) + \dots$



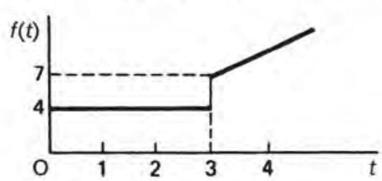
$$\left.\begin{aligned} f(t) &= 2 \quad & 0 < t < 1 \\ &= 0 \quad & 1 < t < 3 \end{aligned}\right\} \qquad f(t) = f(t+3)$$

Further problems 3 (page 110)

1 $f(t) = 3 \cdot u(t) + 2(t-2) \cdot u(t-2) - 2(t-5) \cdot u(t-5)$

2 $f(t) = t \cdot u(t) - (t-1) \cdot u(t-1) + (t-2) \cdot u(t-2) - (t-3) \cdot u(t-3)$

3



$$F(s) = \frac{4}{s} + \frac{3e^{-3s}}{s} + \frac{2e^{-3s}}{s^2}$$

4 (a) $f(t) = t^2 \cdot u(t) - (t^2 - 5t) \cdot u(t-3)$
(b) $f(t) = \cos t \cdot u(t) + (\cos 2t - \cos t) \cdot u(t-\pi) + (\cos 3t - \cos 2t) \cdot u(t-2\pi)$

5 $F(s) = e^{-2s}\left\{\frac{1}{s^2} + \frac{3}{s}\right\} - e^{-3s}\left\{\frac{1}{s^2} + \frac{4}{s}\right\}$

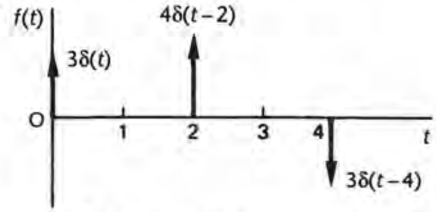
6 (a) $f(t) = t^2 \cdot u(t) - t^2 \cdot u(t-2) + 4 \cdot u(t-2) - 4 \cdot u(t-5)$
(b) $F(s) = \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2} - \frac{4e^{-5s}}{s}$

Test exercise 4 (page 142)

1 $F(s)=\dfrac{2(1-e^{-2s}-2se^{-2s})}{s^2(1-e^{-4s})}$ **2** (a) e^{-6} (b) 0 (c) 11

3 (a) $F(s)=4e^{-3s}$ (b) $F(s)=e^{-2(3+s)}$

4



$F(s)=3+4e^{-2s}-3e^{-4s}$

5 $x=e^{-3t}\{4\sin t-\cos t\}$

6 $x=3e^4e^{-t}\cdot u(t-4)+e^{-2t}\{2\cdot u(t)-3e^8\cdot u(t-4)\}$

7 (a) $f(t)=\sin t$, frequency 1 radian per unit of time, period 2π units of time

(b) $f(t)=\dfrac{18}{\sqrt{53}}e^{-t/6}\sin\left(\dfrac{\sqrt{53}}{6}\right)t$, frequency $\dfrac{\sqrt{53}}{6}$ radian per unit of time,

period $\dfrac{12\pi}{\sqrt{53}}$ units of time

8 Transient solution $\dfrac{e^{-t}}{19}\left(32\sqrt{2}\sin\sqrt{2}t-40\cos\sqrt{2}t\right)$,

steady-state solution $\dfrac{2}{19}e^{5t}$

Further problems 4 (page 143)

2 $L\{f(t)\}=\dfrac{a(1+e^{-\pi s})}{(s^2+1)(1-e^{-\pi s})}$ **3** (a) $F(s)=\dfrac{1}{s^2}-\dfrac{w}{s}\left\{\dfrac{e^{-ws}}{1-e^{-ws}}\right\}$

(b) $F(s)=\dfrac{1-e^{2(1-s)\pi}}{(s-1)(1-e^{-2\pi s})}$ (c) $F(s)=\dfrac{1-e^{-s}(s+1)}{s^2(1-e^{-2s})}$

(d) $F(s)=\dfrac{1}{1-e^{-3s}}\left\{\dfrac{2}{s^3}-\dfrac{2e^{-2s}}{s^3}-\dfrac{4e^{-2s}}{s^2}-\dfrac{4e^{-3s}}{s}\right\}$

4 $x=\dfrac{P}{M\omega}\sin\omega t$ **5** $i=\dfrac{E}{L}\cos\left(\dfrac{t}{\sqrt{LC}}\right)$

6 $x=2e^{-2t}\{1+10e^8\cdot u(t-4)\}-2e^{-3t}\{1+10e^{12}\cdot u(t-4)\}$

7 (a) $f(t)=4\sqrt{3}\sin\dfrac{t}{2\sqrt{3}}-\cos\dfrac{t}{2\sqrt{3}}$, frequency $\dfrac{1}{2\sqrt{3}}$ radian per unit of time,

period $4\pi\sqrt{3}$ units of time (b) $f(t)=2\cos 2\sqrt{3}t-\dfrac{1}{2\sqrt{3}}\sin 2\sqrt{3}t$,

frequency $2\sqrt{3}$ radian per unit of time, period $\pi\sqrt{3}$ units of time

8 (a) $f(t)=-4{\cdot}48\sin 0{\cdot}69t+1{\cdot}06\cos 0{\cdot}69t$

(b) $f(t)=\dfrac{\pi}{(3/2)^{\frac{1}{4}}}\sin[(1{\cdot}5)^{\frac{1}{4}}t]$

9 Transient solution $e^{-3t/8}\left(\dfrac{421}{9\sqrt{23}}\sin\dfrac{\sqrt{23}}{8}t-\dfrac{1}{9}\cos\dfrac{\sqrt{23}}{8}t\right)$,

steady-state solution $\dfrac{1}{9}e^{t}$

Test exercise 5 (page 169)

1 $\dfrac{z}{z+1}$ provided $|z|>1$ **2** $-2\dfrac{z^3-4z^2+(2a+1)z}{(z-1)^2(z-a)}$

3 (a) $\dfrac{z(3z-4)}{(z-1)^2}$, $|z|>1$ (b) $\dfrac{25z}{z-5}$, $|z|>5$

4 $\{2k+3-2^{k+1}\}$ **5** $\{3u_k+4k-2^{k+1}\}$ **6** $\dfrac{z\sin T}{z^2-2z\cos T+1}$

Further problems 5 (page 169)

1 $\dfrac{z}{z+a}$ provided $|z|>|a|$ **2** (a) $\left\{\dfrac{1}{12}u_k-\dfrac{3}{4}(-3)^k+\dfrac{2}{3}(-2)^k\right\}$

(b) $\left\{\dfrac{1}{4}u_k-\dfrac{k}{2}+\dfrac{3}{4}(1/3)^k\right\}$ (c) $\left\{\dfrac{2}{3}(3^k)+\dfrac{1}{3}(-3)^k-2k\right\}$

3 $\left\{\dfrac{1}{2}(1+j)(-j)^{k-1}+\dfrac{1}{2}(1-j)(j)^{k-1}\right\}$ **4** (a) $\left\{u_k+\dfrac{3}{2}k(-2)^k\right\}$

(b) $\left\{\dfrac{1}{9}u_k-\dfrac{5}{6}k(-2)^k+\dfrac{8}{9}(-2)^k\right\}$ **5** (a) $\dfrac{z^2}{z^2-1}$ (b) $\dfrac{z}{z^2-1}$

(c) $\dfrac{z^7+z^5+z^4+1}{z^7}$ (d) $\dfrac{z^7+z^6+z^5+z+1}{z^7}$ (e) $\dfrac{z^7+z^6+z^5+z+1}{z^{10}}$

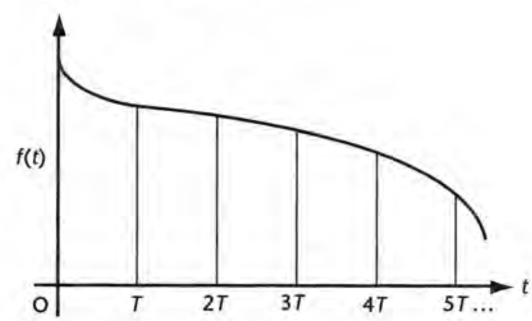
(f) $\dfrac{z^6+z^5+z+1}{z^6}$ **6** (a) $\{x_k\}=\left\{\dfrac{1}{2}((-3)^k-2(-2)^k+(-1)^k)\right\}$ for $k\geq 1$

(b) $\{x_k\}=\left\{\dfrac{1}{2}((-3^{k+1}-(-2)^{k+2}+(-1)^{k+1})\right\}$

(c) $\{x_k\}=\{10(3^k)-7(2^k)\}$ (d) $\{x_k\}=\{6(2^k)-3u_k\}$

9 3 **10** $-\dfrac{2}{7}$ **13** (a) $\dfrac{z\sinh T}{z^2-2z\cosh T+1}$ (b) $\dfrac{z(z-\cosh aT)}{z^2-2z\cosh aT+1}$

(c) $\dfrac{ze^{-aT}(ze^{aT}-\cosh bT)}{z^2-2ze^{-aT}\cosh bT+e^{-2aT}}$



Test exercise 6 (page 227)

1 (a) yes (b) yes (c) no (d) yes (e) no (f) no

2 $f(x)=2\pi-4\{\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ \ldots\}$ **3** (a) odd (b) odd

(c) even (d) neither (e) neither (f) even

4 (a) $f(x)=\dfrac{\pi}{2}+\dfrac{4}{\pi}\left\{\cos x+\dfrac{1}{9}\cos 3x+\dfrac{1}{25}\cos 5x+\ \ldots\right\}$

(b) $f(x) = -2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right\}$ **5** (a) cosine terms only
(b) sine terms only; odd harmonics only (c) odd harmonics only
(d) odd harmonics only

6 $$f(t) = \frac{1}{2} - \frac{1}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \ldots\right\} + \frac{1}{\omega}\left\{\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \ldots\right\} \text{ where } \omega = \pi/2$$

Further problems 6 (page 228)

1 $f(x) = \frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right\}$

2 $f(t) = -1 - \frac{16}{\pi}\left\{\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right\}$ where $\omega = \frac{\pi}{2}$

3 $f(x) = \frac{4}{\pi}\left\{\frac{1}{2} - \frac{1}{1\times 3}\cos 2x - \frac{1}{3\times 5}\cos 4x - \frac{1}{5\times 7}\cos 6x - \ldots\right\}$

4 $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\}$

5 $f(x) = \frac{2A}{\pi}\left\{1 - 2\left(\frac{1}{1\times 3}\cos 2x + \frac{1}{3\times 5}\cos 4x + \ldots\right)\right\}$

6 $$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\} + \left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right\}$$

7 $$i = f(t) = \frac{A}{\pi}\left\{1 + \frac{\pi}{2}\sin\omega t - 2\left(\frac{1}{1\times 3}\cos 2\omega t + \frac{1}{3\times 5}\cos 4\omega t + \frac{1}{5\times 7}\cos 6\omega t + \ldots\right)\right\} \quad \text{where } \omega = \frac{2\pi}{T}$$

8 $f(x) = \frac{3a}{\pi}\left\{\sin 2x + \frac{1}{2}\sin 4x + \frac{1}{4}\sin 8x + \frac{1}{5}\sin 10x + \ldots\right\}$

9 (a) $f(x) = \frac{\pi^2}{6} - \left(\cos 2x + \frac{1}{4}\cos 4x + \frac{1}{9}\cos 6x + \ldots\right)$

(b) $f(x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots\right)$

10 $f(x) = \frac{2}{\pi}\left\{\frac{1}{2} + \frac{\pi}{4}\cos x + \frac{1}{1\times 3}\cos 2x - \frac{1}{3\times 5}\cos 4x + \ldots\right\}$

11 $f(x) = -\frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{3\pi}\cos 2x + \frac{2}{15\pi}\cos 4x - \ldots$

12 $f(x) = \frac{4}{\pi}\left\{\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \ldots\right\}$

13 $f(t) = -\frac{4}{\pi^2}\left\{\cos\pi t + \frac{1}{9}\cos 3\pi t + \ldots\right\} + \frac{2}{\pi}\left\{2\sin\pi t - \frac{1}{2}\sin 2\pi t + \ldots\right\}$

14 $f(x) = \dfrac{\pi^2}{3} - 4\left\{\cos x - \dfrac{1}{4}\cos 2x + \dfrac{1}{9}\cos 3x - \dfrac{1}{16}\cos 4x + \ \dots\right\}$

15 $f(x) = 7 - \dfrac{6}{\pi}\left\{\sin x - \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x - \dfrac{1}{4}\sin 4x + \ \dots\right\}$

16 $f(x) = \dfrac{2}{3} + \dfrac{4}{\pi^2}\left\{\cos \pi t - \dfrac{1}{4}\cos 2\pi t + \dfrac{1}{9}\cos 3\pi t - \ \dots\right\}$

17 $f(x) = -\left\{\sin x + \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x + \dfrac{1}{4}\sin 4x + \ \dots\right\}$

18 $f(t) = -\dfrac{2}{\pi}\left\{\sin \omega t - \sin 2\omega t + \dfrac{1}{3}\sin 3\omega t + \dfrac{1}{5}\sin 5\omega t + \ \dots\right\}$ where $\omega = \pi/2$

19 $f(x) = \dfrac{4\pi^2}{3} + 4\left\{\cos x + \dfrac{1}{4}\cos 2x + \dfrac{1}{9}\cos 3x + \ \dots\right\}$

$$- 4\pi\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ \dots\right\}$$

20 $f(t) = 1 - 1{\cdot}17\cos \omega t + 0{\cdot}328\cos 2\omega t + 0\cos 3\omega t + \ \dots$
$+ 0{\cdot}282\sin \omega t + 0{\cdot}288\sin 2\omega t - 0{\cdot}318\sin 3\omega t + \ \dots$ where $\omega = \pi/3$

Test exercise 7 (page 268)

1 $f(t) = \dfrac{1}{2} + \dfrac{j}{2\pi}\displaystyle\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\dfrac{e^{j2n\pi t}}{n}$ **2** $F(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{(a - j\omega)\sinh(a + j\omega)}{a^2 + \omega^2}$

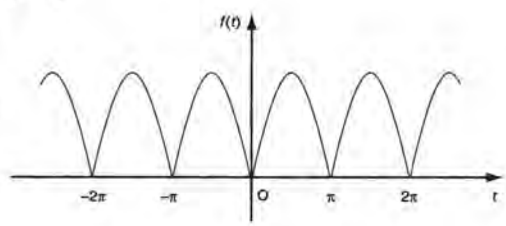
3 $\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sinh a\cos\omega + j\sin\omega\cosh a}{a + j\omega}\right)$ **4** $-\dfrac{j}{2}(F(\omega + \omega_0) - F(\omega - \omega_0))$

5 $2\sqrt{2\pi}(e^t - e^{4t})u(t)$ **6** $F_c(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{k}{k^2\omega^2}$, $F_s(\omega) = \sqrt{\dfrac{2}{\pi}}\dfrac{\omega}{k^2 + \omega^2}$

Further problems 7 (page 268)

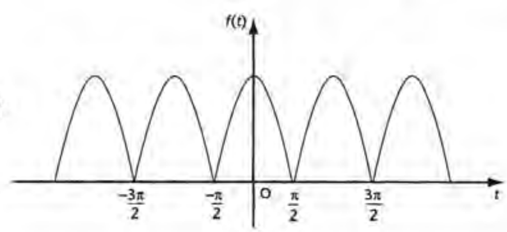
3

$$f(t) = -\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{4n^2 - 1}e^{2\pi nt}$$



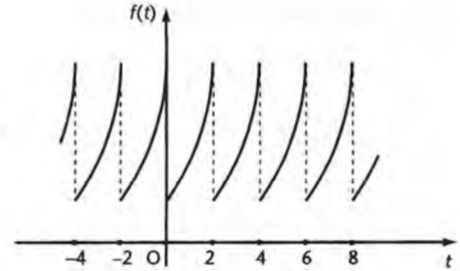
4

$$f(t) = -\frac{4j}{\pi}\sum_{n=-\infty}^{\infty}\frac{n}{4n^2 - 1}e^{j2\pi nt}$$



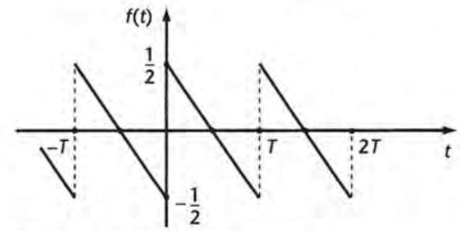
5

$$f(t) = -\frac{e^{2\pi}-1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1+jn}{1+n^2}e^{j\pi nt}$$



6

$$f(t) = \frac{1}{2} + \frac{1}{2\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}e^{j(n\omega_0 t+\pi/2)}$$

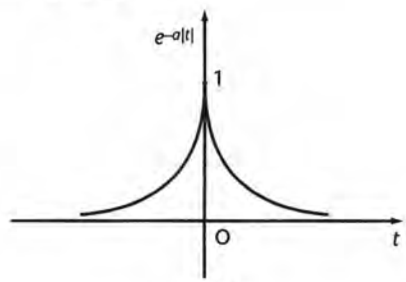


8 $\dfrac{(e^2-1)\cos\omega + \omega(e^2+1)\sin\omega}{\sqrt{2\pi}e(\omega^2+1)}$ **9** $\dfrac{j(\omega(e^2-1)\cos\omega - (e^2+1)\sin\omega)}{\sqrt{2\pi}e(\omega^2+1)}$

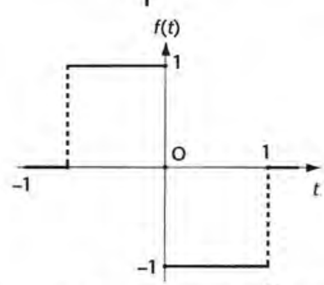
10 $\sqrt{\dfrac{\pi}{2}}\left(\dfrac{1+e^{-j\omega}}{\pi^2-\omega^2}\right)$ **11** $\dfrac{\sqrt{2\pi}\cos(\omega/2)}{\pi^2-\omega^2}$

12

$$F(\omega) = \frac{2a}{a^2+\omega^2}$$



13 (a)



(b) $f(t) = u(t-1) - 2u(t) + u(t+1)$ (c) $F(\omega) = \dfrac{4j}{\omega}\sin^2(\omega/2)$

14 $\dfrac{j}{\sqrt{2\pi}(k^2-\omega^2)}(\omega[1-\cos\pi(k+\omega)] - jk\sin\pi(k-\omega))$

20 $F_s(\omega) = 2\sqrt{\dfrac{2}{\pi}}\dfrac{1}{a^2+\omega^2}\{a\sin\omega\cos a - j\omega\cos\omega\sin a\}$

$F_c(\omega) = 2\sqrt{\dfrac{2}{\pi}}\dfrac{1}{a^2+\omega^2}\{\omega\sin\omega\cos a + ja\cos\omega\sin a\}$

21 $F_s(\omega) = 0$ $F_c(\omega) = 2\cos\omega\,\mathrm{sinc}\,t$

Test exercise 8 (page 324)

2 $y = a_0\left\{1 + \frac{5x^2}{2} + \frac{15x^4}{8} + \frac{5x^6}{16} + \dots\right\} + a_1\left\{x + \frac{4x^3}{3} + \frac{8x^5}{15} + \dots\right\}$

3 (a) $y = A\left\{1 - \frac{x}{1 \times 2} + \frac{x^2}{(1 \times 2)(2 \times 5)} - \frac{x^3}{(1 \times 2)(2 \times 5)(3 \times 8)} + \dots\right\}$

$+ Bx^{\frac{1}{3}}\left\{1 - \frac{x}{1 \times 4} + \frac{x^2}{(1 \times 4)(2 \times 7)} - \frac{x^3}{(1 \times 4)(2 \times 7)(3 \times 10)} + \dots\right\}$

(b) $y = a_0\left\{1 - \frac{x^4}{3 \times 4} + \frac{x^8}{(3 \times 4)(7 \times 8)} + \dots\right\}$

$+ a_1\left\{x - \frac{x^5}{4 \times 5} + \frac{x^9}{(4 \times 5)(8 \times 9)} + \dots\right\}$

(c) $y_A = A\left\{-\frac{1}{2} - \frac{x}{6} - \dots\right\}$

$y_B = B\left\{\ln x\left(-\frac{1}{2} - \frac{x}{6} - \dots\right) + x^{-2}\left(1 - x + \frac{x^2}{4} + \dots\right)\right\}$ **5** $\frac{1}{3}P_0(x) - \frac{4}{3}P_2(x)$

Further problems 8 (page 324)

1 $y_5 = 64e^{4x}\{16x^3 + 60x^2 + 60x + 15\}$

2 $y_n = (-1)^n e^{-x}\{x^3 - 3nx^2 + n(n-1)3x - n(n-1)(n-2)\}$, $n > 3$

3 $y_4 = 480x + 96$ **4** $y_6 = -\{(x^4 - 180x^2 + 360)\cos x + (24x^3 - 480x)\sin x\}$

5 $y_4 = -4e^{-x}\sin x$ **6** $y_3 = 2x(13 + 12\ln x)$ **8** $y_6 = -1018$

10 (a) $y_{2n} = \{x^2 + 2n(2n-1)\}\sinh x + 4nx\cosh x$

(b) $y_{2n} = \{x^3 + 6n(2n-1)x\}\cosh x + \{6nx^2 + 2n(2n-1)(2n-2)\}\sinh x$

11 $y_6 = 2^5 e^{2x}\{2x^3 + 24x^2 + 81x + 75\}$ **12** $y_3 = 2\sqrt{2}a^3 e^{-ax}\{\cos(ax + \pi/4)\}$

14 $y = y_0\left\{1 + \frac{9x^2}{2} + \frac{15x^4}{8} - \frac{7x^6}{16} + \frac{27x^8}{128} + \dots\right\} + y_1\left\{x + \frac{4x^3}{3}\right\}$

15 $y = A(1 + x^2) + Be^{-x}$

16 $y = y_0\left\{1 + \frac{3^2 \times x^2}{2!} + \frac{3^2 \times 5^2 \times x^4}{4!} + \frac{3^2 \times 5^2 \times 7^2 \times x^6}{6!} + \dots\right\}$

$+ y_1\left\{x + \frac{4^2 \times x^3}{3!} + \frac{4^2 \times 6^2 \times x^5}{5!} + \dots\right\}$

17 $y = y_1 x + y_0\left\{1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \dots\right\}$

18 $y = y_0\left\{1 - \frac{2x}{2^2} + \frac{2^2 \times x^4}{2^2 \times 4^2} - \frac{2^3 \times x^6}{2^2 \times 4^2 \times 6^2} + \dots\right\}$

$+ y_1\left\{x - \frac{2x^3}{3^2} + \frac{2^2 \times x^5}{3^2 \times 5^2} - \frac{2^3 \times x^7}{3^2 \times 5^2 \times 7^2} + \dots\right\}$

19 $y = A\left\{1 + x + \frac{x^2}{2 \times 4} + \frac{x^3}{(2 \times 3)(4 \times 7)} + \frac{x^4}{(2 \times 3 \times 4)(4 \times 7 \times 10)} + \dots\right\}$

$+ Bx^{\frac{2}{3}}\left\{1 + \frac{x}{1 \times 5} + \frac{x^2}{(1 \times 2)(5 \times 8)} + \frac{x^3}{(1 \times 2 \times 3)(5 \times 8 \times 11)} + \dots\right\}$

20 $y = a_0\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right\} + a_1\left\{x - \frac{x^3}{3!} + \ldots\right\}$

21 $y = a_0\left\{1 + \frac{x^3}{2\times 3} + \frac{x^6}{(2\times 3)(5\times 6)} + \ldots\right\}$
$+ a_1\left\{x + \frac{x^4}{3\times 4} + \frac{x^7}{(3\times 4)(6\times 7)} + \ldots\right\}$

22 $y = A\left\{1 - \frac{x}{1\times 4} + \frac{x^2}{(1\times 2)(4\times 7)} - \frac{x^3}{(1\times 2\times 3)(4\times 7\times 10)} + \ldots\right\}$
$+ Bx^{-\frac{1}{3}}\left\{1 - \frac{x}{1\times 2} + \frac{x^2}{(1\times 2)(2\times 5)} - \frac{x^3}{(1\times 2\times 3)(2\times 5\times 8)} + \ldots\right\}$

23 $y = a_1x + a_0\left\{1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{3x^6}{6!} - \frac{(3)(5)x^8}{8!} + \ldots\right\}$

24 $y = u + v$ where $u = A\left\{\frac{-x^4}{4!\,3!} + \frac{x^5}{5!\,3!} - \ldots\right\}$
$v = B\left\{\ln x\left(\frac{-x^4}{4!\,3!} + \frac{x^5}{5!\,3!} - \ldots\right) + \left(1 + \frac{x}{1\times 3} + \frac{x^2}{(1\times 2)(2\times 3)} + \ldots\right)\right\}$

25 $y = u + v$ where $u = A\left\{1 + \frac{3x}{1^2} + \frac{3^2\times x^2}{1^2\times 2^2} + \frac{3^3\times x^3}{1^2\times 2^2\times 3^2} + \ldots\right\}$
$v = B\left\{\ln x\left(1 + \frac{3x}{1^2} + \frac{3^2\times x^2}{1^2\times 2^2} + \frac{3^3\times x^3}{1^2\times 2^2\times 3^2} + \ldots\right)\right.$
$\left. - \left(\frac{2\times 3x}{1^2} + \frac{3\times 3^2\times x^2}{1^2\times 2^2} + \frac{11\times 3^3\times x^3}{1^2\times 2^2\times 3^3} + \ldots\right)\right\}$

26 eigenfunctions: $y_n(x) = A_n\cos\sqrt{\lambda_n}x$; eigenvalues: $\lambda_n = \frac{(2n+1)^2\pi^2}{4}$

27 $H_0 = 1$, $H_1 = 2x$, $H_2 = 4x^2 - 2$, $H_3 = 8x^3 - 12x$

28 $L_0 = 1$, $L_1 = 1 - x$, $L_2 = 2 - 4x + x^2$, $L_3 = 6 - 18x + 9x^2 - x^3$

Text exercise 9 (page 367)

1

x	y
0	1·0
0·1	1·1
0·2	1·211
0·3	1·3352
0·4	1·4753
0·5	1·6343

2

x	y
1	0
1·2	0·204
1·4	0·4211
1·6	0·6600
1·8	0·9264
2·0	1·2243

3

x	y
0	1·0
0·1	1·2052
0·2	1·4214
0·3	1·6499
0·4	1·8918
0·5	2·1487

4

x	y
2·0	3·0
2·1	3·005
2·2	3·0195
2·3	3·0427
2·4	3·0736
2·5	3·1117

5

x	y
1·0	0
1·1	0·1052
1·2	0·2215
1·3	0·3401
1·4	0·4717
1·5	0·6180

6

x	y
0·0	1·0000
0·1	1·0101
0·2	1·0202
0·3	1·0305
0·4	1·0408
0·5	1·0513
0·6	1·0619
0·7	1·0726
0·8	1·0834
0·9	1·0943
1·0	1·1053

Further problems 9 (page 368)

1

x	y
0	1·0
0·2	0·8
0·4	0·72
0·6	0·736
0·8	0·8288
1·0	0·9830

2

x	y
0	1·4
0·1	1·596
0·2	0·8707
0·3	2·2607
0·4	2·8318
0·5	3·7136

3

x	y
1·0	2·0
1·2	2·0333
1·4	2·1143
1·6	2·2250
1·8	2·3556
2·0	2·5000

4

x	y
0	0·5
0·1	0·543
0·2	0·5716
0·3	0·5863
0·4	0·5878
0·5	0·5768

5

x	y
0	1·0
0·1	1·1022
0·2	1·2085
0·3	1·3179
0·4	1·4296
0·5	1·5428

6

x	y
1·0	1·0
1·1	1·1871
1·2	1·3531
1·3	1·5033
1·4	1·6411
1·5	1·7688

7

x	y
0	0
0·1	0·1002
0·2	0·2015
0·3	0·3048
0·4	0·4110
0·5	0·5214

8

x	y
0	1·0
0·2	0·8562
0·4	0·8110
0·6	0·8465
0·8	0·9480
1·0	1·1037

9

x	y
0	1·0
0·1	0·9138
0·2	0·8512
0·3	0·8076
0·4	0·7798
0·5	0·7653

10

x	y
0	0·4
0·2	0·4259
0·4	0·4374
0·6	0·4319
0·8	0·4085
1·0	0·3689

11

x	y
1·0	2·0
1·2	2·4197
1·4	2·8776
1·6	3·3724
1·8	3·9027
2·0	4·4677

12

x	y
0	1·0
0·2	1·1997
0·4	1·3951
0·6	1·5778
0·8	1·7358
1·0	1·8540

13

x	y
0	1·0
0·2	1·1679
0·4	1·2902
0·6	1·3817
0·8	1·4497
1·0	1·4983

14

x	y
0	1·0
0·1	1·11
0·2	1·2422
0·3	1·4013
0·4	1·5937
0·5	1·8271

15

x	y
0	3·0
0·1	2·88
0·2	2·5224
0·3	1·9368
0·4	1·1424
0·5	0·1683

16

x	y
0	0
0·2	0·1987
0·4	0·3897
0·6	0·5665
0·8	0·7246
1·0	0·8624

17

x	y
0	1·0
0·2	1·1972
0·4	1·3771
0·6	1·5220
0·8	1·6161
1·0	1·6487

18

x	y
0	2·0
0·1	2·0845
0·2	2·1367
0·3	2·1554
0·4	2·1407
0·5	2·0943

19

x	y
0	1·0
0·2	1·0367
0·4	1·1373
0·6	1·2958
0·8	1·5145
1·0	1·8029

20

x	y
1·0	0
1·2	0·1833
1·4	0·3428
1·6	0·4875
1·8	0·6222
2·0	0·7500

21

x	y
1·0	2·0000
1·2	2·0333
1·4	2·1121
1·6	2·2219
1·8	2·3522
2·0	2·4965

22

x	y
0·0	1·0000
0·2	0·8600
0·4	0·8118
0·6	0·8452
0·8	0·9454
1·0	1·1002

23

x	y
1·0	2·0000
1·2	2·4191
1·4	2·8769
1·6	3·3715
1·8	3·9018
2·0	4·4666

Test exercise 10 (page 411)

2 $145{\cdot}7 \pm 2{\cdot}6$ mm **3** $5{\cdot}8$ m/s **4** $\dfrac{-2(x+y)}{2x+3y};\ \dfrac{-2}{(2x+3y)^3}$

5 $\dfrac{x}{2(x^2-y^2)};\ \dfrac{-y}{4(x^2-y^2)};\ \dfrac{-y}{2(x^2-y^2)};\ \dfrac{x}{4(x^2-y^2)}$

6 (a) $(-1, 1)$, saddle; $(-1, -\frac{4}{3})$, min (b) an infinity of maxima along the line $y = 5x/2$ when $z = 4$ **7** $1{\cdot}10$ m $\times$ $1{\cdot}10$ m $\times$ $0{\cdot}825$ m high

8 $u = \dfrac{8}{7},\ x = \dfrac{6}{7},\ y = -\dfrac{4}{7},\ z = \dfrac{2}{7}$

Further problems 10 (page 412)

1 $(8x\cos x - 6y\sin x)/J$; $-(4x^3\cos y + 6x\sin y)/J$; $J = 4x\cos x\sin y + 2x^2y\sin x\cos y$ **2** $e^{3y}/2(xe^{3y}+e^{-3y})$; $e^{-3y}/2(xe^{3y}+e^{-3y})$; $-1/3(xe^{3y}+e^{-3y})$; $x/3(xe^{3y}+e^{-3y})$

5 $(2e^{-x}\sinh 2x\sin 3y + 3ye^{-x}\cosh 2x\cos 3y)/(1+3y^2)$; $\{-4ye^x\sinh 2x\sin 3y + 3e^x(1+y^2)\cosh 2x\cos 3y\}/2(1+3y^2)$

7 (a) $(4,\ -4,\ -11)$, min (b) $(1,\ -2,\ 4)$, saddle (c) $(\frac{10}{7},\ \frac{6}{7},\ \frac{97}{7})$, max

8 (0, 0), saddle; (2, 0), min; (−2, 0), min **9** (2, 1), max; $(-\frac{2}{3},\ -\frac{1}{3})$, min

10 (0, 0); (3, 3); (−3, −3), all saddle points

11 (a) (1, 0), saddle; (1, 1), min; $(-2,\ \frac{1}{2})$, saddle; $(-\frac{7}{5},\ \frac{1}{5})$, max
(b) (0, 0), max; (1, 1); (1, −1); (−1, 1); (−1, −1), all four saddle points

12 (a) A point of inflexion at the origin (b) An infinity of maxima along the line $y = x/4$ when $z = 6$ (c) The value of z ranges from −1 to 1 and has an infinity of stationary points lying on the circles $x^2 + y^2 = n\pi$. When n is even the stationary points are maxima and when n is odd the stationary points are minima. There is also a single maximum at (0, 0, 1)

13 $x = 66{\cdot}7$ mm; $\theta = \dfrac{\pi}{3}$ **14** $l = h = \dfrac{1}{5\pi}\sqrt[3]{60\pi^2 V}$; $d = l\sqrt{5}$ **15** $l = 1{\cdot}00$ cm; $d = 4.48$ cm; $\theta = 48°\ 11'$ **16** cube of side $\dfrac{2r}{\sqrt{3}}$; $V_{\max} = \dfrac{8r^3}{3\sqrt{3}}$

17 (a) $u = \dfrac{64}{27}$; $x = y = z = \pm\dfrac{2}{\sqrt{3}}$; $u = \dfrac{9}{7}$, $x = y = \pm\dfrac{3}{\sqrt{14}}$ (b) $u = 9$; $x = \pm\dfrac{3}{\sqrt{2}}$, $y = \mp\dfrac{3}{\sqrt{2}}$

Test exercise 11 (page 448)

1 (a) $u = 2x^4(t-2) + 4xt + e^{2t}$ (b) $u = 2\sin 2x\cdot(e^y - 1) + \sin x + y^2$

2 $u(x,t) = \dfrac{16}{\pi^2}\displaystyle\sum_{r=1}^{\infty}\dfrac{1}{r^2}\cdot\sin\dfrac{r\pi}{2}\cdot\sin\dfrac{r\pi x}{10}\cdot\cos\dfrac{r\pi t}{10}$

3 $u(x,t) = \dfrac{100}{\pi}\displaystyle\sum_{r=1}^{\infty}(-1)^{r+1}\cdot\dfrac{1}{r}\sin\dfrac{\lambda x}{c}\cdot e^{-\lambda^2 t}$ where $\lambda = \dfrac{r\pi c}{2}$

4 $u(x,y) = \displaystyle\sum_{r=1}^{\infty}\dfrac{20}{r\pi}\cdot\operatorname{cosech} r\pi\cdot\sin\dfrac{r\pi x}{2}\cdot\sinh\dfrac{r\pi y}{2}$ with $r = 1, 3, 5, \ldots$

5 $v(r,\ \theta) = 5r^3\cos 3\theta$

Further problems 11 (page 449)

2 $u(x,t) = \dfrac{32}{\pi^3}\displaystyle\sum_{r=1}^{\infty}\dfrac{1}{r^3}\cdot\sin\dfrac{r\pi x}{2}\cdot\cos\dfrac{3r\pi t}{2}$ (r odd)

3 $u(x,t) = \dfrac{2}{25\pi^2}\displaystyle\sum_{r=1}^{\infty}\dfrac{1}{r^2}\cdot\sin\dfrac{r\pi}{2}\cdot\sin\dfrac{r\pi x}{4}\cdot\cos\dfrac{5r\pi t}{2}$

4 $u(x,t) = \dfrac{25}{2\pi^2}\displaystyle\sum_{r=1}^{\infty}\dfrac{1}{r^2}\cdot\sin\dfrac{r\pi}{5}\cdot\sin\dfrac{r\pi x}{10}\cdot\cos\dfrac{cr\pi t}{10}$

5 $u(x,t) = \dfrac{800}{\pi^3}\displaystyle\sum_{r=1}^{\infty}\dfrac{1}{r^3}\cdot\sin\dfrac{r\pi x}{10}\cdot e^{-4\lambda^2 t}$ with $r = 1, 3, 5, \ldots$ where $\lambda = \dfrac{r\pi}{10}$

6 $u(x,t) = \dfrac{16}{\pi^2}\sum_{r=1}^{\infty}\dfrac{1}{r^2}\cdot\sin\dfrac{r\pi}{2}\cdot\sin\dfrac{r\pi x}{10}\cdot e^{-r^2c^2\pi^2t/100}$ with $r = 1, 3, 5, \ldots$

7 $u(x,y) = \dfrac{128}{\pi^3}\sum_{r=1}^{\infty}\dfrac{1}{r^3}\cdot\text{cosech}\,\dfrac{r\pi}{2}\cdot\sinh\dfrac{r\pi}{4}(2-y)\cdot\sin\dfrac{r\pi x}{4}$ with $r = 1, 3, 5, \ldots$

8 $u(x,y) = \dfrac{200}{\pi^3}\sum_{r=1}^{\infty}\dfrac{1}{r^3}\cdot\text{cosech}\,\dfrac{2r\pi}{5}\cdot\sin\dfrac{r\pi x}{5}\cdot\sinh\dfrac{r\pi}{5}(y-2)$ with $r = 1, 3, 5, \ldots$

9 $v(r,\theta) = -4r\cos\theta + r^2\sin 2\theta$ **10** $v(r,\theta) = 3(1 - r^2\cos 2\theta)$

Test exercise 12 (page 513)

1 (a) solutions unique (b) infinite number of solutions **2** $x_1 = -4$, $x_2 = 2$, $x_3 = -3$ **3** $x_1 = -2$, $x_2 = 2$, $x_3 = 3$ **4** $x_1 = -3$, $x_2 = 4$, $x_3 = -2$

5 $x_1 = 1$, $x_2 = -2$, $x_3 = 2$ **6** $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$; $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ **7** $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$; $\mathbf{M}^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$

$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$ **8** $f_1(x) = -\dfrac{10}{3}e^{6x} + \dfrac{1}{3}e^{3x}$; $f_2(x) = \dfrac{5}{3}e^{6x} + \dfrac{1}{3}e^{3x}$

9 $f_1(x) = \dfrac{1}{3}\cos\sqrt{5}x + \dfrac{4}{3\sqrt{5}}\sin\sqrt{5}x + \dfrac{2}{3}\cosh 2x + \dfrac{1}{3}\sinh 2x$

$f_2(x) = -\dfrac{1}{3}\cos\sqrt{5}x - \dfrac{4}{3\sqrt{5}}\sin\sqrt{5}x + \dfrac{1}{3}\cosh 2x + \dfrac{1}{6}\sinh 2x$

10 (a) $\begin{bmatrix} -8 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7{\cdot}196 \\ -0{\cdot}464 \end{bmatrix}$

Further problems 12 (page 514)

1 $x_1 = 1$, $x_2 = -4$, $x_3 = 3$ **2** (a) $x_1 = 3$, $x_2 = 1$, $x_3 = -4$ (b) $x_1 = 4$, $x_2 = -2$, $x_3 = -1$ **3** (a) $x_1 = 4$, $x_2 = 2$, $x_3 = 5$, $x_4 = 3$ (b) $x_1 = 5$, $x_2 = -4$, $x_3 = 1$, $x_4 = 3$ (c) $x_1 = 3$, $x_2 = -2$, $x_3 = 0$, $x_4 = 5$

4 (a) $x_1 = -3$, $x_2 = 1$, $x_3 = 3$ (b) $x_1 = 5$, $x_2 = 2$, $x_3 = -1$ (c) $x_1 = 4$, $x_2 = 3$, $x_3 = -1$, $x_4 = -2$ **5** (a) $\lambda_1 = 2$, $\lambda_2 = 7$; $\mathbf{x}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$; $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $\lambda_1 = 1$,

$\lambda_2 = -3$; $\mathbf{x}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$; $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\lambda_1 = -8$, $\lambda_2 = 4$; $\mathbf{x}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$; $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) $\lambda_1 = 4$, $\lambda_2 = -6$; $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{x}_2 = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$ (e) $\lambda_1 = 1$, $\lambda_2 = 3$; $\lambda_3 = 9$;

$\mathbf{x}_1 = \begin{bmatrix} 7 \\ -1 \\ -5 \end{bmatrix}$; $\mathbf{x}_2 = \begin{bmatrix} 7 \\ 1 \\ -7 \end{bmatrix}$; $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ (f) $\lambda = 1, 2, 4$; $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

(g) $\lambda = -1, -3, 7$; $\mathbf{x} = \begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 27 \\ 10 \end{bmatrix}$ (h) $\lambda = -2, 4, 7$;

$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$

6 (a) $f_1(x) = \frac{1}{4}(5e^x - e^{-3x})$; $f_2(x) = \frac{1}{4}(e^x - e^{-3x})$
(b) $f_1(x) = \frac{9}{5}(e^{-6x} - e^{4x})$; $f_2(x) = -\frac{1}{5}(e^{-6x} + 9e^{4x})$
(c) $f_1(x) = \frac{1}{2}(5e^{4x} - 3e^{2x})$; $f_2(x) = \frac{2}{3}e^x - \frac{3}{2}e^{2x} + \frac{5}{6}e^{4x}$;
$f_3(x) = 4e^x - \frac{9}{2}e^{2x} + \frac{5}{2}e^{4x}$ (d) $f_1(x) = 3e^{-2x} - e^{4x} + 2e^{7x}$;
$f_2(x) = -e^{-2x} - \frac{4}{3}e^{4x} + \frac{1}{3}e^{7x}$; $f_3(x) = e^{-2x} - \frac{5}{3}e^{4x} - \frac{1}{3}e^{7x}$ **7** $\lambda = 0, 7, 13$

8 $I_1 = 2, I_2 = -3, I_3 = 2$ **9** $k = 2$; $x_1 = -2, x_2 = \frac{1}{2}, x_3 = 1$

10 (a) $f_1(x) = \frac{3}{5}\cosh\sqrt{2}x + \frac{9}{5\sqrt{2}}\sinh\sqrt{2}x + \frac{2}{5}\cosh\sqrt{7}x + \frac{11}{5\sqrt{7}}\sinh\sqrt{7}x$;
$f_2(x) = -\frac{2}{5}\cosh\sqrt{2}x - \frac{6}{5\sqrt{2}}\sinh\sqrt{2}x + \frac{2}{5}\cosh\sqrt{7}x + \frac{11}{5\sqrt{7}}\sinh\sqrt{7}x$
(b) $f_1(x) = -\frac{5}{12}\cos 2\sqrt{2}x + \frac{5}{12\sqrt{2}}\sin 2\sqrt{2}x + \frac{5}{12}\cosh 2x + \frac{1}{12}\sinh 2x$;
$f_2(x) = \frac{1}{6}\cos 2\sqrt{2}x - \frac{1}{6\sqrt{2}}\sin 2\sqrt{2}x + \frac{5}{6}\cosh 2x + \frac{1}{6}\sinh 2x$
(c) $f_1(x) = -\frac{35}{16}\cosh x + \frac{7}{16}\sinh x + \frac{35}{12}\cosh\sqrt{3}x - \frac{7}{12\sqrt{3}}\sinh\sqrt{3}x$
$+\frac{13}{48}\cosh 3x + \frac{7}{144}\sinh 3x$; $f_2(x) = \frac{5}{16}\cosh x - \frac{1}{16}\sinh x + \frac{5}{12}\cosh\sqrt{3}x$
$-\frac{1}{12\sqrt{3}}\sinh\sqrt{3}x + \frac{13}{48}\cosh 3x + \frac{7}{144}\sinh 3x$; $f_3(x) = \frac{25}{16}\cosh x - \frac{5}{16}\sinh x$
$-\frac{35}{12}\cosh\sqrt{3}x + \frac{7}{12\sqrt{3}}\sinh\sqrt{3}x + \frac{65}{48}\cosh 3x + \frac{35}{144}\sinh 3x$
(d) $f_1(x) = -\frac{9}{8}\cos x + \frac{9}{4}\sin x + \frac{19}{10}\cos\sqrt{3}x - \frac{12}{5\sqrt{3}}\sin\sqrt{3}x + \frac{9}{40}\cosh\sqrt{7}x$
$+\frac{3}{20\sqrt{7}}\sinh\sqrt{7}x$; $f_2(x) = \frac{15}{16}\cos x - \frac{15}{8}\sin x - \frac{19}{20}\cos\sqrt{3}x + \frac{6}{5\sqrt{3}}\sin\sqrt{3}x$
$+\frac{81}{80}\cosh\sqrt{7}x + \frac{27}{40\sqrt{7}}\sinh\sqrt{7}x$; $f_3(x) = -\frac{3}{8}\cos x + \frac{3}{4}\sin x + \frac{3}{8}\cosh\sqrt{7}x$
$+\frac{1}{4\sqrt{7}}\sinh\sqrt{7}x$

Test exercise 13 (page 560)

1 $f(1/4, 1/3) = -19/12$, $f(1/2, 1/3) = -5/6$, $f(3/4, 1/3) = -1/12$,
$f(1/4, 2/3) = 1/12$, $f(1/2, 2/3) = 5/6$, $f(3/4, 2/3) = 19/12$
2 $f(1/3, 1/3) = 4$, $f(2/3, 1/3) = 17/3$, $f(1, 1/3) = 26/3$, $f(1/3, 2/3) = 2/3$,
$f(2/3, 2/3) = 3$, $f(1, 2/3) = 16/3$ **3** (a) parabolic (b) hyperbolic
(c) parabolic (d) hyperbolic (e) elliptic **4** $f(1/3, 1/3) = -1{\cdot}61728$,
$f(2/3, 1/3) = -1{\cdot}18519$, $f(1, 1/3) = -0{\cdot}82716$, $f(1/3, 2/3) = -1{\cdot}61728$,
$f(2/3, 2/3) = -1{\cdot}18519$, $f(1, 2/3) = -0{\cdot}82716$

5

t\x	0·0	0·2	0·4	0·6	0·8	1·0	1·2
0·00	0·00000	0·04000	0·16000	0·36000	0·64000	1·00000	0·89000
0·02	0·00000	0·08000	0·20000	0·40000	0·68000	0·76500	0·93000
0·04	0·00000	0·10000	0·24000	0·44000	0·58250	0·80500	0·83250
0·06	0·00000	0·12000	0·27000	0·41125	0·62250	0·70750	0·87250
0·08	0·00000	0·13500	0·26563	0·44625	0·55938	0·74750	0·80938
0·10	0·00000	0·13281	0·29063	0·41250	0·59688	0·68438	0·84688
0·12	0·00000	0·14531	0·27266	0·44375	0·54844	0·72188	0·79844
0·14	0·00000	0·13633	0·29453	0·41055	0·58281	0·67344	0·83281
0·16	0·00000	0·14727	0·27344	0·43867	0·54199	0·70781	0·79199

6

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·000	1·000000	0·840000	0·760000	0·760000	0·840000	1·000000
0·040	1·000000	0·898182	0·832727	0·832727	0·898182	1·000000
0·080	1·000000	0·929917	0·886942	0·886942	0·929917	1·000000
0·120	1·000000	0·952517	0·923125	0·923125	0·952517	1·000000
0·160	1·000000	0·967729	0·94779	0·94779	0·967729	1·000000
0·200	1·000000	0·978081	0·964533	0·964533	0·978081	1·000000

Further problems 13 (page 561)

1

x\y	0·00	0·33	0·67	1·00
0·00	−3·0000	−2·3333	−1·6667	−1·0000
0·25	−2·7500	−2·0833	−1·4167	−0·7500
0·50	−2·5000	−1·8333	−1·1667	−0·5000
0·75	−2·2500	−1·5833	−0·9167	−0·2500
1·00	−2·0000	−1·3333	−0·6667	0·0000

2

x\y	0·00	0·33	0·67	1·00
0·00	4·0000	7·3333	10·6667	14·0000
0·33	6·3333	9·6667	13·0000	16·3333
0·67	8·6667	12·0000	15·3333	18·6667
1·00	11·0000	14·3333	17·6667	21·0000

3

x\y	0·00	0·33	0·67	1·00
0·00	−1·0000	−1·0000	−1·0000	−1·0000
0·33	−0·6667	−0·7500	−0·8000	−0·8333
0·67	−0·3333	−0·5000	−0·6000	−0·6667
1·00	0·0000	−0·2500	−0·4000	−0·5000

4

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·25	0·0000	−0·0069	−0·0694	−0·1875
0·50	0·0000	0·0278	−0·0556	−0·2500
0·75	0·0000	0·1042	0·0417	−0·1875
1·00	0·0000	0·2222	0·2222	0·0000

5

x\y	0·00	0·33	0·67	1·00
0·00	15·0000	16·6667	18·3333	20·0000
0·33	17·3333	19·0000	20·6667	22·3333
0·67	19·6667	21·3333	23·0000	24·6667
1·00	22·0000	23·6667	25·3333	27·0000

6

x\y	0·00	0·33	0·67	1·00
0·00	21·0000	20·0000	19·0000	18·0000
0·33	22·6667	21·6667	20·6667	19·6667
0·67	24·3333	23·3333	22·3333	21·3333
1·00	26·0000	25·0000	24·0000	23·0000

7

x\y	0·00	0·33	0·67	1·00
0·00	4·0000	4·0000	4·0000	4·0000
0·33	4·2222	4·1111	3·7778	3·2222
0·67	4·8889	4·6667	4·0000	2·8889
1·00	6·0000	5·6667	4·6667	3·0000

8

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·0000	0·0000	−0·0741	−0·2963
0·67	0·0000	0·0741	0·0000	−0·3704
1·00	0·0000	0·2963	0·3704	0·0000

9

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	−0·5556	−2·2222	−5·0000
0·33	0·3333	−0·2222	−1·8889	−4·6667
0·67	1·3333	0·7778	−0·8889	−3·6667
1·00	3·0000	2·4444	0·7778	−2·0000

10

x\y	0·00	0·33	0·67	1·00
0·00	−1·0000	−1·0000	−1·0000	−1·0000
0·33	−1·0000	−0·7037	−0·3333	0·1111
0·67	−1·0000	−0·3333	0·4815	1·4444
1·00	−1·0000	0·1111	1·4444	3·0000

11

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·1111	0·1050	0·0873	0·0600
0·67	0·4444	0·4200	0·3493	0·2401
1·00	1·0000	0·9450	0·7859	0·5403

12

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0370	0·2963	1·0000
0·33	0·0370	0·1481	0·5556	1·4815
0·67	0·2963	0·5556	1·1852	2·4074
1·00	1·0000	1·4815	2·4074	4·0000

13

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·0000	0·1111	0·2222	0·3333
0·67	0·0000	0·2222	0·4444	0·6667
1·00	0·0000	0·3333	0·6667	1·0000

14

x\y	0·00	0·33	0·67	1·00
0·00	0·0000	0·0000	0·0000	0·0000
0·33	0·0000	0·0000	−0·0741	−0·2222
0·67	0·0000	0·0741	0·0000	−0·2222
1·00	0·0000	0·2222	0·2222	0·0000

15

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	0·0000	−0·1600	−0·2400	−0·2400	−0·1600	0·0000
0·02	0·0400	−0·1200	−0·2000	−0·2000	−0·1200	0·0400
0·04	0·0800	−0·0800	−0·1600	−0·1600	−0·0800	0·0800
0·06	0·1200	−0·0400	−0·1200	−0·1200	−0·0400	0·1200
0·08	0·1600	0·0000	−0·0800	−0·0800	0·0000	0·1600
0·10	0·2000	0·0400	−0·0400	−0·0400	0·0400	0·2000
0·12	0·2400	0·0800	0·0000	0·0000	0·0800	0·2400
0·14	0·2800	0·1200	0·0400	0·0400	0·1200	0·2800
0·16	0·3200	0·1600	0·0800	0·0800	0·1600	0·3200
0·18	0·3600	0·2000	0·1200	0·1200	0·2000	0·3600
0·20	0·4000	0·2400	0·1600	0·1600	0·2400	0·4000

16

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	0·0000	0·1987	0·3894	0·5646	0·7174	0·8415
0·02	0·0000	0·1983	0·3886	0·5635	0·7159	0·8398
0·04	0·0000	0·1979	0·3879	0·5624	0·7145	0·8381
0·06	0·0000	0·1975	0·3871	0·5613	0·7131	0·8364
0·08	0·0000	0·1971	0·3863	0·5601	0·7116	0·8348
0·10	0·0000	0·1967	0·3855	0·5590	0·7102	0·8331
0·12	0·0000	0·1963	0·3848	0·5579	0·7088	0·8314
0·14	0·0000	0·1959	0·3840	0·5568	0·7074	0·8298
0·16	0·0000	0·1955	0·3832	0·5557	0·7060	0·8281
0·18	0·0000	0·1951	0·3825	0·5546	0·7046	0·8265
0·20	0·0000	0·1947	0·3817	0·5535	0·7032	0·8248

17

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	0·0000	0·3830	0·7596	1·1239	1·4698	1·7916
0·02	0·0000	0·3798	0·7534	1·1147	1·4578	1·7770
0·04	0·0000	0·3767	0·7473	1·1056	1·4459	1·7624
0·06	0·0000	0·3736	0·7412	1·0966	1·4341	1·7481
0·08	0·0000	0·3706	0·7351	1·0876	1·4223	1·7338
0·10	0·0000	0·3676	0·7291	1·0787	1·4107	1·7196
0·12	0·0000	0·3646	0·7232	1·0699	1·3992	1·7056
0·14	0·0000	0·3616	0·7173	1·0612	1·3878	1·6916
0·16	0·0000	0·3586	0·7114	1·0525	1·3764	1·6778
0·18	0·0000	0·3557	0·7056	1·0439	1·3652	1·6641
0·20	0·0000	0·3528	0·6998	1·0354	1·3541	1·6505

18

t\x	0·00	0·20	0·40	0·60	0·80	1·00
0·00	−1·0000	−0·7600	−0·4400	−0·0400	0·4400	1·0000
0·04	−0·9200	−0·6800	−0·3600	0·0400	0·5200	1·0800
0·08	−0·8400	−0·6000	−0·2800	0·1200	0·6000	1·1600
0·12	−0·7600	−0·5200	−0·2000	0·2000	0·6800	1·2400
0·16	−0·6800	−0·4400	−0·1200	0·2800	0·7600	1·3200
0·20	−0·6000	−0·3600	−0·0400	0·3600	0·8400	1·4000
0·24	−0·5200	−0·2800	0·0400	0·4400	0·9200	1·4800
0·28	−0·4400	−0·2000	0·1200	0·5200	1·0000	1·5600
0·32	−0·3600	−0·1200	0·2000	0·6000	1·0800	1·6400
0·36	−0·2800	−0·0400	0·2800	0·6800	1·1600	1·7200
0·40	−0·2000	0·0400	0·3600	0·7600	1·2400	1·8000
0·44	−0·1200	0·1200	0·4400	0·8400	1·3200	1·8800
0·48	−0·0400	0·2000	0·5200	0·9200	1·4000	1·9600
0·52	0·0400	0·2800	0·6000	1·0000	1·4800	2·0400
0·56	0·1200	0·3600	0·6800	1·0800	1·5600	2·1200
0·60	0·2000	0·4400	0·7600	1·1600	1·6400	2·2000

19

t\x	0·00	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00
0·00	0·0000	−0·9000	−1·6000	−2·1000	−2·4000	−2·5000	−2·4000	−2·1000	−1·6000	−0·9000	0·0000
0·02	0·4000	−0·5000	−1·2000	−1·7000	−2·0000	−2·1000	−2·0000	−1·7000	−1·2000	−0·5000	0·4000
0·04	0·8000	−0·1000	−0·8000	−1·3000	−1·6000	−1·7000	−1·6000	−1·3000	−0·8000	−0·1000	0·8000
0·06	1·2000	0·3000	−0·4000	−0·9000	−1·2000	−1·3000	−1·2000	−0·9000	−0·4000	0·3000	1·2000
0·08	1·6000	0·7000	0·0000	−0·5000	−0·8000	−0·9000	−0·8000	−0·5000	0·0000	0·7000	1·6000
0·10	2·0000	1·1000	0·4000	−0·1000	−0·4000	−0·5000	−0·4000	−0·1000	0·4000	1·1000	2·0000
0·12	2·4000	1·5000	0·8000	0·3000	0·0000	−0·1000	0·0000	0·3000	0·8000	1·5000	2·4000
0·14	2·8000	1·9000	1·2000	0·7000	0·4000	0·3000	0·4000	0·7000	1·2000	1·9000	2·8000

20

t\x	0·00	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00
0·00	0·0000	30·9017	58·7785	80·9017	95·1057	100·0000	95·1057	80·9017	58·7785	30·9017	0·0000
0·04	0·0000	20·8224	39·6065	54·5136	64·0846	67·3825	64·0846	54·5136	39·6065	20·8224	0·0000
0·08	0·0000	14·0306	26·6878	36·7327	43·1818	45·4041	43·1818	36·7327	26·6878	14·0306	0·0000
0·12	0·0000	9·4542	17·9829	24·7514	29·0970	30·5944	29·0970	24·7514	17·9829	9·4542	0·0000
0·16	0·0000	6·3705	12·1174	16·6781	19·6063	20·6153	19·6063	16·6781	12·1174	6·3705	0·0000
0·20	0·0000	4·2926	8·1650	11·2381	13·2112	13·8911	13·2112	11·2381	8·1650	4·2926	0·0000
0·24	0·0000	2·8925	5·5018	7·5725	8·9021	9·3602	8·9021	7·5725	5·5018	2·8925	0·0000
0·28	0·0000	1·9490	3·7072	5·1026	5·9984	6·3071	5·9984	5·1026	3·7072	1·9490	0·0000
0·32	0·0000	1·3133	2·4980	3·4382	4·0419	4·2499	4·0419	3·4382	2·4980	1·3133	0·0000
0·36	0·0000	0·8849	1·6832	2·3168	2·7235	2·8637	2·7235	2·3168	1·6832	0·8849	0·0000
0·40	0·0000	0·5963	1·1342	1·5611	1·8352	1·9296	1·8352	1·5611	1·1342	0·5963	0·0000
0·44	0·0000	0·4018	0·7643	1·0519	1·2366	1·3002	1·2366	1·0519	0·7643	0·4018	0·0000
0·48	0·0000	0·2707	0·5150	0·7088	0·8332	0·8761	0·8332	0·7088	0·5150	0·2707	0·0000
0·52	0·0000	0·1824	0·3470	0·4776	0·5615	0·5904	0·5615	0·4776	0·3470	0·1824	0·0000
0·56	0·0000	0·1229	0·2338	0·3218	0·3783	0·3978	0·3783	0·3218	0·2338	0·1229	0·0000
0·60	0·0000	0·0828	0·1576	0·2169	0·2549	0·2680	0·2549	0·2169	0·1576	0·0828	0·0000

Test exercise 14 (page 614)

1 (a) $dz = 4x^3\cos 3y\,dx - 3x^4\sin 3y\,dy$ (b) $dz = 2e^{2y}\{2\cos 4x\,dx + \sin 4x\,dy\}$ (c) $dz = xw^2\{2yw\,dx + xw\,dy + 3xy\,dw\}$ **2** (a) $z = x^3y^4 + 4x^2 - 5y^3$ (b) $z = x^2\cos 4y + 2\cos 3x + 4y^2$ (c) not exact differential

3 9 square units **4** (a) 278·6 (b) $\pi/2$ (c) 22·5 (d) 48 (e) -21 (f) -54π **5** Area $= \frac{5}{12}$ square units **6** (a) 2 (b) 0

Further problems 14 (page 615)

1 14 **2** 1·6 **3** $\dfrac{\pi}{36}\{9 - 4\sqrt{3}\}$ **4** $\dfrac{1}{2}\{\pi^4 + 4\}$ **5** $\dfrac{9\pi}{256}$ **6** $\dfrac{1}{2}\cdot\ln 2$

7 $2 - \pi/2$ **8** $\dfrac{1}{8}$ **9** 14 **10** (a) 39·24 (b) 0 **11** $\dfrac{2}{3}$

Test exercise 15 (page 658)

1 $4\sqrt{2}\pi$ **2** $a(\pi/2)^2$ **3** (a) (1) (4·47, 0·464, 3) (2) (5·92, 0·564, 0·322) (b) (1) (3·54, 3·54, 3) (2) $(-0{\cdot}832, 1{\cdot}82, 3{\cdot}46)$ **4** 12π

5 $a^3(8 - 3a)\pi/12$ **6** (a) $I = \displaystyle\iint v(1+u)(1+u+v)\,dv\,dv$

(b) $I = \displaystyle\iiint \frac{(2u+v)(v-4w)}{vw}\,du\,dv\,dw$

Further problems 15 (page 658)

1 $4\sqrt{5}\pi$ **2** $\left(\dfrac{a}{2}, \dfrac{a}{2}, \dfrac{a}{2}\right)$ **3** $10\sqrt{61}$ **4** $\dfrac{4\sqrt{22}\pi}{3}$ **5** $\dfrac{\pi}{24}(5\sqrt{5} - 1)$

6 $\pi\sqrt{5}$ **7** $16a^2$ **8** $2a^2(\pi - 2)$ **9** $4\pi(a+b)\sqrt{a^2 - b^2}$ **10** 45π

11 $\frac{11}{30}$ **12** $\frac{\pi a^4}{2}$ **13** $2\left(\pi - \frac{4}{3}\right)$ **14** $\bar{x} = \bar{y} = \bar{z} = \frac{3a}{8}$

15 $\frac{\pi a^3}{3}\{4\sqrt{2} - 3\}$ **16** $\frac{4\pi abc}{3}$ **17** $\frac{2a^3}{3}$ **18** $\frac{1}{4}\int\int (u^2 + v^2)\,du\,dv$

19 $u^2 v\,du\,dv\,dw$ **20** $\bar{z} = -\frac{a}{5}$ **21** $\frac{7}{18}$ **22** $2 - \frac{\pi}{2}$ **23** $\frac{1}{4}(\sqrt{2} - 1)$

Test exercise 16 (page 694)

1 (a) $\frac{20}{3}$ (b) $\frac{2}{3}$ (c) -2 (d) 120 (e) $\frac{15\sqrt{\pi}}{2048}$ **2** (a) $\frac{256}{315}$ (b) $\frac{1}{40}$ (c) $\frac{2}{105}$

3 (a) $\frac{1}{\sqrt{2}} \cdot K\left(\frac{1}{\sqrt{2}}\right)$ (b) $\frac{1}{2} \cdot F\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **4** (a) 0 (b) 1 **5** (a) $F\left(\sqrt{2}, \frac{\pi}{4}\right)$

(b) $\frac{1}{2}F\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Further problems 16 (page 694)

1 (a) 6 (b) $-\frac{1}{2}$ (c) 0·4 (d) 24 (e) $\frac{315}{4}$ **2** (a) 6 (b) $\frac{8}{81}$ (c) $\frac{\sqrt{2\pi}}{16}$

(d) 4 **4** (a) $\frac{1}{8960}$ (b) $\frac{\sqrt{2}\pi}{64}$ (c) $\frac{8}{315}$ (d) $\frac{2}{7}$ (e) $\frac{1}{63}$ (f) $\frac{\pi}{432} = 0{\cdot}00727$

8 (a) $\sqrt{5} \cdot E\left(\frac{2}{\sqrt{5}}\right)$ (b) $\sqrt{2} \cdot K\left(\frac{1}{\sqrt{2}}\right) = 2{\cdot}622$ (c) $2 \cdot E\left(\frac{1}{2}, 1\right) = 2{\cdot}935$

(d) $\frac{1}{4} \cdot F\left(\frac{3}{4}, \frac{2}{3}\right) = 0{\cdot}193$ (e) $\frac{1}{\sqrt{5}} \cdot F\left(\frac{2}{\sqrt{5}}, 1\right)$ (f) $\frac{1}{\sqrt{2}} \cdot F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right)$

(g) $\frac{1}{\sqrt{2}} \cdot \left\{F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right) - F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)\right\}$ **9** $\frac{1}{2} \cdot \left\{F\left(\frac{\sqrt{3}}{2}, \frac{\pi}{2}\right) - F\left(\frac{\sqrt{3}}{2}, \frac{\pi}{4}\right)\right\}$

10 (a) $\frac{1}{\sqrt{3}} \cdot F\left(\frac{1}{\sqrt{3}}, \frac{1}{2}\right) = 0{\cdot}307$ (b) $\frac{1}{\sqrt{3}} \cdot \left\{F\left(\frac{1}{\sqrt{3}}, 1\right) - F\left(\frac{1}{\sqrt{3}}, \frac{1}{2}\right)\right\}$

(c) $\frac{1}{\sqrt{34}} \cdot K\left(\frac{3}{\sqrt{34}}\right) = 0{\cdot}2905$ (d) $\frac{1}{\sqrt{7}}\left\{F\left(\sqrt{\frac{3}{7}}, \frac{\pi}{2}\right) - F\left(\sqrt{\frac{3}{7}}, \frac{\pi}{6}\right)\right\}$

Text exercise 17 (page 741)

1 (a) -15 (b) $-16\mathbf{i} + 10\mathbf{j} + 17\mathbf{k}$ **2** (a) 9 (b) $-(47\mathbf{i} + 17\mathbf{j} + 29\mathbf{k})$

3 $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{0}$ $\therefore$ vectors coplanar **4** (a) $4\mathbf{i} - 4\mathbf{j} + 24\mathbf{k}$

(b) $2\mathbf{i} - 2\mathbf{j} + 24\mathbf{k}$ (c) 24·66 **5** $\mathbf{T} = \frac{1}{\sqrt{66}}(4\mathbf{i} + \mathbf{j} + 7\mathbf{k})$

6 $\frac{8}{5}(25\mathbf{i} - 6\mathbf{j} - 15\mathbf{k})$ **7** 5·08 **8** $\frac{1}{\sqrt{101}}(2\mathbf{i} + 4\mathbf{j} + 9\mathbf{k})$

9 (a) $14\mathbf{i} - 12\mathbf{j} - 30\mathbf{k}$ (b) 8 (c) $5\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$ (d) $7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
(e) $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Further problems 17 (page 741)

1 61 **2** $29\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$ **3** (a) $22\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}$ (b) $-2\mathbf{i} + 14\mathbf{j} - 22\mathbf{k}$

4 (a) $2x\mathbf{i} + 3\mathbf{j} + \cos x\,\mathbf{k}$ (b) $2\mathbf{i} - \sin x\,\mathbf{k}$ (c) $(4x^2 + 9 + \cos^2 x)^{1/2}$
(d) $34 + \sin 2$ **5** (a) $2 - 2u - 9u^2$
(b) $(3u^2 + 4u + 3)\mathbf{i} + (3u^2 + 6)\mathbf{j} + (1 - 2u)\mathbf{k}$ (c) $\mathbf{i} - 2\mathbf{j} + (3 - 2u)\mathbf{k}$

6 $\frac{1}{5\sqrt{21}}(2\mathbf{i} - 20\mathbf{j} + 11\mathbf{k})$ **7** $\frac{-1}{\sqrt{129}}(10\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$

8 $\dfrac{-1}{\sqrt{126}}(5\mathbf{i}-\mathbf{j}+10\mathbf{k})$ **9** $\dfrac{-1}{\sqrt{601}}(12\mathbf{i}+4\mathbf{j}-21\mathbf{k})$ **10** $-8{\cdot}285$
11 $-9{\cdot}165$ **12** (a) 15 (b) -33 (c) 7 **13** (a) $-6\mathbf{i}+4\mathbf{j}-7\mathbf{k}$
(b) $62\mathbf{i}+10\mathbf{j}-38\mathbf{k}$ (c) $18\mathbf{i}-21\mathbf{j}+10\mathbf{k}$ **14** (a) $12\mathbf{i}-4\mathbf{j}+4\mathbf{k}$
(b) $24\mathbf{i}-4\mathbf{j}$ (c) 144 **15** (a) $(2\sin 2)\mathbf{i}+2e^3\mathbf{j}+(\cos 2+e^3)\mathbf{k}$
(b) $(4\sin^2 2+\cos^2 2+2e^3\cos 2+5e^6)^{1/2}$ **16** $-5{\cdot}014$
17 $p=\dfrac{1}{\sqrt{29}}(3\mathbf{i}+2\mathbf{j}-4\mathbf{k})$; $q=\dfrac{1}{\sqrt{38}}(6\mathbf{i}-\mathbf{j}+\mathbf{k})$; $\theta=68^\circ\ 48'$
18 (a) $(2t+3)\mathbf{i}-(6\cos 3t)\mathbf{j}+6e^{2t}\mathbf{k}$ (b) $2\mathbf{i}+(18\sin 3t)\mathbf{j}+12e^{2t}\mathbf{k}$ (c) $12{\cdot}17$
20 $-4x\mathbf{i}+4z\mathbf{k}$ **21** $(2\cos 5{\cdot}5)\mathbf{i}-(6\sin 5{\cdot}5)\mathbf{j}-(6\sin 5{\cdot}5)\mathbf{k}$ **22** $p=6$
23 (a) (1) $p=15/4$ (2) $p=-33$ (b) $\dfrac{1}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$

Test exercise 18 (page 791)

1 $3\mathbf{i}+\dfrac{18}{7}\mathbf{j}-\dfrac{81}{8}\mathbf{k}$ **2** 12 **3** $18\pi(2\mathbf{i}+\mathbf{j})$ **4** $24(\mathbf{i}+\mathbf{j})$ **5** $8+\dfrac{4\pi}{3}$
6 all conservative **7** $36\left(\dfrac{\pi}{4}+1\right)$ **8** 0

Further problems 18 (page 792)

1 (a) $576\mathbf{k}$ (b) $\dfrac{576}{5}(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$ **2** $1771\mathbf{i}+1107\mathbf{j}+830{\cdot}4\mathbf{k}$
3 $416{\cdot}1\mathbf{i}+718{\cdot}5\mathbf{j}+5679\mathbf{k}$ **4** $46{\cdot}9$ **5** $-4{\cdot}18$ **6** 8π **7** $\dfrac{16\pi}{3}(\mathbf{i}+\mathbf{k})$
8 $\dfrac{1}{3}(48\mathbf{i}+64\mathbf{j}-24\mathbf{k})$ **9** $64\left(\dfrac{\pi}{4}-\dfrac{1}{3}\right)(6\mathbf{i}+4\mathbf{j})$
10 $\dfrac{9}{2}\{(\pi+2)\mathbf{i}+(\pi+2)\mathbf{j}+4\mathbf{k}\}$ **11** $\dfrac{12}{5}(32\mathbf{j}+15\mathbf{k})$ **12** -1 **13** $\dfrac{250}{3}\pi$
14 $\dfrac{1}{6}(117\pi+256-28\sqrt{7})=91{\cdot}58$ **15** -80 **16** 96π **17** -2 **18** 12π
19 $-\dfrac{a^3}{3}$ **20** $\dfrac{81\pi}{4}$

Test exercise 19 (page 819)

1 yes, an orthogonal set **2** $h_u=1,\ h_v=2v,\ h_\theta=2u$ **3** $4\mathbf{I}+\mathbf{K}$
4 (a) $(2\cos\phi+2\cos 2\phi+1)$ (b) $(2\sin 2\phi+\sin\phi)\mathbf{K}$
5 (a) $(\mathrm{d}s)^2=(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2+r^2\sin^2\theta\,(\mathrm{d}\phi)^2$ (b) $\mathrm{d}V=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$
6 $-10{\cdot}5$

Further problems 19 (page 820)

1 (a) yes (b) no **2** $-50{\cdot}5$ **3** $2\dfrac{5}{18}$
5 (a) $\nabla^2 V=\dfrac{\partial^2 V}{\partial\rho^2}+\dfrac{1}{\rho}\cdot\dfrac{\partial V}{\partial\rho}+\dfrac{1}{\rho^2}\cdot\dfrac{\partial^2 V}{\partial\phi^2}+\dfrac{\partial^2 V}{\partial z^2}$
(b) $\nabla^2 V=\dfrac{1}{r^2}\cdot\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\cdot\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right)+\dfrac{1}{r^2\sin^2\theta}\cdot\dfrac{\partial^2 V}{\partial\phi^2}$
6 (b) $h_u=h_v=\sqrt{u^2+v^2};\ h_w=1$

(c) $\operatorname{div} F = \dfrac{1}{u^2+v^2}\left\{\dfrac{\partial}{\partial u}\left(\sqrt{u^2+v^2}\cdot\dfrac{\partial F_u}{\partial u}\right)+\dfrac{\partial}{\partial v}\left(\sqrt{u^2+v^2}\cdot\dfrac{\partial F_v}{\partial v}\right)\right\}+\dfrac{\partial F_w}{\partial w}$

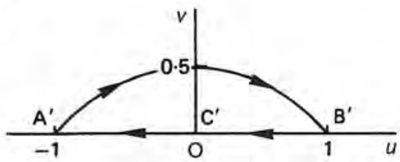
(d) $\nabla^2 V = \dfrac{1}{u^2+v^2}\left\{\dfrac{\partial^2 V}{\partial u^2}+\dfrac{\partial^2 V}{\partial v^2}\right\}+\dfrac{\partial^2 V}{\partial w^2}$

Test exercise 20 (page 858)

1 (a) $w = 6 - j2$ (b) $w = 3 - j2$ (c) $w = j3$ (d) $w = 2$

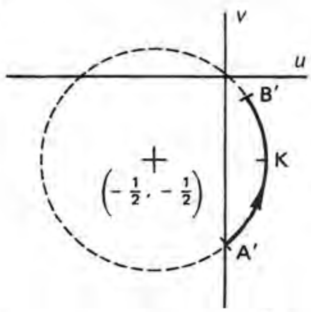
2 Magnification $= 2{\cdot}236$; rotation $= 63^\circ 26'$; translation $= 1$ unit to right, 3 units downwards

3



$v = \dfrac{1}{2}(1-u^2)$

4



Minor arc of circle, centre $\left(-\dfrac{1}{2}, -\dfrac{1}{2}\right)$, radius $\dfrac{1}{\sqrt{2}} = 0{\cdot}7071$, between A′ $(-j)$ and B $(0{\cdot}12 - j0{\cdot}16)$

5 (a) centre $\left(u = 0,\ v = \dfrac{2}{3}\right)$ (b) radius $\dfrac{1}{3}$

6 centre $\left(u = \dfrac{2}{3},\ v = 0\right)$; radius $\dfrac{2}{3}$

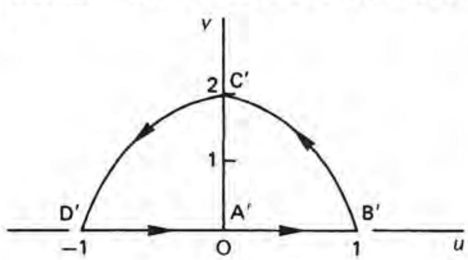
Further problems 20 (page 859)

1 Triangle A′B′C′ with A′ $(-1+j2)$, B′ $(5+j2)$, C′ $(2+j5)$

2 (a) A′ $(-8+j9)$; B′ $(23+j14)$
(b) Magnification $= \sqrt{29} = 5{\cdot}385$; rotation $= 68^\circ 12'$; translation = nil

3 Straight line joining A′ $(5-j7)$ to B′ $(-3-j)$; magnification $= 3{\cdot}162$; rotation $= 161^\circ 34'$ anticlockwise; translation $= 2$ to right, 4 upwards

4



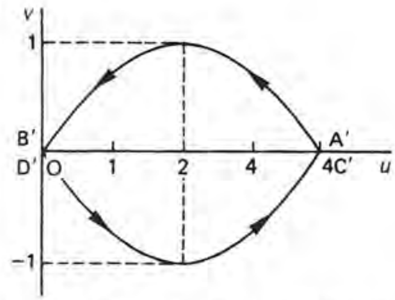
A′ B′: $v = 0$

B′C′: $u = 1 - \dfrac{v^2}{4}$

C′D′: $u = \dfrac{v^2}{4} - 1$

D′A′: $v = 0$

5



A'B' and C'D': $v = \frac{1}{4}(4u - u^2)$

B'C' and D'A': $v = \frac{1}{4}(u^2 - 4u)$

6 A' $(1 - j2)$; B' $(-23 + j10)$; C' $(1 - j8)$ A'B': $u = 2 - \frac{v^2}{4}$; B'C': $v = \frac{(u-1)^2}{32} - 8$; C'A': $u = 1$ **7** circle, centre $\left(\frac{1}{2} - j\frac{2}{3}\right)$, radius $\frac{7}{6}$

8 (a) circle, centre $\left(\frac{1}{3} - j0\right)$, radius $\frac{2}{3}$ (b) region outside the circle in (a)

9 circle, centre $\left(\frac{3}{2} + j0\right)$, radius 1; clockwise development

10 circle, $u^2 + v^2 - \frac{22u}{5} + \frac{8}{5} = 0$, centre $\left(\frac{11}{5} + j0\right)$, radius $\frac{9}{5}$

11 circle, $u^2 + v^2 - \frac{u}{2} = 0$, centre $\left(\frac{1}{4} + j0\right)$, radius $\frac{1}{4}$; region inside this circle

12 circle, centre $\left(-\frac{7}{3} + j0\right)$, radius $\frac{5}{3}$

13 (a) circle, centre $\left(\frac{3}{5}, 0\right)$, radius $\frac{2}{5}$, developed clockwise

(b) region outside the circle in (a)

14 $v = -\frac{u}{3}$

Test exercise 21 (page 906)

1 (a) regular at all points (b) $z = -5$ (c) regular at all points (d) $z = -1$ and $z = 4$ (e) $z = 0$, where $z = x + jy$
2 (a) $v(x, y) = \cosh x \cos y + C$ (b) $v(x, y) = 6(y^2 - x^2) - 4x + C$ **4** $j4\pi$
5 (a) $z = 0$ (b) $z = \pm 1$ (c) no critical point (d) $z = \pm\sqrt{2}$ (e) $z = 0$ (f) no critical point **6** $w = \cosh\frac{\pi z}{4}$; D': $w = 1$

Further problems 21 (page 907)

3 circle, centre $(5, -2)$, radius $\sqrt{2}$ **4** circle, centre $\left(-\frac{1}{3}, 0\right)$, radius $\frac{2}{3}$, anticlockwise **5** (a) $v(x, y) = 2y(x - 1) + C$
(b) $v(x, y) = 3x^2y - y^3 - 2xy + y + C$ (c) $v(x, y) = x^2 - 2x - y^2 + C$
(d) $v(x, y) = e^{x^2 - y^2}\sin 2xy + C$ **6** (a) $j10\pi$ (b) $j6\pi$ **7** (a) 0 (b) $j4\pi$
(c) $j10\pi$ **9** $j2\pi$ **10** $j10\pi$ **11** (a) (1) $z = 0$ (2) $z = \pm 1$
(b) ellipse, centre $(0, 0)$, semi major axis $\frac{5}{2}$, semi minor axis $\frac{3}{2}$

12 (a) $u^2+v^2=1$ (b) $u^2+(v-1)^2=2; \theta=45°$. **13** Unit circle becomes the real axis on the w-plane. Region within the circle maps onto the upper half plane **14** $w=\sin\dfrac{z\pi}{2a}$

Test exercise 22 (page 936)

1 (a) $f(z)=1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\ldots+\dfrac{z^n}{n!}+\ldots$ valid for $|z|<\infty$

(b) $f(z)=4z-\dfrac{(4z)^2}{2}+\dfrac{(4z)^3}{3}-\ldots+\dfrac{(-1)^{n+1}(4z)^n}{n}+\ldots$ valid for $|z|<1/4$ **2** (a) pole of order 5 at $z=-1$ (b) essential singularity at $z=0$ (c) essential singularity at $z=0$ (d) removable singularity at $z=0$

3 $$f(z)=\frac{1}{\sqrt{2}}\left\{1+(z-\pi/4)-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+\frac{(z-\pi/4)^4}{4!}+\frac{(z-\pi/4)^5}{5!}-\ldots\right\};$$ valid for $|z|<\infty$

4 (a) $f(z)=-(z+3)+8+\dfrac{1}{2(z+3)}-\dfrac{4}{(z+3)^2}-\dfrac{1}{24(z+3)^3}+\dfrac{1}{3(z+3)^4}+\ldots;$ essential singularity

(b) $f(z)=\dfrac{3}{z+3}-\dfrac{1}{z+1}=\ldots-\dfrac{1}{z^3}+\dfrac{1}{z^2}-\dfrac{1}{z}+1-\dfrac{z}{3}+\dfrac{z^2}{9}-\dfrac{z^3}{27}+\ldots$

(c) $f(z)=\dfrac{1}{8(z-2)^2}-\dfrac{3}{16(z-2)}+\dfrac{3}{16}-\dfrac{5(z-2)}{32}+\dfrac{15(z-2)^2}{64}+\ldots;$ pole of order 2 **5** double pole at $z=0$; residue -4, double pole at $z=-1$, residue $7/2$, single pole at $z=1$, residue $1/2$ **6** (a) $-\pi/6$ (b) $2\pi/\sqrt{3}$ (c) $2\pi e^{-3}$

Further problems 22 (page 937)

1 (a) $z+\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+\ldots+\dfrac{z^{2n+1}}{(2n+1)!}+\ldots,\ |z|<\infty$

(b) $z+\dfrac{z^3}{3}+\dfrac{2z^5}{15}+\dfrac{17z^7}{315}+\ldots,\ |z|<\pi/2$

(c) $2\left\{z+\dfrac{z^3}{3}+\dfrac{z^5}{5}+\ldots+\dfrac{z^{2n+1}}{2n+1}+\ldots\right\},\ |z|<1$

(d) $1+z\ln a+\dfrac{z^2(\ln a)^2}{2!}+\dfrac{z^3(\ln a)^3}{3!}+\ldots+\dfrac{z^n(\ln a)^n}{n!}+\ldots,\ |z|<\infty$

(e) $\dfrac{3z^2}{25}+\dfrac{27z^3}{125}+\dfrac{162z^4}{625}+\dfrac{810z^5}{3125}+\ldots,\ |z|<5/3;$

$-\dfrac{5}{9z}-\dfrac{25}{9z^2}-\dfrac{250}{27z^3}-\dfrac{6250}{243z^4}-\ldots,\ |z|>5/3$ **3** (b) $-\dfrac{z}{(z+1)^2}, \dfrac{z(z-1)}{(z+1)^3}$

4 (a) convergent for $|z|<\infty$ (b) convergent for $|z|<1$ (c) convergent for $|z|<1$ (d) convergent for $|z|<1$ (e) convergent for $|z|<\infty$

5 (a) $e^2\left\{1+(z-2)+\dfrac{(z-2)^2}{2!}+\dfrac{(z-2)^3}{3!}+\ldots+\dfrac{(z-2)^n}{n!}+\ldots\right\}$

(b) $\dfrac{\sqrt{3}}{2}-\dfrac{(z-\pi/6)}{2}-\dfrac{\sqrt{3}(z-\pi/6)^2}{2\times 2!}+\dfrac{(z-\pi/6)^3}{2\times 3!}+\dfrac{\sqrt{3}(z-\pi/6)^4}{2\times 4!}+\ldots$

(c) $(z-3)\sin 6 + (z-3)^2\cos 6 - \dfrac{(z-3)^3\sin 6}{2!} - \dfrac{(z-3)^4\cos 6}{3!}$
$+\dfrac{(z-3)^5\sin 6}{4!} + \dots$ (d) $-\left\{\dfrac{3}{13} + 2\left(\dfrac{3}{13}\right)^2(z-1/3)\right.$
$\left.+4\left(\dfrac{3}{13}\right)^3(z-1/3)^2 + \dots + 2^n\left(\dfrac{3}{13}\right)^{n+1}(z-1/3)^n + \dots\right\}$
(e) $1 - 2(z-3) + 4(z-3)^2 + \dots + (-2)^n(z-3)^n + \dots$

6 $(z-1) + \dfrac{(z-1)^2}{1\times 2} - \dfrac{(z-1)^3}{2\times 3} + \dfrac{(z-1)^4}{3\times 4} - \dfrac{(z-1)^5}{4\times 5} + \dots$ **7** (a) $z=\infty$ (b) $|z| = \sqrt{6}$ (c) $|z-5| = 1$ (d) $z = \infty$ **8** (a) poles of order 2 at $z=0$ and $z=-1$, removable singularity at $z=\pm 1$ (b) essential singularity at $z=0$ **9** (a) $\dfrac{1}{z^2} - \dfrac{1}{z^4 3!} + \dfrac{1}{z^6 5!} - \dfrac{1}{z^8 7!} + \dots, |z| > 0$
(b) $\dfrac{1}{2}\left(z - \dfrac{3}{2}\right)^{-1}, |2z-3| > 0$
(c) $\dfrac{3}{z-3} - 2\{1 - (z-3) + (z-3)^2 - (z-3)^3 + \dots\}, 0 < |z-3| < 1$

10 (a) $\dots + \dfrac{8}{z^4} - \dfrac{4}{z^3} + \dfrac{2}{z^2} - \dfrac{1}{z} + \dfrac{2}{5} - \dfrac{2z}{25} + \dfrac{2z^2}{125} - \dfrac{2z^3}{625} + \dots$
(b) $\dfrac{1}{z} - \dfrac{8}{z^2} + \dfrac{46}{z^3} - \dfrac{242}{z^4} + \dots$ (c) $-\dfrac{1}{10} + \dfrac{17z}{100} - \dfrac{109z^2}{1000} + \dfrac{593z^3}{10000} - \dots$

11 (a) $2\pi/\sqrt{3}$ (b) $\dfrac{2\pi}{\sqrt{\alpha^2-\beta^2}}$ (c) $\dfrac{2\pi}{|\alpha^2 - 1|}$ (d) $\pi/4$ (e) $\pi/2$ (f) $\pi/2$
(g) $\pi\sqrt{\sqrt{13}/8 - 3/8}$ (h) $\pi/4$ (i) $2\pi/\sqrt{3}$ (j) $2\pi/3$ (k) 0 (l) 0

Test exercise 23 (page 983)

1 $P_{max} = 10$ $(x=4,\ y=3)$ **2** $P_{max} = 13$ $(x=1,\ y=4)$
3 $P_{max} = 188$ $(x=10, y=4, z=6)$ **4** $P_{max} = 296$ $(x=4,\ y=6)$
5 $P_{min} = 16$ $(x=5,\ y=12)$ **6** (a) 13 type A + 4 type B (b) £11,800

Further problems 23 (page 984)

1 $P_{max} = 32$ $(x=4,\ y=9/2)$ **2** $P_{max} = 64$ $(x=0,\ y=8)$
3 $P_{max} = 40$ $(x=6,\ y=5/2)$ **4** $P_{max} = 15$ $(x=6,\ y=3)$
5 $P_{max} = 9$ $(x=1,\ y=3)$ **6** $P_{max} = 10$ $(x=2,\ y=4)$
7 $P_{max} = 10$ $(x=2,\ y=4)$ **8** $P_{max} = 37$ $(x=0,\ y=8,\ z=1)$
9 $P_{max} = 67$ $(x=4,\ y=10,\ z=5)$ **10** $P_{max} = 65$ $(x=5,\ y=10,\ z=5)$
11 $P_{max} = 11{\cdot}568$ $(x=29/22,\ y=14/11,\ z=0)$ to 3 *s.f.*
12 $P_{max} = 340$ $(x=30,\ y=20)$ **13** $P_{max} = 112$ $(x=4,\ y=8)$
14 $P_{max} = 108$ $(x=16,\ y=15)$ **15** $P_{min} = 138$ $(x=12,\ y=18)$
16 $P_{max} = 240$ $(x=9,\ y=15)$ **17** $P_{max} = 4400$ $(x=201,\ y=53)$
18 $P_{max} = 100$ $(x=20,\ y=10)$ **19** $P_{max} = 410$ $(x=9,\ y=5,\ z=2)$
20 $P_{max} = 1560$ $(x=11,\ y=10,\ z=18)$
21 $P_{max} = 660$ $(x=60,\ y=30,\ z=30)$ **22** $P_{min} = -14$ $(x=5,\ y=2)$
23 $P_{min} = 56$ $(x=8,\ y=12)$ **24** $P_{min} = 16$ $(x=8,\ y=6)$
25 $P_{min} = 40$ $(x=4,\ y=4)$ **26** $P_{min} = -10$ $(x=6,\ y=13,\ z=14)$
27 $P_{min} = -75$ $(x=8,\ y=12,\ z=21)$
28 (a) 10 type A + 35 type B (b) £2150
29 (a) 22 type A + 44 type B + 48 type C (b) £12,580
30 (a) 129 type A + 0 type B + 185 type C; (b) £8955

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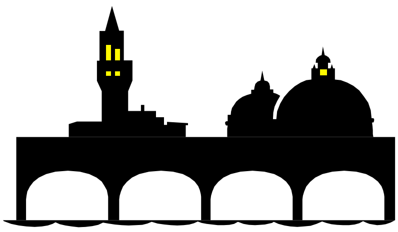
THE SIGNS BEFORE THE DAY OF JUDGEMENT

Author: Al-Haafidh Ibn Katheer Hadeeth
Checking: Abu Talhah Daawood Burbank

Ayaat and Ahaadeeth about the Hour



"My Lord! Increase me in knowledge!" - Qur'an 20:114"

Allaah (subhaanahu wa ta'aala) said:
"They ask you (O Muhammad sal-Allaahu 'alayhe wa sallam) about the Hour, - when will be its appointed time? You have no knowledge to say anything about it, To your Lord belongs (the knowledge of) the term thereof? You (O Muhammad sal-Allaahu 'alayhe wa sallam) are only a warner for those who fear it, The Day they see it, (it will be) as if they had not tarried (in this world) except an afternoon or a morning." [An-Nazi'aat (79):42-46]

"They ask you about the Hour (Day of Resurrection): 'When will be its appointed time?' Say: 'The knowledge thereof is with my Lord (Alone). None can reveal its time but He. Heavy is its burden through the heavens and the earth. It shall not come upon you except all of a sudden.' They ask you as if you have a good knowledge of it. Say: 'The knowledge thereof is with Allaah (Alone) but most of mankind know not.'" [Al-A'raaf (7):187]

There are many Ayaat and Ahaadeeth concerning this subject: Allaah said:
"The Hour has drawn near, and the moon has been cleft asunder." [Al-Qamar (54):1]
The Prophet (sal-Allaahu 'alayhe wa sallam) said, whilst pointing with his index and middle fingers,
"The time of my advent and the Hour are like these two fingers." [1]
In another report he said,
"The Hour almost came before me."

This indicates how close we are, relatively speaking, to the Hour.
Allaah said:
"Draws near for mankind their reckoning, while they turn away in heedlessness." [Al-Anbiyaa' (21):1]
"The Event (the Hour or the punishment of disbelievers and polytheists or the Islaamic laws or commandments), ordained by Allaah will come to pass, so seek not to hasten it. Glorified and Exalted be He above all that they associate as partners with Him." [An-Nahl (16):1]

"Those who believe not therein seek to hasten it, while those who believe are fearful of it, and know that it is the very truth. Verily, those who dispute concerning the Hour are certainly in error far away." [Ash-Shooraa (42):18]
In Saheeh al-Bukhaaree, there is a Hadeeth which states that a Bedouin asked the Prophet (sal-Allaahu 'alayhe wa sallam) about the Hour.

He (sal-Allaahu 'alayhe wa sallam) said,
"It will surely come to pass. What have you prepared for it?" The man said, "O Messenger of Allaah, I have not prepared much in the way of prayer and good works, but I love Allaah and His Messenger." The Prophet (sal-Allaahu 'alayhe wa sallam) said, "You will be with those you love." The Muslims had never rejoiced as much they did when they heard this Hadeeth. [2]
Some Ahaadeeth report that the Prophet (sal-Allaahu 'alayhe wa sallam) was asked about the Hour. He looked towards a young boy and said,
"If he lives, he will not grow very old before he sees your Last Hour coming to you." [3]

By this he meant their death and entering the Hereafter, because everyone who dies enters the Hereafter; some people say that when a person has died, his judgment has begun. This Hadeeth with this meaning is "correct" (Saheeh).
Some heretics comment on this Hadeeth and give it an incorrect meaning. The exact timing of the Great Hour (as-Saa'at al-'Uzmaa) is something which Allaah alone knows and which He has not revealed to anyone, as is clear from the Hadeeth in which the Prophet (sal-Allaahu 'alayhe wa sallam) said:
"There are five things which nobody knows except Allaah;" then he recited, "Verily, Allaah! With Him (Alone) is the knowledge of the Hour, He sends down the rain, and knows that which is in the wombs. No person knows what he will earn tomorrow, and no person knows in what land he will die. Verily, Allaah is All Knower, All Aware (of things)."" [Luqman (31):34] [4]
When Gabriel (Jibreel) (alayhi-salaam) came to the Prophet (sal-Allaahu 'alayhe wa sallam) in the guise of a Bedouin, he asked him about Islaam, Eemaan (faith) and Ihsaan (excellence of faith); and the Prophet (sal-Allaahu 'alayhe wa sallam) answered his questions. But when he asked him about the Hour, he said,

"The one questioned about it knows no better than the questioner." Jibreel said, "Tell me about its signs." Then the Prophet (sal-Allaahu 'alayhe wa sallam) described them, as we shall see later when we quote this Hadeeth and others in full.
Hudhayfah said:
"The Prophet (sal-Allaahu 'alayhe wa sallam) stood up one day to speak to us, and told us everything that was going to happen until the Hour, and left nothing unsaid. Some of the listeners learnt it by heart, and some forgot it; these friends of mine learnt it. I do not remember it completely, but sometimes it springs to mind, just as one might remember and recognise the face of a man whom one had forgotten, when one sees him." [Abu Daawood, Muslim] [5]
Imaam Ahmad reported via Abu Nudrah that Abu Sa'eed said:
"One day the Prophet (sal-Allaahu 'alayhe wa sallam) led us in praying the afternoon prayer (Salaat al-'Asr). Then he stood and addressed us until sunset. He mentioned everything that was to happen until the Day of Resurrection, and left nothing unsaid. Some of us remembered it, and some of us forgot it. One of the things he said was:
'O people, this world is full of attractive temptations. Allaah has appointed you as vicegerents (Khaleefah) in this world, and He will see how you will act. So guard yourselves against the temptations of this world and of women.'
Towards the end of this speech, he said,
'The sun is about to set, and what remains of this world, compared to what has passed, is like what remains of this day compared to what has passed.' [6]"
'Alee ibn Zayd ibn Jad'an al-Timi narrated some Gharib and Munkar Ahaadeeth - which could bring into question the validity of this Hadeeth. But there are some reports which are similar to this Hadeeth, and which were transmitted with different isnaads. Part of this Hadeeth is in Saheeh Muslim, through Abu Nudrah on the authority of Abu Sa'eed. This Hadeeth refers to something which is beyond any doubt: what remains of this world, compared to what has passed, is very little. In spite of that, no-one can know exactly how much time is left except Allaah, and no-one can know exactly how much time has passed, except Allaah.

Footnotes
[1] Bukhaaree, Kitaab at-Tafsir, commentary on Surat an-Nazi'aat, 6/206.
[2] A similar Hadeeth was narrated by Bukhaaree in Kitaab al-Adab.
[3] See Bukhaaree, Kitaab al-Adab; Muslim, Kitaab al-Fitan wa Ashraat

al-Sa'ah.
[4] Bukhaaree, Kitaab at-Tafsir, commentary on Luqman 31:34. A longer Hadeeth is narrated by Muslim in Kitaab al-Eemaan.
[5] Muslim, Kitaab al-Fitan wa Ashraaat al-Sa'ah; Abu Daawood, Kitaab al-Fitan wa'l-Malahim.
[6] The whole speech is narrated by Imaam Ahmad in his Musnad, 2/61. Checker's Note: Da`eef, al-Musnad (3/61) [or 2/61 as stated in the book??], at-Tirmidhee and al-Hakim (4/505). Da`eef due to `Alee ibn Zayd ibn Jad'an. al-Mishkat: 5145 of al-Albaanee. Parts of the complete speech are authentic, see Ibn Kathir's comments.
General Description of the Fitan (Tribulations)
Hudhayfah ibn al-Yaman said,
"People used to ask the Prophet (sal-Allaahu 'alayhe wa sallam) about good things, but I used to ask him about bad things because I was afraid that they might overtake me. I said, 'O Messenger of Allaah, we were lost in ignorance (Jahiliyyah) and evil, then Allaah brought this good (i.e. Islaam). Will some evil come after this good thing?' He said, 'Yes' I asked, 'And will some good come after that evil?' He said, 'Yes, but it will be tainted with some evil' I asked, 'How will it be tainted?' He said, 'There will be some people who will lead others on a path different from mine. You will see good and bad in them." I asked, 'Will some evil come after that good?' He said, 'Some people will be standing and calling at the gates of Hell; whoever responds to their call, they will throw him into the Fire.' I said, 'O Messenger of Allaah, describe them for us.' He said, 'They will be from our own people, and will speak our language.' I asked, 'What do you advise me to do if I should live to see that?' He said, 'Stick to the main body (jamaa'ah) of the Muslims and their leader (Imaam). I asked, What if there isno main body and no leader?' He said 'Isolate yourself from all of these sects, even if you have to eat the roots of trees until death overcomes you while you are in that state.' [1]"
'Abd Allaah ibn Mas'ood said:
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Islaam began as something strange, and it will revert to being strange as it was in the beginning, so good tidings for the strangers.' Someone asked, 'Who are the strangers?' He said, 'The ones who break away from their people (literally, 'tribes') for the sake of Islaam.'" This Hadeeth was narrated by Ibn Majah on the authority of Anas and Abu Hurairah. [2]

Footnotes
[1] Bukhaaree, Kitaab al-Fitan, 9/65.

[2] Muslim, Kitaab al-Eemaan, 1/90: Ibn Majah, Kitaab al-Fitan (Hadeeth 3988), 2/1320.

Divisions Within the Main Religious Groups

Abu Hurairah reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said:

"The Jews have split into seventy-one sects, and my Ummah will divide into seventy-three." [1]

'Awf ibn Malik reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said:

"The Jews split into seventy-one sects: one will enter Paradise and seventy will enter Hell. The Christians split into seventy-two sects: seventy-one will enter Hell and one will enter Paradise. By Him in Whose hand is my soul, my Ummah will split into seventy-three sects: one will enter Paradise and seventy-two will enter Hell." Someone asked, "O Messenger of Allaah, who will they be?" He replied, "The main body of the Muslims (al-Jamaa'ah)."

'Awf ibn Malik is the only one who reported this Hadeeth, and its isnaad is acceptable. [2]

Anas ibn Malik said,

"I shall tell you a Hadeeth which I heard from the Messenger of Allaah, and which no-one will tell you after me. I heard him say, 'Among the signs of the Hour will be the disappearance of knowledge and the appearance of ignorance. Adultery will be prevalent and the drinking of wine will be common. The number of men will decrease and the number of women will increase until there will be fifty women to be looked after by one man.'" This Hadeeth was reported in the two Saheehs from the Hadeeth of 'Abd Rabbihi. [3]

'Abd Allaah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Just before the Hour, there will be days in which knowledge will disappear and ignorance will appear, and there will be much killing.' " [Ibn Maajah; also narrated by Bukhaaree and Muslim, from the Hadeeth of al-A'mash] [4]

Hudhayfah ibn al-Yaman said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Islaam will become worn out like clothes are, until there will be no-one who knows what fasting, prayer, charity and rituals are. The Qur'aan will disappear in one night, and no Ayah will be left on earth. Some groups of old people will be left who will say, 'We heard our fathers saying La ilaha illa Allaah, so we repeated it.' Silah asked Hudhayfah, "What will saying La ilaha illa Allaah do for them when they do not know what prayer, fasting, ritual and charity

are?" Hudhayfah ignored him; then Silah repeated his question three times, and each time Hudayfah ignored him. Finally he answered, "O Silah, it will save them from Hell", and said it three times. [Ibn Maajah] [5]
This indicates that in the last days, knowledge will be taken from the people, and even the Qur'aan will disappear from the Mushafs and from people's hearts. People will be left without knowledge. Only the old people will tell them that they used to hear people saying La ilaha illa Allaah; and they will repeat it to feel close to Allaah, so it will give them some blessing, even if they do not have any good deeds or beneficial knowledge. Knowledge will be taken away from men and ignorance will increase during the last days, and their ignorance and misguidance will increase until the end, as in the Hadeeth of the Prophet (sal-Allaahu 'alayhe wa sallam):
"The Hour will not come upon anyone who says, 'Allaah, Allaah'; it will only come upon the most evil of men." [6]

Footnotes
[1] Ibn Maajah, Kitaab al-Fitan (Hadeeth 3991), 2/1321.
Checker's Note: Saheeh, Saheeh Ibn Maajah: 3225
[2] Abu Daawood, Kitaab as-Sunnah, (Hadeeth 4572, 4573), 12/1340-2. "The main body of the Muslims (al-Jamaa'ah)" means the people of the Qur'aan, Hadeeth, Fiqh and other sciences, who have agreed to follow the Traditions of the Prophet (sal-Allaahu 'alayhe wa sallam) in all circumstances without introducing any changes or imposing their own confused ideas.
Checker's Note: Saheeh, Saheeh al-Jami as-Sagheer: 1082, the narration of `Awf ibn Malik is also reported by Ibn Maajah (no. 4992), Ibn Abee `Asim in as-Sunnah (no. 63) and al-Lalikai in Sharh us-Sunnah (no. 1492).
[3] A similar Hadeeth was narated by al-Bukhaaree in Kitaab al-'Ilm, 1/30,31; and by Muslim.
Checker's Note: Saheeh, al-Bukhari and Muslim.
[4] Bukhaaree, Kitaab al-Fitan, 9/61; Msulim, Kitaab al-'Ilm, 8/58.
[5] Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 87.
[6]The first part of it was related by Muslim in Kitab al-Eemaan, 1/91, and the second part in Kitab al-Fitan wa Ashraat al-Sa'ah, 8/208.

The Evils Which Will Befall the Muslim Ummah During The Last Days

'Abd Allaah ibn 'Umar said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) came to us and said, 'O Muhaajiroon, (emigrants from Makkah to al-Madinah) you may be

afflicted by five things; God forbid that you should live to see them. If fornication should become widespread, you should realise that this has never happened without new diseases befalling the people which their forebears never suffered. If people should begin to cheat in weighing out goods, you should realise that this has never happened without drought and famine befalling the people, and their rulers oppressing them. If people should withhold Zakaat, you should realise that this has never happened without the rain being stopped from falling; and were it not for the animals' sake, it would never rain again. If people should break their covenant with Allaah and His Messenger, you should realise that his has never happened without Allaah sending an enemy against them to take some of their possessions by force. If the leaders do not govern according to the Book of Allaah, you should realise that this has never happened without Allaah making them into groups and making them fight one another.'" [Ibn Maajah] [1]
'Alee ibn Abee Taalib said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said: 'If my Ummah bears fifteen traits, tribulation will befall it.' Someone asked, 'What are they, O Messenger of Allaah?' He said, 'When any gain is shared out only among the rich, with no benefit to the poor; when a trust becomes a means of making a profit; when paying Zakaat becomes a burden; when a man obeys his wife and disobeys his mother; and treats his friend kindly whilst shunning his father; when voices are raised in the mosques; when the leader of a people is the worst of them; when people treat a man with respect because they fear some evil he may do; when much wine is drunk; when men wear silk; when female singers and musical instruments become popular; when the last ones of this Ummah curse the first ones - then let them expect a red wind, or the earth to swallow them, or to be transformed into animals.'" [Tirmidhee] [2]
'Alee ibn Abee Taalib said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) led us in praying Salaat al-Fajr (the morning prayer). When he had finished, a man called to him: 'When will the Hour be? The Prophet (sal-Allaahu 'alayhe wa sallam) reprimanded him and said 'Be quiet!' After a while he raised his eyes to the sky and said, 'Glorified be the One Who raised it and is taking care of it.' Then he lowered his gaze to the earth and said, 'Glory be to the One Who has outspread it and has created it.' Then the Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Where is the one who asked me about the Hour?' The man knelt down and said, 'I asked you.' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will come when leaders are

oppressors, when people believe in the stars and reject al-Qadar (the Divine Decree of destiny) when a trust becomes a way of making a profit, when people give to charity (Sadaqah) reluctantly, when adultery becomes widespread - when this happens, then your people will perish.'" [3]
'Imran ibn Husayn said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Some people of this Ummah will be swallowed up by the earth, some will be transformed into animals, and some will be bombarded with stones.' One of the Muslims asked, 'When will that be, O Messenger of Allaah?' He said, 'When singers and musical instruments will become popular, and much wine will be drunk.'"[4]

Footnotes
[1] Narrated by Ibn Maajah, Kitaab al-Fitan (Hadeeth 4019), 2/1332.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 106-107.
[2] Tirmidhee, Abwaab al-Fitan (Hadeeth 308), 6/4620-458.
Checker's Note: Da`eef, Da`eef al-Jami as-Sagheer: 608.
[3] al-Haythami, Kitaab al-Fitan.
Checker's Note: al-Haythami, says, "al-Bazzar reports it and it contains narrators I do not know." We do not know the status of the hadeeth as no muhaddith (as far as we know) has given a definite verdict.
[4] Narrated by at-Tirmidhee.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 1604.

The Greater Signs of the Hour

After the lesser signs of the Hour appear and increase, mankind will have reached a stage of great suffering. Then the awaited Mahdee will appear; he is the first of the greater, and clear, signs of the Hour. There will be no doubt about his existence, but this will only be clear to the knowledgeable people. The Mahdee will rule until the False Messiah (al-Maseeh ad-Dajjaal) appears, who will spread oppression and corruption. The only ones who will know him well and avoid his evil will be those who have great knowledge and Eemaan (faith).
The false Messiah will remain for a while, destroying mankind completely, and the earth will witness the greatest Fitnah (tribulation) in its history. Then the Messiah Jesus (alayhi salam) will descend, bringing justice from heaven. He will kill the Dajjaal, and there will be years of safety and security.
Then the appearance of Ya'jooj and Ma'jooj (Gog and Magog) will take mankind by surprise, and corruption will overtake them again. In answer

to Jesus' faithful prayer to Allaah (subhanahu wa ta'aala), they will die, and safety, security, justice and stability will return.
This state of affairs will continue for some years, until the death of Jesus.
The Ulamaa differ concerning the order in which the other greater signs of the Hour will come about. They are:
- The destruction of the Ka'bah and the recovery of its treasure.
- The rising of the sun from the west.
- The emergence of the Beast from the earth.
- The smoke.
- A wind will take the souls of the believers.
- The Qur'aan will be taken up into heaven.
- A fire will drive the people to their last gathering place.
- The Trumpet will be sounded: at the first sound everyone will feel terror; at the second sound all will be struck down; at the last sound all will be resurrected.

The Mahdee

The Mahdee will come at the end of time; he is one of the Rightly-Guided Caliphs and Imaams. He is not the Mahdee who is expected by the Shee'ah, who they claim will appear from a tunnel in Saamarraa. This claim of theirs has no basis in reality nor in any reliable source. They allege that his name is Muhammad ibn al-Hasan ibn al-Askaree, and that he went into the tunnel when he was five years old.
The matter we intend to discuss has been proven by Ahaadeeth narrated from the Prophet (sal-Allaahu 'alayhe wa sallam): that the Mahdee will appear at the end of time. I believe that he will appear before Jesus the son of Mary comes down, as the Ahaadeeth indicate.
Hajjaaj said that he heard Alee say,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, "Even if there were only one day left for the world, Allaah would send a man from among us to fill the world with justice, just as it had been filled with oppression and justice." [Ahmad] [1]
Alee said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, "The Mahdee is one of us, from among the people of my household. In one night Allaah will inspire him and prepare him to carry out his task successfully." [Ahmad and Ibn Maajah] [2]
Alee said, whilst looking at his son al-Hasan,
This son of mine is a Sayyid (master), as the Prophet (sal-Allaahu 'alayhe wa sallam) named him. Among his descendants there will be man named after your Prophet (sal-Allaahu 'alayhe wa sallam). He will resemble him

in behaviour but not in looks. Then he told them the report which mentions that the earth will be filled with justice. [Abu Daawood] [3]
Abu Daawood devoted a chapter of his Sunan to the subject of the Mahdee. At the beginning of this chapter he quoted the Hadeeth of Jaabir ibn Samrah, in which the Prophet (sal-Allaahu 'alayhe wa sallam) said, "This religion will remain steadfast until twelve caliphs have ruled over you." (According to another report he said, "This religion will remain strong until twelve caliphs have ruled over you.") Jaabir said, "The people cheered and shouted Allaahu Akbar! Then the Prophet (sal-Allaahu 'alayhe wa sallam) whispered something. I asked my father 'What did he say?' My father said, 'He said, All ofthem will be from Quraysh.'" Another report says that when the Prophet (sal-Allaahu 'alayhe wa sallam) returned to his house, Quraysh came to him and asked, "What will happen after that?" He said, "Then there will be tribulation and killing." [4]
Abu Daawood reported a Hadeeth from Abd Allaah ibn Masood:
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, "If there were only one day left for the world, that day would be lengthened until a man from among my descendants or from among the people of my household, was sent; his name will be the same as my name, and his fathers name will be the same as my fathers name. He will fill the earth with justice and fairness, just as it will have been filled with injustice and oppression. The world will not end until a man of my household, whose name is the same as mine, holds sway." [5]
Abd Allaah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'A man from my household, whose name is like mine, will take power.'" [Tirmidhee] [6]
In another report, from Abu Hurairah, the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"If there were only one day left for this world, Allaah would lengthen it until he took power." [7]
Aboo Sa'eed said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Mahdee will be one of my descendants; he will have a high forehead and a hooked nose. He will fill the earth with justice and fairness just as it was filled with injustice and oppression, and he will rule for seven years.'" [Abu Daawood] [8]
Umm Salamah said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say,'The Mahdee will be one of my descendants, from the children of Faatimah.'" [Abu Daawood] [9]

Umm Salamah reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"People will begin to differ after the death of a Khaleefah. A man from the people of Madinah will flee to Makkah. Some of the people of Makkah will come to him and drag him out against his will; they will swear allegiance to him between al-Rukn and al-Maqam. An army will be sent against him from Syria; it will be swallowed up in the desert between Makkah and Madinah. When the people see this, groups of people from Syria and Iraq will come and swear allegiance to him. Then a man from Quraysh whose mother is from Kalb will appear and send an army against them, and will defeat them; this will be known as the Battle of Kalb. Whoever does not witness the spoils of this battle will miss much! The Mahdee will distribute the wealth, and will rule the people according to the Sunnah of the Prophet (sal-Allaahu 'alayhe wa sallam). Then he will die, and the Muslims will pray for him." [Abu Daawood] [10]
Alee said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'A man named al-Haarith ibn Hirath will come from Transoxania. His army will be led by a man named Mansoor. He will pave the way for and establish the government of the family of Muhammad, just as Quraysh established the government of the Messenger of Allaah. Every believer will be obliged to support him.'" [Abu Daawood] [11]
Abd Allaah ibn al-Haarith ibn Juz' al-Zubaydee said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'A people will come out of the East who will pave the way for the Mahdee.'" [Ibn Maajah] [12]
Abd Allaah said,
"Whilst we were with the Prophet (sal-Allaahu 'alayhe wa sallam), some young men from Banu Hashim approached us. When the Prophet (sal-Allaahu 'alayhe wa sallam) saw them, his eyes filled with tears and the colour of his face changed. I said, 'We can see something has changed in your face, and it upsets us.' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'We are the people of a Household for whom Allaah has chosen the Hereafter rather than this world. The people of my Household (Ahl al-Bayt) will suffer a great deal after my death, and will be persecuted until a people carrying black banners will come out of the east. They will instruct the people to do good, but the people will refuse; they will fight until they are victorious, and the people do as they asked, but they will not accept it from them until they hand over power to a man from my household. Then the earth will be filled with fairness, just as it had been filled with injustice. If any of you live to see this, you should go

to him even if you have to crawl across ice.'" [13]
This text refers to the rule of the Abbasids, as we have mentioned above in the text referring to the beginningof their rule in 132 AH. It also indicates that the Mahdee will appear after the Abbasids, and that he will be one of the Ahl al-Bayt, a descendant of Faatimah, the daughter of the Prophet (sal-Allaahu 'alayhe wa sallam), through Hasan, not Husayn, as mentioned in the Hadeeth from Alee ibn Abee Taalib; and Allaah knows best. [14]
Thwaban said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said,'Three men will be killed at the place where your treasure is. Each of them will be the son of a Khaleefah, and none of them will get hold of the treasure. Then the black banners will come out of the East, and they will slaughter you in a way which has never been seen before.' Then he said something which I do not remember; then, 'If you see him, go and give him your allegiance, even if you have to crawl over ice, because he is the Khaleefah of Allaah, the Mahdee.'" [Ibn Maajah] [15]
The treasure referred to in this text is the treasure of the Ka'bah. Towards the end of time, three of the sons of the Khaleefahs will fight to get hold of it, until the Mahdee appears. He will appear from the East, not from the tunnel of Saamarraa, as the Shee'ah claim; they believe that he is in this tunnel now, and they are waiting for him to emerge at the end of time. There is no evidence for it in any book or Saheeh tradition, and there is no benefit in believing this.
The truth of the matter is that the Mahdee whose coming is promised at the end of time will appear from the East,and people will swear allegiance to him at the Ka'bah, as some Ahaadeeth indicate.
At the time of the Mahdee, there will be peace and prosperity, with abundant crops and wealth, strong rulers, and Islaam will be well-established.
Aboo Sa'eed said,
"By Allaah every ruler we have had has been worse than the previous one, and every year has been worse than the year before, but I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'Among your rulers will be one who will give out wealth freely, without counting it. A man will come and ask him for money, and he will say, "Take"; the man will spread his cloak out and the ruler will pour money into it.' The Prophet (sal-Allaahu 'alayhe wa sallam) spread out a thick cloak he had been wearing, to demonstrate the mans actions; then he gathered it up by its corners and said, 'Then the man will take it and leave.'" [Ahmad] [16]

Footnotes
[1] Ahmad, al-Musnad; similar Hadeeth in Abu Daawood, Kitaab Awwal al-Mahdee.
Checker's Note: Saheeh, Ahmad and Abu Daud, Saheeh al-Jami as-Sagheer: 5305.
[2] Ahmad, al-Musnad and Ibn Maajah, Kitaab al-Fitan.
Checker's Note: Saheeh Saheeh al-Jami as-Sagheer: 6735. It has been mistranslated slightly, it should be "In one night, Allaah will prepare him," the mention of inspiration and the carrying out of the task is explanatory.
[3] See Abu Daawood, Kitaab al-Mahdee.
Checker's Note: Da`eef Munqati' (chain of narration is broken), al-Mishkat, 1st checking: 5462.
[4] Checker's Note: "... returned to his house" is a da`eef addition. The rest of the hadeeth is saheeh.
[5] See Abu Daawood, Kitaab al-Mahdee.
Checker's Note: Saheeh, Saheeh Abee Daawood: 3601.
[6] Checker's Note: "A man from my household ..." Hasan, Saheeh al-Jami as-Sagheer: 8160.
[7] at-Tirmidhee, in his chapters dealing with al-Fitan.
Checker's Note: Saheeh, Saheeh at-Tirmidhee: 1819.
[8] Abu Daawood, Kitaab al-Mahdee.
Checker's Note: Hasan, Saheeh Abee Daawood: 3604. Translation states, "... a hooked nose." Correct translation is "... an aquiline nose."
[9] Abu Daawood, Kitaab al-Mahdee.
Checker's Note: Saheeh, Saheeh Abee Daawood: 3603.
[10] Abu Daawood, Kitaab al-Mahdee.
Checker's Note: Da`eef, al-Mishkat, 1st checking: 5456.
[11] Checker's Note: Da`eef, Da`eef al-Jami as-Sagheer: 6418.
[12] Ibn Maajah, Kitaab al-Fitan (Hadeeth 3088).
Checker's Note: Da`eef, Silsilatul Ahaadieeh ad-Da`eefah: 4826.
[13] Ibn Maajah, ibid., (Hadeeth 4082).
Checker's Note: Da`eef, Da`eef Ibn Maajah: 886.
[14] See Ahaadeeth above. (Ref. 1, 2 & 3)
Checker's Note: Footnote refers to hadeeth no. 3.
[15] Narrated by Ibn Maajah, op. cit., Hadeeth 4084.
Checker's Note: Da`eef Munkar (the isnaad is weak and moreover the meaning is wrong). In this case, the mention of "Khaleefah of Allaah" is munkar because Allaah cannot have a successor or vicegerent in His absence, Silsilatul Ahaadeeth ad-Da`eefah: 85.

[16] Ahmad, Musnad, 3/98.
Checker's Note: Da`eef. Its isnaad contains Mujahid ibn Sa'eed
Different Kinds of Fitan (Tribulations)
Zaynab bint Jahsh said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) got up from his sleep; his face was flushed and he said, 'There is no god but Allaah. Woe to the Arabs, for a great evil which is nearly approaching them. Today a gap has been made in the wall of Gog and Magog like this (Sufyan illustrated this by forming the number of 90 or 100 with his fingers).' Someone asked, 'Shall we be destroyed even though there are righteous people among us?' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Yes, if evil increases.'" [Bukhaaree] [1]
Umm Salamah, the wife of the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"One night the Prophet (sal-Allaahu 'alayhe wa sallam) got up and said, 'SubhanAllaah! How many tribulations have come down tonight, and how many treasures have been disclosed! Go and wake the dwellers of these apartments (i.e. his wives) for prayer. A well-dressed soul in this world may be naked in the Hereafter.'" [Bukhaaree] [2]
Usamah ibn Zayd said,
"Once the Prophet (sal-Allaahu 'alayhe wa sallam) stood over one of the battlements of al-Madinah and asked the people, 'Do you see what I see?' They said, 'No.' He said, 'I see afflictions falling upon your houses as rain drops fall.'" [Bukhaaree and Muslim] [3]
Abu Hurairah (radiallahu anhu) said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Time will pass rapidly, knowledge will decrease, miserliness will become widespread in peoples hearts, afflictions will appear, and there will be much Harj.' The people asked, 'O Messenger of Allaah, what is Harj?' He said, 'Killing, killing!'" [Bukhaaree] [4]
Al-Zubayr ibn Adee narrated,
"We went to Anas ibn Malik and complained about the wrong we were suffering at the hands of al-Hajjaaj. Anas ibn Malik said, 'Be patient, "For no time will come but that the time following it will be worse, until you meet your Lord." I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say that.'" [Bukhaaree]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There will come a time of afflictions when one who sits will be better than one who stands; one who stands will be better than one who walks; and one who walks

will be better than one who runs. Whoever exposes himself to these afflictions, they will destroy him. So whoever can find a place of protection or refuge from them, should take shelter in it.'" [Bukhaaree and Muslim] [5]
Hudhayfah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) told us two Hadeeth, one of which I have seen fulfilled, and I am waiting for the fulfilment of the other. The Prophet (sal-Allaahu 'alayhe wa sallam) told us that honesty came down into mens hearts (from Allaah); then they learnt it from the Qur'aan, and then from the Sunnah. The Prophet (sal-Allaahu 'alayhe wa sallam) told us that honesty would be taken away. He said, 'Man will be overtaken by sleep, during which honesty will be taken away from his heart, and only its trace will remain, like traces of a dark spot. Then man will be overtaken by slumber again, during which honesty will decrease still further, until its trace will resemble a blister such as is caused when an ember is dropped onto ones foot: it swells, but there is nothing inside. People will be carrying on with their trade, but there will hardly be any trustworthy persons. People will say, There is an honest man in such-and-such a tribe. Later they will say about some man, What a wise, polite and strong man he is! - although he will not have faith even the size of a mustard-seed in his heart.' Indeed, there came a time when I did not mind dealing with any one of you, for if he were a Muslim his Islaam would compel him to pay whatever he owed me, and if he were a Christian, the Muslim official would compel him to pay it. But now I do not deal with anyone except so-and-so and so-and-so." [Bukhaaree] [6]
Ibn Umar said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) stood beside the pulpit, facing the east, and said, 'Afflictions will verily emerge from here, where the top of Satans head will appear.'" [Bukhaaree] [7]
Abu Hurairah said that he had heard the Prophet (sal-Allaahu 'alayhe wa sallam) say,
"The Hour will not come until a man passes by someone's grave and says, 'Would that I were in his place!'" [Bukhaaree] [8]
Abu Hurairah said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'The Hour will not come until the buttocks of the women of Daws move whist going around Dhoo l-Khalasah.'" Dhoo l-Khalasah was an idol worshipped by the tribe of Daws during the Jaahiliyyah. [Hadeeth from Bukhaaree] [9]
Abu Hurairah also said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Euphrates will

disclose a golden treasure. Whoever is present at that time should not take anything of it.'" [10]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come before the Euphrates uncovers a mountain of gold, for which people will fight. Ninety-nine out of every hundred will die, but every one among them will say that perhaps he will be the one who will survive (and thus possess the gold).'" [Muslim] [11]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam)said, 'The Hour will not come until the following events have come to pass: two large groups will fight the another, and there will be many casualties; they will both be following the same religious teaching. Nearly thirty Dajjaals will appear, each of them falsely claiming to be a Messenger from Allaah. Knowledge will disappear, earthquakes will increase, time will pass quickly, afflictions will appear, and Harj (ie killing) will increase. Wealth will increase, so that a wealthy man will worry lest no-one accept his Zakaat, and when he offers it to anyone, that person will say, "I am not in need of it." People will compete in constructing high buildings. When a man passes by someones grave, he will say, "Would that I were in his place!" The sun will rise from the west; when it rises and the people see it, they will believe, but,
"No good will it do to a soul to believe in them then, if it believed not before nor earned righteousness through its faith..." [Al-An'aam (6):158]
"The Hour will come suddenly: when a man has milked his she-camel and taken away the milk, but he will not have time to drink it; before a man repairing a tank for his livestock will be able to put water in it for his animals; and before a man who has raised a morsel of food to his mouth will be able to eat it." [Bukhaaree] [12]
Hudhayfah ibn al-Yaman said,
"Of all the people, I know most about every tribulation which is going to happen between now and the Hour. This is not because the Prophet (sal-Allaahu 'alayhe wa sallam) told me something in confidence which he did not tell anyone else; it is because I was present among a group of people to whom he spoke about the tribulations (al-Fitan). The Prophet (sal-Allaahu 'alayhe wa sallam) mentioned three tribulations which would hardly spare anybody, and some which would be like storms in summer; some would be great and some would be small. Everyone who was present at that gathering has passed away, except me." [Muslim] [13]
Abu Hurairah said,

"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, "If you live for a while, you will see people go out under the wrath of Allaah and come back under His curse, and they will have in their hands whips like the tail of an ox.'" [Ahmad, Muslim] [14]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There are two types among the people of Hell whom I have not yet seen. The first are people who have whips like the tails of oxen, with which they beat people, and the second are women who are naked in spite of being dressed; they will be led astray and will lead others astray, and their heads will look like camels humps. These women will not enter Paradise; they will not even experience the faintest scent of it, even though the fragrance of Paradise can be perceived from such a great distance.'" [15]
Anas ibn Malik said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) was asked, 'O Messenger of Allaah, (what will happen) when we stop enjoining good and forbidding evil?' He said, 'When what happened to the Israelites happens among you: when fornication becomes widespread among your leaders, knowledge is in the hands of the lowest of you, and power passes into the hands of the least of you.'" [Ibn Maajah] [16]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Woe to the Arabs from the great evil which is nearly approaching them: it will be like patches of dark night. A man will wake up as a believer, and be a kafir (unbeliever) by nightfall. People will sell their religion for a small amount of worldly goods. The one who clings to his religion on that day will be as one who is grasping an ember - or thorns.'" [Ahmad] [17]
Abu Hurairah said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) saying to Thawbaan, 'O Thawban, what will you do when the nations call one another to invade you as people call one another to come and eat from one bowl?' Thawbaan said, 'May my father and my mother be sacrificed for you, O Messenger of Allaah! Is it because we are so few?' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'No, on that day you (Muslims) will be many, but Allaah will put weakness (wahn) in your hearts.' The people asked, 'What is that weakness, O Messenger of Allaah?' He said, 'It is love for this world and dislike of fighting.'" [Ahmad] [18]
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"There will be a tribulation in which one who is sleeping will be better than one who is lying down, one who is lying will be better than one who is

sitting, one who is sitting will be better than one who is standing, one who is standing will be better than one who is walking, one who is walking will be better than one who is riding, and one who is riding will be better than one who is running; all of their dead will be in Hell." The Companion of the Prophet (sal-Allaahu 'alayhe wa sallam) who narrated this Hadeeth said, "O Messenger of Allaah, when will that be?" He said, "That will be the days of Harj." He asked, "When will the days of Harj be?" The Prophet (sal-Allaahu 'alayhe wa sallam) said, "When a man will not trust the person to whom he is speaking." The Companion asked, "What do you advise me to do if I live to see that?" He said, "Restrain yourself, and go back to your place of residence." The Companion then asked, "O Messenger of Allaah, what should I do if someone enters my neighbourhood to attack me?" He said, "Go into your house." The Companion asked, "What if he enters my house?" He said, "Go into the place where you pray and do this - and he folded his arms, - and say 'My Lord is Allaah', until you die." [19]

Abu Bakrah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There will be a tribulation during which one who is lying down will be better than one who is sitting, one who is sitting will be better than one who is standing, one who is standing will be better than one who is walking, and one who is walking will be better than one who is running.' Someone asked, 'O Messenger of Allaah, what do you advise me to do?' He said, 'Whoever has camels, let him stay with them, and whoever has land, let him stay in his land.' Someone asked, 'What about someone who does not have anything like that?' He said, 'Then let him take his sword and strike its edge against a stone, then go as far away as possible.'" [Abu Daawood; similar Hadeeth in Muslim] [20]

At the time of the Fitnah of Uthmaan ibn Affaan's Khilaafah (Caliphate), Sa'd ibn Abee Waqqaas said,

"I bear witness that the Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There will come a tribulation during which one who sits will be better than one who stands, one who stands will be better than one who walks, and one who walks will be better than one who runs.' Someone asked, 'What do you advise if someone enters my house to kill me?' He said, 'Be like the son of Adam (i.e. resign yourself).'" [Muslim, Tirmidhee] [21]

Abu Moosaa al-Asharee said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Before the Hour comes, there will be a tribulation like patches of dark night. A man will get up a believer and go to sleep a kaafir, or will go to sleep a believer and

get up a kaafir. The one who sits will be better than one who stands, and one who walks will be better than one who runs. Break your bows, cut their strings, and strike your swords against stones. If someone comes to kill any of you, then be like the better of the twos ons of Adam.'" [Abu Daawood] [22]
Abu Dharr said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) was riding a donkey and sat me behind him. He said, 'O Abu Dharr, if the people were suffering from such severe hunger that you could not even get up from your bed to go to the mosque, what would you do?' I said, 'Allaah and His Messenger know best.' He said, 'Be decent and restrain yourself.' Then he said, 'O Abu Dharr, if the people were suffering from severe death (i.e. if a man were worth no more than a grave), what would you do? If the people were killing one another, until Hajarat al-Zayt (an area of Madinah) were submerged in blood, what would you do?' I said, 'Allaah and His Messenger know best.' He said, 'Stay in your house and lock the door.' I asked, 'What if I am not left alone?' He said, 'Then be one of them.' I said, 'Should I take up my sword?' He said, 'If you did that, you would be joining them in their activities. No - if you fear that the brightness of the shining sword will disturb you, then cover your face with part of your clothing, and let him carry his own sin and your sin.'" [Ahmad] [23]
Abd Allaah ibn Amr said,
"We were on a journey with the Prophet (sal-Allaahu 'alayhe wa sallam)...When the Prophet (sal-Allaahu 'alayhe wa sallam)'s caller called for prayer, I went there. The Prophet (sal-Allaahu 'alayhe wa sallam) was addressing the people, saying: "O people, it has been the duty of every Prophet before me to guide his people to whatever he knew was good for them, and to warn them against whatever he knew was bad for them, but this Ummah has its time of peace and security at the beginning; at the end of its existence it will suffer trials and tribulations, one after the other. Tribulation will come, and the believer will say, "This will finish me", but it will pass. Another tribulation will come, and he will say, "This is it", but it will pass, and a third will come and go likewise. Whoever wishes to be rescued from Hell, and enter Paradise, let him die believing in Allaah and the Last Day, and treat the people as he himself wishes to be treated. If anyone gives allegiance to an Imaam, then let him obey him if he can (or on one occasion he said: as much as you can).'"
Abd al-Rahman (one of the narrators of this Hadeeth) said,
"When I heard that, I put my head between my knees and said, 'But your cousin Mu'aawiyah is ordering us to squander our wealth among

ourselves in vanity, and to kill each other, although Allaah has said,
"O you who believe! Squander not your wealth among yourselves in vanity..." ' [an-Nisaa (4):29]
Abd Allaah (another narrator) put his head in his hands and paused a while, then he raised his head and said,
"Obey him in that which is obedience to Allaah and disobey that which is disobedience to Allaah." I asked him, "Did you hear that from the Prophet (sal-Allaahu 'alayhe wa sallam)?" He said, "Yes, I heard it with my ears and understood it in myheart." [Ahmad, Abu Daawood, an-Nasaa'ee, Ibn Maajah] [24]
Abd Allaah ibn Amr said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'If you see my Ummah fearing a tyrant so much that they dare not tell him that he is a tyrant, then there will be no hope for them.'"
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Among my Ummah, some will be swallowed up by the earth, some bombarded with stones, and some transformed into animals." [Ahmad] [25]

Footnotes
[1] Bukhaaree, Kitaab al-Fitan.
[2] Bukhaaree, Kitaab al-'Ilm.
[3] Bukhaaree, Kitaab al-Fitan, 9/60; Muslim, Baab Nuzuul al-Fitan ka-Mawaaqi' al-Qatar, 8/168.
[4] Bukhaaree, ibid.
[5] op. cit., 9/64; Muslim, Baab Nuzuul al-Fitan ka-Mawaaqi' al-Qatar.
[6] Bukhaaree, op. cit., 9/66.
[7] Bukhaaree, Kitaab Bid' al-Khalq, 4/150.
[8] Bukhaaree, Kitaab al-Fitan, 9/73.
[9] Bukhaaree, Kitaab al-Fitan, 8/132
Daws: a tribe in Yemen; Dhoo l-Khalasah: a house full of idols - it is so called because they believed that whoever worshipped it or went around it would be purified (khallasa). This Hadeeth means that the tribe of Daws will become apostates from Islaam and will go back to idol worshipping; even their women will exert themselves in worshipping the idol and running around it, so that their flesh will quiver.
[10] Bukhaaree, Kitaab al-Fitan, 9/73.
[11] Muslim, Kitaab al-Fitan wa Ashraat al-Saa'ah, 8/174.
[12] Bukhaaree, Kitaab al-Fitan, 9/74.
[13] Muslim, Kitaab al-Fitan wa Ashraat al-Saa'ah, 8/172.

[14] Muslim, Kitaab al-Jannah wa Sifat Na'imihaa wa Ahlihaa, 8/155, 156.
[15] Muslim, ibid.
[16] Ibn Maajah narrated a similar Hadeeth in Kitaab al-Fitan (Hadeeth 4015), 2/1331. Ahmad, Musnad, 3/187.
Checker's Note: Da`eef due to the `an`anah (usage of the word `an (from)) of Makhool, Da`eef Ibn Maajah: 870.
[17] Ahmad, Musnad, 2/390.
Checker's Note: Saheeh. Weak isnaad but supported by other narrations similar in meaning. Musnad Imaam Ahmad, checking by Ahmad Shakir: 2/390.
[18] Ahmad, Musnad, 2/359.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 958. Wording should be "dislike of death".
[19] Ahmad, Musnad, 1/448.
Checker's Note: Saheeh. Musnad Imaam Ahmad, checking by Ahmad Shakir: [??] Ahmad's narration contains an unnamed narrator who is named in the narration of Abdur Razzaaq, no. 20727 and al-Hakim, 4/320, and he is reliable.
[20] Abu Daawood, ibid. (Hadeeth 4236). Muslim, Kitaab al-Fitan, 8/169.
Checker's Note: Saheeh, Saheeh Abee Daawood: 3580.
[21] Tirmidhee, Abwaab al-Fitan, (Hadeeth 2290), 6/436, 438. Ahmad, Musnad, 1/185.
Checker's Note: Saheeh, al-Irwa, no. 2451 of Shaykh al-Albaanee. It is saheeh but has not been reported by Muslim, as main text states.
[22] Abu Daawood, Abwaab al-Fitan wa'l-Malaahim, (Hadeeth 4139), 11/337.
Checker's Note: Saheeh, Abu Daawood and Ibn Maajah, Silsilatul Ahadeeth as-Saheehah: 1535.
[23] Ahmad, Musnad, 5/149; similar Hadeeth in Abu Daawood, Abwaab al-Fitan wa'l-Malaahim, (Hadeeth 4241), 11-340, 343.
Checker's Note: Saheeh, al-Irwa, no: 2451.
[24] Ahmad, Musnad, 5/149. Muslim, Kitaab al-Imaarah, 6-18. Ibn Maajah, Kitaab al-Fitan (Hadeeth 3956), 2/1306, 1037. an-Nasaa'ee, Kitaab al-Bay'ah (shorter version), 7-152,153. Abu Daawood, Kitab al-Fitan (shorted version), Hadeeth 4429, 11-319.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 241.
[25] Ahmad, Musnad. 2/163.
Checker's Note: Da`eef, Silsilatul Ahaadeeth ad-Da`eefah: 1264. Its isnaad contains the `an`anah of Abu Zubair, but its last part ("Among my Ummah ...") is supported. See Silsilatul Ahadeeth as-Saheehah: 1787

The Signs and Portents of the Hour
'Abd Allaah ibn 'Amr said,
"I went to the Prophet (sal-Allaahu 'alayhe wa sallam) one day whilst he was performing Wudoo' (ablution) slowly and carefully. He raised his head, looked at me and said, 'Six things will happen to this Ummah: the death of your Prophet - 'and when I heard that I was aghast,' - this is the first. The second is that your wealth will increase so much that if a man were given ten thousand, he would still not be content with it. The third is that tribulation will enter the house of every one of you. The fourth is that sudden death will be widespread. The fifth is a peace-treaty between you and the Romans: they will gather troops against you for nine months - like a woman's period of childbearing - then they will be the first to break the treaty. The sixth is the conquest of a city.' I asked, 'O Messenger of Allaah, which city?' He said, 'Constantinople.' " [Ahmad] [1]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Hasten to do good deeds before six things happen: the rising of the sun from the West, the smoke, the Dajjaal, the beast, the (death) of one of you, or general tribulation.' " [Ahmad, Muslim] [2]
Hudhayfah ibn 'Ubayd said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) came upon us whilst, we were busy in discussion. He asked us, 'What are you talking about?' We said, 'We are discussing the Hour.' He said, 'It will not come until you see ten signs: the smoke, the Dajjaal, the beast, the sun rising from the West, the descent of Jesus son of Mary, Gog and Magog, and three land-slides - one in the East, one in the West, and one in Arabia, at the end of which fire will burst forth from the direction of Aden (Yemen) and drive peopl eto the place of their final assembly.' " [Ahmad] [3]

Footnotes
[1] Ahmad, Musnad, 2/174.
Checker's Note: Da`eef, Musnad Imaam Ahmad, checking by Ahmad Shakir: 6623. Weak hadeeth due to Abra lamah al-Kalbi.
[2] Muslim, Kitaab al-Fitan, 8/207, Ahmad, Musnad, 2/337, 372.
[3] Muslim, Kitaab al-Fitan, 8/179.
Checker's Note: Wrong spelling, Ibn `Ubayd should be Ibn Usayd. Reported by Muslim, not by Ahmad as main text states.
The Battle with the Romans
After the battle with the Romans, which ended with the conquest of Constantinople, the Dajjaal will appear, and Jesus son of Mary will

descend from Heaven to the earth, to the white minaret in the east of Damascus, at the time of Salaat al-Fajr (the morning prayer), as we shall see in the Saheeh Traditions.

Dhoo Mukhammar said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'You will make a peace-treaty with the Romans, and together you will invade an enemy beyond Rome. You will be victorious and take much booty. Then you will camp in a hilly pasture; one of the Roman men will come and raise a cross and say "Victory to the Cross", so one of the Muslims will come and kill him. Then the Romans will break the treaty, and there will be a battle. They will gather an army against you and come against you with eighty banners, each banner followed by 10,000 men.' " [Ahmad, Abu Daawood, Ibn Maajah] [1]

Yusayr ibn Jaabir said,

"Once there was a red storm in Kufah. A man came who had nothing to say except, 'O 'Abd Allaah ibn Mas'ood, has the Hour come?' 'Abd Allaah was sitting reclining against something, and said, 'The Hour will not come until people will not divide inheritance, nor rejoice over booty.' Pointing towards Syria, he said, 'An enemy will gather forces against the Muslims and the Muslims will gather forces against them.' I asked, 'Do you mean the Romans?' He said, 'Yes. At that time there will be very heavy fighting. The Muslims will prepare a detachment to fight to the death; they will not return unless they are victorious. They will fight until night intervenes. Both sides will return without being victorious; then many will be killed on both sides. On the fourth day, the Muslims who are left will return to the fight, and Allaah will cause the enemy to be routed. There will be a battle the like of which has never been seen, so that even if a bird were to pass their ranks, it would fall down dead before it reached the end of them. Out of a family of one hundred, only one man will survive, so how could he enjoy the booty or divide any inheritance? While they are in this state, they will hear of an even worse calamity. A cry will reach them: "The Dajjaal has taken your place among your offspring." So they will throw away whatever is in their hands and go forward, sending 10 horsemen as a scouting party. The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'I know their names, and the names of their fathers, and the colours of their horses. They will be the best horsemen on the face of the earth on that day.' " [Ahmad, Muslim] [2]

Abu Hurayrah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come until the Romans camp at al-A'mash or Dabeeq. An army, composed of the best people on earth at that time, will come out from Madinah to meet them. When they have arranged themselves in ranks, the Romans will say, 'Do not stand between us and those who took prisoners from amongst us. Let us fight with them.' The Muslims will say, 'No, by Allaah, we will never stand aside from you and our brothers.' Then they will fight. One-third will runaway, and Allaah will never forgive them. One-third will be killed, and they will be the best of martyrs in Allaah's sight. One-third, who will never be subjected to trials or tribulations, will win, and will conquer Constantinople. Whilst they are sharing out the booty, after hanging their swords on the olive-trees, Satan will shout to them that the Dajjaal has taken their place among their families. When they come to Syria, the Dajjaal will appear, while they are preparing for battle and drawing up the ranks. When the time for prayer comes, Jesus the son of Mary will descend and lead them in prayer. When the enemy of Allaah (i.e. the Dajjaal) sees him, he will start to dissolve like salt in water, but Allaah will kill him.' " [Muslim] [3]

The Prophet (sal-Allaahu 'alayhe wa sallam) said,

"The Hour will not come until the furthest border of the Muslims will be in Bulaa." Then he said, "O 'Alee!" 'Alee said, "May my father and mother be sacrificed for you!" The Prophet (sal-Allaahu 'alayhe wa sallam) said, "You will fight the Romans, and those who come after you will fight them, until the best people among the Muslims, the people of al-Hijaaz, will go out to fight them, fearing nothing but Allaah. They will conquer Constantinople with Tasbeeh and Takbeer (saying "Subhaan Allaah" and "Allaahu Akbar"), and they will obtain booty the like of which has never been seen - they will share it out by scooping it up with their sheilds. Someone will come and say, 'The Dajjaal has appeared in your land', but he will be lying. Anyone who takes notice of him will regret it, and anyone who ignores him will regret it." [Ibn Maajah] [4]

Naafi' ibn 'Utbah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'You will attack Arabia, and Allaah will enable you to conquer it. Then you will attack Persia, and Allaah will enable you to conquer it. Then you will attack Rome, and Allaah will enable you to conquer it. Then you will attack the Dajjaal, and Allaah will enable you to conquer him.' " [Muslim] [5]

When Mustawrid al-Qurashee was sitting with 'Amr ibn al-'Aas, he said, "I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'The Hour will come when the Romans will be in the majority.' 'Amr asked him, "What are you saying?" He said, "I am repeating that which I heard from the Prophet (sal-Allaahu 'alayhe wa sallam)." 'Amr said, "If you say this, it is true, because they have four good characteristics: they are the most able to cope with tribulation, the quickest to recover after disaster and to return to the fight after disaster, and are the best as far as treating the poor, weak and orphans is concerned. They have a fifth characteristic which is very good; they do not allow themselves to be oppressed by their kings." [6]

The Prophet (sal-Allaahu 'alayhe wa sallam) said,

"You will fight the Romans, and Believers from the Hijaaz will fight them after you, until Allaah enables them to conquer Constantinople and Rome with Tasbeeh and Takbeer ("Subhaan Allaah" and "Allaahu Akbar"). Its fortifications will collapse, and they will obtain booty the like of which has never been seen, so that they will share it out by scooping it up with their shields. Then someone will cry, 'O Muslims! The Dajjaal is in your country, with your families', and the people will leave the wealth. Anyone who takes notice will regret it and anyone who ignores it will regret it. They will ask, 'Who shouted?' but they will not know who he is. They will say, 'Send a vanguard to Ilyaa.' If the Dajjaal has appeared, you will hear about his deeds.' So they will go and see, and if they see that everything is normal, they will say, 'No-one would give a shout like that for no reason, so let us go together to Ilyaa'.' If we find the Dajjaal there we will fight him together, until Allaah decides between us and him. If we do not find the Dajjaal, we will go back to our country and our families.' " [7]

Mu'aadh ibn Jabal said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The building of Bayt al-Maqdis (in Jerusalem) will be followed by the destruction of Yathrib (Madeenah), which will be followed by the conquest of Constantinople, which will be followed by the appearance of the Dajjaal.' Then he put his hand on the thigh or the shoulder of the one with whom he was speaking (i.e. Mu'aadh), and said, 'This is as true as the fact that you are here (or as true as you are sitting here).' " [8]

This does not mean that Madeenah will be destroyed completely before the appearance of the Day, but that will happen at the end of time, as we shall see in some authentic Ahaadeeth. But the building of Bayt al-Maqdis will be the cause of the destruction of Madeenah, as it was proven in the Hadeeth that the Dajjaal will not be able to enter Madinah. He will be prevented from doing so because it is surrounded by angles bearing unsheathed swords. [9]

Abu Hurayrah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said about Madeenah: 'Neither plague nor the Dajjaal can enter it.' " [Bukhaaree] [10]

Footnotes
[1] Similar Hadeeth in Abu Daawood, Kitaab al-Malaahim, (Hadeeth 4271), 11/397/399. Ibn Maajah, Kitab al-Fitan (Hadeeth 4089), 2/1369. Ahmad, Musnad, 9104.
Checker's Note: Saheeh, Saheeh al-Jami as-Sagheer: 3612.
[2] Muslim, Kitaab al-Fitan, 8/177, 178. Ahmad, Musnad, 1/384, 385.
[3] Muslim, Kitaab al-Fitan, 8 - 175, 176.
Al-A'mash is a place outside the city, and Dabeeq is a market-place in the city. "The city" refers to Aleppo (some say Damascus).
[4] Ibn Maajah, Kitaab al-Fitan (Hadeeth 4094), 2 - 1370.
The one who believes the liar will regret it, because he will find that the Dajjaal is not there. The one who does not believe him will regret it because the Dajjaal will appear soon afterwards.
Checker's Note: Fabricated. The reason is Kathir ibn `Abdillah, who is declared a liar by Abu Daawood and Shaafi'ee. He narrates a fabricated collection of hadeeth.
[5] Muslim, Kitaab al-Fitan wa Ashtaar al-Saa'ah, 8 - 178. Ibn Maajah, Baab al-Malaahim (Hadeeth 4091), 2 - 1380.
[6] Muslim, Kitaab al-Fitaan, 8 - 176.
[7] Al-Haythamee, Majma' al-Zawaa'id. Ibn Maajah, 7:248.
Checker's Note: Fabricated. Same reasons as the previous one. Reported by al-Bazzar, Kashful Astar bi Zawa'id al-Bazzar of al-Haythamee: 3386.
[8] Hasan, al-Mishkat, 1st checking: 5424.
[9] Abu Daawood, Kitaab al-Malaahim (Hadeeth 4273), 11 - 400, 401; Ahmad, Musnad, 5/245.
Checker's Note: Hasan.
[10] Bukhaaree, Kitaab al-Fitan, 9/76.

The Appearance of the Dajjaal

First of all, we will quote the reports which mention the liars and "dajjaals" who will precede the coming of the Dajjaal, or Antichrist, who will be the last of them; may Allaah curse them and punish them with Hell-fire.
Jaabir ibn Samurah said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'Just before the Hour there will be many liars.' " Jaabir said, "Be on your guard against them." [Muslim] [1]
Jaabir said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'Just before the Hour there will be many liars; among them is the one in al-Yamamah, the

'Ansi in San'a', the one in Himyar, and the Dajjaal. This will be the greatest fitnah.' " [Ahmad] [2]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come ... until nearly 30 "dajjaals" (liars) appear, each one claiming to be a messenger from Allaah.' " [Bukhaaree and Muslim] [3]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come until 30 "dajjaals" appear, each of them claiming to be a messenger from Allaah, wealth increases, tribulations appear and al-Harj increases.' Someone asked, 'What is al-Harj?' He said, 'Killing, killing.' " [Ahmad] [4]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come until 30 "dajjaals" appear, all of them lying about Allaah and His Messenger.' " [Abu Daawood] [5]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Just before the Hour, there will be thirty "dajjaals", each of whom will say, I am a Prophet.' " [Ahmad] [6]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There will be "dajjaals" and liars among my Ummah. They will tell you something new, which neither you nor your forefathers have heard. Be on your guard against them, and do not let them lead you astray.' " [Ahmad] [7]
Thawbaan said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There will be thirty liars among my Ummah. Each one will claim that he is a prophet; but I am the last of the Prophets (Seal of the Prophets), and there will be no Prophet after me.' " [Ahmad] [8]
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Verily before the Day of Resurrection there will appear the Dajjaal, and thirty or more liars." [Ahmad] [9]
Ibn 'Umar said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'Among my Ummah there will be more than seventy callers, each of whom will be calling people to Hell-fire. If I wished, I could tell you their names and tribes.' " [10]
Abu Bakrah said,
"The people spoke a great deal against Musaylimah before the Prophet

(sal-Allaahu 'alayhe wa sallam) said anything about him. Then the Prophet (sal-Allaahu 'alayhe wa sallam) got up to give a speech and said: '...as for this man about whom you have spoken so much - he is one of the thirty liars who will appear before the Hour, and there is no town which will not feel the fear of the Antichrist.' " [Ahmad] [11]
In another version of this report, the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"He is one of the thirty liars who will appear before the Dajjaal. There is no town which will escape the fear of the Dajjaal, apart from Madeenah. At that time there will be two angels at every entrance of Madeenah, warding off the fear of the Antichrist." [12]
Anas ibn Malik said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The time of the Dajjaal will be years of confusion. People will believe a liar, and disbelieve one who tells the truth. People will distrust one who is trustworthy, and trust one who is treacherous; and the Ruwaybidah will have a say.' Someone asked, 'Who are the Ruwaybidah?' He said, 'Those who rebel against Allaah and will have a say in general affairs.' " [Ahmad] [13]

Footnotes
[1] Muslim, Kitaab al-Fitan, 8/189.
[2] Ahmad, Musnad, 3/345.
Checker's Note: Ahmad's isnaad contains Ibn Lahi'ah who reports by way of `an, as does Abu Zubair. Al-Bazzar's isnaad contains `Abdur-Rahman ibn Maghra and Mujaahid ibn Sa`eed. Majma' az-Zawaid of al-Haythamee: 7/335. The status of the hadeeth is unknown as no muhaddith, as far as we know, has given a definite verdict.
[3] See Part 7, where the Hadeeth is quoted in full.
[4] Ahmad, Musnad, 2 - 457.
Checker's Note: Saheeh.
[5] Ahmad, Musnad, 2/450.
Checker's Note: Hasan, Saheeh Abee Daawood: 3643.
[6] Ahmad, Musnad, 2/429.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 1683.
[7] Ahmad, Musnad, 20/349.
Checker's Note: Da`eef isnaad, but Muslim reports similar wording.
[8] Ahmad, Musnad, 5/46.
Checker's Note: Saheeh. Wrong reference, Ahmad, Musnad, 5/46, should be Ahmad, Musnad, 5/278.

[9] Ahmad, Musnad, 2/95.
Checker's Note: Hasan, Silsilatul Ahadeeth as-Saheehah: 1683, Musnad Imaam Ahmad, checking by Ahmad Shakir: 5694.
[10] Da`eef. Reported by Abu Ya'la. Its isnaad contains Laith ibn Abee Sulaim.
[11] Ahmad, Musnad, 5/41, 46.
Checker's Note: Saheeh, Majma' az-Zawaid of al-Haythamee: 7/335.
[12] Ahmad, Musnad, 5/46.
Checker's Note: Saheeh, Musnad Imaam Ahmad, checking by Ahmad Shakir: 5/41. It is part of no. 10 above, which is not given in full in the book.
[13] Ahmad, Musnad, 3/220.
Checker's Note: Saheeh. Its isnaad is hasan and is further supported, Silsilatul Ahadeeth as-Saheehah: 1887, Kashful Astar bi Zawa'id al-Bazzar of al-Haythamee: 3373.

Ahaadeeth about the Dajjaal

'Abd Allaah ibn 'Umar said,

"'Umar ibn al-Khattab went along with the Prophet (sal-Allaahu 'alayhe wa sallam) and a group of people to Ibn Sayyaad, and found him playing with some children near the battlement of Banoo Maghaalah. At that time Ibn Sayyaad was on the threshold of adolescence; he did not realise that anybody was near until the Prophet (sal-Allaahu 'alayhe wa sallam) struck him on the back. The Prophet (sal-Allaahu 'alayhe wa sallam) said to him: 'Do you bear witness than I am the Messenger of Allaah?' Ibn Sayyaad looked at him and said, 'I bear witness that you are the Prophet (sal-Allaahu 'alayhe wa sallam) of the unlettered.' Then Ibn Sayyaad said to the Prophet (sal-Allaahu 'alayhe wa sallam), 'Do you bear witness that I am the Messenger of Allaah?' The Prophet (sal-Allaahu 'alayhe wa sallam) dismissed this and said, 'I believe in Allaah and His Messengers.' Then the Prophet (sal-Allaahu 'alayhe wa sallam) asked him, 'What do you see?' Ibn Sayyaad said, 'Sometimes a truthful person comes to me, and sometimes a liar.' The Prophet (sal-Allaahu 'alayhe wa sallam) said to him, 'You are confused', then he said, 'I am hiding something from you.' Ibn Sayyaad said, 'It is Dukh.' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Silence! You will not be able to go beyond your rank.' 'Umar ibn al-Khattab said, 'O Messenger of Allaah, shall I cut off his head?' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'If he is (the Dajjaal) you will not be able to overpower him, and if he is not, then killing will not do you any good.' "

Saleem ibn 'Abd Allaah said,

"I heard 'Abd Allaah ibn 'Umar say, 'After that, the Prophet (sal-Allaahu 'alayhe wa sallam) and Ubayy ibn Ka'b went along to the palm trees where Ibn Sayyaad was. The Prophet (sal-Allaahu 'alayhe wa sallam) started to hide behind a tree, with the intention of hearing something from Ibn Sayyaad before Ibn Sayyaad saw him. The Prophet (sal-Allaahu 'alayhe wa sallam) saw him lying on a bed, murmuring beneath a blanket. Ibn Sayyaad's mother saw the Prophet (sal-Allaahu 'alayhe wa sallam) hiding behind a tree, and said to her son, "O Saf (Ibn Sayyaad's first name), here is Muhammad!" Ibn Sayyad jumped up, and the Prophet (sal-Allaahu 'alayhe wa sallam) said, "If you had left him alone, he would have explained himself." "
Saleem said,
"Abd Allaah ibn'Umar said, The Prophet (sal-Allaahu 'alayhe wa sallam) stood up to address the people. He praised Allaah as He deserved to be praised, then he spoke about the Dajjaal: 'I warn you against him; there is no Prophet who has not warned his people against him, even Noah warned his people against him. But I will tell you something which no other Prophet has told his people. You must know that the Dajjaal is one-eyed, and Allaah is not one-eyed.' "
Ibn Shihaab said:
"'Umar ibn Thaabit al-Ansaaree told me that some of the Companions of the Prophet (sal-Allaahu 'alayhe wa sallam) told him that on the day when he warned the people about the Dajjaal, the Prophet (sal-Allaahu 'alayhe wa sallam) said: 'There will be written between his eyes the word Kaafir (unbeliever). Everyone who resents his bad deeds - or every believer - will be able to read it.' He also said, 'You must know that no one of you will be able to see his Lord until he dies.' " [Muslim, Bukhaaree] [1]
Ibn 'Umar said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) mentioned the Dajjaal to the people. He said, 'Allaah is not one-eyed, but the Dajjaal is blind in his right eye, and his eye is like a floating grape.' " [Muslim] [2]
Anas ibn Malik said,
"TheProphet (sal-Allaahu 'alayhe wa sallam) said, 'There has never been a Prophet who did not warn his people against that one-eyed liar. Verily he is one-eyed and your Lord is not one-eyed. On his forehead will be written the letter Kaf, Fa, Ra (Kaafir).' " [Muslim, Bukhaaree] [3]
Hudhayfah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'I know more about the powers which the Dajjaal will have than he will know himself. He will have two flowing rivers: one will appear to be pure water, and the other

will appear to be flaming fire. Whosoever lives to see that, let him choose the river which seems to be fire, then let him close his eyes, lower his head and drink from it, for it will be cold water. The Dajjaal will be one-eyed; the place where one eye should be will be covered by a piece of skin. On his forehead will be written the word Kaafir,and every believer, whether literate or illiterate, will be able to read it.' " [Muslim] [4]
Abu Hurayrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Shall I tell you something about the Dajjaal which no Prophet has ever told his people before me? The Dajjaal is one-eyed and will bring with him something which will resemble Paradise and Hell; but that which he calls Paradise will in fact be Hell. I warn you against him as Noah warned his people against him.' " [Bukhaaree, Muslim] [5]
Muhammad ibn Munkadir said:
"I saw Jaabir ibn 'Abd Allaah swearing by Allaah that Ibn Sayyaad was the Dajjaal, so I asked him, 'Do you swear by Allaah?' He said, 'I heard 'Umar swear to that effect in the presence of the Prophet (sal-Allaahu 'alayhe wa sallam), and the Prophet (sal-Allaahu 'alayhe wa sallam) did not disapprove of it.' " [6]
Some 'ulamaa' (scholars) say that some of the Sahaabah (Companions of the Prophet) believed Ibn Sayyaad to be the greater Dajjaal, but that is not the case: Ibn Sayyaad was a lesser dajjal.
Ibn Sayyaad travelled between Makkah and Madeenah with Abu Sa'eed, and complained to him about the way that people were saying that he was the Dajjaal. Then he said to Abu Sa'eed,
"Did not the Prophet (sal-Allaahu 'alayhe wa sallam) say that the Dajjaal would not enter Madinah? I was born there. Did not he say that he would not have any children? - I have children. Did not he say that he would be a Kaafir? - I have embraced Islaam. Of all the people, I know the most about him: I know where he is now. If I were given the opportunity to be in his place, I would not resent it.' " [Bukhaaree, Muslim]
There are many Ahaadeeth about Ibn Sayyaad, some of which are not clear as to whether he was the Dajjaal or not. We shall see Ahaadeeth which indicates that the Dajjaal is not Ibn Sayyaad, as in the Hadeeth of Faatimah Bint Qays al-Fahriyyah, although this does not mean that he was not one of the lesser Dajjaal; but Allaah knows best.

Footnotes
[1] Bukhaaree, Kitaab al-Adab, 8/49, 50. Muslim, Kitaab al-Fitan, 8/192, 193.

Ibn Sayyaad's [or Ibn Sa'eed's] first name was Saaf. He had some characteristics similar to those ascribed to the Dajjaal. When he was young, he was like a Kaahin (soothsayer) - sometimes he spoke the truth, sometimes he lied. When he grew up, he embraced Islaam and displayed some good characteristics, but later he changed, and it was said that his behaviour might indicate that he was the Dajjaal. But the Prophet (sal-Allaahu 'alayhe wa sallam) had not received any Wahy (revelation) to that effect, so he told 'Umar: "If he is [the Dajjaal] you will not be able to overpower him."
Banoo Maghaalah: if you stand facing the Masjid al-Nabawee, (the Prophet's Mosque in Madeenah), everything on your right is the territory of Banoo Maghaalah.
Dukh: i.e. al-Dukhaan (smoke). The Prophet (sal-Allaahu 'alayhe wa sallam) was thinking of the Aayah,
"Then wait you for the Day when the sky will bring forth a visible smoke." [ad-Dukhaan (44):10]
"… with the intention of hearing something…" in other words, the Prophet (sal-Allaahu 'alayhe wa sallam) wanted to eavesdrop on Ibn Sayyaad so that he and his companions could find out whether he was a soothsayer (kaahin) or a sorcerer.
"if you had left him alone, he would have had explained himself": i.e., if his mother had not told him that the Prophet (sal-Allaahu 'alayhe wa sallam) was there, then the Prophet (sal-Allaahu 'alayhe wa sallam) would have found out what he was - a soothsayer or a sorcerer.
[2] Muslim, Kitaab al-Fitan, 8/194, 195.
"His eye is like a floating grape" - this means that, his eye will protrude and there will be some kind of brightness in it.
[3] Muslim, ibid. Bukhaaree, Kitaab al-Fitan, 9/75, 76.
"On his forehead will be written the letters Kaaf, Faa', Raa' " - this indicates that he will call people to Kufr, not the right path, so we must avoid him. The fact that Muslims will be able to identify him as a Kaafir is a great blessing from Allaah to this Ummah.
[4] Muslim, ibid. Shorter version in Bukhaaree, Kitaab al-Fitan, 9/75.
[5] Bukhaaree, Kitaab al-Anbiyaa, 4/75. Muslim, Kitaab al-Fitan, 8/196.
[6] Muslim, Kitaab al-Fitan, 8/192.
The Hadeeth of Faatimah bint Qays
'Aamir ibn Sharaaheel Sha'bi Sha'b Hamdaan reported that he asked Faatimah bint Qays, the sister of Dahhaak ibn Qays,who was one of the first Muhaajiraat,
"Tell me a Hadeeth which you heard directly from the Prophet (sal-

Allaahu 'alayhe wa sallam) with no narrator in between." She said, "I can tell you if you like." He said, "Yes, please tell me." She said,
"I married Ibn al-Mugheerah, who was one of the best of the youth of Quraysh in those days. But he fell in the first Jihaad on the side of the Prophet (sal-Allaahu 'alayhe wa sallam).
"When I became a widow, 'Abd al-Rahmaan ibn 'Awf, one of the companions of the Prophet (sal-Allaahu 'alayhe wa sallam) sent me a proposal of marriage. The Prophet (sal-Allaahu 'alayhe wa sallam) also sent me a proposal of marriage on behalf of his freedman Usaamah ibn Zayd. I had been told that the Prophet (sal-Allaahu 'alayhe wa sallam) had said, 'He who loves me should also love Usaamah.' When the Prophet (sal-Allaahu 'alayhe wa sallam) spoke to me, I said, 'It is up to you: marry me to whomever you wish.'
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Go and stay with Umm Shareek.' Umm Shareek was a rich Ansaaree (Muslim originally from Madeenah) woman, who spent much in the way of Allaah ande ntertained many guests. I said, 'I will do as you wish.' Then he said, 'Don't go. Umm Shareek has many guests, and I would not like it if your head or leg were to become uncovered accidentally and people saw something you would not wish them to see. It is better if you go and stay with your cousin 'Abd Allaah ibn 'Amr ibn Umm Maktoom' (Abd Allaah was of the Banoo Fihr of Quraysh, the same tribe as that to which Faatimah belonged).
"So I went to stay with him, and when I had completed my 'Iddah (period of waiting), I heard the Prophet's (sal-Allaahu 'alayhe wa sallam) announcer calling for congregational prayer. I went out to the mosque, and prayed behind the Prophet (sal-Allaahu 'alayhe wa sallam). I was in the women's row, which was at the back of the congregation. When the Prophet (sal-Allaahu 'alayhe wa sallam) had finished his prayer, he sat on the pulpit, smiling, and said, 'Everyone should stay in his place.' Then he said, 'Do you know why I had asked you to assemble?' The people said, 'Allaah and His Messenger know best.'
"He said, 'By Allaah, I have not gathered you here to give you an exhortation or a warning. I have kept you here because Tameem ad-Daaree, a Christian man who has come and embraced Islaam, told me something which agrees with that which I have told you about the Dajjaal. He told me that he had sailed in a ship with thirty men from Banoo Lakhm and Banoo Judhaam. The waves had tossed them about for a month, then they were brought near to an island, at the time of sunset. They landed on the island, and were met by a beast who was so hairy that they

could not tell its front from its back. They said, "Woe to you! What are you?" It said, "I am al-Jassaasah." They said, "What is al-Jassaasah?" It said, "O people, go to this man in the monastery, for he is very eager to know about you." Tameem said that when it named a person to us, we were afraid lest it be a devil.

"Tamim said, 'We quickly went to the monastery. There we found a huge man with his hands tied up to his neck and with iron shackles between his legs up to the ankles. We said, "Woe to you, who are you?" He said, "You will soon know about me. Tell me who you are." We said, "We are people from Arabia. We sailed in a ship, but the waves have been tossing us about for amonth, and they brought us to your island, where we met a beast who was so hairy that we could not tell its front from its back. We said to it, "Woe to you! What are you?" and it said, "I am al-Jassaasah." We asked, "Whatis al-Jassaasah?" and it told us, "Go to this man in the monastery, for he is very eager to know about you." So we came to you quickly, fearing that it might be a devil.'

"The man said, 'Tell me about the date-palms of Baysaan.' We said, 'What do you want to know about them?' He said, 'I want to know whether these trees bear fruit or not.' We said, 'Yes.' He said, 'Soon they will not bear fruit.' Then he said, 'Tell me about the lake of at-Tabariyyah [Tiberias, in Palestine].' We said, 'What do you want to know about it?' He asked, 'Is there water in it?' We said, 'There is plenty of water in it.' He said, 'Soon it will become dry.' Then he said, 'Tell me about the spring of Zughar.' We said, 'What do you want to know about it?' He said, 'Is there water in it, and does it irrigate the land?' We said, 'Yes, there is plenty of water in it, and the people use it to irrigate the land.'

"Then he said, 'Tell me about the unlettered Prophet - what has he done?' We said, 'He has left Makkah and settled in Yathrib.' He asked, 'Do the Arabs fight against him?' We said, 'Yes.' He said, 'How does he deal with them?' So we told him that the Prophet (sal-Allaahu 'alayhe wa sallam) had overcome the Arabs around him and that they had followed him. He asked, 'Has it really happened?' We said, 'Yes.' He said, 'It is better for them if they follow him. Now I will tell you about myself. Iam the Dajjaal. I will soon be permitted to leave this place: I will emerge and travel about the earth. In forty nights I will pass through every town, except Makkah and Madeenah, for these have been forbidden to me. Every time I try to enter either of them, I will be met by an angel bearing an unsheathed sword, who will prevent me from entering. There will be angels guarding them at every passage leading to them.'

Faatimah said, "The Prophet (sal-Allaahu 'alayhe wa sallam), striking the

pulpit with his staff, said: 'This is Tayyibah, this is Tayyibah, this is Tayyibah, [i.e. Madeenah]. Have I not told you something like this?' The people said, 'Yes.' He said, 'I liked the account given to me by Tameem because it agrees with that which I have told you about the Dajjaal, and about Makkah and Madeenah. Indeed he is in the Syrian sea or the Yemen sea. No, on the contrary, he is in the East, he is in the East, he is in the East' and he pointed towards the East. Faatimah said: I memorised this from the Prophet (sal-Allaahu 'alayhe wa sallam)." [Muslim] [1]
'Abd Allaah ibn 'Umar said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'While I was asleep, I saw myself in a dream performing Tawaaf (circumambulation) around the Ka'bah. I saw a ruddy man with lank hair and water dripping from his head. I said, "Who is he?" and they said, "The son of Mary." Then I turned around and saw another man with a huge body, red complexion, curly hair and one eye. His other eye looked like a floating grape. They said, "This is the Dajjaal." The one who most resembles him is Ibn Qatan, a man from the tribe of al-Khuzaa'ah'.' " [Bukhaaree, Muslim] [2]
Jaabir ibn 'Abd Allaah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Dajjaal will appear at the end of time, when religion is taken lightly. He will have forty days in which to travel throughout the earth. One of these days will be like a year, another will be like a month, a third will be like a week, and the rest will be like normal days. He will be riding a donkey; the width between its ears will be forty cubits. He will say to the people: "I am your lord." He is one-eyed, but your Lord is not one-eyed. On his forehead will be written the word Kaafir, and every believer, literate or illiterate, will be able to read it. He will go everywhere except Makkah and Madeenah, which Allaah has forbidden to him; angels stand at their gates. He will have a mountain of bread, and the people will face hardship, except for those who follow him. He will have two rivers, and I know what is in them. He will call one Paradise and one Hell. Whoever enters the one he calls Paradise will find that it is Hell, and whoever enters the one he calls Hell will find that it is Paradise. Allaah will send with him devils who will speak to the people. He will bring a great tribulation; he will issue a command to the sky and it will seem to the people as if it is raining. Then he will appear to kill someone and bring him back to life. After that he will no longer have this power. The people will say, "Can anybody do something like this except the Lord?" The Muslims will flee to Jabal al-Dukhaan in Syria, and the Dajjaal will come and besiege them. The siege will intensify and they will suffer great hardship. Then Jesus son of Mary will

descend, and will call the people at dawn: "O people, what prevented you from coming out to fight this evil liar?" They will answer, "He is a Jinn." Then they will go out, and find Jesus son of Mary. The time for prayer will come, and the Muslims will call on Jesus to lead the prayer, but he will say, "Let your Imaam lead the prayer." Their Imaam will lead them in praying Salaat al-Subh (Morning prayer), then they will go out to fight the Dajjaal. When the liar sees Jesus, he will dissolve like salt in water. Jesus will go to him and kill him, and he will not let anyone who followed him live.' " [Ahmad] [3]

Footnotes
[1] Muslim, Kitaab al- Fitan, 8/203-205.
Al-Jassaasah is so called because he spies on behalf of the Dajjaal (from jassa - to try to gain information, to spy out, etc.).
Baysaan - a village in Palestine.
'Ayn Zughar (the spring of Zughar) is a town in Palestine.
[2] Bukhaaree, Kitaab al-Fitan, 9/75. Muslim, Kitaab al-Imaan, 1/108.
[3] Ahmad, Musnad, 3/367, 368.
Checker's Note: Da`eef. Silsilatul Ahaadieeh ad-Da`eefah: 1969. It contains the `an`anah of Abu Zubair. Parts are found in authentic narrations. Refer to Dajjaal section in Saheeh Muslim.
The Hadeeth of Al-Nuwas Ibn Sam'an Al-Kilabi
Al-Nuwas ibn Saman said,
"One morning the Prophet (sal-Allaahu 'alayhe wa sallam) spoke about the Dajjaal. Sometimes he described him as insignificant, and sometimes he described him as so dangerous that we thought he was in the clump of date-palms nearby. When we went to him later on, he noticed that fear in our faces, and asked, 'What is the matter with you?' We said, 'O Messenger of Allaah, this morning you spoke of the Dajjaal; sometimes you described him as insignificant, and sometimes you described him as being so dangerous that we thought he was in the clump of date-palms nearby.
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'I fear for you in other matters besides the Dajjaal. If he appears whilst I am among you, I will contend with him on your behalf. But if he appears while I am not among you, then each man must contend with him on his own behalf, and Allaah will take care of every Muslim on my behalf. The Dajjaal will be a young man, with short, curly hair, and one eye floating. I would liken him to 'Abd al-Uzzaa ibn Qatan. Whoever amongst you lives to see him should recite the opening Ayaat of Surat al-Kahf. He will appear on the way between

Syria and Iraq, and will create disaster left and right. O servants of Allaah, adhere to the Path of Truth.' "
"We said, 'O Messenger of Allaah, for the day which is like a year, will one days prayers be sufficient?' He said, 'No, you must make an estimate of the time, and then observe the prayers.' "
"We asked, 'O Messenger of Allaah, how quickly will he walk upon the earth?' He said, 'Like a cloud driven by the wind. He will come to the people and call them (to a false religion), and they will believe in him and respond to him. He will issue a command to the sky, and it will rain; and to the earth, and it will produce crops. After grazing on these crops, their animals will return with their udders full of milk and their flanks stretched. Then he will come to another people and will call them (to a false religion), but they will reject his call. He will depart from them; they will suffer famine and will possess nothing in the form of wealth. Then he will pass through the wasteland and will say, Bring forth your treasures, and the treasures will come forth, like swarms of bees. Then he will call a man brimming with youth; he will strike him with a sword and cut him in two, then place the two pieces at the distance between an archer and his target. Then he will call him, and the young man will come running and laughing.' "
"At that point, Allaah will send the Messiah, son of Mary, and he will descend to the white minaret in the east of Damascus, wearing two garments dyed with saffron, placing his hands on the wings of two angels. When he lowers his head, beads of perspiration will fall from it, and when he raises his head, beads like pearls will scatter from it. Every Kaafir who smells his fragrance will die, and his breath will reach as far as he can see. He will search for the Dajjaal until he finds him at the gate of Ludd, where he will kill him."
"Then a people whom Allaah has protected will come to Jesus son of Mary, and he will wipe their faces (i.e. wipe the traces of hardship from their faces) and tell them of their status in Paradise. At that time Allaah will reveal to Jesus: "I have brought forth some of My servants whom no-one will be able to fight. Take My servants safely to at-Toor."
"Then Allaah will send Gog and Magog, and they will swarm down from every slope. The first of them will pass by the Lake of Tiberias, and will drink some of its water; the last of them will pass by it and say, "There used to be water here." Jesus, the Prophet of Allaah, and his Companions will be besieged until a bull's head will be dearer to them than one hundred dinars are to you nowadays."
"Then Jesus and his Companions will pray to Allaah, and He will send

insects who will bite the people of Gog and Magog on their necks, so that in the morning they will all perish as one. Then Jesus and his Companions will come down and will not find any nook or cranny on earth which is free from their putrid stench. Jesus and his Companions will again pray to Allaah, Who will send birds like the necks of camels; they will seize the bodies of Gog and Magog and throw them wherever Allaah wills. Then Allaah will send rain which no house or tent will be able to keep out, and the earth will be cleansed, until it will look like a mirror. Then the earth will be told to bring forth its fruit and restore its blessing. On that day, a group of people will be able to eat from a single pomegranate and seek shelter under its skin (i.e. the fruit would be so big). A milch camel will give so much milk that a whole party will be able to drink from it; a cow will give so much milk that a whole tribe will be able to drink from it; and a milch-sheep will give so much milk that a whole family will be able to drink from it. At that time, Allaah will send a pleasant wind which will soothe them even under their armpits, and will take the soul of every Muslim. Only the most wicked people will be left, and they will fornicate like asses; then the Last Hour will come upon them." "
[Muslim] [1]

Footnote
[1] Muslim, Kitaab al-Fitan wa Ashraat al-Saa'ah, 8/196-199.
Ludd: the biblical Lydda, now known as Lod, site of the zionist state's major airport
Hadeeth narrated from Abu Umaamah Al-Baahilee
Abu Umaamah al-Baahilee said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) delivered a speech to us, most of which dealt with the Dajjaal and warned us against him. He said, 'No tribulation on earth since the creation of Adam will be worse than the tribulation of the Dajjaal. Allaah has never sent a Prophet who did not warn his Ummah against the Dajjaal. I am the last of the Prophets, and you are the last Ummah. The Dajjaal is emerging among you and it is inevitable. If he appears while I am still among you, I will contend with him on behalf of every Muslim. But if he appears after I am gone, then every person must contend with him on his own behalf. He will appear on the way between Syria and Iraq, and will spread disaster right and left. O servants of Allaah adhere to the path of Truth. I shall describe him for you in a way that no Prophet has ever done before.
He will start by saying that he is a Prophet, but there will be no Prophet after me. Then he will say, "I am your Lord," but you will never see your

Lord until you die. The Dajjaal is one-eyed, but your Lord, glorified be He, is not one-eyed. On his forehead will be written the word Kaafir, which every Muslim, literate or illiterate, will be able to read. Among that which he will bring will be the Paradise and Hell he will offer; but that which he calls Hell will be Paradise, and that which he calls Paradise will be Hell. Whoever enters his Hell, let him seek refuge with Allaah and recite the opening Ayaat of Surat ul-Kahf, and it will become cool and peaceful for him, as the fire became cool and peaceful for Abraham.

"He will say to a Bedouin, What do you think if I bring your father and mother back to life for you? Will you bear witness that I am your lord? The Bedouin will say Yes, so two devils will assume the appearance of his father and mother, and will say, "O my son, follow him for he is your lord."

"He will be given power over one person, whom he will kill and cut in two with a saw. Then he will say, Look at this slave of mine, now I will resurrect him, but he will still claim that he has a Lord other than me. Allaah will resurrect him, and this evil man (the Dajjaal) will say to him, Who is your Lord? The man will answer, 'My Lord is Allaah, and you are the enemy of Allaah. You are the Dajjaal. By Allaah, I have never been more sure of this than I am today.' "

Aboo Sa'eed said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'That man will have the highest status among my Ummah in Paradise.' "

Abu Sa'eed said,

"By Allaah, we never thought that that man would be any other than 'Umar ibn al-Khattab, until he passed away."

Al-Muhaaribee said:

"Then we referred to the Hadeeth of Abu Raafi'ee, which said, 'Part of his Fitnah will be the fact that he will pass through an area whose people will deny him, and none of their livestock will remain alive. Then he will pass through a second area whose people will believe in him; he will order the sky to rain and the earth to bring forth crops, and their flocks will return from grazing fatter than they have ever been, with their flanks stretched, their udders full. He will pass through every place on earth - except Makkah and Madeenah, which he will never enter, for there are angels guarding every gate of them with unsheathed swords - until he reaches al-Zareeb al-Ahmar and camps at the edge of the salt-marsh. Madeenah will be shaken by three tremors, after which every Munaafiq (hypocrite) will leave it, and it will be cleansed of evil, as iron is cleansed of dross. That day will be called Yawm al-Khalaas (The Day of Purification).' "

Umm Shareek bint Abee'l-'Akr said,
"O Messenger of Allaah, where will the Arabs be at that time?" He said, "At that time they will be few; most of them will be in Bayt al-Maqdis (Jerusalem), and their Imaam will be a righteous man. Whilst their Imaam is going forward to lead the people in praying Salaat al-Subh (themorning prayer), Jesus son of Mary will descend. The Imaam will step back, to let Jesus lead the people in prayer, but Jesus will place his hand between the man's shoulders and say, 'Go forward and lead the prayer, for the Iqaamah was made for you.' So the Imaam will lead the people in prayer, and afterwards Jesus (alayhi-salam) will say, 'Open the gate.' The gate will be opened, and behind it will be the Dajjaal and a thousand Jews, each of them bearing a sword and shield. When the Dajjaal sees Jesus, he will begin to dissolve like salt in water, and will run away. Jesus will say, 'You will remain alive until I strike you with my sword.' He will catch up with him at the eastern gate of Ludd and will kill him. The Jews will be deflated with the help of Allaah. There will be no place for them to hide; they will not be able to hide behind any stone, wall, animal or tree - except the boxthorn (al-Gharqarah) - without it saying, 'O Muslim servant of Allaah! here is a Jew, come and kill him!' " The Prophet (sal-Allaahu 'alayhe wa sallam) said, "The time of the Dajjaal will be forty years; one year like half a year, one year like a month, and one month like a week. The rest of his days will pass so quickly that if one of you were at one of the gates of Madeenah, he would not reach the other gate before evening fell."
Someone asked,
"O Messenger of Allaah, how will wepray in those shorter days?" He said, "Workout the times of prayer in the same way that you do in these longer days, and then pray." The Prophet (sal-Allaahu 'alayhe wa sallam) said, "Jesus son of Mary will be a just administrator and leader of my Ummah. He will break the cross, kill the pigs, and abolish the Jizyah (tax on non-Muslims). He will not collect the Sadaqah, so he will not collect sheep and camels. Mutual enmity and hatred will disappear. Every harmful animal will be made harmless, so that a small boy will be able to put his hand into a snakes mouth without being harmed,a small girl will be able to make a lion run away from her, and a wolf will go among sheep as if he were a sheepdog. The earth will be filled with peace as a container is filled with water. People will be in complete agreement, and only Allaah will be worshipped. Wars will cease, and the authority of Quraysh will be taken away. The earth will be like a silver basin, and will produce fruits so abundantly that a group of people will gather to eat a bunch of grapes or

on epomegranate and will be satisfied. A bull will be worth so much money, but a horse will be worth only a few dirhams."
Someone asked,
"O Messenger of Allaah, why will a horse be so cheap?" He said, "Because it will never be ridden in war." He was asked, "Why will the bull be so expensive?" He said, "Because it will plough the earth. For three years before the Dajjaal emerges, the people will suffer severe hunger. In the first year, Allaah will order the sky to withhold a third of its rain, and the earth to withhold two-thirds of its fruits. In the third year, He will order the sky to withhold all of its rain, and the earth to withhold all of its fruits, so that nothing green will grow. Every cloven-hoofed creature will die except for whatever Allaah wills." Someone asked, "How will the people live at that time?" He said, "By saying Laa ilaaha illaa Allaah, Allaahu Akbar, Subhaan Allaah and Al-Hamdu-lillaah. This will be like food for them." [1]
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Dajjaal will come forth, and one of the Believers will go towards him. The armed men of the Dajjaal will ask him, 'Where are you going?' He will say, 'I am going to this one who has come forth.' They will say, 'Kill him!' Then some of them will say to the others, 'Hasn't your lord [ie the Dajjaal] forbidden you to kill anyone without his permission?' So they will take him to the Dajjaal, and when the Believer sees him, he will say, 'O People, this is the Dajjaal whom the Prophet (sal-Allaahu 'alayhe wa sallam) told us about.' Then the Dajjaal will order them to seize him and wound him in the head; they will inflict blows all over, even in his back and stomach. The Dajjaal will ask him, 'Don't you believe in me?' He will say, 'You are a false Messiah.' The Dajjaal will order that he be sawn in two from the parting of his hair to his legs; then he will walk between the two pieces. Then he will say 'Stand!' and the man will stand up. The Dajjaal will say to him, 'Don't you believe in me?' The believer will say, 'It has only increased my understanding that you are the Dajjaal.' Then he will say, 'O people! he will not treat anyone else in such a manner after me.' The Dajjaal will seize him to slaughter him, but the space between his neck and collar-bone will be turned into copper, and the Dajjaal will not be able to do anything to him. He will take the man by his arms and legs and throw him away; the people will believe that he has been thrown into Hell, whereas in fact he will have been thrown into Paradise." The Prophet (sal-Allaahu 'alayhe wa sallam) said, "He will be the greatest of martyrs in the sight of Allaah, the Lord of the Worlds." [Muslim] [2]

Footnotes
[1] Ibn Maajah, Kitaab al-Fitan, (Hadeeth 4077), 2:1363.
"He will break the cross and kill pigs", i.e. Christianity will be annulled.
"He will not collect the Sadaqah" (i.e. Zakaat) - because there will be so much wealth, and no-one will be in need of Sadaqah.
Checker's Note: Da`eef. Da`eef Ibn Maajah: 884. Parts are supported.
[2] Muslim, Kitaab al-Fitan wa Ashtaar al-Saa'ah, 8/199, 200
The Hadeeth of Al-Mugheerah Ibn Shu'bah
Al-Mugheerah ibn Shu'bah said,
"No-one asked the Prophet (sal-Allaahu 'alayhe wa sallam) more questions about the Dajjaal than I did. He said, 'You should not worry about him, because he will not be able to harm you.' I said, 'But they say that he will have much food and water!' He said, 'He is too insignificant in the sight of Allaah to have all that.' " [Muslim]
Al-Mugheerah ibn Shu'bah said,
"No-one asked the Prophet (sal-Allaahu 'alayhe wa sallam) more questions about the Dajjaal than I did." One of the narrators said, "What did you ask him?" Al-Mugheerah said, "I said, 'They say that the Dajjaal will have a mountain of bread and meat, and a river of water. The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'He is too insignificant in the sight of Allaah to have all that.' " [Muslim]
From these Ahaadeeth, we can see that Allaah will test His servants with the Dajjaal and by the miracles which he will be permitted to perform: as we have already mentioned, the Dajjaal will order the sky to rain for those who accept him, and will order the earth to bring forth its fruits so that they and their livestock will eat of it, and their flocks will return fat and with their udders full of milk. Those who reject the Dajjaal and refuse to believe in him will suffer drought and famine; people and livestock will die, and wealth and supplies of food will be depleted. People will follow the Dajjaal like swarms of bees, and he will kill a young man and bring him back to life.
This is not a kind of magic; it will be something real with which Allaah will test His servants at the end of time. Many will be led astray, and many will be guided by it. Those who doubt will disbelieve, but those who believe will be strengthened in their faith.
Al-Qaadi 'Iyaad and others interpreted the phrase "He is too insignificant in the sight of Allaah to have all that" as meaning that the Dajjaal is too insignificant to have anything that could lead the true believers astray, because he is obviously evil and corrupt. Even if he brings great terror, the word Kaafir will be clearly written between his eyes; one report

explains that it will be written "Kaaf, Faa, Raa," from which we can understand that it will be written perceptibly, not abstractly, as some people say.
One of his eyes will be blind, protruding and repulsive; this is the meaning of the Hadeeth: "...as if it were a grape floating on the surface of the water." Other reports say that it is "dull, with no light in it," or "like white spittle on a wall," i.e. it will look ugly.
Some reports say that it is his right eye which will be blind; others say that it is his left eye.
He could be partly blind in both eyes, or there could be a fault in both eyes. This interpretation could be supported by the Hadeeth narrated by at-Tabaraanee, in which he reports that Ibn 'Abbaas said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Dajjaal is curly-haired and white-skinned. His head is like the branch of a tree; his left eye is blind, and the other eye looks like a floating grape.' "
One may ask: if the Dajjaal is going to cause such widespread evil and his claim to be a "lord" will be so widely believed - even though he is obviously a liar, and all the Prophet's have warned against him - why does the Qur'aan not mention him by name and warn us against his lies and stubbornness?
The answer is:-
The Dajjaal was referred to in the Ayah:
"...The day that some of the Signs of your Lord do come, no good will it do to a person to believe then, if he believed not before, nor earned good (by performing deeds of righteousness) through his Faith..." [al-An'aam (6):158]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'There are three things which, when they appear, no good will it do a soul to believe in them then, if it believed not before nor earned righteousness through its Faith. They are: The Dajjaal, the Beast, and the rising of the sun from the west.' "
Jesus son of Mary will descend from Heaven and kill the Dajjaal, as we have already mentioned. The descent of Jesus is mentioned in the Ayaat:
"And because of their saying (in boast), "We killed Messiah 'Eesaa (Jesus), son of Maryam (Mary), the Messenger of Allaah," - but they killed him not, nor crucified him, but the resemblance of 'Eesaa (Jesus) was put over another man (and they killed that man), and those who differ therein are full of doubts. They have no (certain) knowledge, they follow nothing but conjecture. For surely; they killed him not [i.e. 'Eesaa

(Jesus), son of Maryam (Mary) alayhi-salam]:
But Allaah raised him ['Eesaa (Jesus)] up (with his body and soul) unto Himself (and he alayhi-salam is in the heavens). And Allaah is Ever All Powerful, All Wise.
And there is none of the people of the Scripture (Jews and Christians), but must believe in him ['Eesaa (Jesus), son of Maryam (Mary), as only a Messenger of Allaah and a human being], before his ['Eesaa (Jesus) alayhi-salam or a Jew's or a Christian's] death (at the time of the appearance of the angel of death). And on the Day of Resurrection, he ['Eesaa (Jesus)] will be a witness against them." [an-Nisaa' (4):157-9]
We think that the Tafseer (interpretation) of this Aayah is that the pronoun in "before his death" (qabla mawtihi) refers to Jesus; i.e, he will descend and the People of the Book who differed concerning him will believe in him. The Christians claimed that he was divine, while the Jews made a slanderous accusation, i.e. that he was born from adultery. When Jesus descends before the Day of Judgment, he will correct all these differences and lies.
On this basis, the reference to the descent of the Messiah Jesus son of Mary also includes a reference to the Dajjaal (false Mesor Antichrist), who is the opposite of the true Messiah, because sometimes the Arabs refer to one of two opposites and not the other, but mean both.
The Dajjaal is not mentioned by name in the Qur'aan because he is so insignificant: he claims to be divine, but he is merely a human being. His affairs are too contemptible to be mentioned in the Qur'aan. But the Prophet's, out of loyalty to Allaah, warned their people about the Dajjaal and the tribulations and misguiding miracles he would bring. It is enough for us to know the reports of the Prophet's and the many reports from the Prophet Muhammad (sal-Allaahu 'alayhe wa sallam).
One could argue that Allaah has mentioned Pharaoh and his false claims, such as:
"I am your lord, most high" [an-Nazi'aat (79):24] and "O chiefs! I know not that you have an ilaah (a god) other than me..." [al-Qasas (28):38]
This can be explained by the fact that Pharaoh and his deeds are in the past, and his lies are clear to every believer. But the Dajjaal is yet to come, in the future; it will be a Fitnah and a test for all people. So the Dajjaal is not mentioned in the Qur'aan because he is contemptible; and the fact that he is not mentioned means that it will be a great test.
The facts about the Dajjaal and his lies are obvious and do not need further emphasis. This is often the case when something is very clear. For example, when the Prophet (sal-Allaahu 'alayhe wa sallam) was

terminally ill, he wanted to write a document confirming that Abu Bakr would be the Khaleefah after him. Then he abandoned this idea, and said,
"Allaah and the believers will not accept anyone other than Abu Bakr."
He decided not to write the document because he knew of Abu Bakr's high standing among the Sahaabah (Companions) and was sure that they would not choose anyone else. Similarly, the facts about the Dajjaal are so clear that they did not need to be mentioned in the Qur'aan.
Allaah did not mention the Dajjaal in the Qur'aan because He (subhaanahu wa ta'aala) knew that the Dajjaal would not be able to lead His true servants astray; he would only increase their faith, their submission to Allaah and His Messenger, their belief in the Truth, and their rejection of falsehood. For this reason the believer whom the Dajjaal overpowers will say, when he revives him,
"By Allaah, it has only increased my understanding that you are the one-eyed liar about whom the Prophet (sal-Allaahu 'alayhe wa sallam) spoke."
More Ahaadeeth about the Dajjaal
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Dajjaal will emerge in a land in the east called Khurasan. His followers will be people with faces like hammered shields." [1]
Asmaa' bint Yazeed al-Ansaariyyah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'During the three years just before the Dajjaal comes, there will be one year when the sky will withhold one third of its rain and the earth one-third of its fruits. In the second year the sky will withhold two-thirds of its rain, and the earth two-thirds of its fruits. In the third year the sky will withhold all of its rain, and the earth all of its fruits, and all the animals will die. It will be the greatest tribulation: the Dajjaal will bring a Bedouin and say to him, "What if I bring your camels to life for you? Will you agree that I am your lord?" The Bedouin will say "Yes." So devils will assume the forms of his camels, with the fullest udders and the highest humps. Then he will bring a man whose father and brother have died, and will ask him, "What do you think if I bring your father and brother back to life? Will you agree that I am your lord?" The man will say "Yes," so the devils will assume the forms of his father and brother.' Then the Prophet (sal-Allaahu 'alayhe wa sallam) went out for something, and then returned. The people were very concerned about what he had told them. He stood in the doorway and asked, 'What is wrong, Asma'? I said, 'O Messenger of Allaah, you have terrified us with what you said about the Dajjaal.' He said, 'He will

certainly appear. If I am still alive, I will contend with him on your behalf; otherwise Allaah will take care of every Muslim on my behalf.' I said, 'O Messenger of Allaah, we do not bake our dough until we are hungry, so how will it be for the believers at that time?' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The glorification of Allaah which suffices the people of Heaven will be sufficient for them.' " [2]
Abu Hurairah (radi-Allahu 'anhu) said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come until the Muslims fight the Jews and kill them. When a Jew hides behind a rock or a tree, it will say, "O Muslim, O servant of Allaah! There is a Jew behind me, come and kill him!" All the trees will do this except the box-thorn (al-Gharqad), because it is the tree of the Jews.' " [Ahmad] [3]

Footnotes
[1] Tirmidhee Abwaab al-Fitan, Hadeeth 2338, 6/465. Ibn Maajah, Kitaab al-Fitan, Hadeeth 4072, 1354, 2/1353, 1354. Ahmad, Musnad 1/7.
Checker's Note: Saheeh, Saheeh Ibn Maajah: 4072.
[2] Ahmad, Musnad, 6/455, 456.
Checker's Note: Da`eef, al-Mishkat, 1st checking: 5491. Its isnaad contains Sharh ibn Hawshab, who is weak.
[3] Muslim, Kitaab al-Fitan wa Ashraat al-Saa'ah, 8/388. Ahmad, Musnad, 2/417.
Protection Against The Dajjaal
1. Seeking refuge with Allaah from his tribulation.
It is proven in the Saheeh (authentic) Ahaadeeth that the Prophet (sal-Allaahu 'alayhe wa sallam) used to seek refuge with Allaah from the tribulation of the Dajjaal in his prayers, and that he commanded his Ummah to dol ikewise:
"Allaahumma innaa na'oodhu bika min 'adhaabi jahannam, wa min 'adhaabi 'l-qabr, wa min fitnatee 'l-mahyaa'i wa'l-mamaat, wa min fitnatee' l-maseehi 'd-dajjaal."
"O Allaah! We seek refuge with You from the punishment of Hell, from the punishment of the grave, from the tribulations of life and death, and from the tribulation of the False Messiah (Dajjaal)." [1]
This Hadeeth was narrated by many Sahabah, including Anas, Abu Hurairah, 'Aa'ishah, Ibn 'Abbaas, and Sa'd (radi-Allaahu 'anhum).
2. Memorising certain Aayaat from Surat al-Kahf.
Al-Haafidh ad-Dhahabee said,
"Seeking refuge with Allaah from the Dajjaal is mentioned in many

Mutawaatir Ahaadeeth (those with numerous lines of narrators). One way of doing this is to memorise ten Ayaat from Surat al-Kahf."
Abul-Diraa reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said, "Whoever memorises the first ten Aayaat of Surat al-Kahf will be protected from the Dajjaal." [Abu Daawood] [2]
3. Keeping away from the Dajjaal.
One way to be protected from the tribulation of the Dajjaal is to live in Madeenah or Makkah.
Abu Hurairah reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"There are angels standing at the gates of Madeenah; neither plague nor the Dajjaal can enter it." [Bukhaaree, Muslim]
Abu Bakr reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said, "The terror caused by the Dajjaal will not enter Madeenah. At that time it will have seven gates; there will be two angels guarding every gate." [Bukhaaree] [3]
Anas said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Dajjaal will come to Madeenah, and he will find angels guarding it. Neither plague nor the Dajjaal will enter it, inshaa-Allaah.' " [Tirmidhee, Bukhaaree]
It has been proven in the Saheeh Ahaadeeth that the Dajjaal will not enter Makkah or Madeenah, because the angels will prevent him from entering these two places which are sanctuaries and are safe from him. When he camps at the salt-marsh (Sabkhah) of Madeenah, it will be shaken by three tremors - either physically or metaphorically - and every hypocrite will go out to join the Dajjaal. On that day, Madeenah will be cleansed of its dross and will be refined and purified; and Allaah knows best.

Footnotes
[1] Bukhaaree, Kitaab al-Janaa'iz, 2/124. Muslim, Kitaab al-Masaajid, 2/93.
[2] Muslim, Kitaab al-Musaafirun, 2/199. Abu Daawood, Kitaab al-Malaahim, (Hadeeth 4301), 11/401, 402.
[3] Bukhaaree, Baab Haram al-Madeenah, 1/28.

The Life And Deeds Of The Dajjaal

The Dajjaal will be a man, created by Allaah to be a test for people at the end of time. Many will be led astray through him, and many will be guided through him; only the sinful will be led astray.
Al-Haafidh Ibn 'Alee al-Aabaar wrote in his book of history (at-Taareekh)

that the Dajjaal's Kunyah (nickname or paternal title) would be Abu Yoosuf. [1]
Abu Bakrah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Dajjaal's parents will remain childless for thirty years, then a one-eyed child will be born to them. He will be very bad and will cause a great deal of trouble. When he sleeps, his eyes will be closed but his heart (or mind) will still be active.' Then he described his parents: 'His father will be a tall and bulky man, with a long nose like a beak; his mother will be a huge, heavy-breasted woman.' " [2]
Abu Bakrah said,
"We heard that a child had been born to some of the Jews in Madeenah. Az-Zubayr ibn al-'Awaam and I went see his parents, and found that they matched the description given by the Prophet (sal-Allaahu 'alayhe wa sallam). We saw the boy lying in the sun, covered with a blanket, murmuring to himself. We asked his parents about him, and they said, 'We remained childless for thirty years, then this one-eyed boy was born to us. He is very bad and causes a great deal of trouble.' When we went out, we passed the boy. He asked us, 'What were you doing?' We said, 'Did you hear us?' He said, 'Yes; when I sleep, my eyes are closed but my heart (mind) is still active.' That boy was Ibn Sayyaad." [Ahmad, Tirmidhee; this Hadeeth is not very strong] [3]
As we have already seen in the Saheeh Ahaadeeth, Maalik and others think that Ibn Sayyaad was not the Dajjaal; he was one of a number of "lesser dajjaals." Later he repented and embraced Islaam; Allaah knows best his heart and deeds.
The "greater" Dajjaal is the one mentioned in the Hadeeth of Faatimah bint Qays, which she narrated from the Prophet (sal-Allaahu 'alayhe wa sallam), from Tameem ad-Daaree, and which includes the story of the Jassaasah. The Dajjaal will be permitted to appear at the end of time, after the Muslims have conquered a Roman city called Constantinople. He will first appear in Isfahaan, in an area known as the Jewish quarter (al-Yahoodiyyah). He will be followed by seventy thousand Jews from that area, all of them armed. Seventy thousand Tatars and many people from Khuraasaan will also follow him. At first he will appear as a tyrannical king, then he will claim to be a prophet, then a lord. Only the most ignorant of men will follow him; the righteous and those guided by Allaah will reject him. He will start to conquer the world country by country, fortress by fortress, region by region, town by town; no place will remain unscathed except Makkah and Madeenah. The length of his stay

on earth will be forty days: one day like a year, one day like a month, one day like a week, and the rest of the days like normal days, i.e. his stay will be approximately one year and two and a half months. Allaah will grant him many miracles, through which whoever He wills will be astray, and the faith of the believers will be strengthened. The descent of Jesus son of Mary, the true Messiah, will happen at the time of the Dajjaal, the false messiah. He will descend to the minaret in the east of Damascus. The believers and true servants of Allaah will gather to support him, and the Messiah Jesus son of Mary will lead them against the Dajjaal, who at that time will be heading for Bayt al-Maqdis (Jerusalem). He will catch up with him at 'Aqabah 'Afeeq. The Dajjaal will runaway from him, but Jesus will catch up with him at the gate of Ludd, and will kill him with his spear just as he is entering it. He will say to him, "I have to deal you a blow; you cannot escape." When the Dajjaal faces him, he will begin to dissolve like salt in water. So Jesus will kill him with his spear at the gate of Ludd, and he will die there, as many Saheeh Ahaadeeth indicate.
Majma' ibn Jaariyah is reported to have said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) saying, 'The son of Mary will kill the Dajjaal at the gate of Ludd.' " [Tirmidhee] [4]

Footnotes
[1] Checker's Note: Reported as the saying of ash-Sha'bi. The reporter, Mujaahid ibn Sa'eed is weak.
[2] Checker's Note: Da`eef, al-Mishkat, 1st checking: 6445.
[3] Tirmidhee, Abwaab al-Fitan (Hadeeth 2350), 6/522, 523. Ahmad, Musnad, 5/40.
"When he sleeps, his eyes will be closed but his heart (or mind) will still be active" - means that his evil ideas will still come to him even while he is asleep.
Checker's Note: Da`eef due to Alee ibn Zaid ibn Jad'an.
[4] Tirmidhee, Abwaab al-Fitan (Hadeeth 2345), 6/513, 514.
Checker's Note: Saheeh, Saheeh at-Tirmidhee: 1829.
The Descent of Jesus at the End of Time
Abd Allaah ibn 'Amr said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Dajjaal will appear in my Ummah, and will remain for forty - "I cannot say whether he meant forty days, forty months or forty years." - Then Allaah will send Jesus (alayhi-salam), the son of Mary, who will resemble 'Urwah ibn Mas'ood. He will chase the Dajjaal and kill him. Then the people will live for seven years during which there will be no enmity between any two

persons. Then Allaah will send a cold wind from the direction of Syria, which will take the soul of everyone who has the slightest speck of good or faith in his heart. Even if one of you were to enter the heart of a mountain, the wind would reach him there and take his soul. Only the most wicked people will be left; they will be as careless as birds, with the characteristics of beasts, and will have no concern for right and wrong. Satan will come to them in the form of man and will say, "Don't you respond?" They will say, "What do you order us to do?" He will order them to worship idols, and in spite of that they will have sustenance in abundance, and lead comfortable lives.

"Then the Trumpet will be blown, and everyone will tilt their heads to hear it. The first one to hear it will be a man busy repairing a trough for his camels. He and everyone else will be struck down. Then Allaah will send (or send down) rain like dew, and the bodies of the people (i.e. the dead) will grow out of it. Then the trumpet will be sounded again, and the people will get up and look around. Then it will be said, "O people, go to your Lord and account for yourselves." It will be said, "Bring out the people of Hell," and it will be asked, "How many are there?" - the answer will come: "Nine hundred and ninety-nine out of every thousand." "On that day a child will grow old and the shin will be laid bare." [al-Qalam (68):42] " [Muslim] [1]

Abu Hurairah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The son of Mary will come down as a just leader. He will break the cross, and kill the pigs. Peace will prevail and people will use their swords as sickles. Every harmful beast will be made harmless; the sky will send down rain in abundance,and the earth will bring forth its blessings. A child will play with a fox and not come to any harm; a wolf will graze with sheep and a lion with cattle, without harming them.' " [Ahmad] [2]

Abu Hurairah said,

"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'By Him in Whose hand is my soul, surely the son of Mary will come down among you as a just ruler. He will break the cross, kill the pigs and abolish the Jizyah. Wealth will be in such abundance that no-one will care about it, and a single prostration in prayer will be better than the world and all that is in it.' "

Abu Hurairah said,

'If you wish, recite the Aayah: "And there is none of the People of the Book but must believe in him before his death; and on the Day of Judgement he will be a witness against them..." [an-Nisaa' (4):159] '

[Bukhaaree, Muslim] [3]
Abu Hurairah reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Prophets are like brothers; they have different mothers but their religion is one. I am the closest of all the people to Jesus son of Mary, because there is no other Prophet between him and myself. He will come again, and when you see him, you will recognise him. He is of medium height and his colouring is reddish-white. He will be wearing two garments, and his hair will look wet. He will break the cross, kill the pigs, abolish the Jizyah and call the people to Islaam. During his time, Allaah will end every religion and sect other than Islaam, and will destroy the Dajjaal. Then peace and security will prevail on earth, so that lions will graze with camels, tigers with cattle, and wolves with sheep; children will be able to play with snakes without coming to any harm. Jesus will remain for forty years, then die, and the Muslims will pray for him."
[Ahmad] [4]
Ibn Masood reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"On the night of the Israa' (night journey), I met my father Abraham, Moses and Jesus, and they discussed the Hour. The matter was referred first to Abraham, then to Moses, and both said, 'I have no knowledge of it.' Then it was referred to Jesus, who said, 'No-one knows about its timing except Allaah; what my Lord told me was that the Dajjaal will appear, and when he sees me he will begin to melt like lead. Allaah will destroy him when he sees me. The Muslims will fight against the Kaafirs, and even the trees and rocks will say, "O Muslim, there is a Kaafir hiding beneath me - come and kill him!" Allaah will destroy the Kaafirs, and the people will return to their own lands. Then Gog and Magog will appear from all directions, eating and drinking everything they find. The people will complain to me, so I will pray to Allaah and He will destroy them, so that the earth will be filled with their stench. Allaah will send rain which will wash their bodies into the sea. My Lord has told me that when that happens, the Hour will be very close, like a pregnant woman whose time is due, but her family do not know exactly when she will deliver.' "
[Ahmad, Ibn Maajah] [5]

Footnotes
[1] Muslim, Kitaab al-Fitan wa Ashraat al-Saa'ah, 8/201, 202.
[2] Ahmad, Musnad, 2/482, 483.
Checker's Note: The Tabi'e is Ziyad ibn Sa'd, who is mentioned by Ibn

Hibban in ath-Thiqat al-Bukhari and Abu Hatim did not speak for nor against him. Ibn Kathir says the report is good and strong.
[3] Bukhaaree, Kitaab al-Anbiyaa, 204, 205. Muslim, Kitaab al-Imaan, 1/93, 94.
[4] Ahmad, Musnad, 2/406.
Checker's Note: Saheeh as declared by al-Hakim and adh-Dhahabi. Qatadah's hearing it is reported in a later narration in al-Musnad 2/436.
[5] Ahmad, Musnad, 1/375. Similar Hadeeth in Ibn Maajah, Kitaab al-Fitan (Hadeeth 4081), 2/1365, 1366.
Checker's Note: Da`eef, Da`eef al-Jami as-Sagheer: 4709. See Saheeh Muslim translation, no. 6924
Description of the Messiah Jesus son of Mary, Messenger of Allaah
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'On the night of the Israa' (miraculous journey to Jerusalem) I met Moses - he was a slim man with wavy hair, and looked like a man from the Shanoo'ah tribe. I also met Jesus - he was of medium height and of a red complexion, as if he had just come out of the bath' " [Bukhaaree, Muslim] [1]
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Whilst I was asleep, I saw myself (in a dream) making Tawaaf around the Ka'bah. I saw a brown-skinned man, with straight hair, being supported by two men, and with water dripping from his head. I said, 'Who is this?' They said, 'The son of Mary' I turned around and saw a fat, ruddy man, with curly hair, who was blind in his right eye; his eye looked like a floating grape. I asked, 'Who is this?' They said, 'The Dajjaal.' The one who most resembles him is Ibn Qatan." Az-Zuhri explained: Ibn Qatan was a man from Khuzaa'ah who died during the Jaahiliyyah (before the coming of Islaam). [Bukhaaree] [2]

Footnotes
[1] Bukhaaree, Kitaab al-Anbiyaa, 4/302. Muslim, Kitaab al-Imaan, 1/106, 107.
[2] Bukhaaree, ibid.
The Appearance of Gog and Magog
They (two tribes or peoples) will appear at the time of Jesus son of Mary, after the Dajjaal. Allaah will destroy them all in one night, in response to the supplication of Jesus.
Abu Hurairah reported that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Everyday, Gog and Magog are trying to dig a way out through the

barrier. When they begin to see sunlight through it, the one who is in charge of them says, 'Go back; you can carry on digging tomorrow,' and when they come back, the barrier is stronger than it was before. This will continue until their time comes and Allaah wishes to send them forth. They will dig until they begin to see sunlight, then the one who is in charge of them will say, 'Go back; you can carry on digging tomorrow, inshaa'-Allaah.' In this case he will make an exception by saying inshaa'-Allaah, thus relating the matter to the Will of Allaah. They will return on the following day, and find the hole as they left it. They will carry on digging and come out against the people. They will drink all the water, and the people will entrench themselves in their fortresses. Gog and Magog will fire their arrows into the sky, and they will fall back to earth with something like blood on them. Gog and Magog will say, 'We have defeated the people of earth, and overcome the people of heaven.' Then Allaah will send a kind of worm in the napes of their necks, and they will be killed by it...' 'By Him in Whose hand is the soul of Muhammad, the beasts of the earth will become fat.' " [1]

Gog and Magog are two groups of Turks, descended from Yaafith (Japheth), the father of the Turks, one of the sons of Noah. At the time of Abraham (alayhi-salam), there was a king called Dhu'l-Qarnayn. He performed Tawaaf around the Ka'bah with Abraham (alayhi-salam) when he first built it; he believed and followed him. Dhu'l-Qarnayn was a good man and a great king; Allaah gave him great power and he ruled the east and west. He held sway over all kings and countries, and travelled far and wide in both east and west. He travelled eastwards until he reached a pass between two mountains, through which people were coming out. They did not understand anything, because they were so isolated; they were Gog and Magog. They were spreading corruption through the earth, and harming the people, so the people sought help from Dhu'l-Qarnayn. They asked him to build a barrier between them and Gog and Magog. He asked them to help him to build it, so together they built a barrier by mixing iron, copper and tar.

Thus Dhu'l-Qarnayn restrained Gog and Magog behind the barrier. They tried to penetrate the barrier, or to climb over it, but to no avail. They could not succeed because the barrier is so huge and smooth. They began to dig, and they have been digging for centuries; they will continue to do so until the time when Allaah decrees that they come out. At that time the barrier will collapse, and Gog and Magog will rush out in all directions, spreading corruption, uprooting plants, killing people. When Jesus (alayhi-salam) prays against them, Allaah will send a kind of worm in the napes of their necks, and they will be killed by it.

Footnotes
[1] Tirmidhee, Abwaab al-Tafsir: Soorat al-Kahf (Hadeeth 5160), 8/597-99. Ibn Maajah, Kitaab al-Fitan, (Hadeeth 4080), 2/1364. Ahmad, Musnad, 2/510, 511.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 1735.
The destruction of the Ka'bah
At the end of time, Dhu'l-Suwayqatayn, who will come from Abyssinia (al-Habash), will destroy the Ka'bah in order to steal its treasure and Kiswah (cloth covering). The Ka'bah is the ancient building which was built by Abraham, and whose foundations were laid by Adam.
As Tafseer (interpretation) of the Aayah "Until the Gog and Magog (people) are let through (their barrier)" [al-Anbiyaa' (21):96], it was reported from Ka'b al-Ahbar that Dhu'l-Suwayqatayn will first emerge at the time of Jesus, son of Mary. Allaah will send Jesus at the head of a vanguard of between seven and eight hundred. While they are marching towards Dhu'l-Suwayqatayn, Allaah will send a breeze from the direction of Yemen, which will take the soul of every believer. Only the worst of people will be left, and they will begin to copulate like animals. Ka'b said: "At that time, the Hour will be close at hand." [1]
'Abd Allaah ibn 'Amr said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'Dhu'l-Suwayqatayn from Abyssinia will destroy the Ka'bah and steal its treasure and Kiswah. It is as if I could see him now: he is bald-headed and has a distortion in his wrists. He will strike the Ka'bah with his spade and pick-axe.' " [Ahmad] [2]
It was reported from 'Abd Allaah ibn 'Umar that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Leave the Abyssinians alone so long as they do not disturb you, for no-one will recover the treasure of the Ka'bah except Dhu'l-Suwayqatayn from Abyssinia." [Abu Daawood, in the chapter on the prohibition of provoking the Abyssinians] [3]
Ibn 'Abbaas narrated that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"It is as if I can see him now: he is black and his legs are widely spaced. He will destroy the Ka'bah stone by stone." [Ahmad] [4]
The Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Hour will not come until a man from Qahtaan appears and rules the people." [Muslim; similar Hadeeth in Bukhaaree]

This man could be Dhu'l-Suwayqatayn, or someone else, because this man comes from Qahtaan, while other reports say that Dhu'l-Suwayqatayn comes from Abyssinia; and Allaah knows best.
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Day and night will not come to an end until a freed man called Jahjaah holds sway.' " [Ahmad] [5]
This could be the name of Dhul-Suwayqatayn from Abyssinia; and Allaah knows best.
'Umar ibn al-Khattab reported that he heard the Prophet (sal-Allaahu 'alayhe wa sallam) say:
"The people of Makkah will leave, and only a few people will pass through it. Then it will be resettled and rebuilt; then the people will leave it again, and no-one will ever return." [6]

Footnotes
[1] Concerning Dhu'l-Suwayqatayn, see: Bukhaaree: Kitaab al-Hajj, Baab Hadam al-Ka'bah (The Book of Pilgrimage, Chapter of the Destruction of the Ka'bah), 2/183. Muslim, Kitaab al-Fitan wa Ashraat al-Saa'ah, 8/183. Dhu'l-Suwayqatayn: al-Suwayqatayn is the dimunitive of al-Saaqayn (legs); his legs are described as being "small" because they are thin. Thin legs are, in general, a characteristic of the Sudanese and people of the Horn of Africa.
[2] Ahmad, Musnad, 2/220.
Checker's Note: Da`eef isnaad. It contains the `an`anah of Abu Zubair. The first part, up to 'treasure' is hasan, Silsilatul Ahadeeth as-Saheehah: 771.
[3] Abu Daawood, Kitaab al-Malaahim, (Hadeeth 4287), 11/423.
Checker's Note: Hasan, Saheeh Abee Daawood.
[4] Ahmad, Musnad, 1/227.
Checker's Note: Saheeh, also reported in al-Bukhaaree.
[5] Checker's Note: Saheeh, Ahmad, Muslim and at-Tirmidhee. Silsilatul Ahadeeth as-Saheehah: 7684.
[6] Ahmad, Musnad, 1/23.
Checker's Note: Da`eef, Da`eef al-Jami as-Sagheer: 3298. Da`eef due to Ibn Lahi'ah and `an`anah of Abu Zubair
Madeenah will remain inhabited at the time of the Dajjaal
It has been proven in the Saheeh Ahaadeeth, as already stated, that the Dajjaal will not be able to enter Makkah and Madeenah, and that there will be angels at the gates of Makkah to ward him off and prevent him

from entering.
It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Neither the Dajjaal nor plague will be able to enter Madeenah."
As mentioned above, the Dajjaal will camp outside Madeenah, and it will be shaken by three tremors. Every hypocrite and sinner will go out to join the Dajjaal, and every believer and Muslim will stay. That day will be called the Day of Purification (Yawm al-Khalaas). Most of those who go out to join the Dajjaal will be women. As the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"Verily it (Madeenah) is good; its evil will be eliminated and its goodness will be obvious."
Allaah (subhaanahu wa ta'aala) said:
"Women impure are for men impure, and men impure for women impure, and women of purity are for men of purity, and men of purity are for women of purity..." [an-Noor (24):26)
Madeenah will remain inhabited during the days of the Dajjaal, and during the time of Jesus son of Mary (alayhi-salam), until he dies and is buried there. Then it will be destroyed.
'Umar ibn al-Khattab said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'A rider will go around Madeenah and say, There used to be many Muslims here.' " [Ahmad] [1]

Footnotes
[1] Ahmad, Musnad, 1/20.
Checker's Note: Da`eef isnaad due to Ibn Lahi'ah and `an`anah of Abu Zubair. The fact that Madinah will be deserted is reported from Abu Hurairah by al-Bukhaaree.
The Emergence of the Beast
Among the signs of the Hour will be the emergence of a beast from the earth. It will be very strange inappearance, and extremely huge; one cannot even imagine what it will look like. It will emerge from the earth and shake the dust from its head. It will have with it the ring of Solomon and the rod of Moses. People will be terrified of it and will try to run away, but they will not be able to escape, because such will be the decree of Allaah. It will destroy the nose of every unbeliever with the rod, and write the word "Kaafir" on his forehead; it will adorn the face of every believer and write the word "Mu'min" (true believer) on his forehead, and it will speak to people.

Allaah (subhaanahu wa ta'aala) said:
"And when the Word is fulfilled against them (the unjust), We shall produce from the earth a Beast to (face) them: it will speak to them..." [an-Naml (27):82]
Ibn 'Abbaas, al-Hasan and Qataadah said that "It will speak to them" (tukallimuhum) means that it will address them. Ibn Jareer suggested that it means that the Beast will address them with the words "...for that mankind did not believe with assurance in Our Signs..." [an-Naml (27):82 - latter part of the Ayah]
Ibn Jareer reported this from 'Alee and 'Ata'. It was reported from Ibn 'Abbaas that tukallimuhum means that the Beast will cut them, i.e., it will write the word "Kaafir" on the forehead of the unbeliever. It was also reported from Ibn 'Abbaas that he will both address them and cut them; this suggestion incorporates both of the previous suggestions; and Allaah knows best.
We have already mentioned the Hadeeth of Hudhayfah ibn Usayd, in which the Prophet (sal-Allaahu 'alayhe wa sallam) is reported to have said,
"The Hour will not come until you see ten signs: the smoke; the Dajjaal; the Beast; the sun rising from the West; the descent of Jesus son of Mary; Gog and Magog; and three land-slides - one in the East, one in the West, and one in Arabia, at the end of which fire will burst forth from the direction of Aden (Yemen) and drive people to the place of their final assembly."
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Hasten to do good deeds before six things happen: the rising of the sun from the West, the smoke, the Dajjaal, the Beast, the (death) of one of you or general tribulation.' " [Muslim]
Bareedah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) took me to a place in the desert, near Makkah. It was a dry piece of land surrounded by sand. The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Beast will emerge from this place.' It was a very small area." [Ibn Maajah] [1]
It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Beast of the Earth will emerge, and will have with it the rod of Moses and the ring of Solomon."
It was also reported that he said,
"(The Beast) will destroy the noses of the unbelievers with the ring, - so

that people seated around one table will begin to address one another with the words "O Believer!" or "O Unbeliever!" (i.e., everyone's status will become clear) [Ibn Maajah] [2]
'Abd Allaah ibn 'Amr said,
"I memorised a Hadeeth from the Prophet (sal-Allaahu 'alayhe wa sallam) which I have not forgotten since. I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'The first of the signs (of the Hour) to appear will be the rising of the sun from the West and the appearance of the Beast before the people in the forenoon. Whichever of these two events happens first, the other will follow immediately.' " [Muslim] [3]
That is to say, these will be the first extraordinary signs. The Dajjaal, the descent of Jesus (alayhi-salam), the emergence of Gog and Magog, are less unusual in that they are all human beings. But the emergence of the Beast, whose form will be very strange, its addressing the people and classifying them according to their faith or unbelief, is something truly extraordinary. This is the first of the earthly signs, as the rising of the sun from the West is the first of the heavenly signs.

Footnotes
[1] Ibn Maajah, Kitaab al-Fitan, (Hadeeth 4267), 2/1352.
Checker's Note: Da`eef Jiddan (very weak). Ibn Maajah, no. 4067, Da`eef Ibn Maajah: 882.
[2] Similar Hadeeth narrated by Ibn Maajah in Kitaab al-Fitan, (Hadeeth 4061), 2 - 1351, 1352. Ahmad, Musnad, 2 - 295.
Checker's Note: Da`eef, Da`eef al-Jami as-Sagheer: 2413.
[3] Muslim, Kitaab al-Fitan, 8/202.
The Rising of the Sun from the West
Allaah (subhaanahu wa ta'aala) says:
"Do they then wait for anything other than that the angels should come to them, or that your Lord should come, or that some of the Signs of your Lord should come (i.e. portents of the Hour e.g., arising of the sun from the west)! The day that some of the Signs of your Lord do come, no good will it do to a person to believe then, if he believed not before, nor earned good (by performing deeds of righteousness) through his Faith. Say: 'Wait you! we (too) are waiting.'" [al-An'aam (6):158]
It was reported from Abu Sa'eed al-Khudree that the Prophet (sal-Allaahu 'alayhe wa sallam) explained,
"The day that some of the Signs of your Lord do come, no good will it do to a person to believe then...", referring to the rising of the sun from the West. [Ahmad] [1]

Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'The Hour will not come until the sun rises from the West. When the people see it, whoever is living on earth will believe, but that will be the time when - No good will it do to a person to believe then, if he believed not before then' " [Bukhaaree] [2]
It was also reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Hour will not come until the sun rises from the West. When it rises and the people see it, they will all believe. But that will be the time when 'No good will it do to a person to believe then.' " [Bukhaaree] [3]
It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"There are three things which, if they appear, 'No good will it do to a person to believe then, if he believed not before, nor earned good (by performing deeds of righteousness) through his Faith' They are: the rising of the sun from the West, the Dajjaal, and the Beast of the Earth." [Ahmad, Muslim, Tirmidhee] [4]
Abu Dharr said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) asked me, 'Do you know where the sun goes when it sets?' I said, 'I do not know' He said, 'It travels until it prostrates itself beneath the Throne, and asks for permission to rise again. But a time will come when it will be told, 'Go back whence you came.' That will be the time when 'No good will it do to a person to believe then, if he believed not before, nor earned good (by performing deeds of righteousness) through his Faith' " [Bukhaaree] [5]
'Amr ibn Jareer said,
"Three Muslims were sitting with Marwaan in Madeenah, and heard him say, whilst talking about the Signs of the Hour, that the first of them would be the appearance of the Dajjaal. The three went to 'Abd Allaah ibn 'Amr, and told him what they had heard Marwaan say concerning the Signs. 'Abd Allaah said, 'Marwan has not said much. I memorised a Hadeeth like that from the Prophet (sal-Allaahu 'alayhe wa sallam) which I have not forgotten since. I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say:
"The first of the signs will be the rising of the sun from the West, and the emergence of the Beast in the forenoon. Whichever of the two comes first, the other will follow immediately"
Then 'Abd Allaah, who was widely-read, said,
"I think that the first to happen will be the rising of the sun from the West.

Every time it sets, it goes beneath the Throne, prostrates itself, and seeks permission to rise again. A time will come when three times it will seek permission and will receive no reply, until, when part of the night has passed and it realises that even if it were given permission, it would not be able to rise on time, it will say: 'O my Lord, how far the rising-point is from me! What can I do for the people now?' Then it will seek permission to go back, and it will be told: 'Rise from where you are now' - and it will rise from the West." Then 'Abd Allaah recited the Aayah: "No good will it do to a person to believe then, if he believed not before, nor earned good (by performing deeds of righteousness) through his Faith." [Ahmad] [6]

Footnotes
[1] Ahmad, Musnad, 3/31.
Checker's Note: Saheeh, Saheeh at-Tirmidhee: 2455.
[2] Bukhaaree, Kitaab al-Tafsir, 6/73.
[3] Bukhaaree, ibid.
[4] Muslim, Kitaab al-Imaan, 1/96. Tirmidhee, Abwaab al-Tafsir (Hadeeth 5067), 8/449, 450. Ahmad, Musnad, 2/455, 446.
[5] Similar Hadeeth in Bukhaaree, Kitaab Bid' al-Khalq, 4/131.
[6] Ahmad, Musnad, 2/201.
Checker's Note: Saheeh, Musnad Imaam Ahmad, checking by Ahmad Shakir: 6881.

The Smoke which will appear at the End of Time

Masrooq said:
"While a man was giving a speech among the people of Kindah, he said, 'There will be smoke on the Day of Resurrection which will deprive the hypocrites of their hearing and sight, but the believers will only suffer something like a cold.' We were terrified, so we went to Ibn Masood, who was reclining. When he heard about this, he became angry and sat up, and said: 'O people, whoever knows a thing, let him say it; but whoever does not know, let him say, "Allaah knows best." It is a part of knowledge, when one does not know something, to say "Allaah knows best." Allaah (subhaanahu wa ta'aala) said to His Prophet Muhammad (sal-Allaahu 'alayhe wa sallam):
"Say (O Muhammad sal-Allaahu 'alayhe wa sallam): 'No wage do I ask of you for this (the Qur'aan), nor am I one of the Mutakallifoon (those who pretend and fabricate things which do not exist).'" [Saad (38):86]
"Quraysh were being slow in embracing Islaam, so the Prophet (sal-Allaahu 'alayhe wa sallam) prayed against them, saying, 'O Allaah, help

me against them by sending seven years of famine like those of Joseph.' They were afflicted by a year of famine in which they were destroyed, and ate dead animals and bones. They began to see something like smoke between the sky and the earth. Abu Sufyan came and said, 'O Muhammad! You came to command us to keep good relations with our relatives, and your people have perished, so pray that Allaah may relieve them.'"

Then Ibn Mas'ood recited,

"Then wait you for the Day when the sky will bring forth a visible smoke. Covering the people, this is a painful torment. (They will say): "Our Lord! Remove the torment from us, really we shall become believers!" How can there be for them an admonition (at the time when the torment has reached them), when a Messenger explaining things clearly has already come to them. Then they had turned away from him (Messenger Muhammad sal-Allaahu 'alayhe wa sallam) and said: "One (Muhammad sal-Allaahu 'alayhe wa sallam) taught (by a human being), a madman!" Verily, We shall remove the torment for a while. Verily! You will revert." [ad-Dukhaan (44):10-15]

Ibn Masood asked:

"Will their punishment in the Hereafter be removed so they can go back to their Kufr?"

Allaah (subhaanahu wa ta'aala) said:

"On the Day when We shall seize you with the greatest grasp. Verily, We will exact retribution." [ad-Dukhaan (44):16]

...and soon will come the inevitable (punishment)!" [al-Furqaan (25):77]

These Ayaat refer to the Day (Battle) of Badr. Allaah (subhaanahu wa ta'aala) said:

"Alif Laam Meem. The Romans have been defeated. In the nearer land (Syria, Iraq, Jordan, and Palestine), and they, after their defeat, will be victorious. Within three to nine years." [ar-Room (30): 1-4] [Bukhaaree] [1]

This speaker's suggestion - that the idea that the smoke would be on the Day or Resurrection was not a good one - made Ibn Masood react angrily. But the smoke will appear before the Day of Resurrection; it will be one of the signs, which are: the Beast, the Dajjaal, the Smoke, and Gog and Magog, as the Ahaadeeth narrated from Abu Sareehah, Abu Hurairah and other Sahaabah indicate.

As mentioned in the Saheeh Ahaadeeth, the fire which will appear before the Day of Resurrection will burst forth from the direction of Aden, and drive the people to the place of their final assembly. It will move with them and halt with them, and will devour any who lag behind.

Footnote
[1] Bukhaaree, Kitaab at-Tafsir - Soorat ar-Room, 6/142, 143

Other Events which will precede the Hour

It was reported from Abu Sa'eed al-Khudree that the Prophet (sal-Allaahu 'alayhe wa sallam) said,

"Thunderbolts will increase so much as the Hour approaches that when a man comes to a people, he will ask, 'Who amongst you was struck by a thunderbolt this morning?' and they will say, 'So-and-so and so-and-so was struck.' " [Ahmad] [1]

Heavy rain before the Day of Resurrection

It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said,

"The hour will not come until there has been rain which will destroy all dwellings except tents." [2]

We have already mentioned many Ahaadeeth about the signs of the Hour. Now we will turn our attention to some Ahaadeeth which could indicate that the Hour is close at hand.

It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said:

"The Hour will not come until the following events have come to pass: people will compete with one another in constructing high buildings; two big groups will fight one another, and there will be many casualties - they will both be following the same religious teaching; earthquakes will increase; time will pass quickly; afflictions and killing will increase; nearly thirty dajjals will appear, each of them claiming to be a messenger from Allaah; a man will pass by a grave and say, 'Would that I were in your place'; the sun will rise from the West; when it rises and the people see it, they will all believe, but that will be the time when 'No good will it do to a person to believe then, if it believed not before...' [al-An'aam (6):158]; and a wealthy man will worry lest no-one accept his Zakaat." [Bukhaaree, Muslim]

It was reported from Anas that the Prophet (sal-Allaahu 'alayhe wa sallam) said:

"Among the signs of the Hour are the following: knowledge will decrease and ignorance will prevail; fornication and the drinking of wine will be common; the number of men will decrease and the number of women will increase, until one man will look after fifty women." [Bukhaaree] [3]

'Aa'ishah said,

"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say, 'Day and night will not pass away until people begin to worship Laat and 'Uzzaa' (two goddesses of pre-Islaamic Arabic). I said, 'O Messenger of Allaah, I thought that when Allaah revealed the Aayah "It is He Who has sent His Messenger (Muhammad sal-Allaahu 'alayhe wa sallam) with guidance and the religion of truth (Islaam), to make it superior over all religions even though the Mushrikoon (polytheists, pagans, idolaters, disbelievers in the Oneness of Allaah) hate (it)" [at-Tawbah (9):33], it implied that (this promise) would be fulfilled.' The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'It will happen as Allaah wishes. Then Allaah will send a pleasant breeze, which will take everyone who has as much faith as a grain of mustard-seed in his heart. Only those with no goodness in them will be left, and they will revert to the religion of their forefathers.' " [Muslim] [4]
It was reported from Abu Hurairah that one day, while the Prophet (sal-Allaahu 'alayhe wa sallam) was sitting with the people, a Bedouin came to him and asked him about Eemaan and Islaam, then he asked,
"O Messenger of Allaah, when will the Hour be?" He said, "The one who is asked about it does not know more than the one who asks, but I tell you about its signs. When a slave gives birth to her mistress, and when the bare-footed and naked become the chiefs of the people - these are among the signs of the Hour. There are five things which no-one knows except Allaah." Then he recited:
"Verily the knowledge of the Hour is with God (alone). It is He Who sends down rain, and He Who knows what is in the wombs. Nor does anyone know what it is that he will earn on the morrow: nor does anyone know in what land he is to die. Verily God is All-Knowing, All Aware." [Luqman (31):34]
Then the man went away, and the Prophet (sal-Allaahu 'alayhe wa sallam) said, "Call him back to me," but when the people went to call him, they could not see anything. The Prophet (sal-Allaahu 'alayhe wa sallam) said, "That was Gabriel, who came to teach the people their religion." [Bukhaaree, Muslim] [5]
"The bare-footed and naked paupers will compete with one another in constructing high buildings" means that they will become the chiefs of people. They will become rich, and their only concern will be to compete in constructing high buildings. This is as in the Hadeeth we shall see later:
"The Hour will not come until the happiest people in the world will be the depraved sons of the depraved."
It was reported from Abu Sa'eed that the Prophet (sal-Allaahu 'alayhe wa

sallam) said:
"The Hour will not come until the time when a man will leave his home, and his shoes or whip or stick will tell what is happening to his family." [Ahmad] [6]
It was reported from Abu Sa'eed that the Prophet (sal-Allaahu 'alayhe wa sallam) said:
"By Him in Whose hand is my soul, the Hour will not come until wild animals talk to men, and a man speaks to his whip or his shoe, and his thigh will tell him about what happened to his family after he left." [7]
Anas said,
"We were discussing the fact that the Hour would not come until there is no rain, the earth does not produce crops, and fifty women will be cared for by one man; and if a woman passes by a man, he will look at her and say, 'This woman once had a husband.' " [Ahmad] [8]
Abu Hurairah said:
"The Prophet (sal-Allaahu 'alayhe wa sallam) said: 'The Hour will not come until time passes so quickly that a year will be like a month, a month like a week, a week like a day, a day like an hour, and an hour like the time it takes for a palm-leaf to burn.' " [Ahmad] [9]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said: 'The world will not pass away until the one who enjoys it the most is the depraved son of the depraved.' " [Ahmad] [10]
Abu Hurairah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) said, 'Before the Hour comes, there will be years of deceit, in which a truthful person will be disbelieved and a liar will be believed; and the insignificant will have a say.' " [Ahmad] [11]
Abu Hurairah said:
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say: 'The Hour will not come until the sheep with horns no longer fights the sheep without horns.' " [Ahmad] [12]
It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Hour will not come until wealth increases so much that a wealthy man will be worried lest no-one accept his Sadaqah; knowledge will be taken away; time will pass quickly; tribulations will appear; and there will be much Harj." The people asked, "What is Harj, O Messenger of Allaah?" He said, "Killing, killing." [Ahmad] [13]
It was reported from Abu Hurairah that the Prophet (sal-Allaahu 'alayhe

wa sallam) said,
"By Him Who sent me with the Truth, this earth will not pass away until people are afflicted with landslides, are pelted with stones, and are transformed into animals." The people asked, "When will that be, O Messenger of Allaah?" He said, "When you see women riding in the saddle, when singers are common, when bearing false witness becomes widespread, and when men lie with men and women with women." [14]
Taariq ibn Shihaab said,
"We were sitting with 'Abd Allaah ibn Mas'ood, when a man came and told us that the time for prayer had come. So we got up and went to the mosque... After the prayer, a man came to 'Abd Allaah ibn Mas'ood and said, 'As-salaam 'alayka (Peace be upon you), O Abu 'Abd ar-Rahmaan'. 'Abd Allaah answered, 'Allaah and His Messenger have spoken the truth.' When we went back, we asked one another, 'Did you hear the answer he gave? Who is going to ask him about it?' I said, 'I will ask him'; so I asked him when he came out. He narrated from the Prophet (sal-Allaahu 'alayhe wa sallam):
'Before the Hour comes, there will be a special greeting for the people of distinction; trade will become so widespread that a woman will help her husband in business; family ties will be cut; the giving of false witness will be common, while truthful witness will be rare; and writing will be widespread.' [Ahmad] [15]

Footnotes
[1] Ahmad, Musnad, 3/64, 65.
Checker's Note: Da`eef isnaad due to Muhammad ibn Mus'ab.
[2] Ahmad, Musnad, 2/262.
Checker's Note: Saheeh, Musnad Imaam Ahmad, checking by Ahmad Shakir: 7554 and Majma' az-Zawaid of al-Haythamee: 7/331.
[3] Bukhaaree, Kitaab al-'Ilm, 1/30, 31.
[4] Muslim, Kitaab al-Fitan 8/182.
[5] Bukhaaree, Kitaab al-Imaan, 1/19; Kitaab at-Tafsir - Soorat Luqmaan, 6/144. Muslim, Kitaab al-Imaan, 1/30, 31.
[6] Ahmad, Musnad, 3/88, 89.
Checker's Note: Saheeh.
[7] Ahmad, Musnad, 3/82, 84.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 122.
[8] Ahmad, Musnad, 3/286.
Checker's Note: Saheeh, reported by Ahmad, Abu Ya'la and al-Bazzar. Majma' az-Zawa'id of al-Haythamee: 7/333 and Kashful Astar bi Zawa'id

al-Bazzar of al-Haythamee: 3415.
[9] Ahmad, Musnad, 3/358.
Checker's Note: Saheeh, Saheeh al-Jami as-Sagheer: 7422.
[10] Ahmad, Musnad, 2/358.
Checker's Note: Saheeh, Saheeh al-Jami as-Sagheer: 7272.
[11] Ahmad, Musnad, 2/238.
Checker's Note: Hasan, Silsilatul Ahadeeth as-Saheehah: 1887.
[12] Ahmad, Musnad, 2/442.
Checker's Note: Da`eef?? [not quite sure] due to as-Salt ibn Quwaid; an-Nasaa'ee said, "His hadith is munkar."
[13] Ahmad, Musnad, 2/313.
Checker's Note: Saheeh, Musnad Imaam Ahmad, checking by Ahmad Shakir: 8120, 8121.
[14] al-Haythamee, Kitaab al-Fitan.
Checker's Note: Da`eef, related by al-Bazzar and at-Tabaraanee in al-Awsat. It is weak due to Sulaiman ibn Daawood al-Yamami who is matrook (abandoned). Majma' az-Zawa'id of al-Haythamee.
[15] Ahmad, Musnad, 1/407.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 648.

Description of the People who will be Alive at the End of Time

It was reported from 'Abd Allaah ibn 'Amr that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Hour will not come until Allaah takes away the best people on earth; only the worst people will be left; they will not know any good or forbid any evil." [Ahmad] [1]
'Abd Allaah ibn Mas'ood said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say: 'Eloquence can be bewitching; the worst of the people are those upon whom the Hour will come while they are still alive, and those who turn graves into mosques.' " [Ahmad] [2]
It was reported from Anas that the Prophet (sal-Allaahu 'alayhe wa sallam) said,
"The Hour will not come until no-one on earth says 'Laa ilaaha illaa Allaah.'" [Ahmad] [3]
It was also reported from Anas that the Prophet (sal-Allaahu 'alayhe wa sallam) said:
"The Hour will not come until no-one on earth says, 'Allaah, Allaah.'" [Ahmad] [4]
There are two suggestions as to the meaning of the phrase, "until no-one on earth says 'Allaah, Allaah'":

It could mean that no-one will forbid evil, or try to correct another if he sees him doing something wrong. We have already come across this in the Hadeeth of 'Abd Allaah ibn 'Amr:
"Only the worst people will be left; they will not know any good or forbid any evil."
It could mean that Allaah will no longer be mentioned, and His Name will not be known; this will be part of the prevalent corruption and Kufr, as in the previous Hadeeth,
"Until no-one on earth says 'Laa ilaaha illaa Allaah."
'Aa'ishah said,
"The Prophet (sal-Allaahu 'alayhe wa sallam) came in, saying 'O 'Aa'ishah, your people will be the first of my Ummah to join me.' When he sat down, I said, 'O Messenger of Allaah, may I be sacrificed for you! When you came in, you were saying something which scared me.' He asked, 'What was that?' I said, 'You said that my people would be the first of your Ummah to join you.' He said, 'Yes.' I asked, 'Why is that?' He said, 'Death will be widespread among them, and their relatives will be jealous of them.' I said, 'How will people be after that?' He said, 'Like locusts: the strong will devour the weak, until the Hour comes.' " [Ahmad] [5]
'Albaa' al-Salamee said,
"I heard the Prophet (sal-Allaahu 'alayhe wa sallam) say: 'The Hour will only come upon the worst of the people.' " [Ahmad] [6]

Footnotes
[1] Ahmad, Musnad, 1/454.
Checker's Note: Da`eef isnaad due to `an`anah of al-Hasan al-Basree.
[2] Ahmad, Musnad, 3/268.
Checker's Note: Saheeh, Musnad Imaam Ahmad, checking by Ahmad Shakir: 4342.
[3] Ahmad, Musnad, 3/268.
Checker's Note: Saheeh, Tahdheerus Saajid of Shaykh al-Albaanee, pg. 119.
[4] Ahmad, Musnad, 3/107.
Checker's Note: Saheeh, Saheeh al-Jami as-Sagheer: 7420.
[5] Ahmad, Musnad, 6/81.
Checker's Note: Saheeh, Silsilatul Ahadeeth as-Saheehah: 1953.
[6] Ahmad, Musnad, 3/499.
Checker's Note: Saheeh, Mustadrak of al-Hakim: 4/496. Declared saheeh by al-Hakim and agreed to by adh-Dhahaabee

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