Answers to Exercises

Microeconomic Analysis

Third Edition

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ANSWERS

Chapter 1. Technology

1.1 False. There are many counterexamples. Consider the technology generated by a production function $f(x) = x^2$. The production set is $Y = \{(y, -x) : y \leq x^2\}$ which is certainly not convex, but the input requirement set is $V(y) = \{x : x \geq \sqrt{y}\}$ which is a convex set.

1.2 It doesn't change.

1.3 $\epsilon_1 = a$ and $\epsilon_2 = b$.

1.4 Let $y(t) = f(t\mathbf{x})$. Then

$$\frac{dy}{dt} = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i,$$

so that

$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{f(\mathbf{x})} \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i.$$

1.5 Substitute tx_i for $i = 1, 2$ to get

$$f(tx_1, tx_2) = [(tx_1)^\rho + (tx_2)^\rho]^{\frac{1}{\rho}} = t[x_1^\rho + x_2^\rho]^{\frac{1}{\rho}} = tf(x_1, x_2).$$

This implies that the CES function exhibits constant returns to scale and hence has an elasticity of scale of 1.

1.6 This is half true: if $g'(x) > 0$, then the function must be strictly increasing, but the converse is not true. Consider, for example, the function $g(x) = x^3$. This is strictly increasing, but $g'(0) = 0$.

1.7 Let $f(\mathbf{x}) = g(h(\mathbf{x}))$ and suppose that $g(h(\mathbf{x})) = g(h(\mathbf{x}'))$. Since g is monotonic, it follows that $h(\mathbf{x}) = h(\mathbf{x}')$. Now $g(h(t\mathbf{x})) = g(th(\mathbf{x}))$ and $g(h(t\mathbf{x}')) = g(th(\mathbf{x}'))$ which gives us the required result.

1.8 A homothetic function can be written as $g(h(\mathbf{x}))$ where $h(\mathbf{x})$ is homogeneous of degree 1. Hence the TRS of a homothetic function has the form

$$\frac{g'(h(\mathbf{x}))\frac{\partial h}{\partial x_1}}{g'(h(\mathbf{x}))\frac{\partial h}{\partial x_2}} = \frac{\frac{\partial h}{\partial x_1}}{\frac{\partial h}{\partial x_2}}.$$

That is, the TRS of a homothetic function is just the TRS of the underlying homogeneous function. But we already know that the TRS of a homogeneous function has the required property.

1.9 Note that we can write

$$(a_1 + a_2)^{\frac{1}{\rho}} \left[\frac{a_1}{a_1 + a_2} x_1^\rho + \frac{a_2}{a_1 + a_2} x_2^\rho \right]^{\frac{1}{\rho}}.$$

Now simply define $b = a_1/(a_1 + a_2)$ and $A = (a_1 + a_2)^{\frac{1}{\rho}}$.

1.10 To prove convexity, we must show that for all $\mathbf{y}$ and $\mathbf{y}'$ in Y and $0 \leq t \leq 1$, we must have $t\mathbf{y} + (1-t)\mathbf{y}'$ in Y. But divisibility implies that $t\mathbf{y}$ and $(1-t)\mathbf{y}'$ are in Y, and additivity implies that their sum is in Y. To show constant returns to scale, we must show that if $\mathbf{y}$ is in Y, and $s > 0$, we must have $s\mathbf{y}$ in Y. Given any $s > 0$, let n be a nonnegative integer such that $n \geq s \geq n - 1$. By additivity, $n\mathbf{y}$ is in Y; since $s/n \leq 1$, divisibility implies $(s/n)n\mathbf{y} = s\mathbf{y}$ is in Y.

1.11.a This is closed and nonempty for all $y > 0$ (if we allow inputs to be negative). The isoquants look just like the Leontief technology except we are measuring output in units of $\log y$ rather than y. Hence, the shape of the isoquants will be the same. It follows that the technology is monotonic and convex.

1.11.b This is nonempty but not closed. It is monotonic and convex.

1.11.c This is regular. The derivatives of $f(x_1, x_2)$ are both positive so the technology is monotonic. For the isoquant to be convex to the origin, it is sufficient (but not necessary) that the production function is concave. To check this, form a matrix using the second derivatives of the production function, and see if it is negative semidefinite. The first principal minor of the Hessian must have a negative determinant, and the second principal minor must have a nonnegative determinant.

$$\frac{\partial^2 f(x)}{\partial x_1^2} = -\frac{1}{4} x_1^{-\frac{3}{2}} x_2^{\frac{1}{2}} \qquad \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} = \frac{1}{4} x_1^{\frac{-1}{2}} x_2^{-\frac{1}{2}}$$

$$\frac{\partial^2 f(x)}{\partial x_2^2} = -\frac{1}{4} x_1^{\frac{1}{2}} x_2^{\frac{-3}{2}}$$

$$\text{Hessian} = \begin{bmatrix} -\frac{1}{4}x_1^{-3/2}x_2^{1/2} & \frac{1}{4}x_1^{-1/2}x_2^{-1/2} \\ \frac{1}{4}x_1^{-1/2}x_2^{-1/2} & -\frac{1}{4}x_1^{1/2}x_2^{-3/2} \end{bmatrix}$$

$$D_1 = -\frac{1}{4}x_1^{-3/2}x_2^{1/2} < 0$$
$$D_2 = \frac{1}{16}x_1^{-1}x_2^{-1} - \frac{1}{16}x_1^{-1}x_2^{-1} = 0.$$

So the input requirement set is convex.

1.11.d This is regular, monotonic, and convex.

1.11.e This is nonempty, but there is no way to produce any $y > 1$. It is monotonic and weakly convex.

1.11.f This is regular. To check monotonicity, write down the production function $f(x) = ax_1 - \sqrt{x_1 x_2} + bx_2$ and compute

$$\frac{\partial f(x)}{\partial x_1} = a - \frac{1}{2}x_1^{-1/2}x_2^{1/2}.$$

This is positive only if $a > \frac{1}{2}\sqrt{\frac{x_2}{x_1}}$, thus the input requirement set is not always monotonic.

Looking at the Hessian of f, its determinant is zero, and the determinant of the first principal minor is positive. Therefore f is not concave. This alone is not sufficient to show that the input requirement sets are not convex. But we can say even more: f is *convex*; therefore, all sets of the form

$$\{x_1, x_2 \colon ax_1 - \sqrt{x_1 x_2} + bx_2 \leq y\} \quad \text{for all choices of } y$$

are convex. Except for the border points this is just the complement of the input requirement sets we are interested in (the inequality sign goes in the wrong direction). As complements of convex sets (such that the border line is not a straight line) our input requirement sets can therefore not be themselves convex.

1.11.g This function is the successive application of a linear and a Leontief function, so it has all of the properties possessed by these two types of functions, including being regular, monotonic, and convex.

Chapter 2. Profit Maximization

2.1 For profit maximization, the Kuhn-Tucker theorem requires the following three inequalities to hold

$$\left(p\frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j\right) x_j^* = 0,$$
$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j \leq 0,$$
$$x_j^* \geq 0.$$

Note that if $x_j^* > 0$, then we must have $w_j/p = \partial f(\mathbf{x}^*)/\partial x_j$.

2.2 Suppose that $\mathbf{x}'$ is a profit-maximizing bundle with positive profits $\pi(\mathbf{x}') > 0$. Since

$$f(t\mathbf{x}') > tf(\mathbf{x}'),$$

for $t > 1$, we have

$$\pi(t\mathbf{x}') = pf(t\mathbf{x}') - t\mathbf{w}\mathbf{x}' > t(pf(\mathbf{x}') - \mathbf{w}\mathbf{x}') > t\pi(\mathbf{x}') > \pi(\mathbf{x}').$$

Therefore, $\mathbf{x}'$ could not possibly be a profit-maximizing bundle.

2.3 In the text the supply function and the factor demands were computed for this technology. Using those results, the profit function is given by

$$\pi(p, w) = p\left(\frac{w}{ap}\right)^{\frac{a}{a-1}} - w\left(\frac{w}{ap}\right)^{\frac{1}{a-1}}.$$

To prove homogeneity, note that

$$\pi(tp, tw) = tp\left(\frac{w}{ap}\right)^{\frac{a}{a-1}} - tw\left(\frac{w}{ap}\right)^{\frac{1}{a-1}} = t\pi(p, w),$$

which implies that $\pi(p, w)$ is a homogeneous function of degree 1.

Before computing the Hessian matrix, factor the profit function in the following way:

$$\pi(p, w) = p^{\frac{1}{1-a}} w^{\frac{a}{a-1}} \left(a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right) = p^{\frac{1}{1-a}} w^{\frac{a}{a-1}} \phi(a),$$

where $\phi(a)$ is strictly positive for $0 < a < 1$.

The Hessian matrix can now be written as

$$D^2\pi(p, \omega) = \begin{pmatrix} \frac{\partial^2 \pi(p,w)}{\partial p^2} & \frac{\partial^2 \pi(p,w)}{\partial p \partial w} \\ \frac{\partial^2 \pi(p,w)}{\partial w \partial p} & \frac{\partial^2 \pi(p,w)}{\partial w^2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{a}{(1-a)^2} p^{\frac{2a-1}{1-a}} w^{\frac{a}{a-1}} & -\frac{a}{(1-a)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}} \\ \\ -\frac{a}{(1-a)^2} p^{\frac{a}{1-a}} w^{\frac{1}{a-1}} & \frac{a}{(1-a)^2} p^{\frac{1}{1-a}} w^{\frac{2-a}{a-1}} \end{pmatrix} \phi(a).$$

The principal minors of this matrix are

$$\frac{a}{(1-a)^2} p^{\frac{2a-1}{1-a}} w^{\frac{a}{a-1}} \phi(a) > 0$$

and 0. Therefore, the Hessian is a positive semidefinite matrix, which implies that $\pi(p, w)$ is convex in (p, w).

2.4 By profit maximization, we have

$$|TRS| = \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = \frac{w_1}{w_2}.$$

Now, note that

$$\ln(w_2 x_2 / w_1 x_1) = -(\ln(w_1/w_2) + \ln(x_1/x_2)).$$

Therefore,

$$\frac{d \ln(w_2 x_2 / w_1 x_1)}{d \ln(x_1/x_2)} = \frac{d \ln(w_1/w_2)}{d \ln(x_2/x_1)} - 1 = \frac{d \ln |TRS|}{d \ln(x_2/x_1)} - 1 = 1/\sigma - 1.$$

2.5 From the previous exercise, we know that

$$\ln(w_2 x_2 / w_1 x_1) = \ln(w_2/w_1) + \ln(x_2/x_1),$$

Differentiating, we get

$$\frac{d \ln(w_2 x_2 / w_1 x_1)}{d \ln(w_2/w_1)} = 1 - \frac{d \ln(x_2/x_1)}{d \ln |TRS|} = 1 - \sigma.$$

2.6 We know from the text that $YO \supset Y \supset YI$. Hence for any $\mathbf{p}$, the maximum of $\mathbf{py}$ over YO must be larger than the maximum over Y, and this in turn must be larger than the maximum over YI.

2.7.a We want to maximize $20x - x^2 - wx$. The first-order condition is $20 - 2x - w = 0$.

2.7.b For the optimal x to be zero, the derivative of profit with respect to x must be nonpositive at $x = 0$: $20 - 2x - w < 0$ when $x = 0$, or $w \geq 20$.

2.7.c The optimal x will be 10 when $w = 0$.

2.7.d The factor demand function is $x = 10 - w/2$, or, to be more precise, $x = \max\{10 - w/2, 0\}$.

2.7.e Profits as a function of output are

$$20x - x^2 - wx = [20 - w - x]x.$$

Substitute $x = 10 - w/2$ to find

$$\pi(w) = \left[10 - \frac{w}{2}\right]^2.$$

2.7.f The derivative of profit with respect to w is $-(10 - w/2)$, which is, of course, the negative of the factor demand.

Chapter 3. Profit Function

3.1.a Since the profit function is convex and a decreasing function of the factor prices, we know that $\phi_i'(w_i) \leq 0$ and $\phi_i''(w_i) \geq 0$.

3.1.b It is zero.

3.1.c The demand for factor i is only a function of the i^{th} price. Therefore the marginal product of factor i can only depend on the amount of factor i. It follows that $f(x_1, x_2) = g_1(x_1) + g_2(x_2)$.

3.2 The first-order conditions are $p/x = w$, which gives us the demand function $x = p/w$ and the supply function $y = \ln(p/w)$. The profits from operating at this point are $p\ln(p/w) - p$. Since the firm can always choose $x = 0$ and make zero profits, the profit function becomes $\pi(p, w) = \max\{p\ln(p/w) - p, 0\}$.

3.3 The first-order conditions are

$$a_1 \frac{p}{x_1} - w_1 = 0$$
$$a_2 \frac{p}{x_2} - w_2 = 0,$$

which can easily be solved for the factor demand functions. Substituting into the objective function gives us the profit function.

3.4 The first-order conditions are

$$pa_1 x_1^{a_1 - 1} x_2^{a_2} - w_1 = 0$$
$$pa_2 x_2^{a_2 - 1} x_1^{a_1} - w_2 = 0,$$

which can easily be solved for the factor demands. Substituting into the objective function gives us the profit function for this technology. In order for this to be meaningful, the technology must exhibit decreasing returns to scale, so $a_1 + a_2 < 1$.

3.5 If w_i is strictly positive, the firm will never use more of factor i than it needs to, which implies $x_1 = x_2$. Hence the profit maximization problem can be written as

$$\max \; p x_1^a - w_1 x_1 - w_2 x_2.$$

The first-order condition is

$$p a x_1^{a-1} - (w_1 + w_2) = 0.$$

The factor demand function and the profit function are the same as if the production function were $f(x) = x^a$, but the factor price is $w_1 + w_2$ rather than w. In order for a maximum to exist, $a < 1$.

Chapter 4. Cost Minimization

4.1 Let $\mathbf{x}^*$ be a profit-maximizing input vector for prices $(p, \mathbf{w})$. This means that $\mathbf{x}^*$ must satisfy $pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^* \geq pf(\mathbf{x}) - \mathbf{w}\mathbf{x}$ for all permissible $\mathbf{x}$. Assume that $\mathbf{x}^*$ does not minimize cost for the output $f(\mathbf{x}^*)$; i.e., there exists a vector $\mathbf{x}^{**}$ such that $f(\mathbf{x}^{**}) \geq f(\mathbf{x}^*)$ and $\mathbf{w}(\mathbf{x}^{**} - \mathbf{x}^*) < 0$. But then the profits achieved with $\mathbf{x}^{**}$ must be greater than those achieved with $\mathbf{x}^*$:

$$\begin{aligned} pf(\mathbf{x}^{**}) - \mathbf{w}\mathbf{x}^{**} &\geq pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^{**} \\ &> pf(\mathbf{x}^*) - \mathbf{w}\mathbf{x}^*, \end{aligned}$$

which contradicts the assumption that $\mathbf{x}^*$ was profit-maximizing.

4.2 The complete set of conditions turns out to be

$$\begin{aligned} \left(t\frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j \right) x_j^* &= 0, \\ t\frac{\partial f(\mathbf{x}^*)}{\partial x_j} - \mathbf{w}_j &\leq 0, \\ x_j^* &\geq 0, \\ (y - f(\mathbf{x}^*))\, t &= 0, \\ y - f(\mathbf{x}^*) &\leq 0, \\ t &\geq 0. \end{aligned}$$

If, for instance, we have $x_i^* > 0$ and $x_j^* = 0$, the above conditions imply

$$\frac{\dfrac{\partial f(\mathbf{x}^*)}{\partial x_i}}{\dfrac{\partial f(\mathbf{x}^*)}{\partial x_j}} \geq \frac{\mathbf{w}_i}{\mathbf{w}_j}.$$

This means that it would decrease cost to substitute x_i for x_j, but since there is no x_j used, this is not possible. If we have interior solutions for both x_i and x_j, equality must hold.

4.3 Following the logic of the previous exercise, we equate marginal costs to find

$$y_1 = 1.$$

We also know $y_1 + y_2 = y$, so we can combine these two equations to get $y_2 = y - 1$. It appears that the cost function is $c(y) = 1/2 + y - 1 = y - 1/2$. However, on reflection this can't be right: it is obviously better to produce everything in plant 1 if $y_1 < 1$. As it happens, we have ignored the implicit constraint that $y_2 \geq 0$. The actual cost function is

$$c(y) = \begin{cases} y^2/2 & \text{if } y < 1 \\ y - 1/2 & \text{if } y > 1. \end{cases}$$

4.4 According to the text, we can write the cost function for the first plant as $c_1(y) = Ay$ and for the second plant as $c_2(y) = By$, where A and B depend on a, b, w_1, and w_2. It follows from the form of the cost functions that

$$c(y) = \min\{A, B\}y.$$

4.5 The cost of using activity a is $a_1w_1 + a_2w_2$, and the cost of using activity b is $b_1w_1 + b_2w_2$. The firm will use whichever is cheaper, so

$$c(w_1, w_2, y) = y \min\{a_1w_1 + a_2w_2, b_1w_1 + b_2w_2\}.$$

The demand function for factor 1, for example, is given by

$$x_1 = \begin{cases} a_1 y & \text{if } a_1w_1 + a_2w_2 < b_1w_1 + b_2w_2 \\ b_1 y & \text{if } a_1w_1 + a_2w_2 > b_1w_1 + b_2w_2 \\ \text{any amount between} & \\ \quad a_1y \text{ and } b_1y & \text{otherwise.} \end{cases}$$

The cost function will not be differentiable when

$$a_1w_1 + a_2w_2 = b_1w_1 + b_2w_2.$$

4.6 By the now standard argument,

$$c(y) = \min\{4\sqrt{y_1} + 2\sqrt{y_2} \colon y_1 + y_2 \geq y\}.$$

It is tempting to set $MC_1(y_1) = MC_2(y_2)$ to find that $y_1 = y/5$ and $y_2 = 4y/5$. However, if you think about it a minute you will see that this doesn't make sense—you are producing more output in the plant with the higher costs!

It turns out that this corresponds to a constrained *maximum* and not to the desired minimum. Check the second-order conditions to verify this.

Since the cost function is concave, rather than convex, the optimal solution will always occur at a boundary. That is, you will produce all output at the cheaper plant so $c(y) = 2\sqrt{y}$.

4.7 No, the data violate WACM. It costs 40 to produce 100 units of output, but at the same prices it would only cost 38 to produce 110 units of output.

4.8 Set up the minimization problem

$$\min\ x_1 + x_2$$
$$x_1 x_2 = y.$$

Substitute to get the unconstrained minimization problem

$$\min\ x_1 + y/x_1.$$

The first-order condition is

$$1 - y/x_1^2,$$

which implies $x_1 = \sqrt{y}$. By symmetry, $x_2 = \sqrt{y}$. We are given that $2\sqrt{y} = 4$, so $\sqrt{y} = 2$, from which it follows that $y = 4$.

Chapter 5. Cost Function

5.1 The firm wants to minimize the cost of producing a given level of output:

$$c(y) = \min_{y_1, y_2} y_1^2 + y_2^2$$
$$\text{such that } y_1 + y_2 = y.$$

The solution has $y_1 = y_2 = y/2$. Substituting into the objective function yields

$$c(y) = (y/2)^2 + (y/2)^2 = y^2/2.$$

5.2 The first-order conditions are $6y_1 = 2y_2$, or $y_2 = 3y_1$. We also require $y_1 + y_2 = y$. Solving these two equations in two unknowns yields $y_1 = y/4$ and $y_2 = 3y/4$. The cost function is

$$c(y) = 3\left[\frac{y}{4}\right]^2 + \left[\frac{3y}{4}\right]^2 = \frac{3y^2}{4}.$$

5.3 Consider the first technique. If this is used, then we need to have $2x_1 + x_2 = y$. Since this is linear, the firm will typically specialize and set $x_2 = y$ or $x_1 = y/2$ depending on which is cheaper. Hence the cost function for this technique is $y \min\{w_1/2, w_2\}$. Similarly, the cost function for the other technique is $y \min\{w_3, w_4/2\}$. Since both techniques must be used to produce y units of output,

$$c(w_1, w_2, y) = y \left[\min\{w_1/2, w_2\} + \min\{w_3, w_4/2\}\right].$$

5.4 The easiest way to answer this question is to sketch an isoquant. First draw the line $2x_1 + x_2 = y$ and then the line $x_1 + 2x_2 = y$. The isoquant is the upper northeast boundary of this "cross." The slope is -2 to the left of the diagonal and $-1/2$ to the right of the diagonal. This means that when $w_1/w_2 < 1/2$, we have $x_1 = 0$ and $x_2 = y$. When $w_1/w_2 < 1/2$, we have $x_1 = y$ and $x_2 = 0$. Finally, when $2 > w_1/w_2 > 1/2$, we have $x_1 = x_2 = y/3$. The cost function is then

$$c(w_1, w_2, y) = \min\{w_1, w_2, (w_1 + w_2)/3\}y.$$

5.5 The input requirement set is not convex. Since $y = \max\{x_1, x_2\}$, the firm will use whichever factor is cheaper; hence the cost function is $c(w_1, w_2, y) = \min\{w_1, w_2\}y$. The factor demand function for factor 1 has the form

$$x_1 = \begin{cases} y & \text{if } w_1 < w_2 \\ \text{either } 0 \text{ or } y & \text{if } w_1 = w_2 \\ 0 & \text{if } w_1 > w_2 \end{cases}.$$

5.6 We have $a = 1/2$ and $c = -1/2$ by homogeneity, and $b = 3$ since $\partial x_1/\partial w_2 = \partial x_2/\partial w_1$.

5.7 Set up the minimization problem

$$\min\ x_1 + x_2$$
$$x_1 x_2 = y.$$

Substitute to get the unconstrained minimization problem

$$\min\ x_1 + y/x_1.$$

The first-order condition is

$$1 - y/x_1^2,$$

which implies $x_1 = \sqrt{y}$. By symmetry, $x_2 = \sqrt{y}$. We are given that $2\sqrt{y} = 4$, so $\sqrt{y} = 2$, from which it follows that $y = 4$.

5.8 If $p = 2$, the firm will produce 1 unit of output. If $p = 1$, the first-order condition suggests $y = 1/2$, but this yields negative profits. The firm can get zero profits by choosing $y = 0$. The profit function is $\pi(p) = \max\{p^2/4 - 1, 0\}$.

5.9.a $d\pi/d\alpha = py > 0$.

5.9.b $dy/d\alpha = p/c''(y) > 0$.

5.9.c $p'(\alpha) = n[y + \alpha p/c'']/[D'(p) - n\alpha/c''] < 0$.

5.10 Let $y(p, \mathbf{w})$ be the supply function. Totally differentiating, we have

$$dy = \sum_{i=1}^{n} \frac{\partial y(p, \mathbf{w})}{\partial w_i} dw_i = -\sum_{i=1}^{n} \frac{\partial x_i(p, \mathbf{w})}{\partial p} dw_i = -\sum_{i=1}^{n} \frac{\partial x_i(\mathbf{w}, y)}{\partial y} \frac{\partial y(p, \mathbf{w})}{\partial p} dw_i.$$

The first equality is a definition; the second uses the symmetry of the substitution matrix; the third uses the chain rule and the fact that the unconditional factor demand, $x_i(p, \mathbf{w})$, and the conditional factor demand, $x_i(\mathbf{w}, y)$, satisfy the identity $x_i(\mathbf{w}, y(p, \mathbf{w})) = x_i(p, \mathbf{w})$. The last expression on the right shows that if there are no inferior factors then the output of the firm must increase.

5.11.a $\mathbf{x} = (1, 1, 0, 0)$.

5.11.b $\min\{w_1 + w_2, w_3 + w_4\}y$.

5.11.c Constant returns to scale.

5.11.d $\mathbf{x} = (1, 0, 1, 0)$.

5.11.e $c(w, y) = [\min\{w_1, w_2\} + \min\{w_3, w_4\}]y$.

5.11.f Constant.

5.12.a The diagram is the same as the diagram for an inferior good in consumer theory.

5.12.b If the technology is CRS, then conditional factor demands take the form $x_i(\mathbf{w}, 1)y$. Hence the derivative of a factor demand function with respect to output is $x_i(\mathbf{w}) \geq 0$.

5.12.c The hypothesis can be written as

$$\partial c(\mathbf{w}, y)^2/\partial y \partial w_i < 0.$$

But

$$\partial c(\mathbf{w}, y)^2/\partial y \partial w_i = \partial c(\mathbf{w}, y)^2/\partial w_i \partial y = \partial x_i(\mathbf{w}, y)/\partial y.$$

5.13.a Factor demand curves slope downward, so the demand for unskilled workers must decrease when their wage increases.

5.13.b We are given that $\partial l/\partial p < 0$. But by duality, $\partial l/\partial p = -\partial^2\pi/\partial p\partial w = -\partial^2\pi/\partial w\partial p = -\partial y/\partial w$. It follows that $\partial y/\partial w > 0$.

5.14 Take a total derivative of the cost function to get:

$$dc = \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i + \frac{\partial c}{\partial y} dy.$$

It follows that

$$\frac{\partial c}{\partial y} = \frac{dc - \sum_{i=1}^{n} \frac{\partial c}{\partial w_i} dw_i}{dy}.$$

Now substitute the first differences for the dy, dc, dw_i terms and you're done.

5.15 By the linearity of the function, we know we will use either x_1, or a combination of x_2 and x_3 to produce y. By the properties of the Leontief function, we know that if we use x_2 and x_3 to produce y, we must use 3 units of both x_2 and x_3 to produce one unit of y. Thus, if the cost of using one unit of x_1 is less than the cost of using one unit of both x_2 and x_3, then we will use only x_1, and conversely. The conditional factor demands can be written as:

$$x_1 = \begin{cases} 3y & \text{if } w_1 < w_2 + w_3 \\ 0 & \text{if } w_1 > w_2 + w_3 \end{cases}$$

$$x_2 = \begin{cases} 0 & \text{if } w_1 < w_2 + w_3 \\ 3y & \text{if } w_1 > w_2 + w_3 \end{cases}$$

$$x_3 = \begin{cases} 0 & \text{if } w_1 < w_2 + w_3 \\ 3y & \text{if } w_1 > w_2 + w_3 \end{cases}$$

if $w_1 = w_2 + w_3$, then any bundle (x_1, x_2, x_3) with $x_2 = x_3$ and $x_1 + x_2 = 3y$ (or $x_1 + x_3 = 3y$) minimizes cost.

The cost function is

$$c(w, y) = 3y \min(w_1, w_2 + w_3).$$

5.16.a *Homogeneous*:

$$\begin{aligned} c(t\mathbf{w}, y) &= y^{1/2}(tw_1 tw_2)^{3/4} \\ &= t^{3/2}(y^{1/2}(w_1 w_2)^{3/4}) \\ &= t^{3/2}c(\mathbf{w}, y) \quad \text{No.} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = \frac{3}{4} y^{1/2} w_1^{-1/4} w_2^{3/4} > 0 \qquad \frac{\partial c}{\partial w_2} = \frac{3}{4} y^{1/2} w_1^{3/4} w_2^{-1/4} > 0 \quad \text{Yes.}$$

Concave:

$$\text{Hessian } = \begin{bmatrix} -\frac{3}{16} y^{1/2} w_1^{-5/4} w_2^{3/4} & \frac{9}{16} y^{1/2} w_1^{-1/4} w_2^{-1/4} \\ \frac{9}{16} y^{1/2} w_1^{-1/4} w_2^{-1/4} & -\frac{3}{16} y^{1/2} w_1^{3/4} w_2^{-5/4} \end{bmatrix}$$

$$|H_1| < 0$$

$$\begin{aligned} |H_2| &= \frac{9}{256} y w_1^{-1/2} w_2^{-1/2} - \frac{81}{256} y w_1^{-1/2} w_2^{-1/2} \\ &= -\frac{72}{256} \frac{y}{\sqrt{w_1 w_2}} < 0 \quad \text{No} \end{aligned}$$

Continuous: Yes

5.16.b *Homogeneous*:

$$\begin{aligned} c(t\mathbf{w}, y) &= y(tw_1 + \sqrt{tw_1 tw_2} + tw_2) \\ &= ty(w_1 + \sqrt{w_1 w_2} + w_2) \\ &= tc(y, \overline{w}) \quad \text{Yes} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = y\left(1 + \frac{1}{2}\sqrt{\frac{w_2}{w_1}}\right) > 0 \quad \frac{\partial c}{\partial w_2} = y\left(1 + \frac{1}{2}\sqrt{\frac{w_1}{w_2}}\right) > 0 \quad \text{Yes}$$

Concave:

$$H = \begin{bmatrix} -\frac{1}{4} y w_2^{1/2} w_1^{-3/2} & \frac{1}{4} y w_2^{-1/2} w_1^{-1/2} \\ \frac{1}{4} y w_2^{-1/2} w_1^{-1/2} & -\frac{1}{4} y w_2^{-3/2} w_1^{1/2} \end{bmatrix}$$

$$|H_1| < 0$$

$$|H_2| = \frac{1}{16} y w_2^{-1} w_1^{-1} - \frac{1}{16} y w_2^{-1} w_1^{-1} = 0 \quad \text{Yes}$$

Continuous: Yes

Production Function:

$$x_1(\mathbf{w}, y) = y\left(1 + \frac{1}{2}\sqrt{\frac{w_2}{w_1}}\right) \tag{1}$$

$$x_2(\mathbf{w}, y) = y\left(1 + \frac{1}{2}\sqrt{\frac{w_1}{w_2}}\right) \tag{2}$$

Rearranging these equations:

$$x_1 - y = \frac{y}{2}\sqrt{\frac{w_2}{w_1}} \tag{1'}$$

$$x_2 - y = \frac{y}{2}\sqrt{\frac{w_1}{w_2}} \tag{2'}$$

Multiply (1′) and (2′): $(x_1 - y)(x_2 - y) = \frac{y^2}{4}$. This is a quadratic equation which gives $y = \frac{2}{3}(x_2 + x_1) \pm \frac{2}{3}\sqrt{x_1^2 + x_2^2 + 2 - x_1 x_2}$.

5.16.c *Homogeneous*:

$$\begin{aligned} c(t\mathbf{w}, y) &= y(tw_1 e^{-tw_1} + tw_2) \\ &= ty(w_1 e^{-tw_1} + w_2) \\ &\neq tc(\mathbf{w}, y) \quad \text{No} \end{aligned}$$

Monotone:

$$\frac{\partial c}{\partial w_1} = y(-w_1 e^{-w_1} + e^{-w_1}) = ye^{-w_1}(1 - w_1)$$

This is positive only if $w_1 < 1$.

$$\frac{\partial c}{\partial w_2} = y > 0 \quad \text{No}$$

Concave:

$$H = \begin{bmatrix} y(w_1 - 2)e^{-w_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$|H_1| = y(w_1 - 2)e^{-w_1}$$

This is less than zero only if $w_1 < 2$.

$$|H_2| = 0 \qquad \text{No}$$

Continuous: Yes

5.16.d *Homogeneous:*

$$
\begin{aligned}
c(t\mathbf{w}, y) &= y(tw_1 - \sqrt{tw_1tw_2} + tw_2 \\
&= ty(w_1 - \sqrt{w_1w_2} + w_2) \\
&= tc(\mathbf{w}, y) \qquad \text{Yes}
\end{aligned}
$$

Monotone:

$$\frac{\partial c}{\partial w_1} = y(1 - \frac{1}{2}\sqrt{\frac{w_2}{w_1}})$$

This is greater than 0 only if $1 > \frac{1}{2}\sqrt{\frac{w_2}{w_1}}$

$$\frac{\partial c}{\partial w_2} = y(1 - \frac{1}{2}\sqrt{\frac{w_1}{w_2}})$$

This is greater than 0 only if $2 > \sqrt{\frac{w_2}{w_1}}$

$$w_2 > \frac{1}{4}w_1 \quad \text{(by symmetry)} \quad 2\sqrt{w_1} > \sqrt{w_2}$$

or

$$w_1 < 4w_2 \qquad w_1 > \frac{1}{4}w_2$$

Monotone only if $\frac{1}{4}w_2 < w_1 < 4w_2$. No.

Concave:

$$H = \begin{bmatrix} \frac{1}{4}yw_1^{-3/2}w_2^{1/2} & -\frac{1}{4}yw_1^{-1/2}w_2^{-1/2} \\ -\frac{1}{4}yw_1^{-1/2}w_2^{-1/2} & \frac{1}{4}yw_1^{1/2}w_2^{-1/2} \end{bmatrix}$$

$$
\begin{aligned}
|H_1| &= \frac{1}{4}yw_1^{-3/2}w_2^{1/2} > 0 \\
|H_2| &= 0 \qquad \text{No (it is convex)}
\end{aligned}
$$

Continuous: Yes

5.16.e
Homogeneous:

$$
\begin{aligned}
c(t\mathbf{w}, y) &= (y + \frac{1}{y}\sqrt{tw_1tw_2}) \\
&= tc(y, \overline{w}) \qquad \text{Yes}
\end{aligned}
$$

Monotone in w:

$$\frac{\partial c}{\partial w_1} = \frac{1}{2}(y + \frac{1}{y})\sqrt{\frac{w_2}{w_1}} > 0 \qquad \frac{\partial c}{\partial w_2} = \frac{1}{2}(y + \frac{1}{2})\sqrt{\frac{w_1}{w_2}} > 0 \qquad \text{Yes}$$

Concave:

$$H = \begin{bmatrix} -\frac{1}{4}(y + \frac{1}{y}w_1^{-3/2}w_2^{1/2} & \frac{1}{4}(y + \frac{1}{y})w_1^{-1/2}w_2^{-1/2} \\ \frac{1}{4}(y, \frac{1}{y})w_1^{-1/2}w_2^{-1/2} & -\frac{1}{4}(y + \frac{1}{y})w_1^{1/2}w_2^{-3/2} \end{bmatrix} \qquad \text{But not in } y!$$

$$\begin{matrix} |H_1| < 0 \\ |H_2| = 0 \end{matrix} \qquad \text{Yes}$$

Continuous: Not for $y = 0$.

5.17.a $y = \sqrt{ax_1 + bx_2}$

5.17.b Note that this function is exactly like a linear function, except that the linear combination of x_1 and x_2 will produce y^2, rather than just y. So, we know that if x_1 is relatively cheaper, we will use all x_1 and no x_2, and conversely.

5.17.c The cost function is $c(w, y) = y^2 \min(\frac{w_1}{a}, \frac{w_2}{b})$.

Chapter 6. Duality

6.1 The production function is $f(x_1, x_2) = x_1 + x_2$. The conditional factor demands have the form

$$x_i = \begin{cases} y & \text{if } w_i < w_j \\ 0 & \text{if } w_i > w_j \\ \text{any amount between 0 and } y & \text{if } w_i = w_j. \end{cases}$$

6.2 The conditional factor demands can be found by differentiating. They are $x_1(w_1, w_2, y) = x_2(w_1, w_2, y) = y$. The production function is

$$f(x_1, x_2) = \min\{x_1, x_2\}.$$

6.3 The cost function must be increasing in both prices, so a and b are both nonnegative. The cost function must be concave in both prices, so a and b are both less than 1. Finally, the cost function must be homogeneous of degree 1, so $a = 1 - b$.

Chapter 7. Utility Maximization

7.1 The preferences exhibit local nonsatiation, *except* at $(0,0)$. The consumer will choose this consumption point when faced with positive prices.

7.2 The demand function is

$$x_1 = \begin{cases} m/p_1 & \text{if } p_1 < p_2 \\ \text{any } x_1 \text{ and } x_2 \text{ such that } p_1 x_2 + p_2 x_2 = m & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

The indirect utility function is $v(p_1, p_2, m) = \max\{m/p_1, m/p_2\}$, and the expenditure function is $e(p_1, p_2, u) = u \min\{p_1, p_2\}$.

7.3 The expenditure function is $e(p_1, p_2, u) = u \min\{p_1, p_2\}$. The utility function is $u(x_1, x_2) = x_1 + x_2$ (or any monotonic transformation), and the demand function is

$$x_1 = \begin{cases} m/p_1 & \text{if } p_1 < p_2 \\ \text{any } x_1 \text{ and } x_2 \text{ such that } p_1 x_1 + p_2 x_2 = m & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases}$$

7.4.a Demand functions are $x_1 = m/(p_1 + p_2)$, $x_2 = m/(p_1 + p_2)$.

7.4.b $e(p_1, p_2, u) = (p_1 + p + 2)u$

7.4.c $u(x_1, x_2) = \min\{x_1, x_2\}$

7.5.a Quasilinear preferences.

7.5.b Less than $u(1)$.

7.5.c $v(p_1, p_2, m) = \max\{u(1) - p_1 + m, m\}$

7.6.a Homothetic.

7.6.b $e(\mathbf{p}, u) = u/A(p)$

7.6.c $\mu(\mathbf{p}; \mathbf{q}, m) = mA(\mathbf{q})/A(\mathbf{p})$

7.6.d It will be the same, since this is just a monotonic transformation.

Chapter 8. Choice

8.1 We know that

$$x_j(\mathbf{p}, m) \equiv h_j(\mathbf{p}, v(\mathbf{p}, m)) \equiv \partial e(\mathbf{p}, v(\mathbf{p}, m))/\partial p_j. \tag{0.1}$$

(Note that the partial derivative is taken with respect to the *first* occurrence of p_j.) Differentiating equation (0.1) with respect to m gives us

$$\frac{\partial x_j}{\partial m} = \frac{\partial^2 e(\mathbf{p}, v(\mathbf{p}, m))}{\partial p_j \partial u} \frac{\partial v(p,m)}{\partial m}.$$

Since the marginal utility of income, $\partial v/\partial m$, must be positive, the result follows.

8.2 The Cobb-Douglas demand system with two goods has the form

$$x_1 = \frac{a_1 m}{p_1}$$
$$x_2 = \frac{a_2 m}{p_2}$$

where $a_1 + a_2 = 1$. The substitution matrix is

$$\begin{pmatrix} -a_1 m p_1^{-2} - a_1^2 m p_1^{-2} & -a_1 a_2 m p_1^{-1} p_2^{-1} \\ -a_1 a_2^2 m p_1^{-1} p_2^{-1} & -a_2 m p_2^{-2} - a_2^2 m p_2^{-2} \end{pmatrix}.$$

This is clearly symmetric and negative definite.

8.3 The equation is $d\mu/dt = at + b\mu + \text{c}$. The indirect money metric utility function is

$$\mu(q,p,m) = e^{b(q-p)} \left[m + \frac{c}{b} + \frac{a}{b^2} + \frac{c}{b} p \right] - \frac{c}{b} - \frac{a}{b^2} - \frac{aq}{b}.$$

8.4 The demand function can be written as $x = e^{c+ap+bm}$. The integrability equation is

$$\frac{d\mu}{dt} = e^{at+b\mu+c}.$$

Write this as

$$e^{-b\mu} \frac{d\mu}{dt} = e^c e^{at}.$$

Integrating both sides of this equation between p and q, we have

$$-\frac{e^{-b\mu}}{b} \Big]_p^q = \frac{e^c e^{at}}{a} \Big]_p^q.$$

Evaluating the integrals, we have

$$e^{b\mu(q;p,m)} = e^{-bm} - \frac{be^c}{a} \left[e^{ap} - e^{aq} \right].$$

8.5 Write the Lagrangian

$$\mathcal{L}(\mathbf{x},\lambda) = \frac{3}{2}\ln x_1 + \ln x_2 - \lambda(3x_1 + 4x_2 - 100).$$

(Be sure you understand why we can transform u this way.) Now, equating the derivatives with respect to x_1, x_2, and λ to zero, we get three equations in three unknowns

$$\frac{3}{2x_1} = 3\lambda,$$
$$\frac{1}{x_2} = 4\lambda,$$
$$3x_1 + 4x_2 = 100.$$

Solving, we get

$$x_1(3,4,100) = 20, \text{ and } x_2(3,4,100) = 10.$$

Note that if you are going to interpret the Lagrange multiplier as the marginal utility of income, you must be explicit as to which utility function you are referring to. Thus, the marginal utility of income can be measured in original *'utils'* or in '$\ln utils$'. Let $u^* = \ln u$ and, correspondingly, $v^* = \ln v$; then

$$\lambda = \frac{\partial v^*(\mathbf{p},m)}{\partial m} = \frac{\frac{\partial v(\mathbf{p},m)}{\partial m}}{v(\mathbf{p},m)} = \frac{\mu}{v(\mathbf{p},m)},$$

where μ denotes the Lagrange multiplier in the Lagrangian

$$L(\mathbf{x},\mu) = x_1^{\frac{3}{2}} x_2 - \mu(3x_1 + 4x_2 - 100).$$

Check that in this problem we'd get $\mu = \frac{20^{\frac{3}{2}}}{4}$, $\lambda = \frac{1}{40}$, and $v(3,4,100) = 20^{\frac{3}{2}}10$.

8.6 The Lagrangian for the utility maximization problem is

$$\mathcal{L}(\mathbf{x},\lambda) = x_1^{\frac{1}{2}} x_2^{\frac{1}{3}} - \lambda(p_1x_1 + p_2x_2 - m),$$

taking derivatives,

$$\frac{1}{2}x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}} = \lambda p_1,$$
$$\frac{1}{3}x_1^{\frac{1}{2}} x_2^{-\frac{2}{3}} = \lambda p_2,$$
$$p_1x_1 + p_2x_2 = m.$$

Solving, we get

$$x_1(\mathbf{p},m) = \frac{3}{5}\frac{m}{p_1}, \;\; x_2(\mathbf{p},m) = \frac{2}{5}\frac{m}{p_2}.$$

Plugging these demands into the utility function, we get the indirect utility function

$$v(\mathbf{p}, m) = U(\mathbf{x}(\mathbf{p}, m)) = \left(\frac{3}{5}\frac{m}{p_1}\right)^{\frac{1}{2}} \left(\frac{2}{5}\frac{m}{p_2}\right)^{\frac{1}{3}} = \left(\frac{m}{5}\right)^{\frac{5}{6}} \left(\frac{3}{p_1}\right)^{\frac{1}{2}} \left(\frac{2}{p_2}\right)^{\frac{1}{3}}.$$

Rewrite the above expression replacing $v(\mathbf{p}, m)$ by u and m by $e(\mathbf{p}, u)$. Then solve it for $e(\cdot)$ to get

$$e(\mathbf{p}, u) = 5\left(\frac{p_1}{3}\right)^{\frac{3}{5}} \left(\frac{p_2}{2}\right)^{\frac{2}{5}} u^{\frac{6}{5}}.$$

Finally, since $h_i = \partial e / \partial p_i$, the Hicksian demands are

$$h_1(\mathbf{p}, u) = \left(\frac{p_1}{3}\right)^{-\frac{2}{5}} \left(\frac{p_2}{2}\right)^{\frac{2}{5}} u^{\frac{6}{5}},$$

and

$$h_2(\mathbf{p}, u) = \left(\frac{p_1}{3}\right)^{\frac{3}{5}} \left(\frac{p_2}{2}\right)^{-\frac{3}{5}} u^{\frac{6}{5}}.$$

8.7 Instead of starting from the utility maximization problem, let's now start from the expenditure minimization problem. The Lagrangian is

$$\mathcal{L}(\mathbf{x}, \mu) = p_1 x_1 + p_2 x_2 - \mu((x_1 - \alpha_1)^{\beta_1}(x_2 - \alpha_2)^{\beta_2} - u);$$

the first-order conditions are

$$\begin{aligned} p_1 &= \mu\beta_1(x_1 - \alpha_1)^{\beta_1 - 1}(x_2 - \alpha_2)^{\beta_2}, \\ p_2 &= \mu\beta_2(x_1 - \alpha_1)^{\beta_1}(x_2 - \alpha_2)^{\beta_2 - 1}, \\ (x_1 - \alpha_1)^{\beta_1}(x_2 - \alpha_2)^{\beta_2} &= u. \end{aligned}$$

Divide the first equation by the second

$$\frac{p_1\beta_2}{p_2\beta_1} = \frac{x_2 - \alpha_2}{x_1 - \alpha_1},$$

using the last equation

$$x_2 - \alpha_2 = \left((x_1 - \alpha_1)^{-\beta_1} u\right)^{\frac{1}{\beta_2}};$$

substituting and solving,

$$h_1(\mathbf{p}, u) = \alpha_1 + \left(\frac{p_2\beta_1}{p_1\beta_2} u^{\frac{1}{\beta_2}}\right)^{\frac{\beta_2}{\beta_1 + \beta_2}},$$

and

$$h_2(\mathbf{p}, u) = \alpha_2 + \left(\frac{p_1\beta_2}{p_2\beta_1}u^{\frac{1}{\beta_1}}\right)^{\frac{\beta_1}{\beta_1+\beta_2}}.$$

Verify that

$$\frac{\partial h_1(\mathbf{p}, m)}{\partial p_2} = \left(\frac{u}{\beta_1+\beta_2}\left(\frac{\beta_1}{p_1}\right)^{\beta_2}\left(\frac{\beta_2}{p_2}\right)^{\beta_1}\right)^{\frac{1}{\beta_1+\beta_2}} = \frac{\partial h_2(\mathbf{p}, m)}{\partial p_1}.$$

The expenditure function is

$$e(\mathbf{p}, u) = p_1\left(\alpha_1 + \left(\frac{p_2\beta_1}{p_1\beta_2}u^{\frac{1}{\beta_2}}\right)^{\frac{\beta_2}{\beta_1+\beta_2}}\right) + p_2\left(\alpha_2 + \left(\frac{p_1\beta_2}{p_2\beta_1}u^{\frac{1}{\beta_1}}\right)^{\frac{\beta_1}{\beta_1+\beta_2}}\right).$$

Solving for u, we get the indirect utility function

$$v(\mathbf{p}, m) =$$
$$\left(\frac{\beta_1}{\beta_1+\beta_2}\left(\frac{m-\alpha_2 p_2}{p_1} - \alpha_1\right)\right)^{\beta_1}\left(\frac{\beta_2}{\beta_1+\beta_2}\left(\frac{m-\alpha_1 p_1}{p_2} - \alpha_2\right)\right)^{\beta_2}.$$

By Roy's law we get the Marshallian demands

$$x_1(\mathbf{p}, m) = \frac{1}{\beta_1+\beta_2}\left(\beta_1\alpha_2 + \beta_2\frac{m-\alpha_1 p_1}{p_2}\right),$$

and

$$x_2(\mathbf{p}, m) = \frac{1}{\beta_1+\beta_2}\left(\beta_2\alpha_1 + \beta_1\frac{m-\alpha_2 p_2}{p_1}\right).$$

8.8 Easy—a monotonic transformation of utility doesn't change anything about observed behavior.

8.9 By definition, the Marshallian demands $\mathbf{x}(\mathbf{p}, m)$ maximize $\phi(\mathbf{x})$ subject to $\mathbf{px} = m$. We claim that they also maximize $\psi(\phi(\mathbf{x}))$ subject to the same budget constraint. Suppose not. Then, there would exist some other choice $\mathbf{x}'$ such that $\psi(\phi(\mathbf{x}')) > \psi(\phi(\mathbf{x}(\mathbf{p}, m)))$ and $\mathbf{px}' = m$. But since applying the transformation $\psi^{-1}(\cdot)$ to both sides of the inequality will preserve it, we would have $\phi(\mathbf{x}') > \phi(\mathbf{x}(\mathbf{p}, m))$ and $\mathbf{px}' = m$, which contradicts our initial assumption that $\mathbf{x}(\mathbf{p}, m)$ maximized $\phi(\mathbf{x})$ subject to $\mathbf{px} = m$. Therefore $\mathbf{x}(\mathbf{p}, m) = \mathbf{x}^*(\mathbf{p}, m)$. (Check that the reverse proposition also holds—i.e., the choice that maximizes u^* also maximizes u when the the same budget constraint has to be verified in both cases.)

$$v^*(\mathbf{p}, m) = \psi(\phi(\mathbf{x}^*(\mathbf{p}, m))) = \psi(\phi(\mathbf{x}(\mathbf{p}, m)) = \psi(v(\mathbf{p}, m)),$$

the first and last equalities hold by definition and the middle one by our previous result; now

$$\begin{aligned} e^*(\mathbf{p}, u^*) &= \min\{\mathbf{px} : \psi(\phi(\mathbf{x})) = u^*\} \\ &= \min\{\mathbf{px} : \phi(\mathbf{x}) = \psi^{-1}(u^*)\} \\ &= e(\mathbf{p}, \psi^{-1}(u^*)); \end{aligned}$$

again, we're using definitions at both ends and the properties of $\psi(\cdot)$ — namely that the inverse is well defined since $\psi(\cdot)$ is monotonic— to get the middle equality; finally using definitions and substitutions as often as needed we get

$$\begin{aligned} \mathbf{h}^*(\mathbf{p}, u^*) &= \mathbf{x}^*(\mathbf{p}, e^*(\mathbf{p}, u^*)) = \mathbf{x}(\mathbf{p}, e^*(\mathbf{p}, u^*)) \\ &= \mathbf{x}(\mathbf{p}, e(\mathbf{p}, \psi^{-1}(u^*))) = \mathbf{h}(\mathbf{p}, \psi^{-1}(u^*)). \end{aligned}$$

8.10.a Differentiate the identity $h_j(\mathbf{p}, u) \equiv x_j(\mathbf{p}, e(\mathbf{p}, u))$ with respect to p_i to get

$$\frac{\partial h_j(\mathbf{p}, u)}{\partial p_i} = \frac{\partial x_j(\mathbf{p}, m)}{\partial p_i} + \frac{\partial x_j(\mathbf{p}, e(\mathbf{p}, u))}{\partial m} \frac{\partial e(\mathbf{p}, u)}{\partial p_i}.$$

We must be careful with this last term. Look at the expenditure minimization problem

$$e(\mathbf{p}, u) = \min\{\mathbf{p}(\mathbf{x} - \overline{\mathbf{x}}) : u(\mathbf{x}) = u\}.$$

By the envelope theorem, we have

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = h_i(\mathbf{p}, u) - \overline{x}_i = x_i(\mathbf{p}, e(\mathbf{p}, u)) - \overline{x}_i.$$

Therefore, we have

$$\frac{\partial h_j(\mathbf{p}, u)}{\partial p_i} = \frac{\partial x_j(\mathbf{p}, m)}{\partial p_i} + \frac{\partial x_j(\mathbf{p}, e(\mathbf{p}, u))}{\partial m}(x_i(\mathbf{p}, m) - \overline{x}_i),$$

and reorganizing we get the Slutsky equation

$$\frac{\partial x_j(\mathbf{p}, m)}{\partial p_i} = \frac{\partial h_j(\mathbf{p}, u)}{\partial p_i} + \frac{\partial x_j(\mathbf{p}, e(\mathbf{p}, u))}{\partial m}(\overline{x}_i - x_i(\mathbf{p}, m)).$$

8.10.b Draw a diagram, play with it and verify that Dave is better off when p_2 goes down and worse off when p_1 goes down. Just look at the sets of allocations that are strictly better or worse than the original choice—i.e., the sets $SB(\mathbf{x}) = \{\mathbf{z} : \mathbf{z} \succ \mathbf{x}\}$ and $SW(\mathbf{x}) = \{\mathbf{z} : \mathbf{z} \prec \mathbf{x}\}$. When p_1 goes down the new budget set is contained in $SW(\mathbf{x})$, while when p_2 goes down there's a region of the new budget set that lies in $SB(\mathbf{x})$.

8.10.c The rate of return—also known as "own rate of interest"—on good x is $(p_1/p_2) - 1$

8.11 No, because his demand behavior violates GARP. When prices are $(2, 4)$ he spends 10. At these prices he could afford the bundle $(2, 1)$, but rejects it; therefore, $(1, 2) \succ (2, 1)$. When prices are $(6, 3)$ he spends 15. At these prices he could afford the bundle $(1, 2)$ but rejects it; therefore, $(2, 1) \succ (1, 2)$.

8.12 Inverting, we have $e(p, u) = u/f(p)$. Substituting, we have

$$\mu(p; q, y) = v(q, y)/f(p) = f(q)y/f(p).$$

8.13.a Draw the lines $x_2 + 2x_1 = 20$ and $x_1 + 2x_2 = 20$. The indifference curve is the northeast boundary of this X.

8.13.b The slope of a budget line is $-p_1/p_2$. If the budget line is steeper than 2, $x_1 = 0$. Hence the condition is $p_1/p_2 > 2$.

8.13.c Similarly, if the budget line is flatter than $1/2$, x_2 will equal 0, so the condition is $p_1/p_2 < 1/2$.

8.13.d If the optimum is unique, it must occur where $x_2 - 2x_1 = x_1 - 2x_2$. This implies that $x_1 = x_2$, so that $x_1/x_2 = 1$.

8.14.a This is an ordinary Cobb-Douglas demand: $S_1 = \frac{\alpha}{\alpha+\beta+\gamma}Y$ and $S_2 = \frac{\beta}{\alpha+\beta+\gamma}Y$.

8.14.b In this case the utility function becomes $U(C, S_1, L) = S_1^\alpha L^\beta C^\gamma$. The L term is just a constant, so applying the standard Cobb-Douglas formula $S_1 = \frac{\alpha}{\alpha+\gamma}Y$.

8.15 Use Slutsky's equation to write: $\dfrac{\partial L}{\partial w} = \dfrac{\partial L^s}{\partial w} + (\overline{L} - L)\dfrac{\partial L}{\partial m}$. Note that the substitution effect is always negative, $(\overline{L} - L)$ is always positive, and hence if leisure is inferior, $\dfrac{\partial L}{\partial w}$ is necessarily negative. Thus the slope of the labor supply curve is positive.

8.16.a True. With the grant, the consumer will maximize $u(x_1, x_2)$ subject to $x_1 + x_2 \leq m + g_1$ and $x_1 \geq g_1$. We know that when he maximizes his utility subject to $x_1 + x_2 \leq m$, he chooses $x_1^* \geq g_1$. Since x_1 is a normal good, the amount of good 1 that he will choose if given an unconstrained grant of g_1 is some number $x_1' > x_1^* \geq g_1$. Since this choice satisfies the constraint $x_1' \geq g_1$, it is also the choice he would make when forced to spend g_1 on good 1.

8.16.b False. Suppose for example that $g_1 = x_1^*$. Then if he gets an unconstrained grant of g_1, since good 1 is inferior, he will choose to reduce his consumption to less than $x_1^* = g_1$. But with the constrained grant, he must consume at least g_1 units of good 1. Incidentally, he will accept the grant, since with the grant he can always consume at least as much of both goods as without the grant.

8.16.c If he got an unconstrained grant of g_1, he would spend $(48 + g_1)/4$ on good 1. This is exactly what he will spend if $g_1 \le (48 + g_1)4$. But if $g_1 > (48+g_1)/4$, he will spend g_1 on good 1. The curve therefore has slope $1/4$ if $g_1 < 16$ and slope 1 if $g_1 > 16$. Kink is at $g_1 = 16$.

Chapter 9. Demand

9.1 If preferences are homothetic, demand functions are linear in income, so we can write $x_i(\mathbf{p})m$ and $x_j(\mathbf{p})m$. Applying Slutsky symmetry, we have

$$\frac{\partial x_i(\mathbf{p})}{\partial p_j} + x_i(\mathbf{p})x_j(\mathbf{p})m = \frac{\partial x_j}{\partial p_i} + x_j(\mathbf{p})x_i(\mathbf{p})m.$$

Subtracting $x_i(\mathbf{p})x_j(\mathbf{p})m$ from each side of the equation establishes the result.

9.2 Note that p is the relative price of good x with respect to the other good which we'll call z. Also, let m be income measured in units of z. Thus the consumer's budget constraint is $px + z = m$. How do you know that there must be another good around?

From

$$\frac{d\mu(p; q, m)}{dp} = a + bp,$$

we find

$$\mu(p; q, m) = ap + \frac{bp^2}{2} + C.$$

Here C is a constant of integration. Since $\mu(q; q, m) = m$, note that

$$C = m - aq - \frac{bq^2}{2};$$

therefore,

$$\mu(p; q, m) = ap + \frac{bp^2}{2} + m - aq - \frac{bq^2}{2}.$$

A money metric utility function behaves like an indirect utility function with respect to q and m when holding p fixed. Therefore, an indirect utility function consistent with the demand function given above is

$$v(q, m) = m - aq - \frac{bq^2}{2}.$$

We can drop the terms in p since they are just constants. (Use Roy's identity to check that this indirect utility is indeed consistent with our original demand function.)

To get the direct utility function* we must solve

$$u(x, z) = \min_q \{v(q, m) \ : \ qx + z = m\} = \min_q \{m - aq - \frac{bq^2}{2} \ : \ qx + z = m\};$$

use the budget constraint to eliminate m from the objective function and get the optimal value of q

$$q^* = \frac{x - a}{b}.$$

Thus

$$u(x, z) = z + \frac{(x - a)^2}{2b}.$$

This is, of course, a quasilinear utility function.

On the back of an envelope, solve $\max_{x,z}\{u(x, z) : px + z = m\}$ and check that you get precisely the original demand function for x. What's the demand for z?

9.3 We have to solve

$$\frac{d\mu(p; q, m)}{dp} = a + bp + c\mu(p; q, m).$$

The homogeneous part has a solution of the form Ae^{cp}. A particular solution to the nonhomogeneous equation is given by

$$\overline{\mu} = -\frac{(a + bp)c + b}{c^2}.$$

Therefore the general solution to the differential equation is given by

$$\mu(p; q, m) = Ae^{cp} - \frac{(a + bp)c + b}{c^2}.$$

Since $\mu(q; q, m) = m$ we get

$$\mu(p; q, m) = \left(m + \frac{(a + bq)c + b}{c^2}\right) e^{c(p-q)} - \frac{(a + bp)c + b}{c^2}.$$

Hence, the indirect utility function is

$$v(q, m) = \left(m + \frac{(a + bq)c + b}{c^2}\right) e^{-cq}.$$

* Strictly speaking, we should be saying "*a* utility function consistent with the given demand," but we'll just say "*the* utility function" with the understanding that any monotonic transformation of it would also generate the same demand function.

(Verify that using Roy's identity we get the original demand function.)
To get the direct utility function, we must solve

$$\min_q \left(m + \frac{(a+bq)c+b}{c^2} \right) e^{-cq}$$
$$\text{such that } qx + z = m.$$

The optimal value is given by

$$q^* = \frac{x - cz - a}{b + cx},$$

which implies that

$$u(x, z) = \frac{b + cx}{c^2} \exp\left\{ \frac{ac - cx + c^2 z}{b + cx} \right\}.$$

(Again, substitute z by $m - px$ above, equate the derivative of the resulting expression with respect to x to zero, solve for x and recover the original demand function.)

9.4 Now the budget constraint is given by $z + p_1 x_1 + p_2 x_2 = m$. The symmetry of the substitution effects implies

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} \Longrightarrow b_{12} = b_{21}.$$

The negative semidefiniteness of the substitution matrix implies $b_1 < 0$ and $b_1 b_2 - b^2 > 0$. (Prove that these two conditions together imply that $b_2 < 0$ must also hold.)
We have to solve the following system of partial differential equations

$$\frac{\partial \mu(\mathbf{p}; \mathbf{q}, m)}{\partial p_1} = a_1 + b_1 p_1 + b p_2,$$

and

$$\frac{\partial \mu(\mathbf{p}; \mathbf{q}, m)}{\partial p_2} = a_2 + b p_1 + b_2 p_2.$$

The first equation implies

$$\mu(\mathbf{p}; \mathbf{q}, m) = a_1 p_1 + \frac{b_1}{2} p_1^2 + b p_1 p_2 + C_1,$$

where C_1 is a constant of integration. The second implies

$$\mu(\mathbf{p}; \mathbf{q}, m) = a_2 p_2 + \frac{b_2}{2} p_2^2 + b p_1 p_2 + C_2.$$

Therefore, we must have

$$\begin{aligned}\mu(\mathbf{p};\mathbf{q},m) &= a_1p_1 + \frac{b_1}{2}p_1^2 + bp_1p_2 + a_2p_2 + \frac{b_2}{2}p_2^2 + C \\ &= [p_1,p_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \frac{1}{2}[p_1,p_2]\begin{bmatrix} b_1 & b \\ b & b_2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + C.\end{aligned}$$

Using $\mu(\mathbf{q};\mathbf{q},m) = m$, we have

$$\begin{aligned}\mu(\mathbf{p};\mathbf{q},m) = m &+ a_1(p_1 - q_1) + \frac{b_1}{2}(p_1^2 - q_1^2) + b(p_1p_2 - q_1q_2) \\ &+ a_2(p_2 - q_2) + \frac{b_2}{2}(p_2^2 - q_2^2).\end{aligned}$$

The indirect utility function is given by

$$\begin{aligned}v(\mathbf{q},m) &= m - a_1q_1 - \frac{b_1}{2}q_1^2 - bq_1q_2 - a_2q_2 - \frac{b_2}{2}q_2^2 \\ &= m - [q_1,q_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \frac{1}{2}[q_1,q_2]\begin{bmatrix} b_1 & b \\ b & b_2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.\end{aligned}$$

9.5 To get the direct utility function we must solve

$$u(\mathbf{x},z) = \min_{\mathbf{q}}\{v(\mathbf{q},m) : z + q_1x_1 + q_2x_2 = m\}.$$

After a few minutes of algebraic fun, we get

$$q_1^* = \frac{b_2(x_1 - a_1) - b(x_2 - a_2)}{b_1b_2 - b^2},$$

and

$$q_2^* = \frac{b_1(x_2 - a_2) - b(x_1 - a_1)}{b_1b_2 - b^2}.$$

Substituting these values back into $v(\cdot)$, we get

$$\begin{aligned}&u(\mathbf{x},z) \\ &= z + \frac{b_2(x_1 - a_1)^2 + b_1(x_2 - a_2)^2}{2(b_1b_2 - b^2)} + \frac{b(a_1x_2 + a_2x_1 - x_1x_2 - a_1a_2)}{b_1b_2 - b^2} \\ &= z + \frac{1}{2(b_1b_2 - b^2)}[x_1 - a_1, x_2 - a_2]\begin{bmatrix} b_2 & -b \\ -b & b_1 \end{bmatrix}\begin{bmatrix} x_1 - a_1 \\ x_2 - a_2 \end{bmatrix}.\end{aligned}$$

9.6 Write the indirect utility function as $v(\mathbf{p}) = v(\mathbf{q}/m)$ and differentiate with respect to q_i and m:

$$\begin{aligned}\frac{\partial v}{\partial q_i} &= \frac{\partial v}{\partial p_i}\frac{1}{m} \\ \frac{\partial v}{\partial m} &= -\sum_{i=1}^{k}\frac{\partial v}{\partial p_i}\frac{q_i}{m^2} \\ &= -\sum_{i=1}^{k}\frac{\partial v}{\partial p_i}p_i\frac{1}{m}.\end{aligned}$$

Dividing $\frac{\partial v}{\partial q_i}$ by $\frac{\partial v}{\partial m}$ yields the result.

9.7 The function is weakly separable, and the subutility for the z-good consumption is $z_2^b z_3^c$. The conditional demands for the z-goods are Cobb-Douglas demands:

$$\begin{aligned} z_1 &= \frac{b}{b+c}\frac{m_z}{p_2} \\ z_2 &= \frac{c}{b+c}\frac{m_z}{p_3}. \end{aligned}$$

9.8.a $\frac{d\mu}{dp} = a - bp + c\mu$

9.8.b $\mu(q, q, y) \equiv y$

9.9.a The function $V(x, y) = \min\{x, y\}$, and $U(V, z) = V + z$.

9.9.b The demand function for the z-good is $z = m/p_z$ if $p_z < p_x + p_y$. If $p_z > p_x + p_y$, then the demand for the x-good and the y-good is given by $x = y = m/(p_x + p_y)$. If $p_z = p_x + p_y$, then take any convex combination of these demands.

9.9.c The indirect utility function is

$$v(p_x, p_y, p_z, m) = \max\left\{\frac{m}{p_x + p_y}, \frac{m}{p_z}\right\}.$$

9.11.a There are a variety of ways to solve this problem. The easiest is to solve for the indirect utility function to get $v_1(p_1, p_2, m_1) = m_1(p_1 p_2)^{-1/2}$. Now use Roy's identity to calculate:

$$\begin{aligned} x_1 &= \frac{1}{2}\frac{m_1}{p_1} \\ x_2 &= \frac{1}{2}\frac{m_1}{p_2}. \end{aligned}$$

Note that these are Cobb-Douglas demands.

Recognizing that person 2 has Cobb-Douglas utility, we can write down the demands immediately:

$$\begin{aligned} x_1 &= \frac{3}{3+a}\frac{m_2}{p_1} \\ x_2 &= \frac{a}{3+a}\frac{m_2}{p_2}. \end{aligned}$$

9.11.b We must have the marginal propensity to consume each good the same for each consumer. This means that

$$\frac{1}{2} = \frac{3}{3+a},$$

which implies that $a = 3$.

Chapter 10. Consumers' Surplus

10.1 We saw that in this case the indirect utility function takes the form $v(\mathbf{p})+m$. Hence the expenditure function takes the form $e(\mathbf{p},u) = u-v(\mathbf{p})$. The expenditure function is necessarily a concave function of prices, which implies that $v(\mathbf{p})$ is a convex function.

10.2 Ellsworth's demand functions for the x-good and the y-good take the form

$$x = y = \frac{150}{p_x + p_y}.$$

Plugging this into the utility function, we find that the indirect utility function takes the form

$$v(p_x, p_y, 150) = \frac{150}{p_x + p_y}.$$

Hence A is the solution to

$$\frac{150 - A}{1+1} = \frac{150}{1+2}$$

and B is the solution to

$$\frac{150}{1+1} = \frac{150 + B}{1+2}.$$

Solving, we have $A = 50$ and $B = 75$.

Chapter 11. Uncertainty

11.1 The proof of Pratt's theorem established that

$$\pi(t) \approx \frac{1}{2} r(w)\sigma^2 t^2.$$

But the $\sigma^2 t^2$ is simply the variance of the gamble $t\tilde{\epsilon}$.

11.2 If risk aversion is constant, we must solve the differential equation $u''(x)/u'(x) = -r$. The answer is $u(x) = -e^{-rx}$, or any affine transformation of this. If relative risk aversion is constant, the differential equation is $u''(x)x/u'(x) = -r$. The solution to this is $u(x) = x^{1-r}/(1-r)$ for $r \neq 1$ and $u(x) = \ln x$ for $r = 1$.

11.3 We have seen that investment in a risky asset will be independent of wealth if risk aversion is constant. In an earlier problem, we've seen that constant absolute risk aversion implies that the utility function takes the form $u(w) = -e^{-rw}$.

11.4 Marginal utility is $u'(w) = 1 - 2bw$; when w is large enough this is a negative number. Absolute risk aversion is $2b/(1 - 2bw)$. This is an increasing function of wealth.

11.5.a The probability of heads occurring for the first time on the j^{th} toss is $(1-p)^{j-1}p$. Hence the expected value of the bet is $\sum_{j=1}^{\infty}(1-p)^{j-1}p2^j = \sum_{j=1}^{\infty} 2^{-j}2^j = \sum_{j=1}^{\infty} 1 = \infty$.

11.5.b The expected utility is

$$\sum_{j=1}^{\infty}(1-p)^{j-1}p\ln(2^j) = p\ln(2)\sum_{j=1}^{\infty} j(1-p)^{j-1}.$$

11.5.c By standard summation formulas:

$$\sum_{j=0}^{\infty}(1-p)^j = \frac{1}{p}.$$

Differentiate both sides of this expression with respect to p to obtain

$$\sum_{j=1}^{\infty} j(1-p)^{j-1} = \frac{1}{p^2}.$$

Therefore,

$$p\ln(2)\sum_{j=1}^{\infty} j(1-p)^{j-1} = \frac{\ln(2)}{p}.$$

11.5.d In order to solve for the amount of money required, we equate the utility of participating in the gamble with the utility of not participating. This gives us:

$$\ln(w_0) = \frac{\ln(2)}{p},$$

Now simply solve this equation for w_0 to find

$$w_0 = e^{\ln(2)/p}.$$

11.6.a Note that

$$E[u(R)] = \int_{-\infty}^{\infty} u(s)\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2\right\} ds = \phi(\mu, \sigma^2).$$

11.6.b Normalize $u(\cdot)$ such that $u(\mu) = 0$. Differentiating, we have

$$\frac{\partial E[u(R)]}{\partial \mu} = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} u(s)(s-\mu) f(s) ds > 0,$$

since the terms $[u(s)(s-\mu)]$ and $f(s)$ are positive for all s.

11.6.c Now we have

$$\begin{aligned}
\frac{\partial E[u(R)]}{\partial \sigma^2} &= \frac{1}{\sigma^3} \int_{-\infty}^{\infty} u(s)((s-\mu)^2 - \sigma^2) f(s)\, ds \\
&< \frac{1}{\sigma^3} \int_{-\infty}^{\infty} u'(\mu)(s-\mu)((s-\mu)^2 - \sigma^2) f(s) ds \\
&= \frac{u'(\mu)}{\sigma^3} \left\{ \int_{-\infty}^{\infty} (s-\mu)^3 f(s) ds - \sigma^2 \int_{-\infty}^{\infty} (s-\mu) f(s) ds \right\} \\
&= 0.
\end{aligned}$$

The first inequality follows from the concavity of $u(\cdot)$ and the normalization imposed; the last equality follows from the fact that R is normally distributed and, hence, $E[(R - E[R])^k] = 0$ for k odd.

11.7 Risk aversion implies a concave utility function. Denote by $\alpha \in [0, 1]$ the proportion of the initial wealth invested in asset 1. We have

$$\begin{aligned}
&E[u(\alpha w_0(1+R_1) + (1-\alpha) w_0(1+R_2))] \\
&\quad = \int \int u(\alpha w_0(1+r_1) + (1-\alpha) w_0(1+r_2)) f(r_1) f(r_2) dr_1 dr_2 \\
&\quad > \int \int [\alpha u(w_0(1+r_1)) + (1-\alpha) u(w_0(1+r_2))] f(r_1) f(r_2) dr_1 dr_2 \\
&\quad = \int u(w_0(1+r_1)) f(r_1) dr_1 = \int u(w_0(1+r_2)) f(r_2) dr_2 = \\
&\quad = E[u(w_0(1+R_1))] = E[u(w_0(1+R_2))].
\end{aligned}$$

The inequality follows from the concavity of $u(\cdot)$.

For part b, proceed as before reversing the inequality since now $u(\cdot)$ is convex.

11.8.a Start by expanding both sides of $E[u(\tilde{w} - \pi_u)] = E[u(\tilde{w} + \tilde{\epsilon})]$:

$$\begin{aligned}
E[u(\tilde{w} - \pi_u)] &= p u(w_1 - \pi_u) + (1-p) u(w_2 - \pi_u) \\
&\approx p(u(w_1) - u'(w_1)\pi_u) + (1-p)(u(w_2) - u'(w_2)\pi_u);
\end{aligned}$$

$$\begin{aligned}
E[u(\tilde{w} + \tilde{\epsilon})] &= \frac{p}{2}(u(w_1 - \epsilon) + u(w_1 + \epsilon)) + (1-p) u(w_2) \\
&\approx p\left(u(w_1) + \frac{u''(w_1)\epsilon^2}{2}\right) + (1-p) u(w_2).
\end{aligned}$$

Combining, we obtain

$$-(pu'(w_1) + (1-p)u'(w_2))\pi_u \approx \frac{1}{2}pu''(w_1)\epsilon^2,$$

or

$$\pi_u \approx \frac{-\frac{1}{2}pu''(w_1)\epsilon^2}{pu'(w_1) + (1-p)u'(w_2)}.$$

11.8.b For these utility functions, the Arrow-Pratt measures are $-u''/u' = a$, and $-v''/v' = b$.

11.8.c We are given $a > b$ and we want to show that a value of $(w_1 - w_2)$ large enough will eventually imply $\pi_v > \pi_u$, thus we want to get

$$\frac{ae^{-aw_1}}{pe^{-aw_1} + (1-p)e^{-aw_2}} < \frac{be^{-bw_1}}{pe^{-bw_1} + (1-p)e^{-bw_2}};$$

cross-multiplying we get

$$ape^{-w_1(a+b)} + a(1-p)e^{-(aw_1+bw_2)} < bpe^{-w_1(a+b)} + b(1-p)e^{-(aw_1+bw_2)},$$

which implies

$$(a-b)\frac{p}{1-p} < be^{a(w_1-w_2)} - ae^{b(w_1-w_2)}.$$

The derivative of the RHS of this last inequality with respect to $w_1 - w_2$ is

$$ab\left(e^{a(w_1-w_2)} - e^{b(w_1-w_2)}\right) > 0$$

whenever $w_1 > w_2$; the LHS does not depend on w_1 or w_2. Therefore, this inequality will eventually hold for $(w_1 - w_2)$ large enough.

According to the Arrow-Pratt measure, u exhibits a higher degree of risk aversion than v. We've shown that v could imply a higher risk premium than u to avoid a fair lottery provided there's an additional risk "big" enough. In this case, higher risk premium would no longer be synonymous with higher absolute risk aversion.

11.9 Initially the person has expected utility of

$$\frac{1}{2}\sqrt{4+12} + \frac{1}{2}\sqrt{4+0} = 3.$$

If he sells his ticket for price p, he needs to get at least this utility. To find the breakeven price we write the equation

$$\sqrt{4+p} = 3.$$

Solving, we have $p = 5$.

11.10 The utility maximization problem is $\max \pi \ln(w+x)+(1-\pi)\ln(w-x)$. The first-order condition is

$$\frac{\pi}{w+x} = \frac{1-\pi}{w-x},$$

which gives us $x = w(2\pi - 1)$. If $\pi = 1/2$, $x = 0$.

11.11 We want to solve the equation

$$\frac{p}{w_1} + \frac{1-p}{w_2} = \frac{1}{w}.$$

After some manipulation we have

$$w = \frac{w_1 w_2}{pw_2 + (1-p)w_1}.$$

11.12.a

$$\max_{\mathbf{x}} \alpha \mathbf{px}$$
$$\text{such that } \mathbf{x} \text{ is in } X.$$

11.12.b $v(\mathbf{p}, \alpha)$ has the same form as the profit function.

11.12.c Just mimic the proof used for the profit function.

11.12.d It must be monotonic and convex, just as in the case of the profit function.

Chapter 13. Competitive Markets

13.1 The first derivative of welfare is $v'(p) + \pi'(p) = 0$. Applying Roy's law and Hotelling's lemma, we have $-x(p) + y(p) = 0$, which is simply the condition that demand equals supply. The second derivative of this welfare measure is $-x'(p)+y'(p)$ which is clearly positive; hence, we have a welfare minimum rather than a welfare maximum.

The intuition behind this is that at any price other than the equilibrium price, the firm wants to supply a different amount than the consumer wants to demand; hence, the "welfare" associated with all prices other than the equilibrium price is not attainable.

13.2 By Hotelling's law we know that $\partial\pi(p, \mathbf{w})/\partial p = y(p)$; therefore,

$$\int_{p_0}^{p_1} y(p)dp = \pi(p_1, \mathbf{w}) - \pi(p_0, \mathbf{w}).$$

13.3.a The average cost curve is just

$$\frac{c(\mathbf{w}, y)}{y} = \frac{y^2+1}{y}w_1 + \frac{y^2+2}{y}w_2.$$

You should verify that it is a convex function that has a unique minimum at

$$y_m = \sqrt{\frac{w_1/w_2+2}{w_1/w_2+1}}.$$

The derivative of y_m with respect to w_1/w_2 is negative, so the minimum of the average cost shifts to the left (right) as w_1/w_2 increases (decreases). In fact it converges to 1 as the ratio approaches ∞ and to $\sqrt{2}$ as it goes down to 0.

13.3.b The marginal cost is

$$\frac{\partial c(\mathbf{w}, y)}{\partial y} = 2y(w_1 + w_2),$$

so short-run supply schedule is given by

$$y(p) = \frac{p}{2(w_1+w_2)}.$$

13.3.c The long-run supply curve is

$$Y(p) = \begin{cases} \text{arbitrarily large amount} & \text{if } p > 2y_m(w_1+w_2) \\ 0 & \text{otherwise.} \end{cases}$$

13.3.d From the cost function we have that $x_1 = y^2 + 1$ and $x_2 = y^2 + 2$. Also, we see that x_1 and x_2 are not substitutes at any degree. Therefore, the input requirement set for an individual firm is

$$V(y) = \left\{(x_1, x_2) \in [1, \infty) \times [2, \infty) \colon y \le \min\left\{\sqrt{x_1 - 1}, \sqrt{x_2 - 2}\right\}\right\}.$$

13.4.a $y(p) = p/2$

13.4.b $Y(p) = 50p$

13.4.c Equating $D(p) = Y(p)$, we get $p^* = 2$ and $y^* = 1$ ($Y^* = 100$).

13.4.d The equilibrium rent on land r must equal the difference between each firm revenues and labor costs at the competitive equilibrium. Therefore,

$$r = 2 - 1 = 1.$$

13.5.a In order to offset the output subsidy, the U.S. should choose a tax of the same size as the subsidy; that is, choose $t(s) = s$.

13.5.b In the case of the capital subsidy, the producers receive $p - t(s)$. If y^* is to remain optimal, we must have $p - t(s) = \partial c(w, r - s, y^*)/\partial y$.

13.5.c Differentiate the above expression to get

$$t'(s) = \frac{\partial^2 c(w, r - s, y^*)}{\partial y \partial r} = \frac{\partial K(w, r - s, y^*)}{\partial y}.$$

13.5.d Since $K(w, r - s, y) = K(w, r - s, 1)y$, the formula reduces to $t'(s) = K(w, r - s, 1)$.

13.5.e In this case $\partial K/\partial y < 0$ so that an increase in the subsidy rate implies a decrease in the tariff.

13.6 Each firm that has marginal cost less than 25 will produce to capacity. What about the firm that has marginal cost equal to 25? If it produces a positive amount, it will just cover its variable cost, but lose the quasifixed cost. Hence it prefers to stay out of business. This means that there will be 24 firms in the market, each producing 12 units of output, giving a total supply of 288.

13.7.a $y_m = 500$

13.7.b $p = 5$

13.7.c $y_c = 50 \times 5 = 250$

13.8.a Price equals marginal cost gives us $p = y$, so $Y = p + p = 2p$.

13.8.b Set demand equal to supply $90 - p = 2p$ to find $p^* = 30$ and $Y^* = 60$.

13.8.c Let p be the price paid by consumers. Then the domestic firms receive a price of p and the foreign firms receive a price of $p - 3$. Demand equals supply gives us

$$90 - p = p + [p - 3].$$

Solving we have $p^* = 31$.

13.8.d The supply of umbrellas by domestic firms is 31 and by foreign firms is 28.

Chapter 14. Monopoly

14.1 The profit-maximizing level of output is 5 units. If the monopolist only has 4 to sell, then it would find it most profitable to charge a price of 6. This is the same as the competitive solution. If, however, the monopolist had 6 units to sell, it would be most profitable to dispose of one unit and only sell 5 units at a price of 5.

14.2 The monopolist has zero marginal costs up until 7 units of output and infinite marginal costs for any output greater than 7 units. The profit-maximizing price is 5 and the profits are 25.

14.3 For this constant elasticity demand function revenue is constant at 10, regardless of the level of output. Hence output should be as small as possible—that is, a profit-maximizing level of output doesn't exist.

14.4 According to the formula given in the text, we must have

$$\frac{1}{2 + yp''(y)/p'(y)} = 1,$$

or

$$yp''(y) = -p'(y).$$

The required inverse demand function is the solution to this differential equation. It turns out that it is given by $p(y) = a - b \ln x$. The direct demand function then takes the form $\ln x = a/b - p/b$, which is sometimes called a semilog demand function.

14.5 The monopolist's profit maximization problem is

$$\max_{y} \ p(y,t)y - cy.$$

The first-order condition for this problem is

$$p(y,t) + \frac{\partial p(y,t)}{\partial y} y - c = 0.$$

According to the standard comparative statics calculations, the sign of dy/dt is the same as the sign of the derivative of the first-order expression with respect to t. That is,

$$\text{sign } \frac{dy}{dt} = \text{sign } \frac{\partial p}{\partial t} + \frac{\partial p^2}{\partial y \partial t} y.$$

For the special case $p(y,t) = a(p) + b(t)$, the second term on the right-hand side is zero.

14.6 There is no profit-maximizing level of output since the elasticity of demand is constant at -1. This means that revenue is independent of output, so reductions in output will lower cost but have no effect on revenue.

14.7 Since the elasticity of demand is -1, revenues are constant at any price less than or equal to 20. Marginal costs are constant at c so the monopolist will want to produce the smallest possible output. This will happen when $p = 20$, which implies $y = 1/2$.

14.8 For this to occur, the derivative of consumer's surplus with respect to quality must be zero. Hence $\partial u/\partial q - \partial p/\partial q x \equiv 0$. Substituting for the definition of the inverse demand function, this means that we must have $\partial u/\partial q \equiv x\partial^2 u/\partial x \partial q$. It is easy to verify that this implies that $u(x,q) = f(q)x$.

14.9 The integral to evaluate is

$$\int_0^x \frac{\partial^2 u(z,q)}{\partial z \partial q}\, dz < \int_0^x \frac{\partial p(x,q)}{\partial q}\, dz.$$

Carrying out the integration gives

$$\frac{\partial u(x,q)}{\partial q} < \frac{\partial p(x,q)}{\partial q} x,$$

which is what is required.

14.10 If the firm produces x units of output which it sells at price $p(x)$, then the most that it can charge for entry is the consumer's surplus, $u(x) - p(x)x$. Once the consumer has chosen to enter, the firm makes a profit of $p(x) - c$ on each unit of output purchased. Thus the profit maximization problem of the firm is

$$\max_x \; u(x) - p(x)x + (p(x) - c(x))x = u(x) - c(x).$$

It follows that the monopolist will choose the efficient level of output where $u'(x) = c'(x)$. The entry fee is set equal to the consumer's surplus.

14.11 The figure depicts the situation where the monopolist has reduced the price to the point where the marginal benefit from further reductions just balance the marginal cost. This is the point where $p_2 = 2p_1$. If the high-demand consumer's inverse demand curve is *always* greater than twice the low-demand consumer's inverse demand curve, this condition cannot be satisfied and the low-demand consumer will be pushed to a zero level of consumption.

14.12 Area B is what the monopolist would gain by selling only to the high-demand consumer. Area A is what the monopolist would lose by doing this.

14.13 This is equivalent to the price discrimination problem with $x = q$ and $w_t = r_t$. All of the results derived there translate on a one-to-one basis; e.g., the consumer who values quality the more highly ends up consuming the socially optimal amount, etc.

14.14 The maximization problem is $\max_p \; py(p) - c(y(p))$. Differentiating, we have

$$py'(p) + y(p) - c'(y)y'(p) = 0.$$

This can also be written as

$$p + y(p)/y'(p) - c'(y) = 0,$$

or

$$p[1 + 1/\epsilon] = c'(y).$$

14.15 Under the *ad valorem* tax we have

$$(1-\tau)P_D = \left(1 + \frac{1}{\epsilon}\right)c.$$

Under the output tax we have

$$P_D - t = \left(1 + \frac{1}{\epsilon}\right).$$

Solve each equation for P_D, set the results equal to each other, and solve for t to find

$$t = \frac{\tau k c}{1-\tau} \qquad k = \frac{1}{1+\frac{1}{\epsilon}}$$

14.16.a The monopolist's profit maximization problem is

$$\max_y \; p(y,t)y - cy.$$

The first-order condition for this problem is

$$p(y,t) + \frac{\partial p(y,t)}{\partial y}y - c = 0.$$

According to the standard comparative statics calculations, the sign of dy/dt is the same as the sign of the derivative of the first-order expression with respect to t. That is,

$$\text{sign}\ \frac{dy}{dt} = \text{sign}\ \frac{\partial p}{\partial t} + \frac{\partial p^2}{\partial y \partial t}y.$$

14.16.b For the special case $p(y,t) = a(y) + b(t)$, the second term on the right-hand side is zero, so that $\partial p/\partial t = \partial b/\partial t$.

14.17.a Differentiating the first-order conditions in the usual way gives

$$\frac{\partial x_1}{\partial t_1} = \frac{1}{p_1' - c_1''} < 0$$
$$\frac{\partial x_2}{\partial t_2} = \frac{1}{2p_2' + p_2'' x_2 - c_2''} < 0.$$

14.17.b The appropriate welfare function is $W = u_1(x_1) + u_2(x_2) - c_1(x_1) - c_2(x_2)$. The total differential is

$$dW = (u_1' - c_1')dx_1 + (u_2' - c_2')dx_2.$$

14.17.c Somewhat surprisingly, we should tax the competitive industry and subsidize the monopoly! To see this, combine the answers to the first two questions to get the change in welfare from a tax policy (t_1, t_2).

$$dW = (p_1 - c_1')\frac{dx_1}{dt_1}dt_1 + (p_2 - c_2')\frac{dx_2}{dt_2}dt_2.$$

The change in welfare from a small tax or subsidy on the competitive industry is zero, since price equals marginal cost. But for the monopolized industry, price exceeds marginal cost, so we want the last term to be positive. But this can only happen if dt_2 is negative—i.e., we subsidize industry 2.

14.18.a The profit maximization problem is

$$\begin{aligned} &\max\ r_1 + r_2 \\ \text{such that } &a_1 x_1 - r_1 \geq 0 \\ &a_2 x_2 - r_2 \geq 0 \\ &a_1 x_1 - r_1 \geq a_1 x_2 - r_2 \\ &a_2 x_2 - r_2 \geq a_2 x_1 - r_1 \\ &x_1 + x_2 \leq 10. \end{aligned}$$

14.18.b The binding constraints will be $a_1 x_1 = r_1$ and $a_2 x_2 - r_2 = a_2 x_1 - r_1$, and $x_1 + x_2 = 10$.

14.18.c The expression is $a_2 x_2 + (2a_1 - a_2)x_1$.

14.18.d Formally, our problem is to solve

$$\max\ a_2x_2 + (2a_1 - a_2)x_1$$

subject to the constraint that $x_1 + x_2 = 10$. Solve the constraint for $x_2 = 10 - x_1$ and substitute into the objective function to get the problem

$$\max_{x_1}\ 10a_2 + 2(a_1 - a_2)x_1.$$

Since $a_2 > a_1$ the coefficient on the second term is negative, which means that $x_1^* = 0$ and, therefore, $x_2^* = 10$. Since $x_2^* = 10$, we must have $r_2^* = 10a_2$. Since $x_1^* = 0$, we must have $r_1^* = 0$.

14.19.a The profit-maximizing choices of p_1 and p_2 are

$$\begin{aligned} p_1 &= a_1/2b_1 \\ p_2 &= a_2/2b_2. \end{aligned}$$

These will be equal when $a_1/b_1 = a_2/b_2$.

14.19.b We must have $p_1(1 - 1/b_1) = c = p_2(1 - 1/b_2)$. Hence $p_1 = p_2$ if and only if $b_1 = b_2$.

14.20.a The first-order condition is $(1-t)[p(x) + p'(x)x] = c'(x)$, or $p(x) + p'(x)x = c'(x)/(1-t)$. This expression shows that the revenue tax is equivalent to an increase in the cost function, which can easily be shown to reduce output.

14.20.b The consumer's maximization problem is $\max_x\ u(x) - m - px + tpx = \max_x\ u(x) - m - (1-t)px$. Hence the inverse demand function satisfies $u'(x) - (1-t)p(x)$, or $p(x) = u'(x)/(1-t)$.

14.20.c Substituting the inverse demand function into the monopolist's objective function, we have

$$(1-t)p(x)x - c(x) = (1-t)u'(x)x/(1-t) - c(x) = u'(x)x - c(x).$$

Since this is independent of the tax rate, the monopolist's behavior is the same with or without the tax.

14.21 Under the *ad valorem* tax we have

$$(1-\tau)P_D = \left(1 + \frac{1}{\epsilon}\right)c.$$

Under the output tax we have

$$P_D - t = \left(1 + \frac{1}{\epsilon}\right).$$

Solve each equation for P_D, set the results equal to each other, and solve for t to find

$$t = \frac{\tau k c}{1-\tau} \qquad k = \frac{1}{1+\frac{1}{\epsilon}}$$

14.22.a Note that his revenue is equal to 100 for any price less than or equal to 20. Hence the monopolist will want to produce as little output as possible in order to keep its costs down. Setting $p = 20$ and solving for demand, we find that $D(20) = 5$.

14.22.b They should set price equal to marginal cost, so $p = 1$.

14.22.c $D(1) = 100$.

14.23.a If $c < 1$, then profits are maximized at $p = 3/2 + c/2$ and the monopolist sells to both types of consumers. The best he can do if he sells only to Type A consumers is to sell at a price of $2 + c/2$. He will do this if $c \geq 1$.

14.23.b If a consumer has utility $ax_1 - x_1^2/2 + x_2$, then she will choose to pay k if $(a-p)^2/2 > k$. If she buys, she will buy $a-p$ units. So if $k < (2-p)^2/2$, then demand is $N(4-p) + N(2-p)$. If $(2-p)^2 < k < (4-p)^2/2$, then demand is $N(4-p)$. If $k > (4-p)^2/2$, then demand is zero.

14.23.c Set $p = c$ and $k = (4-c)^2/2$. The profit will be $N(4-c)^2/2$.

14.23.d In this case, if both types of consumers buy the good, then the profit-maximizing prices will have the Type B consumers just indifferent between buying and not buying. Therefore $k = (2-p)^2/2$. Total profits will then be $N((6-2p)(p-c) + (2-p)^2/2)$. This is maximized when $p = 2(c+2)/3$.

Chapter 15. Game Theory

15.1 There are no pure strategy equilibria and the unique mixed strategy equilibrium is for each player to choose Head or Tails with probability 1/2.

15.2 Simply note that the dominant strategy on the last move is to defect. Given that this is so, the dominant strategy on the next to the last move is to defect, and so on.

15.3 The unique equilibrium that remains after eliminating weakly dominant strategies is (Bottom, Right).

15.4 Since each player bids $v/2$, he has probability v of getting the item, giving him an expected payoff of $v^2/2$.

15.5.a $a \geq e$, $c \geq g$, $b \geq d$, $f \geq h$

15.5.b Only $a \geq e$, $b \geq d$.

15.5.c Yes.

15.6.a There are two pure strategy equilibria, (Swerve, Stay) and (Stay, Swerve).

15.6.b There is one mixed strategy equilibrium in which each player chooses Stay with probability .25.

15.6.c This is $1 - .25^2 = .9375$..

15.7 If one player defects, he receives a payoff of π_d this period and π_c forever after. In order for the punishment strategy to be an equilibrium the payoffs must satisfy

$$\pi_d + \frac{\pi_c}{r} \leq \pi_j + \frac{\pi_j}{r}.$$

Rearranging, we find

$$r \leq \frac{\pi_j - \pi_c}{\pi_d - \pi_j.}$$

15.8.a Bottom.

15.8.b Middle.

15.8.c Right.

15.8.d If we eliminate Right, then Row is indifferent between his two remaining strategies.

15.9.a (Top, Left) and (Bottom, Right) are both equilibria.

15.9.b Yes. (Top, Left) dominates (Bottom, Right).

15.9.c Yes.

15.9.d (Top, Left).

Chapter 16. Oligopoly

16.1 The Bertrand equilibrium has price equal to the *lowest* marginal cost, c_1, as does the competitive equilibrium.

16.2 $\partial F(p,u)/\partial u = 1 - r/p$. Since r is the largest possible price, this expression will be nonpositive. Hence, increasing the ratio of uninformed consumers decreases the probability that low prices will be charged, and increases the probability that high prices will be charged.

16.3 Let $\delta = \beta_1\beta_2 - \gamma^2$. Then by direct calculation: $a_i = (\alpha_i\beta_j - \alpha_j\gamma)/\delta$, $b_i = \beta_j/\delta$, and $c = \gamma/\delta$.

16.4 The calculations are straightforward and may be found in Singh & Vives (1984). Let $\Delta = 4\beta_1\beta_2 - \gamma^2$, and $D = 4b_1b_2 - c^2$. Then it turns out that $p_i^c - p_i^b = \alpha_i\gamma^2/\Delta$ and $q_i^b - q_i^c = a_ic^2/D$, where superscripts refer to Bertrand and Cournot.

16.5 The argument is analogous to the argument given on page 297.

16.6 The problem is that the thought experiment is phrased wrong. Firms in a competitive market would like to *reduce* joint output, not increase it. A conjectural variation of -1 means that when one firm reduces its output by one unit, it believes that the other firm will increase its output by one unit, thereby keeping joint output—and the market price—unchanged.

16.7 In a cartel the firms must equate the marginal costs. Due to the assumption about marginal costs, such an equality can only be established when $y_1 > y_2$.

16.8 Constant market share means that $y_1/(y_1 + y_2) = 1/2$, or $y_1 = y_2$. Hence the conjectural variation is 1. We have seen that the conjectural variation that supports the cartel solution is y_2/y_1. In the case of identical firms, this is equal to 1. Hence, if each firm believes that the other will attempt to maintain a constant market share, the collusive outcome is "stable."

16.9 In the Prisoner's Dilemma, (Defect, Defect) is a dominant strategy equilibrium. In the Cournot game, the Cournot equilibrium is only a Nash equilibrium.

16.10.a $Y = 100$

16.10.b $y_1 = (100 - y_2)/2$

16.10.c $y = 100/3)$

16.10.d $Y = 50$

16.10.e $y_1 = 25$, $y_2 = 50$

16.11.a $P(Y) + P'(Y)y_i = c + t_i$

16.11.b Sum the first order conditions to get $nP(Y) + P'(Y)Y = nc + \sum_{i=1}^{n} t_i$, and note that industry output Y can only depend on the sum of the taxes.

16.11.c Since total output doesn't change, Δy_i must satisfy

$$P(Y) + P'(Y)[y_i + \Delta y_i] = c + t_i + \Delta t_i.$$

Using the original first order condition, this becomes $P'(Y)\Delta y_i = \Delta t_i$, or $\Delta y_i = \Delta t_i / P'(Y)$.

16.12.a $y = p$

16.12.b $y = 50p$

16.12.c $D_m(p) = 1000 - 100p$

16.12.d $y_m = 500$

16.12.e $p = 5$

16.12.f $y_c = 50 \times 5 = 250$

16.12.g $Y = y_m + y_c = 750$.

Chapter 17. Exchange

17.1 In the proof of the theorem, we established that $\mathbf{x}_i^* \sim_i \mathbf{x}_i'$. If $\mathbf{x}_i^*$ and $\mathbf{x}_i'$ were distinct, a convex combination of the two bundles would be feasible and strictly preferred by every agent. This contradicts the assumption that $\mathbf{x}^*$ is Pareto efficient.

17.2 The easiest example is to use Leontief indifference curves so that there are an infinite number of prices that support a given optimum.

17.3 Agent 2 holds zero of good 2.

17.4 $x_A^1 = ay/p_1 = ap_2/p_1, x_B^1 = x_B^2$ so from budget constraint, $(p_1 + p_2)x_B^1 = p_1$, so $x_B^1 = p_1/(p_1 + p_2)$. Choose $p_1 = 1$ an numeraire and solve $ap_2 + 1/(1 + p_2) = 1$.

17.5 There is no way to make one person better off without hurting someone else.

17.6 $x_1^1 = ay_1/p_1, x_2 = by_2/p_1 \qquad y_1 = y_2 = p_1 + p_2$. Solve $x_1^1 + x_2^1 = 2$.

17.7 The Slutsky equation for consumer i is

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j}.$$

17.8 The strong Pareto set consists of 2 allocations: in one person A gets all of good 1 and person B gets all of good 2. The other Pareto efficient allocation is exactly the reverse of this. The weak Pareto set consists of all allocations where one of the consumers has 1 unit of good 1 and the other consumer has at least 1 unit of good units of good 2.

17.9 In equilibrium we must have $p_2/p_1 = x_3^2/x_3^1 = 5/10 = 1/2$.

17.10 Note that the application of Walras' law in the proof still works.

17.11.a The diagram is omitted.

17.11.b We must have $p_1 = p_2$.

17.11.c The equilibrium allocation must give one agent all of one good and the other agent all of the other good.

Chapter 18. Production

18.1.a Consider the following two possibilities. (i) Land is in excess supply. (ii) All land is used. If land is in excess supply, then the price of land is zero. Constant returns requires zero profits in both the apple and the bandanna industry. This means that $p_A = p_B = 1$ in equilibrium. Every consumer will have income of 15. Each will choose to consume $15c$ units of apples and $15(1-c)$ units of bandannas. Total demand for land will be $15cN$. Total demand for labor will be $15N$. There will be excess supply of land if $c < 2/3$. So if $c < 2/3$, this is a competitive equilibrium.

If all land is used, then the total outputs must be 10 units of apples and 5 units of bandannas. The price of bandannas must equal the wage which is 1. The price of apples will be $1 + r$ where r is the price of land. Since preferences are homothetic and identical, it will have to be that each person consumes twice as many apples as bandannas. People will want to consume twice as much apples as bandannas if $p_A/p_B = \frac{c}{(1-c)}(1/2)$. Then it also must be that in equilibrium, $r = (p_A/p_B) - 1 \geq 0$. This last inequality will hold if and only if $c \geq 2/3$. This characterizes equilibrium for $c \geq 2/3$.

18.1.b For $c < 2/3$.

18.1.c For $c < 2/3$.

18.2.a Let the price of oil be 1. Then the zero-profit condition implies that $p_g 2x - x = 0$. This means that $p_g = 1/2$. A similar argument shows that $p_b = 1/3$.

18.2.b Both utility functions are Cobb-Douglas, and each consumer has an endowment worth 10. From this we can easily calculate that $x_1^g = 8$, $x_1^b = 18$, $x_2^g = 10$, $x_2^b = 15$.

18.2.c To make 18 guns, firm 1 needs 9 barrels of oil. To make 33 units of butter, firm 2 needs 11 barrels of oil.

Chapter 19. Time

19.1 See Ingersoll (1987), page 238.

19.2.a Apartments will be profitable to construct as long as the present value of the stream of rents is at least as large as the cost of construction. In equations:

$$p + \frac{(1+\pi)p}{1+r} \geq c.$$

In equilibrium, this condition must be satisfied as an equality, so that

$$p = \frac{1+r}{2+r+\pi}c.$$

19.2.b Now the condition becomes

$$p = \frac{1+r}{2+r+\frac{3}{4}\pi}c.$$

19.2.c Draw the first period demand curve and subtract off the K rent controlled apartments to get the residual demand for new apartments. Look for the intersection of this curve with the two flat marginal cost curves derived above.

19.2.d Fewer.

19.2.e The equilibrium price of new apartments will be higher.

Chapter 20. Asset Markets

20.1 The easiest way to show this is to write the first-order conditions as

$$Eu'(\tilde{C})\tilde{R}_a = Eu'(\tilde{C})R_0$$
$$Eu'(\tilde{C})\tilde{R}_b = Eu'(\tilde{C})R_0$$

and subtract.

20.2 Dividing both sides of the equation by p_a and using the definition $\tilde{R}_a = \tilde{V}_a/p_a$, we have

$$R_a = \overline{R}_0 - R_0\text{cov}(F(\tilde{C}), \tilde{R}_a).$$

Chapter 21. Equilibrium Analysis

21.1 The core is simply the initial endowment.

21.2 Since the income effects are zero, the matrix of derivatives of the Marshallian demand function is equal to the matrix of derivatives of the Hicksian demand function. It follows from the discussion in the text that the index of every equilibrium must be $+1$, which means there can be only one equilibrium.

21.3 Differentiating $V(p)$, we have

$$\begin{aligned}\frac{dV(p)}{dt} &= -2\mathbf{z}(\mathbf{p})\mathbf{D}\mathbf{z}(\mathbf{p})\dot{\mathbf{p}} \\ &= -2\mathbf{z}(\mathbf{p})\mathbf{D}\mathbf{z}(\mathbf{p})\mathbf{D}\mathbf{z}(\mathbf{p})^{-\mathbf{1}}\mathbf{z}(\mathbf{p}) \\ &= -2\mathbf{z}(\mathbf{p})\mathbf{z}(\mathbf{p}) < 0.\end{aligned}$$

Chapter 22. Welfare

22.1 We have the equation

$$\theta x_i = \sum_{j=1}^{k} t_j \frac{\partial h_j}{\partial p_i}.$$

Multiply both sides of this equation by t_i and sum to get

$$\theta R = \theta \sum_i t_i x_i = \sum_{j=1}^{k}\sum_{i=1}^{k} t_i t_j \frac{\partial h_j}{\partial p_i}.$$

The right-hand side of expression is nonpositive (and typically negative) since the Slutsky matrix is negative semidefinite. Hence θ has the same sign as R.

22.2 The problem is

$$\max v(\mathbf{p}, m)$$
$$\text{such that } \sum_{i=1}^{k}(p_i - c_i)x_i(p_i) = F.$$

This is almost the same as the optimal tax problem, where $p_i - c_i$ plays the role of t_i. Applying the inverse elasticity rule gives us the result.

Chapter 23. Public goods

23.1 Suppose that it is efficient to provide the public good together, but neither agent wants to provide it alone. Then any set of bids such that $b_1 + b_2 = c$ and $b_i \leq r_i$ is an equilibrium to the game. However, there are also many *inefficient* equilibria, such as $b_1 = b_2 = 0$.

23.2 If utility is homothetic, the the consumption of each good will be proportional to wealth. Let the demand function for the public good be given by

$$f_i(w) = \frac{a_i}{1 + a_i} w.$$

Then the equilibrium amount of the public good is the same as in the Cobb-Douglas example given in the text.

23.3 Agent 1 will contribute $g_1 = \alpha w_1$. Agent 2's reaction function is $f_2(w_2 + g_1) = \max\{\alpha(w_2 + g_1) - g_1, 0\}$. Solving $f_2(w_2 + \alpha w_1) = 0$ yields $w_2 = (1 - \alpha)w_1$.

23.4 The total amount of the public good with k contributors must satisfy

$$G = \alpha \left(\frac{w}{k} + \frac{G}{k} \right).$$

Solving for G, we have $G = \alpha w/(k - \alpha)$. As k increases, the amount of wealth becomes more equally distributed and the amount of the privately provided public good decreases.

23.5 The allocation is not in general Pareto efficient, since for some patterns of preferences some of the private good must be thrown away. However, the amount of the public good provided will be the Pareto efficient amount: 1 unit if $\sum_i r_i > c$, and 0 units otherwise.

23.6.a

$$\max_{g_i} a_i \ln(G_{-i} + g_i) + w_i - g_i$$
$$\text{such that } g_i \geq 0.$$

23.6.b The first-order condition for an *interior* solution is

$$\frac{a_i}{G} = 1,$$

or $G = a_i$. Obviously, the only agent who will give a positive amount is the one with the maximum a_i.

23.6.c Everyone will free ride except for the agent with the maximum a_i.

23.6.d Since all utility functions are quasilinear, a Pareto efficient amount of the public good can be found by maximizing the sum of the utilities:

$$\sum_{i=1}^{n} a_i \ln G - G,$$

which implies $G^* = \sum_{i=1}^{n} a_i$.

Chapter 24. Externalities

24.1.a Agent 1's utility maximization problem is

$$\max_{x_1} \ u_1(x_1) - p(x_1, x_2)c_1,$$

while the social problem is

$$\max_{x_1, x_2} \ u_1(x_1) + u_2(x_2) - p(x_1, x_2)[c_1 + c_2].$$

Since agent 1 ignores the cost he imposes on agent 2, he will generally choose too large a value of x_1.

24.1.b By inspection of the social problem and the private problem, agent 1 should be charged a fine $t_1 = c_2$.

24.1.c If the optimal fines are being used, then the total costs born by the agents in the case of an accident are $2[c_1 + c_2]$, which is simply twice the total cost of the accident.

24.1.d Agent 1's objective function is

$$(1 - p(x_1, x_2))u_1(x_1) - p(x_1, x_2)c_1.$$

This can also be written as

$$u_1(x_1) - p(x_1, x_2)[u_1(x_1) + c_1].$$

This is just the form of the previous objective function with $u_1(x_1) + c_1$ replacing c_1. Hence the optimal fine for agent 1 is $t_1 = u_2(x_2) + c_2$.

Chapter 25. Information

25.1 By construction we know that $f(u(s)) \equiv s$. Differentiating one time shows that $f'(u)u'(s) = 1$. Since $u'(s) > 0$, we must have $f'(u) > 0$. Differentiating again, we have

$$f'(u)u''(s) + f''(u)u'(s)^2 = 0.$$

Using the sign assumptions on $u'(s)$, we see that $f''(u) > 0$.

25.2 According to the envelope theorem, $\partial V/\partial c_a = \lambda + \mu$ and $\partial V/\partial c_b = \mu$. Thus, the sensitivity of the payment scheme to the likelihood ratio, μ, depends on how big an effect an increase in c_b would have on the principal.

25.3 In this case it is just as costly to undertake the action preferred by the principal as to undertake the alternative action. Hence, the incentive constraint will not be binding, which implies $\mu = 0$. It follows that $s(x_i)$ is constant.

25.4 If c_b decreases, the original incentive scheme (s_i) will still be feasible. Hence, an optimal incentive scheme must do at least as well as the original scheme.

25.5 In this case the maximization problem takes the form

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} (x_i - s_i)\pi_{ib} \\ \text{such that} \quad & \sum_{i=1}^{n} s_i \pi_{ib} - c_b \geq \overline{u} \\ & \sum_{i=1}^{n} s_i \pi_{ib} - c_b \geq \sum_{i=1}^{n} s_i \pi_{ia} - c_a. \end{aligned}$$

Assuming that the participation constraint is binding, and ignoring the incentive-compatibility constraint for a moment, we can substitute into the objective function to write

$$\max \sum_{i=1}^{m} s_i \pi_{ib} - c_b - \overline{u}.$$

Hence, the principal will choose the action that maximizes expected output minus (the agent's) costs, which is the first-best outcome. We can satisfy the incentive-compatibility constraint by choosing $s_i = x_i + F$, and choose F so that the participation constraint is satisfied.

25.6 The participation constraints become $s_t - c(x_t) \geq \overline{u}_t$, which we can write as $s_t - (c(x_t) + \overline{u}_t)$. Define $c_t(x) = c(x) + \overline{u}_t$, and proceed as in the text. Note that the marginal costs of each type are the same, which adds an extra case to the analysis.

25.7 Since $c_2'(x) > c_1'(x)$, we must have

$$\int_{x_1}^{x_2} c_2'(x)\,dx > \int_{x_1}^{x_2} c_1'(x)\,dx.$$

The result now follows from the Fundamental Theorem of Calculus.

25.8 The indifference curves take the form $u_1 = s - c_1(x)$ and $u_2 = s - c_2(x)$. Write these as $s = u_1 + c_1(x)$ and $s = u_2 + c_2(x)$. The difference between these two functions is $d(x) = u_2 - u_1 + c_2(x) - c_1(x)$, and the derivative of this difference is $d'(x) = c_2'(x) - c_1'(x) > 0$. Since the difference function is a monotonic function, it can hit zero at most once.

25.9 For only low-cost workers to be employed, there must be no profitable contract that appeals to the high-cost workers. The most profitable contract to a high-cost worker maximizes $x_2 - x_2^2$, which implies $x_2^* = 1/2$. The cost of this to the worker is $(1/2)^2 = 1/4$. For the worker to find this acceptable, $s_2 - 1/4 \geq \overline{u}_2$, or $s_2 = \overline{u}_2 + 1/4$. For the firm to make a profit, $x_2^* \geq s_2$. Hence we have $1/2 \geq \overline{u}_2 + 1/4$, or $\overline{u}_2 \leq 1/4$.

25.10.a The professor must pay $s = x^2/2$ to get the assistant to work x hours. Her payoff will be $x - x^2/2$. This is maximized where $x = 1$.

25.10.b The TA must get his reservation utility when he chooses the optimal x. This means that $s - x^2/2 = s - 1/2 = 0$, so $s = 1/2$.

25.10.c The best the professor can do is to get Mr. A to work 1 hour and have a utility of zero. Mr. A will work up to the point where he maximizes $ax + b - x^2/2$. Using calculus, we find that Mr. A will choose $x = a$. Therefore he will work one hour if $a = 1$. Then his utility will be 0 if $b = -1$.