CHAPTER 3 REFLECTIVE TRANSFORMATIONS

Abstract. A reflective transformation is a one-to-one transformation of $\mathbf{Q}^n$ such that its inverse is isometrically similar to itself. After basic properties of reflective transformations are given in Section 3.1, any Boolean isometry is proved to be reflective through a Boolean isometry of order 2. Then conditions for reflectiveness are given for [ ]-representations of self-dual transformations. Next, reflective circular transformations are described in Section 3.4. Finally, methods of constructing reflective transformations through face copies and orbit modifications are described.

3.1 Reflective transformations

A self-dual transformation and a circular transformation of $\mathbf{Q}^n$ discussed in Chapter 2 are defined as trans- formations F commutative with the complementation $\bar{\neg}$ and the rotation ρ respectively, that is

$$F = (\bar{\neg})^{-1} F \bar{\neg} \quad \text{and} \quad F = \rho^{-1} F \rho$$

respectively. On the other hand, there is a class of one-to-one transformations F such that

$$F^{-1} = T^{-1} F T \tag{3.1.1}$$

for some Boolean isometry T, that is, F^{-1} is isometrically similar to F. Note that the condition (3.1.1) is equivalent to

$$FTF = T. \tag{3.1.2}$$

We call such a transformation reflective through T. For example, a *binary-reflected Gray code* for dimension n introduced by Gray (1953) is the graph of a transformation reflective through n^- and also $(n-1)^-$. One-to-one *threshold transformations* constructed in the next two chapters are not only self-dual but also often turn out to be reflective. In the next section, we prove that any Boolean isometry is reflective through some Boolean isometry of order 2.

Example 3.1.1 Let $f_1 = \{110\}$, $f_2 = \{110\}$, $f_3 = \{111\}$. Then the graph of $F = [f_1, f_2, f_3]$ consists of loops and one 4-cycle, that is

$$110 \to 000 \to 001 \to 111 \to 110.$$

F is reflective through $\{1,2\}^-$ and also reflective through 3^-.

Proposition 3.1.2 Let F be reflective through T. Then (i) if F is commutative with an isometry S, then F is reflective through ST and TS; (ii) if F is reflective through S, then F is commutative with ST.

Proof. (i) $(ST)^{-1}F(ST) = T^{-1}S^{-1}FST = T^{-1}(S^{-1}SF)T = T^{-1}FT = F^{-1}$. $(TS)^{-1}F(TS) = S^{-1}T^{-1}FTS = S^{-1}F^{-1}S = (FS)^{-1}S = (SF)^{-1}S = F^{-1}S^{-1}S = F^{-1}$. (ii) is similarly proved. $\square$

Corollary 3.1.3 Let $\mathbf{N} = J \cup K$ and $J \cap K = \emptyset$. If F is self-dual and reflective through J^- then F is reflective through K^-.

Proof. Let $T = J^-$ and $S = \bar{\neg}$ in (i) of Proposition 3.1.2. $\square$

Corollary 3.1.4 If F is a transformation of $\mathbf{Q}^n$, then the set of all T such that F is commutative with T or reflective through T is a subgroup of $O(\mathbf{Q}^n)$.

If F is reflective through T, then $U^{-1}FU$ is reflective through $U^{-1}TU$ for any isometry U. If F and G are reflective through T and $FG = GF$, then FG is reflective through T. In particular, if F is reflective through T, then F^i is reflective through T for every positive integer i. Since $\mathbf{Q}^n$ is a finite set, there exists a positive integer k such that $F^k = I$. Therefore, $F^{-1} = F^{k-1}$, hence, if F is reflective through T then F^{-1} is reflective through T. If F is self-dual and reflective through T, then $T^{-1}(\bar{\neg}F)T = \bar{\neg}T^{-1}FT = \bar{\neg}F^{-1} = (\bar{\neg}F)^{-1}$, so that $\bar{\neg}F$ is reflective through T.

Assume that F is reflective through S, G is reflective through T, F is commutative with G and T, and G is commutative with S. Then by Proposition 3.1.2 (i), F is reflective through ST, and G is reflective through ST. Further, since F is commutative with G, FG is reflective through ST. Similarly, FG is reflective through TS. If F is reflective through T, and if S is an isometry of order 2 such that $ST = TS$, then SF is reflective through ST.

If F is reflective through T, then $(TF)^2 = T^2$. In particular, if the order of T is 2, then F is isometrically equivalent to a transformation whose graph consists of 2-cycles and loops. If F is reflective through T then $\mathrm{Orb}_{\langle F \rangle}(Tx) = T(\mathrm{Orb}_{\langle F \rangle}x)$ for every x.

3.2 Boolean isometries are reflective

We prove here that any Boolean isometry is reflective through some Boolean isometry of order 2. For any isometry T of the real n-space, that is, $T \in O(\mathbf{R}^n)$, there exists $U \in O(\mathbf{R}^n)$ of order 2 such that $T^{-1} = U^{-1}TU$. This proposition can be proved using the Jordan canonical form of a matrix representing T. However, in a finite subgroup $\mathbf{G} \subseteq O(\mathbf{R}^n)$, it is not always true that any element and its inverse are conjugate. For example, both the group of the isometries of $\mathbf{Q}^n$ and a group of rotation on $\mathbf{R}^2$ are reflection groups (see e.g. Grove & Benson, 1985), but the latter does not have this property if the order is greater than 2.

Lemma 3.2.1 Let $\rho = (1, 2, .., n)$ and $A \subseteq \mathbf{N}$. Then the equation

$$X \dot{+} \rho X = A. \tag{3.2.1}$$

for an unknown subset $X \subseteq \mathbf{N}$ has a solution iff $|A|$ is even. Further, if X is a solution, then X and its complement X^c are the only solutions.

Proof. Since X and ρX have the same cardinality, their symmetric difference must be even (proving the necessity). For the sufficiency, if $A = \emptyset$, then clearly $X = \emptyset$ and $X = \mathbf{N}$ are the only solutions of (3.2.1). Assume $|A|$ is positive and even. If $A = \{i, j\}$ with $i < j$ then $X = \{i, ..., j-1\}$ is a solution. If $|A|$ is even, then A is the sum ($\dot{+}$) of pairs, so that by (2.4.5) of Chapter 2 there exists a solution X. Suppose Y is also a solution. Then $X \dot{+} \rho X = Y \dot{+} \rho Y$, so that $(X \dot{+} \rho X) \dot{+} (Y \dot{+} \rho Y) = \emptyset$, i.e. $(X \dot{+} Y) \dot{+} \rho(X \dot{+} Y) = \emptyset$. Therefore, $X \dot{+} Y = \emptyset$ or $X \dot{+} Y = \mathbf{N}$, i.e. $Y = X$ or $Y = X^c$. $\square$

Theorem 3.2.2 (Ueda,1998) If T is an isometry of $\mathbf{Q}^n$, then T is reflective through some isometry of order 2 of $\mathbf{Q}^n$.

Proof. By Proposition 2.1.5 of Chapter 2, any isometry $T \in O(\mathbf{Q}^n)$ can be decomposed as the commutative product

$$T = \tau_1 J_1^- \odot ... \odot \tau_k J_k^- \cdot \iota J_{k+1}^-,$$

where τ_i is a cyclic permutation such that $\mathrm{Car}\tau_i$ and $\mathrm{Car}\tau_j$ are disjoint, $J_i \subseteq \mathrm{Car}\tau i$ for every $i = 1, .., k$, and $J_{k+1} \subseteq \mathbf{N} \setminus \bigcup_i \mathrm{Car}\tau_i$. It suffices to prove for each component of the disjoint composition. If $\mathbf{N} \setminus \bigcup_i \mathrm{Car}\tau_i$ is nonempty, then ιJ_{k+1}^- is reflective through ιJ_{k+1}^- and $\iota((\mathbf{N} \setminus \bigcup_i \mathrm{Car}\tau_i) \setminus J_{k+1})^-$, and at least one of them is of order 2. Therefore, it suffices to prove for $T = \tau J^-$, where $\mathrm{Car}\tau = \mathbf{N}$ and $J \subseteq \mathbf{N}$. Therefore, it suffices to prove for $T = \rho A^-$, $A \subseteq \mathbf{N}$, since τJ^- is a conjugate of ρA^- for some A.

Assume that ξ is an unknown permutation of $\mathbf{N}$ and X is an unknown subset of $\mathbf{N}$ such that T is reflective through $U = \xi X^-$. Then $U^{-1} T U = T^{-1}$, i.e. $TUT = U$, i.e., $\rho \xi \rho = \xi$ and $\rho^{-1}(\xi^{-1} A \dot{+} X) \dot{+} A = X$, i.e.

$$\rho \xi \rho = \xi \text{ and } X \dot{+} \rho X = \xi^{-1} A \dot{+} \rho A. \tag{3.2.2}$$

The solutions of the first equation of (3.2.2) are the linear permutations ξ of slope -1, for any one of which, the second equation has two solutions X and X^c by Lemma 3.2.1, since $|A \dot{+} B|$ is even for arbitrary sets A and B of same cardinality modulo 2. Therefore, T is reflective through an isometry $U = \xi X^-$. To prove that U is of order 2, it is sufficient to show that any such U satisfies the equation

$$\xi \xi = \iota \text{ and } \xi^{-1} X \dot{+} X = \emptyset \text{ i.e. } \xi X = X. \tag{3.2.3}$$

The first equation of (3.2.3) is satisfied by any linear permutation of slope -1. Further, $\xi X \dot{+} \rho(\xi X) = \rho \xi \rho X \dot{+} \rho \xi X = \rho \xi(\rho X \dot{+} X) = \rho \xi(\xi^{-1} A \dot{+} \rho A) = \rho A \dot{+} \xi A$. Therefore, ξX is also a solution of the second equation of (3.2.3), so that $\xi X = X$ or $\xi X = X^c$. Suppose the second case holds. Then ξ has no fixed point, so that there exists some $i \in \mathbf{N}$ such that $\xi i = \rho i = j$. Clearly $j \notin \xi A \dot{+} \rho A$. If $i \in X$, then $j \in \rho X$ and $j \in \xi X$, i.e. $j \notin X$. If $i \notin X$, then $j \notin \rho X$ and $j \notin \xi X$ i.e. $j \in X$. Therefore, $j \in X \dot{+} \rho X$, which contradicts the fact that X is a solution of the second equation of (3.2.3). □

3.3 Reflectiveness for [ ]-representations

In this section some necessary or sufficient conditions for a self-dual transformation F to be a reflective transformation are described, when F is represented as $F = [f_1, ..., f_n]$ introduced in Chapter 2.3.

Proposition 3.3.1 Let $F = [f_1, ..., f_n]$ be a self- dual transformation of $\mathbf{Q}^n$ and τ be a permutation of $\mathbf{N}$. F is reflective through τ if and only if $\tau F f_i \subseteq \bar{\neg} f_{\tau i}$ and $\tau F(f_i \cup \bar{\neg} f_i)^c \cap \bar{\neg} f_{\tau i} = \emptyset$ for every $i \in \mathbf{N}$.

Proof. Assume that F is reflective through τ. Then $F \tau F = \tau$. Let $x \in f_i$. Then $x_i = 1$, $(Fx)_i = 0$. Therefore,

$$(\tau F x)_{\tau i} = (\tau^{-1}(\tau F x))_i = (Fx)_i = 0$$

and

$$(F(\tau F x))_{\tau i} = (\tau x)_{\tau i} = (\tau^{-1}(\tau x))_i = x_i = 1,$$

so that $\tau Fx \in \neg f_{\tau i}$. Next, let $x \notin f_i \cup \neg f_i$. Then $x_i = (Fx)_i$, so that $(F(\tau Fx))_{\tau i} = (\tau x)_{\tau i} = (\tau Fx)_{\tau i}$. Therefore, $\tau Fx \notin \neg f_{\tau i}$.

Assume $\tau F f_i \subseteq \neg f_{\tau i}$ and $\tau F(f_i \cup \neg f_i)^c \cap \neg f_{\tau i} = \emptyset$ for every $i \in \mathbf{N}$. To prove that F is reflective through τ, we will show $F\tau F = \tau$. Let $x \in f_i$ for some i. Then $\tau Fx \in \neg f_{\tau i}$, so that $(F\tau Fx)_{\tau i} = 1$. Also, $x_i = 1$, so that $(\tau x)_{\tau i} = (\tau^{-1}\tau x)_i = 1$. Therefore, $(F\tau Fx)_{\tau i} = (\tau x)_{\tau i}$. If $x \in \neg f_j$ for some j, then by the self-duality of F and τ, $(F\tau Fx)_{\tau j} = (\tau x)_{\tau j}$ also. Let $x \notin f_k \cup \neg f_k$ for some k. Then $\tau Fx \notin \neg f_{\tau k} \cup f_{\tau \kappa}$, because $\neg x \notin f_k \cup \neg f_k$. Therefore,

$$(F\tau Fx)_{\tau k} = (\tau Fx)_{\tau k} = (Fx)_k = x_k = (\tau x)_{\tau k}.$$

Consequently, $(F\tau F)x = \tau x$ for every $x \in \mathbf{Q}^n$, so that F is reflective through τ. □

Proposition 3.3.1 can be generalized into the following proposition.

Proposition 3.3.2 Let $F = [f_1, ..., f_n]$ be a self- dual transformation of $\mathbf{Q}^n$ and $T = \tau J^-$ be an isometry of $\mathbf{Q}^n$. F is reflective through T if and only if (i) $TFf_i \subseteq f_{\tau i}$ for every $i \in J$, (ii) $TFf_i \subseteq \neg f_{\tau i}$ for every $i \in J^c$, and (iii) $TF(f_i \cup \neg f_i)^c \cap \neg f_{\tau i} = \emptyset$ for every i.

Proof. Assume that F is reflective through T. Then $FTF = T$. Let $x \in f_i$ and $i \in J$. Then $x_i = 1$ and $(Fx)_i = 0$.

$$(Tfx)_{\tau i} = (\tau^{-1}(TFx))_i = (J^- Fx)_i = 1,$$

and

$$(F(TFx))_{\tau i} = (Tx)_{\tau i} = (\tau^{-1}(Tx))_i = (\tau^{-1}\tau J^- x)_i = (J^- x)_i = 0.$$

Therefore $TFx \in f_{\tau i}$. Next, let $x \in f_i$ and $i \in J^c$. Then $x_i = 1$ and $(Fx)_i = 0$. $(Tfx)_{\tau i} = (J^- Fx)_i = 0$. $(F(TFx))_{\tau i} = (J^- x)_i = 1$. Therefore $TFx \in \neg f_{\tau i}$. Next, let $x \notin f_i \cup \neg f_i$. Then $x_i = (Fx)_i$, so that

$$(F(TFx))_{\tau i} = (Tx)_{\tau i} = (J^- x)_i = (J^- Fx)_i = (Tfx)_{\tau i}.$$

Therefore, $TFx \notin \neg f_{\tau i}$.

Assume the conditions (i), (ii) and (iii). To prove that F is reflective through T, we will show that $FTF = T$. Let $x \in f_i$ for some $i \in J$. Then $TFx \in f_{\tau i}$, so that $(FTFx)_{\tau i} = 0$. Also $x_i = 1$, so that

$$(Tx)_{\tau i} = (\tau^{-1}(Tx))_i = (\tau^{-1}\tau J^- x)_i = (J^- x)_i = 0.$$

Therefore, $(FTFx)_{\tau i} = (Tx)_{\tau i}$. If $x \in \neg f_j$ for some $j \in J$, then by the self-duality of F and T, $(FTFx)_{\tau j} = (Tx)_{\tau j}$ also. Let $x \in f_i$ for some $i \in J^c$. Then $TFx \in \neg f_{\tau i}$, so that $(FTFx)_{\tau i} = 1$. Also $x_i = 1$, so that $(Tx)_{\tau i} = (J^- x)_i = 1$. Therefore, $(FTFx)_{\tau i} = (Tx)_{\tau i}$. If $x \in \neg f_j$ for some $j \in J^c$, then by the self-duality of F and T, $(FTFx)_{\tau j} = (Tx)_{\tau j}$ also. Let $x \notin f_k \cup \neg f_k$ for some k. Then $TFx \notin \neg f_{\tau k} \cup f_{\tau \kappa}$, because $\neg x \notin f_k \cup \neg f_k$. Therefore,

$$(FTFx)_{\tau k} = (Tfx)_{\tau k} = (\tau J^- Fx)_{\tau k} = (J^- Fx)_k = (J^- x)_k = (Tx)_{\tau k}.$$

Consequently, $(FTF)x = Tx$ for every $x \in \mathbf{Q}^n$, so that F is reflective through T. □

Corollary 3.3.3 Let $F = [f_1, ..., f_n]$ be a self-dual transformation of $\mathbf{Q}^n$ and $T = \tau J^-$ be an isometry of $\mathbf{Q}^n$. If F is reflective through T, then $|f_{\tau i}| = |f_i|$ for every i.

3.4 Circular and skew-circular reflective transformations

In this section, we investigate the properties of some circular and skew-circular reflective transformations. First, let us consider an example of circular reflective transformation.

Example 3.4.1 Let $f = p_1 \cdot p_2 \cdot p_3 \cdot (\neg p_4) \cdot (\neg p_5)$ be the transformation of $\mathbf{Q}^5$. Then the graph of $F = \langle f \rangle$ consists of loops and one 10-cycle, that is

$$\begin{aligned} 11100 &\rightarrow 01100 \rightarrow 01110 \rightarrow 00110 \rightarrow 00111 \rightarrow 00011 \\ &\rightarrow 10011 \rightarrow 10001 \rightarrow 11001 \rightarrow 11000 \rightarrow 11100. \end{aligned}$$

F is reflective through any linear permutation of $\mathbf{N}_5$ of slope -1.

The above example illustrates Proposition 3.1.2. In fact, If F is circular and reflective through a linear permutation τ of coefficients (a, b), then F is reflective through any linear permutation of slope a, that is, $\rho^j \tau$ whose coefficients are $(a, b + j)$, since the coefficients of ρ^j are $(1, j)$.

Notation Let λ and $\mu \in \mathrm{SYM}(\mathbf{N})$ denote the linear permutations of coefficients $(-1, 1)$ and $(-1, 2)$ respectively, that is,

$$\begin{aligned} \lambda &= (1, n)(2, n-1) \cdot \cdot (i, n-i+1)([n/2], n-[n/2]+1), \\ \mu &= (2, n) \cdot (3, n-1) \cdot \cdot (i, n-i+2) \cdot \cdot ([(n-1)/2]+1, n+1-[(n-1)/2]). \end{aligned}$$

Proposition 3.4.2 Let $F = \langle f \rangle$ be a self-dual circular transformation of $\mathbf{Q}^n$. F is reflective through any linear permutation of slope -1, if and only if $\mu F f \subseteq \bar{\neg} f$ and $\mu F(f \cup \bar{\neg} f)^c \cap \bar{\neg} f = \emptyset$.

Proof. $F = [f, \rho f, .., \rho^{i-1} f, .., \rho^{n-1} f]$ by definition. Assume $\mu F f \subseteq \bar{\neg} f$ and $\mu F(f \cup bar \neg f)^c \cap \bar{\neg} f = \emptyset$. Let $x \in \rho^{i-1} f$. Then $\rho^{-(i-1)} x \in f$. Therefore, $\mu F \rho^{-(i-1)} x \in \bar{\neg} f$. Therefore,

$$\rho^{i-1} \mu F x = \mu \rho^{-(i-1)} F x = \mu F \rho^{-(i-1)} x \in \bar{\neg} f,$$

so that

$$\mu F x \in \rho^{-(i-1)} \bar{\neg} f = \rho^{n-i+1} \bar{\neg} f = \bar{\neg} \rho^{\mu i-1} f.$$

Similarly, if $x \notin \rho^{i-1} f \cup \bar{\neg} \rho^{i-1} f$ then $\mu F x \notin \bar{\neg} \rho^{\mu i-1} f$. Consequently, by Proposition 3.3.1, F is reflective through μ, so that F is reflective through every linear permutation of slope -1. For the converse, apply Proposition 3.3.1 to μ and $f_1 = f$. $\square$

Proposition 3.4.3 (i) A circular transformation expressed by $F = (F_1, .., \rho^{i-1} F_1, .., \rho^{n-1} F_1)$ is reflective through μ, if and only if

$$F^{-1} = (\mu F_1, \rho \mu F_1, .., \rho^{n-1} \mu F_1). \tag{3.4.1}$$

(ii) A self-dual circular transformation $\langle f \rangle$ is reflective through μ, if and only if

$$\langle f \rangle^{-1} =< \mu f > . \tag{3.4.2}$$

Proof. (i)

$$\begin{aligned}
\mu^{-1}F\mu &= \mu^{-1}F\mu^{-1} \\
&= \mu^{-1}(F_1\mu^{-1},..,(\rho^{i-1}F_1)\mu^{-1},..,(\rho^{n-1}F_1)\mu^{-1}) \\
&= \mu^{-1}(\mu F_1,..,\mu\rho^{i-1}F_1,..,\mu\rho^{n-1}F_1) \\
&= \mu^{-1}(\mu F_1,..,\rho^{-(i-1)}\mu F_1,..,\rho^{-(n-1)}\mu F_1) \\
&= (\mu F_1,..,\rho^{-(\mu i-1)}\mu F_1,..,\rho^{-(\mu n-1)}\mu F_1) \\
&= (\mu F_1,..,\rho^{i-1}\mu F_1,..,\rho^{n-1}\mu F_1).
\end{aligned}$$

Therefore, F is reflective through μ, if and only if (3.4.1) holds.
(ii) Note that $\mu^{-1}F\mu$ is self-dual if F is self-dual. Further if $F = \langle f\rangle$, then

$$p_1 \cdot \neg(\mu F_1) = p_1 \cdot \bar{\neg}(\mu F_1) = p_1 \cdot (\mu\bar{\neg}F_1) = \mu(p_1 \cdot (\bar{\neg}F_1)) = \mu(p_1 \cdot \neg F_1) = \mu f.$$

Therefore $\langle f\rangle$ is reflective through μ if and only if (3.4.2) holds. □

Example 3.4.4 $F = \langle\langle f\rangle\rangle$, $f = p_1 \cdots p_i \cdots p_n$. GRAPH$(F)$ consists of one $2n$-cycle and loops. As seen from the graph given in Example 2.5.5, F is reflective through λ.

If F is skew-circular and reflective through λ, then F is reflective through $(\rho n^-)^j\lambda$ for every j, particularly through $\rho n^-\lambda = \mu 1^-$, since F is commutative with ρn^-.

We used the equation (1.2.4) $\rho\mu = \mu\rho^{-1}$ in the proof of Proposition 3.4.2. Similarly we need

$$(\rho n^-)\lambda = \lambda(\rho n^-)^{-1} \tag{3.4.3}$$

in the proof of the next Proposition. In fact,

$$\begin{aligned}
(\rho n^-)\lambda(x_1,...,x_n) &= \rho n^-(x_n,..,x_1) \\
&= \rho(x_n,..,x_2,\neg x_1) \\
&= (\neg x_1,x_n,..,x_2) \\
&= \lambda(x_2,..x_n,\neg x_1) \\
&= \lambda n^-(x_2,...x_n,x_1) \\
&= \lambda n^-\rho^{-1}(x_1,...,x_n) \\
&= \lambda(\rho n^-)^{-1}(x_1,...,x_n).
\end{aligned}$$

Proposition 3.4.5 Let $F = \langle\langle f\rangle\rangle$ be a self-dual skew-circular transformation of $\mathbf{Q}^n$. F is reflective through $(\rho n^-)^j\lambda$ for every j, if and only if $\lambda Ff \subseteq \bar{\neg}(\rho n^-)^{n-1}f$ and $\lambda F(f \cup \bar{\neg}f)^c \cap \bar{\neg}(\rho n^-)^{n-1}f = \emptyset$.

Proof. $F = [f,\rho n^- f,..,(\rho n^-)^{i-1}f,..,(\rho n^-)^{n-1}f]$ by Proposition 2.5.4. Assume $\lambda Ff \subseteq \bar{\neg}(\rho n^-)^{n-1}f$ and $\lambda F(f \cup \bar{\neg}f)^c \cap \bar{\neg}(\rho n^-)^{n-1}f = \emptyset$. Let $x \in (\rho n^-)^{i-1}f$. Then $(\rho n^-)^{-(i-1)}x \in f$. Therefore, $\lambda F(\rho n^-)^{-(i-1)}x \in \bar{\neg}(\rho n^-)^{n-1}f$. Therefore,

$$(\rho n^-)^{i-1}\lambda Fx = \lambda(\rho n^-)^{-(i-1)}Fx = \lambda F(\rho n^-)^{-(i-1)}x \in \bar{\neg}(\rho n^-)^{n-1}f,$$

so that

$$\lambda Fx \in (\rho n^-)^{-(i-1)}\bar{\neg}(\rho n^-)^{n-1}f = \bar{\neg}(\rho n^-)^{n-i}f = \bar{\neg}(\rho n^-)^{\lambda i-1}f.$$

Similarly, if $x \notin \rho^{i-1}f \cup \bar{\neg}\rho^{i-1}f$ then $\lambda Fx \notin \bar{\neg}\rho^{\lambda i-1}f$. Consequently, by Proposition 3.3.1, F is reflective through λ, so that F is reflective through $(\rho n^-)^j\lambda$ for every j. For the converse, apply Proposition 3.3.1 to λ and $f_1 = f$. □

Proposition 3.4.6 (i) A skew-circular transformation expressed by

$$F = (F_1, .., (\rho n^-)^{i-1} F_1, .., (\rho n^-)^{n-1} F_1)$$

is reflective through $T = \mu 1^-$, if and only if

$$F^{-1} = (\bar{\neg} T F_1, \rho n^- \bar{\neg} T F_1, .., (\rho n^-)^{n-1} \bar{\neg} T F_1). \quad (3.4.3)$$

(ii) A skew-circular transformation $\langle\langle f \rangle\rangle$ is reflective through $T = \mu 1^-$, if and only if

$$\langle\langle f \rangle\rangle^{-1} = \langle\langle \bar{\neg} T f \rangle\rangle. \quad (3.4.4)$$

Proof. (i) $T^{-1} F T = T^{-1} F T^{-1}$

$$\begin{aligned}
&= T^{-1}(F_1 T^{-1}, .., ((\rho n^-)^{i-1} F_1) T^{-1}, .., ((\rho n^-)^{n-1} F_1) T^{-1}) \\
&= T^{-1}(T F_1, .., T((\rho n^-)^{i-1} F_1), .., T((\rho n^-)^{n-1} F_1)) \\
&= T^{-1}(T F_1, .., (\rho^{-1} 1^-)^{i-1} T F_1, .., (\rho^{-1} 1^-)^{n-1} T F_1) \\
&= T^{-1}(T F_1, .., \rho^{-(i-1)} \{1, .., i-1\}^- T F_1, .., \rho^{-(n-1)} \{1, .., n-1\}^- T F_1) \\
&= (\neg T F_1, .., \rho^{-(\mu i - 1)} \{1, .., \mu i - 1\}^- T F_1, .., \rho^{-1} 1^- T F_1) \\
&= (\neg T F_1, .., \rho^{i-1} \{1, .., n-i+1\}^- T F_1, .., \rho^{-1} 1^- T F_1) \\
&= (\bar{\neg} T F_1, .., \rho^{i-1} \{n-i+2, .., n\}^- \bar{\neg} T F_1, .., \rho^{-1} \{2, .., n\}^- \bar{\neg} T F_1) \\
&= (\bar{\neg} T F_1, .., (\rho n^-)^{i-1} \bar{\neg} T F_1, .., (\rho n^-)^{n-1} \bar{\neg} T F_1).
\end{aligned}$$

Therefore, F is reflective through T, if and only if (3.4.3) holds. (ii) If $F = \langle\langle f \rangle\rangle$, then

$$p_1 \cdot \neg(\bar{\neg} T F_1) = (\bar{\neg} T p_1) \cdot (\bar{\neg} T(\neg F_1)) = \bar{\neg} T(p_1 \cdot \neg F_1) = \bar{\neg} T f.$$

Therefore $\langle\langle f \rangle\rangle$ is reflective through T if and only if (3.4.4) holds. □

Theorem 3.4.7 Let F be circular and self-dual. If F is reflective through a linear permutation τ of slope -1, then F is isometrically equivalent to a transformation which is minimal, circular, self-dual, and reflective through τ. If F is not reflective through a linear permutation τ of slope -1, then F is isometrically equivalent to a transformation which is minimal, circular, self-dual, and not reflective through τ.

Proof. If F is circular, and self-dual, then F is isometrically equivalent to a minimal circular self-dual transformation G by Theorem 2.5.2 of Chapter 2. More specifically, in the proof of that theorem, it was shown that there exists a Boolean isometry T such that $G = TF$, and T is a product of transformations selected from ρ and $\bar{\neg}$, so that T and F are commutative. Therefore, if F is reflective through a linear permutation τ of slope -1, then G is also reflective through τ; and if F is not reflective through a linear permutation τ of slope -1, then G is not reflective through τ. □

3.5 Reflectiveness of orbit modifications

Let F be a transformation of $\mathbf{Q}^n$, $M = \{n+1, n+2, .., n+m\}$, and $C \subseteq \mathbf{Q}^M$. Then the direct product $F \times I|C$ can be extended to the transformation $(F \times I|C)^\sim$ of $\mathbf{Q}^{\mathbf{N}_{n+m}}$ by defining

$$(F \times I|C)^\sim x = \begin{cases} (F \times I|C)x & \text{if } P_M x \in C, \\ x & \text{if } P_M x \notin C. \end{cases}$$

$(F \times I|C)^\sim$ is the disjoint composition of $|C|$ transformations $(F \times I|\{c\})^\sim$ for $c \in C$. As in Chapter 2.3, a subset C of $\mathbf{Q}^M$ is called *complete* if $\bar{\neg} C = C$. If

$C \subseteq \mathbf{Q}^M$ is complete, the restriction $n\bar{e}g \mid C$ to C is a transformation of C. If F is self-dual and represented as $F = [f_1, ..., f_n]$, and C is complete, then $(F \times I \mid C)^\sim$ is self-dual and can be expressed as

$$(F \times I \mid C)^\sim = [f_1 \cdot 1_C, .., f_n \cdot 1_C, \emptyset, .., \emptyset],$$

where $1_C x = 1$ if and only if $x \in C$. If $C = \mathbf{Q}^M$, we call $F \times I$ the $(n+1, ..., n+m)$th *face copies* of F.

The following propositions are easily proved using the basic properties of reflective transformations described in Section 3.1. In particular, the method of constructing a new Boolean transformation G from F as shown in Propositions 3.5.3 and 3.5.4 is called *orbit modification*.

Proposition 3.5.1 Let F be a reflective transformation of $\mathbf{Q}^n$ through T, and let $C \subseteq \mathbf{Q}^M$, where $M = \{n+1, n+2, .., n+m\}$. Then, (i) $(F \times I|C)^\sim$ is reflective through $T \times I$; (ii) $(F \times I|C)^\sim$ is reflective through $T \times \bar{\neg}$, if C is complete.

Proposition 3.5.2 Let F be any transformation of $\mathbf{Q}^n$, and let $C \subseteq \mathbf{Q}^M$, such that $C \cap \bar{\neg} C = \emptyset$, where $M = \{n+1, n+2, .., n+m\}$. Then, $(F \times I|C)^\sim \odot (F^{-1} \times I|\bar{\neg} C)^\sim$ is reflective through M^-.

Proposition 3.5.3 Let F be a reflective transformation of $\mathbf{Q}^n$ through T of order 2. Assume that F has two cycles C and D such that $T\mathbf{C} = \mathbf{D}$. Let $c \in \mathbf{C}$ and $d \in \mathbf{D}$. Define the transformation G by

$$\begin{aligned} c &\rightarrow Tc \rightarrow FTc \rightarrow F^2Tc \rightarrow ... \\ &\rightarrow d \rightarrow Td \rightarrow FTd \rightarrow F^2Td \rightarrow ... \rightarrow c, \\ Gx &= x \text{ for any other points } x \in \mathbf{C} \cup \mathbf{D}, \\ Gx &= Fx \text{ for any points } x \in (\mathbf{C} \cup \mathbf{D})^c. \end{aligned}$$

Then G is reflective through T.

Proposition 3.5.4 Let a transformation F of $\mathbf{Q}^n$ be reflective through T of order 2. Assume that F has two cycles C and D such that $T\mathbf{C} = \mathbf{C}$ and $T\mathbf{D} = \mathbf{D}$. Let $c \in \mathbf{C}$ and $d \in \mathbf{D}$. Define the transformation G by

$$\begin{aligned} c &\rightarrow d \rightarrow Fd \rightarrow F^2d \rightarrow ... \\ &\rightarrow Td \rightarrow Tc \rightarrow FTc \rightarrow F^2Tc \rightarrow ... \rightarrow c, \\ Gx &= x \quad \text{for any other points } x \in \mathbf{C} \cup \mathbf{D}, \\ Gx &= Fx \quad \text{for any points } x \in (\mathbf{C} \cup \mathbf{D})^c. \end{aligned}$$

Then G is reflective through T.

Example 3.5.5 The transformation $F^(1) : 1 \rightarrow 0 \rightarrow 1$ for $\mathbf{Q}^1$ is reflective through $I_{\mathbf{Q}^1}$. Therefoe, the 2nd face copies $F^{(1)\prime}$ of $F^{(1)}$:

$$11 \rightarrow 01 \rightarrow 11, 10 \rightarrow 00 \rightarrow 10$$

are reflective through 2^- by Proposition 3.5.1 (ii). Let $c = 11$, and define $F^{(2)}$ by orbit modification as

$$\begin{aligned} c &= 11 \rightarrow 2^-c = 10 \rightarrow F^{(1)\prime}2^-c = 00 = \{1,2\}^-c \\ &\rightarrow 2^-(\{1,2\}^-c) = 01 \rightarrow F^{(1)\prime}2^-(\{1,2\}^-c) = 11. \end{aligned}$$

Then $F^{(2)}$ is reflective through 2^- by proposition 3.5.3. Then the 3rd face copies of $F^{(2)}$:

$$111 \to 101 \to 001 \to 011 \to 111, 110 \to 100 \to 000 \to 010 \to 110$$

are reflective through $\{2,3\}^-$ by Proposition 3.5.1 (ii). Define $F^{(3)}$ by orbit modification as

$$\begin{array}{lcl} 111 & \to & \{2,3\}^-(111) = 100 \to 000 \to \{2,3\}^-(000) = 011 \to 111, \\ 101 & \to & 101, 001 \to 001, 110 \to 110, 010 \to 010. \end{array}$$

In this way, for $\mathbf{Q}^n$, we obtain a reflective transformation $F^{(n)}$ through $\{2,3,...,n\}^-$. GRAPH($F^{(n)}$) consists of loops and one 4-cycle, that is,

$$11\cdots1 \to 100\cdots0 \to 00\cdots0 \to 011\cdots1 \to 11\cdots1.$$

Example 3.5.6 (Binary-reflected Gray code. Gray, 1953) (i)

$$G^{(1)} : 0 \to 1 \to 0.$$

(ii) Construct $G^{(n)}$ from the face copies

$$\begin{array}{rcl} G^{(n-1)} \times I_{\{0\}} & : & 00\cdots00 \to ... \to 00\cdots010 \to 00\cdots00 \\ (G^{(n-1)})^{-1} \times I_{\{1\}} & : & 00\cdots01 \leftarrow ... \leftarrow 00\cdots011 \leftarrow 00\cdots01 \end{array}$$

by orbit modification as

$$G^{(n)} : \begin{array}{ccccc} 00\cdots00 & \to & ... & \to & 00\cdots010 \\ \uparrow & & & & \downarrow \\ 00\cdots01 & \leftarrow & ... & \leftarrow & 00\cdots011 \end{array}$$

From Proposition 3.5.2, $(G^{(n-1)} \times I_{\{0\}})^\sim \odot ((G^{(n-1)})^{-1} \times I_{\{1\}})^\sim$ is reflective through n^- for every n. Therefore, $G^{(n)}$ is reflective through n^- for every n by Proposition 3.5.3.

Therefore, $G^{(n-1)}$ is reflective through $(n-1)^-$ for every n, and hence $(G^{(n-1)})^{-1}$ is also reflective through $(n-1)^-$. Therefore, both $(G^{(n-1)} \times I_{\{0\}})^\sim$ and $((G^{(n-1)})^{-1} \times I_{\{1\}})^\sim$ are reflective through $(n-1)^-$. Therefore, $(G^{(n-1)} \times I_{\{0\}})^\sim \odot ((G^{(n-1)})^{-1} \times I_{\{1\}})$ is reflective through $(n-1)^-$. Therefore, $G^{(n)}$ is reflective through $(n-1)^-$ for every n by Proposition 3.5.4. Consequently, $G^{(n)}$ is commutative with $\{n-1, n\}^-$ by Proposition 3.1.2 (ii). GRAPH($G^{(n)}$) consists of one 2^n-cycle, i.e. a Hamiltonian cycle.