CHAPTER 7 CALCULUS OF ATTRACTION

Abstract. According to the last chapter, if F is a minimal threshold transformation, then $\bar{\neg}F$ is PDNN-definable. Moreover, an attractor of F is easily converted to that of $\bar{\neg}F$. In this chapter, various attractors of minimal threshold transformations are constructed from one-to-one transformations that we have already obtained in chapter 4. Further, given a threshold transformation having an expected attractor, some computational processes of proving the attractiveness are developed. For that purpose, extended representations of self-dual transformations are introduced from their [ ]-representations. In some cases, attractiveness can be proved by decomposition of the transformation.

7.1 Extended representations and neighborhood functions

We have already shown that there exist an attractive loop, 3-cycle, and 4-cycle in our PDNNs. In this chapter we seek the existence of more general attractors. Note that if F is a minimal threshold transformation, then $\bar{\neg}F$ is PDNN- definable for spontaneous firing rate 1/2 by Corollary 6.4.8 of Chapter 6. Moreover, an attractor of F is easily converted to that of $\bar{\neg}F$ by Proposition 6.6.1 of Chapter 6. Therefore, we limit ourselves to minimal transformations. Note that all these transformations and their attractors have their isometrically similar counterparts.

Before we go further, we need a tool for systematic analysis of attractiveness for various transformations. This tool is the extended representation of a Boolean transformation.

Let $d_{SH}(x,S)$ be the *signed Hamming distance* between a point x and a non-empty proper subset S of $\mathbf{Q}^n$ defined by

$$d_{SH}(x,S) = \begin{cases} d_H(x,S) & \text{if } x \notin S, \\ 1 - d_H(x,S^c) & \text{if } x \in S. \end{cases}$$

Definition 7.1.1 Let x be an element of $\mathbf{Q}^n$ and $F = [f_1, ..., f_n]$. Then the extended representation $F^{\#}$ of F is a function from $\mathbf{Q}^n$ to $\mathbf{Z}^n$ defined by

$$(F^{\#}x)_i = \begin{cases} d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}f_i) & \text{if } x_i = 1 \\ d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}\bar{\neg}f_i)) & \text{if } x_i = 0. \end{cases}$$

Clearly $|(F^{\#}x)_i| \le n-1$ for every i for every x. In general, $x \in f_i$ or $x \in \bar{\neg}f_i$, if and only if $(F^{\#}x)_i \le 0$. That is,

$$x_i \ne (Fx)_i \quad \text{iff } (F^{\#}x)_i \le 0. \tag{7.1.1}$$

For example, let

$$f = p_1 \cdot S_4\{p_2, p_3, p_4, \neg p_6, \neg p_7, \neg p_8\},$$

and $F = \langle f \rangle$ be a transformation of $\mathbf{Q}^8$. Let $c = 11110000$. Then $F^{\#}c = (-2, 0, 2, 4, -2, 0, 2, 4)$. Therefore, $Fc = 00111100$.

By Proposition 2.4.3, we have

$$Fk^- = [k^- f_1, .., p_k \cdot (\neg\bar{\neg}(f_k|1)), .., k^- f_n]. \tag{7.1.2}$$

Therefore,

$$(F^{\#}x)_i - 1 \leq ((Fk^-)^{\#}x)_i \leq (F^{\#}x)_i + 1, \quad \text{if } i \neq k. \tag{7.1.3}$$

For $i = k$, if $|f_k| \leq 2^{n-2}$, then by Proposition 4.3.8, $(f_k|1) \subseteq \neg\bar{\neg}(f_k|1)$, so that by (7.1.2),

$$((Fk^-)^{\#}x)_k \leq (F^{\#}x)_k, \quad \text{if } |f_k| \leq 2^{n-2}. \tag{7.1.4}$$

The following example is obtained as $\bar{\neg}G$ for (6.5.3) through (6.5.6) of Chapter 6 and is a basis of our various constructions.

Example 7.1.2 Let $2 \leq k \leq [n/2]$,

$$f = p_1 \cdot S_{n-k}\{p_2, p_3, .., p_n\}$$

and $F = \langle f \rangle$ be a transformation of $\mathbf{Q}^n$. The 2-cycle $C = (1...1, 0...0)$ is the unique attractor, and $U_{k-1}\mathbf{C}$ is the basin of attraction.

Theorem 7.1.3 Let $F = [f_1, ..., f_n]$ be a self- dual minimal threshold transformation, and let $\gamma i = d_H(f_i|1, \bar{\neg}(f_i|1))$. If $(F^{\#}x)_i \leq -1$ or $(F^{\#}x)_i \geq \max(2, \gamma i)$ for every i for any point x on a cycle of F, then that cycle is a strong attractor of F.

Proof. Assume $(F^{\#}x)_i \leq -1$ or $(F^{\#}x)_i \geq \max(2, \gamma i)$ for every i for any point x on a cycle of F.

Let $(Fx)_i = \neg x_i$ for a point x on the cycle, say $x \in f_i$. Then $d_{SH}(x, f_i) = (F^{\#}x)_i \leq -1$. $((Fi^-)^{\#}x)_i \leq (F^{\#}x)_i \leq -1$ by (7.1.4). Therefore, $((Fi^-)x)_i = \neg x_i = (Fx)_i$. Also, $((Fk^-)^{\#}x)_i \leq 0$ for every $k \neq i$ by (7.1.3), so that $((Fk^-)x)_i = \neg x_i = (Fx)_i$.

Let $(Fx)_i = x_i = 1$. Then

$$d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}f_i) = (F^{\#}x)_i \geq d_H(P_{\mathbf{N}\setminus i}f_i, P_{\mathbf{N}\setminus i}(\bar{\neg}f_i)),$$

so that

$$d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}(p_i \cdot \bar{\neg}(f_i|1))) = d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}(\bar{\neg}f_i)) \leq 0.$$

Therefore,

$$((Fi^-)^{\#}x)_i = d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}(p_i \cdot \neg\bar{\neg}(f_i|1))) \geq 1.$$

Therefore, $((Fi^-)x))_i = x_i = (Fx)_i$ by (7.1.2). Also, $(F^{\#}x)_i \geq 2$, so that $((Fk^-)^{\#}x)_i \geq 1$ for every $k \neq i$ by (7.1.3). Therefore, $((Fk^-)x)_i = x_i = (Fx)_i$. Therefore, $((Fk^-)x)_i = (Fx)_i$ for every i for every k. Therefore, the cycle is an strong attractor. □

In Example 7.1.2, F is minimal, and $(F^{\#}l)_i \leq -1$ for every i. (l, o) is obviously a strong attractor but also confirmed by Theorem 7.1.3.

Analogously to Example 7.1.2, we can construct transformations having attractors by modifying some of the one- to-one transformations listed in Chapter 4.4. First we give the following general definition. Let f be a function from $\mathbf{Q}^n$ to $\mathbf{Q}$. Then we identify the ϵ-neighborhood $U_\epsilon f$ of the set f with the function under which the inverse image of 1 is the set $U_\epsilon f$. That is,

$$(U_\epsilon f)^{-1}1 = U_\epsilon f = U_\epsilon(f^{-1}1).$$

The function $U_\epsilon f$ is called a *neighborhood function* of f. Then clearly

$$U_\epsilon(f \vee g) = (U_\epsilon f) \vee (U_\epsilon g).$$

For example, if f is a one-term function, $f = q_{k1} \cdot ... \cdot q_{km}$, where $(k1, .., km)$ is a subsequence of $(1, 2, .., n)$ and $q_{ki} = p_{ki}$ or $\neg p_{ki}$, then

$$U_\epsilon f = S_{m-\epsilon}\{q_{k1}, .., q_{km}\}.$$

We consider hereafter only the case $\epsilon = 1$ for simplicity. Let $F = [f_1, ..., f_n]$ be a self-dual transformation of $\mathbf{Q}^n$. Then let $G = [g_1, ..., g_n]$ be the transformation defined by

$$g_i = p_i \cdot U_1(f_i|1). \tag{7.1.5}$$

Then

$$(G^\# x)_i = (F^\# x)_i - 1 \quad \text{for every } i \text{ for every } x, \tag{7.1.6}$$

The following proposition immediately follows from (7.1.6).

Proposition 7.1.4 A cycle of $F = [f_1, ..., f_n]$ is also a cycle of $G = [g_1, ..., g_n]$ defined by (3.5), if and only if $(F^\# x)_i \neq 1$ for every i for any point x on the cycle.

Proof. $x_i \neq (Gx)_i$ iff $(G^\# x)_i \leq 0$ by (7.1.1). $(G^\# x)_i = (F^\# x)_i - 1$ by (7.1.6). Therefore, $Gx = Fx$ iff $(F^\# x)_i \neq 1$ for every i. □

Recall that a term of degree m is a conjunction $f_1 \cdot f_2 \cdots f_m$ of Boolean functions $f_i : \mathbf{Q}^n \to \mathbf{Q}$, such that there exists an injection $\varphi : \mathbf{N}_m \to \mathbf{N}$ such that $f_i = p_{\varphi i}$ or $\neg p_{\varphi i}$ for each i. For a special case where f consists of one term we obtain the following theorem.

Theorem 7.1.5 Let $F = [f_1, ..., f_n]$ be a self-dual minimal threshold transformation, and let $f_i|1$ consist of one term of degree r_i for $r_i \geq 3$ for every i. Then, if $(F^\# x)_i \leq 0$ or $(F^\# x)_i = r_i$ for every i for any point x on a cycle of F, then that cycle is an strong attractor of $G = [g_1, ..., g_n]$ defined by (7.1.5).

Proof. Assume that $(F^\# x)_i \leq 0$ or $(F^\# x)_i = r \geq 3$ for every i for any point x on a cycle of F. Then that cycle is also a cycle of G by Proposition 7.1.4.

Let x be a point on a cycle of F. Then, $(G^\# x)_i = -1$ or $(G^\# x)_i = r_i - 1 \geq 2$ for any point x on the cycle. Further, we have

$$\gamma i = d_H((g_i|1), \bar{\neg}(g_i|1)) = r_i - 2$$

Therefore, $(G^\# x)_i \leq -1$ or $(G^\# x)_i \geq \max(2, \gamma i)$, so that the cycle is a strong attractor of G by Theorem 7.1.3. □

7.2 Multipe attractors

In this section as in other parts, 1^m denotes the m- vector whose every coordinate is 1, and 0^m denotes the m-vector whose every coordinate is 0.

Example 7.2.1 Let $n = 2m$ for $m \geq 2$, $f = p_1 \cdot p_m \cdot \neg p_{2m}$, and $F = \langle f \rangle$ be a transformation of $mathbfQ^{2m}$. Let

$$\begin{array}{rcl} D & = & \{x \mid x \in \mathbf{Q}^{2m}, \rho^m x = \bar{\neg} x\} \\ \Psi & = & \mathrm{Orb}_\rho D. \end{array} \tag{7.2.1}$$

Clearly $\rho D = D$, so that $\mathrm{Im}\Psi = D$.

Let $x \in D$. Then,

$$(F^\# x)_i = \left\{ \begin{array}{ll} 0 & \text{if } x_i = \neg x_{i-1}, \\ 2 & \text{if } x_i = x_{i-1}, \end{array} \right. \tag{7.2.2}$$

for every i. Therefore, $Fx = \rho x$ for every $x \in D$. Therefore, $\mathrm{Im}\Psi$ is an invariant set of F. We now show that Ψ is an attractor.

Let $x \in D$ and $k \in \mathbf{N}$. It suffices to show $F(k^- x) \in D$. First, by (7.1.4),

$$((Fk^-)^{\#} x)_k \leq 0 \quad \text{if } x_k = \neg x_{k-1}.$$

If $x_k = x_{k-1}$, say $x_k = x_{k-1} = 1$, then $x_{k-1+m} = 0$. By (7.1.2),

$$\begin{aligned} (Fk^-) &= [k^- f_1, .., p_k \cdot (\neg\bar{\neg}(f_k|1)), .., k^- f_n]. \\ p_k \cdot (\neg\bar{\neg}(f_k|1)) &= p_k \cdot \neg\bar{\neg}(p_{k+m-1} \cdot \neg p_{k-1}) \\ &= p_k \cdot \neg(\neg p_{k+m-1} \cdot p_{k-1}) \\ &= p_k \cdot (p_{k+m-1} \vee \neg p_{k-1}). \end{aligned}$$

Therefore,

$$((Fk^-)^{\#} x)_k = 1 \quad \text{if } x_k = x_{k-1}.$$

Therefore, $(F(k^- x))_k = (Fx)_k$.

Let $k \neq i$. If $x_i = x_{i-1}$, then by (7.2.2) and (7.1.3),

$$((Fk^-)^{\#} x)_i \geq 1.$$

Assume $x_i = \neg x_{i-1}$. Then, if $k \notin \{i-1, i+m-1\}$, then

$$((Fk^-)^{\#} x)_i = (F^{\#} x)_i.$$

Further, if $k \in \{i-1, i+m-1\}$, then

$$((Fk^-)^{\#} x)_i = 1, \quad \text{while } (F^{\#} x)_i = 0,$$

but also,

$$((Fk^-)^{\#} x)_{i+m} = 1, \quad \text{while } (F^{\#} x)_{i+m} = 0.$$

Therefore, $F(k^- x) \in D$.

Therefore, Ψ is a strong attractor. We have obtained the following theorem.

Proposition 7.2.2 Ψ defined by (7.2.1) is a strong attractor of F in Example 7.2.1.

However, Ψ is not a minimal attractor. Let $c = l^m 0^m$. Then $c \in D$. we now show that

$$C = \mathrm{Orb}_\rho c.$$

is a (minimal) attractor. Let $x \in U_1\mathbf{C}$. It suffices to consider $x = 1^m(k^- 0^m)$ for $1 \leq k \leq m-1$. Then $Fx = \rho c$, so that $Fx \in \mathbf{C}$. Therefore, C is a strong attractor.

Example 7.2.3 Let $n = 2m$ for $m \geq 2$, $f = p_1 \cdot p_{m-1} \cdot \neg p_{2m-1}$, and $F = \langle f \rangle$ be a transformation of $\mathbf{Q}^{2m}$.

In Example 7.2.3, computational results expect that there exists an attractor corresponding to the above Ψ, but the attractor and the orbits starting at its 1-neighborhood are not simple. Corresponding to the above C, we define

$$c\prime = 1^m 0^{m-1},\ c\prime\prime = 1^{m-1} 0^m, \quad \Phi = \mathrm{Orb}_\rho \{c\prime, c\prime\prime\}. \tag{7.2.3}$$

$Fc\prime = \rho c\prime\prime \in \mathrm{Im}\Phi$ and $F^2 c\prime = \rho c\prime \in \mathrm{Im}\Phi$, so that $\mathrm{Im}\Phi$ is invariant. We will show that $CY(F|\Phi)$ is a (minimal) attractor of F. Let $x \in U_1(\mathrm{Im}\Phi)$. It suffices to consider $x = 1^m(k^- 0^{m-1})$ for some $1 \leq k \leq m-2$. If $k = 1$ then $Fx = \rho c\prime$. If $1 < k \leq m-2$ then $Fx = \rho c\prime\prime$. Therefore, $Fx \in \mathrm{Im}\Phi$, so that $CY(F|\mathrm{Im}\Phi)$ is an attractor. We have obtained the following proposition.

Proposition 7.2.4 $\mathrm{CY}(F|\mathrm{Im}\Phi)$ is a minimal attractor in the FSDS generated by F in Example 7.2.3, where Φ is defined by (7.2.3).

Example 7.2.5 Let $n = 3m$ for $m \geq 2$ and

$$f = p_1 \cdot \neg p_m \cdot \neg p_{2m} \cdot \neg p_{3m}.$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{3m}$ defined by

$$g = p_1 \cdot S_2\{\neg p_m, \neg p_{2m}, \neg p_{3m}\}.$$

Let x be a point such that $\rho^m x = x$. Then

$$(F^{\#}x)_i = \begin{cases} 0 & \text{if } x_i = \neg x_{i-1} \\ 3 & \text{if } x_i = x_{i-1}. \end{cases}$$

for every i. Therefore, $(Fx)_i = \neg x_i$ if and only if $x_i = \neg x_{i-1}$, that is, $(Fx)_i = x_{i-1}$ for every i, so that $Fx = \rho x$. Therefore, $\mathrm{Orb}_\rho x$ is a cycle of F. Further, $\mathrm{Orb}_\rho x$ is also a strong attractor of G by Theorem 7.1.5.

Example 7.2.6 Let $n = 4m$ for $m \geq 2$, and

$$f = p_1 \cdot p_m \cdot \neg p_{2m} \cdot p_{3m} \cdot \neg p_{4m}.$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{4m}$ defined by

$$g = p_1 \cdot S_3\{p_m, \neg p_{2m}, p_{3m}, \neg p_{4m}\}.$$

Let x be a point such that $\rho^m x = \neg x$. Then

$$(F^{\#}x)_i = \begin{cases} 0 & \text{if } x_i = \neg x_{i+m-1} \\ 4 & \text{if } x_i = x_{i+m-1}. \end{cases}$$

for every i. Therefore, $(Fx)_i = x_{i-1}$ for every i, so that $Fx = \rho x$, that is, $\mathrm{Orb}_\rho x$ is a cycle of F. $\mathrm{Orb}_\rho x$ is also a strong attractor of G by Theorem 7.1.5.

Example 7.2.7 Let $n = 4m$ for $m = 4$, and let

$$f = 1 \cdot 4 \cdot \neg 5 \cdot \neg 8 \cdot 12 \cdot \neg 13 \cdot \neg 16.$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^n$ defined by

$$g = 1 \cdot S_5\{4, \neg 5, \neg 8, 12, \neg 13, \neg 16\}.$$

Let $a = (11110000)^2$ and $b = (11010010)^2$. Then

$$\begin{aligned} F^{\#}a &= (0,2,2,2,0,2,2,2,0,2,2,2,0,2,2,2), \\ F^{\#}b &= (0,2,0.0,0,2,0,0,0,2,0.0,0,2,0,0). \end{aligned}$$

Therefore, $\mathrm{Orb}_\rho a$ and $\mathrm{Orb}_\rho b$ are non-loop cycles of F. Also, $\mathrm{Orb}_\rho a$ and $\mathrm{Orb}_\rho b$ are non-loop cycles of G by Proposition 7.2.2. Further calculation proves that $\mathrm{Orb}_\rho a$ and $\mathrm{Orb}_\rho b$ are attractors of G.

7.3 Multi-cycle attractors

In this section and later in this chapter, $[a]$ for an element a in $\mathbf{Q}^n$ denotes the orbit of $\langle \bar{\neg}, \rho\rangle$ containing a if the operating transformation is self-dual and circular; $[A]$ for a subset A of $\mathbf{Q}^n$ denotes the union of the orbits of $\langle \bar{\neg}, \rho\rangle$containing $a \in A$. Similarly, $[a]$ and $[A]$ respectively denote the orbit of$\langle \rho n^-\rangle$ containing a andthe union of orbits of $\langle \rho n-\rangle$ for $a \in A$, if the operating transformation is skew-circular. Note that $\langle \bar{\neg}, \rho n^-\rangle = \langle \rho n^-\rangle$, since $(\rho n^-)^n = \bar{\neg}$. Then, a transformation $F^\sim$ of the orbit set $\{[x] \mid x \in \mathbf{Q}^n\}$ is naturally defined by $F^\sim[x] = [Fx]$.

The condition in Theorem 7.1.3 is very strong. In fact, transformations have attractors under weaker conditions as shown in the following Theorems. As in Section 7.2, $G = \langle g\rangle$ is defined by (7.1.5) from a transformation $F = \langle f\rangle$ of $\mathbf{Q}^n$.

First we consider a circular transformation belonging to the class determined by Theorem 4.4.3. Specifically, for any $h-1$ relatively prime with odd n and $0 < h-1 < n$, there exists $F = \langle f\rangle$ of $\mathbf{Q}^n$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n,$$

such that

$$F = \bar{\neg}\rho^{h-1} \quad \text{on Car}F.$$

Let $f = \{c\}$. Then, $C = \text{Orb}_{\bar{\neg}\rho h-1} c$ is a cycle of F.

Then we get the transformation $G = \langle g\rangle$ of $\mathbf{Q}^n$ defined by

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, .., \alpha_n p_n\}.$$

Referring to (4.4.3), we have

$$\begin{aligned}
(F^{\#}c)_i &= 0 \quad \text{iff } i = 1;\\
(F^{\#}c)_i &= 1 \quad \text{iff } i = h, \text{ since } 1^- c = \bar{\neg}\rho^{h-1}c;\\
(F^{\#}c)_i &= 2 \quad \text{iff } i = 2h-1;\\
(F^{\#}c)_i &\geq 3 \quad \text{for every other } i.
\end{aligned}$$

Therefore, by (7.1.6),

$$\begin{aligned}
&(G^{\#}c)_1 = -1, \qquad (G^{\#}c)_h = 0, \qquad (G^{\#}c)_{2h-1} = 1,\\
&(G^{\#}c)_i \geq 2 \quad \text{for every other } i.
\end{aligned}$$

In particular,

$$Gc = \{1, h\}^- c = \rho^{2(h-1)}c. \tag{7.3.1}$$

Let $k \neq 1$. Then if $k \neq h$, then

$$\begin{aligned}
((Gk^-)^{\#}c)_1 &\leq 0 \quad \text{by (7.1.3)};\\
((Gk^-)^{\#}c)_h &= 1, \quad \text{since } (\bar{\neg}\rho)^{h-1}c = 1^- c \text{ and } k \neq 1;\\
((Gk^-)^{\#}c)_{2h-1} &= 2, \quad \text{since } (\bar{\neg}\rho)^{2h-2}c = \{1, h\}^- c \text{ and } k \notin \{1, h\};\\
((Gk^-)^{\#}c)_i &\geq 1 \quad \text{for every other } i \text{ such that } i \neq k \text{ by (7.1.3)}.
\end{aligned}$$

For $k = h$,

$$\begin{aligned}
((Gh^-)^{\#}c)_1 &\leq 0;\\
((Gh^-)^{\#}c)_h &\leq (G^{\#}c)_h = 0 \quad \text{by (7.1.4)};\\
((Gh^-)^{\#}c)_{2h-1} &= 0, \quad \text{since } (\bar{\neg}\rho)^{2h-2}c = \{1, h\}^- c \text{ and } h \in \{1, h\};\\
((Gh^-)^{\#}c)_i &\geq 1 \quad \text{for every other } i \text{ by (7.1.3)}.
\end{aligned}$$

Therefore,

$$G(k^- c) = \bar{\neg}\rho^{h-1} c,$$

or

$$G(k^- c) = \bar{\neg}\rho^{h-1}((\rho^{-(h-1)} k)^- c),$$

or

$$\begin{aligned} G(k^- c) &= \bar{\neg}\rho^{h-1}((\rho^{-(h-1)}\{h, 2h-1\})^- c) \\ &= \{h, 2h-1\}^- (\bar{\neg}\rho^{h-1} c) \\ &= \{1, h, 2h-1\}^- c \\ &= \rho^{3(h-1)} c. \end{aligned}$$

Therefore, under $G^{\sim}$,

$$[k^- c] \to ... \to [i^- c] \to ... \to [c],$$

or

$$[k^- c] \to ... \to [i^- c] \to ... \to [1^- c].$$

On the other hand, by (7.3.1),

$$\begin{aligned} G(1^- c) &= G(\bar{\neg}\rho^{h-1} c) \\ &= \bar{\neg}\rho^{h-1} G c \\ &= \bar{\neg}\rho^{h-1}\rho^{2(h-1)} c \\ &= \bar{\neg}\rho^{3(h-1)} c. \end{aligned}$$

Thus we obtained:

Theorem 7.3.1 For any $h-1$ relatively prime with odd n and $0 < h-1 < n$, let $F = \langle f \rangle$ of $\mathbf{Q}^n$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n,$$

be a transformation determined by Theorem 4.4.3. Let $G = \langle g \rangle$ be defined by

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, .., \alpha_n p_n\}.$$

Let $f = \{c\}$ and

$$\Phi = \mathrm{Orb}_{\rho 2(h-1)}\{c, \bar{\neg}\rho^{h-1} c\}.$$

Then Φ is an attractor of G.

We also get a double-cycle attarctor, if we take F from the class described in Theorem 4.5.3. Specifically, for any $h-1$ relatively prime with $2n$ and $0 < h-1 < 2n$, there exists $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n,$$

such that

$$F = (\rho n^-)^{h-1} \quad \text{on } \mathrm{Car} F.$$

Let $f = \{c\}$. Then, $C = \mathrm{Orb}_{(\rho n-)h-1} c$ is a cycle of F.

Then we get the transformation $G = \langle\langle g \rangle\rangle$ of $\mathbf{Q}^n$ defined by

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, .., \alpha_n p_n\}.$$

Referring to (4.5.3), we have

$$\begin{aligned}
(F^{\#}c)_i &= 0 \quad \text{iff } i = 1; \\
(F^{\#}c)_i &= 1 \quad \text{iff } i = h, \text{ since } (\rho n^-)^{h-1}c = 1^-c; \\
(F^{\#}c)_i &= 2 \quad \text{iff } i = 1 + 2(h-1) = 2h-1; \\
(F^{\#}c)_i &\geq 3 \quad \text{for every other } i.
\end{aligned}$$

Therefore, by (7.1.6),

$$\begin{aligned}
&(G^{\#}c)_1 = -1, \qquad (G^{\#}c)_h = 0, \qquad (G^{\#}c)_{2h-1} = 1, \\
&(G^{\#}c)_i \geq 2 \quad \text{for every other } i.
\end{aligned}$$

In particular,

$$Gc = \{1, h\}^-c = (\rho n^-)^{2(h-1)}c. \tag{7.3.2}$$

Let $k \neq 1$. Then if $k \neq h$, then

$$\begin{aligned}
((Gk^-)^{\#}c)_1 &\leq 0 \quad \text{by (7.1.3);} \\
((Gk^-)^{\#}c)_h &= 1, \quad \text{since } (\rho n^-)^{h-1}c = 1^-c \text{ and } k \neq 1; \\
((Gk^-)^{\#}c)_{2h-1} &= 2, \quad \text{since } (\rho n^-)^{2h-2}c = \{1, h\}^-c \text{ and } k \notin \{1, h\}; \\
((Gk^-)^{\#}c)_i &\geq 1 \quad \text{for every other } i \text{ such that } i \neq k \text{ by (7.1.3).}
\end{aligned}$$

For $k = h$,

$$\begin{aligned}
((Gh^-)^{\#}c)_1 &\leq 0 \quad \text{by (7.1.3);} \\
((Gh^-)^{\#}c)_h &\leq (G^{\#}c)_h = 0 \quad \text{by (7.1.4);} \\
((Gh^-)^{\#}c)_{2h-1} &= 0, \quad \text{since } (\rho n^-)^{2h-2}c = \{1, h\}^-c \text{ and } h \in \{1, h\}; \\
((Gh^-)^{\#}c)_i &\geq 1 \quad \text{for every other } i \text{ by (7.1.3).}
\end{aligned}$$

Therefore,

$$G(k^-c) = (\rho n^-)^{h-1}c,$$

or

$$G(k^-c) = (\rho n^-)^{h-1}((\rho^{-(h-1)}k)^-c),$$

or

$$\begin{aligned}
G(k^-c) &= (\rho n^-)^{h-1}((\rho^{-(h-1)}\{h, 2h-1\})^-c) \\
&= \rho^{h-1}\rho^{-(h-1)}\{h, 2h-1\})^-(\rho n^-)^{2(h-1)}c \\
&= \{1, h, 2h-1\}^-c \\
&= (\rho n^-)^{3(h-1)}c.
\end{aligned}$$

Therefore, under $G^\sim$,

$$[k^-c] \to ... \to [i^-c] \to ... \to [c],$$

or

$$[k^-c] \to ... \to [i^-c] \to ... \to [1^-c].$$

On the other hand, by (7.3.2)

$$\begin{aligned}
G(1^-c) &= G((\rho n^-)^{(h-1)}c) \\
&= (\rho n^-)^{(h-1)}Gc \\
&= (\rho n^-)^{(h-1)}(\rho n^-)^{2(h-1)}c \\
&= (\rho n^-)^{3(h-1)}c
\end{aligned}$$

Thus we obtained:

Theorem 7.3.2 For any $h-1$ relatively prime with $2n$ and $0 < h-1 < 2n$, let $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n,$$

be a transformation determined by Theorem 4.5.3. Let $G = \langle g \rangle$ be defined by

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, .., \alpha_n p_n\}.$$

Let $f = \{c\}$ and

$$\Phi = \mathrm{Orb}_{(\rho n-)2(h-1)}\{c, (\rho n^-)^{h-1} c\}.$$

Then Φ is an attractor of G.

Example 7.3.3 Let $n = 2m$ for $m \geq 2$, and $F = \langle f \rangle$ for

$$f = p_1 \cdot S_{m-2}\{p_2, p_3, .., p_m\} \cdot \neg p_{2m}.$$

Proposition 7.3.4 Let $a = 1^{m+1}0^{m-1}$, $b = 1^{m-1}0^{m+1}$, and $c = 1^m 0^m$. Then, $\Phi = \mathrm{Orb}_\rho\{a, b, c\}$ is an attractor of F in Example 7.3.3.

Proof. Note that $[a] = [b]$ and $\bar{\neg}\mathrm{Im}\Phi = \mathrm{Im}\Phi$. we have

$$\begin{aligned} F^\# a &= (0, 1, 1, 2, 3, .., m-1, 0, 2, 3, .., m-1), \\ F^\# b &= (0, 2, 3, .., m-1, 0, 1, 1, 2, 3, .., m-1), \\ F^\# c &= (0, 1, 2, .., m-1, 0, 1, 2, .., m-1). \end{aligned}$$

Therefore, $\mathrm{Orb}_\rho a$, $\mathrm{Orb}_\rho b$, and $\mathrm{Orb}_\rho c$ are cycles of F.

Let $2 \leq k \leq m-1$. Then

$$\begin{aligned} ((Fk^{-1})^\# c)_1 &= 0, \\ ((Fk^-)^\# c)_2 &= 2 \quad \text{if } k \neq 2, \\ ((Fk^-)^\# c)_{m+1} &= 0, \\ (Fk^-)^\# c)_{m+2} &= 1, \\ ((Fk^-)^\# c))_i &\geq 1 \quad \text{for every other } i \text{ such that } i \neq k \text{ by (7.1.3).} \end{aligned}$$

Therefore,

$$k^- c \to \rho((k-1)^- c) \quad \text{or} \quad k^- c \to \rho c.$$

Also $1^- c = \rho b$. Therefore,

$$[k^- c] \to ... \to [a] \quad \text{or} \quad [k^- c] \to ... \to [c]. \tag{7.3.3}$$

Since $[k^- c] = [(k-m)^- c]$, (7.3.3) is also true for $m+2 \leq k \leq 2m-1$.

Let $3 \leq k \leq m$. Then

$$\begin{aligned} ((Fk^-)^\# a)_1 &= 0, \\ ((Fk^-)^\# a))_2 &= 1, \\ ((Fk^-)^\# a)_3 &= 1, \\ ((Fk^-)^\# a)_{m+2} &= 0. \\ ((Fk^-)^\# a)_i &\geq 1 \quad \text{for every other } i \text{ such that } i \neq k \text{ by (7.1.3).} \end{aligned}$$

Therefore,

$$k^- a \to \rho((k-1)^- a) \quad \text{or} \quad k^- a \to \rho a.$$

$$
\begin{aligned}
((F2^-)^{\#}a)_1 &= 0\\
(F^{\#}(2^-a))_2 &= m-2,\\
(F^{\#}(2^-a))_3 &= 0,\\
(F^{\#}(2^-a))_{m+2} &= 0.\\
(F^{\#}(2^-a))_i &\geq 1 \quad \text{for every other } i \text{ by (7.1.3).}
\end{aligned}
$$

Therefore,

$$2^-a \to \rho^3 b.$$

Therefore, for $2 \leq k \leq m$,

$$[k^-a] \to ... \to [a],$$

and $(m+1)^-a = c$.

$$
\begin{aligned}
(F((m+2)^-)^{\#}a)_1 &= 0,\\
(F((m+2)^-)^{\#}a)_2 &= 1,\\
(F((m+2)^-)^{\#}a)_3 &= 1.\\
(F((m+2)^-)^{\#}a)_{m+2} &\leq (F^{\#}a)_{m+2} = 0 \quad \text{by (7.1.4).}\\
(F((m+2)^-)^{\#}a)_i &\geq 1 \quad \text{for every other } i \text{ by (7.1.3).}
\end{aligned}
$$

Therefore,

$$(m+2)^-a \to \rho a.$$

Let $m+3 \leq k \leq 2m-1$. Then

$$
\begin{aligned}
((Fk^-)^{\#}a)_1 &= 0,\\
((Fk^-)^{\#}a)_2 &= 1,\\
((Fk^-)^{\#}a)_3 &= 2,\\
((Fk^-)^{\#}a)_{m+2} &= 1.\\
((Fk^-)^{\#}a)_i &\geq 1 \quad \text{for every other } i \neq k.
\end{aligned}
$$

Therefore,

$$k^-a \to \rho((k-1)^-c) \quad \text{or} \quad k^-a \to \rho c.$$

Therefore, by (7.3.3),

$$k^-a \to ... \to [a] \quad \text{or} \quad k^-a \to ... \to [c].$$

Therefore, if $x \in U_1\mathrm{Im}\Phi$ then $x \in U_1\mathrm{Im}\Phi$ and $\omega_F x \in \mathrm{Im}\Phi$. Therefore, Φ is an attractor of F. □

Example 7.3.5 Let $m \geq 4$, and let

$$f = p_1 \cdot p_2 \cdot p_3 \cdot .. \cdot p_{m-1} \cdot \neg p_{m+2} \cdot .. \cdot \neg p_{2m-1}.$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{2m}$ defined by

$$g = p_1 \cdot S_{2m-5}\{p_2, p_3, .., p_{m-1}, \neg p_{m+2}, .., \neg p_{2m-1}\}.$$

Let $c = 1^m 0^m$. Then,

$$F^{\#}c = (0, 0, 2, 4, .., 2m-4, 0, 0, 2, 4, .., 2m-4).$$

Therefore, $Fc = \rho^2 c$, so that $\mathrm{Orb}\rho_2 c$ is a cycle of F. Further, $\mathrm{Orb}\rho_2 c$ is a cycle of G by Proposition 7.1.4. Let

$$\Phi = \mathrm{Orb}_{\rho_2}\{c, \rho c\}.$$

We have

$$\begin{aligned}
\neg\bar{\neg}(f|1) &= \neg(\neg p_2 \cdot \neg p_3 \cdot .. \cdot \neg p_{m-1} \cdot p_{m+2} \cdot .. \cdot p_{2m-1}) \\
&= p_2 \vee p_3 \vee .. \vee p_{m-1} \vee \neg p_{m+2} \vee .. \vee \neg p_{2m-1}. \\
(F1^-)^\# c &= (-, 0, 1, .., 2m-5, 0, 0, 1, 3, .., 2m-5), \\
(F2^-)^\# c &= (1, -, 2, 3, .., 2m-5, 1, 0, 2, 3, .., 2m-5), \\
(F3^-)^\# c &= (1, 1, -, 5, .., 2m-5, 1, 1, 2, 4, 5, .., 2m-5), \\
& ... \\
(F(m-1)^-)^\# c &= (1, 1, 3, .., 2m-7, -, 2m-4, 1, 1, 3, .., 2m-7, 2m-6, 2m-4).
\end{aligned}$$

Therefore, by (7.1.6)

$$\begin{aligned}
(G1^-)^\# c &= (-, -1, 0, .., 2m-6, -1, -1, 0, .., 2m-6), \\
(G2^-)^\# c &= (0, -, 1, 2, .., 2m-6, 0, -1, 1, 2, .., 2m-6), \\
(G3^-)^\# c &= (0, 0, -, 4, .., 2m-6, 0, 0, 1, 3, 4, .., 2m-6), \\
& ... \\
(G(m-1)^-)^\# c &= (0, 0, 2, .., 2m-8, -, 2m-5, 0, 0, 2, .., 2m-8, 2m-7, 2m-5).
\end{aligned}$$

Therefore, under G,

$$\begin{aligned}
1^- c &\rightarrow \rho^3 c, \\
2^- c &\rightarrow \rho^2 c, \\
3^- c &\rightarrow \rho^2(1^- c), \\
& .... \\
(m-1)^- c &\rightarrow \rho^2((m-3)^- c), \\
m^- c &= \rho^{-1}(1^- c) = \rho^{-1}(\rho^3 c) = \rho^2 c.
\end{aligned}$$

Therefore, if $x \in U_1 \mathrm{Im}\Phi$ then $x \in U_1 \mathrm{Im}\Phi$ and $\omega_G x = \mathrm{Im}\Phi$. Therefore, Φ is an attractor of G.

Example 7.3.6 Let $m \geq 2$, and consider a special case of $G = F^2$ for transformations described in Theorem 4.4.3, where $G = \langle g \rangle$,

$$g = p_1 \cdot p_2 \cdots p_m \cdot \neg p_{m+2} \cdots \neg p_{2m+1}.$$

From G we construct $H = \langle h \rangle$,

$$h = p_1 \cdot S_{2m-2}\{p_2, p_3, .., p_m, \neg p_{m+2}, .., \neg p_{2m+1}\},$$

and $H = \langle h \rangle$ be a transformation of $\mathbf{Q}^{2m+1}$.

Let

$$\begin{aligned}
h^{(2)} &= 1\cdot\neg 2\cdot 3\cdot .. \cdot(m+1)\cdot\neg(m+2)\cdot .. \cdot\neg(2m+1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot m\cdot\neg(m+1)\cdot(m+2)\cdot\neg(m+3)\cdot .. \cdot\neg(2m+1),\\
.. &\quad ...\\
h^{(m)} &= 1\cdot 2\cdot .. \cdot(m-1)\cdot\neg m\cdot(m+1)\cdot\neg(m+2).. \cdot\neg(2m-1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot m\cdot\neg(m+1)\cdot .. \cdot\neg(2m-1)\cdot 2m\cdot\neg(2m+1),\\
h^{(m+1)} &= 1\cdot 2\cdot .. \cdot m\cdot\neg(m+1).. \cdot\neg(2m+1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot(m+1)\cdot\neg(m+2)\cdot .. \cdot\neg(2m+1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot m\cdot\neg(m+1)\cdot .. \cdot\neg 2m\cdot(2m+1),\\
h^{(m+2)} &= 1\cdot 2\cdot .. \cdot(m+2)\cdot\neg(m+3)\cdot .. \cdot\neg(2m+1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot(m+1)\cdot\neg(m+2)\cdot ..\neg(2m)\cdot(2m+1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot(m-1)\cdot\neg m\cdot .. \cdot(2m+1),\\
h^{(m+3)} &= 1\cdot 2\cdot .. \cdot(m+1)\cdot\neg(m+2)\cdot(m+3)\cdot\neg(m+4)\cdot .. \cdot\neg(2m+1)\\
&\quad \vee 1\cdot\neg 2\cdot 3\cdot .. \cdot m\cdot\neg(m+1)\cdot .. \cdot\neg(2m+1),\\
.. &\quad ...\\
h^{(2m)} &= 1\cdot 2\cdot .. \cdot(m+1)\cdot\neg(m+2)\cdot .. \cdot\neg(2m-1)\cdot(2m)\cdot\neg(2m+1)\\
&\quad \vee 1\cdot 2\cdot .. \cdot(m-2)\cdot\neg(m-1)\cdot m\cdot\neg(m+1)\cdot .. \cdot\neg(2m-1).
\end{aligned}$$

Then

$$h = h^{(2)} \vee ... \vee h^{(2m)},$$

and

$$h^{(i)}\cdot h^{(k)} = 0 \quad \text{for every } i \neq k.$$

Let $H^{(i)} = \langle h^{(i)}\rangle$. Let $c = 1^{m+1}0^m$. Then

$$H = H^{(2)} + ... + H^{(2m)},$$

and a flow graph of H is

$$\begin{array}{ccccccccc}
[h^{(2m)}] & \to & [h^{(2m-1)}] & \to & .. & \to & [h^{(m+1)}] & = & [c]\partial.\\
 & & & & & & \uparrow & & \\
[h^{(m)}] & \to & [h^{(m-1)}] & \to & .. & \to & [h^{(2)}] & &
\end{array}$$

Further, $Hc = \rho^{-(m-1)}\bar{\neg}c$, so that $H^2c = \rho^{-2(m-1)}c$. We have $\gcd(2m+1, 2(m-1)) = \gcd(2(m-1), 3) = \gcd(m-1, 3)$. If $m-1$ is a multiple of 3, then let

$$\Phi = \mathrm{Orb}_{\rho^{-(m-2)}\bar{\neg}}\{c, \rho c, \rho^2 c\};$$

otherwise, let

$$C = \mathrm{Orb}_{\rho^{-(m-2)}\bar{\neg}}c.$$

Since $\mathrm{Im}\Phi = [c]$,$\mathbf{C} = [c]$, $U_1(\mathrm{Im}\Phi) = [h]$, and $U_1\mathbf{C} = [h]$, Φ and C are attractors of H.

7.4 Single-cycle attractors I

In order to construct a transformation having a single-cycle attractor, we consider the expansion described in Example 5.2.6 deribed from a circular transformation belonging to the class determined by Theorem 4.4.3. Specifically, for any $h-1$ relatively prime with odd n and $0 < h-1 < n$, there exists $E = \langle e \rangle$ of $\mathbf{Q}^{2n}$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n \cdot p_{n+1} \cdot \alpha_2 p_{n+2} \cdot .. \cdot \alpha_n p_{2n},$$

such that

$$E = \bar{\neg}\rho^{h-1} \quad \text{on} \mathrm{Car} E.$$

Then we get the transformation $G = \langle g \rangle$ defined by

$$g = p_1 \cdot S_{2n-3}\{\alpha_2 p_2, .., \alpha_n p_n, p_{n+1}, \alpha_2 p_{n+2}, .., \alpha_n p_{2n}\}.$$

Let $e = \{c\}$ Then, $C = \mathrm{Orb}_{\bar{\neg}\rho^{h-1}} c$ is a cycle of E. $(E^{\#}c)_i$ is even for every i, since $\rho^n x = x$ for every $x \in \mathrm{Car} E$. This is a nice effect of an expansion. Therefore, $\mathrm{Orb}_{\rho^{h-1}} c$ is a cycle of G by Proposition 7.1.4.

We have $(E^{\#}c)_i \le 0$ iff $i = 1$ or $n+1$, since $Ec = \{1, n+1\}^- c$. Since $\bar{\neg}\rho^{h-1}c = Ec = \{1, n+1\}^- c$, $c = \{1, n+1\}^- \bar{\neg}\rho^{n-1} e$. Therefore, referring to (4.4.3), $(E^{\#}c)_i = 2$ iff $i = h$ or $n+h$. Then, by (7.1.6),

$$\begin{array}{lcll} (G^{\#}c)_1 & \le & -1, & (G^{\#}c)_{n+1} \le -1, \\ (G^{\#}c)_i & \ge & 1 & \text{for every other } i, \\ (G^{\#}c)_i & = & 1 & \text{only if } i = h \text{ and } n+h. \end{array} \tag{7.4.1}$$

Let $k \notin \{1, n+1\}$. Then, referring to (4.4.3),

$$\begin{array}{lcll} ((Gk^-)^{\#}c)_h & = & 2 & \text{if } k \ne h, \quad \text{by (7.1.2)}, \\ ((Gk^-)^{\#}c)_{n+h} & = & 2 & \text{if } k \ne n+h \quad \text{by (7.1.2)}. \end{array}$$

Also

$$\begin{array}{l} ((Gk^-)^{\#}c)_1 \le 0 \quad \text{and} \quad ((Gk^-)^{\#}c)_{n+1} \le 0, \quad \text{by (7.1.3)}. \\ ((Gk^-)^{\#}c)_i \ge 1 \quad \text{for every other } i \text{ such that } i \ne k \quad \text{by (7.1.3)}. \end{array}$$

Therefore,

$$G(k^- c) = \bar{\neg}\rho^{h-1} c,$$

or

$$G(k^- c) = \bar{\neg}\rho^{h-1}((\rho^{-(h-1)}k)^- c).$$

Therefore, under $G^{\sim}$,

$$[k^- c] \to ... \to [i^- c] \to ... \to [c],$$

or

$$[k^- c] \to ... \to [i^- c] \to ... \to [1^- c].$$

On the other hand, referring to (7.4.1), we have

$$((G1^-)^{\#}c)_1 \le (G^{\#}c)_1 \le -1 \quad \text{by (7.1.4)}.$$

$$((G1^-)^{\#}c)_{n+1} \le 0 \quad \text{by (7.1.3)}.$$

Further, since $1^-(\bar{\neg}\rho^{h-1}c)$ and c differ only at the 1st coordinate, (7.1.2) implies

$$((G1^-)^{\#}c)_h = 0 \quad \text{and} \quad ((G1^-)^{\#}c)_{n+h} = 0.$$

Also $((G1^-)^{\#}c)_i > 0$ for every other i by (7.1.3) and (7.4.1). Therefore,

$$G(1^- c) = \rho^{2(h-1)} c.$$

Therefore, if $x \in U_1\mathbf{C}$ then $x \in U_1\mathbf{C}$ and $\omega_G x = \mathbf{C}$. Therefore, C is an attractor of G. Thus we obtained

Theorem 7.4.1 Let $E = \langle e \rangle$ of $\mathbf{Q}^{2n}$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n \cdot p_{n+1} \cdot \alpha_2 p_{n+2} \cdot .. \cdot \alpha_n p_{2n}.$$

be the expansion described in Example 5.2.6 and derived from a transformation determined by Theorem 4.4.3. Let $G = \langle g \rangle$,

$$g = p_1 \cdot S_{2n-2}\{\alpha_2 p_2, .., \alpha_n p_n, p_{n+1}, \alpha_2 p_{n+2}, .., \alpha_n p_{2n}\}.$$

Then the $2n$-cycle of E is an attractor of G.

Theorem 4.5.3 determined a class of one-to-one skew-circular threshold transformations. Example 5.2.4 gave their circular expansions. Such a transformation E of $\mathbf{Q}^{2n}$ exists for any h such that $0 < h-1 < 2n$ and $h-1$ is relatively prime with $2n$, that is, $E = \langle e \rangle$,

$$\begin{array}{rcl} e & = & p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n \cdot \neg p_{n+1} \cdot \neg\alpha_2 p_{n+2} \cdot .. \cdot \neg\alpha_n p_{2n}, \\ E & = & \rho^{h-1} \quad \text{on } \mathrm{Car}E. \end{array}$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{2n}$ defined by

$$g = p_1 \cdot S_{2n-2}\{\alpha_2 p_2, .., \alpha_n p_n, \neg p_{n+1}, \neg\alpha_2 p_{n+2}, .., \neg\alpha_n p_{2n}\}.$$

Let $e = \{c\}$ Then,

$$C = \mathrm{Orb}_{\rho h-1} c$$

is a cycle of E. $(E^{\#}c)_i$ is even for every i, since $\rho^n x = \bar{\neg} x$ for every $x \in \mathrm{Car}E$. Therefore, $\mathrm{Orb}_{\rho h-1} c$ is a cycle of G by Proposition 7.1.4.

We have

$$(E^{\#}c)_i \leq 0 \quad \text{iff } i = 1 \text{ or } n+1,$$

since $Ec = \{1, n+1\}^{-}c$. Since $\rho^{h-1}c = Ec = \{1, n+1\}^{-}c$,

$$c = \{1, n+1\}^{-}\rho^{h-1}c.$$

Therefore, referring to (4.5.4),

$$(E^{\#}c)_i = 2 \quad \text{iff } i = h \text{ or } n+h.$$

Then, by (7.1.6),

$$\begin{array}{rcll} (G^{\#}c)_1 & \leq & -1, & (G^{\#}c)_{n+1} \leq -1, \\ (G^{\#}c)_i & \geq & 1 & \text{for every other } i, \\ (G^{\#}c)_i & = & 1 & \text{only if } i = h \text{ and } n+h. \end{array} \tag{7.4.2}$$

Let $k \notin \{1, n+1\}$. Then, referring to (4.5.4),

$$\begin{array}{rcll} ((Gk^{-})^{\#}c)_h & = & 2 \quad \text{if } k \neq h, & \text{by (7.1.2)} \\ ((Gk^{-})^{\#}c)_{n+h} & = & 2 \quad \text{if } k \neq n+h, & \text{by (7.1.2)} \\ ((Gk^{-})^{\#}c)_1 & \leq & 0, & \text{by (7.1.3)} \\ ((Gk^{-})^{\#}c)_{n+1} & \leq & 0, & \text{by (7.1.3)} \\ ((Gk^{-})^{\#}c)_i & \geq & 1 \quad \text{for every other } i \neq k. & \text{by (7.1.3)} \end{array}$$

Therefore,

$$G(k^{-}c) = \rho^{h-1}c,$$

or

$$G(k^{-}c) = \rho^{h-1}((\rho^{-(h-1)}k)^{-}c).$$

Therefore, under $G^{\sim}$,

$$[k^- c] \to ... \to [i^- c] \to ... \to [c],$$

or

$$[k^- c] \to ... \to [i^- c] \to ... \to [1^- c].$$

On the other hand, referring to (7.4.2), we have

$$\begin{array}{lll} ((G1^-)^{\#}c)_1 & \leq & (G^{\#}c)_1 = -1, \qquad \text{by (7.1.4)} \\ ((G1^-)^{\#}c)_{n+1} & \leq & 0. \qquad\qquad\qquad\quad \text{by (7.1.3)} \end{array}$$

Further, since $1^-(\rho^{h-1}c)$ and c differ only at their 1st coordinates, (7.1.2) implies

$$((G1^-)^{\#}c)_h = 0 \quad \text{and} \quad ((G1^-)^{\#}c)_{n+h} = 0.$$

Also $((G1^-)^{\#}c)_i \geq 1$ for every other i by (7.1.3) and (7.4.2). Therefore,

$$G(1^- c) = \rho^{2(h-1)}c.$$

Therefore, if $x \in U_1\mathbf{C}$ then $x \in U_1\mathbf{C}$ and $\omega_G x = \mathbf{C}$. Therefore, C is an attractor of G. Thus we obtained

Theorem 7.4.2 Let $E = \langle e \rangle$ of $\mathbf{Q}^{2n}$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n \cdot \neg p_{n+1} \cdot \neg\alpha_2 p_{n+2} \cdot .. \cdot \neg\alpha_n p_{2n}.$$

be the expansion described in Example 5.2.4 and derived from a transformation determined by Theorem 4.5.3. Let $G = \langle g \rangle$,

$$g = p_1 \cdot S_{2n-2}\{\alpha_2 p_2, .., \alpha_n p_n \cdot \neg p_{n+1} \cdot \neg\alpha_2 p_{n+2} \cdot .. \cdot \neg\alpha_n p_{2n}\}.$$

Then the $2n$-cycle of E is an attractor of G.

Example 7.4.3 Let $n = 2m$ for $m \geq 2$, and

$$f = p_1 \cdot p_2 \cdot p_3 \cdot .. \cdot p_m \cdot \neg p_{m+2} \cdot .. \cdot \neg p_{2m}.$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{2m}$ defined by

$$g = p_1 \cdot S_{2m-3}\{p_2, p_3, .., p_m, \neg p_{m+2}, .., \neg p_{2m}\}.$$

Proposition 7.4.4 Let $c = 1^m 0^m$. Then, $C = \mathrm{Orb}_\rho c$ is an attractor of G in Example 7.4.3.

Proof. We have

$$F^{\#}c = (0, 2, 4, .., 2m-2, 0, 2, 4, .., 2m-2).$$

Therefore, $Fc = \rho c$, so that $\mathrm{Orb}_\rho c$ is a cycle of F. Further, $\mathrm{Orb}_\rho c$ is a cycle of G by Proposition 7.2.2.

Further, by (7.1.6),

$$G^{\#}c = (-1, 1, 3, .., 2m-3, -1, 1, 3, .., 2m-3).$$

Let $2 \leq k \leq m-1$. Then,

$$\begin{array}{rcll} ((Gk^-)^{\#}c)_1 & \leq & 0 & \text{by (7.1.3)}, \\ ((Gk^-)^{\#}c)_{m+1} & \leq & 0 & \text{by (7.1.3)}, \\ ((Gk^-)^{\#}c)_2 & = & 2 & \text{if } k \neq 2, \quad \text{by (7.1.2)}, \\ (Gk^-)^{\#}c)_{m+2} & = & 2 & \text{by (7.1.2)}, \\ ((Gk^-)^{\#}c)_i & \geq & 2 & \text{for every other } i \text{ such that } i \neq k \quad \text{by (7.1.3)}. \end{array}$$

Therefore,

$$k^-c \to \rho((k-1)^-c) \quad \text{or} \quad k^-c \to \rho c.$$

Therefore,

$$[k^-c] \to ... \to [1^-c] \quad \text{or} \quad [k^-c] \to ... \to [c].$$

We have

$$\begin{aligned}
((G1^-)^{\#}c)_1 &\leq (G^{\#}c)_1 = -1 \quad \text{by (7.1.4)},\\
((G1^-)^{\#}c)_{m+1} &\leq 0 \quad \text{by (7.1.3)},\\
((G1^-)^{\#}c)_2 &= 0, \quad \text{by (7.1.2)},\\
((G1^-)^{\#}c)_{m+2} &= 0, \quad \text{by (7.1.2)},\\
((G1^-)^{\#}c)_i &\geq 1 \text{ for every other } i \quad \text{by (7.1.3)}.
\end{aligned}$$

Therefore, $1^-c \to \rho^2 c$. Therefore, if $x \in U_1\mathbf{C}$ then $x \in U_1\mathbf{C}$ and $\omega_G x = \mathbf{C}$. Therefore, C is an attractor of G. □

7.5 Single-cycle attractors II

Another method of constructing single-cycle attractors is to apply partial neighborhood functions. First we consider a circular transformation belonging to the class determined by Theorem 4.4.3. Let $h-1$ be relatively prime with odd n and $0 < h-1 < n$, and let $F = \langle f \rangle$ of $\mathbf{Q}^n$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n,$$

such that

$$F = \bar{\neg}\rho^{h-1} \quad \text{on Car}F.$$

Let $f = \{c\}$. Let $G = \langle g \rangle$,

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, \alpha_3 p_3, ..., \alpha_n p_n\}.$$

Then $Gc = \{1, h\}^- c = \rho^{2(h-1)}c$ by (7.3.1), and G has two cycles $\text{Orb}_{\rho 2(h-1)}\{c, \bar{\neg}\rho^{h-1}c\}$.

In order to preserve the original one cycle such that $Gc = 1^-c$ we remove c from $\bar{\neg}\rho^{h-1}g$, i.e. remove $\bar{\neg}\rho^{-(h-1)}c$ from g. Note

$$(\rho n^-)^{-(h-1)}c = (1-(h-1))^-c = (2-h)^-c,$$

referring to (4.4.3). $(2-h)^-c$ is the only element $x \neq c$ in g such that $x \in [c]$, otherwise $G^{\#}$ would have more than two non-positive coordinates. Further we consider the set

$$\{2^-c, , , n^-c\} \backslash (2-h)^-c$$

and find out some elements $i^-c \neq j^-c$ such that

$$[i^-c] = [j^-c].$$

We have

$$(\bar{\neg}\rho^{h-1})^k i^- c = (\rho^{k(h-1)}i)^-(\bar{\neg}\rho^{h-1})^k c,$$

by (2.1.3), and

$$(\bar{\neg}\rho^{h-1})^k c = (1+(k-1)(h-1))^-(1+(k-2)(h-1))^- ... (1+(h-1))^- 1^- c$$

by (4.5.4). Therefore, $(\bar{\neg}\rho^{h-1})^k i^- c = j^- c$ implies

$$(i+k(h-1))^-(1+(k-1)(h-1))^-(1+(k-2)(h-1))^- ... (1+(h-1))^- 1^- = j^-,$$

so that $k = 2$, and $(i+k(h-1))^- h^- 1^- = j^-$. Therefore,

$$(i+k(h-1)) = 1, \quad j = h;$$

or

$$(i + k(h-1)) = h, \quad j = 1.$$

Since $j > 1$, $j = h$ and $i = 3 - 2h$. We want to remove one of $(3-2h) - c$ and $h^{-}c$ from g, but we must decide which to remove. If we remove $(3-2h) - c$ then $Gh^{-}c = G^{2}c$. If we remove $h^{-}c$, then $G(3-2h) - c = (3-2h)^{-}Gc$. The former is better in view of of continuity. Thus we obtained $V = \langle v \rangle$ defined by

$$v = p_1 \cdot S_{n-4}(\{\alpha_2 p_2, .., \alpha_n p_n\} \backslash \{\alpha_{2-h} p_{2-h}, \alpha_{3-2h} p_{3-2h}\}) \cdot \alpha_{2-h} p_{2-h} \cdot \alpha_{3-2h} p_{3-2h}. \tag{7.5.1}$$

By the above removal, if $x \in v$, then $x \notin \{\rho^{i} v, \bar{\neg}\rho^{i} v\}$ for every $i \neq 0 \bmod n$.

We prove the attractiveness of the cycle by decomposition of the transformations instead of calculating the extended representations of the transformations. For the notion of the sum of transformations, refer to Chapter 1.1. Let

$$\begin{aligned} v^{(1)} &= p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n, \\ v^{(i)} &= v \cdot \neg \alpha_i p_i \quad \text{for } i \in \mathbf{N} \backslash \{1, 2-h, 3-2h\}. \end{aligned}$$

Then

$$\begin{aligned} v &= v^{(1)} \vee ... \vee v^{(n)}, \\ v^{(i)} \cdot v^{(j)} &= 0 \quad \text{for every } i \neq j, \end{aligned}$$

as clear from the above process of removing $(2-h)^{-}c$ and $(3-2h)^{-}c$ from g.

Let $V^{(i)} = \langle v^{(i)} \rangle$. Then, if $x \in v^{(1)}$, then $x = c$ and

$$Vc = 1^{-}c \in [v^{(1)}].$$

If $x \in v^{(i)}$ for $i \in \mathbf{N} \backslash \{1, 2-h, 3-2h\}$, then $x = i^{-}c$, and

$$V(i^{-}c) = V^{(i)}(i^{-}c) = \{1, i\}^{-}c,$$

since $x \notin \{\rho^{i} v, \bar{\neg}\rho^{i} v\}$ for every $i \neq 0 \bmod n$. Therefore,

$$V(i^{-}c) = \bar{\neg}\rho^{h-1}((\rho^{-(h-1)} i)^{-}c \in [v^{(i-h+1)}].$$

Therefore,

$$V = V^{(1)} + V^{(2)} + ... + V^{(n)},$$

and a flow graph of V is

$$[v^{(4-3h)}] \to [v^{(5-4h)}] \to ... \to [v^{(1)}] \to [v^{(1)}] = [c].$$

Let $C = \mathrm{Orb}_{\bar{\neg}\rho h-1} c$. Then C is a cycle of V and $\mathbf{C} = [c]$. Further, $1^{-}c \in [v^{(1)}]$, $(2-h)^{-}c \in v^{(1)}$, and $(3-2h)^{-}c \in [v^{(h)}]$. Therefore, $U_1 \mathbf{C} = [v]$. Therefore, C is an attractor of V. Thus we obtained the following theorem.

Theorem 7.5.1 Let $V = \langle v \rangle$ of $\mathbf{Q}^{n}$ be the transformation defined by (7.5.1) from a transformation determined by Theorem 4.4.3. Then G has a $2n$-cycle attractor.

Next, let $0 < h-1 < 2n$ and $h-1$ be relatively prime with $2n$, let $F = \langle\langle f \rangle\rangle$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n$$

be a transformation determined by Theorem 4.5.3, and let $f = \{c\}$. Let $G = \langle\langle g \rangle\rangle$,

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, \alpha_3 p_3, ..., \alpha_n p_n\}.$$

Then $Gc = \{1, h\}^- c = (\rho n^-)^{2(h-1)} c$ by (7.3.2), and G has two cycles $\mathrm{Orb}_{(\rho n-)2(h-1)}\{c, (\rho n^-)^{h-1} c\}$. In order to preserve the original one cycle such that $Gc = 1^- c$ we remove c from $(\rho n^-)^{h-1} g$, i.e. remove $(\rho n^-)^{-(h-1)} c$ from g. Note

$$(\rho n^-)^{-(h-1)} c = (1 - (h-1))^- c = (2-h)^- c,$$

referring to (4.5.4). $(2-h)^- c$ is the only element $x \neq c$ in g such that $x \in [c]$, otherwise $G^{\#}$ would have more than two non-positive coordinates.

Further we consider the set

$$\{2^- c, , , n^- c\} \backslash (2-h)^- c$$

and find out some elements $i^- c \neq j^- c$ such that

$$[i^- c] = [j^- c].$$

We have

$$(\rho n^-)^{k(h-1)} i^- c = (\rho^{k(h-1)} i)^- (\rho n^-)^{k(h-1)} c,$$

by (2.1.3), and

$$(\rho n^-)^{k(h-1)} c = (1 + (k-1)(h-1))^- (1 + (k-2)(h-1))^- ... (1 + (h-1))^- 1^- c$$

by (4.5.4). Therefore, $(\rho n^-)^{k(h-1)} i^- c = j^- c$ implies

$$(i + k(h-1))^- (1 + (k-1)(h-1))^- (1 + (k-2)(h-1))^- ... (1 + (h-1))^- 1^- = j^-,$$

so that $k = 2$, and $(i + k(h-1))^- h^- 1^- = j^-$. Therefore,

$$(i + k(h-1)) = 1, \quad j = h;$$

or

$$(i + k(h-1)) = h, \quad j = 1.$$

Since $j > 1$, $j = h$ and $i = 3 - 2h$. We want to remove one of $(3-2h) - c$ and $h^- c$ from g, but we must decide which to remove. If we remove $(3-2h) - c$ then $Gh^- c = G^2 c$. If we remove $h^- c$, then $G(3-2h) - c = (3-2h)^- Gc$. The former is better in view of of continuity. Thus we obtained $V = \langle\langle v \rangle\rangle$ defined by

$$v = p_1 \cdot S_{n-4}(\{\alpha_2 p_2, .., \alpha_n p_n\} \backslash \{\alpha_{2-h} p_{2-h}, \alpha_{3-2h} p_{3-2h}\}) \cdot \alpha_{2-h} p_{2-h} \cdot \alpha_{3-2h} p_{3-2h}. \tag{7.5.2}$$

By the above removal, if $x \in v$, then $x \notin (\rho n^-)^i v$ for every $i \neq 0 \bmod 2n$.

We prove the attractiveness of the cycle by decomposition of the transformations instead of calculating the extended representations of the transformations. For the notion of the sum of transformations, refer to Chapter 1.1. Let

$$\begin{aligned} v^{(1)} &= p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n, \\ v^{(i)} &= v \cdot \neg \alpha_i p_i \quad \text{for } i \in \mathbf{N} \backslash \{1, 2-h, 3-2h\}. \end{aligned}$$

Then

$$\begin{aligned} v &= v^{(1)} \vee ... \vee v^{(n)}, \\ v^{(i)} \cdot v^{(j)} &= 0 \quad \text{for every } i \neq j, \end{aligned}$$

as clear from the above process of removing $(2-h)^- c$ and $(3-2h)^- c$ from g.

Let $V^{(i)} = \langle\langle v^{(i)} \rangle\rangle$. Then, if $x \in v^{(1)}$, then $x = c$ and

$$Vc = 1^- c \in [v^{(1)}].$$

If $x \in v^{(i)}$ for $i \in \mathbf{N} \backslash \{1, 2-h, 3-2h\}$, then $x = i^- c$, and

$$V(i^- c) = V^{(i)}(i^- c) = \{1, i\}^- c,$$

since $1^{-}c \in (\rho n-)^{j} v$ for only $j = 0 \bmod 2n$. Therefore,

$$V(i^{-}c) = (\rho n^{-})^{h-1}((\rho^{-(h-1)} i)^{-} c \in [v^{(i-h+1)}].$$

Therefore,

$$V = V^{(1)} + V^{(2)} + ... + V^{(n)},$$

and a flow graph of V is

$$[v^{(4-3h)}] \to [v^{(5-4h)}] \to ... \to [v^{(1)}] \to [v^{(1)}] = [c].$$

Let $C = \mathrm{Orb}_{(\rho n-)h-1} c$. Then C is a cycle of V and $\mathbf{C} = [c]$. Further, $1^{-}c \in [v^{(1)}]$, $(2-h)^{-}c \in v^{(1)}$, and $(3-2h)^{-}c \in [v^{(h)}]$. Therefore, $U_1\mathbf{C} = [v]$. Therefore, C is an attractor of V. Thus we obtained the following theorem.

Theorem 7.5.2 Let $V = \langle v \rangle$ of $\mathbf{Q}^n$ be the transformation defined by (7.5.2) from a transformation determined by Theorem 4.5.3. Then G has a $2n$-cycle attractor.

7.6 Attractors derived from polynomials

So far we have shown attractors in transformations obtained by modifying one-to-one transformations $F = \langle f \rangle$ or $F = \langle\langle f \rangle\rangle$ such that f consists of one term. In this section, we generalize the construction to one- to-one transformations $F = \langle f \rangle$ such that f consists of more than one term. For this purpose, we start with the one-to-one transformation of Example 4.4.7 defined by

$$f = 1 \cdot 2 \cdot \neg 3 \cdot (4 \vee 5) \cdot \neg 6.$$

GRAPH(F) consists of the three 6-cycles:

$$\begin{array}{ccccc} 110110 & \to & 010010 & \to & 011011 \\ \uparrow & & & & \downarrow \\ 100100 & \leftarrow & 101101 & \leftarrow & 001001 \\ 110100 & \to & 010110 & \to & 010011 \\ \uparrow & & & & \downarrow \\ 100101 & \leftarrow & 001101 & \leftarrow & 011001 \\ 110010 & \to & 011010 & \to & 001011 \\ \uparrow & & & & \downarrow \\ 100110 & \leftarrow & 101100 & \leftarrow & 101001 \end{array}$$

We modify f by expanding it to

$$\begin{aligned} g &= 1 \cdot U_1(2 \cdot \neg 3 \cdot (4 \vee 5) \cdot \neg 6) \\ &= 1 \cdot (2 \cdot \neg 3 \cdot \neg 6 \vee S_2\{2, 3, \neg 6\} \cdot (4 \vee 5)). \end{aligned}$$

Here, $2 \cdot \neg 3 \cdot \neg 6 \vee S_2\{2, 3, \neg 6\} \cdot (4 \vee 5)$ is a threshold function by Proposition 4.1.6, so that g is a threshold function by Proposition 4.1.2.

For the point 110110 in the first cycle of F,

$$F^{\#}(110110) = (0, 2, 1, 0, 2, 1).$$

Therefore, the condition in Proposition 7.1.4 is violated, so that the cycle is not preserved in $G = \langle g \rangle$. In fact, by (7.1.6), we have

$$G^{\#}(110110) = (-1, 1, 0, -1, 1, 0),$$

so that $110110 \to 011011$ under G. We have thus obtained the two 3-cycles

$$\begin{array}{ccccccc} 110110 & \rightarrow & 011011 & & 001001 & \rightarrow & 100100 \\ \nwarrow & & \swarrow & & \nwarrow & & \swarrow \\ & 101101 & & & & 0100101 & \end{array}$$

Further calculation shows that the set of these two cycles is an attractor of G.

Next, we take the one-to-one transformation of Example 4.4.17 in Chapter 4. Therefore, we start with f defined recursively as follows,

$$\begin{array}{rcl} f^{(4)} & = & p_1 \cdot p_2 \cdot \neg p_3 \cdot \neg p_4, \\ f^{(5)} & = & p_1 \cdot p_2 \cdot \neg p_4 \cdot \neg p_5, \\ f^{(n)} & = & p_1 \cdot p_2 \cdot \neg p_{n-1} \cdot \neg p_n \vee f^{(n-2)} \cdot \neg p_n \\ & = & p_1 \cdot p_2 \cdot \neg p_n \cdot (\neg p_{n-1} \vee (f^{(n-2)}|11)), \end{array} \tag{7.6.1}$$

where

$$(f^{(n-2)}|11)(x_3, .., x_{n-2}) = f^{(n-2)}(1, 1, x_3, .., x_{n-2})$$

for every $(x_3, .., x_{n-2})$.

We modify $f^{(n)}$ by expanding it to

$$g^{(n)} = p_1 \cdot U_1(p_2 \cdot \neg p_n \cdot (\neg p_{n-1} \cdot \vee (f^{(n-2)}|11))).$$

Then

$$\begin{array}{rcl} g^{(n)} & = & U_1(p_2 \cdot \neg p_n \cdot \neg p_{n-1}) \vee U_1(p_2 \cdot \neg p_n \cdot (f^{(n-2)}|11))) \\ & = & p_2 \cdot \neg p_n \vee \cdot (p_2 \vee \neg p_n) \cdot \neg p_{n-1} \\ & & \quad \vee p_2 \cdot \neg p_n \cdot U_1(f^{(n-2)}|11)) \vee (p_2 \vee \neg p_n)(f^{(n-2)}|11) \\ & = & p_2 \cdot \neg p_n \vee \cdot (p_2 \vee \neg p_n) \cdot \neg p_{n-1} \vee (p_2 \vee \neg p_n)(f^{(n-2)}|11) \\ & = & p_2 \cdot \neg p_n \vee \cdot (p_2 \vee \neg p_n) \cdot (\neg p_{n-1} \vee (f^{(n-2)}|11)) \end{array}$$

Thus we have obtained

$$g = g^{(n)} = p_1 \cdot (p_2 \cdot \neg p_n \vee (p_2 \vee \neg p_n) \cdot (\neg p_{n-1} \cdot \vee (f^{(n-2)}|11))). \tag{7.6.2}$$

and the transformation $G = \langle g \rangle$ of Q^n for $n \geq 6$.

Proposition 7.6.1 g defined by (7.6.1) and (7.6.2) is a threshold function.

Proof. $f^{(n-2)}$ is a threshold function by Proposition 4.4.18. Therefore, $f^{(n-2)}|11$ is a threshold transformation by Proposition 4.1.7. Therefore, $\neg p_{n-1} \cdot \vee (f^{(n-2)}|11)$ is a threshold function by Proposition 4.1.2. Therefore, $p_2 \cdot \neg p_n \vee (p_2 \vee \neg p_n) \cdot (\neg p_{n-1} \cdot \vee (f^{(n-2)}|11))$ is a threshold function by Proposition 4.1.6. Therefore, g is a threshold function by Proposition 4.1.2. □

Computational results expect that the transformations G have non-trivial attractors. However, at present I can give only the following weaker result Proposition 7.6.4. First, we generalize the ϵ-neighborhood of a non-empty proper subset S of Q^n, $U_\epsilon S$, for any integer ϵ as

$$U_\epsilon S = \{x \mid d_{SH}(x, S) \leq \epsilon\}.$$

Here, $d_{SH}(x, S)$ is the signed Hamming distance between the point x and the set S defined in Section 7.1.

As before, o is the n-vector whose every coordinate is 0, l is the n-vector whose every coordinate is 1. Further, $(10)^{n/2} = 1010...10$ is the concatenation of $n/2$ 10, and $(10)^{n/2} = 0101...01$ is the concatenation of $n/2$ 01.

Lemma 7.6.2 Let $A = \{o, l\}$ and $B = \{(10)^{n/2}, (01)n/2\}$. Then Car$G = A^C$ for odd n, and Car$G = (A \cup B)^c$ for even n.

Proof. If x is an element of $A \cup B$, then clearly $x \notin g$ and $x \notin \neg g$, so that $x \notin \rho^{i-1} g$ and $x \notin \neg\rho^{i-1} g$ for every i, so that $x \notin$ Car$G$.

Let $x \in (A \cup B)^c$. Then there exists some i such that (x_i, x_{i+1}, x_{i+2}) is 011 or 100. Let $x_1 = x_2 = 1$, and $x_n = 0$ without loss of generality. Then $x \in g$. Therefore, $x \in$ Car$G$. □

Lemma 7.6.3 Let $Gx = y$. Then if $(x_i, x_{i+1}, x_{i+2}) = 110$ then $(y_{i+1}, y_{i+2}) = 11$. If $(x_i, x_{i+1}, x_{i+2}, x_{i+3}) = 0111$ then $(y_{i+1}, y_{i+2}) = 00$. If $(x_k, x_{k+1}, x_{k+2}) = 011$ then $y_{k+1} = 0$.

Proof. The proof is clear from the definition of g. □

Proposition 7.6.4 Let $G = \langle g \rangle$ be the transformation of Q^n, where g is defined by (7.6.1) and (7.6.2). Then there exists a subset Φ of $CY(G)$ such that

$$\begin{array}{ll} (1) & G(\mathrm{Car}G) \subseteq \mathrm{Car}G, \\ (2) & \omega_G(\mathrm{Car}G) = \mathrm{Im}\Phi, \\ (3) & \mathrm{Im}\Phi \subseteq U_{-1}(\mathrm{Car}G). \end{array}$$

Proof. Assume $x \in$ Car$G$, and $y = Gx$. Then there exists some i such that $(x_i, x_{i+1}, x_{i+2}) = 110$ or $(x_i, x_{i+1}, x_{i+2}) = 001$. Let $(x_1, x_2, x_3) = 110$ without loss of generality. Then $(y_2, y_3) = 11$. Assume there exists some j such that $(x_j, x_{j+1}, x_{j+2}) = 001$. Then $(y_{j+1}, y_{j+2}) = 00$, so that $d_H(y, (\mathrm{Car}G)^c) \geq 2$.

Assume no j such that $(x_j, x_{j+1}, x_{j+2}) = 001$ exists. Assume there exists some j such that $(x_j, x_{j+1}, x_{j+2}, x_{j+3}) = 0111$, then $(y_{j+1}, y_{j+2}) = 00$, so that $d_H(y, (\mathrm{Car}G)^c) \geq 2$. Assume no such j exists. Then $x_n = 0$, so that $y_1 = 0$. Assume there exists no $j \neq n$ such that $(x_j, x_{j+1}, x_{j+2}) = 011$. Then $x = 11(01)^r 0$, so that $y_n = 0$. Therefore, $(y_n, y_1, y_2, y_3) = 0011$, so that $d_H(y, (\mathrm{Car}G)^c) \geq 2$. Assume there exists some $j \neq n$ such that $(x_j, x_{j+1}, x_{j+2}) = 011$, and let (x_j, x_{j+1}, x_{j+2}) be the first such coordinates. Then $x = 11(01)^r 011...0$. Therefore, $y_{j+1} = 0$, so that $(y_1, y_2, y_3, y_{j+1}) = 0110$. Since $j+1$ is even, $d_H(y, (\mathrm{Car}G)^c) \geq 2$.

Therefore, $d_{SH}(y, \mathrm{Car}G) \leq 1 - 2 = -1$ in every case, Therefore, $G(\mathrm{Car}G) \subseteq U_{-1}(\mathrm{Car}G)$. Let Φ be the set of all non-loop cycles. Then the desired properties (1), (2), and (3) are clear. □

7.7 Orbit modification

We have shown that orbit modification provided a powerful tool for constructing new desired transformations in Chapters 3 and 4. In particular, we constructed a McCulloch and Pitts network (6.2.1) that has an attractive unique k-cycle for any $k \leq 2^n$ (Theorems 5.5.2 and 6.6.3). In that network, any state converges to a unique cycle regardless of the initial state. Here we further explore the possibilities of the method to construct some transformations that have the same property.

(1) We start with the skew circular transformation for $n = 2$, $F = [f_1, f_2]$,

$$\begin{aligned} f_1 &= 11 = 1 \cdot 2, \\ f_2 &= 01 = \neg 1 \cdot 2. \end{aligned}$$

GRAPH(F) is

$$11 \to 01 \to 00 \to 10 \to 11.$$

(2) For $n = 3$, we have the 3rd face copies of (1) defined by $F = [f_1, f_2, f_3]$,

$$\begin{array}{rcl} f_1 &=& \{111, 110\} = 1 \cdot 2, \\ f_2 &=& \{011, 010\} = \neg 1 \cdot 2, \\ f_3 &=& \emptyset. \end{array}$$

GRAPH(F) is

$$\begin{array}{ccccccccc} 111 & \to & 011 & \to & 001 & \to & 101 & \to & 111, \\ 110 & \to & 010 & \to & 000 & \to & 100 & \to & 110. \end{array}$$

(3) Modify (2) by first letting $f_3 = \{001\} = \neg 1 \cdot \neg 2 \cdot 3$, and delete the complement of 001, i.e. 110 from f_1. We now have $F = [f_1, f_2, f_3]$,

$$\begin{array}{rcl} f_1 &=& 1 \cdot 2 \cdot 3, \\ f_2 &=& \neg 1 \cdot 2, \\ f_3 &=& \neg 1 \cdot \neg 2 \cdot 3. \end{array}$$

GRAPH(F) is

$$\begin{array}{ccccccccc} 111 & \to & 011 & \to & 001 & & 101 & \to & 111. \\ & & & & \downarrow & & & & \uparrow \\ & & 010 & \to & 000 & \to & 100 & \to & 110 \end{array}$$

This has a unique 6-cycle, $\mathrm{Orb}_{\rho 3-}111$.

(4) For $n = 4$, we have the 4th face copies of (3) $F = [f_1, f_2, f_3, f_4]$,

$$\begin{array}{rcl} f_1 &=& 1 \cdot 2 \cdot 3, \\ f_2 &=& \neg 1 \cdot 2, \\ f_3 &=& \neg 1 \cdot \neg 2 \cdot 3, \\ f_4 &=& \emptyset. \end{array}$$

GRAPH(F) is

$$\begin{array}{ccccccccc} 1111 & \to & 0111 & \to & 0011 & & 1011 & \to & 1111, \\ & & & & \downarrow & & & & \uparrow \\ & & 0101 & \to & 0001 & \to & 1001 & \to & 1101, \\ 1110 & \to & 0110 & \to & 0010 & & 1010 & \to & 1110. \\ & & & & \downarrow & & & & \uparrow \\ & & 0100 & \to & 0000 & \to & 1000 & \to & 1100 \end{array}$$

(5) Modify (2) by first letting $f_4 = \{0001\} = \neg 1 \cdot \neg 2 \cdot \neg 3 \cdot 4$, and delete the complement of 0001, i.e. 1110 from f_1. We now have $F = [f_1, f_2, f_3, f_4]$,

$$\begin{array}{rcl} f_1 &=& 1 \cdot 2 \cdot 3 \cdot 4, \\ f_2 &=& \neg 1 \cdot 2, \\ f_3 &=& \neg 1 \cdot \neg 2 \cdot 3, \\ f_4 &=& \neg 1 \cdot \neg 2 \cdot \neg 3 \cdot 4. \end{array}$$

GRAPH(F) is

$$
\begin{array}{ccccccccccc}
1111 & \to & 0111 & \to & 0011 & & & & 1011 & \to & 1111, \\
 & & & & \downarrow & & & & & & \uparrow \\
 & & 0101 & \to & 0001 & \to & 0000, & & 1001 & \to & 1101, \\
1110 & \to & 1111, & & 0110 & \to & 0010 & & 1010 & \to & 1110. \\
 & & & & & & \downarrow & & & & \uparrow \\
 & & & & 0100 & \to & 0000 & \to & 1000 & \to & 1100
\end{array}
$$

This has a unique 8-cycle, $\mathrm{Orb}_{\rho 4^-}1111$.

For the general dimension n, we obtain

Example 7.7.1 Let $F = [f_1, .., f_n]$,

$$
\begin{array}{rcl}
f_1 & = & p_1 \cdot .. \cdot p_n, \\
f_2 & = & \neg p_1 \cdot p_2, \\
f_3 & = & \neg p_1 \cdot \neg p_2 \cdot p_3, \\
 & & \ldots\ldots \\
f_{n-1} & = & \neg p_1 \cdot .. \cdot \neg p_{n-2} \cdot p_{n-1}, \\
f_n & = & \neg p_1 \cdot .. \cdot \neg p_{n-1} \cdot p_n,
\end{array}
$$

then, GRAPH(F) consists of the unique $2n$-cycle $\mathrm{Orb}_{\rho n^-} l$.

For the next example, we consider the transformation $G = \langle g \rangle$ of Example 7.4.3 for $n = 4$, where $g = 1 \cdot (2 \vee \neg 4)$. GRAPH($G$) is

$$
\begin{array}{ccccccc}
0001 & & & & & & 1000 \\
 & \searrow & & & & \swarrow & \\
1011 & \to & 1100 & \to & 0110 & \leftarrow & 1101 \\
 & & \uparrow & & \downarrow & & \\
0111 & \to & 1001 & \to & 0011 & \leftarrow & 1110 \\
 & \nearrow & & & & \nwarrow & \\
0010 & & & & & & 0100, \\
0000 & \leftrightarrow & 1111, & & 0101 & \leftrightarrow & 1010.
\end{array}
$$

The 5th face copies are defined by $G = [g_1, g_2, g_3, g_4, g_5]$,

$$
\begin{array}{rcl}
g_1 & = & 1 \cdot (2 \vee \neg 4), \\
g_2 & = & 2 \cdot (3 \vee \neg 1), \\
g_3 & = & 1 \cdot (4 \vee \neg 2), \\
g_4 & = & 4 \cdot (1 \vee \neg 3), \\
g_5 & = & \emptyset.
\end{array}
$$

GRAPH(G) is

$$\begin{array}{ccccccc}
00011 & & & & & & 10001 \\
 & \searrow & & & & \swarrow & \\
10111 & \rightarrow & 11001 & \rightarrow & 01101 & \leftarrow & 11011 \\
 & & \uparrow & & \downarrow & & \\
01111 & \rightarrow & 10011 & \rightarrow & 00111 & \leftarrow & 11101 \\
 & \nearrow & & & & \nwarrow & \\
00101 & & & & & & 01001, \\
00001 & \leftrightarrow & 11111, & & 01011 & \leftrightarrow & 10101, \\
00010 & & & & & & 10000 \\
 & \searrow & & & & \swarrow & \\
10110 & \rightarrow & 11000 & \rightarrow & 01100 & \leftarrow & 11010 \\
 & & \uparrow & & \downarrow & & \\
01110 & \rightarrow & 10010 & \rightarrow & 00110 & \leftarrow & 11100 \\
 & \nearrow & & & & \nwarrow & \\
00100 & & & & & & 01000, \\
00000 & \leftrightarrow & 11110, & & 01010 & \leftrightarrow & 10100.
\end{array}$$

Modify the above by letting $f_5 = 1 \cdot 2 \cdot \neg 3 \cdot \neg 4 \cdot 5$. Then following changes in the arcs of GRAPH(G) occur, while the other arcs remain unchanged.

$$\begin{array}{ccc}
11001 & \rightarrow & 01100, \\
00110 & \rightarrow & 10011.
\end{array}$$

Therefore,

$$\begin{array}{ccc}
11001 & \rightarrow & 01100 \\
\uparrow & & \downarrow \\
10011 & \leftarrow & 00110
\end{array}$$

becomes an attractor.

For the general dimension $n = 2m + 1$, we obtain

Example 7.7.2 Let $n = 2m + 1$ for $m \geq 2$, and

$$\begin{aligned}
g_1 &= p_1 \cdot S_{2m-3}\{p_2, p_3, .., p_m, \neg p_{m+2}, .., \neg p_{2m}\}, \\
g_2 &= p_2 \cdot S_{2m-3}\{p_3, p_4, .., p_{m+1}, \neg p_{m+3}, .., \neg p_{2m}, \neg p_1\}, \\
& \ \ \ ... \\
g_{2m} &= p_{2m} \cdot S_{2m-3}\{p_1, p_2, .., p_{m-1}, \neg p_{m+1}, .., \neg p_{2m-1}\}, \\
g_{2m+1} &= p_1 \cdot p_2 \cdot .. \cdot p_m \cdot \neg p_{m+1} \cdot \neg p_{m+2} \cdot .. \cdot \neg p_{2m-1} \cdot p_{2m}.
\end{aligned}$$

Then the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{2m+1}$ has a unique attractive $2m$-cycle, although there are some non-attractive loops or cycles.

Next, we consider the direct product of the transformation $[1 \cdot 2 \cdot 3, \neg 1 \cdot 2, \neg 1 \cdot \neg 2 \cdot 3]$ of Example 7.7.1 for $n = 3$ and the skew-circular transformation $[4 \cdot 5, \neg 4 \cdot 5]$, which are respectively transformations of $\mathbf{Q}^{\{1,2,3\}}$ and $\mathbf{Q}^{\{4,5\}}$. Then

$$G = [1 \cdot 2 \cdot 3, \neg 1 \cdot 2, \neg 1 \cdot \neg 2 \cdot 3, 4 \cdot 5, \neg 4 \cdot 5].$$

GRAPH(G) is

$$
\begin{array}{ccccccccccc}
10110 & \to & 11111 & \to & 01101 & \to & 00100 & \to & 00010 & \leftarrow & 01000 \\
 & & \uparrow & & & & & & \downarrow & & \\
 & & 11010 & & & & & & 10011 & & \\
 & & \uparrow & & & & & & \downarrow & & \\
 & & 10000 & & & & & & 11001 & & \\
 & & \uparrow & & & & & & \downarrow & & \\
01011 & \to & 00001 & \leftarrow & 00111 & \leftarrow & 01110 & \leftarrow & 11100 & \leftarrow & 10101, \\
01001 & \to & 00000 & \to & 10010 & \to & 11011 & \to & 11101 & \leftarrow & 10111 \\
 & & \uparrow & & & & & & \downarrow & & \\
 & & 00101 & & & & & & 01100 & & \\
 & & \uparrow & & & & & & \downarrow & & \\
 & & 01111 & & & & & & 00110 & & \\
 & & \uparrow & & & & & & \downarrow & & \\
10100 & \to & 11110 & \leftarrow & 11000 & \leftarrow & 10001 & \leftarrow & 00011 & \leftarrow & 01010.
\end{array}
$$

We try to unite these two 12-cycles and construct a transformation having a unique 24- cycle. By try and error, we find that Arimoto's orbit modification G at point 11011 is a threshold transformation. In fact,

$$
\begin{array}{rcl}
1\cdot 2\cdot 3 \cup 11011 & = & 1\cdot 2\cdot 3 \vee 1\cdot 2\cdot \neg 3\cdot 4\cdot 5 = 1\cdot 2\cdot (3 \vee 4\cdot 5), \\
\neg 1\cdot 2 \cup 11011 & = & \neg 1\cdot 2 \vee 1\cdot 2\cdot \neg 3\cdot 4\cdot 5 = 2\cdot (\neg 1 \vee \neg 3\cdot 4\cdot 5), \\
\neg 1\cdot \neg 2\cdot 3 \backslash 00100 & = & \neg 1\cdot \neg 2\cdot 3\cdot (4 \vee \cdot 5), \\
4\cdot 5 \backslash 11011 & = & 4\cdot 5\cdot (\neg 1 \vee \neg 2 \vee 3), \\
\neg 4\cdot 5 \cup 11011 & = & \neg 4\cdot 5 \vee 1\cdot 2\cdot \neg 3\cdot 4\cdot 5 = (1\cdot 2\cdot \neg 3 \vee \neg 4)\cdot 5.
\end{array}
$$

Therefore,

$$
\begin{array}{rcl}
G & = & [1\cdot 2\cdot (3 \vee 4\cdot 5), 2\cdot (\neg 1 \vee \neg 3\cdot 4\cdot 5), \neg 1\cdot \neg 2\cdot 3\cdot (4 \vee \cdot 5), \\
 & & \quad 4\cdot 5\cdot (\neg 1 \vee \neg 2 \vee 3), (1\cdot 2\cdot \neg 3 \vee \neg 4)\cdot 5]
\end{array}
$$

is a threshold transformation. G has a unique 24-cycle. In this case, generalization to the general dimension is impossible.

Neither G nor $\neg G$ is PDNN-definable according to Proposition 6.4.7. A striking fact is that unlike attractors obtained by the enhanced Arimoto theorem and the present example, all of the attractors obtained so far in our PDNN model consist of one or a few cycles found in the graphs of Boolean isometries. Whether this observation generally holds or not is an open question.

Open Question If a self-dual minimal transformation F has an attractor Ψ such that $\mathrm{Im}\Psi \subseteq \mathrm{Car}F$, then is it true that $\Psi \subseteq \mathrm{CY}(T)$ for some Boolean isometry T?.

7.8 A temporary review

In this chapter we constructed self-dual minimal threshold transformations having attractors and developed some systematic methods and tools for proving the attractiveness. The transformations and their attractors are easily converted to those for PDNNS defined in Chapter 6. Seeing from what has been described so far about non-loop attractors, we can expect mathematically rich as well as difficult contents as a dynamical system even in the present simplified PDNN model.

However, unlike attractors obtained by the enhanced Arimoto theorem, all non-trivial attractors obtained so far in the class of minimal self-dual transformations consist of one or a few cycles found in the graphs of Boolean isometries. Here, differences from Boolean isometries do not lie in more complex cycles but in the selection of the few cycles from the cycles generated by Boolean isometries. Whether this observation generally holds or not is a mathematical open question.

The present model is autonomous, that is, stable periodic firing patterns that are represented by attractors are completely determined by the efficacy matrices of synaptic connections and the initial states of the neurons at time $t = 0$. However, the dynamics of any biological system depends on information that changes at every unit time and that is input from the outside of the system, from neurons of other nervous systems and/or from external stimulus. In autonomous models, if a minimal attractor consists of more than one cycle, then there are some fluidity of shifting from one pattern to another caused by noise, even with a change in firing rate in some cases. For example, the attractor in Example 3.7 of Chapter 6 consists of two completely different states 1111 and $-1-1-1-1$, and if the state shifts from 1111 to $11-11$, then it converges to $-1-1-1-1$. This problem may be solved only in a non-autonomous model with input from outside the network.

Further, the post-synaptic potential in this model incorporates only spatial summation and no temporal summation (See Kalat, 1995). As a result, the firing rate of any neuron cannot exceed 2 times the spontaneous firing rate. Still further, the rigid synchronization (alignment) of firing for all neurons is unrealistic. Some of these problems can be overcome by modifying the present model and extending these means, although things become more complex. The trade-off is that we have a greater variety of stable periodic patterns represented by attractors. In fact, the present combinatorial approach can be extended to a modified model with incorporation of temporal summation in Chapter 8, and some results on non-autonomous networks will be described in Chapter 9.