CHAPTER 8 ATTRACTORS IN SECOND-ORDER NEURAL NETWORKS

Abstract. Autonomous primitive dynamical neural networks (PDNNs) that incorporate temporal summation and a spontaneous firing rate of 1/3 per unit time are constructed. The PDNNs are second-order neural networks defined as $x(t+1) = \mathrm{Sgn}(Ax(t) + Bx(t-1) - l)$ such that all the diagonal elements of the efficacy matrices A and B are negative, but with additional imaginary delay neurons, they are made to be a class of McCulloch and Pitts networks $x(t) = \mathrm{Sgn}(Ex(t-1) - h)$ on $\{-1,1\}^{2n}$. PDNN- definable threshold transformations are characterized in terms of Boolean functions. The existence of minimal attractors in circular PDNNs are proved for the general dimension n by construction using [ ]-representation of Boolean transformations. These attractors include an attractive loop, an attractive 4-cycle, $3n$-cycle, two $(3/2)n$-cycles, two $2n$-cycles, and six $(2/3)n$-cycles and provide a variety of stable periodic firing patterns.

8.1 PDNNs of spontaneous firing rate 1/3

In Chapter 6, we described a primitive dynamic neural network (PDNN) model of spontaneous firing rate 1/2. Let $\mathbf{N}$ be the residue class ring $\{1, 2, .., n\}$. The state space $\{-1,1\}^{\mathbf{N}}$ denoted by $\{-1,1\}^n$ of this PDNN is a finite metric space with the integer-valued Hamming distance d_H defined by $d_H(x, y) = |\{i \mid x_i \neq y_i\}|$, where $|S|$ denotes the number of elements of the set S. The PDNN is a finite-state dynamical system (FSDS) on the state space $\{-1,1\}^n$ generated by the threshold transformation F of $\{-1,1\}^n$.

$$\begin{aligned} Fx &= \mathrm{Sgn}(Ex), \\ x(t+1) &= F(x(t)), \end{aligned}$$

where E is an $n \times n$ real matrix such that $E_{ii} = -1$ for every i. In this model, every neuron oscillates at a constant spontaneous firing rate 1/2 per unit time in its prototype, in which all non-diagonal elements of E are zero. In other words, when there is no synaptic input from other neurons, each neuron performs a neutral activity represented by a cycle of period 2. This definition of neutral activity enabled us to distinguish significant neural activity from insignificant activity and sort out a great number of loops or 2-cycles often appearing in the McCulloch and Pitts network (6.2.1).

However, the firing rate of any neuron can not exceed two times the spontaneous firing rate in this model, and this problem is due to a greater problem that any state $x(t+1)$ depends only on $x(t)$ for the given efficacy matrix E and time t. Therefore the postsynaptic potential $(Ex(t))_i$ is spatial summation. As suggested by Sherrington's classical experiment on reflex responses that a small amount of rapidly repeated taps on the knee tendon produces tendon reflex, the postsynaptic potential should also include temporal summation. Further, if the instantaneous firing rate per unit time of a neuron is recognized by a neural network, a simple and plausible mechanism is through temporal summation.

According to McCulloch and Pitts (1943, p. 22), "Temporal summation may be replaced by spatial summation. This is obvious: one need merely introduce a suitable sequence of delaying chains." Therefore, temporal summation can be expressed in the McCulloch-Pitts network by creating auxiliary imaginary delay neurons. Let us assume that a single neuron disconnected from any other neuron performs a spontaneous periodic firing of the action potentials with period 3, that is, ... $\to 1 \to -1 \to -1 \to 1 \to -1 \to -1 \to 1 \to$ ... In this sequence, the state of this neuron at time $t+1$ clearly depends on the states of this neuron at t and $t-1$.

If we want a PDNN in which $x(t+1) \in \{-1,1\}^n$ depends on t and $t-1$, and if we assume that the periodic firing of the action potentials with period 3 of any neuron, that is, ... $\to 1 \to -1 \to -1 \to 1 \to -1 \to -1 \to 1 \to$ ... is neutral, then we should create the set of imaginary delay neurons $\mathbf{N\prime} = \{n+1, n+2, ..., 2n\}$. Then the prototype DNN that is to be modified by further synaptic connections can be generated by a threshold transformation H of $\{-1,1\}^{2n}$ defined by

$$Hx = \mathrm{Sgn}(Dx - e), \quad D = \begin{bmatrix} -I & -I \\ I & O \end{bmatrix}, \quad e = \begin{bmatrix} l \\ o \end{bmatrix} \tag{8.1.1}$$

where I is the $n \times n$ identity matrix, O is the $n \times n$ zero matrix, l is the column n-vector whose every component is 1, and o is the column n-vector whose every component is 0. Fig. 1 illustrates the state transition of neurons in the prototype DNN for $n = 4$ in the case where their initial transition is $1100 \to 1010$, that is, 10101100 is the initial state for H.

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Fig. 1 Prototype DNN. | denotes the action potential and · denotes the resting potential. The first 2 columns represent the initial states at $t = 0$ and $t = 1$; The second column onward displays spontaneous firing.

Then any modified threshold transformation H of $\{-1,1\}^{2n}$ obtained by modifying the prototype DNN (8.1.1) is expressed by

$$Hx = \mathrm{Sgn}(Ex - e), \quad E = \begin{bmatrix} A & B \\ I & O \end{bmatrix}, \tag{8.1.2}$$

where A and B are $n \times n$ real matrix such that $A_{ii} = B_{ii} = -1$ for every i.

Further, in order to eliminate the threshold vector e, we can add an extra imaginary neuron $2n+1$ and define a self-dual threshold transformation F of $\{-1,1\}^{2n+1}$ as follows:

$$Fx = \mathrm{Sgn}(E^+ x), \quad E^+ = \begin{bmatrix} A & B & -l \\ I & O & o \\ o^{\mathrm{T}} & o^{\mathrm{T}} & 1 \end{bmatrix}, \tag{8.1.3}$$

where ^{T} denotes the transpose. The relation between H and F is

$$\begin{bmatrix} Hx \\ 1 \end{bmatrix} = F \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

A topology of the state space $\{-1,1\}^{2n}$ can be defined from the Hamming distance d_H on $\{-1,1\}^n$. Thus we can define PDNNs of spontaneous firing rate 1/3 as follows.

Definition 8.1.1 A second-order primitive dynamical neural network (PDNN) of spontaneous firing rate 1/3 is a parametrized FSDS on $\{-1,1\}^{\mathbf{N}_{2n}}$ with parameters

$(\alpha_{12},..,\alpha_{1n},\alpha_{21},\alpha_{23},..,\alpha_{2n},..,\alpha_{n1},..,\alpha_{nn-1},\beta_{12},..,\beta_{1n},\beta_{21},\beta_{23},..,\beta_{2n},..,\beta_{n1},..,\beta_{nn-1})$

$\in \mathbf{R}^{2n(n-1)}$ and generated by the threshold transformation H defined by (8.1.2) or F defined by (8.1.3), where A and B are $n\times n$ real matrices such that $A_{ii} = B_{ii} = -1$ for every i and $A_{ij} = \alpha_{ij}$, $B_{ij} = \beta_{ij}$ for every other i and j. The n is called the *dimension* of the PDNN.

In this definition, any point $uv \in \{-1,1\}^{2n}$ represents a state transition such that $x(t) = u$ and $x(t-1) = v$ of real neurons. The direct recursive equation for x is obtained from (8.1.1) as

$$x(t+1) = \mathrm{Sgn}(Ax(t) + Bx(t-1) - l). \tag{8.1.4}$$

Note that if $A_{ij} \cdot B_{ij} \geq 0$ for every $i \neq j$, then (8.1.4) is a special deterministic case of the discrete model constructed by Bressloff and Taylor (1991, (3.6), p. 793), since the weights $\omega_{ii}(t)$ in that were held fixed at a negative value for each i, and since an equivalent transformation can be defined by the same efficacy matrix E with the change of the state space from $\{-1,1\}^{2n}$ into $\{0,1\}^{2n}$. However, $\omega_{ii}(t)$ were made negative not because of the spontaneous release of chemical transmitters but because of relative refractory periods. Some of the examples given in later sections are in the case where $A_{ij} \cdot B_{ij} < 0$ for some $i \neq j$, which may require a different interpretation.

8.2 PDNN-definable transformations

The result of this section, which is Proposition 8.2.5, is not directly used in later sections, but the readers should familiarize themselves with the concepts, expressions, and operations defined in this section. As in the last chapter, we first give the following definition about PDNN-definable transformations.

Definition 8.2.1 If a transformation F of $\{-1,1\}^{2n+1}$ can be defined by (8.1.3) for some $n \times n$ real matrices A and B such that $A_{ii} = B_{ii} = -1$ for every i, then F is called *PDNN-definable.*

Let F_i be the component function of F defined by $F_i = p_i F$. The following proposition follows from Theorem 4.2 of Chapter 6 and (8.1.3).

Proposition 8.2.2 A self-dual threshold transformation F of $\{-1,1\}^{2n+1}$ is PDNN-definable, if and only if

(1) $\mathrm{Var}(i^- F) \leq \mathrm{Var}(F)$ for every $i \in \mathbf{N}$,
(2) $(i, i+n)F_i = F_i$ and $(i, 2n+1)F_i = F_i$ for every $i \in \mathbf{N}$,
(3) $F_i = p_{i-n}$ for every $i \in \mathbf{N}'$, and
(4) $F_{2n+1} = p_{2n+1}$.

We now describe a PDNN-definable transfpormation of $\mathbf{Q}^{2n}$ or of $\mathbf{Q}^{2n+1}$, which is respectively corresponding to H and F defined by (8.1.2) and (8.1.3). F is self dual, so that F can be represented by

$$f_i = p_i \cdot \neg F_i \tag{8.2.1}$$

for $i \in \mathbf{N}_{2n+1}$. Conversely,

$$F_i = p_i \cdot \neg f_i \vee \bar{\neg} f_i. \tag{8.2.2}$$

Lemma 8.2.3 Let $F = [f_1, ..., f_{2n+1}]$ and let f_i for some i be expressed as

$$f_i = p_i \cdot (p_j \cdot u_i \vee v_i),$$

where $j \neq i$, $u_i, v_i : \mathbf{Q}^{\mathbf{N}_{2n+1} \setminus \{i,j\}} \to \mathbf{Q}$, and $v_i \subseteq u_i$. Then $(i,j)F_i = F_i$ if and only if $\bar{\neg} v_i = \neg v_i$.

Proof. Let $g_i = p_j \cdot u_i \vee v_i$. Then

$$\begin{aligned} \neg g_i &= (\neg p_j \vee \neg u_i) \cdot \neg v_i = \neg p_j \cdot \neg v_i \vee \neg u_i \cdot \neg v_i = p_j \cdot \neg v_i \vee \neg u_i, \\ \bar{\neg} g_i &= \bar{\neg} p_j \cdot \bar{\neg} u_i \vee \bar{\neg} v_i = \neg p_j \cdot \bar{\neg} u_i \vee \bar{\neg} v_i. \end{aligned}$$

Then, by (8.2.2)

$$\begin{aligned} F_i &= p_i \cdot \neg f_i \vee \bar{\neg} f_i = p_i \cdot \neg (p_i \cdot g_i) \vee \bar{\neg}(p_i \cdot g_i) \\ &= p_i \cdot (\neg p_i \vee \neg g_i) \vee \bar{\neg} p_i \cdot \bar{\neg} g_i \\ &= p_i \cdot p_j \cdot \neg u_i \vee p_i \cdot \neg p_j \cdot \neg v_i \vee \neg p_i \cdot p_j \cdot \bar{\neg} v_i \vee \neg p_i \cdot \neg p_j \cdot \bar{\neg} u_i. \end{aligned}$$

Therefore, since $(i,j)u_i = u_i$ and $(i,j)v_i = v_i$,

$$(i,j)F_i = p_i \cdot p_j \cdot \neg u_i \vee p_i \cdot \neg p_j \cdot \bar{\neg} v_i \vee \neg p_i \cdot p_j \cdot \neg v_i \vee \neg p_i \cdot \neg p_j \cdot \bar{\neg} u_i.$$

Therefore, $(i,j)F_i = F_i$ if and only if $\bar{\neg} v_i = \neg v_i$. □

Lemma 8.2.4 Let $v, w : \mathbf{Q}^{\mathbf{N}_{2n+1} \setminus \{k\}} \to \mathbf{Q}$, and $w \subseteq v$. Then $\bar{\neg}(p_k \cdot v \vee w) = \neg(p_k \cdot v \vee w)$ if and only if $w = \neg(\bar{\neg} v)$.

Proof.

$$\begin{aligned} \bar{\neg}(p_k \cdot v \vee w) &= \neg p_k \cdot \bar{\neg} v \vee \bar{\neg} w = \neg p_k \cdot \bar{\neg} v \vee p_k \cdot \bar{\neg} w. \\ \neg(p_k \cdot v \vee w) &= (\neg p_k \vee \neg v) \cdot \neg w = \neg p_k \cdot \neg w \vee \neg v \\ &= \neg p_k \cdot \neg w \vee p_k \cdot \neg v. \end{aligned}$$

Therefore, $\bar{\neg}(p_k \cdot v \vee w) = \neg(p_k \cdot v \vee w)$ if and only if $\bar{\neg} v = \neg w$ and $\bar{\neg} w = \neg v$, that is, $w = \neg(\bar{\neg} v)$. □

Theorem 8.2.5 A self-dual threshold transformation $F = [f_1, .., f_{2n+1}]$ of $\mathbf{Q}^{2n+1}$ is PDNN-definable, if and only if

(1) $f_i = p_i \cdot (p_{i+n} \cdot p_{2n+1} \cdot u_i \vee (p_{i+n} \vee p_{2n+1}) \cdot v_i \vee \neg(\bar{\neg} v_i))$ for $i \in \mathbf{N}$, such that $\neg(\bar{\neg} v_i) \subseteq v_i \subseteq u_i$, where $u_i, v_i : \mathbf{Q}^{\mathbf{N}_{2n+1} \setminus \{i, i+n, 2n+1\}} \to \mathbf{Q}$.
(2) $f_i = p_i \cdot \neg p_{i-n}$ for $i \in \mathbf{N}'$, and
(3) $f_{2n+1} = \emptyset$.

Proof. Assume condition 1 of the present theorem. Then clearly, $(i, i+n)F_i = F_i$ for every $i \in \mathbf{N}$. $(i, 2n+1)F_i = F_i$ for every $i \in \mathbf{N}$ by Lemmas 8.2.3 and 8.2.4. Therefore, condition 2 of Proposition 8.2.2 is satisfied. We have $|\neg \bar{\neg} v_i| = 2^{2n+1} - |v_i|$ and $|v_i| \geq |\neg(\bar{\neg} v_i)|$. Therefore, $|v_i| \geq \times 2^{2n+1}$, so that $|f_i| \geq 1/2 \times 3/4 \times |v_i| + 1/2 \times 1/4 \times (2^{2n+1} - |v_i|) = 2^{2n-2} + 1/4 \times |v_i| \geq 2^{2n-2} + 1/4 \times 1/2 \times 2^{2n+1} = 2^{2n-1}$.

Therefore, $\mathrm{Var}(i^- F) \leq \mathrm{Var}(F)$, so that condition 1 of Proposition 8.2.2 is also satisfied.

Assume F is PDNN-definable. Any threshold function is unate (see Theorem 4.2.5 of Chapter 4). Also, $(i, i+n)F_i = F_i$ by condition 2 of Proposition 8.2.2. Therefore,

$$f_i = p_i \cdot (p_{i+n} \cdot p_{2n+1} \cdot u_i \vee (p_{i+n} \vee p_{2n+1}) \cdot v_i \vee w_i),$$

for every $i \in \mathbf{N}$, or

$$f_i = p_i \cdot (\neg p_{i+n} \cdot \neg p_{2n+1} \cdot u_i \vee (\neg p_{i+n} \vee \neg p_{2n+1}) \cdot v_i \vee w_i),$$

for every $i \in \mathbf{N}$, where $u_i, v_i, w_i : \mathbf{Q}^{\mathbf{N}_{2n+1} \setminus \{i, i+n, 2n+1\}} \to \mathbf{Q}$, and $w_i \subseteq v_i \subseteq u_i$. Suppose f_i takes the second form. Since F is defined by (8.1.3) with the diagonal elements of A and B being -1, $f_i = p_i \cdot (u_i \vee v_i \vee w_i)$, so that f_i takes the first form. Then the condition $(i, i+n)F_i = F_i$ in condition 2 of Proposition 8.2.2 implies $w_i = \neg(\bar{\neg} v_i)$ by Lemmas 8.2.3 and 8.2.4. Therefore, condition 2 of the present proposition is satisfied. Conditions 2 and 3 of the present theorem are respectively equivalent to conditions 3 and 4 of Proposition 8.2.2. □

Let τ denote the permutation $(1, 2, .., n)(n+1, .., 2n)$. A second-order PDNN of spontaneous firing rate 1/3 generated by a PDNN-definable H of $\mathbf{Q}^{2n}$ or F of $\mathbf{Q}^{2n+1}$ respectively defined by (8.1.2) and (8.1.3) is called *circular*, if $F\tau = \tau F$ or $H\tau = \tau H$. The PDNN is circular if and only if $F_i = \tau^{i-1} F_1$, or $f_i = \tau^{i-1} f_1$ for $i = 1, .., n$, where f_i is defined by (8.2.1). Therefore, a circular PDNN generated by F is denoted by $F = \langle f_1 \rangle$. Further, a circular PDNN generated by H is denoted by $H = \langle h^1, h^0 \rangle$, where

$$\begin{array}{rcl} h^1(x_1, .., x_{2n}) & = & f_1(x_1, .., x_{2n}, 1), \\ h^0(x_1, .., x_{2n}) & = & f_1(\neg x_1, .., \neg x_{2n}, 1). \end{array} \tag{8.2.3}$$

If $i \leq n$ and $x_i = 1$, then $(Hx)_i = 0$ if and only if $x \in \tau^{i-1} h^1$. If $i \leq n$ and $x_i = 0$, then $(Hx)_i = 1$ if and only if $x \in \tau^{i-1} h^0$.

A point x of $\mathbf{Q}^{2n}$ is expressed by uv, where u and v are points on $\mathbf{Q}^n$. Let $[uv]$ denote $\mathrm{Orb}_{\langle\tau\rangle} uv$. If D is a subset of $\mathbf{Q}^{2n}$, then let $[D]$ denote $\mathrm{Orb}_{\langle\tau\rangle} D$.

As in Chapter 7, we define the *extended representation* of a Boolean function as follows. For $H = \langle h^1, h^0 \rangle$. let $h^1|1$ and $h^0|0$ be the Boolean functions defined on $\mathbf{Q}^{\mathbf{N} \cup \mathbf{N}\prime \setminus \{i\}}$ such that

$$\begin{array}{rcl} (h^1|1)(x_2, .., x_{2n}) & = & h^1(1, x_2, .., x_{2n}), \\ (h^0|0)(x_2, .., x_{2n}) & = & h^0(0, x_2, .., x_{2n}). \end{array}$$

Definition 8.2.6 Let x be an element of $\mathbf{Q}^{2n}$ and $H = \langle h^1, h^0 \rangle$. Then the extended representation $H^{\#}$ of H is the function from $\mathbf{Q}^n$ to $\mathbf{Z}^n$ defined by

$$(H^{\#} x)_i = \begin{cases} d_{SH}(P_{\mathbf{N} \cup \mathbf{N}\prime \setminus \{i\}} \tau^{-(i-1)} x, h^1|1) & \text{if } x_i = 1 \\ d_{SH}(P_{\mathbf{N} \cup \mathbf{N}\prime \setminus \{i\}} \tau^{-(i-1)} x, h^0|0) & \text{if } x_i = 0 \end{cases}$$

for $i = 1, .., n$. It is clear

$$x_i \neq (Hx)_i \ \text{ if and only if } (H^{\#} x)_i \leq 0. \tag{8.2.4}$$

8.3 Attractive loops

In this section it is assumed that $\mathbf{Q}^{2n}$ is a metric space with the distance d_H on $\mathbf{Q}^{2n}$. In this and later sections we prove the existence of a variety of attractors in circular second-order PDNNs. The situation is more complex than in our first-order PDNNs, since a kind of symmetry due to the self-duality of generating transformations no more exists. In fact, we have to deal with the non-symmetric pair of Boolean functions h^1 and h^0 defined by (8.2.3).

Example 8.3.1 Consider the prototype DNN, where $A_{ij} = B_{ij} = 0$ for every $i \neq j$ in (8.1.2). Let the corresponding transformation of $\mathbf{Q}^{2n+1}$ be also denoted by F. Then $F_i = S_2\{\neg p_i, \neg p_{i+n}, \neg p_{2n+1}\}$, that is, $F = [f_1, .., f_{2n+1}]$,

$$\begin{aligned} f_i &= p_i \cdot (p_{i+n} \vee p_{2n+1}) \text{ for } i \in \mathbf{N}. \\ H &= < h^1, h^0 >, \quad h^1 = p_1, \quad h^0 = \neg p_1 \cdot \neg p_{i+n}. \end{aligned}$$

Example 8.3.2 Let $n \geq 3$, $A_{ij} = \epsilon$, $B_{ij} = \epsilon$ for every $i \neq j$ in (8.1.2), where $\epsilon > 3/2$. Let the corresponding transformation of $\mathbf{Q}^{2n+1}$ be F. Then

$$\begin{aligned} F_1 &= S_n\{p_2, p_3, .., p_n, p_{n+2}, p_{n+3}, .., p_{2n}\} \\ &\vee S_2\{\neg p_1, \neg p_{n+1}, \neg p_{2n+1}\} \cdot S_{n-1}\{p_2, p_3, .., p_n, p_{n+2}, p_{n+3}, .., p_{2n}\}. \end{aligned}$$

Therefore, $F = \langle f \rangle$,

$$\begin{aligned} f &= p_1 \cdot (p_{n+1} \vee p_{2n+1}) \cdot S_{n-1}\{\neg p_2, \neg p_3, .., \neg p_n, \neg p_{n+2}, \neg p_{n+3}, .., \neg p_{2n}\} \\ &\vee p_1 \cdot S_n\{\neg p_2, \neg p_3, .., \neg p_n, \neg p_{n+2}, \neg p_{n+3}, .., \neg p_{2n}\}. \end{aligned}$$

It is confirmed that f satisfies 1 of Theorem 8.2.5. Further, $H =< h^1, h^0 >$,

$$\begin{aligned} h^1 &= p_1 \cdot S_{n-1}\{\neg p_2, \neg p_3, .., \neg p_n, \neg p_{n+2}, \neg p_{n+3}, .., \neg p_{2n}\}, \\ h^0 &= \neg p_1 \cdot S_n\{p_2, p_3, .., p_n, p_{n+1}, p_{n+2}, p_{n+3}, .., p_{2n}\}. \end{aligned}$$

Theorem 8.3.3 In Example 8.3.2, (ll) and (oo) are the only loops of H. Further, they are attractors in the PDNN generated by H.

Proof. Let x be a fixed point of H, and let $x_1 = 1$. Then

$$x \notin p_1 \cdot S_{n-1}\{\neg p_2, \neg p_3, .., \neg p_n, \neg p_{n+2}, \neg p_{n+3}, .., \neg p_{2n}\};$$

otherwise $(Hx)_1 = 0 \neq x_1$. On the other hand, $(Hx)_{i+n} = x_i$ and $(Hx)_{i+n} = x_{i+n}$. Therefore, $x_i = x_{i+n}$ for every $i \in \mathbf{N}$. Therefore, at least $n+2$ of $x_1, ..., x_{2n}$ are 1. Suppose $x_j = x_{j+n} = 0$ for some j. Then $x \in \bar{\neg} f_j$, contrary to the assumption that x is a fixed point of H. Therefore $x_i = 1$ for every $i \in \mathbf{N}_{2n}$. Similarly if x is a fixed point of H and $x_1 = 0$, then $x_i = 0$ for every $i \in \mathbf{N}_{2n}$. Therefore, if x is a fixed point of H, then x is ll or oo. In fact, each of them is a fixed point.

Let $x \in U_{n-2}(ll)$. Then if $x_i = 0$ for $i \leq n$, then $(Hx)_i = 1$. If $x_i = 1$ for $i \leq n$, then $(Hx)_i = 1$. Therefore, $Hx \in U_{n-2}(ll)$ and $H^2x = ll$. Let $U_{n-1}(oo)$. Then if $x_i = 0$ for $i \leq n$, then $(Hx)_i = 0$. If $x_i = 1$ for $i \leq n$, then $(Hx)_i = 0$. Therefore, $Hx \in U_{n-1}(oo)$ and $H^2x = oo$. Therefore, (ll) and (oo) are attractors of H. □

Example 8.3.4 Let $A_{ij} = 1$, $B_{ij} = 1$ for every $i \neq j$ in (8.1.2). This PDNN is structurally stable, since $2n+1$ is odd. Let the corresponding transformation of $\mathbf{Q}^{2n+1}$ be F. Then

$$F_1 = S_{n+1}\{\neg p_1, p_2, .., p_n, \neg p_{n+1}, p_{n+2}, .., p_{2n}, \neg p_{2n+1}\}.$$

Therefore, $F = \langle f \rangle$,

$$\begin{aligned} f &= p_1 \cdot S_{n+1}\{p_1, \neg p_2, .., \neg p_n, p_{n+1}, \neg p_{n+2}, .., \neg p_{2n}, p_{2n+1}\} \\ &= p_1 \cdot \cdot S_n\{\neg p_2, .., \neg p_n, p_{n+1}, \neg p_{n+2}, .., \neg p_{2n}, p_{2n+1}\}. \end{aligned}$$

Therefore, $H =< h^1, h^0 >$,

$$\begin{aligned} h^1 &= p_1 \cdot S_{n-1}\{\neg p_2, \neg p_3, .., \neg p_n, p_{n+1}, \neg p_{n+2}, .., \neg p_{2n}\}, \\ h^0 &= \neg p_1 \cdot S_n\{p_2, p_3, .., p_n, \neg p_{n+1}, p_{n+2}, .., p_{2n}\}. \end{aligned}$$

This example has the same attractors as Example 8.3.2.

8.4 Attractive 4-cycles

In this section it is assumed that $\mathbf{Q}^{2n}$ is a metric space with the distance

$$d(uv, yz) = \max(d_H(u, y), d_H(v, z)),$$

where $u, v, y, z \in \mathbf{Q}^n$. Further, if $a \in \mathbf{Q}^n$, then let

$$U_{ij}(ab) = \{uv \mid u, v \in \mathbf{Q}^n, d_H(u, a) = i, d_H(v, b) = j\}.$$

Clearly, $U_1(ab)$ is a mutually disjoint union of $U_{10}(ab)$, $U_{01}(ab)$, $U_{11}(ab)$, and $\{ab\}$.

Example 8.4.1 Let n be even and $n \geq 4$. By modifying Example 8.3.2, let $H = \langle h^1, h^0 \rangle$,

$$\begin{aligned} h^1 &= p_1 \cdot S_{n-1}\{p_2, \neg p_3, p_4, .., \neg p_{n-1}, p_n, \neg p_{n+2}, p_{n+3}, \neg p_{n+4}, .., p_{2n-1}, \neg p_{2n}\}, \\ h^0 &= \neg p_1 \cdot S_n\{\neg p_2, p_3, \neg p_4, .., p_{n-1}, \neg p_n, \neg p_{n+1}, p_{n+2}, .., \neg p_{2n-1}, p_{2n}\}. \end{aligned}$$

Proposition 8.4.2 Let $c = 1010 \cdots 10 \in \mathbf{Q}^n$. Then

$$cc \to (\bar{\neg} c)c \to (\bar{\neg} c)(\bar{\neg} c) \to c(\bar{\neg} c) \to cc.$$

is a cycle of H defined in Example 8.4.1.

Proof. We prove the proposition for $n = 4$. In this case, $c = 1010$, and

$$\begin{aligned} h^1 &= 1 \cdot S_3\{2, \neg 3, 4, \neg 6, 7, \neg 8\}, \\ h^0 &= \neg 1 \cdot S_4\{-2, 3, -4, -5, 6, -7, 8\}. \end{aligned}$$

Then

$$H^{\#}(cc) = (0, 0, 0, 0), H^{\#}((\bar{\neg} c)c) = (4, 3, 4, 3).$$

Therefore,

$$cc \to (\bar{\neg} c)c \to (\bar{\neg} c)(\bar{\neg} c).$$

Since $\rho c = \bar{\neg} c$,

$$cc \to (\bar{\neg} c)c \to (\bar{\neg} c)(\bar{\neg} c) \to c(\bar{\neg} c) \to cc.$$

□

Theorem 8.4.3 Let $c = 1010 \cdots 10 \in \mathbf{Q}^n$ and $n \geq 4$. Then

$$A = (cc, (\bar{\neg} c)c, (\bar{\neg} c)(\bar{\neg} c), c(\bar{\neg} c), cc)$$

is an attractive cycle of H defined in Example 8.4.1.

Proof. We have $[cc] = \{cc, (\bar{\neg}c)(\bar{\neg}c)\}$ and $[(\bar{\neg}c)c] = \{(\bar{\neg}c)c, c(\bar{\neg}c)\}$. Also, $U_{10}[cc] = [(1^-c)c] \cup [(2^-c)c]$ and $U_{01}[cc] = [c(1^-c)] \cup [c(2^-c)]$. We prove the theorem for $n = 4$ without loss of generality.

(i) Consider a point x of $U_{01}((\bar{\neg}c)c)$. As shown in the proof of Proposition 8.4.2, $(H^{\#}((\bar{\neg}c)c))_i \geq 3$ for every i. Therefore, $(H^{\#}x)_i \geq 2$ for every i, so that $Hx = (\bar{\neg}c)(\bar{\neg}c)$.

(ii) Let $x \in U_{01}(cc)$. It suffices to consider $x = c(1^-c)$ or $c(2^-c)$. First, if i is even then $(H^{\#}x)_i = 1$. Specifically, $H^{\#}(c(1^-c)) = (0, 1, 1, 1)$ and $H^{\#}(c(2^-c)) = (1, 1, 1, 1)$. Therefore, $H(c(1^-c)) = (1^-c)c$ and $H(c(2^-c)) = cc$.

(iii) Let $x = uv \in U_{10}((\bar{\neg}c)c)$. It suffices to consider $x = (1^-\bar{\neg}c)c$ or $(2^-\bar{\neg}c)c$. $H^{\#}(1^-\bar{\neg}c)c) = (-3, 2, 32)$ and $H^{\#}(2^-\bar{\neg}c)c) = (3.-3, 3, 2)$. Therefore, $H((1^-\bar{\neg}c)c) = (\bar{\neg}c)(1^-\bar{\neg}c)$ and $H((2^-\bar{\neg}c)c) = (\bar{\neg}c)(2^-\bar{\neg}c)$. Therefore, $Hx \in U_{01}((\bar{\neg}c)(\bar{\neg}c))$.

(iv) Note that $|(H^{\#}x)_i| \geq 2$ for every i. Therefore, if $x \in U_{11}((\bar{\neg}c)c)$, then $Hx \in U_{01}((\bar{\neg}c)(\bar{\neg}c))$ also.

(v) Let $x \in U_{10}(cc)$. Then it suffices to consider $x = (1^-c)c$ or $(2^-c)c$. $H^{\#}((1^-c)c) = (1, -1, -1, -1)$ and $H^{\#}((2^-c)c) = (-1, 0, -1, -1)$. Therefore, $H((1^-c)c) = (\bar{\neg}c)(1^-c) \in U_{01}((\bar{\neg}c)c)$ and $H((2^-c)c) = (2^-\bar{\neg}c)(2^-c) \in U_{11}((\bar{\neg}c)c)$.

(vi) Let $x \in U_{11}(cc)$. It suffices to consider $x = (1^-c)(j^-c)$ or $(2^-c)(j^-c)$ for some j. $H^{\#}((1^-c)c) = (1, -1, -1, -1)$ by (iv), so that $(H^{\#}((1^-c)(j^-c)))_i \leq 0$ for $i = 2, 3, 4$, so that $(H((1^-c)(j^-c))_i = (H((1^-c)c))_i$ for $i = 2, 3, 4$. $H^{\#}((2^-c)c) = (-1, 0, -1, -1)$ by (iv), so that $(H^{\#}((2^-c)(j^-c)))_i \leq 0$ for $i = 1, 3, 4$, so that $(H((2^-c)(j^-c))_i = (H((2^-c)c))_i$ for $i = 1, 3, 4$. Therefore $Hx \in U_{11}((\bar{\neg}c)c)$.

Therefore, the subsets of $U_1\mathbf{A}$ are mapped as shown in the following flow subgraph.

$$
\begin{array}{ccccc}
U_{11}[cc] \cup [(2^-c)c] & & & & \\
\downarrow & & & & \\
U_{10}[(\bar{\neg}c)c] \cup U_{11}[(\bar{\neg}c)c] & \rightarrow & [c(1^-c)] \cup [c(2^-c)] & & [(\bar{\neg}c)c)] \\
 & \swarrow & & \searrow & \updownarrow \\
[(1^-c)c] & \rightarrow & U_{01}[(\bar{\neg}c)c] & \rightarrow & [cc].
\end{array}
$$

Therefore, $H(U_1\mathbf{A}) \subseteq \mathbf{A}$ and $\omega_H(U_1\mathbf{A}) = \mathbf{A}$. □

Fig. 2 illustrates the state transition of neurons in Example 8.4.1 for $n = 4$ in the case where their initial transition is $0010 \rightarrow 1110$, that is, $11100010 \in U_1\mathbf{A}$ is the initial state for H.

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Fig. 2 Attractor of Example 4.4.1. Any consecutive 4 columns from the sixth column display the attractor.

8.5 Attractive cycles for $n \equiv 2 \bmod 6$

It is assumed hereafter that $\mathbf{Q}^{2n}$ is a metric space with the distance $d(uv, yz) = 2d_H(u, y) + d_H(v, z))$, where $u, v \in \mathbf{Q}^n$. Also $H^\#$ is defined as the same way as the last section. Further, in this and the next section, n is even, $l = 1^{n/2}$, and $o = 0^{n/2}$. The cyclic permutation $(1, 2, .., n)$ is denoted by ρ.

Example 8.5.1 By modifying Example 8.3.4, let $H =< h^1, h^0 >$,

$$\begin{aligned} h^1 &= p_1 \cdot S_{n-1}\{p_2, .., p_{n/2}, \neg p_{n/2+1}, .., \neg p_n, p_{n+1}, .., p_{n+n/2}, \neg p_{n+n/2+1}, .., \neg p_{2n}\}, \\ h^0 &= \neg p_1 \cdot S_n\{\neg p_2, .., \neg p_{n/2}, p_{n/2+1}, .., p_n, \neg p_{n+1}, .., \neg p_{n+n/2}, p_{n+n/2+1}, .., p_{2n}\}. \end{aligned} \quad (8.5.1)$$

Clearly,

$$(H^\#(\bar{\neg} x))_i = \begin{cases} (H^\#(x))_i + 1 & \text{if } x_i = 1, \\ (H^\#(x))_i - 1 & \text{if } x_i = 0. \end{cases},$$

and

$$(H^\#(i^- x))_i = -(H^\#(x))_i.$$

Let $x \in \mathbf{Q}^{2n}$ and $i \le n$. Then, $x_i \cdot (Hx)_i = 0$ if and only if $(H^\# x)_i \le 0$.

Let $a = lo\rho^{-r}(lo)$. Then $(H^\# a)_i = 4(i-1) - (n-2r)$ for $1 \le i \le \in$. Therefore, if $(H^\# a)_r = 0$, then $r = (n+4)/6$, so that $n \equiv 2 \bmod 6$.

Definition 8.5.2 Let $n \equiv 2 \bmod 6$ and $r = (n+4)/6$. Let $a = lo\rho^{-r}(lo)$ and

$$S = [a, (n/2)^- a, (2n - r + 1)^- a].$$

Proposition 8.5.3 If $n \equiv 2 \bmod 6$ and $r = (n+4)/6$, then S is an invariant set of H.

Proof. We prove for $r = 2$ and $n = 8$. $H^\# a = (-4, 0, 3, 4, -3, 1, 4, 5)$, so that $Ha = \tau^2(4^- a)$. $H^\#(4^- a) = (-3, 1, 5, -4, -2, 2, 6, 6)$, so that $H(4^- a) = \tau(15^- a)$. $H^\#(15^- a) = (-5, -1, 2, 5, -4, 0, 3, 6)$, so that $H(15^- a) = \tau^2 a$. Therefore, $HS = S$. □

Theorem 8.5.4 If $n \equiv 2 \bmod 6$ and $n \ge 8$, then $H(U_1 S) \subseteq U_1 S$ and $\omega_H(U_1 S) = S$. Therefore, $\mathrm{Orb}_H S$ is an attractor.

Proof. Let $j \in \mathbf{N}/$. Since $(H^\#(a))_2 = 0$, $(H^\#(j^- a))_2 = 1$ or $(H^\#(j^- a))_2 = -1$. Let $(H^\#(j^- a))_2 = 1$. Then, $(H^\#(j^- a))_6 = 2$ and $(H^\#(j^- a))_i = (H^\# a)_i \pm 1$ for every other i. Therefore, $H(j^- a) = 2^-(Ha) = \tau(\{11, 15\} - a) \in \tau U_1(15^- a)$. Let $(H^\#(j^- a))_2 = -1$. Then, $(H^\#(j^- a))_6 = 0$ and $(H^\#(j^- a))_i = (H^\# a)_i \pm 1$ for every other i. Therefore, $H(j^- a) = 6^-(Ha) = \tau^2 a \in S$.

$(H^\#(j^-(4^- a))_i = (H^\#(4^- a))_i \pm 1$ for every i. Let $(H^\#(j^-(4^- a))_2 = 0$. Then

$$H(j^-(4^- a)) = 2^- H(4^- a) = 2^- \tau(15^- a) = \tau^2(4, 10^- a \in \tau^2 U_1(4^- a),$$

and $(H^\#(10^-(4^- a)))_2 \ne 0$. If $(H^\#(j^-(4^- a)))_2 \ne 0$, then $H(j^-(4^- a)) = H(4^- a) \in S$.

Let $(H^\#(j^-(15^- a)))_2 = 0$. Then $(H^\#(j^-(15^- a)))_6 = 1$ and $(H^\#(j^-(15^- a))_i = (H^\#(15^- a))_i \pm 1$ for every other i. Therefore, $H(j^-(15^- a)) = 6^- H(15^- a) = \tau^2(4^- a) \in S$. If $(H^\#(j^-(15^- a)))_2 \ne 0$, then $H(j^-(15^- a)) = H(15^- a) \in S$. Therefore, subsets of $U_1 S$ are mapped as shown in the following flow subgraph.

$$\begin{array}{ccccc}
\begin{array}{c}[j^-a] \\ (H^\#(j^-a))_2 = 1\end{array} & \rightarrow & [k^-(15^-a)] & \searrow & \\
\begin{array}{c}[j^-(4^-a)] \\ (H^\#(j^-(4^-a)))_2 = 0\end{array} & \rightarrow & \begin{array}{c}[k^-(4^-a)] \\ (H^\#(k^-(4^-a)))_2 \neq 0\end{array} & \rightarrow & S \\
 & & & \nearrow & \\
 & & \begin{array}{c}[j^-a] \\ (H^\#(j^-a))_2 = -1\end{array} & &
\end{array}$$

Therefore, $H(U_1 S) \subseteq S$ and $\omega_H(U_1 S) = S$. □

Proposition 8.5.5 The attractor $\mathrm{Orb}_H S$ consists of one $3n$-cycle if r is even, and two $(3/2)n$-cycles if r is odd.

Proof. $H^3 a = \tau^{3r-1} a$ follows from the proof of Theorem 8.5.4. Further, $\gcd(n, 3r-1) = \gcd(2(3r-2), 3r-1)$. Therefore, $\gcd(n, 3r-1) = 1$ if r is even, 2 if r is odd. □

Computational results for $n = 8$ expect that Theorem 8.5.4 generally holds for $U_i(S)$ for some $i \geq 2$, e.g., $i = 2, 3$ for $n = 8$. we give here the proof for $i = 2$. Details that are the same as in the proof of Proposition 8.5.3 or Theorem 8.5.4 are skipped. Note that the state space $\mathbf{Q}^{2n}$ consists of approximately 268 million 28-bit words for $n = 14$, but $\mathrm{Orb}_H S$ that is expected to be the only attractor consists of only 42 words (oo is a fixed point but not an attractor). The following proposition shows that the basin for the attractor $\mathrm{Orb}_H S$ contains $U_2 S$. The proof is tedious, so that the reader may skip it.

Proposition 8.5.6 If $n \equiv 2 \bmod 6$, then $H(U_2 S) \subseteq U_2 S$ and $\omega_H(U_2 S) = S$.

Proof. We prove for $n = 8$ and $r = 2$.

(i) Let $x = \{j, k\}^- a$ for $j, k \in \mathbf{N}'$. If $(H^\#(j^-a))_2 = 1$ and $(H^\#(k^-a))_2 = 1$, then $Hx = r^-(Ha) \in U_1 S$. If $(H^\#(j^-a))_2 = -1$ and $(H^\#(k^-a))_2 = -1$, then $Hx = 6^-(Ha) \in S$. If $(H^\#(j^-a))_2 = 1$ and $(H^\#(k^-a))_2 = -1$, then $Hx = Ha$.

Let $x = \{j, k\}^-(4^-a)$ for $j, k \in \mathbf{N}'$. If $(H^\#(j^-(4^-a)))_2 \neq 0$ and $(H^\#(k^-(4^-a)))_2 \neq 0$, then $Hx = H(4^-a)$. If $(H^\#(j^-(4^-a)))_2 = 0$ and $(H^\#(k^-(4^-a)))_2 \neq 0$, then $Hx = 2^- H(4^-a) \in U_1 S$. If $(H^\#(j^-(4^-a)))_2 = 0$ and $(H^\#(k^-(4^-a)))_2 = 0$, then $Hx = \{2, 6\}^- H(4^-a) = 10^- a \in U_1 S$.

Let $x = \{j, k\}^-(15^-a)$ for $j, k \in \mathbf{N}'$. If $(H^\#(j^-(15^-a)))_2 \neq 0$ and $(H^\#(k^-(15^-a)))_2 \neq 0$, then $Hx = Ha$. If $(H^\#(j^-(15^-a)))_2 = 0$ and $(H^\#(k^-(15^-a)))_2 \neq 0$, then $Hx = 6^- H(15^-a) \in U_1 S$. If $(H^\#(j^-(15^-a)))_2 = 0$ and $(H^\#(k^-(15^-a)))_2 = 0$, then $Hx = \{2, 6\}^- H(15^-a) = \tau(11^-(15^-a)) \in U_1 S$.

Consequently, if $x = \{j, k\}^- q$, for some $j, k \in \mathbf{N}'$ and $q \in S$, then $Hx \in U_2 S$ and $\omega_H x \subseteq S$ by Theorem 8.5.4.

(ii) Let $x = j^- a$ for $j \in \mathbf{N}$. Let $(H^\#(j^-a))_2 = -1$. Then

$$\begin{aligned} Hx &= \{6, 8+j\}^-(Ha) = \{6, 8+j\}^- \tau^2(4^-a) \\ &= \tau^2(8 + (j-2)\%8)a \in U_1 S. \end{aligned}$$

Let $(H^\#(j^-a))_2 = 1$. Then

$$\begin{aligned} Hx &= \{2, 8+j\}^-(Ha) = \{2, 8+j\}^- \tau^2(4^-a) \\ &= \tau^2\{4, 8, 8 + (j-2)\%8\}^- a = \tau^2 \tau^{-1}\{11, 8 + (j-1)\%8\}^-(15^-a) \in U_2 S, \end{aligned}$$

and $\{11, 8+(j-1)\%8\}^{-}(15^{-}a)$ is in the case (i). Let $j = 2$. Then $(H^{\#}(j^{-}x))_2 = -(H^{\#}(x))_2 = 0$ and $(H^{\#}(j^{-}x))_6 = 2$, so that $Hx = \{2, 10\}^{-}(Ha) = \{2, 8+j\}^{-}(Ha)$ as above.

Let $x = j^{-}(4^{-}a)$ for $j \in \mathbf{N}$. Let $(H^{\#}(j^{-}(4^{-}a)))_2 = 0$. Then $Hx = \{2, 8+j\}^{-}H(4^{-}a) = \{2, 8+j\}^{-}\tau(15^{-}a) = \tau\{1, 8+(j-1)\%8, 15\}^{-}a = \tau^2\{10, 8+(j-2)\%8\}^{-}(4^{-}a) \in U_2S$, and $\{10, 8+(j-2)\%8\}^{-}(4^{-}a)$ is in the case (i). If $j \neq 2$ and $(H^{\#}(j^{-}(4^{-}a)))_2 \neq 0$, then $Hx = (8+j)^{-}H(4^{-}a) \in U_1S$. Let $j = 2$. Then, $(H^{\#}(j^{-}(4^{-}a)))_2 = -(H^{\#}(4^{-}a))_2 = -1$, so that $Hx = (8+j)^{-}H(4^{-}a) \in U_1S$.

Let $x = j^{-}(15^{-}a)$ for $j \in \mathbf{N}$. Let $(H^{\#}(15^{-}a))_2 = 0$. Then $Hx = \{6, 8+j\}^{-}H(15^{-}a) = \{6, 8+j\}^{-}\tau^2 a = \tau^2\{4, 8+(j-2)\%8\}^{-}a = \tau^2(8+(j-2)\%8)^{-}(4^{-}a) \in U_1S$. If $j \neq 2$ and $(H^{\#}(15^{-}a))_2 \neq 0$, then $Hx = (8+j)^{-}H(15^{-}a) \in U_1S$. Let $j = 2$. Then, $(H^{\#}(j^{-}(15^{-}a)))_2 = -(H^{\#}(15^{-}a))_2 = 1$ and $(H^{\#}(j^{-}(15^{-}a)))_6 = 1$, so that $H(j^{-}(15^{-}a)) = 6^{-}H(15^{-}a) = 6^{-}\tau^2 a = \tau^2(4^{-}a) = S$.

Consequently, $if x = j^{-}q$ for some $j \in \mathbf{N}$ and $q \in S$, then $Hx \in U_2S$ and $\omega_H x \subseteq S$ by Theorem 8.5.4. □

8.6 Attractive cycles for $n \equiv 0$ or 4 mod 6

Example 8.6.1 By modifying Example 8.5.1, let $H = < h^1, h^0 >$,

$$h^1 = p_1 \cdot S_{n-1}\{p_2, .., p_{n/2}, \neg p_{n/2+1}, .., \neg p_n, p_{n+1}, .., p_{n+n/2-1}, \neg p_{n+n/2}, .., \neg p_{2n-1}, p_{2n}\},$$
$$h^0 = \neg p_1 \cdot S_n\{p_2, .., p_{n/2}, \neg p_{n/2+1}, .., \neg p_n, p_{n+1}, .., p_{n+n/2-1}, \neg p_{n+n/2}, .., \neg p_{2n-1}, p_{2n}\}.$$

If $a = lo\rho^{-r}(lo)$, then $(H^{\#}a)_i = 4(i-1) - (n-2(r-1))$ for $1 \leq i \leq n$. Therefore, if $(H^{\#}a)_r = 0$, then $r = (n+6)/6$, so that $n \equiv 0$ mod 6. The following theorem is similarly obtained as Theorem 8.5.4 and Proposition 8.5.5. Fig.3 illustrates the periodic firing pattern expressed by the attractor.

Theorem 8.6.2 If $n \equiv 0$ mod 6 and $r = (n+6)/6$, then $\mathrm{Orb}_H S$ defined by Definition 8.5.2 is an attractor of H in Example 8.6.1, and $\mathrm{Orb}_H S$ consists of one $3n$-cycle if r is even and two $(3/2)n$-cycles if r is odd.

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Fig. 3 Attractor of Example 8.6.1 for $n = 6$. Any consecutive 18 columns display the attractor.

Example 8.6.3 By modifying Example 8.5.1, let $H = \langle h^1, h^0 \rangle$,

$$\begin{aligned} h^1 &= p_1 \cdot S_{n-1}\{p_2, .., p_{n/2}, \neg p_{n/2+1}, .., \neg p_n, p_{n+1}, .., \\ &\quad p_{n+n/2-2}, \neg p_{n+n/2-1}, .., \neg p_{2n-2}, p_{2n-1}, p_{2n}\}, \\ h^0 &= \neg p_1 \cdot S_n\{p_2, .., p_{n/2}, \neg p_{n/2+1}, .., \neg p_n, p_{n+1}, .., \\ &\quad p_{n+n/2-2}, \neg p_{n+n/2-1}, .., \neg p_{2n-2}, p_{2n-1}, p_{2n}\}. \end{aligned}$$

If $a = lo\rho^{-r}(lo)$, then $(H^{\#}a)_i = 4(i-1) - (n - 2(r-2))$ for $1 \le i \le n/2$. Therefore, if $(H^{\#}a)_r = 0$. then $r = (n+8)/6$, so that $n \equiv 4 \bmod 6$. The following theorem is similarly obtained as Theorem 8.5.4 and Proposition 8.5.5.

Theorem 8.6.4 If $n \equiv 4 \bmod 6$ and $r = (n+8)/6$, then $\mathrm{Orb}_H S$ defined by Definition 8.5.2 is an attractor of H in Example 8.6.3, and $\mathrm{Orb}_H S$ consists of one $3n$-cycle if r is even and two $(3/2)n$-cycles if r is odd.

8.7 Attractive cycles for odd n

In this section, n is odd, $l = 1^{(n-1)/2}$, and $o = 0^{(n+1)/2}$.

Example 8.7.1 By letting $A_{1j} = B_{1j} = -1$ for every $j \le (n-1)/2$, $A_{1(n+1)/2} = B_{1(n+1)/2} = 0$, and $A_{1j} = B_{1j} = 1$ for every $j \ge (n+2)/2$ in (8.1.2), and making F circular, we obtain $F = \langle f \rangle$,

$$\begin{aligned} f \;=\; & p_1 \cdot p_{n+1} \cdot p_{2n+1} \cdot S_{n-3}\{p_2, .., p_{(n-1)/2}, \neg p_{(n+3)/2}, .., \neg p_n, \\ & p_{n+2}, .., p_{n+(n-3)/2}, \neg p_{n+(n+1)/2}, .., \neg p_{2n-1}, p_{2n}\} \\ & \vee p_1 \cdot (p_{n+1} \vee p_{2n+1}) \cdot S_{n-2}\{p_2, .., p_{(n-1)/2}, \neg p_{(n+3)/2}, .., \neg p_n, \\ & p_{n+2}, .., p_{n+(n-3)/2}, \neg p_{n+(n+1)/2}, .., \neg p_{2n-1}, p_{2n}\} \\ & \vee p_1 \cdot S_{n-1}\{p_2, .., p_{(n-1)/2}, \neg p_{(n+3)/2}, .., \neg p_n, \\ & p_{n+2}, .., p_{n+(n-3)/2}, \neg p_{n+(n+1)/2}, .., \neg p_{2n-1}, p_{2n}\}. \end{aligned}$$

Therefore, $H = \langle h^1, h^0 \rangle$,

$$\begin{aligned} h^1 \;=\; & p_1 \cdot S_{n-2}\{p_2, .., p_{(n-1)/2}, \neg p_{(n+3)/2}, .., \neg p_n, p_{n+1}, \\ & p_{n+2}, .., p_{n+(n-1)/2}, \neg p_{n+(n+3)/2}, .., \neg p_{2n}\}, \\ h^0 \;=\; & \neg p_1 \cdot S_{n-1}\{\neg p_2, .., \neg p_{(n-1)/2}, p_{(n+3)/2}, .., p_n, \neg p_{n+1}, \\ & \neg p_{n+2}, .., \neg p_{n+(n-1)/2}, p_{n+(n+3)/2}, .., p_{2n}\}. \end{aligned} \tag{8.7.1}$$

If $a = lo\rho^{-r}(lo)$, then $(H^{\#}a)_i = 4(i-1) - (n-2r-1)$ for $1 \le i \le (n-1)/2$. Therefore, if $(H^{\#}a)_r = 0$, then $r = (n-1)/6$, so that $n \equiv 1 \bmod 6$. The following theorem is similarly obtained as Theorem 8.5.4 and Proposition 8.5.5.

Definition 8.7.2 Let $a = lo\rho^{-r}(lo)$ and

$$S = [\{a, \{n+2r, 2n-r\}^- a, (2n-r)^- a, ((n+1)/2)^- a\}].$$

Theorem 8.7.3 If $n \equiv 1 \bmod 6$ and $r = (n-1)/6$, then $\mathrm{Orb}_H S$ defined by Definition 8.7.2 is an attractor of H in Example 8.7.1, and $\mathrm{Orb}_H S$ consists of two $2n$-cycles.

Example 8.7.4 By modifying Example 8.7.1, let $H = \langle h^1, h^0 \rangle$.

$$\begin{aligned} h^1 \;=\; & p_1 \cdot S_{n-2}\{p_2, .., p_{(n-1)/2}, \neg p_{(n+3)/2}, .., \neg p_n, p_{n+1}, \\ & p_{n+2}, .., p_{n+(n-3)/2}, \neg p_{n+(n+1)/2}, .., \neg p_{2n-1}, p_{2n}\}, \\ h^0 \;=\; & \neg p_1 \cdot S_{n-1}\{\neg p_2, .., \neg p_{(n-1)/2}, p_{(n+3)/2}, .., p_n, \neg p_{n+1}, \\ & \neg p_{n+2}, .., \neg p_{n+(n-3)/2}, p_{n+(n+1)/2}, .., p_{2n-1}, \neg p_{2n}\}. \end{aligned}$$

If $a = lo\rho^{-r}(lo)$, then $(H^{\#}a)_i = 4(i-1) - (n-2r+1)$ for $1 \le i \le (n-1)/2$. Therefore, if $(H^{\#}a)_r = 0$, then $r = (n+1)/6$, so that $n \equiv 5 \bmod 6$. The following theorem is similarly obtained as Theorem 8.5.4 and 8.5.5.

Theorem 8.7.5 If $n \equiv 5 \bmod 6$ and $r = (n+1)/6$, then $\mathrm{Orb}_H S$ defined by Definition 8.7.2 is an attractor of H in Example 8.7.4, and $\mathrm{Orb}_H S$ consists of two $2n$-cycles.

Example 8.7.6 By modifying Example 8.7.1, let $H = \langle h^1, h^0 \rangle$,

$$\begin{aligned} h^1 &= p_1 \cdot S_{n-2}\{p_2, .., p_{(n-1)/2}, \neg p_{(n+3)/2}, .., \neg p_n, p_{n+1}, \\ &\quad p_{n+2}, .., p_{n+(n-3)/2}, \neg p_{n+(n+1)/2}, .., \neg p_{2n-1}, p_{2n}\}, \\ h^0 &= \neg p_1 \cdot S_{n-1}\{\neg p_2, .., \neg p_{(n-1)/2}, p_{(n+3)/2}, .., p_n, \neg p_{n+1}, \\ &\quad \neg p_{n+2}, .., \neg p_{n+(n-3)/2}, p_{n+(n+1)/2}, .., p_{2n-1}, \neg p_{2n}\}. \end{aligned}$$

If $a = lo\rho^{-r}(lo)$, then $(H^{\#}a)_i = 4(i-1)-(n-(r-1)-(r-2))$ for $1 \leq i \leq (n-1)/2$ and $r \geq 2$. Therefore, if $(H^{\#}a)_r = n-2$, then $r = (n+3)/6$, so that $n \equiv 3 \bmod 6$. The following theorem is similarly obtained as Theorem 8.5.4 and Proposition 8.5.5.

Theorem 8.7.7 If $n \equiv 3 \bmod 6$ and $r = (n+3)/6$, then $\mathrm{Orb}_H S$ defined by Definition 8.7.2 is an attractor of H in Example 8.7.6, and $\mathrm{Orb}_H S$ consists of six $(2/3)n$-cycles if $2r+1$ is divisible by 3 and two $2n$-cycles otherwise.

8.8 A temporary review

The present second-order PDNNs that incorporate a spontaneous firing rate of 1/3 per unit time at least partially incorporate temporal summation in postsynaptic potentials, and the firing rate of a neuron can reach three times the spontaneous firing rate. Yet the PDNNs are still represented in the classical McCulloch and Pitts network. Attractors in the PDNNs provide stable periodic firing patterns that are expressed in terms of pulses of action potentials and have more variety than those in the PDNNs of spontaneous firing rate 1/2. Simplifications, such as putting the firing mechanism in a black box, discrete-time modeling, representation of each neuron's activity by binary values, and synchronization of firing for all neurons, are more or less inevitable for global analysis, in particular, by combinatorial methods, which reveal the richness and difficulties of the finite-state dynamical system due to non-linearity of threshold transformations. The present chapter eased the difficulties by further limiting the global description of exemplary PDNNs to circular ones, but could not get rid of tedious details. The fundamental limitation of the present PDNNs is that they are autonomous, that is, the stable periodic firing patterns that are represented by attractors are completely determined by the efficacy matrices of synaptic connections and the initial states of the neurons at time $t = 0$ and $t = 1$. In autonomous models, if a minimal attractor consists of more than one cycle, then there are some fluidity of shifting from one pattern to another caused by noise, even with a change in firing rate in some cases. This problem may be solved only in a non-autonomous model with input from outside the network.