

REFERENCES, INDEX, & SYMBOLS AND NOTATIONS

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SYMBOLS AND NOTATIONS

1_A	Characteristic function for subset A 1.4
\mathbf{B}	Set of Boolean functions: $\mathbf{Q}^n \rightarrow \mathbf{Q}$ 2.1
$Bool$	Function: $\mathbf{R}^n \rightarrow \mathbf{Q}^n$ 2.1
$CarF$	Carrier of transformation F 1.1
$CS(F)$	Cycle structure of transformation F 2.3
$CY(F)$	Set of all cycles in $GRAPH(F)$ 1.1
$d(x)$	Density of x 1.3
$d(x, y)$	Distance 1.1
$d(x, A)$	<i>Distance between a point and sets</i> 1.1
$d(A, B)$	<i>Distance between sets</i> 1.1
$d_H(x, y)$	Hamming distance 2.1
$d_R(x, y)$	Runs-based distance 7.4
$f a$	Boolean function defined by $(f a)x = f(x, a)$ 1.4
gcd	Greatest common divisor 1.2
$GRAPH(F)$	Graph of transformation F 1.1
I	Identity transformation 2.1
$Im\Psi$	Image of a set Ψ of sequences 6.1
l	Column vector whose every element is 1 4.3
\mathbf{N}_m	Residue class ring with m elements $\{1, 2, \dots, m\}$ 1.2
1^m	m – vector whose every coordinate is 1 5.5, 7.1
0^m	m – vector whose every coordinate is 0 5.5, 7.1
\mathbf{N}	Residue class ring with n elements $\{1, 2, \dots, n\}$, i.e. \mathbf{N}_n 1.2
$O(\mathbf{Q}^n)$	Group of Boolean isometries 2.1
$O(\{-1, 1\}^n)$	Group of orthogonal transformations of $\{-1, 1\}^n$ 2.1
$O(\mathbf{R}^n)$	Real orthogonal group 2.1
o	Column vector whose every coordinate is 0 6.2
$Orb_{\mathbf{G}}x$	Orbit of \mathbf{G} acting on X containing $x \in X$ 1.3
$Orb_{\mathbf{G}}S$	Union of orbits $Orb_{\mathbf{G}}x$ for $x \in S$ 1.3
Orb_Fx	Orbit starting at x in a FSDS generated by F 6.1
Orb_FS	Set of orbits starting at some $x \in S$ in a FSDS generated by F 6.1
$Orb_{F,V}x$	Orbit starting at x in a non-autonomous FSDS generated by F and V 9.2
$Orb_{F,V}S$	Set of orbits starting at some $x \in S$ in a non-autonomous FSDS generated by F and V 9.1
$PO(F, V)$	Set of all periodic orbits in the non-autonomous FSDS generated by F and V 9.2
P_L	Projection function: $\mathbf{Q}^M \rightarrow \mathbf{Q}^L$ 1.4
p_i	Projection function: $\mathbf{Q}^n \rightarrow \mathbf{Q}$ defined by $p_ix = x_i$ 1.4
\mathbf{Q}	Minimal Boolean algebra $\{0, 1\}$ 1.3, 1.4
\mathbf{Q}^n	Set of all functions from \mathbf{N} to \mathbf{Q} , i.e. n -bit binary strings $x_1 \cdot x_2 \cdots x_n$ 1.3
\mathbf{R}^n	Real n -dimensional space 2.1
RP	Number of non-overlapping runs pairs 7.4
S_m	The disjunction of all conjunctions of m Boolean functions or variables selected from $\{\cdot\}$ 4.1
$SCY(F)$	Set of all significant cycles 6.3
Sgn	Function: $\mathbf{R}^n \rightarrow \{-1, 1\}^n$ 2.1

$\text{SYM}(X)$	Symmetric group on set X 1.2
\mathbf{U}_n	Multiplicative abelian group 1.2
$U_\epsilon S$	ϵ -neighborhood of a subset S 6.1
\mathbf{V}	<i>Image of a sequence</i> V 1.1
$\text{Var}(F)$	Variation of transformation F 2.2
X^D	Set of all functions from D to X 1.1
\mathbf{Z}	Set of all integers 1.1
\mathbf{Z}_n	Residue class ring with n elements $\{0, 1, \dots, n-1\}$ 1.2
\mathbf{Z}_+	Set of non-negative integers $\{0, 1, 2, \dots\}$ 1.1
\mathbf{Z}^+	Set of positive integers $\{1, 2, \dots\}$ 1.1
ι	Identity permutation 1.2
ρ	Circular permutation $(1, 2, \dots, n)$ 1.2
$\varphi(n)$	Euler's function 1.2
$\omega_F x$	Limit set of x in the FSDS generated by F 6.1
$\omega_F A$	Union of limit sets of some $x \in A$ in the FSDS generated by F 6.1
$\Omega_{F,V} x$	Limit orbit of x in the non-autonomous FSDS generated by F and V 9.2
$\Omega_{F,V} A$	Union of limit orbits of some $x \in A$ in the FSDS generated by F and V 9.2
$ X $	Number of elements of finite set X 1.1
J^-	Complementation or inversion of coordinates 2.1
j^-	Complementation or inversion of a coordinate 2.1
$\{s, t, \dots, w\}^-$	Complementation or inversion of coordinates 2.1
(F_1, \dots, F_n)	Transformation F from \mathbf{Q}^m to \mathbf{Q}^n such that $F_i = p_i F$ 2.3
$[f_1, \dots, f_n]$	$[\]$ -representation of a self-dual transformation
$\langle \tau, \dots, \omega \rangle$	A subgroup generated by elements τ, \dots, ω of a group 1.2
$\langle f \rangle$	Circular self-dual transformation defined by $f : \mathbf{Q}^n \rightarrow \mathbf{Q}$ such that $p_1 \cdot f = f$ 2.4
$\langle h^1, h^0 \rangle$	Transformation defined by f in circular second-order DNNs 8.3 Transformation defined by h^1 and h^0 in a circular second-order DNN 8.3
$\langle\langle f \rangle\rangle$	Skew-circular transformation defined by $f : \mathbf{Q}^n \rightarrow \mathbf{Q}$ such that $p_1 \cdot f = f$ 2.4
$ f $	Number of elements of the set $f^{-1}1$ for a function f 1.4
$[x]$	The greatest integer not greater than real number x 1.2
$[x]$	$\text{Orb}_{\mathbf{G}} x$ for a given group \mathbf{G} 6.6, 7.1, 7.2, 8.2, 9.5.
$[S]$	$\text{Orb}_{\mathbf{G}} S$ for a given group \mathbf{G} 6.6, 7.1, 7.2, 8.2, 9.5.
(s_1, s_2, \dots, s_m)	Cyclic permutation 1.2
(x, y)	Inner product of x and y 4.1
X^c	Complement of a subset X 1.1
$F : X \rightarrow Y$	F is a function from X to Y 1.1
$A \rightarrow_F B$	B is the image of A under F 1.1
$x \rightarrow y$	(x, y) is an arc of a digraph 1.1
$x\partial$	(x, x) is a loop of a digraph 1.1

$F \mid A$	Transformation F restricted to A	1.1
\setminus	Difference of sets: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$	1.1
$+$	Sum of transformations	1.1
$\dot{+}$	Symmetric difference of sets: $A \dot{+} B = (A \setminus B) \cup (B \setminus A)$	1.1
\times	Cartesian product of sets	1.1
	Direct product of transformations	1.1
	Direct product of groups	1.2
\emptyset	Empty set	1.1
\circ	Composition of functions or transformations	1.1
\odot	Disjoint composition (direct sum) of transformations	1.1
\vee	Disjunction, OR	1.4
\cdot	Conjunction, AND, Product	1.4
\neg	Complementation, Negation	1.4
$(=)$	Equality operation on \mathbf{Q}^n or \mathbf{B}	1.4,
$\bar{=}$	Complementation, Negation	2.1
\sim_G	Equivalence relation with respect to an acting group G	1.3
\sim_f	Equivalence relation	4.2
\sim_F	Equivalence relation	5.2
\succeq_f	Preorder	4.2
$\equiv \text{ mod}$	$a \equiv b \text{ mod } n$ means $a - b$ is divisible by n	1.2
$\%$	$a \% b$ is the remainder obtained by dividing a by b	1.2