ON A CLASS OF THRESHOLD TRANSFORMATIONS HAVING SINGLE-CYCLE ATTRACTORS

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Abstract. The first step is to construct a class of skew-circular one-to-one threshold transformations of the Boolean n-space having single cycles of length $2n$ by means of [ ]- representations. Then $2n$-dimensional threshold transformations having single $2n$-cycle attractors are constructed through expansion and neighborhood functions. The construction of n- dimensional transformations having single $2n$-cycle attractors is carried out through partial neighborhood functions. For the proof of attractiveness, newly-introduced extended representations and decomposition of transformations are employed.

1. Introduction

The mathematical nature of neural networks is fundamentally characterized by threshold transformations. Yet we have few results on the transformations, because of the difficulties of non-linearity [1,3,4,5,6,7,8]. The author's combinatorial approach is based on the [ ]-representations of Boolean transformations introduced in [4]. In particular, [4,5,6] were concerned with construction and analysis of one-to-one threshold transformations. Dynamical systems generated by threshold transformations were the concern of [7], in which it was proved that there existed a transformation having a single-cycle attractor of any length and the basin of attraction was the whole space. In this article we construct a class of n-dimensional threshold transformations having single-cycle attractors of length $2n$. In contrast to [7], the basin of attraction is limited to a close neighborhood of the attractive cycles.

The background concepts of this preliminary section are found in good introductory textbooks on discrete mathematics such as [2,9]. Those directly relevant to this article are described in [8]. Definitions on finite-space dynamical systems are given in Appendix of this article.

The difference of sets is denoted by $\backslash$, and $A\backslash\{q\}$ is denoted by $A\backslash q$ for a one-element set. The cardinality of a set A is denoted by $|A|$. $\mathbf{N} = \{1, 2, ..., n\}$ is the residue class ring with n as the zero element. The symmetric group on $\mathbf{N}$ is denoted by SYM($\mathbf{N}$).

Let $\mathbf{Q}$ be the minimal Boolean algebra $\{0, 1\}$ with the operations, identity $I_{\mathbf{Q}}$, complementation $\neg$, AND $\cdot$, and OR $\vee$. $\mathbf{Q}^{\mathbf{N}}$, the set of functions: $\mathbf{N} \to \mathbf{Q}$, is simply denoted by $\mathbf{Q}^n$ hereafter, and $\mathbf{Q}^{\{i\}}$ is identified with $\mathbf{Q}$. Let $L \subseteq M \subseteq \mathbf{N}$. Then the *projection* $P_L : \mathbf{Q}^M \to \mathbf{Q}^L$ is defined by

$$(p_L x)_j = x_j \quad \text{for every } j \in L \text{ for every } x \in \mathbf{Q}^M.$$

If $i \in M$, then the *projection* $p_i : \mathbf{Q}^M \to \mathbf{Q}$ is defined by

$$p_i x = x_i \quad \text{for every } x \in \mathbf{Q}^M.$$

A function: $\mathbf{Q}^M \to \mathbf{Q}$ is called a *Boolean function*, and a function: $\mathbf{Q}^M \to \mathbf{Q}^M$ is called a *Boolean transformation*. For a set of Boolean functions, let $S_m\{.\}$ denote the disjunction of all conjunctions of m elements of $\{.\}$. For example, $S_2\{p_1, p_2, p_3\} = p_1 \cdot p_2 \vee p_1 \cdot p_3 \vee p_2 \cdot p_3$. The Polya action h of SYM($\mathbf{N}$) on $\mathbf{Q}^n$ associates a permutation τ of $\mathbf{N}$ with a permutation of coordinates of $\mathbf{Q}^n$ by

$$(h\tau)(x_1, x_2, .., x_n) = (x_{\tau^{-1}1}, x_{\tau^{-1}2}, .., x_{\tau^{-1}n}).$$

We omit the Polya action h hereafter and write τx in place of $(h\tau)x$ for an element x of $\mathbf{Q}^n$.

Let J^- for $J = \{s, t, .., w\} \subseteq \mathbf{N}$ denote the complementation of the sth, tth,..,wth coordinates defined by

$$J^- x = (x_1, .., \neg x_s, .., \neg x_t, .., \neg x_w, .., x_n).$$

If J is a one element-set $\{s\}$, J^- is denoted by s^-. Also $\mathbf{N}^-$ is denoted by $\bar{\neg}$. A product τJ^-, where $\tau \in$ SYM($\mathbf{N}$) and J^- is a complementation is an isometry, and the set $O(\mathbf{Q}^n)$ of all isometries is a transformation group of $\mathbf{Q}^n$. For the product of isometries,

$$(\sigma J^-)(\tau K^-) = \sigma\tau(\tau^{-1} J \dot{+} K)^-, \tag{1.1}$$

where $\dot{+}$ is the symmetric difference. Further, the Polya action of $O(\mathbf{Q}^n)$ on the set of Boolean functions defines $Tf = fT^{-1}$ for every Boolean function f and every isometry T.

We refer to the set f for a Boolean function meaning the set $f^{-1}1$, ie. the inverse image of 1. Therefore, $x \in f$ means $fx = 1$. Then $\neg f$ is the set $f^{-1}0$, and $\bar{\neg} f$ is the set of complements of points of $f^{-1}1$. We call the set of all non-fixed points of a transformation F of $\mathbf{Q}^n$ the *carrier* of F and write Car$F$. If Car$F$ and Car$G$ are disjoint, then the *sum* $F + G$ can be defined by

$$(F+G)x = \begin{cases} Fx & \text{if } x \in \mathrm{Car}F, \\ Gx & \text{if } x \in \mathrm{Car}G, \\ x & \text{if } x \in (\mathrm{Car}F) \cup \mathrm{Car}G)^c. \end{cases}$$

A transformation F is called *self-dual*, if $\bar{\neg} F = F \bar{\neg}$. A function $f : \mathbf{Q}^n \to \mathbf{Q}$ is called a threshold function, if the set f and $\neg f$ are separated by a hyperplane in the real n-space R^n. A frequently used property of a threshold function is that it is not 2-*summable*. Here, a function is called 2-*summable*, if there exist some a, b, c, d such that $a \in f$, $b \in f$, $c \in \neg f$, $d \in \neg f$, and $a + b = c + d$ where $+$ is the addition in R^n. A transformation F of $\mathbf{Q}^n$ is called a *threshold transformation* if $p_i F$ is a threshold function for every i.

Assume that the transformation $F = (F_1, ..., F_n)$ of $\mathbf{Q}^n$, where $F_i = p_i F$, is self-dual. For a point $x \in \mathbf{Q}^n$, $x_i = 1$ and $(Fx)_i = 0$ iff $x \in p_i \cdot \neg F_i$. Let f_i be defined by

$$f_i = p_i \cdot \neg F_i \tag{1.2}$$

for every i. Then

$$F_i = p_i \cdot \neg f_i \vee \bar{\neg} f_i. \tag{1.3}$$

Conversely, for any Boolean function f_i such that $f_i = p_i \cdot f_i$ for every i, let F_i be defined by (1.3). Then $F = (F_1, ..., F_n)$ is a self-dual transformation, and (1.2) is satisfied. Consequently, any self-dual transformation F such that $F = (F_1, ..., F_n)$ can be represented by

$$F = [f_1, ..., f_n].$$

Here $x = (x_1, ..., x_n) \in f_i$ if and only if $x_i = 1$ and $(Fx)_i = 0$. Therefore,

$$\mathrm{Car}F = \bigcup_{i=1}^{n} (f_i \cup \bar{\neg} f_i).$$

If F is self-dual and represented by $[f_1, ..., f_n]$, then F is a threshold transformation if and only if f_i is a threshold function for every i [4].

A transformation F of $\mathbf{Q}^n$ is called *circular*, if $F\rho = \rho F$, where ρ is the cyclic permutation $(1, 2, .., n)$ of $\mathbf{N}$. F is called *skew-circular*, if $F(\rho n^-) = (\rho n^-)F$. $F = [f_1, ..., f_n]$ is circular if and only if $f_i = \rho^{i-1} f_1$, and F is denoted by $F = \langle f_1 \rangle$. F is skew-circular, if and only if $f_i = (\rho n^-)^{i-1} f_1$, and F is denoted by $F = \langle\langle f_1 \rangle\rangle$.

Let $\langle \bar{\neg}, \rho \rangle$ and $\langle \rho n^- \rangle$ respectively denote the subgroup of $O(\mathbf{Q}^n)$ generated by $\bar{\neg}$ and ρ, and the subgroup generated by ρn^-. In this article, $[a]$ for an element a in $\mathbf{Q}^n$ denotes the orbit of $\langle \bar{\neg}, \rho \rangle$ containing a if the operating transformation is self-dual and circular; $[A]$ for a subset A of $\mathbf{Q}^n$ denotes the union of the orbits of $\langle \bar{\neg}, \rho \rangle$containing $a \in A$. Similarly, $[a]$ and $[A]$ respectively denote the orbit of $\langle \rho n^- \rangle$ containing a and the union of orbits of $\langle \rho n - \rangle$ for $a \in A$, if the operating transformation is skew-circular. Note that $\langle \bar{\neg}, \rho n^- \rangle = \langle \rho n^- \rangle$, since $(\rho n^-)^n = \bar{\neg}$. Then, a transformation $F^\sim$ of the orbit set $\{[x] \mid x \in \mathbf{Q}^n\}$ is naturally defined by $F^\sim[x] = [Fx]$.

2. Single-cycle one-to-one transformations

The simplest skew-circular one-to-one threshold transformation is

Example 2.1. $F = \langle\langle f \rangle\rangle$, where $f = p_1 \cdot \cdot p_i \cdot \cdot p_n$. The graph of F consists of one $2n$-cycle and loops.

As a generalization of Example 2.1, we now determine a condition for the transformation $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$ such that $f = p_1 \cdot q_2 \cdot ... \cdot q_n$, where $q_i = p_i$ or $\neg p_i$ for every i, to be one-to-one.

Let $f = \{x\}$, a one-element set, where $x = (1, x_2, .., x_n)$. We assume $Fx = (0, x_2, .., x_n)$. In order to determine a condition for $F(\mathrm{Car}F) = \mathrm{Car}F$, suppose $Fx = (\rho n^-)^{h-1} x$ for $0 < h - 1 < 2n$. Then

$$p_1 x = 1, 1^- x = (\rho n^-)^{h-1} x. \tag{2.1}$$

If $0 < h - 1 < n$, then

$$(0, x_2, .., x_n) = (\neg x_{n+2-h}, .., \neg x_n, 1, x_2, .., x_{n+2-(h+1)}),$$

that is,

$$\begin{aligned} \neg x_1 = 0 = \neg x_{n+2-h}, x_2 = \neg x_{n+2-(h-1)}, ..., x_{h-1} = \neg x_n, \\ x_h = 1, x_{h+1} = x_2, ..., x_n = x_{n+2-(h+1)}, \end{aligned}$$

that is,

$$\begin{aligned} x_h = 1, x_{h+(h-1)} = \alpha_h x_h,\ x_{h+2(h-1)} = \alpha_{h+(h-1)} x_{h+(h-1)}, .., \\ x_{n+2-h} = \alpha_{n+2-(h+(h-1))} x_{n+2-(h+(h-1))},\ \neg x_1 = 0 = \neg x_{n+2-h}, \end{aligned} \tag{2.2}$$

where

$$\alpha_i = \begin{cases} \neg & \text{for } n+2-(h-1) \le i \le n, \\ I_{\mathbf{Q}} \quad (\text{identity}) & \text{for } 2 \le i \le n+2-(h+1). \end{cases}$$

In particular, the number of i such that $\alpha_i = \neg$ is $h - 2$.

If $n < h-1 < 2n$, then for $h' = h - n$,

$$(0, x_2, ..., x_n) = (x_{n+2-h'}, .., x_n, 0, \neg x_2, .., \neg x_{n+2-(h'+1)}),$$

that is,

$$\begin{aligned} x_{h'} = 0,\ x_{h'+(h'-1)} = \alpha'_{h'} x_{h'},\ x_{h'+2(h'-1)} = \alpha'_{h'+(h'-1)} x_{h'+(h'-1)}, .., \\ x_{n+2-h'} = \alpha'_{n+2-(h'+(h'-1))} x_{n+2-(h'+(h'-1))},\ \neg x_1 = 0 = x_{n'+2-h'}, \end{aligned} \tag{2.2$'$}$$

where

$$\alpha'_i = \begin{cases} I_{\mathbf{Q}} & \text{for } n+2-(h'-1) \le i \le n, \\ \neg & \text{for } 2 \le i \le n+2-(h'+1), \end{cases}$$

particularly the number of i such that $\alpha_i = \neg$ is $n - h'$.

If $0 < h-1 < n$ is relatively prime with n, the system of equations (2.2) sequentially and uniquely determines x_i from $x_h = 1$ to $\neg x_1 = 0$ by step $h-1$ of their subscripts. Further, the values of x_i change $h-1$ times as the values of x_i are determined from x_h to x_1. Therefore, if $h-1$ is relatively prime with $2n$, then x_i is consistently determined for every i. Similarly, if $0 < h'-1 < n$, $h'-1$ is relatively prime with n, and $n - h'$ is even (i.e. $n < h-1 < 2n$ and $h-1$ is relatively prime with $2n$), then (2.2)' is uniquely solved. Consequently (2.1) is uniquely solved if $0 < h < 2n$ and h is relatively prime with $2n$.

Example 2.2. Let $n = 7$ and $h-1 = 9$. The equation $1^- x = (\rho 7^-)^9 = \bar{\neg}(\rho 7^-)^2 x$ for $x = (1, x_2, .., x_7)$ is

$$\begin{aligned} \neg x_1 = 0 = x_6, x_2 = x_7, x_3 = 0, x_4 = \neg x_2, \\ x_5 = \neg x_3, x_6 = \neg x_4, x_7 = \neg x_5, \end{aligned}$$

that is,

$$\begin{aligned} x_3 = 0, x_5 = \neg x_3, x_7 = \neg x_5, x_2 = x_7, \\ x_4 = \neg x_2, x_6 = \neg x_4, \neg x_1 = 0 = x_6. \end{aligned}$$

The solution is $x = 1001100$.

Now assume $0 < h-1 < 2n$ and $h-1$ is relatively prime with $2n$, and let x be the solution of (2.1). Then

$$\begin{aligned} ((\rho n^-)^{h-1})^2 x &= (\rho n^-)^{h-1}(1^- x) \\ &= (\neg x_{n-h+2}, .., \neg x_n, 0, x_2, .., x_{n-h+1}) \\ &= h^-((\rho n^-)^{h-1} x) \\ &= h^- 1^- x. \end{aligned}$$

In general,

$$(\rho n^-)^{i(h-1)} x = (1+(i-1)(h-1))^-(1+(i-2)(h-1))^- ... (1+(h-1))^- 1^- x \quad \text{for every positive integer } i. \tag{2.3}$$

Therefore, $(\rho n^-)^{i(h-1)} x$ for $i = 1, .., 2n-1$ are all different from x and $(\rho n^-)^{n(h-1)} x = \bar{\neg} x$. Let $f = \{x\}$ and $F = \langle\langle f \rangle\rangle$. Then

$$Fx = 1^- x = (\rho n^-)^{h-1} x,$$

and F is one-to-one with one $2n$-cycle. Thus we obtained the following theorem.

Theorem 2.3. Assume $0 < h-1 < 2n$ is relatively prime with $2n$. Then there exists a one-to-one transformation $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$ such that

$$\begin{aligned} f &= p_1 \cdot \alpha_2 q_2 \cdot \ldots \cdot \alpha_n q_n, \quad \text{where } \alpha_i = I_{\mathbf{Q}} \text{ or } \neg \text{ for every } i, \\ F &= (\rho n^-)^{h-1} \quad \text{on Car}F. \end{aligned}$$

In this case, f is uniquely determined by (2.2) or (2.2)'. F has one $2n$-cycle.

Any transformation F described in Theorem 2.3 is reflective. That is, there exists an isometry of order 1 such that $F^{-1} = T^{-1}FT$. See [8] for the proof. This result is consistent with the author's conjecture given in [6] that any minimal noncompressible one-to-one threshold transformation is reflective.

Example 2.4. For Example 2.2 we have $F = \langle\langle f \rangle\rangle$,

$$f = p_1 \cdot \neg p_2 \cdot \neg p_3 \cdot p_4 \cdot p_5 \cdot \neg p_6 \cdot \neg p_7.$$

The 14-cycle of F is

$$\begin{array}{ccccccccccccc} 1001100 & \rightarrow & 0001100 & \rightarrow & 0011100 & \rightarrow & 0011000 & \rightarrow & 0011001 & \rightarrow & 0111001 & \rightarrow & 0110001 \\ \uparrow & & & & & & & & & & & & \downarrow \\ 1001110 & \leftarrow & 1000110 & \leftarrow & 1100110 & \leftarrow & 1100111 & \leftarrow & 1100011 & \leftarrow & 1110011 & \leftarrow & 0110011 \end{array}$$

3. Extended representations and neighborhood functions

Before we go further, we need a tool for systematic analysis of attractiveness for various transformations. This tool is the *extended representation* of a Boolean transformation.

The *Hamming distance* d_H is defined on $\mathbf{Q}^n$ by

$$d_H(x, y) = |\{i \mid x_i \neq y_i\}|.$$

Let $d_{SH}(x, S)$ be the *signed Hamming distance* between a point x and a non-empty proper subset S of $\mathbf{Q}^n$ defined by

$$d_{SH}(x, S) = \begin{cases} d_H(x, S) & \text{if } x \notin S, \\ 1 - d_H(x, S^c) & \text{if } x \in S. \end{cases}$$

Definition 3.1. Let x be an element of $\mathbf{Q}^n$ and $F = [f_1, \ldots, f_n]$. Then the extended representation $F^\#$ of F is a function from $\mathbf{Q}^n$ to $\mathbf{Z}^n$ defined by

$$(F^\# x)_i = \begin{cases} d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}f_i) & \text{if } x_i = 1 \\ d_{SH}(P_{\mathbf{N}\setminus i}x, P_{\mathbf{N}\setminus i}\bar{\neg} f_i)) & \text{if } x_i = 0. \end{cases}$$

Clearly $|(F^\# x)_i| \leq n-1$ for every i for every x. In general, $x \in f_i$ or $x \in \bar{\neg} f_i$, if and only if $(F^\# x)_i \leq 0$. That is,

$$x_i \neq (Fx)_i \quad \text{iff } (F^\# x)_i \leq 0. \tag{3.1}$$

For example, let

$$f = p_1 \cdot S_4\{p_2, p_3, p_4, \neg p_6, \neg p_7, \neg p_8\},$$

and $F = \langle f \rangle$ be a transformation of $\mathbf{Q}^8$. Let $c = 11110000$. Then $F^\# c = (-2, 0, 2, 4, -2, 0, 2, 4)$. Therefore, $Fc = 00111100$.

For $F = [f_1, \ldots, f_n]$, let $f_i|1$ be the Boolean function defined on $\mathbf{Q}^{\mathbf{N}\setminus i}$ by

$$(f_i|1)(x_1, .., x_{i-1}, x_{i+1}, .., x_n) = f_i(x_1, .., x_{i-1}, 1, x_{i+1}, .., x_n).$$

Proposition 3.2. If $F=[f_1,...,f_n]$ then

$$Fk^- = [k^- f_1, .., p_k \cdot (\neg\bar{\neg}(f_k|1)), .., k^- f_n]. \tag{3.2}$$

Proof. Let $Fk- = [g_1, .., g_k, .., g_n]$. If $i \neq k$, then

$$\begin{array}{rcll} g_i & = & p_i \cdot \neg(Fk^-)_i & \\ & = & p_i \cdot \neg p_i(Fk^-) & \\ & = & (p_i \cdot \neg(p_i F))k^- & \\ & = & f_i k^- & \text{by (1.2)} \\ & = & k^- f_i. & \text{Polya action} \\ g_k & = & p_k \cdot \neg(Fk^-)_k & \\ & = & p_k \cdot (\neg p_k(Fk^-)) & \\ & = & p_k \cdot (\neg(p_k \cdot \neg f_k \vee \bar{\neg} f_k)k^-) & \text{by (1.3)} \\ & = & p_k \cdot ((\neg(p_k \cdot \neg f_k) \cdot \neg\bar{\neg} f_k)k^-) & \\ & = & p_k \cdot (((\neg p_k \vee f_k) \cdot \neg\bar{\neg} f_k)k^-) & \\ & = & p_k \cdot ((\neg p_k \cdot \neg\bar{\neg} f_k \vee f_k \cdot \neg\bar{\neg} f_k)k^-). & \end{array}$$

Since

$$p_k \cdot (f_k k^-) = p_k \cdot ((p_k \cdot f_k)k^-) = p_k \cdot (p_k k^-) \cdot (f_k k^-) = p_k \cdot (\neg p_k) \cdot (f_k k^-) = 0,$$

$$\begin{array}{rcll} g_k & = & p_k \cdot ((\neg p_k \cdot \neg\bar{\neg} f_k)k^-) & \\ & = & p_k \cdot p_k \cdot ((\neg\bar{\neg} f_k)k^-) & \\ & = & p_k \cdot (\neg f_k \bar{\neg})k^-) & \text{Polya action} \\ & = & p_k \cdot (\neg f_k (\mathbf{N}\backslash k)^-) & \\ & = & p_k \cdot (\neg(p_k \cdot (f_k|1))(\mathbf{N}\backslash k)^-) & \\ & = & p_k \cdot ((\neg p_k \vee \cdot(\neg(f_k|1)))(\mathbf{N}\backslash k)^-) & \\ & = & p_k \cdot (\neg p_k \vee \neg(f_k|1)\bar{\neg}) & \\ & = & p_k \cdot (\neg(f_k|1)\bar{\neg}) & \\ & = & p_k \cdot (\neg\bar{\neg}(f_k|1)). & \text{Polya action} \end{array}$$

□

Proposition 3.3. If $|f_i| \leq 2^{n-2}$ for a threshold transformation $F=[f_1,...,f_n]$, then

$$(f_i|1) \subseteq \neg\bar{\neg}(f_i|1).$$

Proof. Assume that $(f_k|1) \subseteq \neg\bar{\neg}(f_k|1)$, i.e. $\neg(f_k|1) \supseteq \bar{\neg}(f_k|1)$ does not hold. Then there exists some x such that $P_{\mathbf{N}\backslash k}x \in \bar{\neg}(f_k|1)$ i.e. $\bar{\neg}P_{\mathbf{N}\backslash k}x \in (f_k|1)$ and $P_{\mathbf{N}\backslash k}x \in (f_k|1)$. Then the 2-asummability condition for $(f_k|1)f$ requires that for any $r \in \mathbf{Q}^{\mathbf{N}\backslash k}$, either r or $\bar{\neg}r$ or both belong to $(f_k|1)f$. Therefore

$$|f_k| = |(f_k|1)| \geq 2^{n-2}+1.$$

□

The above two propositions imply the following two inequalities for $((Fk^-)^\# x)_i$.

$$(F^\# x)_i - 1 \leq ((Fk^-)^\# x)_i \leq (F^\# x)_i + 1, \qquad \text{if } i \neq k. \tag{3.3}$$

$$((Fk^-)^\# x)_k \leq (F^\# x)_k, \qquad \text{if } |f_k| \leq 2^{n-2}, \tag{3.4}$$

since $(f_k|1) \subseteq \neg\bar{\neg}(f_k|1)$. The following example suggests our construction of attractors.

Example 3.4. Let $2 \leq k \leq [n/2]$,

$$f = p_1 \cdot S_{n-k}\{p_2, p_3, .., p_n\}$$

and $F = \langle f \rangle$ be a transformation of $\mathbf{Q}^n$. The 2-cycle $C = (1...1, 0...0)$ is the unique attractor, and $U_{k-1}\mathbf{C}$ is the basin of attraction.

Analogously, we can construct transformations having attractors by modifying some of the one-to-one transformations. First we give the following general definition. Let f be a function from $\mathbf{Q}^n$ to $\mathbf{Q}$. Then we identify the ϵ-neighborhood $U_\epsilon f$ of the set f with the function under which the inverse image of 1 is the set $U_\epsilon f$. That is,

$$(U_\epsilon f)^{-1}1 = U_\epsilon f = U_\epsilon(f^{-1}1).$$

The function $U_\epsilon f$ is called a *neighborhood function* of f. Then clearly

$$U_\epsilon(f \vee g) = (U_\epsilon f) \vee (U_\epsilon g).$$

For example, if f is a one-term function, $f = q_{k1} \cdot ... \cdot q_{km}$, where $(k1, .., km)$ is a subsequence of $(1, 2, .., n)$ and $q_{ki} = p_{ki}$ or $\neg p_{ki}$, then

$$U_\epsilon f = S_{m-\epsilon}\{q_{k1}, .., q_{km}\}.$$

We consider hereafter only the case $\epsilon = 1$ for simplicity. Let $F = [f_1, ..., f_n]$ be a self-dual transformation of $\mathbf{Q}^n$. Then let $G = [g_1, ..., g_n]$ be the transformation defined by

$$g_i = p_i \cdot U_1(f_i|1). \tag{3.5}$$

Then

$$(G^\# x)_i = (F^\# x)_i - 1 \quad \text{for every } i \text{ for every } x, \tag{3.6}$$

The following proposition immediately follows from (3.6).

Proposition 3.5. A cycle of $F = [f_1, ..., f_n]$ is also a cycle of $G = [g_1, ..., g_n]$ defined by (3.5), if and only if $(F^\# x)_i \neq 1$ for every i for any point x on the cycle.

Proof. $x_i \neq (Gx)_i$ iff $(G^\# x)_i \leq 0$ by (3.1). $(G^\# x)_i = (F^\# x)_i - 1$ by (3.6). Therefore, $Gx = Fx$ iff $(F^\# x)_i \neq 1$ for every i. □

4. Construction through expansion

In Section 2 we determined a class of one-to-one skew-circular threshold transformations having single cycles. Although neighborhood functions are useful for constructing attractors, transformations generated by $p_1 \cdot U_1(f|1)$ usually do not preserve the cycles of original transformations generated by f.

In our case, let $0 < h - 1 < 2n$ and $h - 1$ be relatively prime with $2n$, and let $F = \langle\langle f \rangle\rangle$,

$$f = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n$$

be a transformation determined by Theorem 2.3, and let $f = \{c\}$. Let $G = \langle\langle g \rangle\rangle$,

$$g = p_1 \cdot S_{n-2}\{\alpha_2 p_2, \alpha_3 p_3, ..., \alpha_n p_n\}. \tag{4.1}$$

Referring to (2.3), we have

$$\begin{array}{rcll} (F^{\#}c)_i & = & 0 & \text{iff } i = 1; \\ (F^{\#}c)_i & = & 1 & \text{iff } i = h, \quad \text{since } (\rho n^-)^{h-1}c = 1^- c; \\ (F^{\#}c)_i & = & 2 & \text{iff } i = 1 + 2(h-1) = 2h-1; \\ (F^{\#}c)_i & \geq & 3 & \text{for every other } i. \end{array}$$

Therefore, by (3.6),

$$\begin{array}{ll} (G^{\#}c)_1 = -1, & (G^{\#}c)_h = 0, \quad (G^{\#}c)_{2h-1} = 1, \\ (G^{\#}c)_i \geq 2 \quad \text{for every other } i. & \end{array}$$

Therefore,

$$Gc = \{1, h\}^- c = (\rho n^-)^{2(h-1)} c, \tag{4.2}$$

and G has two cycles $\mathrm{Orb}_{(\rho n-)^{2(h-1)}}\{c, (\rho n^-)^{h-1}c\}$. This set of two cycles is an attractor. See [8] for the proof. However, we are here concerned with attractors consisting of single cycles.

One method of constructing such attractors is through expansion, which was introduced in [4]. Corresponding to any function in the class described in Theorem 2.3, we have its circular expansion E of $\mathbf{Q}^{2n}$. Specifically, for any h such that $0 < h-1 < 2n$ and $h-1$ is relatively prime with $2n$, let $E = \langle e \rangle$,

$$e = p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n \cdot \neg p_{n+1} \cdot \neg\alpha_2 p_{n+2} \cdot .. \cdot \neg\alpha_n p_{2n}, \tag{4.3}$$

corresponding to f in Theorem 2.3. Then

$$E = \rho^{h-1} \qquad \text{on } \mathrm{Car}E.$$

Then we get the transformation $G = \langle g \rangle$ of $\mathbf{Q}^{2n}$ defined by

$$g = p_1 \cdot S_{2n-2}\{\alpha_2 p_2, .., \alpha_n p_n, \neg p_{n+1}, \neg\alpha_2 p_{n+2}, .., \neg\alpha_n p_{2n}\}. \tag{4.4}$$

Let $e = \{c\}$. Then

$$C = \mathrm{Orb}_{\rho^{h-1}} c$$

is a cycle of E. $(E^{\#}c)_i$ is even for every i, since $\rho^n x = \bar{\neg} x$ for every $x \in \mathrm{Car}E$. Therefore, C is a cycle of G by Proposition 3.5.

We have

$$(E^{\#}c)_i \leq 0 \qquad \text{iff } i = 1 \text{ or } n+1,$$

since $Ec = \{1, n+1\}^- c$. Since $(\rho n^-)^{h-1}c = Ec = \{1, n+1\}^- c$,

$$c = \{1, n+1\}^- (\rho n^-)^{h-1} c.$$

Therefore, referring to (2.3),

$$(E^{\#}c)_i = 2 \quad \text{iff } i = h \text{ or } n+h.$$

Then, by (3.6),

$$\begin{array}{rcll} (G^{\#}c)_1 & \leq & -1, & (G^{\#}c)_{n+1} \leq -1, \\ (G^{\#}c)_i & \geq & 1 & \text{for every other } i, \\ (G^{\#}c)_i & = & 1 & \text{only if } i = h \text{ and } n+h. \end{array} \tag{4.5}$$

Let $k \notin \{1, n+1\}$. Then, referring to (2.3),

$$\begin{array}{lcll} ((Gk^-)^\# c)_h & = & 2 \quad \text{if } k \neq h, & \text{by (3.2)} \\ ((Gk^-)^\# c)_{n+h} & = & 2 \quad \text{if } k \neq n+h, & \text{by (3.2)} \\ ((Gk^-)^\# c)_1 & \leq & 0, & \text{by (3.3)} \\ ((Gk^-)^\# c)_{n+1} & \leq & 0, & \text{by (3.3)} \\ ((Gk^-)^\# c)_i & \geq & 1 \quad \text{for every other } i \neq k. & \text{by (3.3)} \end{array}$$

Therefore,

$$G(k^- c) = \rho^{h-1} c,$$

or

$$G(k^- c) = \rho^{h-1}((\rho^{-(h-1)}k)^- c).$$

Therefore, under $G^\sim$,

$$[k^- c] \to ... \to [i^- c] \to ... \to [c],$$

or

$$[k^- c] \to ... \to [i^- c] \to ... \to [1^- c].$$

On the other hand, referring to (4.5), we have

$$\begin{array}{lcll} ((G1^-)^\# c)_1 & \leq & (G^\# c)_1 = -1, & \text{by (3.4)} \\ ((G1^-)^\# c)_{n+1} & \leq & 0. & \text{by (3.3)} \end{array}$$

Further, since $1^-(\rho^{h-1}c)$ and c differ only at their 1st coordinates, (3.2) implies

$$((G1^-)^\# c)_h = 0 \quad \text{and} \quad ((G1^-)^\# c)_{n+h} = 0.$$

Also $((G1^-)^\# c)_i \geq 1$ for every other i by (3.3) and (4.5). Therefore,

$$G(1^- c) = \rho^{2(h-1)} c.$$

Therefore, if $x \in U_1\mathbf{C}$ then $x \in U_1\mathbf{C}$ and $\omega_G x = \mathbf{C}$. Therefore, C is an attractor of G. Thus we obtained

Theorem 4.1. Let $G = \langle g \rangle$ of $\mathbf{Q}^{2n}$ be the transformation defined by (4.4) through the expansion (4.3) of a transformation determined by Theorem 2.3. Then G has a $2n$-cycle attractor.

5. Through partial neighborhood functions

Another method of constructing single-cycle attractors is to apply partial neighborhood functions. Let $0 < h-1 < 2n$ and $h-1$ be relatively prime with $2n$, and let $G = \langle\langle g \rangle\rangle$ be defined by (4.1). Then $Gc = \{1, h\}^- c = (\rho n^-)^{2(h-1)} c$ by (4.2), and G has two cycles $\mathrm{Orb}_{(\rho n-)^{2(h-1)}}\{c, (\rho n^-)^{h-1} c\}$. In order to preserve the original one cycle such that $Gc = 1^- c$ we remove c from $(\rho n^-)^{h-1} g$, i.e. remove $(\rho n^-)^{-(h-1)} c$ from g. Note

$$(\rho n^-)^{-(h-1)} c = (1-(h-1))^- c = (2-h)^- c,$$

referring to (2.3). $(2-h)^- c$ is the only element $x \neq c$ in g such that $x \in [c]$, otherwise $G^\#$ would have more than two non-positive coordinates.

Further we consider the set

$$\{2^- c, , , n^- c\} \backslash (2-h)^- c$$

and find out some elements $i^- c \neq j^- c$ in the set such that

$$[i^- c] = [j^- c].$$

We have

$$(\rho n^-)^{k(h-1)} i^- c = (\rho^{k(h-1)} i)^- (\rho n^-)^{k(h-1)} c$$

by (1.1), and

$$(\rho n^-)^{k(h-1)} c = (1 + (k-1)(h-1))^- (1 + (k-2)(h-1))^- ... (1 + (h-1))^- 1^- c$$

by (2.3). Therefore, $(\rho n^-)^{k(h-1)} i^- c = j^- c$ implies

$$(i + k(h-1))^- (1 + (k-1)(h-1))^- (1 + (k-2)(h-1))^- ... (1 + (h-1))^- 1^- = j^-,$$

so that $k = 2$, and $(i + k(h-1))^- h^- 1^- = j^-$. Therefore,

$$(i + k(h-1)) = 1, j = h$$

or

$$(i + k(h-1)) = h, j = 1.$$

Since $j > 1$, $j = h$ and $i = 3 - 2h$. We want to remove one of $(3 - 2h)^- c$ and $h^- c$ from g, but we must decide which to remove. If we remove $(3 - 2h)^- c$ then $Gh^- c = G^2 c$. If we remove $h^- c$, then $G(3 - 2h)^- c = (3 - 2h)^- Gc$. The former is better in view of continuity.

Thus we obtained $V = \langle\langle v \rangle\rangle$ defined by

$$v = p_1 \cdot S_{n-4}(\{\alpha_2 p_2, .., \alpha_n p_n\} \backslash \{\alpha_{2-h} p_{2-h}, \alpha_{3-2h} p_{3-2h}\}) \cdot \alpha_{2-h} p_{2-h} \cdot \alpha_{3-2h} p_{3-2h}. \tag{5.1}$$

By the above removal, if $x \in v$, then $x \notin (\rho n^-)^i v$ for every $i \neq 0 \bmod 2n$.

We prove the attractiveness of the cycle by decomposition of the transformations instead of calculating the extended representations of the transformations. For the notion of the sum of transformations, refer to Section 1.

Let

$$\begin{aligned} v^{(1)} &= p_1 \cdot \alpha_2 p_2 \cdot .. \cdot \alpha_n p_n, \\ v^{(i)} &= v \cdot \neg \alpha_i p_i \quad \text{for } i \in \mathbf{N} \backslash \{1, 2-h, 3-2h\}. \end{aligned}$$

Then

$$\begin{aligned} &v = v^{(1)} \vee ... \vee v^{(n)}, \\ &v^{(i)} \cdot v^{(j)} = 0 \quad \text{for every } i \neq j, \end{aligned}$$

as clear from the above process of removing $(2-h)^- c$ and $(3-2h)^- c$ from g.

Let $V^{(i)} = \langle\langle v^{(i)} \rangle\rangle$. Then, if $x \in v^{(1)}$ then $x = c$ and

$$Vc = 1^- c \in [v^{(1)}].$$

If $x \in v^{(i)}$ for $i \in \mathbf{N} \backslash \{1, 2-h, 3-2h\}$, then $x = i^- c$, and

$$V(i^- c) = V^{(i)}(i^- c) = \{1, i\}^- c,$$

since $i^- c \in (\rho n -)^j v^{(i)}$ for only $j = 0 \bmod 2n$. Therefore,

$$V(i^- c) = (\rho n^-)^{h-1}((\rho^{-(h-1)} i)^- c \in [v^{(i-h+1)}].$$

Therefore,

$$V = V^{(1)} + V^{(2)} + ... + V^{(n)},$$

and under $V^\sim$,

$$[v^{(4-3h)}] \to [v^{(5-4h)}] \to ... \to [v^{(1)}] \to [v^{(1)}] = [c].$$

Let $C = \mathrm{Orb}_{(\rho n-)^{h-1}}c$. Then C is a cycle of V and $C = [c]$. Further, $1^{-}c \in [v^{(1)}]$, $(2-h)^{-}c \in v^{(1)}$, and $(3-2h)^{-}c \in [v^{(h)}]$. Therefore, $U_1 C = [v]$. Therefore, C is an attractor of V. Thus we obtained the following theorem.

Theorem 5.1 Let $V = \langle\langle v \rangle\rangle$ of $\mathbf{Q}^n$ be the transformation defined by (5.1) from a transformation determined by Theorem 2.3. Then G has a $2n$-cycle attractor.

6. Another class

Assume $0 < h-1 < n$ is relatively prime with odd n. Then there exists a one-to-one reflective transformation $F = \langle f \rangle$ of $\mathbf{Q}^n$ such that

$$\begin{aligned} f &= p_1 \cdot q_2 \cdot \ldots \cdot q_n, \qquad \text{where } q_i = p_i \text{ or } \neg p_i \text{ for every } i, \\ F &= \bar{\neg}\rho^{h-1} \qquad \text{on } \mathrm{Car}F. \end{aligned}$$

F has one $2n$-cycle. Starting with the class of such transformations, we can construct a class of threshold transformations having single-cycle attractors in the same methods developed here. See [8] for details.

Appendix: Finite-state dynamical system (FSDS)

Let X be a finite-element metric space with an integer-valued distance d. The distance between a point x and a non-empty subset S of X is defined by

$$d(x, S) = \min\{d(x, y) \mid y \in S\},$$

where min denotes the minimum element. If S is a non-empty subset of X then the ϵ-neighborhood of S, $U_\epsilon S$ for a positive integer ϵ is defined by

$$U_\epsilon S = \{x \mid d(x, S) \le \epsilon\}.$$

Let φ be a function from $X \times \mathbf{Z}_+$ to X. For each $t \in \mathbf{Z}_+$ (set of non-negative integers) a transformation $\varphi_t : X \to X$ is defined by $\varphi_t x = \varphi(x, t)$ for every $x \in X$. If φ_t satisfies

(1) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbf{Z}_+$,
(2) $\varphi_0 = I_X$ (the identity transformation of X),

then φ is called a *finite-state dynamical system* (FSDS) on the *statespace* X. If F is a transformation of X, F defines a function $\varphi : X \times \mathbf{Z}_+ \to X$ by

$$\varphi(x, t) = F^t x, x \in X, t \in \mathbf{Z}_+.$$

Then φ is an FSDS on X such that $\varphi_t = F^t$ and called the FSDS *generated by* F.

A sequence $V = (v_0, v_1, ...)$ in X is a function $V : \mathbf{Z}_+ \to X$. The image of V, that is, the set $\{x \mid x = v_i \text{ for some } i\}$ is denoted by $\mathbf{V}$. If Ψ is a set of sequences in X, then $\mathrm{Im}\Psi$, the image of Ψ, is $\bigcup_{V \in \Psi} \mathbf{V}$, that is, the union of the images of the sequences belonging to Ψ. A sequence $A = (a_0, a_1, ...)$ is called *cyclic*, if there exists some k such that $a_i = a_j$ for every i and j such that $i = j \bmod k$ and $a_i \neq a_j$ for every i and j such that $i \neq j \bmod k$.

The sequence $(x, Fx, F^2x, ...)$ is called the *orbit* starting at x and denoted by $\mathrm{Orb}_F x$. A cyclic orbit is identified with an element of $\mathrm{CY}(F)$, ie. the set of all cycles in the graph of F (a loop is a 1-cycle). That is one cyclic orbit obtained from another by shifting the starting point is regarded as the same. For a subset

S of X, $\mathrm{Orb}_F S$ is the set of all orbits $\mathrm{Orb}_F x$ such that $x \in S$. The limit set $\omega_F x$ x is defined by

$$\omega_F x = \{y \quad | \quad \text{For any } k \in \mathbf{Z}_+, \text{ there exists} \\ \text{some } t > k \text{ such that } y = F^t x\}.$$

$\omega_F S$ is the union of all limit sets $\omega_F x$ such that $x \in S$.

Definition A subset Φ of $\mathrm{CY}(F)$ is called *attractive* or an *attractor* in the FSDS generated by F, if there exists some ϵ-neighborhood $U_\epsilon(\mathrm{Im}\Phi)$ satisfying

(1) $F(U_\epsilon(\mathrm{Im}\Phi)) \subseteq U_\epsilon(\mathrm{Im}\Phi)$;
(2) $\omega_F(U_\epsilon(\mathrm{Im}\Phi)) = \mathrm{Im}\Phi$.

In particular, if Φ consists of one cycle, the cycle is called attractive. The *basin of attraction* for an attractor Φ is the set of all points x such that $F^k x \in \mathrm{Im}\Phi$ for some k.

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