University of Waterloo

Waterloo, Ontario

Mathematics 237

Mid-Term Test – Winter Term 2000

Duration: 1 hour 5 minutes

Date: 8 February, 2000

NO AIDS PERMITTED

NO CALCULATORS ALLOWED

Family Name: ______________________ Initials: ___ Id. Number: ___________

Signature: ________________________________

Instructor: C. B. Collins

Instructions:

1. Complete the information above.
2. Attempt all questions, in the space provided. If you require more space, use the reverse of the **preceding** page.
3. The marks for each question are indicated. Marks will be deducted for negligently presented work. Your grade will be influenced by how clearly you express your ideas, and by how well you organize your solutions.
   Justification should be provided by referring to definitions and theorems where appropriate.
4. This examination has **seven** pages. The last page is for rough work.
5. Up to 2 marks will be deducted for improper completion of this title page and for ignoring instructions.

FOR EXAMINERS' USE ONLY		
		Mark
Title page/instructions		
Question	Maximum	
1	14	
2	13	
3	13	
Total	40	

[14] 1. (i) (a) Give the precise definition of the directional derivative of a function $f(x, y)$ at the point (a, b), in a direction given by a unit vector (u, v).

(b) State a theorem which relates the directional derivative to the gradient vector.

(c) Obtain the directional derivative of $f(x, y) = 4x^3y - 3x^2y^2$ at the point $(1, 1)$, in the direction of the vector $(2, 2)$.

(d) If $g(x) = f(x, x)$, where f is defined in part (c), determine $g'(1)$.

(e) Explain why the value for $g'(1)$ obtained in part (d) does not coincide with the value of the directional derivative of part (c).

[continued...]

(ii) Show that, under the transformation given by $x = e^s, y = e^t$,

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2},$$

where on the left $u = u(x, y)$ and on the right $u = u(e^s, e^t)$.

[13] 2. Let $f(x,y) = (xy)^{6/7}$.

(a) Use the definition of derivative to show that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

(b) Deduce that f is differentiable at $(0,0)$. As part of your answer, state the definition that a function of two variables be differentiable at a point (a,b).

(c) Obtain $f_x(x,y)$ where $x \neq 0$. Show that $f_x(x,y)$ has a limiting value of $6/7$ as $(x,y) \to (0,0)$ along the curve $x = y^6$.

(d) Combining the results of parts (a) and (c), what conclusions may be drawn about the continuity of f_x at $(0,0)$?

(e) State a theorem which relates the continuity of partial derivatives to the differentiability of a function. Can this theorem be used to deduce part (b) from part (d), or part (d) from part (b)?

[13] 3. (i) Define what is meant by the statement 'the function $f(x, y)$ is continuous at (a, b)'.

(ii) Evaluate $\lim_{(x,y)\to(1,0)} (x-y)e^{x+y}$.

(iii) Consider the function, f, given by

$$f(x, y) = \frac{e^y \sin x}{x}.$$

Determine the set of points of $\mathbb{R}^2$ at which this function is defined, and explain how the definition of f may be extended to that of a function which is continuous on $\mathbf{R}^2$.

(iv) Show that the function $g(x, y) = e^{xy}$ satisfies the equation $x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} = 0$.

THIS PAGE IS FOR ROUGH WORK